Equal Covering Sets

Definition. Call a set C of sets to be an *equal covering set* of S if the elements of C are all the same size and each element of S is contained an equal number of times throughout the sets of C. We say that an equal covering set is *of order* k if all elements of C are of size k.

> **Problem.** How many equal covering sets are there of order k for the set $\{1, 2, \ldots, n\}$? How many total equal covering sets are there for the set?

For convenience, let $f(n, k)$ denote the function that counts this value.

Example. We can write down a few trivial cases:

- $f(n, 0) = 2$ (given by $\{\}$ and $\{\{\}\}$)
- $f(n, 1) = 2$
- $f(n, n) = 2$
- $f(n, k) = f(n, n-k)$
- $f(3, 2) = 2$
- $f(4, 2) = 8$

Using abbreviated notation, we may list out the actual sets for $f(4, 2)$ to be $\{\}$, $\{12, 34\}$, $\{13, 24\}$, $\{14, 23\}$, $\{12, 34, 14, 23\}$, $\{13, 24, 14, 23\}$, $\{12, 34, 13, 24\}$, $\{12, 13, 14, 23, 24, 34\}$.

Observation. We can see that the covering sets are governed by the following equation, which gives us some intuition towards counting them:

$$\begin{aligned}(\text{total elements}) &= (\text{number of sets}) \cdot (\text{size of sets}) \\ &= (\text{number of distinct elements}) \cdot (\text{times elements are included}).\end{aligned}$$

It should be noted that each tuple doesn't uniquely describe a single covering set, as there may be different permutations used. We shall customarily denote a tuple of these four elements respectively by (s, k, n, m).

Observation. The greatest number of sets contained in an equal covering set of order k is given by $\binom{n}{k}$. This tells us that the maximum number of times an element is included, denoted by m, is given by

$$\binom{n}{k} k = nm \implies m = \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

The lower value for the number of inclusions is simply just 0. This may motivate going casewise on inclusions for counting these sets.

Observation. For all valid tuples (s, k, n, m), m must be a multiple of $k' = k/\gcd(n, k)$.

Proof. We have that

$$sk = nm,$$

and letting $k' = k/\gcd(n, k)$, $n' = n/\gcd(n, k)$, we have that

$$s = \frac{n'm}{k'}.$$

Since $\gcd(n', k') = 1$ and s is a natural number, clearly $k' \mid m$. ■

This allows us to decompose the value of $f(n, k)$ into a sum that goes casewise based on m:

$$f(n, k) = \sum_{\substack{0 \leq m \leq \binom{n-1}{k-1}, \\ k' \mid m}} g(nm/k, k, n, m),$$

for some (unknown) function $g(s, k, n, m)$.

Observation. We may brute force these values; although as the inputs grow, the time complexity grows rather quickly. In particular, we may generate all $\binom{n}{k}$ possible k-sized subsets of $\{1, 2, \ldots, n\}$. Then any equal covering set is a valid s-sized subset of these $\binom{n}{k}$ sets. A naive approach is to generate all of these subsets and only calculate the valid ones. Thus the time complexity is given by

$$g(s, k, n, m) \in O\left(\binom{\binom{n}{k}}{s} sk\right),$$

which is rather ugly but it shall do. We shall sum this over all possible values of s (determined earlier).

n	$f(n, k)$'s	Sum of $f(n, k)$
1	$2, 2$	4
2	$2, 2, 2$	6
3	$2, 2, 2, 2$	8
4	$2, 2, 8, 2, 2$	16
5	$2, 2, 14, 14, 2, 2$	36
6	$2, 2, 172, 3436, 172, 2, 2$	3788

Figure 1. Calculated values for small values of n and all values of k.

Observation. If we're counting the number of distinct equal covering sets, then summing all values of $f(n,k)$ over k is actually incorrect because we count $\{\}$ like k times. So the real value should be the sum minus $k-1$.

Observation. There exists a natural correspondence between equal covering sets and symmetric polynomials in n variables. In particular, we can see that $f(n,k)$ is equal to the number of symmetric polynomials of the form

$$\sum_{\sigma\in P} x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(k)} \in R(x_1, x_2, \dots, x_n),$$

where P is some set of permutations such that the resulting sum is subject to the constraint that each x_i appears an equal number of times. While this doesn't really change the overall structure, it is potentially a good mental model. This does beg the question of whether there is a more general structure that has this property because we can also find other examples of things that have this sort of flavor (for example, bases of roots of unity adding to 0).

Idea. Can we show that P must be a subgroup of S_n? Perhaps it would help to work out some examples.

Example. Let's go through all equal covering sets for $n = 4$ and turn them into