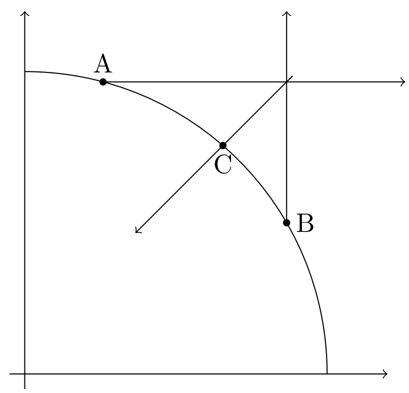
Convergent Circle Corners

Circle Areas

Given two points on the first quadrant of unit circle A, B, we may define a new unique point $C = A \oplus B$ through the following: extend a horizontal line through the y-coordinate of the point with the smaller x-coordinate. Next, extend a vertical line through the x-coordinate of the point with the larger x-coordinate. Find the point of intersection between these two lines and draw a line with slope 1 running through this point. The new point C shall be the intersection of this line with the unit circle in the first quadrant.

More explicitly, suppose that $A = (x_1, y_1), B = (x_2, y_2), x_1 \leqslant x_2$. Then we have that

$$C = \left(\frac{x_2 - y_1 + \sqrt{2 - (x_2 - y_1)^2}}{2}, \frac{y_1 - x_2 + \sqrt{2 - (y_1 - x_2)^2}}{2} \right).$$



With this, we can begin to build up our circle corner converging series. We can build this recursively, starting with an initial condition. For $n \geqslant 0$, define S_n to be the ordered set (according to x-coordinate) of all corner points lying on the unit circle quadrant. We let $S_0 = ((0,1),(1,0))$ to start with and then use the following recursive definition: suppose $S_n = (P_1, P_2, \ldots, P_k)$; we have

$$S_{n+1} = (P_1, P_1 \oplus P_2, P_2, \ldots, P_{k-1}, P_{k-1} \oplus P_k, P_k).$$

One can confirm that for $n \geqslant 1$, there are $k = 2^n + 1$ points in S_n.

We now desire the area of the converging corners shape. Fix a specific n and look at the ordered set of points $S_n = ((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k))$. We then have that the total area is

$$A_n = 4 \sum_{i=1}^{k-1} y_i (x_{i+1} - x_i).$$

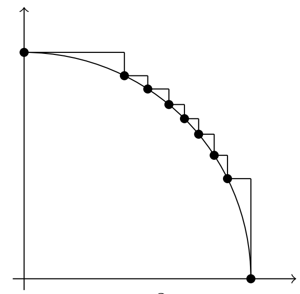
We assert that this area approaches π as $n \to \infty$, but I can't properly show this right now.

Now that we have mathematically formulated this, perhaps it would be a good exercise to code it. As for how to formally show that this approaches π, I'll have to think about it (and perhaps try a different approach). For now, it will do good to verify it empirically though. Certainly the previous series looks suspiciously like an integral (in fact it is called the Darboux integral), so perhaps our proof shall rely upon this.

In coding a rough example, we can see empirically that the area does roughly approach π, albeit a bit slowly considering the number of points. We have

n	Points	Area
1	3	3.6568542494923806
2	5	3.447477652576853
3	9	3.324791968290031
4	17	3.2536764558450173
5	33	3.2121295100035105
$\vdots$	$\vdots$	$\vdots$
20	1048577	3.1430271877389186

I'm not quite sure how to express a good closed form for any of this, but certainly we can prove that the maximum difference between the x coordinates of the points goes to 0, but I'm not sure whether this is any help for proving that the Darboux integral approaches the Riemann integral or anything of the sort.



$n = 3$

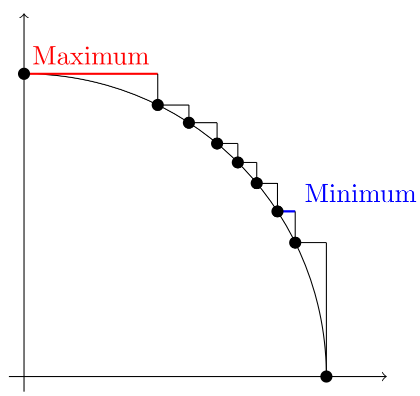
Darboux goes to Riemann?

The quickest explanation of the Darboux integral is that it is equivalent to the definition of a Riemann integral except for the fact that the x differences between each point don't necessarily have to be equal. We instead define a partition of the interval that we're integrating over and tile this interval with not necessary equal lengths. In practice, this makes them easier to work with in real analysis to prove whether things are Riemann integrable, but we're using them for a slightly different purpose it seems.

As for whether the values of Darboux integrals and Riemann integrals converge to each other as $n \to \infty$, it's a bit hard to say. Intuitively, one would think that they would, but this isn't the first time that real analysis has Looney Tooned people, so I'm going to be a bit wary. Wikipedia was a bit vague in stating this fact as well, with the citation coming from a large real analysis textbook that might be a bit above my pay grade, so we're going to do some street mathematics.

In order to satisfy me that these two sums are equal, I'm going to see what the ratio of the maximum x distance and the mininum x distance in the partition at $n \to \infty$ looks like. Supposing that the ratio is 1, one could use the sandwich theorem to then assert that all the distances would be equal, thus giving us a Riemann integral. So now all that's left to determine is this ratio.

I state without much formal proof that the maximum x difference found in our partition is found between the first and second points in the partition. This one perhaps makes a bit of intuitive sense due to how the points are spread out.



The minimum case is a bit more wacky, and I had to test it out empirically to find out whether there was a real pattern. Doing so, one can build the following table for n.

n	Index of Min
1	1
2	2
3	6
4	12
5	24
6	56
7	112
8	224
9	448
10	960
11	1920
12	3840
13	7936
14	15872
15	31744
16	63488
17	129024
18	258048
19	516096
20	1040384

Every so often (it appears to be when $n \equiv 3, 6 \pmod 7$), the index of the minimum jumps around, while for the others it simply doubles (corresponding to the same point as the previous n given the doubling nature of the points through each iteration). The size of these jumps appeared to be powers of 8, but the $n = 20$ case destroys this pattern.

Even if one were to be able to determine the exact index of the minimum, it seems that it would be quite cumbersome to find the actual minimum length described by it and the point in front of it, which is really not nice of it.

While we certainly can't do this for the minimum length, we can try it on some easier to work with lengths. This doesn't really prove a lot (we can't use the same sandwich theorem consequence really), but it seems to be the only way to proceed from here, so it might help to try it. As such, pick our "minimum" to be the distance between the exact middle point (which is at $(\cos(\pi/4), \sin(\pi/4))$) and the point in front of it. This makes things a bit easier to calculate and work with.

Denote G_n the second point in the ordered set. We have the following recursion:

$$G_{n+1} = (0, 1) \oplus G_n.$$

Something similar follows for our other extreme. Denote L_n the point after the midway point in the ordered set. We have:

$$L_{n+1} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \oplus L_n.$$

We can easily extract the extreme lengths now by simply subtracting the x coordinates of these points with that of the point directly before them. Let g_n denote the x component of G_n and l_n denote the x component of L_n minus $1/\sqrt{2}$. We desire the following limit:

$$\lim_{n\to\infty} \frac{l_n}{g_n}.$$

Now we must solve the recurrences.

Obtaining g_n, l_n

Let us use the explicit definition of the "cornering" operator. Suppose $G_n = (x_n, y_n)$; then

$$(x_{n+1}, y_{n+1}) = \left(\frac{x_n - 1 + \sqrt{2 - (x_n - 1)^2}}{2}, \frac{1 - x_n + \sqrt{2 - (1 - x_n)^2}}{2}\right),$$

where $(x_0, y_0) = (1, 0)$ and $g_n = x_n$. We may attempt to solve this recurrence as follows:

$$\begin{aligned}
g_{n+1} &= \frac{1}{\sqrt{2}}\left(\frac{g_n - 1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sqrt{1 - \left(\frac{g_n - 1}{\sqrt{2}}\right)^2} \\
\sqrt{2}\sin(\theta_{n+1}) + 1 &= \frac{1}{\sqrt{2}}\sin(\theta_n) + \frac{1}{\sqrt{2}}\cos(\theta_n) \\
&= \sin(\theta_n + \pi/4) \\
\implies \sin(\theta_{n+1}) &= \frac{1}{\sqrt{2}}\left(\sin(\theta_n + \pi/4) - 1\right),
\end{aligned}$$

where on the second line we make the trig substitution $\sin(\theta_n) = (g_n - 1)/\sqrt{2}$. From here, it's hard to proceed. The same case happens for when we try to solve for l_n, unfortunately, so I'm rather at a loss for how to do this. It seems that proving that the Darboux integral goes to the Riemann integral with this approach isn't looking quite promising.

Empirical Limits

That's right, we're back to the even more street mathematics called applied mathematics where mathematical results are as good as true if we see a pattern in what the funky computer box outputs. This mathematics has gone so much to the streets, that I would like to affectionately call it gang mathematics.

That being said, the ratio doesn't seem to approach 1 as n increases (in fact, it seems to approach 0 if anything), which is certainly rather confusing. Perhaps this may be a consequence of me only being able to input $n = 20$ before the program drastically slows down, but I can't say for certain.

Conclusion to Darboux and Riemann Integrals

I don't know a whole lot about real analysis, but I guess one has to assume that this specific Darboux integral value eventually converges onto Riemann integral value as $n \to \infty$.

Conclusion

It seems unlikely that a better closed form will be possible other than the sum for A_n (or the integral of $\sqrt{1 - x^2}$ if you truly believe the sum approaches the integral), but ultimately this view may be a consequence of how I tackled the problem. There could be some clever trick to reduce the problem to something easier to solve, but so far I cannot deduce anything of such likes.

This was quite a cool problem though.