Local Graph Derangements

I came up with this graph theory problem a while ago, so I thought I'd revisit it or at the very least just write it down to archive it better.

Definition. Given a graph $G = (V, E)$, we call a *local derangement* to be a graph isomorphism $\phi\colon V \to V$ in which for every vertex $v \in V$, v is adjacent to (but never equal to) $\phi(v)$.

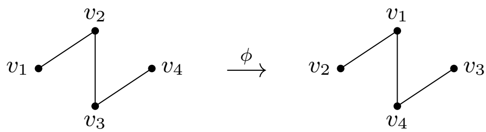

Figure 1. An example of a local derangement.

Definition. We call the *local derangement count* of an undirected graph G to be the number of distinct local derangements that exist for G. We denote this by $\pi(G)$.

We shall define the local derangement count of the empty graph to be 0.

> **Problem.** For some graph G, what is its local derangement count? Is there a general algorithm to find it?

Observation. The local derangement count of G is the product of the local derangement counts of its connected components. In other words, since we can decompose any graph G into its maximal connected components $G_1, G_2, \ldots, G_n$, we have that

$$\pi(G) = \pi(G_1 \cup G_2 \cup \cdots \cup G_n) = \prod_{i=1}^{n} \pi(G_i).$$

With this assertion, we need only concern ourselves with connected graphs now. In my previous work, I wanted to go through specific families of graphs and gave a closed form for each of them. Perhaps I will do this once I have time, but I'm more interested in the question of the general case, as it seems like it would be a fun programming problem.