# Circle Projection of Polygons

**Definition.** We call a polygon *simple* if it is non-degenerate and non-intersecting.

**Definition.** Suppose we have a simple polygon $P = \overline{A_1 A_2 \ldots A_n}$ and a circle $\omega$ with center $O$. Define the *circle projection* of $P$ onto $\omega$ by

$$\Gamma(P, \omega) = \overline{B_1 B_2 \ldots B_n},$$

where $B_k = \overrightarrow{OA_k} \cap \omega$.

> **Problem.** For every simple polygon $P$, does there exist a circle $\omega$ such that $\Gamma(P, \omega)$ is convex and simple?

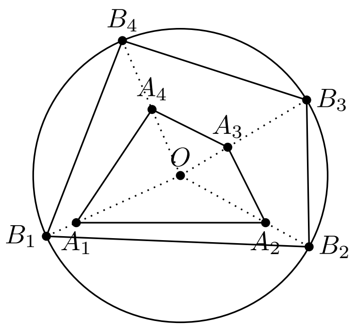


**Figure 1.** A diagram showing the principle idea of the circle projection.

I'm not all too much of a geo guy (not by preference but just by what I've experience in), but this was an interesting surprise of a problem that I thought of while going to bed after USACO.

The key question one must tackle here is the case when $P$ is concave (if $P$ is convex, then trivially the circle projection is convex too, although I

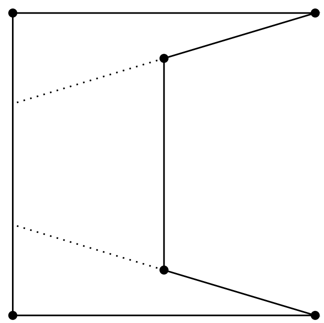

**Figure 2.** An example of a construction such that $\Gamma(P, \omega)$ cannot possibly be non-intersecting.

can't formalize this at the moment perhaps due to a lack of experience). We can pretty assuredly answer this in the negative with a following example, so the goal narrows down to being able to prove as such with a counterexample.

The reason why we can answer in the negative is contained in Figure 2. The intuition behind this (intuition is all I really can give at this stage because I'd need to think a whole lot more on how to actually formalize this; man I really don't know enough geometry) is that, in order for the projected polygon to be non-intersecting, the placement of the center $O$ (the radius actually does not matter at all as far as I can tell) needs to be such that the clockwise/counterclockwise order of the vertices from $O$ should correspond to the actual order of the points (or like a modular shift of them but you probably get the idea).

When we have a concave dip in the shape, it closes off a region (given by the extension of the segments connected to that concave dip point) that the point must be in, otherwise we can't go in clockwise/counterclockwise order. By creating two concave dips with disjoint regions, we immediately win.

Perhaps I shall attempt an actual proof when I learn more.