A Small Little Three Variable Inequality

Problem. For real numbers a, b, c such that $a \leq b \leq c$, prove that

$$a^2 + ac + c^2 \geq 3b(a - b + c).$$

Proof. Bring everything over to one side. We must prove that

$$a^2 + c^2 + 3b^2 + ac - 3bc - 3ab \geq 0.$$

From the trivial inequality, one has that $(b - a)^2 = b^2 - 2ab + a^2 \geq 0$ and also $(c - b)^2 = c^2 - 2bc + b^2 \geq 0$. Substituting these into the inequality we get that

$$(b - a)^2 + (c - b)^2 - bc + b^2 - ab + ac \geq 0.$$

From here we can factor a little bit. Notice that $-bc + b^2 = -b(c - b)$ and $-ab + ac = a(c-b)$. Adding these we get that $-bc+b^2-ab+ac = (a-b)(c-b) = -(b-a)(c-b)$, so

$$(b - a)^2 + (c - b)^2 \geq (b - a)(c - b).$$

But this follows directly from AM-GM, where we have

$$(b - a)^2 + (c - b)^2 \geq 2\sqrt{(b - a)^2(c - b)^2} = 2(b - a)(c - b).$$

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One must remark that the inequality in the problem actually isn't needed, which is rather odd.