Binomial Matrices

> **Idea.** Consider $(n+1) \times (n+1)$ matrices of the form $A_{ij} = \binom{i-1}{j-1}$, with the convention that $\binom{i}{j} = 0$ for $j > i$. Denote these matrices by B_n. What kind of properties do these matrices (and perhaps similar ones) have?

Example. The first few such matrices are

$$B_0 = \begin{bmatrix} 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Observation. Immediately we can see that by definition such matrices are lower unitriangular, as for any such matrix A, $A_{ij} = 0$ for $j > i$ and $A_{ii} = \binom{i-1}{i-1} = 1$. This is also obvious from the examples.

With this structure comes a few properties:

- $\det B_n = 1$.
- $\det(B_n - \lambda I) = (\lambda - 1)^{n+1}$.
- B_n is not diagonalizable.

Question. What is $B_n B_n^T$?

Idea. Using Mathematica, we can gain the intuition that the answer should be $(B_n B_n^T)_{ij} = \binom{i+j-2}{i-1} = \binom{i+j-2}{j-1}$. We can prove this is true by observing that

$$\begin{aligned} (B_n B_n^T)_{ij} &= \sum_{k=0}^{n} (B_n)_{ik} (B_n^T)_{kj} \\ &= \sum_{k=1}^{n} \binom{i-1}{k-1} \binom{j-1}{k-1} \\ &= \sum_{k=0}^{\min(i-1,j-1)} \binom{i-1}{k} \binom{j-1}{j-1-k}. \end{aligned}$$

If we WLOG let $i \geq j$, then Vandermonde's identity tells us that $(B_n B_n^T)_{ij}$ indeed is $\binom{i+j-2}{j-1}$.

Question. What is the inverse of B_n?

Idea. Since B_n is unitriangular, it's very easy to work out some small examples, so let us do so. For B_3, we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 1 & 1 & 0 & 0 & b_2 \\ 1 & 2 & 1 & 0 & b_3 \\ 1 & 3 & 3 & 1 & b_4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 - b_1 \\ 1 & 2 & 1 & 0 & b_3 \\ 1 & 3 & 3 & 1 & b_4 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & 0 & b_3 - 2b_2 + b_1 \\ 1 & 3 & 3 & 1 & b_4 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & 0 & b_3 - 2b_2 + b_1 \\ 0 & 0 & 0 & 1 & b_4 - 3b_3 + 3b_2 - b_1 \end{array}\right],$$

which I suppose is somewhat obvious. This tells us that

$$B_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix},$$

which allows one to guess that $(B_n^{-1})_{ij} = (-1)^{i+j}(B_n)_{ij} = (-1)^{i+j}\binom{i-1}{j-1}$. Let's try and prove this. We have that

$$(B_n B_n^{-1})_{ij} = \delta_{ij},$$

so we must have that

$$\begin{aligned} (B_n B_n^{-1})_{ij} &= \sum_{k=1}^{n+1} (B_n)_{ik}(B_n^{-1})_{kj} \\ &= \sum_{k=1}^{n+1} (-1)^{k+j} \binom{i-1}{k-1}\binom{k-1}{j-1} \\ &= \sum_{k=0}^{n} (-1)^{k+1+j} \binom{i-1}{k}\binom{k}{j-1} \\ &= \binom{i-1}{j-1}(-1)^{1+j} \sum_{k=0}^{n} (-1)^k \binom{i-1-(j-1)}{k-(j-1)}. \end{aligned}$$

For $j > i$, this is trivially 0, so we'll assume from now on that $j \leq i$.

$$\begin{aligned}
(B_n B_n^{-1})_{ij} &= \binom{i-1}{j-1}(-1)^{1+j} \sum_{k=j-1}^{i-1} (-1)^k \binom{i-1-(j-1)}{k-(j-1)} \\
&= \binom{i-1}{j-1}(-1)^{1+j} \sum_{k=0}^{i-j} (-1)^{k+j-1} \binom{i-j}{k} \\
&= \binom{i-1}{j-1} \sum_{k=0}^{i-j} (-1)^k \binom{i-j}{k} \\
&= \binom{i-1}{j-1}(1-1)^{i-j},
\end{aligned}$$

which is 0 for $j < i$ and 1 only for $j = i$, which is what we wished to show.

Remark. Knowing the inverse actually allows one to calculate a formula for the number of derangements of n objects without using the inclusion-exclusion principle, denoted by D_n. In particular,

$$\begin{bmatrix} D_0 \\ D_1 \\ \vdots \\ D_n \end{bmatrix} = B_n^{-1} \begin{bmatrix} 0! \\ 1! \\ \vdots \\ n! \end{bmatrix},$$

which shows that

$$\begin{aligned}
D_n &= \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$