Drawing Equipotential Curves

This is a small little description of how I would try to draw equipotential curves, specifically for **two dimensional** vector fields. We shall first start by laying the groundwork of vector fields and potentials.

> **Definition.** Let $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. We call the field's *potential function* $P\colon \mathbb{R}^n \to \mathbb{R}$ the unique function such that:
>
> $$\mathbf{F} = -\nabla P.$$

It is from this definition that the usual physics properties of the potential arise, and it also gives one a nice geometric intuition (field lines follow the potential downhill). In particular, we have that

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = \int_{\mathbf{b}}^{\mathbf{a}} \nabla P \cdot d\mathbf{l} = P(a) - P(b),$$

which follows from the Gradient Theorem.

It is also from this definition that another key property of potentials arises, one that is very useful in attempting to draw them.

> **Claim.** At all points across an equipotential surface, the field lines are orthogonal to the surface.

Proof. Suppose we have a potential function P and we are looking in the neighborhood of some point $\mathbf{r}$. Any point on the equipotential surface in the neighborhood of this point is by definition a point for which the potential does not differ from $P(\mathbf{r})$. In other words, we are looking for any point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that

$$x_1 \frac{\partial P(\mathbf{r})}{\partial x_1} + x_2 \frac{\partial P(\mathbf{r})}{\partial x_2} + \cdots + x_n \frac{\partial P(\mathbf{r})}{\partial x_n} = \mathbf{x} \cdot \nabla P(\mathbf{r}) = 0.$$

Observe that $\nabla P = \mathbf{F}$, so this reduces to any point $\mathbf{x}$ on the equipotential surface in the neighborhood of $\mathbf{r}$ having the property that

$$\mathbf{F} \cdot \mathbf{x} = 0,$$

which means that the vector tangent to the equipotential surface is orthogonal to the field lines. ■

Realizing this, we can use this to find and draw points along equipotential curves in two dimensions. Note that, for higher dimensions, this strategy fails to work given a greater number of points needed to define a valid surface as well as having a greater number of directions. That being said, this algorithm could be adapted to higher dimensions if given enough thought.

Suppose we start at some point $\mathbf{r}$ and we wish to find some further point $\mathbf{r}''$ on the equipotential surface. We may first move along the vector orthogonal to the field line at the point by some amount. That is to say,

$$\mathbf{r}' \leftarrow \mathbf{r} + dt \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\mathbf{F}(\mathbf{r})}{\|\mathbf{F}(\mathbf{r})\|},$$

where the matrix represents a clockwise 90° rotation and dt represents some step size.

Practically, however, this also changes the potential we are at, so we must normalize the vector in a sense. We shall choose to normalize the vector along the field vector $\mathbf{F}(\mathbf{r}')$, as going along the surface does not change the potential value. We can measure the potential value at $\mathbf{r}'$ and then compare this to the value of $P(\mathbf{r})$ we wish to stay at in order to determine how far to step. In particular, using the shorthand that $\mathbf{F} = \mathbf{F}(\mathbf{r}')$, let

$$\Delta P = \frac{\partial}{\partial t} P(\mathbf{r}' + t\mathbf{F}) \bigg|_{t=0}$$

so that $(\Delta P)t = P(\mathbf{r}) - P(\mathbf{r}')$. Then our final approximation for the next point along the equipotential surface is given by

$$\mathbf{r}'' \leftarrow \mathbf{r}' + t\mathbf{F}.$$

Note that for several steps along the equipotential surface, the value of $P(\mathbf{r})$ should be stored and held constant in order to not diverge from the equipotential.

As a final remark, we shall calculate the value of ΔP for an electrostatic system of n particles with respective charges $q_1, q_2, \ldots, q_n$ and positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. We have that

$$P(\mathbf{r}) = \sum_{i=1}^{n} \frac{kq_i}{\|\mathbf{r} - \mathbf{r}_i\|}.$$

This then tells us that

$$\frac{\partial}{\partial t} P(\mathbf{r}' + t\mathbf{F}) \bigg|_{t=0} = -\sum_{i=1}^{n} \frac{kq_i}{\|\mathbf{r}' + t\mathbf{F} - \mathbf{r}_i\|^3} (\mathbf{r}' + t\mathbf{F} - \mathbf{r}_i) \cdot \mathbf{F} \bigg|_{t=0} = -\mathbf{F} \cdot \sum_{i=1}^{n} \frac{kq_i(\mathbf{r}' - \mathbf{r}_i)}{\|\mathbf{r}' - \mathbf{r}_i\|^3}.$$