

Drawing Equipotential Curves

This is a small little description of how I would try to draw equipotential curves, specifically for **two dimensional** vector fields. We shall first start by laying the groundwork of vector fields and potentials.

Definition. Let $\mathbf{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field. We call the field's *potential function* $P: \mathbb{R}^n \rightarrow \mathbb{R}$ the unique function such that:

$$\mathbf{F} = -\nabla P.$$

It is from this definition that the usual physics properties of the potential arise, and it also gives one a nice geometric intuition (field lines follow the potential downhill). In particular, we have that

$$\int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_b^a \nabla P \cdot d\mathbf{l} = P(a) - P(b),$$

which follows from the Gradient Theorem.

It is also from this definition that another key property of potentials arises, one that is very useful in attempting to draw them.

Claim. At all points across an equipotential surface, the field lines are orthogonal to the surface.

Proof. Suppose we have a potential function P and we are looking in the neighborhood of some point \mathbf{r} . Any point on the equipotential surface in the neighborhood of this point is by definition a point for which the potential does not differ from $P(\mathbf{r})$. In other words, we are looking for any point $\mathbf{x} = (x_1, x_2, \dots, x_n)$ such that

$$x_1 \frac{\partial P(\mathbf{r})}{\partial x_1} + x_2 \frac{\partial P(\mathbf{r})}{\partial x_2} + \dots + x_n \frac{\partial P(\mathbf{r})}{\partial x_n} = \mathbf{x} \cdot \nabla P(\mathbf{r}) = 0.$$

Observe that $\nabla P = -\mathbf{F}$, so this reduces to any point \mathbf{x} on the equipotential surface in the neighborhood of \mathbf{r} having the property that

$$\mathbf{F} \cdot \mathbf{x} = 0,$$

which means that the vector tangent to the equipotential surface is orthogonal to the field lines. ■

Realizing this, we can use this to find and draw points along equipotential curves in two dimensions. Note that, for higher dimensions, this strategy fails to work given a greater number of points needed to define a valid surface as well as having a greater number of directions. That being said, this algorithm could be adapted to higher dimensions if given enough thought.

Suppose we start at some point \mathbf{r} and we wish to find some further point \mathbf{r}'' on the equipotential surface. We may first move along the vector orthogonal to the field line at the point by some amount. That is to say,

$$\mathbf{r}' \leftarrow \mathbf{r} + dt \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\mathbf{F}(\mathbf{r})}{\|\mathbf{F}(\mathbf{r})\|},$$

where the matrix represents a clockwise 90° rotation and dt represents some step size.

Practically, however, this also changes the potential we are at, so we must normalize the vector in a sense. We shall choose to normalize the vector along the field vector $\mathbf{F}(\mathbf{r}')$, as going along the surface does not change the potential value. We can measure the potential value at \mathbf{r}' and then compare this to the value of $P(\mathbf{r})$ we wish to stay at in order to determine how far to step. In particular, using the shorthand that $\mathbf{F} = \mathbf{F}(\mathbf{r}')$, let

$$\Delta P = \left. \frac{\partial}{\partial t} P(\mathbf{r}' + t\mathbf{F}) \right|_{t=0}$$

so that $(\Delta P)t = P(\mathbf{r}) - P(\mathbf{r}')$. Then our final approximation for the next point along the equipotential surface is given by

$$\mathbf{r}'' \leftarrow \mathbf{r}' + t\mathbf{F}.$$

Note that for several steps along the equipotential surface, the value of $P(\mathbf{r})$ should be stored and held constant in order to not diverge from the equipotential.

As a final remark, we shall calculate the value of ΔP for an electrostatic system of n particles with respective charges q_1, q_2, \dots, q_n and positions $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$. We have that

$$P(\mathbf{r}) = \sum_{i=1}^n \frac{kq_i}{\|\mathbf{r} - \mathbf{r}_i\|}.$$

This then tells us that

$$\left. \frac{\partial}{\partial t} P(\mathbf{r}' + t\mathbf{F}) \right|_{t=0} = - \sum_{i=1}^n \frac{kq_i}{\|\mathbf{r}' + t\mathbf{F} - \mathbf{r}_i\|^3} (\mathbf{r}' + t\mathbf{F} - \mathbf{r}_i) \cdot \mathbf{F} \Big|_{t=0} = -\mathbf{F} \cdot \sum_{i=1}^n \frac{kq_i(\mathbf{r}' - \mathbf{r}_i)}{\|\mathbf{r}' - \mathbf{r}_i\|^3}.$$