Project Euler: Problem #483

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Problem

Description

The set of natural numbers from 1 to n, namely $[n] = \{1, 2, \ldots, n\}$, can be permuted in $n!$ different ways. If we treat these permutations as operations upon this set, we can compose them together in interesting ways.

Suppose we select a permutation P_i from one of the $n!$ permutations. Let $f(P_i)$ denote the minimum number of times we must compose , or repeatedly apply, P_i to return back to the identity element.

Further, let $g(n)$ denote the average value of $f^2(P_i)$ over all $n!$ permutations of $[n]$, calculated as

$$g(n) = \frac{1}{n!} \sum_{i=1}^{n!} f^2(P_i).$$

Our task is to find $g(350)$ to at least 10 digits of accuracy.

Examples

Let us take a look at the case given to us when $n = 3$. The set $\{1, 2, 3\}$ has the following $3! = 6$ permutations:

$$P_1\colon \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \qquad P_3\colon \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \qquad P_5\colon \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$$

$$P_2\colon \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \qquad P_4\colon \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \qquad P_6\colon \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

Explained in human language:

- P_1 doesn't change anything.
- P_2 swaps the first and second element.
- P_3 swaps the first and third element.
- P_4 swaps the second and third element.
- P_5 rotates all elements to the right.
- P_6 rotates all elements to the left.

This is better illustrated when we find the values of $f(P_i)$. We start with arrangement $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and apply each permutation until we arrive back at this original value.

- $f(P_1) = 1$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- $f(P_2) = 2$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- $f(P_3) = 2$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- $f(P_4) = 2$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- $f(P_5) = 3$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- $f(P_6) = 3$ with $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

With this in hand, we can now calculate $g(3)$ to be

$$\begin{aligned} g(3) &= \frac{1}{6} \sum_{i=1}^{6} f^2(P_i) \\ &= \left(1^2 + 2^2 + 2^2 + 2^2 + 3^3 + 3^3\right)/6 \\ &= 31/6 \\ &\approx 5.166666667 \end{aligned}$$

In addition to this worked-out example, we are also given the following

- $g(5) = 2081/120 \approx 1.734166667 \times 10$
- $g(20) = 12422728886023769167301/2432902008176640000 \approx 5.106136147 \times 10^3$

Solution

Background

Exploration

Solution