USAJMO 2017 Solutions

> **Problem 1.** Prove that there are infinitely many distinct pairs (a, b) of relatively prime positive integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

Solution. In other words, we must look for pairs such that

$$a^b + b^a \equiv a^b + (-a)^a \equiv 0 \pmod{a+b}.$$

Suppose a is odd and $b = a + 2$. We then have that

$$\iff a^{a+2} - a^a \equiv a^a(a-1)(a+1) \equiv 0 \pmod{2a+2},$$

but notice that since $a - 1$ is even, this is always the case, so there are infinitely many solutions. One may also verify that a and b are coprime by observing that $\gcd(a, a+2) = \gcd(a, 2) = 1$, which follows from the fact that a is odd.