Miscellaneous Problems

> **Problem** (MathDash Round 3 Silver #8). Draw triangle $\triangle ABC$ with side lengths $AB = 160, BC = 170, CA = 180$. Point P is inside triangle $\triangle ABC$ such that $\angle PAB = \angle PAC$ and $\angle ABC = 90°$. If M is the midpoint of BC, find PM.

Solution. We can first make a few natural simplifying observations about the angle conditions on P:

- Let E denote the intersection of the angle bisector of $\angle BAC$ and BC. By the first condition, P lies on the angle bisector of $\angle BAC$ and thus on AE.
- Let O be the midpoint of AB and let ω be the circle centered O. Then by the second angle condition, P lies inside $\triangle ABC$ and on ω.

Combining these, we get that $P = AE \cap OM$. Inspired by the diagram, we make the following claim:

Claim. We have that O, P, M are collinear.

Proof. To prove this, we shall use a bit of coordinate geometry, choosing A to be the origin. This tells us that

$$O = (80 \cos \alpha, 80 \sin \alpha),$$

where we have denoted $\angle BAC$ as α for convenience.

Since OM is obviously parallel to AC, we see that proving P lies on OM is equivalent to showing that

$$P = (80 \cos \alpha + 80, 80 \sin \alpha) = (80(1 + \cos \alpha), 80 \sin \alpha).$$

Since the RHS is guaranteed to fall on ω it suffices to then show that P lies on the angle bisector AM. Equivalently

$$\begin{aligned} \tan(\alpha/2) &= \frac{\sin \alpha}{1 + \cos \alpha} \\ \frac{1}{\cos^2(\alpha/2)} - 1 &= \frac{1 - \cos^2 \alpha}{(1 + \cos \alpha)^2} \\ \frac{2}{\cos \alpha + 1} - 1 &= \frac{1 - \cos^2 \alpha}{(1 + \cos \alpha)^2}. \end{aligned}$$

Additionally, since the Law of Cosines tells us that

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos\alpha \\ \implies \cos\alpha &= \frac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \frac{97}{192}. \end{aligned}$$

We can substitute this into the previous equation and verify that it does in fact hold, thus verifying the proof. ■

Now we can finish trivially, as $\triangle OBM \sim \triangle ABC$ and M is the midpoint of BC. This tell us that $OM = \frac{1}{2}AC = 90$ and since $OP + PM = OM$ and $OP = 80$, $PM = \boxed{10}$.