Student: Rushil Surti
Username: chirpyboat
ID#: 42195

USA Mathematical Talent Search

Year	Round	Problem
35	1	1

Solution 1. Using the condition that only one neighboring cell shares the same number, we can solve the puzzle similar to sudoku or minesweeper.

5	3	3	5	5	3	5
1	5	1	1	3	1	5
1	3	3		5	1	3
	5	1	1	3	5	3
3	3	5	3		1	
5		1	1	5	1	5
5	3	3	5	3	3	5

Solution 2. No; this number can never be prime.

Suppose the sequence $p_0, p_1, \ldots, p_{100}$ is some permutation of the list of numbers 101 numbers $2024, 2025, \ldots, 2124$. We can express all concatenations of the numbers as the following sum, with the order determined by the sequence p_k:

$$P = \sum_{k=0}^{100} p_k 10^{4k} = \sum_{k=0}^{100} p_k 100^{2k}.$$

Take P modulo 101.

$$\begin{aligned} P &\equiv \sum_{k=0}^{100} p_k (-1)^{2k} \pmod{101} \\ &\equiv \sum_{k=0}^{100} p_k \pmod{101} \\ &\equiv \sum_{k=2024}^{2124} k \pmod{101}. \end{aligned}$$

That is, regardless of the ordering of the numbers, P is equivalent to the sum of list of 101 numbers when modulo 101. Using the formula for the sum of the first n natural numbers, we can clearly see that

$$\begin{aligned} P &\equiv \frac{2124 \cdot 2125}{2} - \frac{2023 \cdot 2024}{2} \pmod{101} \\ &\equiv 1062 \cdot 2125 - 2023 \cdot 1012 \pmod{101} \\ &\equiv 52 \cdot 4 - 3 \cdot 2 \pmod{101} \\ &\equiv 6 - 3 \cdot 2 \pmod{101} \\ &\equiv 0 \pmod{101}. \end{aligned}$$

Any way of concatenating the numbers will end up with the same result, meaning that all numbers of the questioned form are divisible by 101 and thus not prime.

We claim that the minimum number and maximum number of roof-friendly pairs for a given $n \geq 2$ are $n-1$ and $2n-3$ respectively.

The minimum case is trivial. Given that we consider each pair of adjacent buildings to be roof-friendly regardless of height, there necessarily must be at least $n-1$ roof-friendly pairs. It remains to pick a sequence in which no other "non-trivial" roof-friendly pairs exist. This sequence is simply a sorted list. Observe that because the following list

$$1, 2, \ldots, n$$

is strictly increasing, there shall be no pairs of numbers for which those contained between will be shorter than the pair edges (other than the $n-1$ pairs for which there exist no inner buildings).

For the maximum case, we must choose a sequence that maximizes the number of pairs for which they contain all smaller numbers. Notice that if we have roof-friendly pairs that are disjoint from each other, they interfere in a way and decrease the possible number of pairs. This motivates a sequence such as the following which maximizes pairs without interfering with each other

$$n-1, n-2, \ldots, 2, 1, n.$$

Apart from the $n-1$ trivial pairs, this makes an additional $n-2$ pairs between all numbers $2, 3, \ldots, n-1$ and n, giving us $2n-3$ pairs.

Solution 4. [Incomplete] Motivated by the structure of the left-hand side of the equations, we can transform the question of whether there exists a real solution (a, b, c) to whether the following cubic polynomial equation has all real roots:

$$P(z) := z^3 - Az^2 + Bz - C = 0,$$

where

$$\begin{aligned} A &:= a + b + c = \frac{x + x^2 + x^4 + y + y^2 + y^4}{2}, \\ B &:= ab + bc + ac = \frac{x^3 + x^5 + x^6 + y^3 + y^5 + y^6}{2}, \\ C &:= abc = \frac{x^7 + y^7}{2}. \end{aligned}$$

In order to determine that this polynomial has all real roots (given the boundary conditions), we shall make liberal use of Descartes' rule of signs.

Observe that

$$\begin{aligned} P(z + k) &= z^3 + (3k - A)z^2 + (-2Ak + B + 3k^2)z + (-Ak^2 + Bk - C + k^3), \\ P(-z + k) &= -z^3 + (3k - A)z^2 - (-2Ak + B + 3k^2)z + (-Ak^2 + Bk - C + k^3). \end{aligned}$$

We will use these equations to shift the polynomial and isolate the real roots. Specifically, we claim the following:

For $k = 0$, the polynomial the following has coefficient signs:

	z^3	z^2	z	1	Changes
$z + k$	+	-	+	-	3
$-z + k$	-	-	-	-	0

For $k = \max\{x, y\}$, the polynomial the following has coefficient signs:

	z^3	z^2	z	1	Changes
$z + k$	+	-	+	+	2
$-z + k$	-	-	-	+	1

For $k = (\max\{x, y\})^2$, the polynomial has the following coefficient signs:

	z^3	z^2	z	1	Changes
$z + k$	+	-	-	-	1
$-z + k$	-	-	+	-	2

Indeed, if we have these sign changes, we can easily prove that all roots are real. From the negative z case of $k = \max\{x, y\}$ and the positive z case of $k = (\max\{x, y\})^2$, we can see that there is a real root in the interval $(-\infty, \max\{x, y\}]$ and another real root in $[(\max\{x, y\})^2, \infty)$. Because the coefficients of $P(z)$ are real, and $P(z)$ has at least two real roots, the last root cannot be complex (by the complex conjugate root theorem). In short, all three roots are real, and thus there is a real solution (a, b, c).

Thus, now it is left to us to prove that these signs are in fact true. It is sufficient to prove that the $z + k$ coefficients are true, as the $-z + k$ trivially follow through flipping the sign of z^3 and z.

For $k = 0$, the inequalities are trivial. Notice that $A, B, C \geq 0$, so the polynomial $P(z) = z^3 - Az^2 + Bz - C$ has signs $+ - + -$.

The shifted cases are a bit more unwieldy. We shall tackle them casewise.

Case $x \geq y$: We have two cases to figure out: $k = x$ and $k = x^2$. Starting with $k = x$, we must prove

$$3x - A \leq 0 \tag{1}$$

$$-2Ax + B + 3x^2 \geq 0 \tag{2}$$

$$-Ax^2 + Bx - C + x^3 \geq 0 \tag{3}$$

Showing that (1) holds is rather simple, using AM-GM and that $y \geq \sqrt{x}$:

$$\begin{aligned} A &\geq 3\sqrt{x^7y^7} \geq 3x^{21/4} \\ \implies 3x - A &\leq 3x - 3x^{21/4} \\ \implies 3x - A &\leq 3x(1 - x^{17/4}) \leq 0. \end{aligned}$$

Showing that (3) holds also isn't too bad. Notice that if we view the LHS as a polynomial in y, we have the roots $y = x$ and $y = \sqrt{x}$. We can divide these out (and flipping the sign because $\sqrt{x} \leq y \leq x$, so $(y - x)(y - \sqrt{x})$ is negative) to get

$$\begin{aligned} &-Ax^2 + Bx - C + x^3 \geq 0 \\ \iff &\frac{1}{2}\left(-y^5 - x^{1/2}y^4 + xy + x^{3/2}\right) \leq 0, \end{aligned}$$

but since $-y^5 - x^{1/2}y^4 + xy + x^{3/2} = -(y^4 - x)(y + x^{1/2})$, this always holds.

It's unfortunately in showing (2) and the other cases (such as $k = x^2$) that I struggled to make progress and subsequently ran out of time. Thus, this solution is incomplete.

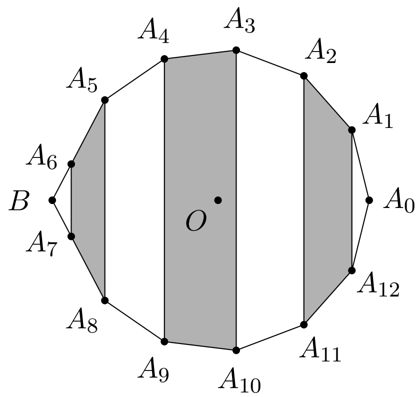

Figure 1: The polygon turned on its side and reindexed.

Solution 5. [Incomplete] While perhaps not the most elegant method, we shall proceed by using coordinate geometry. In order to facilitate this, we shall flip the entire polygon on its side and set the origin to be the center of the 13-gon (not including the triangle with point B). Next, reindex the points, sending $A_k \mapsto A_{k-1}$ so that the indices align with the multiples of $2\pi/13$. In this way, we can describe the 13-gon points to be

$$A_k = \left(r\cos\left(2\pi k/13\right), r\sin\left(2\pi k/13\right)\right).$$

WLOG, we can set $r = 1$. Our first order of business is determining the x-coordinate of B, as its y-coordinate is simply 0. We may do so by finding the intersection between the line $y = 0$ and the line running through A_5 and A_6. Doing so, we see that

$$\begin{aligned}\frac{\sin\left(12\pi/13\right) - \sin\left(10\pi/13\right)}{\cos\left(12\pi/13\right) - \cos\left(10\pi/13\right)}\left(x - \cos\left(12\pi/13\right)\right) &= 0 - \sin\left(12\pi/13\right)\\ -\cot\left(11\pi/13\right)\left(x - \cos\left(12\pi/13\right)\right) &= -\sin\left(12\pi/13\right),\end{aligned}$$

giving us that $x = \cos\left(12\pi/13\right) + \sin\left(12\pi/13\right)\tan\left(11\pi/13\right)$.

With this, we can now go about solving. Denote $[P_1P_2\ldots P_n]$ the area contained by the polygon $P_1P_2\ldots P_n$. We shall use the following tools:

$$\begin{aligned}[ABCD] &= [ABC] + [ADC] = [ABC] + [ACD],\\ [ABC] &= \frac{1}{2}\left|(A_x - B_x)(C_y - B_y) - (A_y - B_y)(C_x - B_x)\right|\end{aligned}$$

One can also see that for a triangle $A_jA_kA_l$, we know that

$$\begin{aligned}[A_iA_jA_k] = 2\bigg|&\sin\left(\frac{(k+j)\pi}{13}\right)\sin\left(\frac{(k-j)\pi}{13}\right)\sin\left(\frac{(i-j)\pi}{13}\right)\cos\left(\frac{(i+j)\pi}{13}\right)\\ &-\sin\left(\frac{(i+j)\pi}{13}\right)\sin\left(\frac{(i-j)\pi}{13}\right)\sin\left(\frac{(k-j)\pi}{13}\right)\cos\left(\frac{(k+j)\pi}{13}\right)\bigg|,\end{aligned}$$

by trig identities.

We now must prove that

$$[A_1A_2A_{11}A_{12}] + [A_3A_4A_9A_{10}] + [A_5A_6A_7A_8] = [A_0A_1A_2A_3A_4A_5A_6B].$$

Finding the value of the right hand side is rather simple enough. Going from A_0 to A_6, there are 6 triangles with identical area to $[A_0OA_1]$, and there is one smaller triangle containing B. In other words,

$$\begin{aligned} H &:= [A_0A_1A_2A_3A_4A_5A_6B] = 6[A_0OA_1] + [A_6OB] \\ &= 3\sin(2\pi/13) + \frac{1}{2}\left|\cos(12\pi/13)\sin(12\pi/13) + \sin^2(12\pi/13)\tan(11\pi/13)\right| \\ &= 3\sin(2\pi/13) + \frac{1}{4}\sin(2\pi/13) + \frac{1}{2}\sin^2(\pi/13)\tan(2\pi/13). \end{aligned}$$

Now, we'll move onto the left hand side. Taking the first trapezoid, we have

$$\begin{aligned} T_1 &:= [A_1A_2A_{11}A_{12}] = [A_1A_2A_{11}] + [A_1A_{12}A_{11}] \\ &= 2\left|\sin(\pi/13)\sin(3\pi/13)\sin(9\pi/13)\right| + 2\left|\sin(\pi/13)\sin(11\pi/13)\sin(23\pi/13)\right| \\ &= 2\sin(\pi/13)\sin(3\pi/13)\sin(4\pi/13) + 2\sin(\pi/13)\sin(2\pi/13)\sin(3\pi/13). \end{aligned}$$

For the second trapezoid, we have

$$\begin{aligned} T_2 &:= [A_3A_4A_9A_{10}] = [A_3A_4A_9] + [A_3A_{10}A_9] \\ &= 2\left|\sin(\pi/13)\sin(5\pi/13)\sin(7\pi/13)\right| + 2\left|\sin(\pi/13)\sin(7\pi/13)\sin(19\pi/13)\right| \\ &= 2\sin(\pi/13)\sin(5\pi/13)\sin(6\pi/13) + 2\sin(\pi/13)\sin^2(6\pi/13). \end{aligned}$$

For the third trapezoid, we have

$$\begin{aligned} T_3 &:= [A_5A_6A_7A_8] = [A_5A_6A_7] + [A_5A_8A_7] \\ &= 2\left|\sin^2(\pi/13)\sin(11\pi/13)\right| + 2\left|\sin(\pi/13)\sin(3\pi/13)\sin(15\pi/13)\right| \\ &= 2\sin^2(\pi/13)\sin(2\pi/13) + 2\sin(\pi/13)\sin(2\pi/13)\sin(3\pi/13). \end{aligned}$$

We now must show that

$$\begin{aligned} & H - T_1 - T_2 - T_3 = 0 \\ \iff & \cos(2\pi/13)H - \cos(2\pi/13)T_1 - \cos(2\pi/13)T_2 - \cos(2\pi/13)T_3 = 0, \end{aligned}$$

where we multiply by $\cos(2\pi/13)$ in order to simplify the tangent term in H.

In order to get terms to cancel well, we'll want to write all these trigonometric values in terms of smaller ones, in particular $\sin(\pi/13)$ and $\cos(\pi/13)$, using the common trigonometric addition rules:

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b), \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b). \end{aligned}$$

Mathematica is quite helpful at this stage. For typesetting purposes, let $S = \sin(\pi/13)$ and $C = \cos(\pi/13)$. We may rewrite $\cos(2\pi/13)H$ to be

$$\cos(2\pi/13)H = \frac{13}{2}SC - 12S^3C.$$

The other expressions are perhaps less elegant. It is here where I realized this method of bashing turns out to be almost a nightmare to try and simplify and work with.