Project Euler: Problem 743

> **Problem.** Let $A(n, k)$ be the number of $2 \times n$ matrices such the sum of the entries in each contiguous $2 \times k$ submatrix is k.
>
> Determine $A(10^{16}, 10^8)$ modulo $10^9 + 7$.

Solution. A key observation to make for this problem is that one can choose the entries of the first initial $2 \times k$ submatrix, and by a sliding window technique, we see that this choice determines the rest of the entire $2 \times n$ matrix up to reflections of each row. In particular, knowing the ith row tells us some very helpful information about all rows of the form $i + mk$ for integer m:

- If the row is $(0, 0)$, then so are all the rows.
- If the row is $(1, 1)$, then again so are all the rows.
- If the row is $(0, 1)$ or $(1, 0)$, then all of the aforementioned rows can be either $(0, 1)$ or $(1, 0)$.

This motivates one to go casewise on the number of each type of row that appear in the initial chosen submatrix. In particular, we shall index a submatrix by the tuple $(a_{00}, a_{11}, a_{01}, a_{10})$. Finding the number of total matrices then amounts to summing over all cases and then determining the number of $2 \times n$ matrices with the parameters of the initially chosen $2 \times k$ submatrix.

We shall note that for a valid tuple, there are

$$\binom{k}{a_{00}, a_{11}, a_{01}, a_{10}},$$

initial submatrices, and thus since the submatrix gets copied over $n/k - 1$ times, there are

$$\binom{k}{a_{00}, a_{11}, a_{01}, a_{10}} 2^{(n/k-1)(a_{01}+a_{10})},$$

full size matrices. This means that our answer should be

$$A(n, k) = \sum_{2a_{11}+a_{01}+a_{10}=k} \binom{k}{a_{00}, a_{11}, a_{01}, a_{10}} 2^{(n/k-1)(a_{01}+a_{10})}.$$

We shall index instead by $l = a_{01} + a_{10}$, noting that it must be the same parity as k, which transforms our sum into

$$\begin{aligned}
A(n,k) &= \sum_{\substack{l=0,\\ l\equiv_2 k}}^{k} \binom{k}{l}\binom{k-l}{m-l}\sum_{i=0}^{l}\binom{l}{i}2^{(n/k-1)l} \\
&= \sum_{\substack{l=0,\\ l\equiv_2 k}}^{k} \binom{k}{l}\binom{k-l}{(k-l)/2}2^{nl/k} \\
&= \sum_{\substack{l=0,\\ l\equiv_2 k}}^{k} \frac{k!}{l!\cdot((k-l)/2)!^2}2^{nl/k}.
\end{aligned}$$

Precomputing factorials, powers, and modular inverses, this should be rather quick to compute, running maybe in roughly $O(k \log k)$ time (just eyeballing it).