AIME II 2011 Solutions

> **Problem** (2019 AIME II #9). Let $x_1, x_2, \ldots, x_6$ be non-negative real numbers such that
>
> $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1,$$
>
> and
>
> $$x_1 x_3 x_5 + x_2 x_4 x_6 \geq \frac{1}{540}.$$
>
> Let p and q be positive relatively prime positive integers such that p/q is the maximum possible value of
>
> $$\sum_{cyc} x_1 x_2 x_3.$$
>
> Find $p + q$.

Solution. By AM-GM we have that

$$\frac{1}{27}(x_1 + x_3 + x_5)^3 \geq x_1 x_3 x_5,$$
$$\frac{1}{27}(x_2 + x_4 + x_6)^3 \geq x_2 x_4 x_6.$$

Motivated by this, let $A := x_1 + x_3 + x_5$ and $B := x_2 + x_4 + x_6$ so that we have

$$A + B = 1,$$
$$A^3 + B^3 \geq \frac{1}{20}.$$

Cubing our first relation, we get that

$$1 = A^3 + 3A^2B + 3AB^2 + B^3 = A^3 + B^3 + 3AB \geq 3AB + \frac{1}{20},$$

so $AB \leq 19/60$.

Note that

$$\begin{aligned} AB &= (x_1 + x_3 + x_5)(x_2 + x_4 + x_6) \\ &= x_1x_2 + x_1x_4 + x_1x_6 + x_2x_3 + x_3x_4 + x_3x_6 + x_2x_5 + x_4x_5 + x_5x_6. \end{aligned}$$

Consider the cyclic sum given in the problem, which we shall denote S. Observe that

$$\begin{aligned} S = {} & (x_1 + x_3 + x_5)(x_2x_3 + x_4x_5 + x_1x_6) \\ & + (x_2 + x_4 + x_6)(x_3x_4 + x_5x_6 + x_1x_2). \end{aligned}$$