Codeforces Problem 1948E

> **Problem.** Suppose we have a graph with n vertices and a permutation a of the numbers 1 through n, and we add an edge on the graph for every pair of vertices (i, j) if
> $$|i - j| + |a_i - a_j| \leq k.$$
> Determine the permutation a for which the number of distinct cliques in the graph is a minimum.

Remark. A *clique* is defined to be a complete subgraph of a graph.

Constraints. We have that

- $1 \leq t \leq 1600$,
- $2 \leq n \leq 40$,
- $1 \leq k \leq 2n$

Solution. We shall first tackle the problem of creating a clique of size m. Immediately, we see that $m \leq \min\{n, k\}$.

Claim. A clique of size m is constructible with cost m at the very least.

Proof. Observe first that any cost for a clique of size m cannot be lower than m because the maximum value of $|i - j|$ is achieved when $i = 1$ and $j = m$, and $|a_i - a_j| \geq 1$, so their sum must be greater than or equal to m for some pair of vertices (i, j).

We provide the following construction for a permutation that costs only m and results in a clique of size m:

$$a : \lceil m/2 \rceil, (\lceil m/2 \rceil - 1), \ldots, 1, m, (m - 1), \ldots, (m + 1 - \lceil m/2 \rceil),$$

though because the math is a tad gruesome, we shall not verify here that the sequence does in fact fulfill the cost conditions stated above. ■

Now since we have that a clique of size m is constructible and $m \leq \min\{n, k\}$, the solution is rather obvious (simply form as many cliques of greatest size that we can). We also trivially have that when $k \geq n$, the minimum clique number is 1, which seems to be a rather good sanity check. In particular, we also have a closed form for the minimum number of cliques needed:

$$\text{cliques} = \left\lceil \frac{n}{\min\{n, k\}} \right\rceil.$$