Codeforces Problem 1763D

> **Problem.** We call a permutation *bitonic* if all the elements increase until a certain index k, where $2 \leq k \leq n-1$, and then decrease until the end.
>
> Count modulo $10^9 + 7$ the number of bitonic permutations of length n such that $B_i = x$ and $B_j = y$, where $i < j$ and $x \neq y$.

Solution. Given the definition of a bitonic permutation, we are motivated to go case-wise on k when counting. Notice that since the elements increase up to k and decrease afterwards, B_k must be equal to the maximum value, n.

Observation. Without the for B_i and B_j, the number of total bitonic permutations A on n numbers is given by

$$A = \sum_{k=2}^{n-1} \binom{n-1}{k-1} = 2^{n-1} - 2.$$

This works because we must set $B_k = n$, giving us $n-1$ elements left to choose from. We can then choose any $k-1$ of the remaining elements because there is only one way to increasingly sort each choice, and the choice of $k-1$ elements uniquely determines the decreasing elements.

With this idea in mind, we can start looking at the constrained problem. There are two main cases that we have to consider:

- Either $B_i = x = n$ or $B_j = y = n$. Having this condition forces k to be either i or j. This case is actually easier, however, as we only have to consider one value. Consulting the following diagram,

  $$\bullet \ \overset{i-1}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ B_i = B_k\ \overset{j-i-1}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ B_j\ \overset{n-j}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ \bullet$$

  $$\bullet \ \overset{i-1}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ B_i\ \overset{j-i-1}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ B_j = B_k\ \overset{n-j}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}\ \bullet$$

  we see that the answers for each case respectively should be

  $$\binom{n-y-1}{j-i-1}\binom{y-1}{n-j}, \text{ and } \binom{n-x-1}{j-i-1}\binom{x-1}{i-1},$$

  since we must handle the sorting condition and since choosing two of the sections uniquely determines the choice of the third.

- Otherwise we must go casewise on all possible values of k. In particular, though, we can group these cases of k into three further cases: $k < i$, $i < k < j$, and $k > j$.

We shall now tackle these cases:

- Case $k < i$. We must have that $x > y$, and we have the following diagram:

$$\bullet \quad \overset{k-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_k \quad \overset{i-k-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_i \quad \overset{j-i-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_j \quad \overset{n-j}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad \bullet$$

  For this case, it's clear to see that the answer is

$$\binom{n-x-1}{i-k-1}\binom{x-y-1}{j-i-1}\binom{y-1}{n-j}.$$

- Case $i < k < j$. We have the following diagram:

$$\bullet \quad \overset{i-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_i \quad \overset{k-i-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_k \quad \overset{j-k-1}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad B_j \quad \overset{n-j}{\vdash\!\!\!\longrightarrow\!\!\dashv} \quad \bullet$$

  This case is slightly more complicated, as there is an overlap of elements in the inner two sections. With a bit of thinking, we see that the condition $n - \min\{x, y\} - 1 \geq j - i - 2$ must hold (otherwise there is no possible permutation). For convenience, we shall say that WLOG $x < y$, simplifying the condition to $n - x - 1 \geq j - i - 2$.

  We shall first place the elements in range $y+1$ to $n-1$. We must choose $j-k-1$ of them to place in the third interval, leaving the others to be uniquely placed in the second interval. We shall then choose the remaining $k - i - 1 - (n - y - 1 - (j - k - 1)) = j - i - 1 + y - n$ numbers out of the $y - x - 1$ we have. All that is left is to choose the first interval out of simplicity, which then uniquely determines the fourth.

  Thus, if $x < y$, the answer is

$$\binom{n-y-1}{j-k-1}\binom{y-x-1}{j-i-1+y-n}\binom{x-1}{i-1},$$

  and if $x > y$, the answer is

$$\binom{n-x-1}{k-i-1}\binom{x-y-1}{j-i-1+x-n}\binom{y-1}{n-j}.$$

- Case $k > j$. We must have that $x < y$, and we have the following diagram:

$$\bullet \quad \overset{i-1}{\vdash\!\!\relbar\!\!\relbar\!\!\relbar\!\!\dashv} \quad B_i \quad \overset{j-i-1}{\vdash\!\!\relbar\!\!\relbar\!\!\relbar\!\!\dashv} \quad B_j \quad \overset{k-j-1}{\vdash\!\!\relbar\!\!\relbar\!\!\relbar\!\!\dashv} \quad B_k \quad \overset{n-k}{\vdash\!\!\relbar\!\!\relbar\!\!\relbar\!\!\dashv} \quad \bullet$$

  It is also apparent for this case that the answer is

$$\binom{n-y-1}{k-j-1}\binom{y-x-1}{j-i-1}\binom{x-1}{i-1}.$$