Codeforces Problem 1758D

Problem. We must find n distinct integers such that

$$\max(a_1, a_2, \ldots, a_n) - \min(a_1, a_2, \ldots, a_n) = \sqrt{a_1 + a_2 + \cdots + a_n}.$$

Observation. WLOG, let $a_1 < a_2 < \cdots < a_n$. The condition then turns into

$$\begin{aligned} a_n - a_1 &= \sqrt{a_1 + a_2 + \cdots + a_n} \\ (a_n - a_1)^2 &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

We shall assume this ordering for the sequence moving forward.

Solution (Invalid, Integer Overflow)**.** In fact, suppose that we turn this into an arithemtic sequence. Let

$$a_i = a_1 + (i-1)d.$$

This transforms our condition into

$$\begin{aligned} (n-1)^2 d^2 &= n a_1 + \frac{n(n-1)}{2} \cdot d \\ \implies a_1 &= \frac{(n-1)^2 d^2}{n} - \frac{(n-1)d}{2} \\ &= (n-1)d \left(\frac{(n-1)d}{n} - \frac{1}{2} \right). \end{aligned}$$

If we take d to be equal to $2n$, then we get

$$a_1 = n(n-1)(4n-5),$$

which is always an integer for $n \geq 2$, which is rather reassuring.

The only problem is that this sequence of integers quickly blows up for larger n. Unfortunately, the problem gives the constraint that $1 \leq a_i \leq 10^9$, which means that we'll have to play around a bit more.

Solution. Let us step away from arithmetic progressions and be a bit more general in our reasoning. Let us consider sequences such that $a_n = a_1 + k$ for some integer $k \geq n - 1$. Thus, the main condition turns into

$$k^2 = a_1 + a_2 + \cdots + a_n.$$

Let K be the integer value such that

$$k^2 = na_1 + K + k,$$

which tells us that

$$a_1 = (k^2 - k - K)/n,$$

so all we have to do is find valid values of k, K such that $n \mid k^2 - k - K$, and then we win.

In order to get an idea of the values we can give to K, we may bound it as follows:

$$\frac{(n-1)(n-2)}{2} \leq K \leq (k-1)+(k-2)+\cdots+(k-n+2) = (n-2)k - \frac{(n-1)(n-2)}{2}.$$

This gives us an interval of length

$$(n-2)k - (n-1)(n-2) + 1,$$

which is greater than n for $k \geq n+1$ for all $n \geq 3$ (so I suppose we can just hardcode the case for $n = 2$).

Motivated by this, let's suppose that $k \equiv 1 \pmod{n}$ and $K \equiv 0 \pmod{n}$, so that the divisibility condition always hold true. With the above work, we can see that $k = n+1$, which allows us to determine the valid value of K through examining the interval.