USAMO 2004 Solutions

> **Problem 2.** Suppose $a_1, a_2, \dots, a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:
>
> **a.** For each a_i, $a_i \in S$.
>
> **b.** For each pair (a_i, a_j) (where the indices are not neessarily distinct), $a_i - a_j \in S$.
>
> **c.** For each pair (x, y) where $x, y \in S$, $x + y \in S \implies x - y \in S$.

Solution. We may first easily prove that $0, 1 \in S$.

- For the first case of $0 \in S$, this trivially holds because we can take $a_1 - a_1 = 0$, which is in S by the second rule.
- For the second case of $1 \in S$, we may see that, sincethe greatest common divisor of all elements is 1, there must exist an odd element in $a_1, a_2, \dots, a_n$, which is also in S by the first condition. We shall write this odd number in the form $2k+1$ and use the third condition to see that since $(k+1) + k = 2k+1 \in S$, then $(k+1) - k = 1 \in S$. Wait no this doesn't work because you can't know that $k, k+1 \in S$ whoops.

We shall also show that $x \in S \implies -x \in S$. This follows trivially from the fact that $x = 0 + x \implies 0 - x = -x \in S$.

From here, finishing is trivial, as we may induct up. More specifically, we have that $x \in S \implies x + 2 \in S$, and since we have both 0 and 1 in S, this reaches all integers after reflection (because $x \in S \implies -x \in S$). We may prove this proposition by observing that

$$x = (x+1) + (-1) \in S \implies (x+1) - (-1) = x + 2 \in S,$$

which finishes the proof when plugging in $x = 0, 1, \dots$ and so on.