Fun Kattis Math Problem

Problem. Define the following function $g(n)$ to be

$$g(n) = \{g(k) \colon 0 \leq k < n\}.$$

where $g(0) = \{\}$.

Define $f(n)$ to be the number of braces and commas required to write out $g(n)$. Derive a closed form expression for $f(n)$ (i.e. a form that is computable in $O(1)$ time).

Solution. We shall start with the obvious: we shall decompose $f(n)$ into the sum of the number braces and the number of the commas, and then we shall derive a recurrence for both of them. In particular, write

$$f(n) = 2b(n) + c(n),$$

where $b(n)$ is the number of open brackets and $c(n)$ the number of commas.

We shall start with deriving a closed form for $b(n)$. Observe that

$$b(n) = 1 + \sum_{k=0}^{n-1} b(k), \quad b(0) = 1.$$

This then tells us that $b(n) = 2^n$.

For $c(n)$, we have that

$$c(n) = n - 1 + \sum_{k=0}^{n-1} c(k), \quad c(0) = c(1) = 0.$$

This then tells us that $c(n) = 2^{n-1} - 1$ for $n \geq 1$.

So our final answer is simply

$$f(0) = 2, \quad f(n) = 2^{n+1} + 2^{n-1} - 1$$