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**Problem 1.** Using some casework, inequalities, and some time investment, one can obtain the following solution:

| | | | | | |
|---|---|---|---|---|---|
| 6 | 1 | 3 | 5 | 4 | 2 |
| 2 | 6 | 5 | 3 | 1 | 4 |
| 3 | 5 | 1 | 4 | 2 | 6 |
| 4 | 2 | 6 | 1 | 5 | 3 |
| 1 | 3 | 4 | 2 | 6 | 5 |
| 5 | 4 | 2 | 6 | 3 | 1 |

**Problem 2.** We shall search for possible unordered pairs $(a, b, c)$ and display all possible products at the end of the answer.

Notice that, of all $abc$ cubes, $(a-2)(b-2)(c-2)$ are unpainted. Thus we are looking for all unordered $(a, b, c)$ such that

$$f(a, b, c) := \frac{(a-2)(b-2)(c-2)}{abc} = \frac{1}{5},$$

where $a, b, c \geq 3$.

This is a standard looking Diophantine equation, to which we seemingly want to apply SFFT and then look at prime factorizations for solutions. Treating $c$ as a constant and removing the denominators, we achieve

$$\begin{aligned}
5(a-2)(b-2)(c-2) &= abc \\
(2c-5)ab - 5(c-2)a - 5(c-2)b + 10(c-2) &= 0 \\
(2c-5)^2ab - 5(c-2)(2c-5)a - 5(c-2)(2c-5)b + 10(c-2)(2b-5) &= 0 \\
((2c-5)a - 5(c-2))\,((2c-5)b - 5(c-2)) &= 5c(c-2).
\end{aligned}$$

With this, we must go through values of $c$ to find solutions in $a$ and $b$, which makes the proof slightly inelegant and a little bashy, but oh well. First, however, we must prove a very handy claim, which allows us to stop looking after $c = 20$.

**Claim.** WLOG suppose $c = \max\{a, b, c\}$. There exists no $a, b, c$ for which $f(a, b, c) = 1/5$ when $c > 20$.

*Proof.* Suppose $c > 20$. Observe that

$$f(a, b, 20) < f(a, b, c) < \lim_{c' \to \infty} f(a, b, c'),$$

so if $f(a, b, c) = 1/5$, then we must have that

$$\frac{9}{10} \cdot \frac{(a-2)(b-2)}{ab} < \frac{1}{5} < \frac{(a-2)(b-2)}{ab}.$$

If both $a \geq 4$ and $b \geq 4$, then this condition cannot possibly be true. Thus, WLOG we set $b = 3$ to look for any remaining cases that may have a solution. Doing so, we get that

$$\frac{a-2}{a} > \frac{3}{5} \implies a > 5$$

and simulatenously

$$\frac{a-2}{a} < \frac{2}{3} \implies a < 6,$$

which cannot be true.

Because we've covered all cases, there exists no solution for $c \geq 20$ (or any of the dimensions being greater than 20). $\square$

The only real bashy part of the proof comes into play with checking all cases where $c = 3, 4, \ldots, 20$. Luckily, we can use a bit of Python to do the work for us:

```
from sympy import divisors

solutions = set()

for c in range(3, 21):
    N = 5 * c * (c - 2)
    A = 2 * c - 5
    B = 5 * (c - 2)

    print(f"Case c = {c}: ({A}a - {B})({A}b - {B}) = {N}")
    found = False

    for d in divisors(N):
        g = N // d
        if g < d:
            break
        if (d + B) % A == 0 and (g + B) % A == 0:
            a, b = (d + B) // A, (g + B) // A
            S = a * b * c
            if S not in solutions:
                print(f"    ({(d + B) // A}, {(g + B) // A}, {c}) -> {S}")
                solutions.add(S)
                found = True

    if not found:
        print("    No new solutions")

    print()
```

This outputs

```
Case c = 3: (1a - 5)(1b - 5) = 15
    (6, 20, 3) -> 360
    (8, 10, 3) -> 240

Case c = 4: (3a - 10)(3b - 10) = 40
    (4, 10, 4) -> 160
    (5, 6, 4) -> 120

Case c = 5: (5a - 15)(5b - 15) = 75
```

```
    No new solutions

Case c = 6: (7a - 20)(7b - 20) = 120
    No new solutions

Case c = 7: (9a - 25)(9b - 25) = 175
    No new solutions

Case c = 8: (11a - 30)(11b - 30) = 240
    No new solutions

Case c = 9: (13a - 35)(13b - 35) = 315
    No new solutions

Case c = 10: (15a - 40)(15b - 40) = 400
    No new solutions

Case c = 11: (17a - 45)(17b - 45) = 495
    No new solutions

Case c = 12: (19a - 50)(19b - 50) = 600
    No new solutions

Case c = 13: (21a - 55)(21b - 55) = 715
    No new solutions

Case c = 14: (23a - 60)(23b - 60) = 840
    No new solutions

Case c = 15: (25a - 65)(25b - 65) = 975
    No new solutions

Case c = 16: (27a - 70)(27b - 70) = 1120
    No new solutions

Case c = 17: (29a - 75)(29b - 75) = 1275
    No new solutions

Case c = 18: (31a - 80)(31b - 80) = 1440
    No new solutions

Case c = 19: (33a - 85)(33b - 85) = 1615
```

```
    No new solutions

Case c = 20: (35a - 90)(35b - 90) = 1800
    No new solutions
```

Thus, the only cases that yield any new solutions are $c = 3$ and $c = 4$. (that is, there solutions for $c = 5, 6, \ldots, 20$, but they do not yield a different product from any of the cases found in cases $c = 3$ and $c = 4$).

So, the only possible numbers of cubes are the products of the values in each unordered pair: $\boxed{120, 160, 240, 360}$.

**Problem 3.** After a little bit of playing around and perhaps some computer simulation (a small exercise in DP actually), one gets a very good indicator of the answer, so now it's left to actually make it rigorous.

First, notice that we can represent the state of the game as a pair $(t, w)$, where $t$ represents the number of 2's on the whiteboard and $w$ represents the number of 1's.

**Definition.** We call a state $(t, w)$ *winning* if, no matter what sequence of moves one plays, the player who's turn it is with that state can win. Similarly, we call a state *losing* if, no matter what sequence of moves one plays, the player who's turn it is with the state can never win.

In order to determine whether or not Lizzie has a winning strategy, it suffices to determine for which $n$ the state $(n, 0)$ is winning. We shall show a slightly stronger result to do so:

**Claim.** We have the following:

- If $n \equiv 0 \pmod 3$, then $(n, 0)$ is losing and $(n, w)$ is winning for $w \geq 1$.
- If $n \equiv 1 \pmod 3$, then $(n, 0)$ is winning, $(n, 1)$ is losing, and $(n, w)$ is winning for $w \geq 2$.
- If $n \equiv 2 \pmod 3$, then $(n, w)$ is winning for $w \geq 0$.

*Proof.* We shall prove this through strict induction.

Suppose our claim holds true for $n = 0, 1, 2, \ldots, 3k-3, 3k-2, 3k-1$. We shall then prove our claim for $n = 3k, 3k+1, 3k+2$, tackling these cases in order:

- We must first prove that $(3k, 0)$ is losing. First, observe that playing the first move type, which yields the state $(3k-1, 0)$ is winning for the other player by our inductive assumption. Next, notice that $(3k-w, w)$ for $w \geq 1$ (which represents all game states reachable by the second move type) is always winning for the other player because $(3k-1, 1)$ is winning by our inductive hypothesis and $(n, w)$ for $w \geq 2$ is always winning. Thus, there is no way for the current player to reach a position where the opponent loses, so $(3k, 0)$ loses.

  Notice then that, if one considers positions $(3k, w)$ for $w \geq 1$, the current player can simply subtract $w$ 1's from to force the opponent to $(3k, 0)$, which is losing for them. Therefore, all states of the form $(3k, w)$ for $w \geq 1$ are winning.

- Next, we shall look at $(3k+1, 0)$. This is trivially winning because, if we remove a single 2, the position becomes $(3k, 0)$, which we've proved is losing for the other player.

  Now we consider the position $(3k+1, 1)$. From this position, we can play a move of the first type to reach $(3k, 1)$, but we have already proven that this is winning for the opponent. Using the second move type, we may arrive at positions of the form $(3k+1-t, 1+t)$ and $(3k+1-t, t)$ for $t \geq 1$, but because all positions (with a lesser number of twos of course) of the form $(a, b)$ for $b \geq 2$ are winning for the opponent, so we need only look at $(3k+1-1, 1)$, which we've already proven to be winning for the opponent. Thus, $(3k+1, 1)$ is losing.

  It follows that, if the state is of the form $(3k+1, w)$, where $w \geq 2$, the current player may subtract away $(w-1)$ 1's in order to reach $(3k+1, 1)$, so all of these positions are winning.

- Finally, consider the state $(3k+2,0)$. Since we can turn one of the 2's into a 1, the opponent receives $(3k+1,1)$, which we've proven to be losing, so $(3k+2,0)$ is winning.

  For the added 1's cases, we can do something of a very similar nature. Consider the states $(3k+2,w)$ for $w \geq 1$. By subtracting one from a single 2 and taking away all other $w$ 1's, we can still reach $(3k+2-1, w-w+1) = (3k+1,1)$, which means that these positions are winning too.

Now for our base case, it suffices to prove that $(0,0)$ is losing and states $(1,0),(2,0)$ are winning, with the states with extra 1's appended on following exactly according to the logic detailed in the above three cases.

- For $(0,0)$, notice that if the current player reaches this position, then the other player must necessarily have reached the state where everything is 0 first, so this means that the state is losing.
- The state $(1,0)$ is trivially winning, for one may simply take away the 2 to win.
- The state $(2,0)$ is also winning. Once again using logic previously detailed in the top three cases, we can see that $(1,1)$ is losing, so we can subtract one from a single 2 to send the opponent to a losing position.

$\square$

Thus, it follows through induction that, after proving this slightly stronger result, the only $n$ for which Lizzie can always win are $n$ of the form $3k+1$ and $3k+2$.

**Solution 4.** [Incomplete]

a)

**Claim.** We claim that no such "concave-to-convex" function $f$ exists.

[Incomplete]

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b)

**Claim.** We claim that no such "convex-to-concave" function $f$ exists.

*Proof.* Suppose for contradiction some $f$ did exist. We shall now prove that this cannot be possible.

Consider a set of $n$ distinct points $S := \{S_1, S_2, \ldots, S_n\}$ and the image of this set under $f$, $P$. That is, let $P_k := f(S_k)$ for $1 \leq k \leq n$ and $P := \{P_1, P_2, \ldots, P_n\}$ (in order to avoid self-intersections, one can order the points by their angle about the centroid going in the counterclockwise direction). We shall in particular be concerned with all quadrilaterals formed by taking 4 points from $P$, and we shall call the set of these quadrilaterals $Q$. Notice that, by the non-degenerate condition, at most one point from $S$ can be mapped to a non-distinct point in $P$ (if this was not the case, we could take the quadrilateral containg these greater than one non-distinct points mapped under $f$ and observe that it would form either a line or point, which would contradict the definition of $f$). This implies $|P| \geq n - 1$.

Additionally, we observe that any quadrilateral trivially can only have one reflex angle (that is, an angle that is between 180° and 360° exclusive), and that a quadrilateral is concave iff it has a reflex angle. Thus, it is sufficient to prove that some quadrilateral in $Q$ does not have a reflex angle.

For convenience of handling both cases of the value of $|P|$, choose $n = 6$ and, if $|P| = 5$, we can consider all the points, else if $|P| = 6$, we may consider the first five points (following the ordering chosen above). Thus we have points $P_1, P_2, P_3, P_4, P_5$ to work with, which we shall alias as $A, B, C, D, E$ for typesetting purposes.

Clearly we can form 5 quadrilaterals: $ABCD, ABCE, ACDE, ABDE, BCDE$. We may go through these and determine which angles are reflex angles and which are not.

- Consider quadrilateral $ABCD$. Without loss of generality, we can say that $\angle ABC$ is a reflex angle. Thus,
$$\angle BCD, \angle CDA, \angle DAB$$
are all not reflex angles.

- Consider quadrilateral $ABCE$. We have that $\angle ABC$ is a reflex angle from the previous quadrilateral. Thus,
$$\angle BCE, \angle CEA, \angle EAB$$
are all not reflex angles.
- Consider quadrilateral $ACDE$ (yes, this order of quadrilaterals is intentional). We already know that $\angle ACD$ is not a reflex angle by ordering. In addition, $\angle CDA = \angle CDE + \angle EDA$ and $\angle DAC = \angle EAC + \angle EAD$ (where both $\angle CDA$ and $\angle DAC$ are not reflex angles by previous quadrilaterals and ordering respectively), so
$$\angle ACD, \angle CDE, \angle EAC$$
are all not reflex angles. Thus $\angle DEA$ is a reflex angle.
- Consider quadrilateral $ABDE$. Since $\angle DEA$ is a reflex angle,
$$\angle ABD, \angle BDE, \angle EAB$$
are all not reflex angles.
- Consider quadrilateral $BCDE$. We already know that $\angle BCD$ and $\angle CDE$ are not reflex angles. Observe that the conjugate of $\angle DEB$ is a reflex angle by our point ordering (perhaps it would have been better to use signed angles) the conjugate of $\angle EBC$ is also a reflex angle by similar reasoning, angles $\angle DEB$ and $\angle EBC$ are not reflex angles. This implies that $BCDE$ is not concave.

As we've reached a contradiction, $f$ cannot exist. $\square$