Chapter 2

Sequences and Series

2.1 Sequences and limits

Analysis is essentially about taking limits. The most basic type of a limit is a limit of a sequence of real numbers. We have already seen sequences used informally. Let us give the formal definition.

Definition 2.1. A *sequence* (of real numbers) is a function $x\colon \mathbb{N} \to \mathbb{R}$. Instead of $x(n)$, we usually denote the nth element in the sequence by x_n. We use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty},$$

to denote a sequence.

A sequence $\{x_n\}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that

$$|x_n| \leq B \qquad \text{for all } n \in \mathbb{N}.$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded, or equivalently when it is bounded as a function.

When we need to give a concrete sequence we often give each term as a formula in terms of n. For example, $\{\frac{1}{n}\}_{n=1}^{\infty}$, or simply $\{\frac{1}{n}\}$, stands for the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$. The sequence $\{\frac{1}{n}\}$ is a bounded sequence ($B = 1$ suffices). On the other hand the sequence $\{n\}$ stands for $1, 2, 3, 4, \ldots$, and this sequence is not bounded (why?).

While the notation for a sequence is similar[1] to that of a set, the notions are distinct. For example, the sequence $\{(-1)^n\}$ is the sequence $-1, 1, -1, 1, -1, 1, \ldots$, whereas the set of values, the *range of the sequence*, is just the set $\{-1, 1\}$. We can write this set as $\{(-1)^n : n \in \mathbb{N}\}$. When ambiguity can arise, we use the words *sequence* or *set* to distinguish the two concepts.

Another example of a sequence is the so-called *constant sequence*. That is a sequence $\{c\} = c, c, c, c, \ldots$ consisting of a single constant $c \in \mathbb{R}$ repeating indefinitely.

We now get to the idea of a *limit of a sequence*. We will see in Proposition 2.6 that the notation below is well-defined. That is, if a limit exists, then it is unique. So it makes sense to talk about *the* limit of a sequence.

[1] [1] use the notation (x_n) to denote a sequence instead of $\{x_n\}$, which is what [2] uses. Both are common.

Definition 2.2. A sequence $\{x_n\}$ is said to *converge* to a number $x \in \mathbb{R}$ if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq M$. The number x is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n := x.$$

A sequence that converges is said to be *convergent*. Otherwise, we say the sequence *diverges* or that it is *divergent*.

It is good to know intuitively what a limit means. It means that eventually every number in the sequence is close to the number x. More precisely, we can get arbitrarily close to the limit, provided we go far enough in the sequence. It does not mean we ever reach the limit. It is possible, and quite common, that there is no x_n in the sequence that equals the limit x. We illustrate the concept in Figure 2.1. In the figure we first think of the sequence as a graph, as it is a function of $\mathbb{N}$. Secondly we also plot it as a sequence of labeled points on the real line.

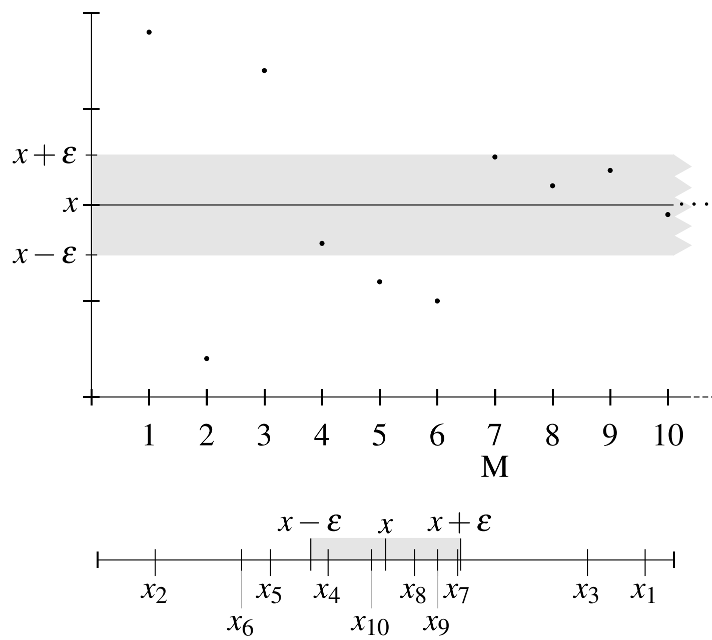

Figure 2.1: Illustration of convergence. On top, we show the first ten points of the sequence as a graph with M and the interval around the limit x marked. On bottom, the points of the same sequence are marked on the number line.

When we write $\lim x_n = x$ for some real number x, we are saying two things: first, that $\{x_n\}$ is convergent, and second, that the limit is x.

The definition above is one of the most important definitions in analysis, and it is necessary to understand it perfectly. The key point in the definition is that given *any* $\epsilon > 0$, we can find an M. The M can depend on ϵ, so we only pick an M once we know ϵ. Let us illustrate convergence on a few examples.

Example 2.3: The constant sequence $1, 1, 1, 1, \ldots$ is convergent and the limit is 1. For every $\epsilon > 0$, we pick $M = 1$.

Example 2.4: Claim: *The sequence* $\{\frac{1}{n}\}$ *is convergent and*

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proof: Given an $\epsilon > 0$, we find an $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \epsilon$ (Archimedean property at work). Then for all $n \geq M$,

$$|x_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{M} < \epsilon.$$

Example 2.5: The sequence $\{(-1)^n\}$ is divergent. Proof: If there were a limit x, then for $\epsilon = \frac{1}{2}$ we expect an M that satisfies the definition. Suppose such an M exists. Then for an even $n \geq M$ we compute

$$\frac{1}{2} > |x_n - x| = |1 - x| \qquad \text{and} \qquad \frac{1}{2} > |x_{n+1} - x| = |-1 - x| \,.$$

But

$$2 = |1 - x - (-1 - x)| \leq |1 - x| + |-1 - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

and that is a contradiction.

Proposition 2.6. *A convergent sequence has a unique limit.*

The proof of this proposition exhibits a useful technique in analysis. Many proofs follow the same general scheme. We want to show a certain quantity is zero. We write the quantity using the triangle inequality as two quantities, and we estimate each one by arbitrarily small numbers.

Proof. Suppose the sequence $\{x_n\}$ has limits x and y. Take an arbitrary $\epsilon > 0$. From the definition find an M_1 such that for all $n \geq M_1$, $|x_n - x| < \frac{\epsilon}{2}$. Similarly, find an M_2 such that for all $n \geq M_2$, we have $|x_n - y| < \frac{\epsilon}{2}$. Now take an n such that $n \geq M_1$ and also $n \geq M_2$, and estimate

$$\begin{aligned} |y - x| &= |x_n - x - (x_n - y)| \\ &\leq |x_n - x| + |x_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As $|y - x| < \epsilon$ for all $\epsilon > 0$, then $|y - x| = 0$ and $y = x$. Hence the limit (if it exists) is unique. □

Proposition 2.7. *A convergent sequence* $\{x_n\}$ *is bounded.*

Proof. Suppose $\{x_n\}$ converges to x. Thus there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $|x_n - x| < 1$. Let $B_1 := |x| + 1$ and note that for $n \geq M$,

$$\begin{aligned} |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x| = B_1. \end{aligned}$$

The set $\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}$ is a finite set and hence let

$$B_2 := \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}.$$

Let $B := \max\{B_1, B_2\}$. Then for all $n \in \mathbb{N}$,

$$|x_n| \leq B. \qquad \square$$

The sequence $\{(-1)^n\}$ shows that the converse does not hold. A bounded sequence is not necessarily convergent.

Example 2.8: Let us show the sequence $\left\{\frac{n^2+1}{n^2+n}\right\}$ converges and

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+n} = 1.$$

Given $\epsilon > 0$, find $M \in \mathbb{N}$ such that $\frac{1}{M} < \epsilon$. Then for all $n \geq M$,

$$\begin{aligned}
\left| \frac{n^2+1}{n^2+n} - 1 \right| = \left| \frac{n^2+1-(n^2+n)}{n^2+n} \right| &= \left| \frac{1-n}{n^2+n} \right| \\
&= \frac{n-1}{n^2+n} \\
&\leq \frac{n}{n^2+n} = \frac{1}{n+1} \\
&\leq \frac{1}{n} \leq \frac{1}{M} < \epsilon.
\end{aligned}$$

Therefore, $\lim \frac{n^2+1}{n^2+n} = 1$. This example shows that sometimes to get what you want, you must throw away some information to get a simpler estimate.

2.1.1 Monotone sequences

The simplest type of a sequence is a monotone sequence. Checking that a monotone sequence converges is as easy as checking that it is bounded. It is also easy to find the limit for a convergent monotone sequence, provided we can find the supremum or infimum of a countable set of numbers.

Definition 2.9. A sequence $\{x_n\}$ is *monotone increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}$ is *monotone decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If a sequence is either monotone increasing or monotone decreasing, we can simply say the sequence is *monotone*.[2]

For example, $\{n\}$ is monotone increasing, $\{\frac{1}{n}\}$ is monotone decreasing, the constant sequence $\{1\}$ is both monotone increasing and monotone decreasing, and $\{(-1)^n\}$ is not monotone. First few terms of a sample monotone increasing sequence are shown in Figure 2.2.

[2]Some authors use the word *monotonic*.

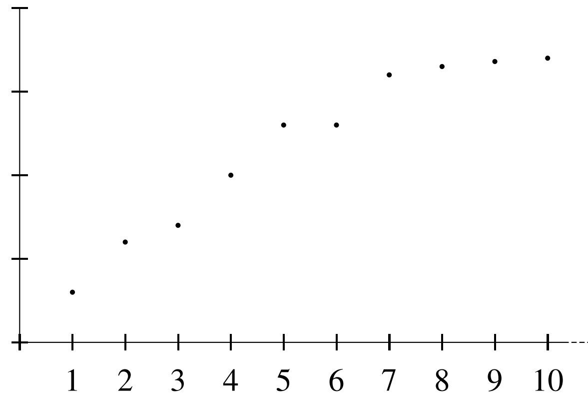

Figure 2.2: First few terms of a monotone increasing sequence as a graph.

Proposition 2.10. *A monotone sequence $\{x_n\}$ is bounded if and only if it is convergent.*

Furthermore, if $\{x_n\}$ is monotone increasing and bounded, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

If $\{x_n\}$ is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Proof. Consider a monotone increasing sequence $\{x_n\}$. Suppose the sequence is bounded, that is, the set $\{x_n : n \in \mathbb{N}\}$ is bounded. Let

$$x := \sup\{x_n : n \in \mathbb{N}\}.$$

Let $\epsilon > 0$ be arbitrary. As x is the supremum, then there must be at least one $M \in \mathbb{N}$ such that $x_M > x - \epsilon$ (because x is the supremum). As $\{x_n\}$ is monotone increasing, then it is easy to see (by induction) that $x_n \geq x_M$ for all $n \geq M$. Hence for all $n \geq M$,

$$|x_n - x| = x - x_n \leq x - x_M < \epsilon.$$

Therefore, the sequence converges to x, so a bounded monotone increasing sequence converges. For the other direction, we have already proved that a convergent sequence is bounded.

The proof for monotone decreasing sequences is left as an exercise. □

Example 2.11: Take the sequence $\{\frac{1}{\sqrt{n}}\}$.

The sequence is bounded below as $\frac{1}{\sqrt{n}} > 0$ for all $n \in \mathbb{N}$. Let us show that it is monotone decreasing. We start with $\sqrt{n+1} \geq \sqrt{n}$ (why is that true?). From this inequality we obtain

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}.$$

So the sequence is monotone decreasing and bounded below (hence bounded). Proposition 2.10 says that that the sequence is convergent and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \inf\left\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\right\}.$$

We already know that the infimum is greater than or equal to 0, as 0 is a lower bound. Take a number $b \geq 0$ such that $b \leq \frac{1}{\sqrt{n}}$ for all n. We square both sides to obtain

$$b^2 \leq \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

We have seen before that this implies that $b^2 \leq 0$ (a consequence of the Archimedean property). As $b^2 \geq 0$ as well, we have $b^2 = 0$ and so $b = 0$. Hence, $b = 0$ is the greatest lower bound, and $\lim \frac{1}{\sqrt{n}} = 0$.

Example 2.12: A word of caution: We must show that a monotone sequence is bounded in order to use Proposition 2.10 to conclude a sequence converges. The sequence $\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\}$ is a monotone increasing sequence that grows very slowly. We will see, once we get to series, that this sequence has no upper bound and so does not converge. It is not at all obvious that this sequence has no upper bound.

A common example of where monotone sequences arise is the following proposition. The proof is left as an exercise.

Proposition 2.13. *Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in S$ and*

$$\sup S = \lim_{n\to\infty} x_n \qquad \textit{and} \qquad \inf S = \lim_{n\to\infty} y_n.$$

2.1.2 Tail of a sequence

Definition 2.14. For a sequence $\{x_n\}$, the K*-tail* (where $K \in \mathbb{N}$), or just the *tail*, of $\{x_n\}$ is the sequence starting at $K+1$, usually written as

$$\{x_{n+K}\}_{n=1}^{\infty} \qquad \text{or} \qquad \{x_n\}_{n=K+1}^{\infty}.$$

For example, the 4-tail of $\{\frac{1}{n}\}$ is $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$. The 0-tail of a sequence is the sequence itself. The convergence and the limit of a sequence only depends on its tail.

Proposition 2.15. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then the following statements are equivalent:*

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(ii) The K-tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.

(iii) The K-tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

Furthermore, if any (and hence all) of the limits exist, then for all $K \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}.$$

Proof. It is clear that ii implies iii. We will therefore show first that i implies ii, and then we will show that iii implies i. That is,

$$\begin{tikzcd} & \text{ii} \arrow[dr, Rightarrow] & \\ \text{i} \arrow[ur, Rightarrow, "\text{to prove}"] & & \text{iii} \arrow[ll, Rightarrow, "\text{to prove}"] \end{tikzcd}$$

In the process we will also show that the limits are equal.

We start with i implies ii. Suppose $\{x_n\}$ converges to some $x \in \mathbb{R}$. Let $K \in \mathbb{N}$ be arbitrary, and define $y_n := x_{n+K}$. We wish to show that $\{y_n\}$ converges to x. Given an $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x - x_n| < \epsilon$ for all $n \geq M$. Note that $n \geq M$ implies $n + K \geq M$. Therefore, for all $n \geq M$, we have

$$|x - y_n| = |x - x_{n+K}| < \epsilon.$$

Consequently, $\{y_n\}$ converges to x.

Let us move to iii implies i. Let $K \in \mathbb{N}$ be given, define $y_n := x_{n+K}$, and suppose that $\{y_n\}$ converges to $x \in \mathbb{R}$. That is, given an $\epsilon > 0$, there exists an $M' \in \mathbb{N}$ such that $|x - y_n| < \epsilon$ for all $n \geq M'$. Let $M := M' + K$. Then $n \geq M$ implies $n - K \geq M'$. Thus, whenever $n \geq M$, we have

$$|x - x_n| = |x - y_{n-K}| < \epsilon.$$

Therefore, $\{x_n\}$ converges to x. $\square$

At the end of the day, the limit does not care about how the sequence begins, it only cares about the tail of the sequence. The beginning of the sequence may be arbitrary.

For example, the sequence defined by $x_n := \frac{n}{n^2+16}$ is decreasing if we start at $n = 4$ (it is increasing before). That is: $\{x_n\} = \frac{1}{17}, \frac{1}{10}, \frac{3}{25}, \frac{1}{8}, \frac{5}{41}, \frac{3}{26}, \frac{7}{65}, \frac{1}{10}, \frac{9}{97}, \frac{5}{58}, \ldots$, and

$$\frac{1}{17} < \frac{1}{10} < \frac{3}{25} < \frac{1}{8} > \frac{5}{41} > \frac{3}{26} > \frac{7}{65} > \frac{1}{10} > \frac{9}{97} > \frac{5}{58} > \ldots .$$

If we throw away the first 3 terms and look at the 3-tail, it is decreasing. The proof is left as an exercise. Since the 3-tail is monotone and bounded below by zero, it is convergent, and therefore the sequence is convergent.

2.1.3 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $\{x_n\}$ is a sequence that contains only some of the numbers from $\{x_n\}$ in the same order.

Definition 2.16. Let $\{x_n\}$ be a sequence. Let $\{n_i\}$ be a strictly increasing sequence of natural numbers, that is, $n_i < n_{i+1}$ for all i (in other words $n_1 < n_2 < n_3 < \cdots$). The sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a *subsequence* of $\{x_n\}$.

So the subsequence is the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$. Consider the sequence $\{\frac{1}{n}\}$. The sequence $\{\frac{1}{3n}\}$ is a subsequence. To see how these two sequences fit in the definition, take $n_i := 3i$. The numbers in the subsequence must come from the original sequence. So $1, 0, \frac{1}{3}, 0, \frac{1}{5}, \ldots$ is not a subsequence of $\{\frac{1}{n}\}$. Similarly, order must be preserved. So the sequence $1, \frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \ldots$ is not a subsequence of $\{\frac{1}{n}\}$.

A tail of a sequence is one special type of a subsequence. For an arbitrary subsequence, we have the following proposition about convergence.

Proposition 2.17. *If $\{x_n\}$ is a convergent sequence, then every subsequence $\{x_{n_i}\}$ is also convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

Proof. Suppose $\lim_{n\to\infty} x_n = x$. So for every $\epsilon > 0$ there is an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$|x_n - x| < \epsilon.$$

It is not hard to prove (do it!) by induction that $n_i \geq i$. Hence $i \geq M$ implies $n_i \geq M$. Thus, for all $i \geq M$,

$$|x_{n_i} - x| < \epsilon,$$

and we are done. □

Example 2.18: Existence of a convergent subsequence does not imply convergence of the sequence itself. Take the sequence $0, 1, 0, 1, 0, 1, \ldots$. That is, $x_n = 0$ if n is odd, and $x_n = 1$ if n is even. The sequence $\{x_n\}$ is divergent; however, the subsequence $\{x_{2n}\}$ converges to 1 and the subsequence $\{x_{2n+1}\}$ converges to 0. Compare Proposition 2.39.

2.1.4 Exercises

In the following exercises, feel free to use what you know from calculus to find the limit, if it exists. But you must *prove* that you found the correct limit, or prove that the series is divergent.

***Exercise* 2.1:** *Is the sequence $\{3n\}$ bounded? Prove or disprove.*

***Exercise* 2.2:** *Is the sequence $\{n\}$ convergent? If so, what is the limit?*

***Exercise* 2.3:** *Is the sequence $\left\{\dfrac{(-1)^n}{2n}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.4:** *Is the sequence $\{2^{-n}\}$ convergent? If so, what is the limit?*

***Exercise* 2.5:** *Is the sequence $\left\{\dfrac{n}{n+1}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.6:** *Is the sequence $\left\{\dfrac{n}{n^2+1}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.7:** *Let $\{x_n\}$ be a sequence.*

a) Show that $\lim x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim |x_n| = 0$.

b) Find an example such that $\{|x_n|\}$ converges and $\{x_n\}$ diverges.

***Exercise* 2.8:** *Is the sequence $\left\{\dfrac{2^n}{n!}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.9:** *Show that the sequence $\left\{\dfrac{1}{\sqrt[3]{n}}\right\}$ is monotone and bounded. Then use Proposition 2.10 to find the limit.*

***Exercise* 2.10:** *Show that the sequence $\left\{\dfrac{n+1}{n}\right\}$ is monotone and bounded. Then use Proposition 2.10 to find the limit.*

Exercise* 2.11*: *Finish the proof of Proposition 2.10 for monotone decreasing sequences.*

Exercise* 2.12*: *Prove Proposition 2.13.*

Exercise* 2.13*: *Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

Exercise* 2.14*: *Find a convergent subsequence of the sequence $\{(-1)^n\}$.*

Exercise* 2.15*: *Let $\{x_n\}$ be a sequence defined by*

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

a) Is the sequence bounded? (prove or disprove)

b) Is there a convergent subsequence? If so, find it.

Exercise* 2.16*: *Let $\{x_n\}$ be a sequence. Suppose there are two convergent subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$. Suppose*

$$\lim_{i\to\infty} x_{n_i} = a \qquad \text{and} \qquad \lim_{i\to\infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}$ is not convergent, without using Proposition 2.17.

Exercise* 2.17** (Tricky):*** *Find a sequence $\{x_n\}$ such that for every $y \in \mathbb{R}$, there exists a subsequence $\{x_{n_i}\}$ converging to y.*

Exercise* 2.18** (Easy):*** *Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$. Suppose for every $\epsilon > 0$, there is an M such that $|x_n - x| \leq \epsilon$ for all $n \geq M$. Show that $\lim x_n = x$.*

Exercise* 2.19** (Easy):*** *Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$ such that there exists a $k \in \mathbb{N}$ such that for all $n \geq k$, $x_n = x$. Prove that $\{x_n\}$ converges to x.*

Exercise* 2.20*: *Let $\{x_n\}$ be a sequence and define a sequence $\{y_n\}$ by $y_{2k} := x_{k^2}$ and $y_{2k-1} := x_k$ for all $k \in \mathbb{N}$. Prove that $\{x_n\}$ converges if and only if $\{y_n\}$ converges. Furthermore, prove that if they converge, then $\lim x_n = \lim y_n$.*

Exercise* 2.21*: *Show that the 3-tail of the sequence defined by $x_n := \frac{n}{n^2+16}$ is monotone decreasing. Hint: Suppose $n \geq m \geq 4$ and consider the numerator of the expression $x_n - x_m$.*

Exercise* 2.22*: *Suppose that $\{x_n\}$ is a sequence such that the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$, and $\{x_{3n}\}$ all converge. Show that $\{x_n\}$ is convergent.*

Exercise* 2.23*: *Suppose that $\{x_n\}$ is a monotone increasing sequence that has a convergent subsequence. Show that $\{x_n\}$ is convergent. Note: So Proposition 2.17 is an "if and only if" for monotone sequences.*

2.2 Facts about limits of sequences

In this section we go over some basic results about the limits of sequences. We start by looking at how sequences interact with inequalities.

2.2.1 Limits and inequalities

A basic lemma about limits and inequalities is the so-called squeeze lemma. It allows us to show convergence of sequences in difficult cases if we find two other simpler convergent sequences that "squeeze" the original sequence.

Lemma 2.19 (Squeeze lemma). *Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that*

$$a_n \leq x_n \leq b_n \quad \textit{for all } n \in \mathbb{N}.$$

Suppose $\{a_n\}$ and $\{b_n\}$ converge and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n .$$

Then $\{x_n\}$ converges and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n .$$

Proof. Let $x := \lim a_n = \lim b_n$. Let $\epsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have that $|a_n - x| < \epsilon$, and an M_2 such that for all $n \geq M_2$, we have $|b_n - x| < \epsilon$. Set $M := \max\{M_1, M_2\}$. Suppose $n \geq M$. In particular, $x - a_n < \epsilon$, or $x - \epsilon < a_n$. Similarly, $b_n < x + \epsilon$. Putting everything together, we find

$$x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon .$$

In other words, $-\epsilon < x_n - x < \epsilon$ or $|x_n - x| < \epsilon$. So $\{x_n\}$ converges to x. See Figure 2.3. $\square$

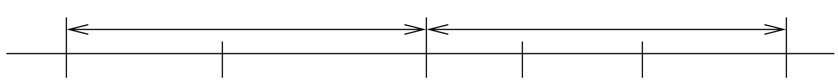

Figure 2.3: Squeeze lemma proof in picture.

Example 2.20: One application of the squeeze lemma is to compute limits of sequences using limits that we already know. For example, consider the sequence $\{\frac{1}{n\sqrt{n}}\}$. Since $\sqrt{n} \geq 1$ for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. We already know $\lim \frac{1}{n} = 0$. Hence, using the constant sequence $\{0\}$ and the sequence $\{\frac{1}{n}\}$ in the squeeze lemma, we conclude

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0 .$$

Limits, when they exist, preserve non-strict inequalities.

Lemma 2.21. *Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences and*

$$x_n \leq y_n,$$

for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. Let $x := \lim x_n$ and $y := \lim y_n$. Let $\epsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \frac{\epsilon}{2}$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \frac{\epsilon}{2}$. In particular, for some $n \geq \max\{M_1, M_2\}$, we have $x - x_n < \frac{\epsilon}{2}$ and $y_n - y < \frac{\epsilon}{2}$. We add these inequalities to obtain

$$y_n - x_n + x - y < \epsilon, \qquad \text{or} \qquad y_n - x_n < y - x + \epsilon.$$

Since $x_n \leq y_n$, we have $0 \leq y_n - x_n$ and hence $0 < y - x + \epsilon$. In other words,

$$x - y < \epsilon.$$

Because $\epsilon > 0$ was arbitrary, we obtain $x - y \leq 0$. Therefore, $x \leq y$. $\square$

The next corollary follows by using constant sequences in Lemma 2.21. The proof is left as an exercise.

Corollary 2.22.

(i) If $\{x_n\}$ is a convergent sequence such that $x_n \geq 0$, then

$$\lim_{n\to\infty} x_n \geq 0.$$

(ii) Let $a, b \in \mathbb{R}$ and let $\{x_n\}$ be a convergent sequence such that

$$a \leq x_n \leq b,$$

for all $n \in \mathbb{N}$. Then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

In Lemma 2.21 and Corollary 2.22 we cannot simply replace all the non-strict inequalities with strict inequalities. For example, let $x_n := \frac{-1}{n}$ and $y_n := \frac{1}{n}$. Then $x_n < y_n$, $x_n < 0$, and $y_n > 0$ for all n. However, these inequalities are not preserved by the limit operation as $\lim x_n = \lim y_n = 0$. The moral of this example is that strict inequalities may become non-strict inequalities when limits are applied; if we know $x_n < y_n$ for all n, we may only conclude

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

This issue is a common source of errors.

2.2.2 Continuity of algebraic operations

Limits interact nicely with algebraic operations.

Proposition 2.23. *Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.*

(i) The sequence $\{z_n\}$, where $z_n := x_n + y_n$, converges and

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(ii) The sequence $\{z_n\}$, where $z_n := x_n - y_n$, converges and

$$\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n.$$

(iii) The sequence $\{z_n\}$, where $z_n := x_n y_n$, converges and

$$\lim_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} z_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

(iv) If $\lim y_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{z_n\}$, where $z_n := \frac{x_n}{y_n}$, converges and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} z_n = \frac{\lim x_n}{\lim y_n}.$$

Proof. We start with i. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n + y_n$. Let $x := \lim x_n$, $y := \lim y_n$, and $z := x + y$.

Let $\epsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \frac{\epsilon}{2}$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \frac{\epsilon}{2}$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$, we have

$$\begin{aligned} |z_n - z| &= |(x_n + y_n) - (x + y)| \\ &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore i is proved. Proof of ii is almost identical and is left as an exercise.

Let us tackle iii. Suppose again that $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n y_n$. Let $x := \lim x_n$, $y := \lim y_n$, and $z := xy$.

Let $\epsilon > 0$ be given. Let $K := \max\{|x|, |y|, \frac{\epsilon}{3}, 1\}$. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \frac{\epsilon}{3K}$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \frac{\epsilon}{3K}$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$, we have

$$\begin{aligned} |z_n - z| &= |(x_n y_n) - (xy)| \\ &= |(x_n - x + x)(y_n - y + y) - xy| \\ &= |(x_n - x)y + x(y_n - y) + (x_n - x)(y_n - y)| \\ &\leq |(x_n - x)y| + |x(y_n - y)| + |(x_n - x)(y_n - y)| \\ &= |x_n - x|\,|y| + |x|\,|y_n - y| + |x_n - x|\,|y_n - y| \\ &< \frac{\epsilon}{3K}K + K\frac{\epsilon}{3K} + \frac{\epsilon}{3K}\frac{\epsilon}{3K} \qquad \text{(now notice that } \tfrac{\epsilon}{3K} \leq 1 \text{ and } K \geq 1\text{)} \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Finally, we examine iv. Instead of proving iv directly, we prove the following simpler claim:

Claim: *If $\{y_n\}$ is a convergent sequence such that* $\lim\, y_n \neq 0$ *and* $y_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\{\frac{1}{y_n}\}$ *converges and*

$$\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{\lim\, y_n}.$$

Once the claim is proved, we take the sequence $\{\frac{1}{y_n}\}$, multiply it by the sequence $\{x_n\}$ and apply item iii.

Proof of claim: Let $\epsilon > 0$ be given. Let $y := \lim\, y_n$. As $|y| \neq 0$, then $\min\left\{|y|^2 \frac{\epsilon}{2}, \frac{|y|}{2}\right\} > 0$. Find an M such that for all $n \geq M$, we have

$$|y_n - y| < \min\left\{|y|^2 \frac{\epsilon}{2}, \frac{|y|}{2}\right\}.$$

For all $n \geq M$, we have $|y - y_n| < \frac{|y|}{2}$, and so

$$|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \frac{|y|}{2} + |y_n|.$$

Subtracting $\frac{|y|}{2}$ from both sides we obtain $\frac{|y|}{2} < |y_n|$, or in other words,

$$\frac{1}{|y_n|} < \frac{2}{|y|}.$$

We finish the proof of the claim:

$$\begin{aligned}
\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y - y_n}{y y_n}\right| \\
&= \frac{|y - y_n|}{|y|\,|y_n|} \\
&\leq \frac{|y - y_n|}{|y|} \frac{2}{|y|} \\
&< \frac{|y|^2 \frac{\epsilon}{2}}{|y|} \frac{2}{|y|} = \epsilon.
\end{aligned}$$

And we are done. □

By plugging in constant sequences, we get several easy corollaries. If $c \in \mathbb{R}$ and $\{x_n\}$ is a convergent sequence, then for example

$$\lim_{n\to\infty} c x_n = c \left(\lim_{n\to\infty} x_n\right) \qquad \text{and} \qquad \lim_{n\to\infty} (c + x_n) = c + \lim_{n\to\infty} x_n.$$

Similarly, we find such equalities for constant subtraction and division.

As we can take limits past multiplication we can show (exercise) that $\lim\, x_n^k = (\lim\, x_n)^k$ for all $k \in \mathbb{N}$. That is, we can take limits past powers. Let us see if we can do the same with roots.

Proposition 2.24. *Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$. Then*

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}.$$

Of course, to even make this statement, we need to apply Corollary 2.22 to show that $\lim x_n \geq 0$, so that we can take the square root without worry.

Proof. Let $\{x_n\}$ be a convergent sequence and let $x := \lim x_n$. As we just mentioned, $x \geq 0$.

First suppose $x = 0$. Let $\epsilon > 0$ be given. Then there is an M such that for all $n \geq M$, we have $x_n = |x_n| < \epsilon^2$, or in other words, $\sqrt{x_n} < \epsilon$. Hence,

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \sqrt{x_n} < \epsilon.$$

Now suppose $x > 0$ (and hence $\sqrt{x} > 0$).

$$\begin{aligned}
\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\
&= \frac{1}{\sqrt{x_n} + \sqrt{x}} \left|x_n - x\right| \\
&\leq \frac{1}{\sqrt{x}} \left|x_n - x\right|.
\end{aligned}$$

We leave the rest of the proof to the reader. □

A similar proof works for the kth root. That is, we also obtain $\lim x_n^{1/k} = (\lim x_n)^{1/k}$. We leave this to the reader as a challenging exercise.

We may also want to take the limit past the absolute value sign. The converse of this proposition is not true, see Exercise 2.7 part b).

Proposition 2.25. *If $\{x_n\}$ is a convergent sequence, then $\{|x_n|\}$ is convergent and*

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|.$$

Proof. We simply note the reverse triangle inequality

$$\big|\, |x_n| - |x| \,\big| \leq |x_n - x|.$$

Hence if $|x_n - x|$ can be made arbitrarily small, so can $\big|\, |x_n| - |x| \,\big|$. Details are left to the reader. □

Let us see an example putting the propositions above together. Since $\lim \frac{1}{n} = 0$, then

$$\lim_{n\to\infty} \left| \sqrt{1 + \frac{1}{n}} - \frac{100}{n^2} \right| = \left| \sqrt{1 + (\lim \frac{1}{n})} - 100(\lim \frac{1}{n})(\lim \frac{1}{n}) \right| = 1.$$

That is, the limit on the left-hand side exists because the right-hand side exists. You really should read the equality above from right to left.

On the other hand you must apply the propositions carefully. For example, by rewriting the expression with common denominator first we find

$$\lim_{n\to\infty} \left(\frac{n^2}{n+1} - n \right) = -1.$$

However, $\left\{\frac{n^2}{n+1}\right\}$ and $\{n\}$ are not convergent, so $\left(\lim \frac{n^2}{n+1}\right) - \left(\lim n\right)$ is nonsense.

2.2.3 Recursively defined sequences

Now that we know we can interchange limits and algebraic operations, we can compute the limits of many sequences. One such class are recursively defined sequences, that is, sequences where the next number in the sequence is computed using a formula from a fixed number of preceding elements in the sequence.

Example 2.26: Let $\{x_n\}$ be defined by $x_1 := 2$ and

$$x_{n+1} := x_n - \frac{x_n^2 - 2}{2x_n}.$$

We must first find out if this sequence is well-defined; we must show we never divide by zero. Then we must find out if the sequence converges. Only then can we attempt to find the limit.

So let us prove x_n exists and $x_n > 0$ for all n (so the sequence is well-defined and bounded below). Let us show this by induction. We know that $x_1 = 2 > 0$. For the induction step, suppose $x_n > 0$. Then

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

It is always true that $x_n^2 + 2 > 0$, and as $x_n > 0$, then $\frac{x_n^2+2}{2x_n} > 0$ and hence $x_{n+1} > 0$.

Next let us show that the sequence is monotone decreasing. If we show that $x_n^2 - 2 \geq 0$ for all n, then $x_{n+1} \leq x_n$ for all n. Obviously $x_1^2 - 2 = 4 - 2 = 2 > 0$. For an arbitrary n, we have

$$x_{n+1}^2 - 2 = \left(\frac{x_n^2 + 2}{2x_n}\right)^2 - 2 = \frac{x_n^4 + 4x_n^2 + 4 - 8x_n^2}{4x_n^2} = \frac{x_n^4 - 4x_n^2 + 4}{4x_n^2} = \frac{\left(x_n^2 - 2\right)^2}{4x_n^2}.$$

Since squares are nonnegative, $x_{n+1}^2 - 2 \geq 0$ for all n. Therefore, $\{x_n\}$ is monotone decreasing and bounded ($x_n > 0$ for all n), and so the limit exists. It remains to find the limit.

Write

$$2x_n x_{n+1} = x_n^2 + 2.$$

Since $\{x_{n+1}\}$ is the 1-tail of $\{x_n\}$, it converges to the same limit. Let us define $x := \lim x_n$. Take the limit of both sides to obtain

$$2x^2 = x^2 + 2,$$

or $x^2 = 2$. As $x_n > 0$ for all n we get $x \geq 0$, and therefore $x = \sqrt{2}$.

You may have seen the sequence above before. It is *Newton's method*[3] for finding the square root of 2. This method comes up often in practice and converges very rapidly. We used the fact that $x_1^2 - 2 > 0$, although it was not strictly needed to show convergence by considering a tail of the sequence. The sequence converges as long as $x_1 \neq 0$, although with a negative x_1 we would arrive at $x = -\sqrt{2}$. By replacing the 2 in the numerator we obtain the square root of any positive number. These statements are left as an exercise.

You should, however, be careful. Before taking any limits, you must make sure the sequence converges. Let us see an example.

[3]Named after the English physicist and mathematician Isaac Newton (1642–1726/7).

Example 2.27: Suppose $x_1 := 1$ and $x_{n+1} := x_n^2 + x_n$. If we blindly assumed that the limit exists (call it x), then we would get the equation $x = x^2 + x$, from which we might conclude $x = 0$. However, it is not hard to show that $\{x_n\}$ is unbounded and therefore does not converge.

The thing to notice in this example is that the method still works, but it depends on the initial value x_1. If we set $x_1 := 0$, then the sequence converges and the limit really is 0. An entire branch of mathematics, called dynamics, deals precisely with these issues. See Exercise 2.37.

2.2.4 Some convergence tests

It is not always necessary to go back to the definition of convergence to prove that a sequence is convergent. We first give a simple convergence test. The main idea is that $\{x_n\}$ converges to x if and only if $\{|x_n - x|\}$ converges to zero.

Proposition 2.28. *Let $\{x_n\}$ be a sequence. Suppose there is an $x \in \mathbb{R}$ and a convergent sequence $\{a_n\}$ such that*

$$\lim_{n\to\infty} a_n = 0$$

and

$$|x_n - x| \leq a_n \qquad \text{for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ converges and $\lim x_n = x$.

Proof. Let $\epsilon > 0$ be given. Note that $a_n \geq 0$ for all n. Find an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $a_n = |a_n - 0| < \epsilon$. Then, for all $n \geq M$, we have

$$|x_n - x| \leq a_n < \epsilon. \qquad \square$$

As the proposition shows, to study when a sequence has a limit is the same as studying when another sequence goes to zero. In general, it may be hard to decide if a sequence converges, but for certain sequences there exist easy to apply tests that tell us if the sequence converges or not. Let us see one such test. First, let us compute the limit of a certain specific sequence.

Proposition 2.29. *Let $c > 0$.*

(i) If $c < 1$, then

$$\lim_{n\to\infty} c^n = 0.$$

(ii) If $c > 1$, then $\{c^n\}$ is unbounded.

Proof. First consider $c < 1$. As $c > 0$, then $c^n > 0$ for all $n \in \mathbb{N}$ by induction. As $c < 1$, then $c^{n+1} < c^n$ for all n. So $\{c^n\}$ is a decreasing sequence that is bounded below. Hence, it is convergent. Let $L := \lim c^n$. The 1-tail $\{c^{n+1}\}$ also converges to L. Taking the limit of both sides of $c^{n+1} = c \cdot c^n$, we obtain $L = cL$, or $(1-c)L = 0$. It follows that $L = 0$ as $c \neq 1$.

Now consider $c > 1$. Let $B > 0$ be arbitrary. As $\frac{1}{c} < 1$, then $\left\{\left(\frac{1}{c}\right)^n\right\}$ converges to 0. Hence for some large enough n, we get

$$\frac{1}{c^n} = \left(\frac{1}{c}\right)^n < \frac{1}{B}.$$

In other words, $c^n > B$, and B is not an upper bound for $\{c^n\}$. As B was arbitrary, $\{c^n\}$ is unbounded. $\square$

In the proposition above, the ratio of the $(n+1)$th term and the nth term is c. We generalize this simple result to a larger class of sequences. The following lemma will come up again once we get to series.

Lemma 2.30 (Ratio test for sequences)**.** *Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n and such that the limit*

$$L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \qquad \textit{exists.}$$

(i) If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$.

(ii) If $L > 1$, then $\{x_n\}$ is unbounded (hence diverges).

If L exists, but $L = 1$, the lemma says nothing. We cannot make any conclusion based on that information alone. For example, the sequence $\{\frac{1}{n}\}$ converges to zero, but $L = 1$. The constant sequence $\{1\}$ converges to 1, not zero, and $L = 1$. The sequence $\{(-1)^n\}$ does not converge at all, and $L = 1$ as well. Finally, the sequence $\{n\}$ is unbounded, yet again $L = 1$. The statement of the lemma may be strengthened somewhat, see exercises 2.36 and 2.54.

Proof. Suppose $L < 1$. As $\frac{|x_{n+1}|}{|x_n|} \geq 0$ for all n, then $L \geq 0$. Pick r such that $L < r < 1$. We wish to compare the sequence $\{x_n\}$ to the sequence $\{r^n\}$. The idea is that while the ratio $\frac{|x_{n+1}|}{|x_n|}$ is not going to be less than L eventually, it will eventually be less than r, which is still less than 1. The intuitive idea of the proof is illustrated in Figure 2.4.

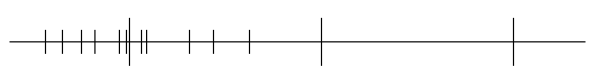

Figure 2.4: Proof of ratio test in picture. The short lines represent the ratios $\frac{|x_{n+1}|}{|x_n|}$ approaching L.

As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore, for $n \geq M$,

$$\frac{|x_{n+1}|}{|x_n|} - L < r - L \qquad \text{or} \qquad \frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M + 1$) write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| \, rr \cdots r = |x_M| \, r^{n-M} = (|x_M| \, r^{-M}) r^n.$$

The sequence $\{r^n\}$ converges to zero and hence $|x_M|\, r^{-M} r^n$ converges to zero. By Proposition 2.28, the M-tail of $\{x_n\}$ converges to zero and therefore $\{x_n\}$ converges to zero.

Now suppose $L > 1$. Pick r such that $1 < r < L$. As $L - r > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < L - r.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} > r.$$

Again for $n > M$, write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} > |x_M|\, rr \cdots r = |x_M|\, r^{n-M} = (|x_M|\, r^{-M}) r^n.$$

The sequence $\{r^n\}$ is unbounded (since $r > 1$), and so $\{x_n\}$ cannot be bounded (if $|x_n| \leq B$ for all n, then $r^n < \frac{B}{|x_M|} r^M$ for all $n > M$, which is impossible). Consequently, $\{x_n\}$ cannot converge. □

Example 2.31: A simple application of the lemma above is to prove

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

Proof: Compute

$$\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \frac{2}{n+1}.$$

It is not hard to see that $\left\{\frac{2}{n+1}\right\}$ converges to zero. The conclusion follows by the lemma.

Example 2.32: A more complicated (and useful) application of the ratio test is to prove

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Proof: Let $\epsilon > 0$ be given. Consider the sequence $\left\{\frac{n}{(1+\epsilon)^n}\right\}$. Compute

$$\frac{(n+1)/(1+\epsilon)^{n+1}}{n/(1+\epsilon)^n} = \frac{n+1}{n} \frac{1}{1+\epsilon}.$$

The limit of $\frac{n+1}{n} = 1 + \frac{1}{n}$ as $n \to \infty$ is 1, and so

$$\lim_{n\to\infty} \frac{(n+1)/(1+\epsilon)^{n+1}}{n/(1+\epsilon)^n} = \frac{1}{1+\epsilon} < 1.$$

Therefore, $\left\{\frac{n}{(1+\epsilon)^n}\right\}$ converges to 0. In particular, there exists an M such that for $n \geq M$, we have $\frac{n}{(1+\epsilon)^n} < 1$, or $n < (1+\epsilon)^n$, or $n^{1/n} < 1 + \epsilon$. As $n \geq 1$, then $n^{1/n} \geq 1$, and so $0 \leq n^{1/n} - 1 < \epsilon$. Consequently, $\lim n^{1/n} = 1$.

2.2.5 Exercises

Exercise* 2.24*: *Prove Corollary 2.22. Hint: Use constant sequences and Lemma 2.21.*

Exercise* 2.25*: *Prove part ii of Proposition 2.23.*

Exercise* 2.26*: *Prove that if $\{x_n\}$ is a convergent sequence, $k \in \mathbb{N}$, then*

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Hint: Use induction.

Exercise* 2.27*: *Suppose $x_1 := \frac{1}{2}$ and $x_{n+1} := x_n^2$. Show that $\{x_n\}$ converges and find $\lim x_n$. Hint: You cannot divide by zero!*

Exercise* 2.28*: *Let $x_n := \frac{n-\cos(n)}{n}$. Use the squeeze lemma to show that $\{x_n\}$ converges and find the limit.*

Exercise* 2.29*: *Let $x_n := \frac{1}{n^2}$ and $y_n := \frac{1}{n}$. Define $z_n := \frac{x_n}{y_n}$ and $w_n := \frac{y_n}{x_n}$. Do $\{z_n\}$ and $\{w_n\}$ converge? What are the limits? Can you apply Proposition 2.23? Why or why not?*

Exercise* 2.30*: *True or false, prove or find a counterexample. If $\{x_n\}$ is a sequence such that $\{x_n^2\}$ converges, then $\{x_n\}$ converges.*

Exercise* 2.31*: *Show that*

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

Exercise* 2.32*: *Suppose $\{x_n\}$ is a sequence and suppose for some $x \in \mathbb{R}$, the limit*

$$L := \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Show that $\{x_n\}$ converges to x.

Exercise* 2.33** (Challenging):*** *Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ and $k \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} x_n^{1/k} = \left(\lim_{n\to\infty} x_n\right)^{1/k}.$$

Hint: Find an expression q such that $\frac{x_n^{1/k} - x^{1/k}}{x_n - x} = \frac{1}{q}$.

Exercise* 2.34*: *Let $r > 0$. Show that starting with an arbitrary $x_1 \neq 0$, the sequence defined by*

$$x_{n+1} := x_n - \frac{x_n^2 - r}{2x_n}$$

converges to $\sqrt{r}$ if $x_1 > 0$ and $-\sqrt{r}$ if $x_1 < 0$.

Exercise* 2.35*:

a) Suppose $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0. Show that $\{a_n b_n\}$ converges to 0.

b) Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_n b_n\}$ is not convergent.

c) Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_n b_n\}$ is not convergent.

Exercise* 2.36** (Easy):*** *Prove the following stronger version of Lemma 2.30, the ratio test. Suppose $\{x_n\}$ is a sequence such that $x_n \neq 0$ for all n.*

a) Prove that if there exists an $r < 1$ and $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\frac{|x_{n+1}|}{|x_n|} \leq r,$$

then $\{x_n\}$ converges to 0.

b) Prove that if there exists an $r > 1$ and $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\frac{|x_{n+1}|}{|x_n|} \geq r,$$

then $\{x_n\}$ is unbounded.

Exercise* 2.37*: *Suppose $x_1 := c$ and $x_{n+1} := x_n^2 + x_n$. Show that $\{x_n\}$ converges if and only if $-1 \leq c \leq 0$, in which case it converges to 0.*

Exercise* 2.38*: *Prove $\lim\limits_{n\to\infty} (n^2 + 1)^{1/n} = 1$.*

Exercise* 2.39*: *Prove that $\left\{(n!)^{1/n}\right\}$ is unbounded. Hint: Show that $\left\{\frac{C^n}{n!}\right\}$ converges to zero for all $C > 0$.*

2.3 Limit superior, limit inferior, and Bolzano–Weierstrass

In this section we study bounded sequences and their subsequences. In particular, we define the so-called limit superior and limit inferior of a bounded sequence and talk about limits of subsequences. Furthermore, we prove the Bolzano–Weierstrass theorem[4], an indispensable tool in analysis, showing the existence of convergent subsequences.

We have seen that every convergent sequence is bounded, although there exist many bounded divergent sequences. For example, the sequence $\{(-1)^n\}$ is bounded, but divergent. All is not lost, however, and we can still compute certain limits with a bounded divergent sequence.

2.3.1 Upper and lower limits

There are ways of creating monotone sequences out of any sequence, and in this fashion we get the so-called *limit superior* and *limit inferior*. These limits always exist for bounded sequences.

If a sequence $\{x_n\}$ is bounded, then the set $\{x_k : k \in \mathbb{N}\}$ is bounded. For every n, the set $\{x_k : k \geq n\}$ is also bounded (as it is a subset), so we take its supremum and infimum.

Definition 2.33. Let $\{x_n\}$ be a bounded sequence. Define the sequences $\{a_n\}$ and $\{b_n\}$ by $a_n := \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$. Define, if the limits exist,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n,$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n.$$

For a bounded sequence, liminf and limsup always exist (see below). It is possible to define liminf and limsup for unbounded sequences if we allow ∞ and $-\infty$, and we do so later in this section. It is not hard to generalize the following results to include unbounded sequences; however, we first restrict our attention to bounded ones.

Proposition 2.34. *Let $\{x_n\}$ be a bounded sequence. Let a_n and b_n be as in the definition above.*

(i) The sequence $\{a_n\}$ is bounded monotone decreasing and $\{b_n\}$ is bounded monotone increasing. In particular, $\liminf x_n$ and $\limsup x_n$ exist.

(ii) $\limsup\limits_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\}$ and $\liminf\limits_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\}$.

(iii) $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$.

Proof. Let us see why $\{a_n\}$ is a decreasing sequence. As a_n is the least upper bound for $\{x_k : k \geq n\}$, it is also an upper bound for the subset $\{x_k : k \geq (n+1)\}$. Therefore a_{n+1}, the least upper bound for $\{x_k : k \geq (n+1)\}$, has to be less than or equal to a_n, the least upper bound for $\{x_k : k \geq n)\}$. That is, $a_n \geq a_{n+1}$ for all n. Similarly (an exercise), $\{b_n\}$ is an increasing sequence. It is left as an exercise to show that if $\{x_n\}$ is bounded, then $\{a_n\}$ and $\{b_n\}$ must be bounded.

[4]Named after the Czech mathematician Bernhard Placidus Johann Nepomuk Bolzano (1781–1848), and the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897).

The second item follows as the sequences $\{a_n\}$ and $\{b_n\}$ are monotone and bounded.

For the third item, note that $b_n \leq a_n$, as the inf of a nonempty set is less than or equal to its sup. The sequences $\{a_n\}$ and $\{b_n\}$ converge to the limsup and the liminf respectively. Apply Lemma 2.21 to obtain

$$\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n. \qquad \square$$

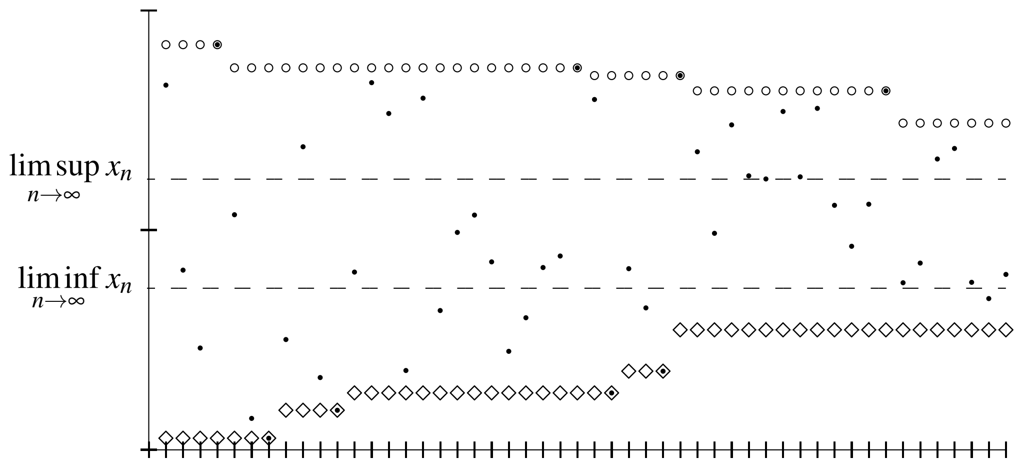

Figure 2.5: First 50 terms of an example sequence. Terms x_n of the sequence are marked with dots ($\bullet$), a_n are marked with circles ($\circ$), and b_n are marked with diamonds ($\diamond$).

Example 2.35: Let $\{x_n\}$ be defined by

$$x_n := \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Let us compute the lim inf and lim sup of this sequence. See also Figure 2.6. First the limit inferior:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \bigl(\inf\{x_k : k \geq n\}\bigr) = \lim_{n\to\infty} 0 = 0.$$

For the limit superior, we write

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \bigl(\sup\{x_k : k \geq n\}\bigr).$$

It is not hard to see that

$$\sup\{x_k : k \geq n\} = \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ \frac{n+2}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

We leave it to the reader to show that the limit is 1. That is,

$$\limsup_{n\to\infty} x_n = 1.$$

Do note that the sequence $\{x_n\}$ is not a convergent sequence.

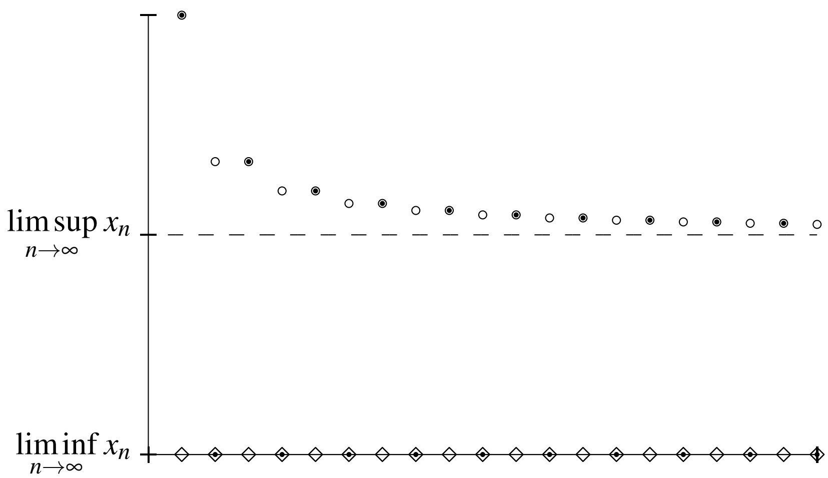

Figure 2.6: First 20 terms of the sequence in Example 2.35. The marking is the same as in Figure 2.5.

We associate certain subsequences with lim sup and lim inf. It is important to notice that $\{a_n\}$ and $\{b_n\}$ are not necessarily subsequences of $\{x_n\}$, nor do they have to even consist of the same numbers. For example, for the sequence $\{\frac{1}{n}\}$, $b_n = 0$ for all $n \in N$.

Theorem 2.36. *If $\{x_n\}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

Similarly, there exists a (perhaps different) subsequence $\{x_{m_k}\}$ such that

$$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n.$$

Proof. Define $a_n := \sup\{x_k : k \geq n\}$. Write $x := \limsup x_n = \lim a_n$. We define the subsequence inductively. Let $n_1 := 1$ and suppose we have defined the subsequence until n_k for some k. Pick some $m > n_k$ such that

$$a_{(n_k+1)} - x_m < \frac{1}{k+1}.$$

We can do this as $a_{(n_k+1)}$ is a supremum of the set $\{x_n : n \geq n_k + 1\}$ and hence there are elements of the sequence arbitrarily close (or even possibly equal) to the supremum. Set $n_{k+1} := m$. The subsequence $\{x_{n_k}\}$ is defined. Next we need to prove that it converges and has the right limit.

For all $k \geq 2$, we have $a_{(n_{k-1}+1)} \geq a_{n_k}$ (why?) and $a_{n_k} \geq x_{n_k}$. Therefore, for every $k \geq 2$,

$$\begin{aligned} |a_{n_k} - x_{n_k}| &= a_{n_k} - x_{n_k} \\ &\leq a_{(n_{k-1}+1)} - x_{n_k} \\ &< \frac{1}{k}. \end{aligned}$$

Let us show that $\{x_{n_k}\}$ converges to x. Note that the subsequence need not be monotone. Let $\epsilon > 0$ be given. As $\{a_n\}$ converges to x, the subsequence $\{a_{n_k}\}$ converges to x. Thus there exists an $M_1 \in \mathbb{N}$ such that for all $k \geq M_1$, we have

$$|a_{n_k} - x| < \frac{\epsilon}{2}.$$

Find an $M_2 \in \mathbb{N}$ such that

$$\frac{1}{M_2} \leq \frac{\epsilon}{2}.$$

Take $M := \max\{M_1, M_2, 2\}$ and compute. For all $k \geq M$, we have

$$\begin{aligned} |x - x_{n_k}| &= |a_{n_k} - x_{n_k} + x - a_{n_k}| \\ &\leq |a_{n_k} - x_{n_k}| + |x - a_{n_k}| \\ &< \frac{1}{k} + \frac{\epsilon}{2} \\ &\leq \frac{1}{M_2} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We leave the statement for lim inf as an exercise. □

2.3.2 Using limit inferior and limit superior

The advantage of lim inf and lim sup is that we can always write them down for any (bounded) sequence. If we could somehow compute them, we could also compute the limit of the sequence if it exists, or show that the sequence diverges. Working with lim inf and lim sup is a little bit like working with limits, although there are subtle differences.

Proposition 2.37. *Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ converges if and only if*

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Furthermore, if $\{x_n\}$ converges, then

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Proof. Let a_n and b_n be as in Definition 2.33. In particular, for all $n \in \mathbb{N}$,

$$b_n \leq x_n \leq a_n.$$

If $\liminf x_n = \limsup x_n$, then we know that $\{a_n\}$ and $\{b_n\}$ both converge to the same limit. By the squeeze lemma (Lemma 2.19), $\{x_n\}$ converges and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n.$$

Now suppose $\{x_n\}$ converges to x. By Theorem 2.36, there exists a subsequence $\{x_{n_k}\}$ that converges to $\limsup x_n$. As $\{x_n\}$ converges to x, every subsequence converges to x and therefore $\limsup x_n = \lim x_{n_k} = x$. Similarly, $\liminf x_n = x$. □

Limit superior and limit inferior behave nicely with subsequences.

Proposition 2.38. *Suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Then*

$$\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Proof. The middle inequality has been proved already. We will prove the third inequality, and leave the first inequality as an exercise.

We want to prove that $\limsup x_{n_k} \leq \limsup x_n$. Define $a_j := \sup\{x_k : k \geq j\}$ as usual. Also define $c_j := \sup\{x_{n_k} : k \geq j\}$. It is not true that $\{c_j\}$ is necessarily a subsequence of $\{a_j\}$. However, as $n_k \geq k$ for all k, we have that $\{x_{n_k} : k \geq j\} \subset \{x_k : k \geq j\}$. A supremum of a subset is less than or equal to the supremum of the set and therefore

$$c_j \leq a_j \qquad \text{for all } j.$$

Lemma 2.21 gives

$$\lim_{j\to\infty} c_j \leq \lim_{j\to\infty} a_j,$$

which is the desired conclusion. □

Limit superior and limit inferior are the largest and smallest subsequential limits. If the subsequence $\{x_{n_k}\}$ in the previous proposition is convergent, then $\liminf x_{n_k} = \lim x_{n_k} = \limsup x_{n_k}$. Therefore,

$$\liminf_{n\to\infty} x_n \leq \lim_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Similarly, we get the following useful test for convergence of a bounded sequence. We leave the proof as an exercise.

Proposition 2.39. *A bounded sequence $\{x_n\}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}$ converges to x.*

2.3.3 Bolzano–Weierstrass theorem

While it is not true that a bounded sequence is convergent, the Bolzano–Weierstrass theorem tells us that we can at least find a convergent subsequence. The version of Bolzano–Weierstrass that we present in this section is the Bolzano–Weierstrass for sequences of real numbers.

Theorem 2.40 (Bolzano–Weierstrass)**.** *Suppose a sequence $\{x_n\}$ of real numbers is bounded. Then there exists a convergent subsequence $\{x_{n_i}\}$.*

Proof. We use Theorem 2.36. It says that there exists a subsequence whose limit is $\limsup x_n$. □

The reader might complain right now that Theorem 2.36 is strictly stronger than the Bolzano–Weierstrass theorem as presented above. That is true. However, Theorem 2.36 only applies to the real line, but Bolzano–Weierstrass applies in more general contexts (that is, in $\mathbb{R}^n$) with pretty much the exact same statement.

As the theorem is so important to analysis, we present an explicit proof. The idea of the following proof also generalizes to different contexts.

Alternate proof of Bolzano–Weierstrass. As the sequence is bounded, then there exist two numbers $a_1 < b_1$ such that $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$. We will define a subsequence $\{x_{n_i}\}$ and two sequences $\{a_i\}$ and $\{b_i\}$, such that $\{a_i\}$ is monotone increasing, $\{b_i\}$ is monotone decreasing, $a_i \leq x_{n_i} \leq b_i$ and such that $\lim a_i = \lim b_i$. That x_{n_i} converges then follows by the squeeze lemma.

We define the sequences inductively. We will always have that $a_i < b_i$, and that $x_n \in [a_i, b_i]$ for infinitely many $n \in \mathbb{N}$. We have already defined a_1 and b_1. We take $n_1 := 1$, that is $x_{n_1} = x_1$. Suppose that up to some $k \in \mathbb{N}$, we have defined the subsequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$, and the sequences $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$. Let $y := \frac{a_k + b_k}{2}$. Clearly $a_k < y < b_k$. If there exist infinitely many $j \in \mathbb{N}$ such that $x_j \in [a_k, y]$, then set $a_{k+1} := a_k$, $b_{k+1} := y$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [a_k, y]$. If there are not infinitely many j such that $x_j \in [a_k, y]$, then it must be true that there are infinitely many $j \in \mathbb{N}$ such that $x_j \in [y, b_k]$. In this case pick $a_{k+1} := y$, $b_{k+1} := b_k$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [y, b_k]$.

We now have the sequences defined. What is left to prove is that $\lim a_i = \lim b_i$. The limits exist as the sequences are monotone. In the construction, $b_i - a_i$ is cut in half in each step. Therefore, $b_{i+1} - a_{i+1} = \frac{b_i - a_i}{2}$. By induction,

$$b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

Let $x := \lim a_i$. As $\{a_i\}$ is monotone,

$$x = \sup\{a_i : i \in \mathbb{N}\}.$$

Let $y := \lim b_i = \inf\{b_i : i \in \mathbb{N}\}$. Since $a_i < b_i$ for all i, then $x \leq y$. As the sequences are monotone, then for all i, we have (why?)

$$y - x \leq b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

Because $\frac{b_1 - a_1}{2^{i-1}}$ is arbitrarily small and $y - x \geq 0$, we have $y - x = 0$. Finish by the squeeze lemma. □

Yet another proof of the Bolzano–Weierstrass theorem is to show the following claim, which is left as a challenging exercise. *Claim: Every sequence has a monotone subsequence.*

2.3.4 Infinite limits

Just as for infima and suprema, it is possible to allow certain limits to be infinite. That is, we write $\lim x_n = \infty$ or $\lim x_n = -\infty$ for certain divergent sequences.

Definition 2.41. We say $\{x_n\}$ *diverges to infinity*[5] if for every $K \in \mathbb{R}$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n > K$. In this case we write

$$\lim_{n\to\infty} x_n := \infty.$$

Similarly, if for every $K \in \mathbb{R}$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n < K$, we say $\{x_n\}$ *diverges to minus infinity* and we write

$$\lim_{n\to\infty} x_n := -\infty.$$

[5]Sometimes it is said that $\{x_n\}$ *converges to infinity*.

With this definition and allowing ∞ and $-\infty$, we can write $\lim x_n$ for any monotone sequence.

Proposition 2.42. *Suppose $\{x_n\}$ is a monotone unbounded sequence. Then*

$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \text{if } \{x_n\} \text{ is increasing,} \\ -\infty & \text{if } \{x_n\} \text{ is decreasing.} \end{cases}$$

Proof. The case of monotone increasing follows from Exercise 2.53 part c) below. Let us do monotone decreasing. Suppose $\{x_n\}$ is decreasing and unbounded, that is, for every $K \in \mathbb{R}$, there is an $M \in \mathbb{N}$ such that $x_M < K$. By monotonicity $x_n \leq x_M < K$ for all $n \geq M$. Therefore, $\lim x_n = -\infty$. □

Example 2.43:

$$\lim_{n\to\infty} n = \infty, \qquad \lim_{n\to\infty} n^2 = \infty, \qquad \lim_{n\to\infty} -n = -\infty.$$

We leave verification to the reader.

We may also allow $\liminf$ and $\limsup$ to take on the values ∞ and $-\infty$, so that we can apply $\liminf$ and $\limsup$ to absolutely any sequence, not just a bounded one. Unfortunately, the sequences $\{a_n\}$ and $\{b_n\}$ are not sequences of real numbers but of extended real numbers. In particular, a_n can equal ∞ for some n, and b_n can equal $-\infty$. So we have no definition for the limits. But since the extended real numbers are still an ordered set, we can take suprema and infima.

Definition 2.44. Let $\{x_n\}$ be an unbounded sequence of real numbers. Define sequences of extended real numbers by $a_n := \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$. Define

$$\limsup_{n\to\infty} x_n := \inf\{a_n : n \in \mathbb{N}\}, \qquad \text{and} \qquad \liminf_{n\to\infty} x_n := \sup\{b_n : n \in \mathbb{N}\}.$$

This definition agrees with the definition for bounded sequences whenever $\lim a_n$ or $\lim b_n$ makes sense including possibly ∞ and $-\infty$.

Proposition 2.45. *Let $\{x_n\}$ be an unbounded sequence. Define $\{a_n\}$ and $\{b_n\}$ as above. Then $\{a_n\}$ is decreasing, and $\{b_n\}$ is increasing. If a_n is a real number for every n, then $\limsup x_n = \lim a_n$. If b_n is a real number for every n, then $\liminf x_n = \lim b_n$.*

Proof. As before, $a_n = \sup\{x_k : k \geq n\} \geq \sup\{x_k : k \geq n+1\} = a_{n+1}$. So $\{a_n\}$ is decreasing. Similarly, $\{b_n\}$ is increasing.

If the sequence $\{a_n\}$ is a sequence of real numbers, then $\lim a_n = \inf\{a_n : n \in \mathbb{N}\}$. This follows from Proposition 2.10 if $\{a_n\}$ is bounded and Proposition 2.42 if $\{a_n\}$ is unbounded. We proceed similarly with $\{b_n\}$. □

The definition behaves as expected with $\limsup$ and $\liminf$, see exercises 2.52 and 2.53.

Example 2.46: Suppose $x_n := 0$ for odd n and $x_n := n$ for even n. Then $a_n = \infty$ for all n, since for every M, there exists an even k such that $x_k = k \geq M$. On the other hand, $b_n = 0$ for all n, as for every n, the set $\{b_k : k \geq n\}$ consists of 0 and nonnegative numbers. So,

$$\lim_{n\to\infty} x_n \quad \text{does not exist}, \qquad \limsup_{n\to\infty} x_n = \infty, \qquad \liminf_{n\to\infty} x_n = 0.$$

2.3.5 Exercises

Exercise* 2.40*: *Suppose $\{x_n\}$ is a bounded sequence. Define a_n and b_n as in Definition 2.33. Show that $\{a_n\}$ and $\{b_n\}$ are bounded.*

Exercise* 2.41*: *Suppose $\{x_n\}$ is a bounded sequence. Define b_n as in Definition 2.33. Show that $\{b_n\}$ is an increasing sequence.*

Exercise* 2.42*: *Finish the proof of Proposition 2.38. That is, suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Prove $\liminf\limits_{n\to\infty} x_n \leq \liminf\limits_{k\to\infty} x_{n_k}$.*

Exercise* 2.43*: *Prove Proposition 2.39.*

Exercise* 2.44*:

a) Let $x_n := \dfrac{(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

b) Let $x_n := \dfrac{(n-1)(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

Exercise* 2.45*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences such that $x_n \leq y_n$ for all n. Then show that*

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

and

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n.$$

Exercise* 2.46*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.*

a) Show that $\{x_n + y_n\}$ is bounded.

b) Show that

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq \liminf_{n\to\infty} (x_n + y_n).$$

Hint: Find a subsequence $\{x_{n_i} + y_{n_i}\}$ of $\{x_n + y_n\}$ that converges. Then find a subsequence $\{x_{n_{m_i}}\}$ of $\{x_{n_i}\}$ that converges. Then apply what you know about limits.

c) Find an explicit $\{x_n\}$ and $\{y_n\}$ such that

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) < \liminf_{n\to\infty} (x_n + y_n).$$

Hint: Look for examples that do not have a limit.

Exercise* 2.47*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences (from the previous exercise we know that $\{x_n + y_n\}$ is bounded).*

a) Show that

$$(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) \geq \limsup_{n\to\infty} (x_n + y_n).$$

Hint: See previous exercise.

b) Find an explicit $\{x_n\}$ and $\{y_n\}$ such that

$$(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) > \limsup_{n\to\infty} (x_n + y_n).$$

Hint: See previous exercise.

Exercise* 2.48*: *If $S \subset \mathbb{R}$ is a set, then $x \in \mathbb{R}$ is a* cluster point *if for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap S \setminus \{x\}$ is not empty. That is, if there are points of S arbitrarily close to x. For example, $S := \{\frac{1}{n} : n \in \mathbb{N}\}$ has a unique (only one) cluster point 0, but $0 \notin S$. Prove the following version of the Bolzano–Weierstrass theorem:*

Theorem. Let $S \subset \mathbb{R}$ be a bounded infinite set, then there exists at least one cluster point of S.

Hint: If S is infinite, then S contains a countably infinite subset. That is, there is a sequence $\{x_n\}$ of distinct numbers in S.

Exercise* 2.49** (Challenging):***

a) *Prove that every sequence contains a monotone subsequence. Hint: Call $n \in \mathbb{N}$ a* peak *if $a_m \leq a_n$ for all $m \geq n$. There are two possibilities: Either the sequence has at most finitely many peaks, or it has infinitely many peaks.*

b) *Conclude the Bolzano–Weierstrass theorem.*

Exercise* 2.50*: *Prove a stronger version of Proposition 2.39. Suppose $\{x_n\}$ is a sequence such that every subsequence $\{x_{n_i}\}$ has a subsequence $\{x_{n_{m_i}}\}$ that converges to x.*

a) *First show that $\{x_n\}$ is bounded.*

b) *Now show that $\{x_n\}$ converges to x.*

Exercise* 2.51*: *Let $\{x_n\}$ be a bounded sequence.*

a) *Prove that there exists an s such that for every $r > s$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n < r$.*

b) *If s is a number as in a), then prove* $\limsup x_n \leq s$.

c) *Show that if S is the set of all s as in a), then* $\limsup x_n = \inf S$.

Exercise* 2.52** (Easy):*** *Suppose $\{x_n\}$ is such that* $\liminf x_n = -\infty$, $\limsup x_n = \infty$.

a) *Show that $\{x_n\}$ is not convergent, and also that neither* $\lim x_n = \infty$ *nor* $\lim x_n = -\infty$ *is true.*

b) *Find an example of such a sequence.*

Exercise* 2.53*: *Let $\{x_n\}$ be a sequence.*

a) *Show that* $\lim x_n = \infty$ *if and only if* $\liminf x_n = \infty$.

b) *Then show that* $\lim x_n = -\infty$ *if and only if* $\limsup x_n = -\infty$.

c) *If $\{x_n\}$ is monotone increasing, show that either* $\lim x_n$ *exists and is finite or* $\lim x_n = \infty$. *In either case,* $\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$.

Exercise* 2.54*: *Prove the following stronger version of Lemma 2.30, the ratio test. Suppose $\{x_n\}$ is a sequence such that $x_n \neq 0$ for all n.*

a) *Prove that if*

$$\limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} < 1,$$

then $\{x_n\}$ converges to 0.

b) Prove that if

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} > 1,$$

then $\{x_n\}$ *is unbounded.*

Exercise 2.55: *Suppose* $\{x_n\}$ *is a bounded sequence,* $a_n := \sup\{x_k : k \geq n\}$ *as before. Suppose that for some* $\ell \in \mathbb{N}$, $a_\ell \notin \{x_k : k \geq \ell\}$. *Then show that* $a_j = a_\ell$ *for all* $j \geq \ell$, *and hence* $\limsup x_n = a_\ell$.

Exercise 2.56: *Suppose* $\{x_n\}$ *is a sequence, and* $a_n := \sup\{x_k : k \geq n\}$ *and* $b_n := \sup\{x_k : k \geq n\}$ *as before.*

a) Prove that if $a_\ell = \infty$ *for some* $\ell \in \mathbb{N}$, *then* $\limsup x_n = \infty$.

b) Prove that if $b_\ell = -\infty$ *for some* $\ell \in \mathbb{N}$, *then* $\liminf x_n = -\infty$.

Exercise 2.57: *Suppose* $\{x_n\}$ *is a sequence such that both* $\liminf x_n$ *and* $\limsup x_n$ *are finite. Prove that* $\{x_n\}$ *is bounded.*

Exercise 2.58: *Suppose* $\{x_n\}$ *is a bounded sequence, and* $\epsilon > 0$ *is given. Prove that there exists an* M *such that for all* $k \geq M$,

$$x_k - \left(\limsup_{n\to\infty} x_n\right) < \epsilon \qquad \textit{and} \qquad \left(\liminf_{n\to\infty} x_n\right) - x_k < \epsilon.$$

Exercise 2.59: *Extend Theorem 2.36 to unbounded sequences: Suppose that* $\{x_n\}$ *is a sequence. If* $\limsup x_n = \infty$, *then prove that there exists a subsequence* $\{x_{n_i}\}$ *converging to* ∞. *Then prove the same result for* $-\infty$, *and then prove both statements for* $\liminf$.

2.4 Cauchy sequences

Often we wish to describe a certain number by a sequence that converges to it. In this case, it is impossible to use the number itself in the proof that the sequence converges. It would be nice if we could check for convergence without knowing the limit.

Definition 2.47. A sequence $\{x_n\}$ is a *Cauchy sequence*[6] if for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$, we have

$$|x_n - x_k| < \epsilon.$$

Informally, being Cauchy means that the terms of the sequence are eventually all arbitrarily close to each other. We might expect such a sequence to be convergent, and we would be correct due to $\mathbb{R}$ having the least-upper-bound property. Before we prove this fact, we look at some examples.

Example 2.48: The sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Proof: Given $\epsilon > 0$, find M such that $M > \frac{2}{\epsilon}$. Then for $n, k \geq M$, we have $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{k} < \frac{\epsilon}{2}$. Therefore, for $n, k \geq M$, we have

$$\left|\frac{1}{n} - \frac{1}{k}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{k}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Example 2.49: The sequence $\{\frac{n+1}{n}\}$ is a Cauchy sequence.

Proof: Given $\epsilon > 0$, find M such that $M > \frac{2}{\epsilon}$. Then for $n, k \geq M$, we have $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{k} < \frac{\epsilon}{2}$. Therefore, for $n, k \geq M$, we have

$$\begin{aligned}
\left|\frac{n+1}{n} - \frac{k+1}{k}\right| &= \left|\frac{k(n+1) - n(k+1)}{nk}\right| \\
&= \left|\frac{kn + k - nk - n}{nk}\right| \\
&= \left|\frac{k-n}{nk}\right| \\
&\leq \left|\frac{k}{nk}\right| + \left|\frac{-n}{nk}\right| \\
&= \frac{1}{n} + \frac{1}{k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Proposition 2.50. *A Cauchy sequence is bounded.*

Proof. Suppose $\{x_n\}$ is Cauchy. Pick an M such that for all $n, k \geq M$, we have $|x_n - x_k| < 1$. In particular, for all $n \geq M$,

$$|x_n - x_M| < 1.$$

By the reverse triangle inequality, $|x_n| - |x_M| \leq |x_n - x_M| < 1$. Hence for $n \geq M$,

$$|x_n| < 1 + |x_M|.$$

Let

$$B := \max\bigl\{|x_1|, |x_2|, \ldots, |x_{M-1}|, 1 + |x_M|\bigr\}.$$

Then $|x_n| \leq B$ for all $n \in \mathbb{N}$. □

[6]Named after the French mathematician Augustin-Louis Cauchy (1789–1857).

Theorem 2.51. *A sequence of real numbers is Cauchy if and only if it converges.*

Proof. Let $\epsilon > 0$ be given and suppose $\{x_n\}$ converges to x. Then there exists an M such that for $n \geq M$,

$$|x_n - x| < \frac{\epsilon}{2}.$$

Hence for $n \geq M$ and $k \geq M$,

$$|x_n - x_k| = |x_n - x + x - x_k| \leq |x_n - x| + |x - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Alright, that direction was easy. Now suppose $\{x_n\}$ is Cauchy. We have shown that $\{x_n\}$ is bounded. For a bounded sequence, liminf and limsup exist, and this is where we use the least-upper-bound property. If we show that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n,$$

then $\{x_n\}$ must be convergent by Proposition 2.37.

Define $a := \limsup x_n$ and $b := \liminf x_n$. By Theorem 2.36, there exist subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$, such that

$$\lim_{i\to\infty} x_{n_i} = a \qquad \text{and} \qquad \lim_{i\to\infty} x_{m_i} = b.$$

Given an $\epsilon > 0$, there exists an M_1 such that $|x_{n_i} - a| < \frac{\epsilon}{3}$ for all $i \geq M_1$ and an M_2 such that $|x_{m_i} - b| < \frac{\epsilon}{3}$ for all $i \geq M_2$. There also exists an M_3 such that $|x_n - x_k| < \frac{\epsilon}{3}$ for all $n, k \geq M_3$. Let $M := \max\{M_1, M_2, M_3\}$. If $i \geq M$, then $n_i \geq M$ and $m_i \geq M$. Hence,

$$\begin{aligned} |a - b| &= |a - x_{n_i} + x_{n_i} - x_{m_i} + x_{m_i} - b| \\ &\leq |a - x_{n_i}| + |x_{n_i} - x_{m_i}| + |x_{m_i} - b| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

As $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$ and the sequence converges. □

Remark 2.52. The statement of this proposition is sometimes used to define the completeness property of the real numbers. We say a set is *Cauchy-complete* (or sometimes just *complete*) if every Cauchy sequence converges. Above, we proved that as $\mathbb{R}$ has the least-upper-bound property, then $\mathbb{R}$ is Cauchy-complete. One can construct $\mathbb{R}$ via "completing" $\mathbb{Q}$ by "throwing in" just enough points to make all Cauchy sequences converge (we omit the details). The resulting field has the least-upper-bound property. The advantage of using Cauchy sequences to define completeness is that this idea generalizes to more abstract settings such as metric spaces, see chapter **??**.

The Cauchy criterion is stronger than $|x_{n+1} - x_n|$ (or $|x_{n+j} - x_n|$ for a fixed j) going to zero as n goes to infinity. When we get to the partial sums of the harmonic series (see Example 2.63 in the next section), we will have a sequence such that $x_{n+1} - x_n = \frac{1}{n}$, yet $\{x_n\}$ is divergent. In fact, for that sequence, $\lim_{n\to\infty} |x_{n+j} - x_n| = 0$ for every $j \in \mathbb{N}$ (confer Exercise 2.79). The key point in the definition of Cauchy is that n and k vary independently and can be arbitrarily far apart.

2.4.1 Exercises

Exercise* 2.60*: *Prove that* $\{\frac{n^2-1}{n^2}\}$ *is Cauchy using directly the definition of Cauchy sequences.*

Exercise* 2.61*: *Let* $\{x_n\}$ *be a sequence such that there exists a positive* $C < 1$ *and for all* n*,*

$$|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|\,.$$

Prove that $\{x_n\}$ *is Cauchy. Hint: You can freely use the formula (for* $C \neq 1$*)*

$$1 + C + C^2 + \cdots + C^n = \frac{1 - C^{n+1}}{1 - C}.$$

Exercise* 2.62** (Challenging):*** *Suppose* F *is an ordered field that contains the rational numbers* $\mathbb{Q}$*, such that* $\mathbb{Q}$ *is dense, that is: Whenever* $x, y \in F$ *are such that* $x < y$*, then there exists a* $q \in \mathbb{Q}$ *such that* $x < q < y$*. Say a sequence* $\{x_n\}_{n=1}^{\infty}$ *of rational numbers is Cauchy if given every* $\epsilon \in \mathbb{Q}$ *with* $\epsilon > 0$*, there exists an* M *such that for all* $n, k \geq M$*, we have* $|x_n - x_k| < \epsilon$*. Suppose every Cauchy sequence of rational numbers has a limit in* F*. Prove that* F *has the least-upper-bound property.*

Exercise* 2.63*: *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences such that* $\lim y_n = 0$*. Suppose that for all* $k \in \mathbb{N}$ *and for all* $m \geq k$*, we have*

$$|x_m - x_k| \leq y_k.$$

Show that $\{x_n\}$ *is Cauchy.*

Exercise* 2.64*: *Suppose a Cauchy sequence* $\{x_n\}$ *is such that for every* $M \in \mathbb{N}$*, there exists a* $k \geq M$ *and an* $n \geq M$ *such that* $x_k < 0$ *and* $x_n > 0$*. Using simply the definition of a Cauchy sequence and of a convergent sequence, show that the sequence converges to* 0*.*

Exercise* 2.65*: *Suppose* $|x_n - x_k| \leq \frac{n}{k^2}$ *for all* n *and* k*. Show that* $\{x_n\}$ *is Cauchy.*

Exercise* 2.66*: *Suppose* $\{x_n\}$ *is a Cauchy sequence such that for infinitely many* n*,* $x_n = c$*. Using only the definition of Cauchy sequence prove that* $\lim x_n = c$*.*

Exercise* 2.67*: *True or false, prove or find a counterexample: If* $\{x_n\}$ *is a Cauchy sequence, then there exists an* M *such that for all* $n \geq M$*, we have* $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$*.*

2.5 Series

A fundamental object in mathematics is that of a series. In fact, when the foundations of analysis were being developed, the motivation was to understand series. Understanding series is important in applications of analysis. For example, solutions to differential equations are often given as series, and differential equations are the basis for understanding almost all of modern science.

2.5.1 Definition

Definition 2.53. Given a sequence $\{x_n\}$, we write the formal object

$$\sum_{n=1}^{\infty} x_n \qquad \text{or sometimes just} \qquad \sum x_n$$

and call it a *series*. A series *converges* if the sequence $\{s_k\}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + \cdots + x_k,$$

converges. The numbers s_k are called *partial sums*. If $x := \lim\, s_k$, we write

$$\sum_{n=1}^{\infty} x_n = x.$$

In this case, we cheat a little and treat $\sum_{n=1}^{\infty} x_n$ as a number.

If the sequence $\{s_k\}$ diverges, we say the series is *divergent*. In this case, $\sum x_n$ is simply a formal object and not a number.

In other words, for a convergent series, we have

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} \sum_{n=1}^{k} x_n.$$

We only have this equality if the limit on the right actually exists. If the series does not converge, the right-hand side does not make sense (the limit does not exist). Therefore, be careful as $\sum x_n$ means two different things (a notation for the series itself or the limit of the partial sums), and you must use context to distinguish.

Remark 2.54. It is sometimes convenient to start the series at an index different from 1. For instance, we can write

$$\sum_{n=0}^{\infty} r^n = \sum_{n=1}^{\infty} r^{n-1}.$$

The left-hand side is more convenient to write.

Remark 2.55. It is common to write the series $\sum x_n$ as

$$x_1 + x_2 + x_3 + \cdots$$

with the understanding that the ellipsis indicates a series and not a simple sum. We do not use this notation as it is the sort of informal notation that leads to mistakes in proofs.

Example 2.56: The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges and the limit is 1. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n} = 1.$$

Proof: First we prove the following equality

$$\left(\sum_{n=1}^{k} \frac{1}{2^n} \right) + \frac{1}{2^k} = 1.$$

The equality is immediate when $k = 1$. The proof for general k follows by induction, which we leave to the reader. See Figure 2.7 for an illustration.

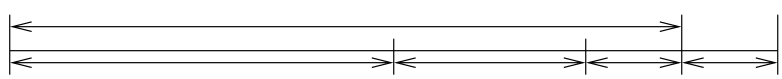

Figure 2.7: The equality $\left(\sum_{n=1}^{k} \frac{1}{2^n} \right) + \frac{1}{2^k} = 1$ illustrated for $k = 3$.

Let s_k be the partial sum. We write

$$|1 - s_k| = \left| 1 - \sum_{n=1}^{k} \frac{1}{2^n} \right| = \left| \frac{1}{2^k} \right| = \frac{1}{2^k}.$$

The sequence $\left\{ \frac{1}{2^k} \right\}$, and therefore $\left\{ |1 - s_k| \right\}$, converges to zero. So, $\{s_k\}$ converges to 1.

Proposition 2.57. *Suppose $-1 < r < 1$. Then the* geometric series $\sum_{n=0}^{\infty} r^n$ *converges, and*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Details of the proof are left as an exercise. The proof consists of showing

$$\sum_{n=0}^{k-1} r^n = \frac{1-r^k}{1-r},$$

and then taking the limit as k goes to ∞. Geometric series is one of the most important series, and in fact it is one of the few series for which we can so explicitly find the limit.

As for sequences we can talk about a *tail of a series.*

Proposition 2.58. *Let $\sum x_n$ be a series. Let $M \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} x_n \quad \textit{converges if and only if} \quad \sum_{n=M}^{\infty} x_n \quad \textit{converges.}$$

Proof. We look at partial sums of the two series (for $k \geq M$)

$$\sum_{n=1}^{k} x_n = \left(\sum_{n=1}^{M-1} x_n\right) + \sum_{n=M}^{k} x_n.$$

Note that $\sum_{n=1}^{M-1} x_n$ is a fixed number. Use Proposition 2.23 to finish the proof. □

2.5.2 Cauchy series

Definition 2.59. A series $\sum x_n$ is said to be *Cauchy* or a *Cauchy series* if the sequence of partial sums $\{s_n\}$ is a Cauchy sequence.

A sequence of real numbers converges if and only if it is Cauchy. Therefore, a series is convergent if and only if it is Cauchy. The series $\sum x_n$ is Cauchy if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$, such that for every $n \geq M$ and $k \geq M$, we have

$$\left|\left(\sum_{j=1}^{k} x_j\right) - \left(\sum_{j=1}^{n} x_j\right)\right| < \epsilon.$$

Without loss of generality we assume $n < k$. Then we write

$$\left|\left(\sum_{j=1}^{k} x_j\right) - \left(\sum_{j=1}^{n} x_j\right)\right| = \left|\sum_{j=n+1}^{k} x_j\right| < \epsilon.$$

We have proved the following simple proposition.

Proposition 2.60. *The series $\sum x_n$ is Cauchy if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$, we have*

$$\left|\sum_{j=n+1}^{k} x_j\right| < \epsilon.$$

2.5.3 Basic properties

Proposition 2.61. *Let $\sum x_n$ be a convergent series. Then the sequence $\{x_n\}$ is convergent and*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. Let $\epsilon > 0$ be given. As $\sum x_n$ is convergent, it is Cauchy. Thus we find an M such that for every $n \geq M$, we have

$$\epsilon > \left|\sum_{j=n+1}^{n+1} x_j\right| = |x_{n+1}|.$$

Hence for every $n \geq M + 1$, we have $|x_n| < \epsilon$. □

Example 2.62: If $r \geq 1$ or $r \leq -1$, then the geometric series $\sum_{n=0}^{\infty} r^n$ diverges.

Proof: $|r^n| = |r|^n \geq 1^n = 1$. So the terms do not go to zero and the series cannot converge.

So if a series converges, the terms of the series go to zero. The implication, however, goes only one way. Let us give an example.

Example 2.63: The series $\sum \frac{1}{n}$ diverges (despite the fact that $\lim \frac{1}{n} = 0$). This is the famous *harmonic series*[7].

Proof: We will show that the sequence of partial sums is unbounded, and hence cannot converge. Write the partial sums s_n for $n = 2^k$ as:

$$
\begin{aligned}
s_1 &= 1, \\
s_2 &= (1) + \left(\frac{1}{2}\right), \\
s_4 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right), \\
s_8 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right), \\
&\vdots \\
s_{2^k} &= 1 + \sum_{j=1}^{k} \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right).
\end{aligned}
$$

Notice $\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. More generally

$$
\sum_{m=2^{k-1}+1}^{2^k} \frac{1}{m} \geq \sum_{m=2^{k-1}+1}^{2^k} \frac{1}{2^k} = (2^{k-1})\frac{1}{2^k} = \frac{1}{2}.
$$

Therefore,

$$
s_{2^k} = 1 + \sum_{j=1}^{k} \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right) \geq 1 + \sum_{j=1}^{k} \frac{1}{2} = 1 + \frac{k}{2}.
$$

As $\{\frac{k}{2}\}$ is unbounded by the Archimedean property, that means that $\{s_{2^k}\}$ is unbounded, and therefore $\{s_n\}$ is unbounded. Hence $\{s_n\}$ diverges, and consequently $\sum \frac{1}{n}$ diverges.

Convergent series are linear. That is, we can multiply them by constants and add them and these operations are done term by term.

Proposition 2.64 (Linearity of series)**.** *Let $\alpha \in \mathbb{R}$ and $\sum x_n$ and $\sum y_n$ be convergent series. Then*

(i) $\sum \alpha x_n$ is a convergent series and

$$
\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n.
$$

[7] The divergence of the harmonic series was known long before the theory of series was made rigorous. The proof we give is the earliest proof and was given by Nicole Oresme (1323?–1382).

(ii) $\sum (x_n + y_n)$ *is a convergent series and*

$$\sum_{n=1}^{\infty} (x_n + y_n) = \left(\sum_{n=1}^{\infty} x_n \right) + \left(\sum_{n=1}^{\infty} y_n \right).$$

Proof. For the first item, we simply write the kth partial sum

$$\sum_{n=1}^{k} \alpha x_n = \alpha \left(\sum_{n=1}^{k} x_n \right).$$

We look at the right-hand side and note that the constant multiple of a convergent sequence is convergent. Hence, we take the limit of both sides to obtain the result.

For the second item we also look at the kth partial sum

$$\sum_{n=1}^{k} (x_n + y_n) = \left(\sum_{n=1}^{k} x_n \right) + \left(\sum_{n=1}^{k} y_n \right).$$

We look at the right-hand side and note that the sum of convergent sequences is convergent. Hence, we take the limit of both sides to obtain the proposition. □

An example of a useful application of the first item is the following formula. If $|r| < 1$ and $j \in \mathbb{N}$, then

$$\sum_{n=j}^{\infty} r^n = \frac{r^j}{1-r}.$$

The formula follows by using the geometric series and multiplying by r^j:

$$r^j \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} r^{n+j} = \sum_{n=j}^{\infty} r^n.$$

Multiplying series is not as simple as adding, see the next section. It is not true, of course, that we multiply term by term. That strategy does not work even for finite sums: $(a+b)(c+d) \neq ac + bd$.

2.5.4 Absolute convergence

As monotone sequences are easier to work with than arbitrary sequences, it is usually easier to work with series $\sum x_n$, where $x_n \geq 0$ for all n. The sequence of partial sums is then monotone increasing and converges if it is bounded above. Let us formalize this statement as a proposition.

Proposition 2.65. *If $x_n \geq 0$ for all n, then $\sum x_n$ converges if and only if the sequence of partial sums is bounded above.*

As the limit of a monotone increasing sequence is the supremum, then when $x_n \geq 0$ for all n, we have the inequality

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{\infty} x_n.$$

If we allow infinite limits, the inequality still holds even when the series diverges to infinity, although in that case it is not terribly useful.

We will see that the following common criterion for convergence of series has big implications for how the series can be manipulated.

Definition 2.66. A series $\sum x_n$ *converges absolutely* if the series $\sum |x_n|$ converges. If a series converges, but does not converge absolutely, we say it is *conditionally convergent*.

Proposition 2.67. *If the series $\sum x_n$ converges absolutely, then it converges.*

Proof. A series is convergent if and only if it is Cauchy. Hence suppose $\sum |x_n|$ is Cauchy. That is, for every $\epsilon > 0$, there exists an M such that for all $k \geq M$ and all $n > k$, we have

$$\sum_{j=k+1}^{n} |x_j| = \left| \sum_{j=k+1}^{n} |x_j| \right| < \epsilon.$$

We apply the triangle inequality for a finite sum to obtain

$$\left| \sum_{j=k+1}^{n} x_j \right| \leq \sum_{j=k+1}^{n} |x_j| < \epsilon.$$

Hence $\sum x_n$ is Cauchy, and therefore it converges. □

If $\sum x_n$ converges absolutely, the limits of $\sum x_n$ and $\sum |x_n|$ are generally different. Computing one does not help us compute the other. However, the computation above leads to a useful inequality for absolutely convergent series, a series version of the triangle inequality, a proof of which we leave as an exercise:

$$\left| \sum_{j=1}^{\infty} x_j \right| \leq \sum_{j=1}^{\infty} |x_j| \, .$$

Absolutely convergent series have many wonderful properties. For example, absolutely convergent series can be rearranged arbitrarily, or we can multiply such series together easily. Conditionally convergent series on the other hand often do not behave as one would expect. See the next section.

We leave as an exercise to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, although the reader should finish this section before trying. On the other hand, we proved

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Therefore, $\sum \frac{(-1)^n}{n}$ is a conditionally convergent series.

2.5.5 Comparison test and the p-series

We noted above that for a series to converge the terms not only have to go to zero, but they have to go to zero "fast enough." If we know about convergence of a certain series, we can use the following comparison test to see if the terms of another series go to zero "fast enough."

Proposition 2.68 (Comparison test)**.** *Let $\sum x_n$ and $\sum y_n$ be series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$.*

(i) If $\sum y_n$ converges, then so does $\sum x_n$.

(ii) If $\sum x_n$ diverges, then so does $\sum y_n$.

Proof. As the terms of the series are all nonnegative, the sequences of partial sums are both monotone increasing. Since $x_n \leq y_n$ for all n, the partial sums satisfy for all k

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n. \tag{2.1}$$

If the series $\sum y_n$ converges, the partial sums for the series are bounded. Therefore, the right-hand side of (2.1) is bounded for all k; there exists some $B \in \mathbb{R}$ such that $\sum_{n=1}^{k} y_n \leq B$ for all k, and so

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n \leq B.$$

Hence the partial sums for $\sum x_n$ are also bounded. Since the partial sums are a monotone increasing sequence they are convergent. The first item is thus proved.

On the other hand if $\sum x_n$ diverges, the sequence of partial sums must be unbounded since it is monotone increasing. That is, the partial sums for $\sum x_n$ are eventually bigger than any real number. Putting this together with (2.1) we see that for every $B \in \mathbb{R}$, there is a k such that

$$B \leq \sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n.$$

Hence the partial sums for $\sum y_n$ are also unbounded, and $\sum y_n$ also diverges. $\square$

A useful series to use with the comparison test is the p-series[8].

Proposition 2.69 (p-series or the p-test)**.** *For $p \in \mathbb{R}$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

[8] We have not yet defined x^p for $x > 0$ and an arbitrary $p \in \mathbb{R}$. The definition is $x^p := \exp(p \ln x)$. We will define the logarithm and the exponential in §5.4. For now you can just think of rational p where $x^{k/m} = (x^{1/m})^k$. See also Exercise 1.31.

Proof. First suppose $p \leq 1$. As $n \geq 1$, we have $\frac{1}{n^p} \geq \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^p}$ must diverge for all $p \leq 1$ by the comparison test.

Now suppose $p > 1$. We proceed as we did for the harmonic series, but instead of showing that the sequence of partial sums is unbounded, we show that it is bounded. The terms of the series are positive, so the sequence of partial sums is monotone increasing and converges if it is bounded above. Let s_n denote the nth partial sum.

$$\begin{aligned}
s_1 &= 1, \\
s_3 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right), \\
s_7 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right), \\
&\vdots \\
s_{2^k-1} &= 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right).
\end{aligned}$$

Instead of estimating from below, we estimate from above. As p is positive, then $2^p < 3^p$, and hence $\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$. Similarly, $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$. Therefore, for all $k \geq 2$,

$$\begin{aligned}
s_{2^k-1} &= 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right) \\
&< 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{(2^j)^p} \right) \\
&= 1 + \sum_{j=1}^{k-1} \left(\frac{2^j}{(2^j)^p} \right) \\
&= 1 + \sum_{j=1}^{k-1} \left(\frac{1}{2^{p-1}} \right)^j.
\end{aligned}$$

As $p > 1$, then $\frac{1}{2^{p-1}} < 1$. Proposition 2.57 says that

$$\sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j$$

converges. Thus,

$$s_{2^k-1} < 1 + \sum_{j=1}^{k-1} \left(\frac{1}{2^{p-1}} \right)^j \leq 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j.$$

For every n there is a $k \geq 2$ such that $n \leq 2^k - 1$, and as $\{s_n\}$ is a monotone sequence, $s_n \leq s_{2^k-1}$. So for all n,

$$s_n < 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j$$

Thus the sequence of partial sums is bounded, and the series converges. □

Neither the p-series test nor the comparison test tell us what the sum converges to. They only tell us that a limit of the partial sums exists. For instance, while we know that $\sum \frac{1}{n^2}$ converges, it is far harder to find[9] that the limit is $\frac{\pi^2}{6}$. If we treat $\sum \frac{1}{n^p}$ as a function of p, we get the so-called Riemann ζ function. Understanding the behavior of this function contains one of the most famous unsolved problems in mathematics today and has applications in seemingly unrelated areas such as modern cryptography.

Example 2.70: The series $\sum \frac{1}{n^2+1}$ converges.

Proof: First, $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$. The series $\sum \frac{1}{n^2}$ converges by the p-series test. Therefore, by the comparison test, $\sum \frac{1}{n^2+1}$ converges.

2.5.6 Ratio test

Suppose $r > 0$. The ratio of two subsequent terms in the geometric series $\sum r^n$ is $\frac{r^{n+1}}{r^n} = r$, and the series converges whenever $r < 1$. Just as for sequences, this fact can be generalized to more arbitrary series as long as we have such a ratio "in the limit." We then compare the tail of a series to the geometric series.

Proposition 2.71 (Ratio test)**.** *Let $\sum x_n$ be a series, $x_n \neq 0$ for all n, and such that*

$$L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \qquad \textit{exists.}$$

(i) If $L < 1$, then $\sum x_n$ converges absolutely.

(ii) If $L > 1$, then $\sum x_n$ diverges.

Although the test as stated is often sufficient, it can be strengthened a bit, see Exercise 2.73.

Proof. If $L > 1$, then Lemma 2.30 says that the sequence $\{x_n\}$ diverges. Since it is a necessary condition for the convergence of series that the terms go to zero, we know that $\sum x_n$ must diverge.

Thus suppose $L < 1$. We will argue that $\sum |x_n|$ must converge. The proof is similar to that of Lemma 2.30. Of course $L \geq 0$. Pick r such that $L < r < 1$. As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M + 1$), write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| \, rr \cdots r = |x_M| \, r^{n-M} = (|x_M| \, r^{-M}) r^n.$$

[9] Demonstration of this fact is what made the Swiss mathematician Leonhard Paul Euler (1707–1783) famous.

For $k > M$, write the partial sum as

$$\begin{aligned}\sum_{n=1}^{k} |x_n| &= \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} |x_n|\right) \\ &< \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} (|x_M| \, r^{-M}) r^n\right) \\ &= \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M| \, r^{-M}) \left(\sum_{n=M+1}^{k} r^n\right).\end{aligned}$$

As $0 < r < 1$, the geometric series $\sum_{n=0}^{\infty} r^n$ converges, so $\sum_{n=M+1}^{\infty} r^n$ converges as well. We take the limit as k goes to infinity on the right-hand side above to obtain

$$\begin{aligned}\sum_{n=1}^{k} |x_n| &< \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M| \, r^{-M}) \left(\sum_{n=M+1}^{k} r^n\right) \\ &\leq \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M| \, r^{-M}) \left(\sum_{n=M+1}^{\infty} r^n\right).\end{aligned}$$

The right-hand side is a number that does not depend on k. Hence the sequence of partial sums of $\sum |x_n|$ is bounded and $\sum |x_n|$ is convergent. Thus $\sum x_n$ is absolutely convergent. $\square$

Example 2.72: The series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges absolutely.

Proof: We write

$$\lim_{n\to\infty} \frac{2^{(n+1)}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Therefore, the series converges absolutely by the ratio test.

2.5.7 Exercises

Exercise 2.68: *Suppose the kth partial sum of $\sum_{n=1}^{\infty} x_n$ is $s_k = \frac{k}{k+1}$. Find the series, that is find x_n, prove that the series converges, and then find the limit.*

Exercise 2.69: *Prove Proposition 2.57, that is for $-1 < r < 1$ prove*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Hint: See Example **??**.

Exercise 2.70: *Decide the convergence or divergence of the following series.*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$ *b)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ *c)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *d)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *e)* $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Exercise* 2.71*:

a) *Prove that if* $\sum_{n=1}^{\infty} x_n$ *converges, then* $\sum_{n=1}^{\infty} (x_{2n} + x_{2n+1})$ *also converges.*

b) *Find an explicit example where the converse does not hold.*

Exercise* 2.72*: *For* $j = 1, 2, \ldots, n$, *let* $\{x_{j,k}\}_{k=1}^{\infty}$ *denote* n *sequences. Suppose that for each* $j \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} x_{j,k}$$

is convergent. Prove

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} x_{j,k} \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n} x_{j,k} \right).$$

Exercise* 2.73*: *Prove the following stronger version of the ratio test: Let* $\sum x_n$ *be a series.*

a) *If there is an* N *and a* $\rho < 1$ *such that* $\frac{|x_{n+1}|}{|x_n|} < \rho$ *for all* $n \geq N$, *then the series converges absolutely. (Remark: Equivalently the condition can be stated as* $\limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} < 1$.*)*

b) *If there is an* N *such that* $\frac{|x_{n+1}|}{|x_n|} \geq 1$ *for all* $n \geq N$, *then the series diverges.*

Exercise* 2.74** (Challenging):*** *Suppose* $\{x_n\}$ *is a decreasing sequence and* $\sum x_n$ *converges. Prove* $\lim_{n\to\infty} n x_n = 0$.

Exercise* 2.75*: *Show that* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges. Hint: Consider the sum of two subsequent entries.*

Exercise* 2.76*:

a) *Prove that if* $\sum x_n$ *and* $\sum y_n$ *converge absolutely, then* $\sum x_n y_n$ *converges absolutely.*

b) *Find an explicit example where the converse does not hold.*

c) *Find an explicit example where all three series are absolutely convergent, are not just finite sums, and* $(\sum x_n)(\sum y_n) \neq \sum x_n y_n$. *That is, show that series are not multiplied term-by-term.*

Exercise* 2.77*: *Prove the triangle inequality for series: If* $\sum x_n$ *converges absolutely, then*

$$\left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n| .$$

Exercise* 2.78*: *Prove the* limit comparison test. *That is, prove that if* $a_n > 0$ *and* $b_n > 0$ *for all* n, *and*

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty,$$

then either $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*

Exercise* 2.79*: *Let* $x_n := \sum_{j=1}^{n} \frac{1}{j}$. *Show that for every* k, *we get* $\lim_{n\to\infty} |x_{n+k} - x_n| = 0$, *yet* $\{x_n\}$ *is not Cauchy.*

Exercise* 2.80*: *Let s_k be the kth partial sum of $\sum x_n$.*

a) Suppose that there exists an $m \in \mathbb{N}$ such that $\lim\limits_{k\to\infty} s_{mk}$ exists and $\lim x_n = 0$. Show that $\sum x_n$ converges.

b) Find an example where $\lim\limits_{k\to\infty} s_{2k}$ exists and $\lim x_n \neq 0$ (and therefore $\sum x_n$ diverges).

c) (Challenging) Find an example where $\lim x_n = 0$, and there exists a subsequence $\{s_{k_j}\}$ such that $\lim\limits_{j\to\infty} s_{k_j}$ exists, but $\sum x_n$ still diverges.

Exercise* 2.81*: *Suppose $\sum x_n$ converges and $x_n \geq 0$ for all n. Prove that $\sum x_n^2$ converges.*

***Exercise* 2.82** (Challenging)*: Suppose $\{x_n\}$ is a decreasing sequence of positive numbers. The proof of convergence/divergence for the p-series generalizes. Prove the so-called* Cauchy condensation principle*:*

$$\sum_{n=1}^{\infty} x_n \qquad \textit{converges if and only if} \qquad \sum_{n=1}^{\infty} 2^n x_{2^n} \qquad \textit{converges.}$$

Exercise* 2.83*: *Use the Cauchy condensation principle (see Exercise 2.82) to decide the convergence of*

a) $\sum \frac{\ln n}{n^2}$ *b)* $\sum \frac{1}{n \ln n}$ *c)* $\sum \frac{1}{n(\ln n)^2}$ *d)* $\sum \frac{1}{n(\ln n)(\ln \ln n)^2}$

For the series to be well-defined you need to start some of the series at $n = 2$. Note that only the tails of some of these series satisfy the hypotheses of the principle; you should argue why that is sufficient.
Hint: Feel free to use the identity $\ln(2^n) = n \ln 2$.

***Exercise* 2.84** (Challenging)*: Prove* Abel's theorem*:*

Theorem. Suppose $\sum x_n$ is a series whose partial sums are a bounded sequence, $\{\lambda_n\}$ is a sequence with $\lim \lambda_n = 0$, and $\sum |\lambda_{n+1} - \lambda_n|$ is convergent. Then $\sum \lambda_n x_n$ is convergent.

2.6 More on series

2.6.1 Root test

A test similar to the ratio test is the so-called *root test*. In fact, the proof of this test is similar and somewhat easier. Again, the idea is to generalize what happens for the geometric series.

Proposition 2.73 (Root test)**.** *Let $\sum x_n$ be a series and let*

$$L := \limsup_{n\to\infty} |x_n|^{1/n}.$$

(i) If $L < 1$, then $\sum x_n$ converges absolutely.

(ii) If $L > 1$, then $\sum x_n$ diverges.

Proof. If $L > 1$, then there exists[10] a subsequence $\{x_{n_k}\}$ such that $L = \lim_{k\to\infty} |x_{n_k}|^{1/n_k}$. Let r be such that $L > r > 1$. There exists an M such that for all $k \geq M$, we have $|x_{n_k}|^{1/n_k} > r > 1$, or in other words $|x_{n_k}| > r^{n_k} > 1$. The subsequence $\{|x_{n_k}|\}$, and therefore also $\{|x_n|\}$, cannot possibly converge to zero, and so the series diverges.

Now suppose $L < 1$. Pick r such that $L < r < 1$. By definition of limit supremum, there is an M such that for all $n \geq M$,

$$\sup\left\{|x_k|^{1/k} : k \geq n\right\} < r.$$

Therefore, for all $n \geq M$,

$$|x_n|^{1/n} < r, \qquad \text{or in other words} \qquad |x_n| < r^n.$$

Let $k > M$, and estimate the kth partial sum:

$$\sum_{n=1}^{k} |x_n| = \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} |x_n|\right) \leq \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} r^n\right).$$

As $0 < r < 1$, the geometric series $\sum_{n=M+1}^{\infty} r^n$ converges to $\frac{r^{M+1}}{1-r}$. As everything is positive,

$$\sum_{n=1}^{k} |x_n| \leq \left(\sum_{n=1}^{M} |x_n|\right) + \frac{r^{M+1}}{1-r}.$$

Thus the sequence of partial sums of $\sum |x_n|$ is bounded, and the series converges. Therefore, $\sum x_n$ converges absolutely. □

[10] In case $L = \infty$, see Exercise 2.59. Alternatively, note that if $L = \infty$, then $\left\{|x_n|^{1/n}\right\}$ and thus $\{x_n\}$ is unbounded.

2.6.2 Alternating series test

The tests we have seen so far only addressed absolute convergence. The following test gives a large supply of conditionally convergent series.

Proposition 2.74 (Alternating series)**.** *Let $\{x_n\}$ be a monotone decreasing sequence of positive real numbers such that* $\lim x_n = 0$. *Then*

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

converges.

Proof. Let $s_m := \sum_{k=1}^{m} (-1)^k x_k$ be the mth partial sum. Then write

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k x_k = (-x_1 + x_2) + \cdots + (-x_{2n-1} + x_{2n}) = \sum_{k=1}^{n} (-x_{2k-1} + x_{2k}).$$

The sequence $\{x_k\}$ is decreasing and so $(-x_{2k-1} + x_{2k}) \leq 0$ for all k. Therefore, the subsequence $\{s_{2n}\}$ of partial sums is a decreasing sequence. Similarly, $(x_{2k} - x_{2k+1}) \geq 0$, and so

$$s_{2n} = -x_1 + (x_2 - x_3) + \cdots + (x_{2n-2} - x_{2n-1}) + x_{2n} \geq -x_1.$$

The sequence $\{s_{2n}\}$ is decreasing and bounded below, so it converges. Let $a := \lim s_{2n}$.

We wish to show that $\lim s_m = a$ (and not just for the subsequence). Notice

$$s_{2n+1} = s_{2n} + x_{2n+1}.$$

Given $\epsilon > 0$, pick M such that $|s_{2n} - a| < \frac{\epsilon}{2}$ whenever $2n \geq M$. Since $\lim x_n = 0$, we also make M possibly larger to obtain $x_{2n+1} < \frac{\epsilon}{2}$ whenever $2n \geq M$. If $2n \geq M$, we have $|s_{2n} - a| < \frac{\epsilon}{2} < \epsilon$, so we just need to check the situation for s_{2n+1}:

$$|s_{2n+1} - a| = |s_{2n} - a + x_{2n+1}| \leq |s_{2n} - a| + x_{2n+1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Notably, there exist conditionally convergent series where the absolute values of the terms go to zero arbitrarily slowly. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

converges for arbitrarily small $p > 0$, but it does not converge absolutely when $p \leq 1$.

2.6.3 Rearrangements

Absolutely convergent series behave as we imagine they should. For example, absolutely convergent series can be summed in any order whatsoever. Nothing of the sort holds for conditionally convergent series (see Example 2.76 and Exercise 2.87).

Consider a series

$$\sum_{n=1}^{\infty} x_n.$$

Given a bijective function $\sigma\colon \mathbb{N} \to \mathbb{N}$, the corresponding rearrangement is the following series:

$$\sum_{k=1}^{\infty} x_{\sigma(k)}.$$

We simply sum the series in a different order.

Proposition 2.75. *Let $\sum x_n$ be an absolutely convergent series converging to a number x. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum x_{\sigma(n)}$ is absolutely convergent and converges to x.*

In other words, a rearrangement of an absolutely convergent series converges (absolutely) to the same number.

Proof. Let $\epsilon > 0$ be given. As $\sum x_n$ is absolutely convergent, take M such that

$$\left| \left(\sum_{n=1}^{M} x_n \right) - x \right| < \frac{\epsilon}{2} \qquad \text{and} \qquad \sum_{n=M+1}^{\infty} \lvert x_n \rvert < \frac{\epsilon}{2}.$$

As σ is a bijection, there exists a number K such that for each $n \leq M$, there exists $k \leq K$ such that $\sigma(k) = n$. In other words $\{1,2,\ldots,M\} \subset \sigma\bigl(\{1,2,\ldots,K\}\bigr)$.

For $N \geq K$, let $Q := \max \sigma\bigl(\{1,2,\ldots,N\}\bigr)$. Compute

$$\begin{aligned}
\left| \left(\sum_{n=1}^{N} x_{\sigma(n)} \right) - x \right| &= \left| \left(\sum_{n=1}^{M} x_n + \sum_{\substack{n=1 \\ \sigma(n)>M}}^{N} x_{\sigma(n)} \right) - x \right| \\
&\leq \left| \left(\sum_{n=1}^{M} x_n \right) - x \right| + \sum_{\substack{n=1 \\ \sigma(n)>M}}^{N} \left\lvert x_{\sigma(n)} \right\rvert \\
&\leq \left| \left(\sum_{n=1}^{M} x_n \right) - x \right| + \sum_{n=M+1}^{Q} \lvert x_n \rvert \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So $\sum x_{\sigma(n)}$ converges to x. To see that the convergence is absolute, we apply the argument above to $\sum \lvert x_n \rvert$ to show that $\sum \left\lvert x_{\sigma(n)} \right\rvert$ converges. $\square$

Example 2.76: Let us show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$, which does not converge absolutely, can be rearranged to converge to anything. The odd terms and the even terms diverge to plus infinity and minus infinity respectively (prove this!):

$$\sum_{m=1}^{\infty} \frac{1}{2m-1} = \infty, \qquad \text{and} \qquad \sum_{m=1}^{\infty} \frac{-1}{2m} = -\infty.$$

Let $a_n := \frac{(-1)^{n+1}}{n}$ for simplicity, let an arbitrary number $L \in \mathbb{R}$ be given, and set $\sigma(1) := 1$. Suppose we have defined $\sigma(n)$ for all $n \leq N$. If

$$\sum_{n=1}^{N} a_{\sigma(n)} \leq L,$$

then let $\sigma(N+1) := k$ be the smallest odd $k \in \mathbb{N}$ that we have not used yet, that is, $\sigma(n) \neq k$ for all $n \leq N$. Otherwise, let $\sigma(N+1) := k$ be the smallest even k that we have not yet used.

By construction $\sigma\colon \mathbb{N} \to \mathbb{N}$ is one-to-one. It is also onto, because if we keep adding either odd (resp. even) terms, eventually we pass L and switch to the evens (resp. odds). So we switch infinitely many times.

Finally, let N be the N where we just pass L and switch. For example, suppose we have just switched from odd to even (so we start subtracting), and let $N' > N$ be where we first switch back from even to odd. Then

$$L + \frac{1}{\sigma(N)} \geq \sum_{n=1}^{N-1} a_{\sigma(n)} > \sum_{n=1}^{N'-1} a_{\sigma(n)} > L - \frac{1}{\sigma(N')}.$$

And similarly for switching in the other direction. Therefore, the sum up to $N' - 1$ is within $\frac{1}{\min\{\sigma(N), \sigma(N')\}}$ of L. As we switch infinitely many times we obtain that $\sigma(N) \to \infty$ and $\sigma(N') \to \infty$, and hence

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} \frac{(-1)^{\sigma(n)+1}}{\sigma(n)} = L.$$

Here is an example to illustrate the proof. Suppose $L = 1.2$, then the order is

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{4} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \frac{1}{8} + \cdots .$$

At this point we are no more than $\frac{1}{8}$ from the limit.

2.6.4 Multiplication of series

As we have already mentioned, multiplication of series is somewhat harder than addition. If at least one of the series converges absolutely, then we can use the following theorem. For this result, it is convenient to start the series at 0, rather than at 1.

Theorem 2.77 (Mertens' theorem[11]). *Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series, converging to A and B respectively. If at least one of the series converges absolutely, then the series $\sum_{n=0}^{\infty} c_n$ where*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{j=0}^{n} a_j b_{n-j},$$

converges to AB.

The series $\sum c_n$ is called the *Cauchy product* of $\sum a_n$ and $\sum b_n$.

Proof. Suppose $\sum a_n$ converges absolutely, and let $\epsilon > 0$ be given. In this proof instead of picking complicated estimates just to make the final estimate come out as less than ϵ, let us simply obtain an estimate that depends on ϵ and can be made arbitrarily small.

[11] Proved by the German mathematician Franz Mertens (1840–1927).

Write

$$A_m := \sum_{n=0}^{m} a_n, \qquad B_m := \sum_{n=0}^{m} b_n.$$

We rearrange the mth partial sum of $\sum c_n$:

$$\begin{aligned}
\left| \left(\sum_{n=0}^{m} c_n \right) - AB \right| &= \left| \left(\sum_{n=0}^{m} \sum_{j=0}^{n} a_j b_{n-j} \right) - AB \right| \\
&= \left| \left(\sum_{n=0}^{m} B_n a_{m-n} \right) - AB \right| \\
&= \left| \left(\sum_{n=0}^{m} (B_n - B) a_{m-n} \right) + BA_m - AB \right| \\
&\leq \left(\sum_{n=0}^{m} \left| B_n - B \right| \left| a_{m-n} \right| \right) + \left| B \right| \left| A_m - A \right|
\end{aligned}$$

We can surely make the second term on the right-hand side go to zero. The trick is to handle the first term. Pick K such that for all $m \geq K$, we have $|A_m - A| < \epsilon$ and also $|B_m - B| < \epsilon$. Finally, as $\sum a_n$ converges absolutely, make sure that K is large enough such that for all $m \geq K$,

$$\sum_{n=K}^{m} |a_n| < \epsilon.$$

As $\sum b_n$ converges, then we have that $B_{\max} := \sup\{|B_n - B| : n = 0, 1, 2, \ldots\}$ is finite. Take $m \geq 2K$, then in particular $m - K + 1 > K$. So

$$\begin{aligned}
\sum_{n=0}^{m} \left| B_n - B \right| \left| a_{m-n} \right| &= \left(\sum_{n=0}^{m-K} \left| B_n - B \right| \left| a_{m-n} \right| \right) + \left(\sum_{n=m-K+1}^{m} \left| B_n - B \right| \left| a_{m-n} \right| \right) \\
&\leq \left(\sum_{n=K}^{m} |a_n| \right) B_{\max} + \left(\sum_{n=0}^{K-1} \epsilon \left| a_n \right| \right) \\
&\leq \epsilon B_{\max} + \epsilon \left(\sum_{n=0}^{\infty} |a_n| \right).
\end{aligned}$$

Therefore, for $m \geq 2K$, we have

$$\begin{aligned}
\left| \left(\sum_{n=0}^{m} c_n \right) - AB \right| &\leq \left(\sum_{n=0}^{m} \left| B_n - B \right| \left| a_{m-n} \right| \right) + \left| B \right| \left| A_m - A \right| \\
&\leq \epsilon B_{\max} + \epsilon \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \, \epsilon = \epsilon \left(B_{\max} + \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \right).
\end{aligned}$$

The expression in the parenthesis on the right-hand side is a fixed number. Hence, we can make the right-hand side arbitrarily small by picking a small enough $\epsilon > 0$. So $\sum_{n=0}^{\infty} c_n$ converges to AB. $\square$

Example 2.78: If both series are only conditionally convergent, the Cauchy product series need not even converge. Suppose we take $a_n = b_n = (-1)^n \frac{1}{\sqrt{n+1}}$. The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$ converges by the alternating series test; however, it does not converge absolutely as can be seen from the p-test. Let us look at the Cauchy product.

$$c_n = (-1)^n \left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3(n-1)}} + \cdots + \frac{1}{\sqrt{n+1}} \right) = (-1)^n \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n-j+1)}}.$$

Therefore,

$$|c_n| = \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n-j+1)}} \geq \sum_{j=0}^{n} \frac{1}{\sqrt{(n+1)(n+1)}} = 1.$$

The terms do not go to zero and hence $\sum c_n$ cannot converge.

2.6.5 Power series

Fix $x_0 \in \mathbb{R}$. A *power series* about x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

A power series is really a function of x, and many important functions in analysis can be written as a power series. We use the convention that $0^0 = 1$ (if $x = x_0$ and $n = 0$).

We say that a power series is *convergent* if there is at least one $x \neq x_0$ that makes the series converge. If $x = x_0$, then the series always converges since all terms except the first are zero. If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Example 2.79: The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

is absolutely convergent for all $x \in \mathbb{R}$ using the ratio test: For any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\bigl(1/(n+1)!\bigr)\, x^{n+1}}{(1/n!)\, x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

Recall from calculus that this series converges to e^x.

Example 2.80: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

converges absolutely for all $x \in (-1, 1)$ via the ratio test:

$$\lim_{n\to\infty} \left| \frac{\bigl(1/(n+1)\bigr)\, x^{n+1}}{(1/n)\, x^n} \right| = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| < 1.$$

The series converges at $x = -1$, as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test. But the power series does not converge absolutely at $x = -1$, because $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. The series diverges at $x = 1$. When $|x| > 1$, then the series diverges via the ratio test.

Example 2.81: The series

$$\sum_{n=1}^{\infty} n^n x^n$$

diverges for all $x \neq 0$. Let us apply the root test

$$\limsup_{n\to\infty} \left|n^n x^n\right|^{1/n} = \limsup_{n\to\infty} n\,|x| = \infty.$$

Therefore, the series diverges for all $x \neq 0$.

Convergence of power series in general works analogously to one of the three examples above.

Proposition 2.82. *Let $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ be a power series. If the series is convergent, then either it converges at all $x \in \mathbb{R}$, or there exists a number ρ, such that the series converges absolutely on the interval $(x_0-\rho, x_0+\rho)$ and diverges when $x < x_0 - \rho$ or $x > x_0 + \rho$.*

The number ρ is called the *radius of convergence* of the power series. We write $\rho = \infty$ if the series converges for all x, and we write $\rho = 0$ if the series is divergent. At the endpoints, that is if $x = x_0 + \rho$ or $x = x_0 - \rho$, the proposition says nothing, and the series might or might not converge. See Figure 2.8. In Example 2.80 the radius of convergence is $\rho = 1$. In Example 2.79 the radius of convergence is $\rho = \infty$, and in Example 2.81 the radius of convergence is $\rho = 0$.

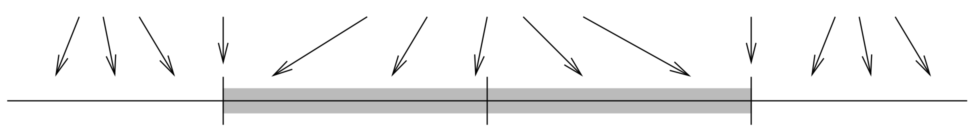

Figure 2.8: Convergence of a power series.

Proof. Write

$$R := \limsup_{n\to\infty} |a_n|^{1/n}.$$

We use the root test to prove the proposition:

$$L = \limsup_{n\to\infty} \left|a_n (x-x_0)^n\right|^{1/n} = |x-x_0| \limsup_{n\to\infty} |a_n|^{1/n} = |x-x_0|\,R.$$

In particular, if $R = \infty$, then $L = \infty$ for every $x \neq x_0$, and the series diverges by the root test. On the other hand, if $R = 0$, then $L = 0$ for every x, and the series converges absolutely for all x.

Suppose $0 < R < \infty$. The series converges absolutely if $1 > L = R\,|x-x_0|$, or in other words when

$$|x-x_0| < \frac{1}{R}.$$

The series diverges when $1 < L = R\,|x - x_0|$, or

$$|x - x_0| > \frac{1}{R}.$$

Letting $\rho := \frac{1}{R}$ completes the proof. $\square$

It may be useful to restate what we have learned in the proof as a separate proposition.

Proposition 2.83. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series, and let*

$$R := \limsup_{n\to\infty} |a_n|^{1/n}.$$

If $R = \infty$, the power series is divergent. If $R = 0$, then the power series converges everywhere. Otherwise, the radius of convergence $\rho = \frac{1}{R}$.

Often, radius of convergence is written as $\rho = \frac{1}{R}$ in all three cases, with the understanding of what ρ should be if $R = 0$ or $R = \infty$.

Convergent power series can be added and multiplied together, and multiplied by constants. The proposition has a straight forward proof using what we know about series in general, and power series in particular. We leave the proof to the reader.

Proposition 2.84. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n (x - x_0)^n$ be two convergent power series with radius of convergence at least $\rho > 0$ and $\alpha \in \mathbb{R}$. Then for all x such that $|x - x_0| < \rho$, we have*

$$\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n\right) + \left(\sum_{n=0}^{\infty} b_n (x - x_0)^n\right) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n,$$

$$\alpha\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n\right) = \sum_{n=0}^{\infty} \alpha a_n (x - x_0)^n,$$

and

$$\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (x - x_0)^n\right) = \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

That is, after performing the algebraic operations, the radius of convergence of the resulting series is at least ρ. For all x with $|x - x_0| < \rho$, we have two convergent series so their term by term addition and multiplication by constants follows by what we learned in the last section. For multiplication of two power series, the series are absolutely convergent inside the radius of convergence and that is why for those x we can apply Mertens' theorem. Note that after applying an algebraic operation the radius of convergence could increase. See the exercises.

Let us look at some examples of power series. Polynomials are simply finite power series. That is, a polynomial is a power series where the a_n are zero for all n large enough. We expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x - x_0)$. For example, $2x^2 - 3x + 4$ as a power series around $x_0 = 1$ is

$$2x^2 - 3x + 4 = 3 + (x - 1) + 2(x - 1)^2.$$

We can also expand *rational functions* (that is, ratios of polynomials) as power series, although we will not completely prove this fact here. Notice that a series for a rational function only defines the function on an interval even if the function is defined elsewhere. For example, for the *geometric series*, we have that for $x \in (-1,1)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

The series diverges when $|x| > 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions as power series around x_0, as long as the denominator is not zero at x_0. We state without proof that this is always possible, and we give an example of such a computation using the geometric series.

Example 2.85: Let us expand $\frac{x}{1+2x+x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

Write $1 + 2x + x^2 = (1+x)^2 = \bigl(1-(-x)\bigr)^2$, and suppose $|x| < 1$. Compute

$$\begin{aligned}
\frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\
&= x\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)^2 \\
&= x\left(\sum_{n=0}^{\infty} c_n x^n\right) \\
&= \sum_{n=0}^{\infty} c_n x^{n+1}.
\end{aligned}$$

Using the formula for the product of series, we obtain $c_0 = 1$, $c_1 = -1-1 = -2$, $c_2 = 1+1+1 = 3$, etc. Hence, for $|x| < 1$,

$$\frac{x}{1+2x+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^n.$$

The radius of convergence is at least 1. We leave it to the reader to verify that the radius of convergence is exactly equal to 1.

You can use the method of partial fractions you know from calculus. For example, to find the power series for $\frac{x^3+x}{x^2-1}$ at 0, write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n.$$

2.6.6 Exercises

Exercise 2.85: *Decide the convergence or divergence of the following series.*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}}$ *b)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{n}$ *c)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/10}}$ *d)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}}$

***Exercise* 2.86:** *Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Show that the product series, $\sum_{n=0}^{\infty} c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, also converges absolutely.*

***Exercise* 2.87** (Challenging)**:** *Let $\sum a_n$ be conditionally convergent. Show that given an arbitrary $x \in \mathbb{R}$ there exists a rearrangement of $\sum a_n$ such that the rearranged series converges to x. Hint: See Example 2.76.*

***Exercise* 2.88:**

a) Show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ has a rearrangement such that whenever $x < y$, there exists a partial sum s_n of the rearranged series such that $x < s_n < y$.

b) Show that the rearrangement you found does not converge. See Example 2.76.

c) Show that for every $x \in \mathbb{R}$, there exists a subsequence of partial sums $\{s_{n_k}\}$ of your rearrangement such that $\lim s_{n_k} = x$.

***Exercise* 2.89:** *For the following power series, find if they are convergent or not, and if so find their radius of convergence.*

a) $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ *b)* $\displaystyle\sum_{n=0}^{\infty} n x^n$ *c)* $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$ *d)* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}(x-10)^n$ *e)* $\displaystyle\sum_{n=0}^{\infty} x^{2n}$ *f)* $\displaystyle\sum_{n=0}^{\infty} n!\, x^{n!}$

***Exercise* 2.90:** *Suppose $\sum a_n x^n$ converges for $x = 1$.*

a) What can you say about the radius of convergence?

b) If you further know that at $x = 1$ the convergence is not absolute, what can you say?

***Exercise* 2.91:** *Expand $\frac{x}{4-x^2}$ as a power series around $x_0 = 0$ and compute its radius of convergence.*

***Exercise* 2.92:**

a) Find an example where the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are both 1, but the radius of convergence of the sum of the two series is infinite.

b) (Trickier) Find an example where the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are both 1, but the radius of convergence of the product of the two series is infinite.

***Exercise* 2.93:** *Figure out how to compute the radius of convergence using the ratio test. That is, suppose $\sum a_n x^n$ is a power series and* $R := \lim \frac{|a_{n+1}|}{|a_n|}$ *exists or is ∞. Find the radius of convergence and prove your claim.*

***Exercise* 2.94:**

a) Prove that $\lim n^{1/n} = 1$ *using the following procedure: Write $n^{1/n} = 1 + b_n$ and note $b_n > 0$. Then show that $(1 + b_n)^n \geq \frac{n(n-1)}{2} b_n^2$ and use this to show that* $\lim b_n = 0$.

b) Use the result of part a) to show that if $\sum a_n x^n$ is a convergent power series with radius of convergence R, then $\sum n a_n x^n$ is also convergent with the same radius of convergence.

There are different notions of summability (convergence) of a series than just the one we have seen. A common one is *Cesàro summability*[12]. Let $\sum a_n$ be a series and let s_n be the nth partial sum. The series is said to be Cesàro summable to a if

$$a = \lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

[12]Named for the Italian mathematician Ernesto Cesàro (1859–1906).

Exercise* 2.95** (Challenging):***

a) If $\sum a_n$ is convergent to a (in the usual sense), show that $\sum a_n$ is Cesàro summable (see above) to a.

b) Show that in the sense of Cesàro $\sum (-1)^n$ is summable to $\frac{1}{2}$.

c) Let $a_n := k$ when $n = k^3$ for some $k \in \mathbb{N}$, $a_n := -k$ when $n = k^3+1$ for some $k \in \mathbb{N}$, otherwise let $a_n := 0$. Show that $\sum a_n$ diverges in the usual sense, (partial sums are unbounded), but it is Cesàro summable to 0 (seems a little paradoxical at first sight).

Exercise* 2.96** (Challenging):*** *Show that the monotonicity in the alternating series test is necessary. That is, find a sequence of positive real numbers $\{x_n\}$ with $\lim x_n = 0$ but such that $\sum (-1)^n x_n$ diverges.*

Exercise* 2.97*: *Find a series such that $\sum x_n$ converges but $\sum x_n^2$ diverges. Hint: Compare Exercise 2.81.*

Exercise* 2.98*: *Suppose $\{c_n\}$ is a sequence. Prove that for every $r \in (0,1)$, there exists a strictly increasing sequence $\{n_k\}$ of natural numbers ($n_{k+1} > n_k$) such that*

$$\sum_{k=1}^{\infty} c_k x^{n_k}$$

converges absolutely for all $x \in [-r, r]$.