Chapter 1

Real Numbers

1.1 Basic properties

The main object we work with in analysis is the set of real numbers. As this set is so fundamental, often much time is spent on formally constructing the set of real numbers. However, we take an easier approach here and just assume that a set with the correct properties exists. We start with the definitions of those properties.

Definition 1.1. An *ordered set* is a set S together with a relation $<$ such that

(i) *(trichotomy)* For all $x, y \in S$, exactly one of $x < y$, $x = y$, or $y < x$ holds.

(ii) *(transitivity)* If $x, y, z \in S$ are such that $x < y$ and $y < z$, then $x < z$.

We write $x \leq y$ if $x < y$ or $x = y$. We define $>$ and $\geq$ in the obvious way.

The set of rational numbers $\mathbb{Q}$ is an ordered set by letting $x < y$ if and only if $y - x$ is a positive rational number, that is if $y - x = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Similarly, $\mathbb{N}$ and $\mathbb{Z}$ are also ordered sets.

There are other ordered sets than sets of numbers. For example, the set of countries can be ordered by landmass, so India $>$ Lichtenstein. A typical ordered set that you have used since primary school is the dictionary. It is the ordered set of words where the order is the so-called lexicographic ordering. Such ordered sets often appear, for example, in computer science. In this book we will mostly be interested in ordered sets of numbers.

Definition 1.2. Let $E \subset S$, where S is an ordered set.

(i) If there exists a $b \in S$ such that $x \leq b$ for all $x \in E$, then we say E is *bounded above* and b is an *upper bound* of E.

(ii) If there exists a $b \in S$ such that $x \geq b$ for all $x \in E$, then we say E is *bounded below* and b is a *lower bound* of E.

(iii) If there exists an upper bound b_0 of E such that whenever b is an upper bound for E we have $b_0 \leq b$, then b_0 is called the *least upper bound* or the *supremum* of E. See Figure 1.1. We write

$$\sup E := b_0.$$

(iv) Similarly, if there exists a lower bound b_0 of E such that whenever b is a lower bound for E we have $b_0 \geq b$, then b_0 is called the *greatest lower bound* or the *infimum* of E. We write

$$\inf E := b_0.$$

When a set E is both bounded above and bounded below, we say simply that E is *bounded*.

The notation $\sup E$ and $\inf E$ is justified as the supremum (or infimum) is unique (if it exists): If b and b' are suprema of E, then $b \leq b'$ and $b' \leq b$, because both b and b' are the least upper bounds, so $b = b'$.

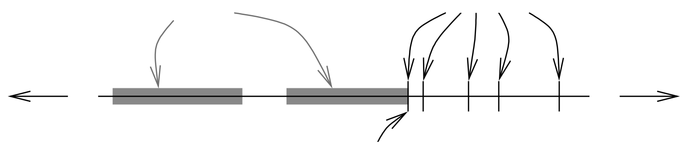

Figure 1.1: A set E bounded above and the least upper bound of E.

A simple example: Let $S := \{a, b, c, d, e\}$ be ordered as $a < b < c < d < e$, and let $E := \{a, c\}$. Then c, d, and e are upper bounds of E, and c is the least upper bound or supremum of E.

A supremum or infimum for E (even if it exists) need not be in E. The set $E := \{x \in \mathbb{Q} : x < 1\}$ has a least upper bound of 1, but 1 is not in the set E itself. The set $G := \{x \in \mathbb{Q} : x \leq 1\}$ also has an upper bound of 1, and in this case $1 \in G$. The set $P := \{x \in \mathbb{Q} : x \geq 0\}$ has no upper bound (why?) and therefore it cannot have a least upper bound. The set P does have a greatest lower bound: 0.

Definition 1.3. An ordered set S has the *least-upper-bound property* if every nonempty subset $E \subset S$ that is bounded above has a least upper bound, that is $\sup E$ exists in S.

The *least-upper-bound property* is sometimes called the *completeness property* or the *Dedekind completeness property*[1]. As we will note in the next section, the real numbers have this property.

Example 1.4: The set $\mathbb{Q}$ of rational numbers does not have the least-upper-bound property. The subset $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum in $\mathbb{Q}$. We will see later (Example 1.14) that the supremum is $\sqrt{2}$, which is not rational[2]. Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. Write $x = \frac{m}{n}$ in lowest terms. So $\left(\frac{m}{n}\right)^2 = 2$ or $m^2 = 2n^2$. Hence, m^2 is divisible by 2, and so m is divisible by 2. Write $m = 2k$ and so $(2k)^2 = 2n^2$. Divide by 2 and note that $2k^2 = n^2$, and hence n is divisible by 2. But that is a contradiction as $\frac{m}{n}$ is in lowest terms.

[1] Named after the German mathematician Julius Wilhelm Richard Dedekind (1831–1916).

[2] This is true for all other roots of 2, and interestingly, the fact that $\sqrt[k]{2}$ is never rational for $k > 1$ implies no piano can ever be perfectly tuned in all keys. See for example: `https://youtu.be/1Hqm0dYKUx4`.

That $\mathbb{Q}$ does not have the least-upper-bound property is one of the most important reasons why we work with $\mathbb{R}$ in analysis. The set $\mathbb{Q}$ is just fine for algebraists. But us analysts require the least-upper-bound property to do any work. We also require our real numbers to have many algebraic properties. In particular, we require that they are a field.

Definition 1.5. A set F is called a *field* if it has two operations defined on it, addition $x + y$ and multiplication xy, and if it satisfies the following axioms:

(A1) If $x \in F$ and $y \in F$, then $x + y \in F$.

(A2) *(commutativity of addition)* $x + y = y + x$ for all $x, y \in F$.

(A3) *(associativity of addition)* $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) There exists an element $0 \in F$ such that $0 + x = x$ for all $x \in F$.

(A5) For every element $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) If $x \in F$ and $y \in F$, then $xy \in F$.

(M2) *(commutativity of multiplication)* $xy = yx$ for all $x, y \in F$.

(M3) *(associativity of multiplication)* $(xy)z = x(yz)$ for all $x, y, z \in F$.

(M4) There exists an element $1 \in F$ (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.

(M5) For every $x \in F$ such that $x \neq 0$ there exists an element $\frac{1}{x} \in F$ such that $x(\frac{1}{x}) = 1$.

(D) *(distributive law)* $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Example 1.6: The set $\mathbb{Q}$ of rational numbers is a field. On the other hand $\mathbb{Z}$ is not a field, as it does not contain multiplicative inverses. For example, there is no $x \in \mathbb{Z}$ such that $2x = 1$, so (M5) is not satisfied. You can check that (M5) is the only property that fails[3].

We will assume the basic facts about fields that are easily proved from the axioms. For example, $0x = 0$ is easily proved by noting that $xx = (0 + x)x = 0x + xx$, using (A4), (D), and (M2). Then using (A5) on xx, along with (A2), (A3), and (A4), we obtain $0 = 0x$.

Definition 1.7. A field F is said to be an *ordered field* if F is also an ordered set such that

(i) For $x, y, z \in F$, $x < y$ implies $x + z < y + z$.

(ii) For $x, y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$.

If $x > 0$, we say x is *positive*. If $x < 0$, we say x is *negative*. We also say x is *nonnegative* if $x \geq 0$, and x is *nonpositive* if $x \leq 0$.

It can be checked that the rational numbers $\mathbb{Q}$ with the standard ordering is an ordered field.

[3] An algebraist would say that $\mathbb{Z}$ is an ordered ring, or perhaps more precisely a commutative ordered ring.

Proposition 1.8. *Let F be an ordered field and $x, y, z, w \in F$. Then*

(i) If $x > 0$, then $-x < 0$ (and vice versa).

(ii) If $x > 0$ and $y < z$, then $xy < xz$.

(iii) If $x < 0$ and $y < z$, then $xy > xz$.

(iv) If $x \neq 0$, then $x^2 > 0$.

(v) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

(vi) If $0 < x < y$, then $x^2 < y^2$.

(vii) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.

Note that iv implies in particular that $1 > 0$.

Proof. Let us prove i. The inequality $x > 0$ implies by item i of definition of ordered fields that $x + (-x) > 0 + (-x)$. Apply the algebraic properties of fields to obtain $0 > -x$. The "vice versa" follows by similar calculation.

For ii, notice that $y < z$ implies $0 < z - y$ by item i of the definition of ordered fields. Apply item ii of the definition of ordered fields to obtain $0 < x(z - y)$. By algebraic properties, $0 < xz - xy$. Again by item i of the definition, $xy < xz$.

Part iii is left as an exercise.

To prove part iv first suppose $x > 0$. By item ii of the definition of ordered fields, $x^2 > 0$ (use $y = x$). If $x < 0$, we use part iii of this proposition, where we plug in $y = x$ and $z = 0$.

To prove part v, notice that $\frac{1}{x}$ cannot be equal to zero (why?). Suppose $\frac{1}{x} < 0$, then $\frac{-1}{x} > 0$ by i. Apply part ii (as $x > 0$) to obtain $x(\frac{-1}{x}) > 0x$ or $-1 > 0$, which contradicts $1 > 0$ by using part i again. Hence $\frac{1}{x} > 0$. Similarly, $\frac{1}{y} > 0$. Thus $(\frac{1}{x})(\frac{1}{y}) > 0$ by definition of ordered field and by part ii

$$(\frac{1}{x})(\frac{1}{y})x < (\frac{1}{x})(\frac{1}{y})y.$$

By algebraic properties we get $\frac{1}{y} < \frac{1}{x}$.

Parts vi and vii are left as exercises. ☐

The product of two positive numbers (elements of an ordered field) is positive. However, it is not true that if the product is positive, then each of the two factors must be positive. For instance, $(-1)(-1) = 1 > 0$.

Proposition 1.9. *Let $x, y \in F$, where F is an ordered field. If $xy > 0$, then either both x and y are positive, or both are negative.*

Proof. We show the contrapositive: If either one of x or y is zero, or if x and y have opposite signs, then xy is not positive. If either x or y is zero, then xy is zero and hence not positive. Hence assume that x and y are nonzero and have opposite signs. Without loss of generality suppose $x > 0$ and $y < 0$. Multiply $y < 0$ by x to get $xy < 0x = 0$. ☐

Example 1.10: The reader may also know about the *complex numbers*, usually denoted by $\mathbb{C}$. That is, $\mathbb{C}$ is the set of numbers of the form $x + iy$, where x and y are real numbers, and i is the imaginary number, a number such that $i^2 = -1$. The reader may remember from algebra that $\mathbb{C}$ is also a field; however, it is not an ordered field. While one can make $\mathbb{C}$ into an ordered set in some way, it is not possible to put an order on $\mathbb{C}$ that would make it an ordered field: In every ordered field, $-1 < 0$ and $x^2 > 0$ for all nonzero x, but in $\mathbb{C}$, $i^2 = -1$.

Finally, an ordered field that has the least-upper-bound property has the corresponding property for greatest lower bounds.

Proposition 1.11. *Let F be an ordered field with the least-upper-bound property. Let $A \subset F$ be a nonempty set that is bounded below. Then $\inf\, A$ exists.*

Proof. Let $B := \{-x : x \in A\}$. Let $b \in F$ be a lower bound for A: If $x \in A$, then $x \geq b$. In other words, $-x \leq -b$. So $-b$ is an upper bound for B. Since F has the least-upper-bound property, $c := \sup\, B$ exists, and $c \leq -b$. As $y \leq c$ for all $y \in B$, then $-c \leq x$ for all $x \in A$. So $-c$ is a lower bound for A. As $-c \geq b$, $-c$ is the greatest lower bound of A. $\square$

1.1.1 Exercises

Exercise 1.1: *Prove part iii of Proposition 1.8. That is, let F be an ordered field and $x, y, z \in F$. Prove If $x < 0$ and $y < z$, then $xy > xz$.*

Exercise 1.2: *Let S be an ordered set. Let $A \subset S$ be a nonempty finite subset. Then A is bounded. Furthermore, $\inf\, A$ exists and is in A and $\sup\, A$ exists and is in A. Hint: Use induction.*

Exercise 1.3: *Prove part vi of Proposition 1.8. That is, let $x, y \in F$, where F is an ordered field, such that $0 < x < y$. Show that $x^2 < y^2$.*

Exercise 1.4: *Let S be an ordered set. Let $B \subset S$ be bounded (above and below). Let $A \subset B$ be a nonempty subset. Suppose all the infs and sups exist. Show that*

$$\inf\, B \leq \inf\, A \leq \sup\, A \leq \sup\, B.$$

Exercise 1.5: *Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup\, A$.*

Exercise 1.6: *Let S be an ordered set. Let $A \subset S$ be nonempty and bounded above. Suppose $\sup\, A$ exists and $\sup\, A \notin A$. Show that A contains a countably infinite subset.*

Exercise 1.7: *Find a (nonstandard) ordering of the set of natural numbers $\mathbb{N}$ such that there exists a nonempty proper subset $A \subsetneq \mathbb{N}$ and such that $\sup\, A$ exists in $\mathbb{N}$, but $\sup\, A \notin A$. To keep things straight it might be a good idea to use a different notation for the nonstandard ordering such as $n \prec m$.*

Exercise 1.8: *Let $F := \{0, 1, 2\}$.*

a) Prove that there is exactly one way to define addition and multiplication so that F is a field if 0 and 1 have their usual meaning of (A4) and (M4).

b) Show that F cannot be an ordered field.

Exercise* 1.9*: *Let S be an ordered set and A is a nonempty subset such that* $\sup A$ *exists. Suppose there is a $B \subset A$ such that whenever $x \in A$ there is a $y \in B$ such that $x \leq y$. Show that* $\sup B$ *exists and* $\sup B = \sup A$.

Exercise* 1.10*: *Let D be the ordered set of all possible words (not just English words, all strings of letters of arbitrary length) using the Latin alphabet using only lower case letters. The order is the lexicographic order as in a dictionary (e.g.* $\mathrm{aa} < \mathrm{aaa} < \mathrm{dog} < \mathrm{door}$*). Let A be the subset of D containing the words whose first letter is 'a' (e.g.* $\mathrm{a} \in A$, $\mathrm{abcd} \in A$*). Show that A has a supremum and find what it is.*

Exercise* 1.11*: *Let F be an ordered field and $x, y, z, w \in F$.*

a) Prove part vii of Proposition 1.8. That is, if $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.

b) Prove that if $x < y$ and $z \leq w$, then $x + z < y + w$.

Exercise* 1.12*: *Prove that any ordered field must contain a countably infinite set.*

Exercise* 1.13*: *Let $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, where elements of $\mathbb{N}$ are ordered in the usual way amongst themselves, and $k < \infty$ for every $k \in \mathbb{N}$. Show $\mathbb{N}_\infty$ is an ordered set and that every subset $E \subset \mathbb{N}_\infty$ has a supremum in $\mathbb{N}_\infty$ (make sure to also handle the case of an empty set).*

Exercise* 1.14*: *Let $S := \{a_k : k \in \mathbb{N}\} \cup \{b_k : k \in \mathbb{N}\}$, ordered such that $a_k < b_j$ for every k and j, $a_k < a_m$ whenever $k < m$, and $b_k > b_m$ whenever $k < m$.*

a) Show that S is an ordered set.

b) Show that every subset of S is bounded (both above and below).

c) Find a bounded subset of S that has no least upper bound.

1.2 The set of real numbers

1.2.1 The set of real numbers

We finally get to the real number system. To simplify matters, instead of constructing the real number set from the rational numbers, we simply state their existence as a theorem without proof. Notice that $\mathbb{Q}$ is an ordered field.

Theorem 1.12. *There exists a unique*[4] *ordered field* $\mathbb{R}$ *with the least-upper-bound property such that* $\mathbb{Q} \subset \mathbb{R}$.

Note that also $\mathbb{N} \subset \mathbb{Q}$. We saw that $1 > 0$. By induction (exercise) we can prove that $n > 0$ for all $n \in \mathbb{N}$. Similarly, we verify simple statements about rational numbers. For example, we proved that if $n > 0$, then $\frac{1}{n} > 0$. Then $m < k$ implies $\frac{m}{n} < \frac{k}{n}$.

Let us prove one of the most basic but useful results about the real numbers. The following proposition is essentially how an analyst proves an inequality.

Proposition 1.13. *If* $x \in \mathbb{R}$ *is such that* $x \leq \epsilon$ *for all* $\epsilon \in \mathbb{R}$ *where* $\epsilon > 0$*, then* $x \leq 0$.

Proof. If $x > 0$, then $0 < \frac{x}{2} < x$ (why?). Taking $\epsilon = \frac{x}{2}$ obtains a contradiction. Thus $x \leq 0$. □

Another useful version of this idea is the following equivalent statement for nonnegative numbers: *If* $x \geq 0$ *is such that* $x \leq \epsilon$ *for all* $\epsilon > 0$*, then* $x = 0$. And to prove that $x \geq 0$ in the first place, an analyst might prove that all $x \geq -\epsilon$ for all $\epsilon > 0$. From now on, when we say $x \geq 0$ or $\epsilon > 0$, we automatically mean that $x \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$.

A related simple fact is that any time we have two real numbers $a < b$, then there is another real number c such that $a < c < b$. Take, for example, $c = \frac{a+b}{2}$ (why?). In fact, there are infinitely many real numbers between a and b. We will use this fact in the next example.

The most useful property of $\mathbb{R}$ for analysts is not just that it is an ordered field, but that it has the least-upper-bound property. Essentially, we want $\mathbb{Q}$, but we also want to take suprema (and infima) willy-nilly. So what we do is take $\mathbb{Q}$ and throw in enough numbers to obtain $\mathbb{R}$.

We mentioned already that $\mathbb{R}$ contains elements that are not in $\mathbb{Q}$ because of the least-upper-bound property. Let us prove it. We saw there is no rational square root of two. The set $\{x \in \mathbb{Q} : x^2 < 2\}$ implies the existence of the real number $\sqrt{2}$, although this fact requires a bit of work. See also Exercise 1.28.

Example 1.14: Claim: *There exists a unique positive* $r \in \mathbb{R}$ *such that* $r^2 = 2$*. We denote* r *by* $\sqrt{2}$.

Proof. Take the set $A := \{x \in \mathbb{R} : x^2 < 2\}$. We first show that A is bounded above and nonempty. The equation $x \geq 2$ implies $x^2 \geq 4$ (see Exercise 1.3), so if $x^2 < 2$, then $x < 2$, and A is bounded above. As $1 \in A$, the set A is nonempty. We can therefore find the supremum.

[4] Uniqueness is up to isomorphism, but we wish to avoid excessive use of algebra. For us, it is simply enough to assume that a set of real numbers exists. See Rudin [2] for the construction and more details.

Let $r := \sup A$. We will show that $r^2 = 2$ by showing that $r^2 \geq 2$ and $r^2 \leq 2$. This is the way analysts show equality, by showing two inequalities. We already know that $r \geq 1 > 0$.

In the following, it may seem we are pulling certain expressions out of a hat. When writing a proof such as this we would, of course, come up with the expressions only after playing around with what we wish to prove. The order in which we write the proof is not necessarily the order in which we come up with the proof.

Let us first show that $r^2 \geq 2$. Take a positive number s such that $s^2 < 2$. We wish to find an $h > 0$ such that $(s+h)^2 < 2$. As $2 - s^2 > 0$, we have $\frac{2-s^2}{2s+1} > 0$. Choose an $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$. Furthermore, assume $h < 1$. Estimate,

$$\begin{aligned} (s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) && \bigl(\text{since } h < 1\bigr) \\ &< 2 - s^2 && \bigl(\text{since } h < \tfrac{2-s^2}{2s+1}\bigr). \end{aligned}$$

Therefore, $(s+h)^2 < 2$. Hence $s + h \in A$, but as $h > 0$, we have $s + h > s$. So $s < r = \sup A$. As s was an arbitrary positive number such that $s^2 < 2$, it follows that $r^2 \geq 2$.

Now take a positive number s such that $s^2 > 2$. We wish to find an $h > 0$ such that $(s-h)^2 > 2$ and $s - h$ is still positive. As $s^2 - 2 > 0$, we have $\frac{s^2-2}{2s} > 0$. Let $h := \frac{s^2-2}{2s}$, and check $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$. Estimate,

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh && \bigl(\text{since } h^2 > 0 \text{ as } h \neq 0\bigr) \\ &= s^2 - 2 && \bigl(\text{since } h = \tfrac{s^2-2}{2s}\bigr). \end{aligned}$$

By subtracting s^2 from both sides and multiplying by -1, we find $(s-h)^2 > 2$. Therefore, $s - h \notin A$.

Moreover, if $x \geq s - h$, then $x^2 \geq (s-h)^2 > 2$ (as $x > 0$ and $s - h > 0$) and so $x \notin A$. Thus, $s-h$ is an upper bound for A. However, $s-h < s$, or in other words, $s > r = \sup A$. Hence, $r^2 \leq 2$.

Together, $r^2 \geq 2$ and $r^2 \leq 2$ imply $r^2 = 2$. The existence part is finished. We still need to handle uniqueness. Suppose $s \in \mathbb{R}$ such that $s^2 = 2$ and $s > 0$. Thus $s^2 = r^2$. However, if $0 < s < r$, then $s^2 < r^2$. Similarly, $0 < r < s$ implies $r^2 < s^2$. Hence $s = r$. $\square$

The number $\sqrt{2} \notin \mathbb{Q}$. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational* numbers. We just saw that $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. Not only is it nonempty, we will see later that it is very large indeed.

Using the same technique as above, we can show that a positive real number $x^{1/n}$ exists for all $n \in \mathbb{N}$ and all $x > 0$. That is, for each $x > 0$, there exists a unique positive real number r such that $r^n = x$. The proof is left as an exercise.

1.2.2 Archimedean property

As we have seen, there are plenty of real numbers in any interval. But there are also infinitely many rational numbers in any interval. The following is one of the fundamental facts about the real numbers. The two parts of the next theorem are actually equivalent, even though it may not seem like that at first sight.

Theorem 1.15.

(i) (Archimedean property)[5] *If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that*

$$nx > y.$$

(ii) ($\mathbb{Q}$ is dense in $\mathbb{R}$) *If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. Let us prove i. Divide through by x. Then i says that for every real number $t := \frac{y}{x}$, we can find $n \in \mathbb{N}$ such that $n > t$. In other words, i says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Let $b := \sup \mathbb{N}$. The number $b - 1$ cannot possibly be an upper bound for $\mathbb{N}$ as it is strictly less than b (the least upper bound). Thus there exists an $m \in \mathbb{N}$ such that $m > b - 1$. Add one to obtain $m + 1 > b$, contradicting b being an upper bound.

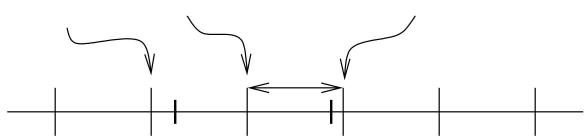

Figure 1.2: Idea of the proof of the density of $\mathbb{Q}$: Find n such that $y - x > \frac{1}{n}$, then take the least m such that $\frac{m}{n} > x$.

Let us tackle ii. See Figure 1.2 for a picture of the idea behind the proof. First assume $x \geq 0$. Note that $y - x > 0$. By i, there exists an $n \in \mathbb{N}$ such that

$$n(y - x) > 1 \qquad \text{or} \qquad y - x > \frac{1}{n}.$$

Again by i the set $A := \{k \in \mathbb{N} : k > nx\}$ is nonempty. By the well ordering property of $\mathbb{N}$, A has a least element m, and as $m \in A$, then $m > nx$. Divide through by n to get $x < \frac{m}{n}$. As m is the least element of A, $m - 1 \notin A$. If $m > 1$, then $m - 1 \in \mathbb{N}$, but $m - 1 \notin A$ and so $m - 1 \leq nx$. If $m = 1$, then $m - 1 = 0$, and $m - 1 \leq nx$ still holds as $x \geq 0$. In other words,

$$m - 1 \leq nx \qquad \text{or} \qquad m \leq nx + 1.$$

On the other hand, from $n(y - x) > 1$ we obtain $ny > 1 + nx$. Hence $ny > 1 + nx \geq m$, and therefore $y > \frac{m}{n}$. Putting everything together we obtain $x < \frac{m}{n} < y$. So take $r = \frac{m}{n}$.

Now assume $x < 0$. If $y > 0$, then just take $r = 0$. If $y \leq 0$, then $0 \leq -y < -x$, and we find a rational q such that $-y < q < -x$. Then take $r = -q$. □

Let us state and prove a simple but useful corollary of the Archimedean property.

[5] Named after the Ancient Greek mathematician Archimedes of Syracuse (c. 287 BC – c. 212 BC). This property is Axiom V from Archimedes' "On the Sphere and Cylinder" 225 BC.

Corollary 1.16. $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.

Proof. Let $A := \{\frac{1}{n} : n \in \mathbb{N}\}$. Obviously A is not empty. Furthermore, $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, and so 0 is a lower bound, and $b := \inf A$ exists. As 0 is a lower bound, then $b \geq 0$. Take an arbitrary $a > 0$. By the Archimedean property there exists an n such that $na > 1$, or in other words $a > \frac{1}{n} \in A$. Therefore, a cannot be a lower bound for A. Hence $b = 0$. □

1.2.3 Using supremum and infimum

Suprema and infima are compatible with algebraic operations. For a set $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ define

$$x + A := \{x + y \in \mathbb{R} : y \in A\},$$
$$xA := \{xy \in \mathbb{R} : y \in A\}.$$

For example, if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition 1.17. *Let $A \subset \mathbb{R}$ be nonempty.*

(i) If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$.

(ii) If $x \in \mathbb{R}$ and A is bounded below, then $\inf(x + A) = x + \inf A$.

(iii) If $x > 0$ and A is bounded above, then $\sup(xA) = x(\sup A)$.

(iv) If $x > 0$ and A is bounded below, then $\inf(xA) = x(\inf A)$.

(v) If $x < 0$ and A is bounded below, then $\sup(xA) = x(\inf A)$.

(vi) If $x < 0$ and A is bounded above, then $\inf(xA) = x(\sup A)$.

Do note that multiplying a set by a negative number switches supremum for an infimum and vice versa. Also, as the proposition implies that supremum (resp. infimum) of $x + A$ or xA exists, it also implies that $x + A$ or xA is nonempty and bounded above (resp. below).

Proof. Let us only prove the first statement. The rest are left as exercises.

Suppose b is an upper bound for A. That is, $y \leq b$ for all $y \in A$. Then $x + y \leq x + b$ for all $y \in A$, and so $x + b$ is an upper bound for $x + A$. In particular, if $b = \sup A$, then

$$\sup(x + A) \leq x + b = x + \sup A.$$

The opposite inequality is similar. If b is an upper bound for $x + A$, then $x + y \leq b$ for all $y \in A$ and so $y \leq b - x$ for all $y \in A$. So $b - x$ is an upper bound for A. If $b = \sup(x + A)$, then

$$\sup A \leq b - x = \sup(x + A) - x.$$

The result follows. □

Sometimes we need to apply supremum or infimum twice. Here is an example.

Proposition 1.18. *Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and* $\sup A \leq \inf B$.

Proof. Any $x \in A$ is a lower bound for B. Therefore $x \leq \inf B$ for all $x \in A$, so $\inf B$ is an upper bound for A. Hence, $\sup A \leq \inf B$. $\square$

We must be careful about strict inequalities and taking suprema and infima. Note that $x < y$ whenever $x \in A$ and $y \in B$ still only implies $\sup A \leq \inf B$, and not a strict inequality. This is an important subtle point that comes up often. For example, take $A := \{0\}$ and take $B := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$. However, $\sup A = 0$ and $\inf B = 0$.

The proof of the following often used elementary fact is left to the reader. A similar statement holds for infima.

Proposition 1.19. *If $S \subset \mathbb{R}$ is nonempty and bounded above, then for every $\epsilon > 0$ there exists an $x \in S$ such that $(\sup S) - \epsilon < x \leq \sup S$.*

To make using suprema and infima even easier, we may want to write $\sup A$ and $\inf A$ without worrying about A being bounded and nonempty. We make the following natural definitions.

Definition 1.20. Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\sup A := -\infty$.

(ii) If A is not bounded above, then $\sup A := \infty$.

(iii) If A is empty, then $\inf A := \infty$.

(iv) If A is not bounded below, then $\inf A := -\infty$.

For convenience, ∞ and $-\infty$ are sometimes treated as if they were numbers, except we do not allow arbitrary arithmetic with them. We make $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ into an ordered set by letting

$$-\infty < \infty \quad \text{and} \quad -\infty < x \quad \text{and} \quad x < \infty \quad \text{for all } x \in \mathbb{R}.$$

The set $\mathbb{R}^*$ is called the set of *extended real numbers*. It is possible to define some arithmetic on $\mathbb{R}^*$. Most operations are extended in an obvious way, but we must leave $\infty - \infty$, $0 \cdot (\pm\infty)$, and $\frac{\pm\infty}{\pm\infty}$ undefined. We refrain from using this arithmetic, it leads to easy mistakes as $\mathbb{R}^*$ is not a field. Now we can take suprema and infima without fear of emptiness or unboundedness. In this book, we mostly avoid using $\mathbb{R}^*$ outside of exercises, and leave such generalizations to the interested reader.

1.2.4 Maxima and minima

By Exercise 1.2, a finite set of numbers always has a supremum or an infimum that is contained in the set itself. In this case we usually do not use the words supremum or infimum.

When a set A of real numbers is bounded above, such that $\sup A \in A$, then we can use the word *maximum* and the notation $\max A$ to denote the supremum. Similarly for infimum: When a set A is bounded below and $\inf A \in A$, then we can use the word *minimum* and the notation $\min A$. For example,

$$\max\{1, 2.4, \pi, 100\} = 100,$$
$$\min\{1, 2.4, \pi, 100\} = 1.$$

While writing sup and inf may be technically correct in this situation, max and min are generally used to emphasize that the supremum or infimum is in the set itself.

1.2.5 Exercises

Exercise* 1.15*: *Prove that if $t > 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n^2} < t$.*

Exercise* 1.16*: *Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.*

Exercise* 1.17*: *Finish the proof of Proposition 1.17.*

Exercise* 1.18*: *Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.*

Exercise* 1.19*: *Show that $\sqrt{3}$ is irrational.*

Exercise* 1.20*: *Let $n \in \mathbb{N}$. Show that either $\sqrt{n}$ is either an integer or it is irrational.*

Exercise* 1.21*: *Prove the* arithmetic-geometric mean inequality*. That is, for two positive real numbers x, y, we have*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Furthermore, equality occurs if and only if $x = y$.

Exercise* 1.22*: *Show that for every pair of real numbers x and y such that $x < y$, there exists an irrational number s such that $x < s < y$. Hint: Apply the density of $\mathbb{Q}$ to $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$.*

Exercise* 1.23*: *Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that*

$$\sup C = \sup A + \sup B \qquad \textit{and} \qquad \inf C = \inf A + \inf B.$$

Exercise* 1.24*: *Let A and B be two nonempty bounded sets of nonnegative real numbers. Define the set $C := \{ab : a \in A, b \in B\}$. Show that C is a bounded set and that*

$$\sup C = (\sup A)(\sup B) \qquad \textit{and} \qquad \inf C = (\inf A)(\inf B).$$

Exercise* 1.25** (Hard):*** *Given $x > 0$ and $n \in \mathbb{N}$, show that there exists a unique positive real number r such that $x = r^n$. Usually r is denoted by $x^{1/n}$.*

Exercise* 1.26** (Easy):*** *Prove Proposition 1.19.*

Exercise* 1.27*: *Prove the so-called* Bernoulli's inequality[6]*: If $1 + x > 0$, then for all $n \in \mathbb{N}$, we have $(1 + x)^n \geq 1 + nx$.*

Exercise* 1.28*: *Prove* $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sup\{x \in \mathbb{R} : x^2 < 2\}$.

[6]Named after the Swiss mathematician Jacob Bernoulli (1655–1705).

Exercise* 1.29*:

a) Prove that given $y \in \mathbb{R}$, we have $\sup\{x \in \mathbb{Q} : x < y\} = y$.

b) Let $A \subset \mathbb{Q}$ be a set that is bounded above such that whenever $x \in A$ and $t \in \mathbb{Q}$ with $t < x$, then $t \in A$. Further suppose $\sup A \notin A$. *Show that there exists a $y \in \mathbb{R}$ such that $A = \{x \in \mathbb{Q} : x < y\}$. A set such as A is called a* Dedekind cut.

c) Show that there is a bijection between $\mathbb{R}$ and Dedekind cuts.

Note: Dedekind used sets as in part b) in his construction of the real numbers.

Exercise* 1.30*: *Prove that if $A \subset \mathbb{Z}$ is a nonempty subset bounded below, then there exists a least element in A. Now describe why this statement would simplify the proof of Theorem 1.15 part ii so that you do not have to assume $x \geq 0$.*

Exercise* 1.31*: *Let us suppose we know $x^{1/n}$ exists for every $x > 0$ and every $n \in \mathbb{N}$ (see Exercise 1.25 above). For integers p and $q > 0$ where $\frac{p}{q}$ is in lowest terms, define $x^{p/q} := (x^{1/q})^p$.*

a) Show that the power is well-defined even if the fraction is not in lowest terms: If $\frac{p}{q} = \frac{m}{k}$ where m and $k > 0$ are integers, then $(x^{1/q})^p = (x^{1/m})^k$.

b) Let x and y be two positive numbers and r a rational number. Assuming $r > 0$, show $x < y$ if and only if $x^r < y^r$. Then suppose $r < 0$ and show: $x < y$ if and only if $x^r > y^r$.

c) Suppose $x > 1$ and r, s are rational where $r < s$. Show $x^r < x^s$. If $0 < x < 1$ and $r < s$, show that $x^r > x^s$. Hint: Write r and s with the same denominator.

d) (Challenging)[7] *For an irrational $z \in \mathbb{R} \setminus \mathbb{Q}$ and $x > 1$ define $x^z := \sup\{x^r : r \leq z, r \in \mathbb{Q}\}$, for $x = 1$ define $1^z = 1$, and for $0 < x < 1$ define $x^z := \inf\{x^r : r \leq z, r \in \mathbb{Q}\}$. Prove the two assertions of part b) for all real z.*

[7] In §5.4 we will define exponential and the logarithm and define $x^z := \exp(z \ln x)$. We will then have sufficient machinery to make proofs of these assertions far easier. At this point, however, we do not yet have these tools.

1.3 Absolute value and bounded functions

A concept we will encounter over and over is the concept of *absolute value*. You want to think of the absolute value as the "size" of a real number. Let us give a formal definition.

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Let us give the main features of the absolute value as a proposition.

Proposition 1.21.

(i) $|x| \geq 0$, *moreover,* $|x| = 0$ *if and only if* $x = 0$.

(ii) $|-x| = |x|$ *for all* $x \in \mathbb{R}$.

(iii) $|xy| = |x|\,|y|$ *for all* $x, y \in \mathbb{R}$.

(iv) $|x|^2 = x^2$ *for all* $x \in \mathbb{R}$.

(v) $|x| \leq y$ *if and only if* $-y \leq x \leq y$.

(vi) $-|x| \leq x \leq |x|$ *for all* $x \in \mathbb{R}$.

Proof. i: First suppose $x \geq 0$. Then $|x| = x \geq 0$. Also, $|x| = x = 0$ if and only if $x = 0$. On the other hand, if $x < 0$, then $|x| = -x > 0$, and $|x|$ is never zero.

ii: If $x > 0$, then $-x < 0$ and so $|-x| = -(-x) = x = |x|$. Similarly when $x < 0$, or $x = 0$.

iii: If x or y is zero, then the result is immediate. When x and y are both positive, then $|x|\,|y| = xy$. xy is also positive and hence $xy = |xy|$. If x and y are both negative, then xy is still positive and $xy = |xy|$, and $|x|\,|y| = (-x)(-y) = xy$. Next assume $x > 0$ and $y < 0$. Then $|x|\,|y| = x(-y) = -(xy)$. Now xy is negative and hence $|xy| = -(xy)$. Similarly if $x < 0$ and $y > 0$.

iv: Immediate if $x \geq 0$. If $x < 0$, then $|x|^2 = (-x)^2 = x^2$.

v: Suppose $|x| \leq y$. If $x \geq 0$, then $x \leq y$. It follows that $y \geq 0$, leading to $-y \leq 0 \leq x$. So $-y \leq x \leq y$ holds. If $x < 0$, then $|x| \leq y$ means $-x \leq y$. Negating both sides we get $x \geq -y$. Again $y \geq 0$ and so $y \geq 0 > x$. Hence, $-y \leq x \leq y$.

On the other hand, suppose $-y \leq x \leq y$ is true. If $x \geq 0$, then $x \leq y$ is equivalent to $|x| \leq y$. If $x < 0$, then $-y \leq x$ implies $(-x) \leq y$, which is equivalent to $|x| \leq y$.

vi: Apply v with $y = |x|$. □

A property used frequently enough to give it a name is the so-called *triangle inequality*.

Proposition 1.22 (Triangle Inequality)**.** $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof. Proposition 1.21 gives $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Add these two inequalities to obtain

$$-\bigl(|x| + |y|\bigr) \leq x + y \leq |x| + |y|.$$

Apply Proposition 1.21 again to find $|x + y| \leq |x| + |y|$. □

There are other often applied versions of the triangle inequality.

Corollary 1.23. *Let $x, y \in \mathbb{R}$.*

(i) (reverse triangle inequality) $\big|(|x| - |y|)\big| \leq |x - y|$.

(ii) $|x - y| \leq |x| + |y|$.

Proof. Let us plug in $x = a - b$ and $y = b$ into the standard triangle inequality to obtain

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

or $|a| - |b| \leq |a - b|$. Switching the roles of a and b we find $|b| - |a| \leq |b - a| = |a - b|$. Applying Proposition 1.21, we obtain the reverse triangle inequality.

The second version of the triangle inequality is obtained from the standard one by just replacing y with $-y$, and noting $|-y| = |y|$. $\square$

Corollary 1.24. *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof. We proceed by induction. The conclusion holds trivially for $n = 1$, and for $n = 2$ it is the standard triangle inequality. Suppose the corollary holds for n. Take $n + 1$ numbers $x_1, x_2, \ldots, x_{n+1}$ and first use the standard triangle inequality, then the induction hypothesis

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|. \end{aligned}$$

$\square$

Let us see an example of the use of the triangle inequality.

Example 1.25: Find a number M such that $|x^2 - 9x + 1| \leq M$ for all $-1 \leq x \leq 5$.

Using the triangle inequality, write

$$|x^2 - 9x + 1| \leq |x^2| + |9x| + |1| = |x|^2 + 9|x| + 1.$$

The expression $|x|^2 + 9|x| + 1$ is largest when $|x|$ is largest (why?). In the interval provided, $|x|$ is largest when $x = 5$ and so $|x| = 5$. One possibility for M is

$$M = 5^2 + 9(5) + 1 = 71.$$

There are, of course, other M that work. The bound of 71 is much higher than it need be, but we didn't ask for the best possible M, just one that works.

The last example leads us to the concept of bounded functions.

Definition 1.26. Suppose $f\colon D \to \mathbb{R}$ is a function. We say f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for all $x \in D$.

In the example, we proved $x^2 - 9x + 1$ is bounded when considered as a function on $D = \{x : -1 \leq x \leq 5\}$. On the other hand, if we consider the same polynomial as a function on the whole real line $\mathbb{R}$, then it is not bounded.

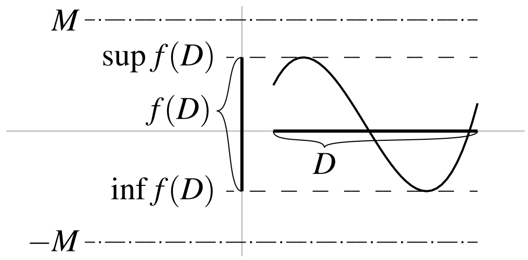

Figure 1.3: Example of a bounded function, a bound M, and its supremum and infimum.

For a function $f\colon D \to \mathbb{R}$, we write (see Figure 1.3 for an example)

$$\sup_{x\in D} f(x) := \sup\, f(D),$$
$$\inf_{x\in D} f(x) := \inf\, f(D).$$

We also sometimes replace the "$x \in D$" with an expression. For example if, as before, $f(x) = x^2 - 9x + 1$, for $-1 \leq x \leq 5$, a little bit of calculus shows

$$\sup_{x\in D} f(x) = \sup_{-1\leq x\leq 5} (x^2 - 9x + 1) = 11, \qquad \inf_{x\in D} f(x) = \inf_{-1\leq x\leq 5} (x^2 - 9x + 1) = \frac{-77}{4}.$$

Proposition 1.27. *If $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ (D nonempty) are bounded*[8] *functions and*

$$f(x) \leq g(x) \qquad \text{for all } x \in D,$$

then

$$\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x) \qquad \text{and} \qquad \inf_{x\in D} f(x) \leq \inf_{x\in D} g(x). \tag{1.1}$$

Be careful with the variables. The x on the left side of the inequality in (1.1) is different from the x on the right. You should really think of, say, the first inequality as

$$\sup_{x\in D} f(x) \leq \sup_{y\in D} g(y).$$

Let us prove this inequality. If b is an upper bound for $g(D)$, then $f(x) \leq g(x) \leq b$ for all $x \in D$, and hence b is also an upper bound for $f(D)$, or $f(x) \leq b$ for all $x \in D$. Take the least upper bound of $g(D)$ to get that for all $x \in D$

$$f(x) \leq \sup_{y\in D} g(y).$$

Therefore, $\sup_{y\in D} g(y)$ is an upper bound for $f(D)$ and thus greater than or equal to the least upper bound of $f(D)$.

$$\sup_{x\in D} f(x) \leq \sup_{y\in D} g(y).$$

The second inequality (the statement about the inf) is left as an exercise (Exercise 1.35).

[8]The boundedness hypothesis is for simplicity, it can be dropped if we allow for the extended real numbers.

A common mistake is to conclude

$$\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y). \tag{1.2}$$

The inequality (1.2) is not true given the hypothesis of the proposition above. For this stronger inequality we need the stronger hypothesis

$$f(x) \leq g(y) \qquad \text{for all } x \in D \text{ and } y \in D.$$

The proof as well as a counterexample is left as an exercise (Exercise 1.36).

1.3.1 Exercises

***Exercise* 1.32:** *Show that* $|x - y| < \epsilon$ *if and only if* $x - \epsilon < y < x + \epsilon$.

***Exercise* 1.33:** *Show:* *a)* $\max\{x, y\} = \frac{x+y+|x-y|}{2}$ *b)* $\min\{x, y\} = \frac{x+y-|x-y|}{2}$

***Exercise* 1.34:** *Find a number* M *such that* $|x^3 - x^2 + 8x| \leq M$ *for all* $-2 \leq x \leq 10$.

***Exercise* 1.35:** *Finish the proof of Proposition 1.27. That is, prove that given a set* D*, and two bounded functions* $f \colon D \to \mathbb{R}$ *and* $g \colon D \to \mathbb{R}$ *such that* $f(x) \leq g(x)$ *for all* $x \in D$*, then*

$$\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

***Exercise* 1.36:** *Let* $f \colon D \to \mathbb{R}$ *and* $g \colon D \to \mathbb{R}$ *be functions (*D *nonempty).*

a) Suppose $f(x) \leq g(y)$ *for all* $x \in D$ *and* $y \in D$*. Show that*

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

b) Find a specific D*,* f*, and* g*, such that* $f(x) \leq g(x)$ *for all* $x \in D$*, but*

$$\sup_{x \in D} f(x) > \inf_{x \in D} g(x).$$

***Exercise* 1.37:** *Prove Proposition 1.27 without the assumption that the functions are bounded. Hint: You need to use the extended real numbers.*

***Exercise* 1.38:** *Let* D *be a nonempty set. Suppose* $f \colon D \to \mathbb{R}$ *and* $g \colon D \to \mathbb{R}$ *are bounded functions.*

a) Show

$$\sup_{x \in D} \bigl(f(x) + g(x)\bigr) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x) \qquad \text{and} \qquad \inf_{x \in D} \bigl(f(x) + g(x)\bigr) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x).$$

b) Find examples where we obtain strict inequalities.

***Exercise* 1.39:** *Suppose* $f \colon D \to \mathbb{R}$ *and* $g \colon D \to \mathbb{R}$ *are bounded functions and* $\alpha \in \mathbb{R}$.

a) Show that $\alpha f \colon D \to \mathbb{R}$ *defined by* $(\alpha f)(x) := \alpha f(x)$ *is a bounded function.*

b) Show that $f + g \colon D \to \mathbb{R}$ *defined by* $(f + g)(x) := f(x) + g(x)$ *is a bounded function.*

***Exercise* 1.40:** *Let* $f \colon D \to \mathbb{R}$ *and* $g \colon D \to \mathbb{R}$ *be functions,* $\alpha \in \mathbb{R}$*, and recall what* $f + g$ *and* αf *means from the previous exercise.*

a) Prove that if $f + g$ *and* g *are bounded, then* f *is bounded.*

b) Find an example where f *and* g *are both unbounded, but* $f + g$ *is bounded.*

c) Prove that if f *is bounded but* g *is unbounded, then* $f + g$ *is unbounded.*

d) Find an example where f *is unbounded but* αf *is bounded.*

1.4 Intervals and the size of $\mathbb{R}$

You surely saw the notation for intervals before, but let us give a formal definition here. For $a, b \in \mathbb{R}$ such that $a < b$ we define

$$\begin{aligned}
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, \\
(a,b) &:= \{x \in \mathbb{R} : a < x < b\}, \\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\}, \\
[a,b) &:= \{x \in \mathbb{R} : a \leq x < b\}.
\end{aligned}$$

The interval $[a, b]$ is called a *closed interval* and (a, b) is called an *open interval*. The intervals of the form $(a, b]$ and $[a, b)$ are called *half-open intervals*.

The intervals above were all *bounded intervals*, since both a and b were real numbers. We define *unbounded intervals*,

$$\begin{aligned}
[a,\infty) &:= \{x \in \mathbb{R} : a \leq x\}, \\
(a,\infty) &:= \{x \in \mathbb{R} : a < x\}, \\
(-\infty,b] &:= \{x \in \mathbb{R} : x \leq b\}, \\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\}.
\end{aligned}$$

For completeness, we define $(-\infty, \infty) := \mathbb{R}$. The intervals $[a, \infty)$, $(-\infty, b]$, and $\mathbb{R}$ are sometimes called *unbounded closed intervals*, and (a, ∞), $(-\infty, b)$, and $\mathbb{R}$ are sometimes called *unbounded open intervals*.

The proof of the following proposition is left as an exercise. In short, an interval is a set with at least two points that contains all points between any two points.[9]

Proposition 1.28. *A set $I \subset \mathbb{R}$ is an interval if and only if I contains at least 2 points and for all $a, c \in I$ and $b \in \mathbb{R}$ such that $a < b < c$, we have $b \in I$.*

We have already seen that every open interval (a, b) (where $a < b$ of course) must be nonempty. For example, it contains the number $\frac{a+b}{2}$. An unexpected fact is that from a set-theoretic perspective, all intervals have the same "size," that is, they all have the same cardinality. For example the map $f(x) := 2x$ takes the interval $[0, 1]$ bijectively to the interval $[0, 2]$.

Maybe more interestingly, the function $f(x) := \tan(x)$ is a bijective map from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. Hence the bounded interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ has the same cardinality as $\mathbb{R}$. It is not completely straightforward to construct a bijective map from $[0, 1]$ to $(0, 1)$, but it is possible.

And do not worry, there does exist a way to measure the "size" of subsets of real numbers that "sees" the difference between $[0, 1]$ and $[0, 2]$. However, its proper definition requires much more machinery than we have right now.

Let us say more about the cardinality of intervals and hence about the cardinality of $\mathbb{R}$. We have seen that there exist irrational numbers, that is $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. The question is: How many irrational numbers are there? It turns out there are a lot more irrational numbers than rational numbers. We have seen that $\mathbb{Q}$ is countable, and we will show that $\mathbb{R}$ is uncountable. In fact, the cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathcal{P}(\mathbb{N})$, although we will not prove this claim here.

[9] Sometimes single point sets and the empty set are also called intervals, but in this book, intervals have at least 2 points. That is, we only defined the bounded intervals if $a < b$.

Theorem 1.29 (Cantor)**.** $\mathbb{R}$ *is uncountable.*

We give a version of Cantor's original proof from 1874 as this proof requires the least setup. Normally this proof is stated as a contradiction, but a proof by contrapositive is easier to understand.

Proof. Let $X \subset \mathbb{R}$ be a countably infinite subset such that for every pair of real numbers $a < b$, there is an $x \in X$ such that $a < x < b$. Were $\mathbb{R}$ countable, we could take $X = \mathbb{R}$. We will show that X is necessarily a proper subset, and so X cannot equal $\mathbb{R}$, and $\mathbb{R}$ must be uncountable.

As X is countably infinite, there is a bijection from $\mathbb{N}$ to X. We write X as a sequence of real numbers $x_1, x_2, x_3, \ldots$, such that each number in X is given by x_n for some $n \in \mathbb{N}$.

We inductively construct two sequences of real numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$. Let $a_1 := x_1$ and $b_1 := x_1 + 1$. Note that $a_1 < b_1$ and $x_1 \notin (a_1, b_1)$. For some $k > 1$, suppose a_j and b_j have been defined for $j = 1, 2, \ldots, k-1$, suppose the open interval (a_j, b_j) does not contain x_ℓ for $\ell = 1, 2, \ldots, j$, and suppose $a_1 < a_2 < \cdots < a_{k-1} < b_{k-1} < \cdots < b_2 < b_1$.

(i) Define $a_k := x_n$, where n is the smallest $n \in \mathbb{N}$ such that $x_n \in (a_{k-1}, b_{k-1})$. Such an x_n exists by our assumption on X, and $n \geq k$ by the assumption on (a_{k-1}, b_{k-1}).

(ii) Next, define b_k to be some real number in (a_k, b_{k-1}).

Notice that $a_{k-1} < a_k < b_k < b_{k-1}$. Also notice that (a_k, b_k) does not contain x_k and hence does not contain x_j for $j = 1, 2, \ldots, k$. The two sequences are now defined.

Claim: $a_n < b_m$ *for all* n *and* m *in* $\mathbb{N}$. Proof: Let us first assume $n < m$. Then $a_n < a_{n+1} < \cdots < a_{m-1} < a_m < b_m$. Similarly for $n > m$. The claim follows.

Let $A := \{a_n : n \in \mathbb{N}\}$ and $B := \{b_n : n \in \mathbb{N}\}$. By Proposition 1.18 and the claim above,

$$\sup A \leq \inf B.$$

Define $y := \sup A$. The number y cannot be a member of A: If $y = a_n$ for some n, then $y < a_{n+1}$, which is impossible. Similarly, y cannot be a member of B. Therefore, $a_n < y$ for all $n \in \mathbb{N}$ and $y < b_n$ for all $n \in \mathbb{N}$. In other words, for every $n \in \mathbb{N}$, we have $y \in (a_n, b_n)$. By the construction of the sequence, $x_n \notin (a_n, b_n)$, and so $y \neq x_n$. As this was true for all $n \in \mathbb{N}$, we have that $y \notin X$.

We have constructed a real number y that is not in X, and thus X is a proper subset of $\mathbb{R}$. The sequence $x_1, x_2, \ldots$ cannot contain all elements of $\mathbb{R}$ and thus $\mathbb{R}$ is uncountable. □

1.4.1 Exercises

Exercise* 1.41*: *For $a < b$, construct an explicit bijection from $(a, b]$ to $(0, 1]$.*

Exercise* 1.42*: *Suppose $f \colon [0, 1] \to (0, 1)$ is a bijection. Using f, construct a bijection from $[-1, 1]$ to $\mathbb{R}$.*

Exercise* 1.43*: *Prove Proposition 1.28. That is, suppose $I \subset \mathbb{R}$ is a subset with at least 2 elements such that if $a < b < c$ and $a, c \in I$, then $b \in I$. Prove that I is one of the nine types of intervals explicitly given in this section. Furthermore, prove that the intervals given in this section all satisfy this property.*

***Exercise* 1.44** (Hard)*: Construct an explicit bijection from $(0, 1]$ to $(0, 1)$. Hint: One approach is as follows: First map $(\frac{1}{2}, 1]$ to $(0, \frac{1}{2}]$, then map $(\frac{1}{4}, \frac{1}{2}]$ to $(\frac{1}{2}, \frac{3}{4}]$, etc. Write down the map explicitly, that is, write down an algorithm that tells you exactly what number goes where. Then prove that the map is a bijection.*

***Exercise* 1.45** (Hard)*: Construct an explicit bijection from $[0, 1]$ to $(0, 1)$.*

Exercise* 1.46:*

a) Show that every closed interval $[a, b]$ is the intersection of countably many open intervals.

b) Show that every open interval (a, b) is a countable union of closed intervals.

c) Show that an intersection of a possibly infinite family of bounded closed intervals, $\bigcap\limits_{\lambda \in I} [a_\lambda, b_\lambda]$, is either empty, a single point, or a bounded closed interval.

Exercise* 1.47: Suppose S is a set of disjoint open intervals in $\mathbb{R}$. That is, if $(a, b) \in S$ and $(c, d) \in S$, then either $(a, b) = (c, d)$ or $(a, b) \cap (c, d) = \emptyset$. Prove S is a countable set.*

Exercise* 1.48: Prove that the cardinality of $[0, 1]$ is the same as the cardinality of $(0, 1)$ by showing that $|[0, 1]| \leq |(0, 1)|$ and $|(0, 1)| \leq |[0, 1]|$. This proof requires the Cantor–Bernstein–Schröder theorem we stated without proof. Note that this proof does not give you an explicit bijection.*

***Exercise* 1.49** (Challenging)*: A number x is* algebraic *if x is a root of a polynomial with integer coefficients, in other words, $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ where all $a_n \in \mathbb{Z}$.*

a) Show that there are only countably many algebraic numbers.

b) Show that there exist non-algebraic (transcendental) numbers (follow in the footsteps of Cantor, use the uncountability of $\mathbb{R}$).

Hint: Feel free to use the fact that a polynomial of degree n has at most n real roots.

***Exercise* 1.50** (Challenging)*: Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Prove $|\mathbb{R}| < |F|$ using Cantor's Theorem* **??**.[10]

[10]Interestingly, if C is the set of continuous functions, then $|\mathbb{R}| = |C|$.

1.5 Decimal representation of the reals

We often think of real numbers as their *decimal representation*. For a positive integer n, we find the digits $d_K, d_{K-1}, \ldots, d_2, d_1, d_0$ for some K, where each d_j is an integer between 0 and 9, then

$$n = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_2 10^2 + d_1 10 + d_0.$$

We often assume $d_K \neq 0$. To represent n we write the sequence of digits: $n = d_K d_{K-1} \cdots d_2 d_1 d_0$. By a (decimal) *digit*, we mean an integer between 0 and 9.

Similarly, we represent some rational numbers. That is, for certain numbers x, we can find negative integer $-M$, a positive integer K, and digits $d_K, d_{K-1}, \ldots, d_1, d_0, d_{-1}, \ldots, d_{-M}$, such that

$$x = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_2 10^2 + d_1 10 + d_0 + d_{-1} 10^{-1} + d_{-2} 10^{-2} + \cdots + d_{-M} 10^{-M}.$$

We write $x = d_K d_{K-1} \cdots d_1 d_0 \,.\, d_{-1} d_{-2} \cdots d_{-M}$.

Not every real number has such a representation, even the simple rational number $\frac{1}{3}$ does not. The irrational number $\sqrt{2}$ does not have such a representation either. To get a representation for all real numbers, we must allow infinitely many digits.

Let us consider only real numbers in the interval $(0,1]$. If we find a representation for these, adding integers to them obtains a representation for all real numbers. Take an infinite sequence of decimal digits:

$$0.d_1 d_2 d_3 \ldots .$$

That is, we have a digit d_j for every $j \in \mathbb{N}$. We renumbered the digits to avoid the negative signs. We call the number

$$D_n := \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n}.$$

the truncation of x to n decimal digits. We say this sequence of digits represents a real number x if

$$x = \sup_{n \in \mathbb{N}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} \right) = \sup_{n \in \mathbb{N}} D_n.$$

Proposition 1.30.

(i) Every infinite sequence of digits $0.d_1 d_2 d_3 \ldots$ represents a unique real number $x \in [0,1]$, and

$$D_n \leq x \leq D_n + \frac{1}{10^n} \qquad \text{for all } n \in \mathbb{N}.$$

(ii) For every $x \in (0,1]$ there exists an infinite sequence of digits $0.d_1 d_2 d_3 \ldots$ that represents x. There exists a unique representation such that

$$D_n < x \leq D_n + \frac{1}{10^n} \qquad \text{for all } n \in \mathbb{N}.$$

Proof. We start with the first item. Take an arbitrary infinite sequence of digits $0.d_1d_2d_3\ldots$. Use the geometric sum formula to write

$$\begin{aligned} D_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} &\leq \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n} \\ &= \frac{9}{10}\big(1 + \frac{1}{10} + (\frac{1}{10})^2 + \cdots + (\frac{1}{10})^{n-1}\big) \\ &= \frac{9}{10}\left(\frac{1-(\frac{1}{10})^n}{1-\frac{1}{10}}\right) = 1 - (\frac{1}{10})^n < 1. \end{aligned}$$

In particular, $D_n < 1$ for all n. A sum of nonnegative numbers is nonnegative so $D_n \geq 0$, and hence

$$0 \leq \sup_{n\in\mathbb{N}} D_n \leq 1.$$

Therefore, $0.d_1d_2d_3\ldots$ represents a unique number $x := \sup_{n\in\mathbb{N}} D_n \in [0,1]$. As x is a supremum, then $D_n \leq x$. Take $m \in \mathbb{N}$. If $m < n$, then $D_m - D_n \leq 0$. If $m > n$, then computing as above

$$D_m - D_n = \frac{d_{n+1}}{10^{n+1}} + \frac{d_{n+2}}{10^{n+2}} + \frac{d_{n+3}}{10^{n+3}} + \cdots + \frac{d_m}{10^m} \leq \frac{1}{10^n}\big(1 - (\frac{1}{10})^{m-n}\big) < \frac{1}{10^n}.$$

Take the supremum over m to find

$$x - D_n \leq \frac{1}{10^n}.$$

We move on to the second item. Take any $x \in (0,1]$. First let us tackle the existence. For convenience, let $D_0 := 0$. Then, $D_0 < x \leq D_0 + 10^{-0}$. Suppose we defined the digits $d_1, d_2, \ldots, d_n$, and that $D_k < x \leq D_k + 10^{-k}$, for $k = 0, 1, 2, \ldots, n$. We need to define d_{n+1}.

By the Archimedean property of the real numbers, find an integer j such that $x - D_n \leq j10^{-(n+1)}$. Take the least such j and obtain

$$(j-1)10^{-(n+1)} < x - D_n \leq j10^{-(n+1)}. \tag{1.3}$$

Let $d_{n+1} := j - 1$. As $D_n < x$, then $d_{n+1} = j - 1 \geq 0$. On the other hand, since $x - D_n \leq 10^{-n}$, we have that j is at most 10, and therefore $d_{n+1} \leq 9$. So d_{n+1} is a decimal digit. Since $D_{n+1} = D_n + d_{n+1}10^{-(n+1)}$ add D_n to the inequality (1.3) above:

$$\begin{aligned} D_{n+1} = D_n + (j-1)10^{-(n+1)} < x &\leq D_n + j10^{-(n+1)} \\ &= D_n + (j-1)10^{-(n+1)} + 10^{-(n+1)} = D_{n+1} + 10^{-(n+1)}. \end{aligned}$$

And so $D_{n+1} < x \leq D_{n+1} + 10^{-(n+1)}$ holds. We inductively defined an infinite sequence of digits $0.d_1d_2d_3\ldots$.

Consider $D_n < x \leq D_n + 10^{-n}$. As $D_n < x$ for all n, then $\sup\{D_n : n \in \mathbb{N}\} \leq x$. The second inequality for D_n implies

$$x - \sup\{D_m : m \in \mathbb{N}\} \leq x - D_n \leq 10^{-n}.$$

As the inequality holds for all n and 10^{-n} can be made arbitrarily small (see Exercise 1.58), we have $x \leq \sup\{D_m : m \in \mathbb{N}\}$. Therefore, $\sup\{D_m : m \in \mathbb{N}\} = x$.

What is left to show is the uniqueness. Suppose $0.e_1e_2e_3\ldots$ is another representation of x. Let E_n be the n-digit truncation of $0.e_1e_2e_3\ldots$, and suppose $E_n < x \leq E_n + 10^{-n}$ for all $n \in \mathbb{N}$. Suppose for some $K \in \mathbb{N}$, $e_n = d_n$ for all $n < K$, so $D_{K-1} = E_{K-1}$. Then

$$E_K = D_{K-1} + e_K 10^{-K} < x \leq E_K + 10^{-K} = D_{K-1} + e_K 10^{-K} + 10^{-K}.$$

Subtracting D_{K-1} and multiplying by 10^K we get

$$e_K < (x - D_{K-1})10^K \leq e_K + 1.$$

Similarly,

$$d_K < (x - D_{K-1})10^K \leq d_K + 1.$$

Hence, both e_K and d_K are the largest integer j such that $j < (x - D_{K-1})10^K$, and therefore $e_K = d_K$. That is, the representation is unique. □

The representation is not unique if we do not require $D_n < x$ for all n. For example, for the number $\frac{1}{2}$, the method in the proof obtains the representation

$$0.49999\ldots.$$

However, $\frac{1}{2}$ also has the representation $0.50000\ldots$.

The only numbers that have nonunique representations are ones that end either in an infinite sequence of 0s or 9s, because the only representation for which $D_n = x$ is one where all digits past the nth digit are zero. In this case there are exactly two representations of x (see the exercises).

Let us give another proof of the uncountability of the reals using decimal representations. This is Cantor's second proof, and is probably better known. This proof may seem shorter, but it is because we already did the hard part above and we are left with a slick trick to prove that $\mathbb{R}$ is uncountable. This trick is called *Cantor diagonalization* and finds use in other proofs as well.

Theorem 1.31 (Cantor)**.** *The set* $(0, 1]$ *is uncountable.*

Proof. Let $X := \{x_1, x_2, x_3, \ldots\}$ be any countable subset of real numbers in $(0, 1]$. We will construct a real number not in X. Let

$$x_n = 0.d_1^n d_2^n d_3^n \ldots$$

be the unique representation from the proposition, that is, d_j^n is the jth digit of the nth number. Let

$$e_n := \begin{cases} 1 & \text{if } d_n^n \neq 1, \\ 2 & \text{if } d_n^n = 1. \end{cases}$$

Let E_n be the n-digit truncation of $y = 0.e_1e_2e_3\ldots$. Because all the digits are nonzero we get $E_n < E_{n+1} \leq y$. Therefore

$$E_n < y \leq E_n + 10^{-n}$$

for all n, and the representation is the unique one for y from the proposition. For every n, the nth digit of y is different from the nth digit of x_n, so $y \neq x_n$. Therefore $y \notin X$, and as X was an arbitrary countable subset, $(0, 1]$ must be uncountable. See Figure 1.4 for an example. □

$$\begin{array}{rlcccccc} x_1 = & 0. & \boxed{1} & 3 & 2 & 1 & 0 & \cdots \\ x_2 = & 0. & 7 & \boxed{9} & 4 & 1 & 3 & \cdots \\ x_3 = & 0. & 3 & 0 & \boxed{1} & 3 & 4 & \cdots \\ x_4 = & 0. & 8 & 9 & 2 & \boxed{5} & 6 & \cdots \\ x_5 = & 0. & 1 & 6 & 0 & 2 & \boxed{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Number not in the list:
$y = 0.21211\ldots$

Figure 1.4: Example of Cantor diagonalization, the diagonal digits d_n^n marked.

Using decimal digits we can also find lots of numbers that are not rational. The following proposition is true for every rational number, but we give it only for $x \in (0, 1]$ for simplicity.

Proposition 1.32. *If $x \in (0, 1]$ is a rational number and $x = 0.d_1d_2d_3\ldots$, then the decimal digits eventually start repeating. That is, there are positive integers N and P, such that for all $n \geq N$, $d_n = d_{n+P}$.*

Proof. Suppose $x = \frac{p}{q}$ for positive integers p and q. Suppose also that x is a number with a unique representation, as otherwise we have seen above that both its representations are repeating, see also Exercise 1.53. This also means that $x \neq 1$ so $p < q$.

To compute the first digit we take $10p$ and divide by q. Let d_1 be the quotient, and the remainder r_1 is some integer between 0 and $q - 1$. That is, d_1 is the largest integer such that $d_1q \leq 10p$ and then $r_1 = 10p - d_1q$. As $p < q$, then $d_1 < 10$, so d_1 is a digit. Furthermore,

$$\frac{d_1}{10} \leq \frac{p}{q} = \frac{d_1}{10} + \frac{r_1}{10q} \leq \frac{d_1}{10} + \frac{1}{10}.$$

The first inequality must be strict since x has a unique representation. That is, d_1 really is the first digit. What is left is $\frac{r_1}{(10q)}$. This is the same as computing the first digit of $\frac{r_1}{q}$. To compute d_2 divide $10r_1$ by q, and so on. After computing $n - 1$ digits, we have $\frac{p}{q} = D_{n-1} + \frac{r_{n-1}}{(10^n q)}$. To get the nth digit, divide $10r_{n-1}$ by q to get quotient d_n, remainder r_n, and the inequalities

$$\frac{d_n}{10} \leq \frac{r_{n-1}}{q} = \frac{d_n}{10} + \frac{r_n}{10q} \leq \frac{d_n}{10} + \frac{1}{10}.$$

Dividing by 10^{n-1} and adding D_{n-1} we find

$$D_n \leq D_{n-1} + \frac{r_{n-1}}{10^n q} = \frac{p}{q} \leq D_n + \frac{1}{10^n}.$$

By uniqueness we really have the nth digit d_n from the construction.

The new digit depends only the remainder from the previous step. There are at most q possible remainders and hence at some step the process must start repeating itself, and P is at most q. $\square$

The converse of the proposition is also true and is left as an exercise.

Example 1.33: The number

$$x = 0.101001000100001000001\ldots,$$

is irrational. That is, the digits are n zeros, then a one, then $n+1$ zeros, then a one, and so on and so forth. The fact that x is irrational follows from the proposition; the digits never start repeating. For every P, if we go far enough, we find a 1 followed by at least $P+1$ zeros.

1.5.1 Exercises

Exercise* 1.51** (Easy):*** *What is the decimal representation of* 1 *guaranteed by Proposition 1.30? Make sure to show that it does satisfy the condition.*

Exercise* 1.52*: *Prove the converse of Proposition 1.32, that is, if the digits in the decimal representation of x are eventually repeating, then x must be rational.*

Exercise* 1.53*: *Show that real numbers $x \in (0,1)$ with nonunique decimal representation are exactly the rational numbers that can be written as $\frac{m}{10^n}$ for some integers m and n. In this case show that there exist exactly two representations of x.*

Exercise* 1.54*: *Let $b \geq 2$ be an integer. Define a representation of a real number in $[0,1]$ in terms of base b rather than base 10 and prove Proposition 1.30 for base b.*

Exercise* 1.55*: *Using the previous exercise with $b = 2$ (binary), show that cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathcal{P}(\mathbb{N})$, obtaining yet another (though related) proof that $\mathbb{R}$ is uncountable. Hint: Construct two injections, one from $[0,1]$ to $\mathcal{P}(\mathbb{N})$ and one from $\mathcal{P}(\mathbb{N})$ to $[0,1]$. Hint 2: Given a set $A \subset \mathbb{N}$, let the nth binary digit of x be 1 if $n \in A$.*

Exercise* 1.56** (Challenging):*** *Explicitly construct an injection from $[0,1] \times [0,1]$ to $[0,1]$ (think about why this is so surprising[11]). Then describe the set of numbers in $[0,1]$ not in the image of your injection (unless, of course, you managed to construct a bijection). Hint: Consider even and odd digits of the decimal expansion.*

Exercise* 1.57*: *Prove that if $x = \frac{p}{q} \in (0,1]$ is a rational number, $q > 1$, then the period P of repeating digits in the decimal representation of x is in fact less than or equal to $q-1$.*

Exercise* 1.58*: *Prove that if $b \in \mathbb{N}$ and $b \geq 2$, then for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $b^{-n} < \epsilon$. Hint: One possibility is to first prove that $b^n > n$ for all $n \in \mathbb{N}$ by induction.*

Exercise* 1.59*: *Explicitly construct an injection $f \colon \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ using Proposition 1.32.*

[11] With quite a bit more work (or by applying the Cantor–Bernstein–Schröder theorem) one can prove that there is a bijection. When he proved this result, Cantor apparently wrote “I see it but I don’t believe it.”