Chapter 5

The Riemann Integral

5.1 The Riemann integral

An integral is a way to "sum" the values of a function. There is often confusion among students of calculus between *integral* and *antiderivative*. The integral is (informally) the area under the curve, nothing else. That we can compute an antiderivative using the integral is a nontrivial result we have to prove. In this chapter we define the *Riemann integral*[1] using the Darboux integral[2], which is technically simpler than (but equivalent to) the traditional definition of Riemann.

5.1.1 Partitions and lower and upper integrals

We want to integrate a bounded function defined on an interval $[a, b]$. We first define two auxiliary integrals that are defined for all bounded functions. Only then can we talk about the Riemann integral and the Riemann integrable functions.

Definition 5.1. A *partition* P of the interval $[a, b]$ is a finite set of numbers $\{x_0, x_1, x_2, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

We write

$$\Delta x_i := x_i - x_{i-1}.$$

Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Let P be a partition of $[a, b]$. Define

$$\begin{aligned} m_i &:= \inf \big\{ f(x) : x_{i-1} \le x \le x_i \big\}, \\ M_i &:= \sup \big\{ f(x) : x_{i-1} \le x \le x_i \big\}, \\ L(P, f) &:= \sum_{i=1}^{n} m_i \Delta x_i, \\ U(P, f) &:= \sum_{i=1}^{n} M_i \Delta x_i. \end{aligned}$$

We call $L(P, f)$ the *lower Darboux sum* and $U(P, f)$ the *upper Darboux sum*.

[1]Named after the German mathematician Georg Friedrich Bernhard Riemann (1826–1866).
[2]Named after the French mathematician Jean-Gaston Darboux (1842–1917).

The geometric idea of Darboux sums is indicated in Figure 5.1. The lower sum is the area of the shaded rectangles, and the upper sum is the area of the entire rectangles, shaded plus unshaded parts. The width of the ith rectangle is Δx_i, the height of the shaded rectangle is m_i, and the height of the entire rectangle is M_i.

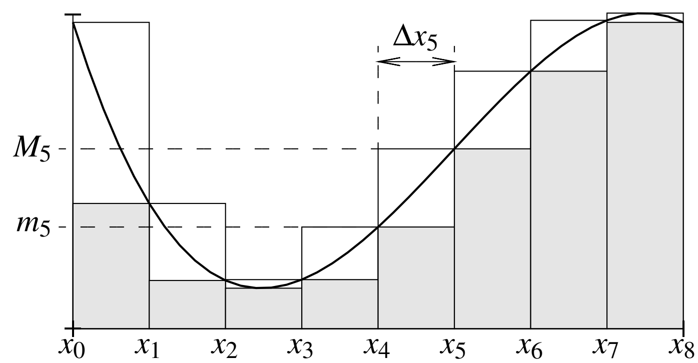

Figure 5.1: Sample Darboux sums.

Proposition 5.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a,b]$, we have $m \leq f(x) \leq M$. Then for every partition P of $[a,b]$,*

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a). \tag{5.1}$$

Proof. Let P be a partition. Note that $m \leq m_i$ for all i and $M_i \leq M$ for all i. Also $m_i \leq M_i$ for all i. Finally, $\sum_{i=1}^n \Delta x_i = (b-a)$. Therefore,

$$m(b-a) = m\left(\sum_{i=1}^n \Delta x_i\right) = \sum_{i=1}^n m \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i \leq$$
$$\leq \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M \Delta x_i = M\left(\sum_{i=1}^n \Delta x_i\right) = M(b-a).$$

Hence we get (5.1). In particular, the set of lower and upper sums are bounded sets. $\square$

Definition 5.3. As the sets of lower and upper Darboux sums are bounded, we define

$$\underline{\int_a^b} f(x)\ dx := \sup\bigl\{L(P,f) : P \text{ a partition of } [a,b]\bigr\},$$
$$\overline{\int_a^b} f(x)\ dx := \inf\bigl\{U(P,f) : P \text{ a partition of } [a,b]\bigr\}.$$

We call $\underline{\int}$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral*. To avoid worrying about the variable of integration, we often simply write

$$\underline{\int_a^b} f := \underline{\int_a^b} f(x)\ dx \qquad \text{and} \qquad \overline{\int_a^b} f := \overline{\int_a^b} f(x)\ dx.$$

If integration is to make sense, then the lower and upper Darboux integrals should be the same number, as we want a single number to call *the integral.* However, these two integrals may differ for some functions.

Example 5.4: Take the Dirichlet function $f\colon [0,1] \to \mathbb{R}$, where $f(x) := 1$ if $x \in \mathbb{Q}$ and $f(x) := 0$ if $x \notin \mathbb{Q}$. Then

$$\underline{\int_0^1} f = 0 \qquad \text{and} \qquad \overline{\int_0^1} f = 1.$$

The reason is that for every i, we have $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1$. Thus

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0, \\ U(P,f) &= \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1. \end{aligned}$$

Remark 5.5. The same definition of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ is used when f is defined on a larger set S such that $[a,b] \subset S$. In that case, we use the restriction of f to $[a,b]$ and we must ensure that the restriction is bounded on $[a,b]$.

To compute the integral, we often take a partition P and make it finer. That is, we cut intervals in the partition into yet smaller pieces.

Definition 5.6. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $\widetilde{P} = \{\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_\ell\}$ be partitions of $[a,b]$. We say $\widetilde{P}$ is a *refinement* of P if as sets $P \subset \widetilde{P}$.

That is, $\widetilde{P}$ is a refinement of a partition if it contains all the numbers in P and perhaps some other numbers in between. For example, $\{0, 0.5, 1, 2\}$ is a partition of $[0,2]$ and $\{0, 0.2, 0.5, 1, 1.5, 1.75, 2\}$ is a refinement. The main reason for introducing refinements is the following proposition.

Proposition 5.7. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function, and let P be a partition of $[a,b]$. Let $\widetilde{P}$ be a refinement of P. Then*

$$L(P,f) \le L(\widetilde{P},f) \qquad \textit{and} \qquad U(\widetilde{P},f) \le U(P,f).$$

Proof. The tricky part of this proof is to get the notation correct. Let $\widetilde{P} = \{\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_\ell\}$ be a refinement of $P = \{x_0, x_1, \ldots, x_n\}$. Then $x_0 = \widetilde{x}_0$ and $x_n = \widetilde{x}_\ell$. In fact, there are integers $k_0 < k_1 < \cdots < k_n$ such that $x_j = \widetilde{x}_{k_j}$ for $j = 0, 1, 2, \ldots, n$.

Let $\Delta \widetilde{x}_p := \widetilde{x}_p - \widetilde{x}_{p-1}$. See Figure 5.2. We get

$$\Delta x_j = x_j - x_{j-1} = \widetilde{x}_{k_j} - \widetilde{x}_{k_{j-1}} = \sum_{p=k_{j-1}+1}^{k_j} \widetilde{x}_p - \widetilde{x}_{p-1} = \sum_{p=k_{j-1}+1}^{k_j} \Delta \widetilde{x}_p.$$

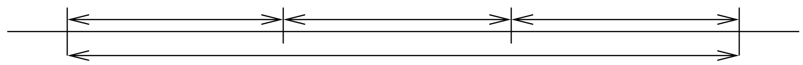

Figure 5.2: Refinement of a subinterval. Notice $\Delta x_j = \Delta\widetilde{x}_{p-2} + \Delta\widetilde{x}_{p-1} + \Delta\widetilde{x}_p$, and also $k_{j-1} + 1 = p - 2$ and $k_j = p$.

Let m_j be as before and correspond to the partition P. Let $\widetilde{m}_j := \inf\{f(x) : \widetilde{x}_{j-1} \leq x \leq \widetilde{x}_j\}$. Now, $m_j \leq \widetilde{m}_p$ for $k_{j-1} < p \leq k_j$. Therefore,

$$m_j \Delta x_j = m_j \sum_{p=k_{j-1}+1}^{k_j} \Delta\widetilde{x}_p = \sum_{p=k_{j-1}+1}^{k_j} m_j \Delta\widetilde{x}_p \leq \sum_{p=k_{j-1}+1}^{k_j} \widetilde{m}_p \Delta\widetilde{x}_p.$$

So

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j \leq \sum_{j=1}^{n} \sum_{p=k_{j-1}+1}^{k_j} \widetilde{m}_p \Delta\widetilde{x}_p = \sum_{j=1}^{\ell} \widetilde{m}_j \Delta\widetilde{x}_j = L(\widetilde{P}, f).$$

The proof of $U(\widetilde{P}, f) \leq U(P,f)$ is left as an exercise. □

Armed with refinements we prove the following. The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

Proposition 5.8. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a,b]$, we have $m \leq f(x) \leq M$. Then*

$$m(b-a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b-a). \tag{5.2}$$

Proof. By Proposition 5.2, for every partition P,

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

The inequality $m(b-a) \leq L(P,f)$ implies $m(b-a) \leq \underline{\int_a^b} f$. The inequality $U(P,f) \leq M(b-a)$ implies $\overline{\int_a^b} f \leq M(b-a)$.

The middle inequality in (5.2) is the main point of this proposition. Let P_1, P_2 be partitions of $[a,b]$. Define $\widetilde{P} := P_1 \cup P_2$. The set $\widetilde{P}$ is a partition of $[a,b]$, which is a refinement of P_1 and a refinement of P_2. By Proposition 5.7, $L(P_1, f) \leq L(\widetilde{P}, f)$ and $U(\widetilde{P}, f) \leq U(P_2, f)$. So

$$L(P_1, f) \leq L(\widetilde{P}, f) \leq U(\widetilde{P}, f) \leq U(P_2, f).$$

In other words, for two arbitrary partitions P_1 and P_2, we have $L(P_1, f) \leq U(P_2, f)$. Recall Proposition 1.18, and take the supremum and infimum over all partitions:

$$\underline{\int_a^b} f = \sup\bigl\{L(P,f) : P \text{ a partition of } [a,b]\bigr\} \leq \inf\bigl\{U(P,f) : P \text{ a partition of } [a,b]\bigr\} = \overline{\int_a^b} f.$$

□

5.1.2 Riemann integral

We can finally define the Riemann integral. However, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Definition 5.9. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function such that

$$\underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x)\ dx.$$

Then f is said to be *Riemann integrable*. The set of Riemann integrable functions on $[a,b]$ is denoted by $\mathcal{R}[a,b]$. When $f \in \mathcal{R}[a,b]$, we define

$$\int_a^b f(x)\ dx := \underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x)\ dx.$$

As before, we often write

$$\int_a^b f := \int_a^b f(x)\ dx.$$

The number $\int_a^b f$ is called the *Riemann integral* of f, or sometimes simply the *integral* of f.

By definition, a Riemann integrable function is bounded. Appealing to Proposition 5.8, we immediately obtain the following proposition. See also Figure 5.3.

Proposition 5.10. *Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Then*

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

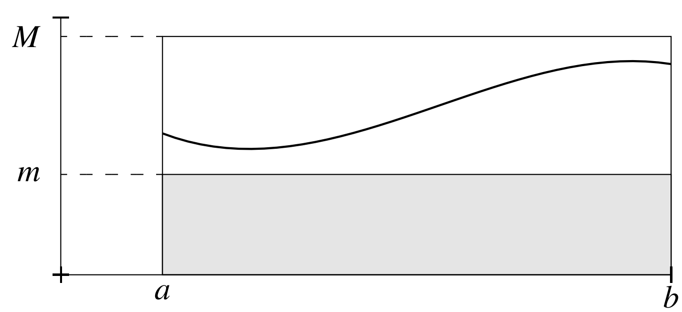

Figure 5.3: The area under the curve is bounded from above by the area of the entire rectangle, $M(b-a)$, and from below by the area of the shaded part, $m(b-a)$.

Often we use a weaker form of this proposition. That is, if $|f(x)| \leq M$ for all $x \in [a,b]$, then

$$\left| \int_a^b f \right| \leq M(b-a).$$

Example 5.11: We integrate constant functions using Proposition 5.8. If $f(x) := c$ for some constant c, then we take $m = M = c$. In inequality (5.2) all the inequalities must be equalities. Thus f is integrable on $[a,b]$ and $\int_a^b f = c(b-a)$.

Example 5.12: Let $f : [0,2] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if } x < 1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

We claim f is Riemann integrable and $\int_0^2 f = 1$.

Proof: Let $0 < \epsilon < 1$ be arbitrary. Let $P := \{0, 1-\epsilon, 1+\epsilon, 2\}$ be a partition. We use the notation from the definition of the Darboux sums. Then

$$\begin{aligned} m_1 &= \inf \bigl\{ f(x) : x \in [0,1-\epsilon] \bigr\} = 1, & M_1 &= \sup \bigl\{ f(x) : x \in [0,1-\epsilon] \bigr\} = 1, \\ m_2 &= \inf \bigl\{ f(x) : x \in [1-\epsilon,1+\epsilon] \bigr\} = 0, & M_2 &= \sup \bigl\{ f(x) : x \in [1-\epsilon,1+\epsilon] \bigr\} = 1, \\ m_3 &= \inf \bigl\{ f(x) : x \in [1+\epsilon,2] \bigr\} = 0, & M_3 &= \sup \bigl\{ f(x) : x \in [1+\epsilon,2] \bigr\} = 0. \end{aligned}$$

Furthermore, $\Delta x_1 = 1-\epsilon$, $\Delta x_2 = 2\epsilon$ and $\Delta x_3 = 1-\epsilon$. See Figure 5.4.

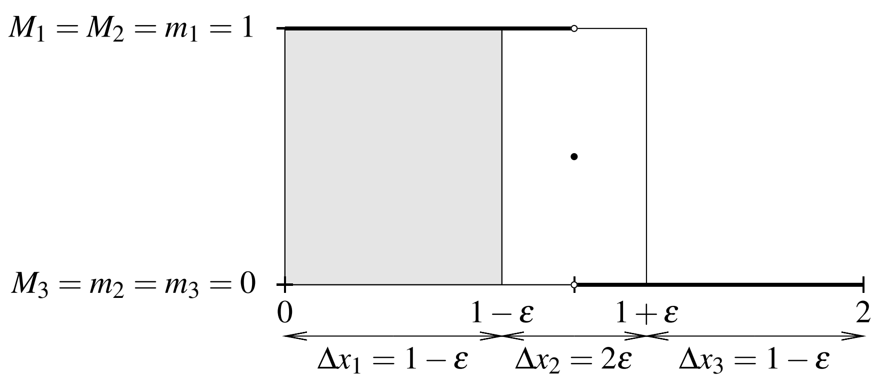

Figure 5.4: Darboux sums for the step function. $L(P,f)$ is the area of the shaded rectangle, $U(P,f)$ is the area of both rectangles, and $U(P,f) - L(P,f)$ is the area of the unshaded rectangle.

We compute

$$L(P,f) = \sum_{i=1}^{3} m_i \Delta x_i = 1 \cdot (1-\epsilon) + 0 \cdot 2\epsilon + 0 \cdot (1-\epsilon) = 1-\epsilon,$$

$$U(P,f) = \sum_{i=1}^{3} M_i \Delta x_i = 1 \cdot (1-\epsilon) + 1 \cdot 2\epsilon + 0 \cdot (1-\epsilon) = 1+\epsilon.$$

Thus,

$$\overline{\int_0^2} f - \underline{\int_0^2} f \le U(P,f) - L(P,f) = (1+\epsilon) - (1-\epsilon) = 2\epsilon.$$

By Proposition 5.8, we have $\underline{\int_0^2} f \leq \overline{\int_0^2} f$. As ϵ was arbitrary, $\overline{\int_0^2} f = \underline{\int_0^2} f$. So f is Riemann integrable. Finally,

$$1 - \epsilon = L(P, f) \leq \int_0^2 f \leq U(P, f) = 1 + \epsilon.$$

Hence, $\left| \int_0^2 f - 1 \right| \leq \epsilon$. As ϵ was arbitrary, we conclude $\int_0^2 f = 1$.

It may be worthwhile to extract part of the technique of the example into a proposition.

Proposition 5.13. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(P, f) - L(P, f) < \epsilon.$$

Proof. If for every $\epsilon > 0$ such a P exists, then

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P, f) - L(P, f) < \epsilon.$$

Therefore, $\overline{\int_a^b} f = \underline{\int_a^b} f$, and f is integrable. □

Example 5.14: Let us show $\frac{1}{1+x}$ is integrable on $[0, b]$ for all $b > 0$. We will see later that continuous functions are integrable, but let us demonstrate how we do it directly.

Let $\epsilon > 0$ be given. Take $n \in \mathbb{N}$ and pick $x_j := \frac{jb}{n}$, to form the partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[0, b]$. We have $\Delta x_j = \frac{b}{n}$ for all j. As f is decreasing, for every subinterval $[x_{j-1}, x_j]$, we obtain

$$m_j = \inf \left\{ \frac{1}{1+x} : x \in [x_{j-1}, x_j] \right\} = \frac{1}{1+x_j}, \qquad M_j = \sup \left\{ \frac{1}{1+x} : x \in [x_{j-1}, x_j] \right\} = \frac{1}{1+x_{j-1}}.$$

Then

$$\begin{aligned} U(P, f) - L(P, f) = \sum_{j=1}^{n} \Delta x_j (M_j - m_j) = \frac{b}{n} \sum_{j=1}^{n} \left(\frac{1}{1 + \frac{(j-1)b}{n}} - \frac{1}{1 + \frac{jb}{n}} \right) = \\ = \frac{b}{n} \left(\frac{1}{1 + \frac{0b}{n}} - \frac{1}{1 + \frac{nb}{n}} \right) = \frac{b^2}{n(b+1)}. \end{aligned}$$

The sum telescopes, the terms successively cancel each other, something we have seen before. Picking n to be such that $\frac{b^2}{n(b+1)} < \epsilon$, the proposition is satisfied, and the function is integrable.

Remark 5.15. A way of thinking of the integral is that it adds up (integrates) lots of local information—it sums $f(x)\,dx$ over all x. The integral sign was chosen by Leibniz to be the long S to mean summation. Unlike derivatives, which are "local," integrals show up in applications when one wants a "global" answer: total distance travelled, average temperature, total charge, etc.

5.1.3 More notation

When $f: S \to \mathbb{R}$ is defined on a larger set S and $[a,b] \subset S$, we say f is Riemann integrable on $[a,b]$ if the restriction of f to $[a,b]$ is Riemann integrable. In this case, we say $f \in \mathcal{R}[a,b]$, and we write $\int_a^b f$ to mean the Riemann integral of the restriction of f to $[a,b]$.

It is useful to define the integral $\int_a^b f$ even if $a \not< b$. Suppose $b < a$ and $f \in \mathcal{R}[b,a]$, then define

$$\int_a^b f := -\int_b^a f.$$

For any function f, define

$$\int_a^a f := 0.$$

At times, the variable x may already have some other meaning. When we need to write down the variable of integration, we may simply use a different letter. For example,

$$\int_a^b f(s)\, ds := \int_a^b f(x)\, dx.$$

5.1.4 Exercises

***Exercise* 5.1:** *Define $f: [0,1] \to \mathbb{R}$ by $f(x) := x^3$ and let $P := \{0, 0.1, 0.4, 1\}$. Compute $L(P,f)$ and $U(P,f)$.*

***Exercise* 5.2:** *Let $f: [0,1] \to \mathbb{R}$ be defined by $f(x) := x$. Show that $f \in \mathcal{R}[0,1]$ and compute $\int_0^1 f$ using the definition of the integral (but feel free to use the propositions of this section).*

***Exercise* 5.3:** *Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}$ of $[a,b]$ such that*

$$\lim_{k\to\infty} \bigl(U(P_k,f) - L(P_k,f)\bigr) = 0.$$

Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k,f) = \lim_{k\to\infty} L(P_k,f).$$

***Exercise* 5.4:** *Finish the proof of Proposition 5.7.*

***Exercise* 5.5:** *Suppose $f: [-1,1] \to \mathbb{R}$ is defined as*

$$f(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that $f \in \mathcal{R}[-1,1]$ and compute $\int_{-1}^1 f$ using the definition of the integral (but feel free to use the propositions of this section).

***Exercise* 5.6:** *Let $c \in (a,b)$ and let $d \in \mathbb{R}$. Define $f: [a,b] \to \mathbb{R}$ as*

$$f(x) := \begin{cases} d & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$$

Prove that $f \in \mathcal{R}[a,b]$ and compute $\int_a^b f$ using the definition of the integral (but feel free to use the propositions of this section).

Exercise* 5.7*: *Suppose $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable. Let $\epsilon > 0$ be given. Then show that there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that for every set of numbers $\{c_1, c_2, \ldots, c_n\}$ with $c_k \in [x_{k-1}, x_k]$ for all k, we have*

$$\left| \int_a^b f - \sum_{k=1}^n f(c_k) \Delta x_k \right| < \epsilon.$$

Exercise* 5.8*: *Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then define $g(x) := f(\alpha x + \beta)$ on the interval $I = [\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}]$. Show that g is Riemann integrable on I.*

Exercise* 5.9*: *Suppose $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ are such that for all $x \in (0,1]$, we have $f(x) = g(x)$. Suppose f is Riemann integrable. Prove g is Riemann integrable and $\int_0^1 f = \int_0^1 g$.*

Exercise* 5.10*: *Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function. Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a uniform partition of $[0,1]$, that is, $x_j = \frac{j}{n}$. Is $\{L(P_n, f)\}_{n=1}^\infty$ always monotone? Yes/No: Prove or find a counterexample.*

Exercise* 5.11** (Challenging):*** *For a bounded function $f\colon [0,1] \to \mathbb{R}$, let $R_n := (\frac{1}{n}) \sum_{j=1}^n f(\frac{j}{n})$ (the uniform right-hand rule).*

a) If f is Riemann integrable show $\int_0^1 f = \lim R_n$.

b) Find an f that is not Riemann integrable, but $\lim R_n$ exists.

Exercise* 5.12** (Challenging):*** *Generalize the previous exercise. Show that $f \in \mathcal{R}[a,b]$ if and only if there exists an $I \in \mathbb{R}$, such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if P is a partition with $\Delta x_i < \delta$ for all i, then $|L(P,f) - I| < \epsilon$ and $|U(P,f) - I| < \epsilon$. If $f \in \mathcal{R}[a,b]$, then $I = \int_a^b f$.*

Exercise* 5.13*: *Using Exercise 5.12 and the idea of the proof in Exercise 5.7, show that Darboux integral is the same as the standard definition of Riemann integral, which you have most likely seen in calculus. That is, show that $f \in \mathcal{R}[a,b]$ if and only if there exists an $I \in \mathbb{R}$, such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition with $\Delta x_i < \delta$ for all i, then $\left|\sum_{i=1}^n f(c_i) \Delta x_i - I\right| < \epsilon$ for every set $\{c_1, c_2, \ldots, c_n\}$ with $c_i \in [x_{i-1}, x_i]$. If $f \in \mathcal{R}[a,b]$, then $I = \int_a^b f$.*

Exercise* 5.14** (Challenging):*** *Construct functions f and g, where $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable, $g\colon [0,1] \to [0,1]$ is one-to-one and onto, and such that the composition $f \circ g$ is not Riemann integrable.*

5.2 Properties of the integral

5.2.1 Additivity

Adding a bunch of things in two parts and then adding those two parts should be the same as adding everything all at once. The corresponding property for integral is called the additive property of the integral. First, we prove the additivity property for the lower and upper Darboux integrals.

Lemma 5.16. *Suppose $a < b < c$ and $f \colon [a,c] \to \mathbb{R}$ is a bounded function. Then*

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f$$

and

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

Proof. If we have partitions $P_1 = \{x_0, x_1, \ldots, x_k\}$ of $[a,b]$ and $P_2 = \{x_k, x_{k+1}, \ldots, x_n\}$ of $[b,c]$, then the set $P := P_1 \cup P_2 = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,c]$. We find

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{k} m_j \Delta x_j + \sum_{j=k+1}^{n} m_j \Delta x_j = L(P_1,f) + L(P_2,f).$$

When we take the supremum of the right-hand side over all P_1 and P_2, we are taking a supremum of the left-hand side over all partitions P of $[a,c]$ that contain b. If Q is a partition of $[a,c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $L(Q,f) \leq L(P,f)$. Therefore, taking a supremum only over the P that contain b is sufficient to find the supremum of $L(P,f)$ over all partitions P, see Exercise 1.9. Finally, recall Exercise 1.23 to compute

$$\begin{aligned}
\underline{\int_a^c} f &= \sup \bigl\{ L(P,f) : P \text{ a partition of } [a,c] \bigr\} \\
&= \sup \bigl\{ L(P,f) : P \text{ a partition of } [a,c], b \in P \bigr\} \\
&= \sup \bigl\{ L(P_1,f) + L(P_2,f) : P_1 \text{ a partition of } [a,b], P_2 \text{ a partition of } [b,c] \bigr\} \\
&= \sup \bigl\{ L(P_1,f) : P_1 \text{ a partition of } [a,b] \bigr\} + \sup \bigl\{ L(P_2,f) : P_2 \text{ a partition of } [b,c] \bigr\} \\
&= \underline{\int_a^b} f + \underline{\int_b^c} f.
\end{aligned}$$

Similarly, for P, P_1, and P_2 as above, we obtain

$$U(P,f) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{k} M_j \Delta x_j + \sum_{j=k+1}^{n} M_j \Delta x_j = U(P_1,f) + U(P_2,f).$$

We wish to take the infimum on the right over all P_1 and P_2, and so we are taking the infimum over all partitions P of $[a,c]$ that contain b. If Q is a partition of $[a,c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $U(Q,f) \geq U(P,f)$. Therefore, taking an infimum only over the P that contain b is sufficient to find the infimum of $U(P, f)$ for all P. We obtain

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f. \qquad \square$$

Proposition 5.17. *Let $a < b < c$. A function $f : [a, c] \to \mathbb{R}$ is Riemann integrable if and only if f is Riemann integrable on $[a, b]$ and $[b, c]$. If f is Riemann integrable, then*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof. Suppose $f \in \mathcal{R}[a, c]$, then $\overline{\int_a^c} f = \underline{\int_a^c} f = \int_a^c f$. We apply the lemma to get

$$\int_a^c f = \underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f \leq \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f = \int_a^c f.$$

Thus the inequality is an equality,

$$\underline{\int_a^b} f + \underline{\int_b^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

As we also know $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ and $\underline{\int_b^c} f \leq \overline{\int_b^c} f$, we conclude

$$\underline{\int_a^b} f = \overline{\int_a^b} f \qquad \text{and} \qquad \underline{\int_b^c} f = \overline{\int_b^c} f.$$

Thus f is Riemann integrable on $[a, b]$ and $[b, c]$ and the desired formula holds.

Now assume f is Riemann integrable on $[a, b]$ and on $[b, c]$. Again apply the lemma to get

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f = \int_a^b f + \int_b^c f = \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f.$$

Therefore, f is Riemann integrable on $[a, c]$, and the integral is computed as indicated. $\square$

An easy consequence of the additivity is the following corollary. We leave the details to the reader as an exercise.

Corollary 5.18. *If $f \in \mathcal{R}[a, b]$ and $[c, d] \subset [a, b]$, then the restriction $f|_{[c,d]}$ is in $\mathcal{R}[c, d]$.*

5.2.2 Linearity and monotonicity

A sum is a linear function of the summands. So is the integral.

Proposition 5.19 (Linearity)**.** *Let f and g be in $\mathcal{R}[a, b]$ and $\alpha \in \mathbb{R}$.*

(i) αf is in $\mathcal{R}[a, b]$ and

$$\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

(ii) $f + g$ is in $\mathcal{R}[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Proof. Let us prove the first item for $\alpha \geq 0$. Let P be a partition of $[a, b]$. Let $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ as usual. Since α is nonnegative, we can move the multiplication by α past the infimum,

$$\inf\bigl\{\alpha f(x) : x \in [x_{i-1}, x_i]\bigr\} = \alpha \inf\bigl\{f(x) : x \in [x_{i-1}, x_i]\bigr\} = \alpha m_i.$$

Therefore,

$$L(P, \alpha f) = \sum_{i=1}^{n} \alpha m_i \Delta x_i = \alpha \sum_{i=1}^{n} m_i \Delta x_i = \alpha L(P, f).$$

Similarly,

$$U(P, \alpha f) = \alpha U(P, f).$$

Again, as $\alpha \geq 0$ we may move multiplication by α past the supremum. Hence,

$$\begin{aligned}
\underline{\int_a^b} \alpha f(x)\ dx &= \sup\bigl\{L(P, \alpha f) : P \text{ a partition of } [a, b]\bigr\} \\
&= \sup\bigl\{\alpha L(P, f) : P \text{ a partition of } [a, b]\bigr\} \\
&= \alpha \sup\bigl\{L(P, f) : P \text{ a partition of } [a, b]\bigr\} \\
&= \alpha \underline{\int_a^b} f(x)\ dx.
\end{aligned}$$

Similarly, we show

$$\overline{\int_a^b} \alpha f(x)\ dx = \alpha \overline{\int_a^b} f(x)\ dx.$$

The conclusion now follows for $\alpha \geq 0$.

To finish the proof of the first item, we need to show that $-f$ is Riemann integrable and $\int_a^b -f(x)\ dx = -\int_a^b f(x)\ dx$. The proof of this fact is left as Exercise 5.15.

The proof of the second item is left as Exercise 5.16. It is not difficult, but it is not as trivial as it may appear at first glance. □

The second item in the proposition does not hold with equality for the Darboux integrals, but we do obtain inequalities. The proof of the following proposition is Exercise 5.30. It follows for upper and lower sums on a fixed partition by Exercise 1.38, that is, supremum of a sum is less than or equal to the sum of suprema and similarly for infima.

Proposition 5.20. *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded functions. Then*

$$\overline{\int_a^b}(f + g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g, \qquad \text{and} \qquad \underline{\int_a^b}(f + g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Adding up smaller numbers should give us a smaller result. That is true for an integral as well.

Proposition 5.21 (Monotonicity)**.** *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded, and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then*

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \qquad \text{and} \qquad \overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

Moreover, if f and g are in $\mathcal{R}[a,b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Then let

$$m_i := \inf \bigl\{ f(x) : x \in [x_{i-1}, x_i] \bigr\} \qquad \text{and} \qquad \widetilde{m}_i := \inf \bigl\{ g(x) : x \in [x_{i-1}, x_i] \bigr\}.$$

As $f(x) \leq g(x)$, then $m_i \leq \widetilde{m}_i$. Therefore,

$$L(P,f) = \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n \widetilde{m}_i \Delta x_i = L(P,g).$$

We take the supremum over all P (see Proposition 1.27) to obtain

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g.$$

Similarly, we obtain the same conclusion for the upper integrals. Finally, if f and g are Riemann integrable all the integrals are equal, and the conclusion follows. $\square$

5.2.3 Continuous functions

Let us show that continuous functions are Riemann integrable. In fact, we can even allow some discontinuities. We start with a function continuous on the whole closed interval $[a,b]$.

Lemma 5.22. *If $f \colon [a,b] \to \mathbb{R}$ is a continuous function, then $f \in \mathcal{R}[a,b]$.*

Proof. As f is continuous on a closed bounded interval, it is uniformly continuous. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \frac{\epsilon}{b-a}$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta$ for all $i = 1, 2, \ldots, n$. For example, take n such that $\frac{b-a}{n} < \delta$, and let $x_i := \frac{i}{n}(b-a) + a$. Then for all $x, y \in [x_{i-1}, x_i]$, we have $|x-y| \leq \Delta x_i < \delta$, and so

$$f(x) - f(y) \leq |f(x)-f(y)| < \frac{\epsilon}{b-a}.$$

As f is continuous on $[x_{i-1}, x_i]$, it attains a maximum and a minimum on this interval. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_i$ and $f(y) = m_i$ in the notation from the definition of the integral. Therefore,

$$M_i - m_i = f(x) - f(y) < \frac{\epsilon}{b-a}.$$

And so

$$\begin{aligned}
\overline{\int_a^b} f - \underline{\int_a^b} f &\leq U(P,f) - L(P,f) \\
&= \left(\sum_{i=1}^{n} M_i \Delta x_i\right) - \left(\sum_{i=1}^{n} m_i \Delta x_i\right) \\
&= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \\
&< \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\
&= \frac{\epsilon}{b-a} (b-a) = \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary,

$$\overline{\int_a^b} f = \underline{\int_a^b} f,$$

and f is Riemann integrable on $[a,b]$. $\square$

The second lemma says that we need the function to only be "Riemann integrable inside the interval," as long as it is bounded. It also tells us how to compute the integral.

Lemma 5.23. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function, $\{a_n\}$ and $\{b_n\}$ be sequences such that $a < a_n < b_n < b$ for all n, with $\lim a_n = a$ and $\lim b_n = b$. Suppose $f \in \mathcal{R}[a_n, b_n]$ for all n. Then $f \in \mathcal{R}[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f.$$

Proof. Let $M > 0$ be a real number such that $|f(x)| \leq M$. As $(b-a) \geq (b_n - a_n)$,

$$-M(b-a) \leq -M(b_n - a_n) \leq \int_{a_n}^{b_n} f \leq M(b_n - a_n) \leq M(b-a).$$

Therefore, the sequence of numbers $\left\{\int_{a_n}^{b_n} f\right\}_{n=1}^{\infty}$ is bounded and by Bolzano–Weierstrass has a convergent subsequence indexed by n_k. Let us call L the limit of the subsequence $\left\{\int_{a_{n_k}}^{b_{n_k}} f\right\}_{k=1}^{\infty}$.

Lemma 5.16 says that the lower and upper integral are additive and the hypothesis says that f is integrable on $[a_{n_k}, b_{n_k}]$. Therefore

$$\underline{\int_a^b} f = \underline{\int_a^{a_{n_k}}} f + \int_{a_{n_k}}^{b_{n_k}} f + \underline{\int_{b_{n_k}}^b} f \geq -M(a_{n_k} - a) + \int_{a_{n_k}}^{b_{n_k}} f - M(b - b_{n_k}).$$

We take the limit as k goes to ∞ on the right-hand side,

$$\underline{\int_a^b} f \geq -M \cdot 0 + L - M \cdot 0 = L.$$

Next we use additivity of the upper integral,

$$\overline{\int_a^b} f = \overline{\int_a^{a_{n_k}}} f + \int_{a_{n_k}}^{b_{n_k}} f + \overline{\int_{b_{n_k}}^b} f \leq M(a_{n_k} - a) + \int_{a_{n_k}}^{b_{n_k}} f + M(b - b_{n_k}).$$

We take the same subsequence $\{\int_{a_{n_k}}^{b_{n_k}} f\}_{k=1}^{\infty}$ and take the limit to obtain

$$\overline{\int_a^b} f \leq M \cdot 0 + L + M \cdot 0 = L.$$

Thus $\overline{\int_a^b} f = \underline{\int_a^b} f = L$ and hence f is Riemann integrable and $\int_a^b f = L$. In particular, no matter what subsequence we chose, the L is the same number.

To prove the final statement of the lemma we use Proposition 2.39. We have shown that every convergent subsequence $\left\{\int_{a_{n_k}}^{b_{n_k}} f\right\}$ converges to $L = \int_a^b f$. Therefore, the sequence $\left\{\int_{a_n}^{b_n} f\right\}$ is convergent and converges to $\int_a^b f$. □

We say a function $f\colon [a, b] \to \mathbb{R}$ has *finitely many discontinuities* if there exists a finite set $S = \{x_1, x_2, \ldots, x_n\} \subset [a, b]$, and f is continuous at all points of $[a, b] \setminus S$.

Theorem 5.24. *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function with finitely many discontinuities. Then $f \in \mathcal{R}[a, b]$.*

Proof. We divide the interval into finitely many intervals $[a_i, b_i]$ so that f is continuous on the interior (a_i, b_i). If f is continuous on (a_i, b_i), then it is continuous and hence integrable on $[c_i, d_i]$ whenever $a_i < c_i < d_i < b_i$. By Lemma 5.23 the restriction of f to $[a_i, b_i]$ is integrable. By additivity of the integral (and induction) f is integrable on the union of the intervals. □

5.2.4 More on integrable functions

Sometimes it is convenient (or necessary) to change certain values of a function and then integrate. The next result says that if we change the values at finitely many points, the integral does not change.

Proposition 5.25. *Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable. Let $g\colon [a, b] \to \mathbb{R}$ be such that $f(x) = g(x)$ for all $x \in [a, b] \setminus S$, where S is a finite set. Then g is a Riemann integrable function and*

$$\int_a^b g = \int_a^b f.$$

Sketch of proof. Using additivity of the integral, we split up the interval $[a, b]$ into smaller intervals such that $f(x) = g(x)$ holds for all x except at the endpoints (details are left to the reader).

Therefore, without loss of generality suppose $f(x) = g(x)$ for all $x \in (a, b)$. The proof follows by Lemma 5.23, and is left as Exercise 5.17. □

Finally, monotone (increasing or decreasing) functions are always Riemann integrable. The proof is left to the reader as part of Exercise 5.28.

Proposition 5.26. *Let $f\colon [a, b] \to \mathbb{R}$ be a monotone function. Then $f \in \mathcal{R}[a, b]$.*

5.2.5 Exercises

Exercise* 5.15*: *Finish the proof of the first part of Proposition 5.19. Let f be in $\mathcal{R}[a,b]$. Prove that $-f$ is in $\mathcal{R}[a,b]$ and*

$$\int_a^b -f(x)\ dx = -\int_a^b f(x)\ dx.$$

Exercise* 5.16*: *Prove the second part of Proposition 5.19. Let f and g be in $\mathcal{R}[a,b]$. Prove, without using Proposition 5.20, that $f+g$ is in $\mathcal{R}[a,b]$ and*

$$\int_a^b \big(f(x)+g(x)\big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$$

Hint: One way to do it is to use Proposition 5.7 to find a single partition P such that $U(P,f) - L(P,f) < \frac{\epsilon}{2}$ and $U(P,g) - L(P,g) < \frac{\epsilon}{2}$.

Exercise* 5.17*: *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable, and $g\colon [a,b] \to \mathbb{R}$ be such that $f(x) = g(x)$ for all $x \in (a,b)$. Prove that g is Riemann integrable and that*

$$\int_a^b g = \int_a^b f.$$

Exercise* 5.18*: *Prove the* mean value theorem for integrals*: If $f\colon [a,b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.*

Exercise* 5.19*: *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all x.*

Exercise* 5.20*: *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function and $\int_a^b f = 0$. Prove that there exists a $c \in [a,b]$ such that $f(c) = 0$. (Compare with the previous exercise.)*

Exercise* 5.21*: *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be continuous functions such that $\int_a^b f = \int_a^b g$. Show that there exists a $c \in [a,b]$ such that $f(c) = g(c)$.*

Exercise* 5.22*: *Let $f \in \mathcal{R}[a,b]$. Let α, β, γ be arbitrary numbers in $[a,b]$ (not necessarily ordered in any way). Prove*

$$\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f.$$

Recall what $\int_a^b f$ means if $b \leq a$.

Exercise* 5.23*: *Prove Corollary 5.18.*

Exercise* 5.24*: *Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and has finitely many discontinuities. Show that as a function of x the expression $|f(x)|$ is bounded with finitely many discontinuities and is thus Riemann integrable. Then show*

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)|\ dx.$$

Exercise* 5.25** (Hard):*** *Show that the Thomae or popcorn function (see Example 3.29) is Riemann integrable. Therefore, there exists a function discontinuous at all rational numbers (a dense set) that is Riemann integrable.*

In particular, define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{k} & \text{if } x = \frac{m}{k} \text{ where } m,k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show $\int_0^1 f = 0$.

If $I \subset \mathbb{R}$ is a bounded interval, then the function

$$\varphi_I(x) := \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$

is called an *elementary step function.*

Exercise* 5.26*: *Let I be an arbitrary bounded interval (you should consider all types of intervals: closed, open, half-open) and $a < b$, then using only the definition of the integral show that the elementary step function φ_I is integrable on $[a,b]$, and find the integral in terms of a, b, and the endpoints of I.*

A function f is called a *step function* if it can be written as

$$f(x) = \sum_{k=1}^{n} \alpha_k \varphi_{I_k}(x)$$

for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and some bounded intervals $I_1, I_2, \ldots, I_n$.

Exercise* 5.27*: *Using Exercise 5.26, show that a step function is integrable on every interval $[a,b]$. Furthermore, find the integral in terms of a, b, the endpoints of I_k and the α_k.*

Exercise* 5.28*: *Let $f \colon [a,b] \to \mathbb{R}$ be a function.*

a) Show that if f is increasing, then it is Riemann integrable. Hint: Use a uniform partition; each subinterval of same length.

b) Use part a) to show that if f is decreasing, then it is Riemann integrable.

c) Suppose[3] $h = f - g$ where f and g are increasing functions on $[a,b]$. Show that h is Riemann integrable.

Exercise* 5.29** (Challenging):*** *Suppose $f \in \mathcal{R}[a,b]$, then the function that takes x to $|f(x)|$ is also Riemann integrable on $[a,b]$. Then show the same inequality as Exercise 5.24.*

Exercise* 5.30*: *Suppose $f \colon [a,b] \to \mathbb{R}$ and $g \colon [a,b] \to \mathbb{R}$ are bounded.*

a) Show $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$ and $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$.

b) Find example f and g where the inequality is strict. Hint: f and g should not be Riemann integrable.

Exercise* 5.31*: *Suppose $f \colon [a,b] \to \mathbb{R}$ is continuous and $g \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Define*

$$h(x) := \int_a^b g(t-x) f(t) \; dt.$$

Prove that h is Lipschitz continuous.

[3] Such an h is said to be of *bounded variation.*

5.3 Fundamental theorem of calculus

In this chapter we discuss and prove the *fundamental theorem of calculus*. The entirety of integral calculus is built upon this theorem, ergo the name. The theorem relates the seemingly unrelated concepts of integral and derivative. It tells us how to compute the antiderivative of a function using the integral and vice versa.

5.3.1 First form of the theorem

Theorem 5.27. *Let $F\colon [a,b] \to \mathbb{R}$ be a continuous function, differentiable on (a,b). Let $f \in \mathcal{R}[a,b]$ be such that $f(x) = F'(x)$ for $x \in (a,b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

It is not hard to generalize the theorem to allow a finite number of points in $[a,b]$ where F is not differentiable, as long as it is continuous. This generalization is left as an exercise.

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. For each interval $[x_{i-1}, x_i]$, use the mean value theorem to find a $c_i \in (x_{i-1}, x_i)$ such that

$$f(c_i)\Delta x_i = F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}).$$

See Figure 5.5, and notice that the area of all three shaded rectangles is $F(x_{i+1}) - F(x_{i-2})$. The idea is that by taking smaller and smaller subintervals we prove that this area is the integral of f.

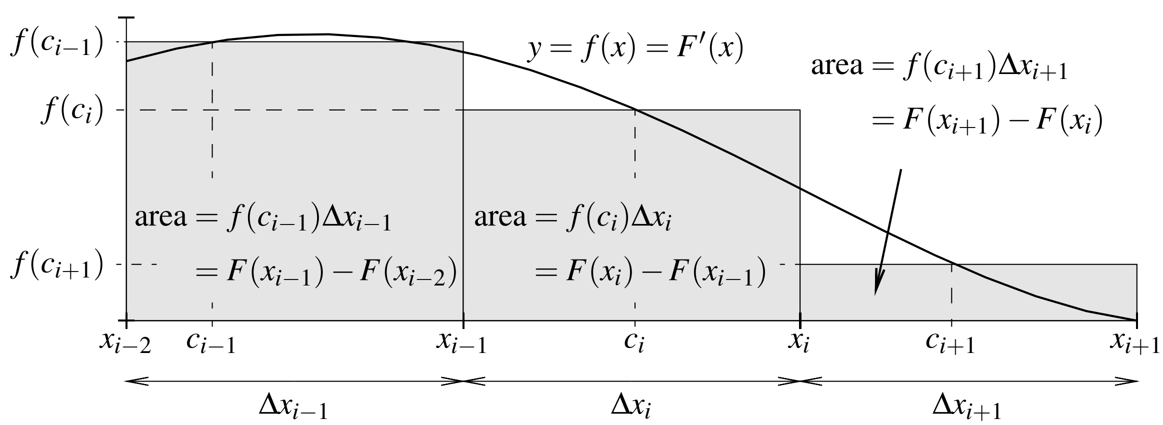

Figure 5.5: Mean value theorem on subintervals of a partition approximating area under the curve.

Using the notation from the definition of the integral, we have $m_i \leq f(c_i) \leq M_i$, and so

$$m_i \Delta x_i \leq F(x_i) - F(x_{i-1}) \leq M_i \Delta x_i.$$

We sum over $i = 1, 2, \ldots, n$ to get

$$\sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n \bigl(F(x_i) - F(x_{i-1})\bigr) \leq \sum_{i=1}^n M_i \Delta x_i.$$

In the middle sum, all the terms except the first and last cancel and we end up with $F(x_n) - F(x_0) = F(b) - F(a)$. The sums on the left and on the right are the lower and the upper sum respectively. So

$$L(P, f) \leq F(b) - F(a) \leq U(P, f).$$

We take the supremum of $L(P, f)$ over all partitions P and the left inequality yields

$$\underline{\int_a^b} f \leq F(b) - F(a).$$

Similarly, taking the infimum of $U(P, f)$ over all partitions P yields

$$F(b) - F(a) \leq \overline{\int_a^b} f.$$

As f is Riemann integrable, we have

$$\int_a^b f = \underline{\int_a^b} f \leq F(b) - F(a) \leq \overline{\int_a^b} f = \int_a^b f.$$

The inequalities must be equalities and we are done. □

The theorem is used to compute integrals. Suppose we know that the function $f(x)$ is a derivative of some other function $F(x)$, then we can find an explicit expression for $\int_a^b f$.

Example 5.28: To compute

$$\int_0^1 x^2 \, dx,$$

we notice x^2 is the derivative of $\frac{x^3}{3}$. The fundamental theorem says

$$\int_0^1 x^2 \, dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

5.3.2 Second form of the theorem

The second form of the fundamental theorem gives us a way to solve the differential equation $F'(x) = f(x)$, where f is a known function and we are trying to find an F that satisfies the equation.

Theorem 5.29. *Let $f \colon [a, b] \to \mathbb{R}$ be a Riemann integrable function. Define*

$$F(x) := \int_a^x f.$$

First, F is continuous on $[a, b]$. Second, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. As f is bounded, there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Suppose $x, y \in [a, b]$ with $x > y$. Then

$$|F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \leq M \, |x - y| \, .$$

By symmetry, the same also holds if $x < y$. So F is Lipschitz continuous and hence continuous.

Now suppose f is continuous at c. Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that for $x \in [a, b]$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. In particular, for such x, we have

$$f(c) - \epsilon < f(x) < f(c) + \epsilon.$$

Thus if $x > c$, then

$$\bigl(f(c) - \epsilon\bigr)(x - c) \leq \int_c^x f \leq \bigl(f(c) + \epsilon\bigr)(x - c).$$

When $c > x$, then the inequalities are reversed. Therefore, assuming $c \neq x$, we get

$$f(c) - \epsilon \leq \frac{\int_c^x f}{x - c} \leq f(c) + \epsilon.$$

As

$$\frac{F(x) - F(c)}{x - c} = \frac{\int_a^x f - \int_a^c f}{x - c} = \frac{\int_c^x f}{x - c},$$

we have

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \epsilon.$$

The result follows. It is left to the reader to see why is it OK that we just have a non-strict inequality. □

Of course, if f is continuous on $[a, b]$, then it is automatically Riemann integrable, F is differentiable on all of $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Remark 5.30. The second form of the fundamental theorem of calculus still holds if we let $d \in [a, b]$ and define

$$F(x) := \int_d^x f.$$

That is, we can use any point of $[a, b]$ as our base point. The proof is left as an exercise.

Let us look at what a simple discontinuity can do. Take $f(x) := -1$ if $x < 0$, and $f(x) := 1$ if $x \geq 0$. Let $F(x) := \int_0^x f$. It is not difficult to see that $F(x) = |x|$. Notice that f is discontinuous at 0 and F is not differentiable at 0. However, the converse in the theorem does not hold. Let $g(x) := 0$ if $x \neq 0$, and $g(0) := 1$. Letting $G(x) := \int_0^x g$, we find that $G(x) = 0$ for all x. So g is discontinuous at 0, but $G'(0)$ exists and is equal to 0.

A common misunderstanding of the integral for calculus students is to think of integrals whose solution cannot be given in closed-form as somehow deficient. This is not the case. Most integrals we write down are not computable in closed-form. Even some integrals that we consider in closed-form are not really such. We define the natural logarithm as the antiderivative of $\frac{1}{x}$ such that $\ln 1 = 0$:

$$\ln x := \int_1^x \frac{1}{s} \, ds.$$

How does a computer find the value of $\ln x$? One way to do it is to numerically approximate this integral. Morally, we did not really "simplify" $\int_1^x \frac{1}{s} \, ds$ by writing down $\ln x$. We simply gave the integral a name. If we require numerical answers, it is possible we end up doing the calculation by approximating an integral anyway. In the next section, we even define the exponential using the logarithm, which we define in terms of the integral.

Another common function defined by an integral that cannot be evaluated symbolically in terms of elementary functions is the erf function, defined as

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds.$$

This function comes up often in applied mathematics. It is simply the antiderivative of $\left(\frac{2}{\sqrt{\pi}}\right) e^{-x^2}$ that is zero at zero. The second form of the fundamental theorem tells us that we can write the function as an integral. If we wish to compute any particular value, we numerically approximate the integral.

5.3.3 Change of variables

A theorem often used in calculus to solve integrals is the change of variables theorem, you may have called it *u-substitution*. Let us prove it now. Recall a function is continuously differentiable if it is differentiable and the derivative is continuous.

Theorem 5.31 (Change of variables)**.** *Let $g \colon [a,b] \to \mathbb{R}$ be a continuously differentiable function, let $f \colon [c,d] \to \mathbb{R}$ be continuous, and suppose $g\big([a,b]\big) \subset [c,d]$. Then*

$$\int_a^b f\big(g(x)\big)\, g'(x) \, dx = \int_{g(a)}^{g(b)} f(s) \, ds.$$

Proof. As g, g', and f are continuous, $f\big(g(x)\big)\, g'(x)$ is a continuous function of $[a,b]$, therefore it is Riemann integrable. Similarly, f is integrable on every subinterval of $[c,d]$.

Define $F \colon [c,d] \to \mathbb{R}$ by

$$F(y) := \int_{g(a)}^{y} f(s) \, ds.$$

By the second form of the fundamental theorem of calculus (see Remark 5.30 and Exercise 5.35), F is a differentiable function and $F'(y) = f(y)$. Apply the chain rule,

$$\big(F \circ g\big)'(x) = F'\big(g(x)\big) g'(x) = f\big(g(x)\big) g'(x).$$

Note that $F\big(g(a)\big) = 0$ and use the first form of the fundamental theorem to obtain

$$\begin{aligned} \int_{g(a)}^{g(b)} f(s) \, ds &= F\big(g(b)\big) = F\big(g(b)\big) - F\big(g(a)\big) \\ &= \int_a^b \big(F \circ g\big)'(x) \, dx = \int_a^b f\big(g(x)\big) g'(x) \, dx. \end{aligned}$$

$\square$

The change of variables theorem is often used to solve integrals by changing them to integrals that we know or that we can solve using the fundamental theorem of calculus.

Example 5.32: The derivative of $\sin(x)$ is $\cos(x)$. Using $g(x) := x^2$, we solve

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^{\pi} \frac{\cos(s)}{2}\,ds = \frac{1}{2}\int_0^{\pi}\cos(s)\,ds = \frac{\sin(\pi)-\sin(0)}{2} = 0.$$

However, beware that we must satisfy the hypotheses of the theorem. The following example demonstrates why we should not just move symbols around mindlessly. We must be careful that those symbols really make sense.

Example 5.33: Consider

$$\int_{-1}^{1} \frac{\ln|x|}{x}\,dx.$$

It may be tempting to take $g(x) := \ln|x|$. Compute $g'(x) = \frac{1}{x}$ and try to write

$$\int_{g(-1)}^{g(1)} s\,ds = \int_0^0 s\,ds = 0.$$

This "solution" is incorrect, and it does not say that we can solve the given integral. First problem is that $\frac{\ln|x|}{x}$ is not continuous on $[-1,1]$. It is not defined at 0, and cannot be made continuous by defining a value at 0. Second, $\frac{\ln|x|}{x}$ is not even Riemann integrable on $[-1,1]$ (it is unbounded). The integral we wrote down simply does not make sense. Finally, g is not continuous on $[-1,1]$, let alone continuously differentiable.

5.3.4 Exercises

***Exercise* 5.32:** *Compute* $\frac{d}{dx}\left(\int_{-x}^{x} e^{s^2}\,ds\right)$.

***Exercise* 5.33:** *Compute* $\frac{d}{dx}\left(\int_{0}^{x^2} \sin(s^2)\,ds\right)$.

***Exercise* 5.34:** *Suppose* $F\colon [a,b]\to\mathbb{R}$ *is continuous and differentiable on* $[a,b]\setminus S$*, where* S *is a finite set. Suppose there exists an* $f\in\mathcal{R}[a,b]$ *such that* $f(x)=F'(x)$ *for* $x\in[a,b]\setminus S$*. Show that* $\int_a^b f = F(b)-F(a)$.

***Exercise* 5.35:** *Let* $f\colon [a,b]\to\mathbb{R}$ *be a continuous function. Let* $c\in[a,b]$ *be arbitrary. Define*

$$F(x) := \int_c^x f.$$

Prove that F *is differentiable and that* $F'(x)=f(x)$ *for all* $x\in[a,b]$.

***Exercise* 5.36:** *Prove* integration by parts*. That is, suppose* F *and* G *are continuously differentiable functions on* $[a,b]$*. Then prove*

$$\int_a^b F(x)G'(x)\,dx = F(b)G(b)-F(a)G(a) - \int_a^b F'(x)G(x)\,dx.$$

***Exercise* 5.37:** *Suppose F and G are continuously[4] differentiable functions defined on $[a,b]$ such that $F'(x) = G'(x)$ for all $x \in [a,b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$.*

The next exercise shows how we can use the integral to "smooth out" a non-differentiable function.

***Exercise* 5.38:** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let $\epsilon > 0$ be a constant. For $x \in [a+\epsilon, b-\epsilon]$, define*

$$g(x) := \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f.$$

a) Show that g is differentiable and find the derivative.

b) Let f be differentiable and fix $x \in (a,b)$ (let ϵ be small enough). What happens to $g'(x)$ as ϵ gets smaller?

c) Find g for $f(x) := |x|$, $\epsilon = 1$ (you can assume $[a,b]$ is large enough).

***Exercise* 5.39:** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$.*

***Exercise* 5.40:** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all rational x in $[a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$.*

***Exercise* 5.41:** *A function f is an* odd function *if $f(x) = -f(-x)$, and f is an* even function *if $f(x) = f(-x)$. Let $a > 0$. Assume f is continuous. Prove:*

a) If f is odd, then $\int_{-a}^a f = 0$.

b) If f is even, then $\int_{-a}^a f = 2\int_0^a f$.

***Exercise* 5.42:**

a) Show that $f(x) := \sin(\frac{1}{x})$ is integrable on every interval (you can define $f(0)$ to be anything).

b) Compute $\int_{-1}^1 \sin(\frac{1}{x})\, dx$ (mind the discontinuity).

***Exercise* 5.43** (uses §3.6)**:**

a) Suppose $f\colon [a,b] \to \mathbb{R}$ is increasing, by Proposition 5.26, f is Riemann integrable. Suppose f has a discontinuity at $c \in (a,b)$, show that $F(x) := \int_a^x f$ is not differentiable at c.

b) In Exercise 3.89, you constructed an increasing function $f\colon [0,1] \to \mathbb{R}$ that is discontinuous at every $x \in [0,1] \cap \mathbb{Q}$. Use this f to construct a function $F(x)$ that is continuous on $[0,1]$, but not differentiable at every $x \in [0,1] \cap \mathbb{Q}$.

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[4]Compare this hypothesis to Exercise 4.24.

5.4 The logarithm and the exponential

We now have the tools required to properly define the exponential and the logarithm that you know from calculus so well. We start with exponentiation. If n is a positive integer, it is obvious to define

$$x^n := \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}.$$

It makes sense to define $x^0 := 1$. For negative integers, let $x^{-n} := \frac{1}{x^n}$. If $x > 0$, define $x^{1/n}$ as the unique positive nth root. Finally, for a rational number $\frac{n}{m}$ (in lowest terms), define

$$x^{n/m} := \left(x^{1/m}\right)^n.$$

It is not difficult to show we get the same number no matter what representation of $\frac{n}{m}$ we use, so we do not need to use lowest terms.

However, what do we mean by $\sqrt{2}^{\sqrt{2}}$? Or x^y in general? In particular, what is e^x for all x? And how do we solve $y = e^x$ for x? This section answers these questions and more.

5.4.1 The logarithm

It is convenient to define the logarithm first. Let us show that a unique function with the right properties exists, and only then will we call it *the* logarithm.

Proposition 5.34. *There exists a unique function $L\colon (0,\infty) \to \mathbb{R}$ such that*

(i) $L(1) = 0$.

(ii) L *is differentiable and* $L'(x) = \frac{1}{x}$.

(iii) L *is strictly increasing, bijective, and*

$$\lim_{x\to 0} L(x) = -\infty, \qquad \textit{and} \qquad \lim_{x\to\infty} L(x) = \infty.$$

(iv) $L(xy) = L(x) + L(y)$ *for all* $x, y \in (0,\infty)$.

(v) If q *is a rational number and* $x > 0$, *then* $L(x^q) = qL(x)$.

Proof. To prove existence, we define a candidate and show it satisfies all the properties. Let

$$L(x) := \int_1^x \frac{1}{t}\, dt.$$

Obviously, i holds. Property ii holds via the second form of the fundamental theorem of calculus (Theorem 5.29).

To prove property iv, we change variables $u = yt$ to obtain

$$L(x) = \int_1^x \frac{1}{t}\, dt = \int_y^{xy} \frac{1}{u}\, du = \int_1^{xy} \frac{1}{u}\, du - \int_1^{y} \frac{1}{u}\, du = L(xy) - L(y).$$

Let us prove iii. Property ii together with the fact that $L'(x) = \frac{1}{x} > 0$ for $x > 0$, implies that L is strictly increasing and hence one-to-one. Let us show L is onto. As $\frac{1}{t} \geq \frac{1}{2}$ when $t \in [1, 2]$,

$$L(2) = \int_1^2 \frac{1}{t} \, dt \geq \frac{1}{2}.$$

By induction, iv implies that for $n \in \mathbb{N}$

$$L(2^n) = L(2) + L(2) + \cdots + L(2) = nL(2).$$

Given $y > 0$, by the Archimedean property of the real numbers (notice $L(2) > 0$), there is an $n \in \mathbb{N}$ such that $L(2^n) > y$. By the intermediate value theorem there is an $x_1 \in (1, 2^n)$ such that $L(x_1) = y$. We get $(0, \infty)$ is in the image of L. As L is increasing, $L(x) > y$ for all $x > 2^n$, and so

$$\lim_{x \to \infty} L(x) = \infty.$$

Next $0 = L(\frac{x}{x}) = L(x) + L(\frac{1}{x})$, and so $L(x) = -L(\frac{1}{x})$. Using $x = 2^{-n}$, we obtain as above that L achieves all negative numbers. And

$$\lim_{x \to 0} L(x) = \lim_{x \to 0} -L(\frac{1}{x}) = \lim_{x \to \infty} -L(x) = -\infty.$$

In the limits, note that only $x > 0$ are in the domain of L.

Let us prove v. Fix $x > 0$. As above, iv implies $L(x^n) = nL(x)$ for all $n \in \mathbb{N}$. We already found that $L(x) = -L(\frac{1}{x})$, so $L(x^{-n}) = -L(x^n) = -nL(x)$. Then for $m \in \mathbb{N}$

$$L(x) = L\Big(\big(x^{1/m}\big)^m\Big) = mL\big(x^{1/m}\big).$$

Putting everything together for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have $L(x^{n/m}) = nL(x^{1/m}) = (\frac{n}{m})L(x)$.

Uniqueness follows using properties i and ii. Via the first form of the fundamental theorem of calculus (Theorem 5.27),

$$L(x) = \int_1^x \frac{1}{t} \, dt$$

is the unique function such that $L(1) = 0$ and $L'(x) = \frac{1}{x}$. □

Having proved that there is a unique function with these properties, we simply define the *logarithm* or sometimes called the *natural logarithm*:

$$\ln(x) := L(x).$$

Mathematicians usually write $\log(x)$ instead of $\ln(x)$, which is more familiar to calculus students. For all practical purposes, there is only one logarithm: the natural logarithm. See Exercise 5.45.

5.4.2 The exponential

Just as with the logarithm we define the exponential via a list of properties.

Proposition 5.35. *There exists a unique function $E: \mathbb{R} \to (0, \infty)$ such that*

(i) $E(0) = 1$.

(ii) E *is differentiable and* $E'(x) = E(x)$.

(iii) E *is strictly increasing, bijective, and*

$$\lim_{x\to-\infty} E(x) = 0, \qquad \textit{and} \qquad \lim_{x\to\infty} E(x) = \infty.$$

(iv) $E(x + y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$.

(v) If $q \in \mathbb{Q}$, *then* $E(qx) = E(x)^q$.

Proof. Again, we prove existence of such a function by defining a candidate and proving that it satisfies all the properties. The $L = \ln$ defined above is invertible. Let E be the inverse function of L. Property i is immediate.

Property ii follows via the inverse function theorem, in particular Lemma 4.26: L satisfies all the hypotheses of the lemma, and hence

$$E'(x) = \frac{1}{L'\big(E(x)\big)} = E(x).$$

Let us look at property iii. The function E is strictly increasing since $E'(x) = E(x) > 0$. As E is the inverse of L, it must also be bijective. To find the limits, we use that E is strictly increasing and onto $(0, \infty)$. For every $M > 0$, there is an x_0 such that $E(x_0) = M$ and $E(x) \geq M$ for all $x \geq x_0$. Similarly, for every $\epsilon > 0$, there is an x_0 such that $E(x_0) = \epsilon$ and $E(x) < \epsilon$ for all $x < x_0$. Therefore,

$$\lim_{n\to-\infty} E(x) = 0, \qquad \text{and} \qquad \lim_{n\to\infty} E(x) = \infty.$$

To prove property iv, we use the corresponding property for the logarithm. Take $x, y \in \mathbb{R}$. As L is bijective, find a and b such that $x = L(a)$ and $y = L(b)$. Then

$$E(x + y) = E\big(L(a) + L(b)\big) = E\big(L(ab)\big) = ab = E(x)E(y).$$

Property v also follows from the corresponding property of L. Given $x \in \mathbb{R}$, let a be such that $x = L(a)$ and

$$E(qx) = E\big(qL(a)\big) = E\big(L(a^q)\big) = a^q = E(x)^q.$$

Uniqueness follows from i and ii. Let E and F be two functions satisfying i and ii.

$$\frac{d}{dx}\Big(F(x)E(-x)\Big) = F'(x)E(-x) - E'(-x)F(x) = F(x)E(-x) - E(-x)F(x) = 0.$$

Therefore, by Proposition 4.16, $F(x)E(-x) = F(0)E(-0) = 1$ for all $x \in \mathbb{R}$. Next, $1 = E(0) = E(x - x) = E(x)E(-x)$. Then

$$0 = 1 - 1 = F(x)E(-x) - E(x)E(-x) = \big(F(x) - E(x)\big)E(-x).$$

Finally, $E(-x) \neq 0$[5] for all $x \in \mathbb{R}$. So $F(x) - E(x) = 0$ for all x, and we are done. □

Having proved E is unique, we define the *exponential* function as

$$\exp(x) := E(x).$$

If $y \in \mathbb{Q}$ and $x > 0$, then

$$x^y = \exp\bigl(\ln(x^y)\bigr) = \exp\bigl(y \ln(x)\bigr).$$

We can now make sense of exponentiation x^y for arbitrary $y \in \mathbb{R}$; if $x > 0$ and y is irrational, define

$$x^y := \exp\bigl(y \ln(x)\bigr).$$

As exp is continuous, then x^y is a continuous function of y. Therefore, we would obtain the same result had we taken a sequence of rational numbers $\{y_n\}$ approaching y and defined $x^y = \lim x^{y_n}$.

Define the number e, sometimes called *Euler's number* or the *base of the natural logarithm*, as

$$e := \exp(1).$$

Let us justify the notation e^x for $\exp(x)$:

$$e^x = \exp\bigl(x \ln(e)\bigr) = \exp(x).$$

The properties of the logarithm and the exponential extend to irrational powers. The proof is immediate.

Proposition 5.36. *Let $x, y \in \mathbb{R}$.*

(i) $\exp(xy) = \bigl(\exp(x)\bigr)^y$.

(ii) If $x > 0$, then $\ln(x^y) = y \ln(x)$.

Remark 5.37. There are other equivalent ways to define the exponential and the logarithm. A common way is to define E as the solution to the differential equation $E'(x) = E(x)$, $E(0) = 1$. See Example **??**, for a sketch of that approach. Yet another approach is to define the exponential function by power series, see Example **??**.

Remark 5.38. We proved the uniqueness of the functions L and E from just the properties $L(1) = 0$, $L'(x) = \frac{1}{x}$ and the equivalent condition for the exponential $E'(x) = E(x)$, $E(0) = 1$. Existence also follows from just these properties. Alternatively, uniqueness also follows from the laws of exponents, see the exercises.

[5] E is a function into $(0, \infty)$ after all. However, $E(-x) \neq 0$ also follows from $E(x)E(-x) = 1$. Therefore, we can prove uniqueness of E given i and ii, even for functions $E \colon \mathbb{R} \to \mathbb{R}$.

5.4.3 Exercises

Exercise* 5.44*: *Given a real number y and $b > 0$, define $f : (0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as $f(x) := x^y$ and $g(x) := b^x$. Show that f and g are differentiable and find their derivative.*

Exercise* 5.45*: *Let $b > 0$, $b \neq 1$ be given.*

a) Show that for every $y > 0$, there exists a unique number x such that $y = b^x$. Define the logarithm base b, $\log_b : (0, \infty) \to \mathbb{R}$, *by $\log_b(y) := x$.*

b) Show that $\log_b(x) = \frac{\ln(x)}{\ln(b)}$.

c) Prove that if $c > 0$, $c \neq 1$, then $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$.

d) Prove $\log_b(xy) = \log_b(x) + \log_b(y)$, and $\log_b(x^y) = y \log_b(x)$.

Exercise* 5.46** (requires §4.3):*** *Use Taylor's theorem to study the remainder term and show that for all $x \in \mathbb{R}$*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hint: Do not differentiate the series term by term (unless you would prove that it works).

Exercise* 5.47*: *Use the geometric sum formula to show (for $t \neq -1$)*

$$1 - t + t^2 - \cdots + (-1)^n t^n = \frac{1}{1+t} - \frac{(-1)^{n+1} t^{n+1}}{1+t}.$$

Using this fact show

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

for all $x \in (-1, 1]$ (note that $x = 1$ is included). Finally, find the limit of the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Exercise* 5.48*: *Show*

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

Hint: Take the logarithm.
Note: The expression $\left(1 + \frac{x}{n}\right)^n$ arises in compound interest calculations. It is the amount of money in a bank account after 1 year if 1 dollar was deposited initially at interest x and the interest was compounded n times during the year. The exponential e^x is the result of continuous compounding.

Exercise* 5.49*:

a) Prove that for $n \in \mathbb{N}$,

$$\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) \leq \sum_{k=1}^{n-1} \frac{1}{k}.$$

b) Prove that the limit

$$\gamma := \lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n) \right)$$

exists. This constant is known as the Euler–Mascheroni constant[6]*. It is not known if this constant is rational or not. It is approximately* $\gamma \approx 0.5772$*.*

Exercise* 5.50*: *Show*

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0.$$

Exercise* 5.51*: *Show that* e^x *is* convex*, in other words, show that if* $a \leq x \leq b$*, then* $e^x \leq e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a}$*.*

Exercise* 5.52*: *Using the logarithm find*

$$\lim_{n\to\infty} n^{1/n}.$$

Exercise* 5.53*: *Show that* $E(x) = e^x$ *is the unique continuous function such that* $E(x+y) = E(x)E(y)$ *and* $E(1) = e$*. Similarly, prove that* $L(x) = \ln(x)$ *is the unique continuous function defined on positive* x *such that* $L(xy) = L(x) + L(y)$ *and* $L(e) = 1$*.*

Exercise* 5.54** (requires §4.3):*** *Since* $(e^x)' = e^x$*, it is easy to see that* e^x *is infinitely differentiable (has derivatives of all orders). Define the function* $f \colon \mathbb{R} \to \mathbb{R}$*.*

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) Prove that for every $m \in \mathbb{N}$,

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^m} = 0.$$

b) Prove that f *is infinitely differentiable.*

c) Compute the Taylor series for f *at the origin, that is,*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Show that it converges, but show that it does not converge to $f(x)$ *for any given* $x > 0$*.*

[6]Named for the Swiss mathematician Leonhard Paul Euler (1707–1783) and the Italian mathematician Lorenzo Mascheroni (1750–1800).

5.5 Improper integrals

Often it is necessary to integrate over the entire real line, or an unbounded interval of the form $[a, \infty)$ or $(-\infty, b]$. We may also wish to integrate unbounded functions defined on a open bounded interval (a, b). For such intervals or functions, the Riemann integral is not defined, but we will write down the integral anyway in the spirit of Lemma 5.23. These integrals are called *improper integrals* and are limits of integrals rather than integrals themselves.

Definition 5.39. Suppose $f\colon [a, b) \to \mathbb{R}$ is a function (not necessarily bounded) that is Riemann integrable on $[a, c]$ for all $c < b$. We define

$$\int_a^b f := \lim_{c \to b^-} \int_a^c f$$

if the limit exists.

Suppose $f\colon [a, \infty) \to \mathbb{R}$ is a function such that f is Riemann integrable on $[a, c]$ for all $c < \infty$. We define

$$\int_a^\infty f := \lim_{c \to \infty} \int_a^c f$$

if the limit exists.

If the limit exists, we say the improper integral *converges*. If the limit does not exist, we say the improper integral *diverges*.

We similarly define improper integrals for the left-hand endpoint, we leave this to the reader.

For a finite endpoint b, if f is bounded, then Lemma 5.23 says that we defined nothing new. What is new is that we can apply this definition to unbounded functions. The following set of examples is so useful that we state it as a proposition.

Proposition 5.40 (p-test for integrals)**.** *The improper integral*

$$\int_1^\infty \frac{1}{x^p}\, dx$$

converges to $\frac{1}{p-1}$ *if* $p > 1$ *and diverges if* $0 < p \leq 1$.

The improper integral

$$\int_0^1 \frac{1}{x^p}\, dx$$

converges to $\frac{1}{1-p}$ *if* $0 < p < 1$ *and diverges if* $p \geq 1$.

Proof. The proof follows by application of the fundamental theorem of calculus. Let us do the proof for $p > 1$ for the infinite right endpoint and leave the rest to the reader. Hint: You should handle $p = 1$ separately.

Suppose $p > 1$. Then using the fundamental theorem,

$$\int_1^b \frac{1}{x^p}\, dx = \int_1^b x^{-p}\, dx = \frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \frac{-1}{(p-1)b^{p-1}} + \frac{1}{p-1}.$$

As $p > 1$, then $p - 1 > 0$. Take the limit as $b \to \infty$ to obtain that $\frac{1}{b^{p-1}}$ goes to 0. The result follows. $\square$

We state the following proposition on "tails" for just one type of improper integral, though the proof is straight forward and the same for other types of improper integrals.

Proposition 5.41. *Let $f\colon [a,\infty) \to \mathbb{R}$ be a function that is Riemann integrable on $[a,b]$ for all $b > a$. For every $b > a$, the integral $\int_b^\infty f$ converges if and only if $\int_a^\infty f$ converges, in which case*

$$\int_a^\infty f = \int_a^b f + \int_b^\infty f.$$

Proof. Let $c > b$. Then

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Taking the limit $c \to \infty$ finishes the proof. □

Nonnegative functions are easier to work with as the following proposition demonstrates. The exercises will show that this proposition holds only for nonnegative functions. Analogues of this proposition exist for all the other types of improper limits and are left to the student.

Proposition 5.42. *Suppose $f\colon [a,\infty) \to \mathbb{R}$ is nonnegative ($f(x) \geq 0$ for all x) and such that f is Riemann integrable on $[a,b]$ for all $b > a$.*

(i)

$$\int_a^\infty f = \sup \left\{ \int_a^x f : x \geq a \right\}.$$

(ii) Suppose $\{x_n\}$ is a sequence with $\lim x_n = \infty$. Then $\int_a^\infty f$ converges if and only if $\lim \int_a^{x_n} f$ exists, in which case

$$\int_a^\infty f = \lim_{n\to\infty} \int_a^{x_n} f.$$

In the first item we allow for the value of ∞ in the supremum indicating that the integral diverges to infinity.

Proof. We start with the first item. As f is nonnegative, $\int_a^x f$ is increasing as a function of x. If the supremum is infinite, then for every $M \in \mathbb{R}$ we find N such that $\int_a^N f \geq M$. As $\int_a^x f$ is increasing, $\int_a^x f \geq M$ for all $x \geq N$. So $\int_a^\infty f$ diverges to infinity.

Next suppose the supremum is finite, say $A := \sup \left\{ \int_a^x f : x \geq a \right\}$. For every $\epsilon > 0$, we find an N such that $A - \int_a^N f < \epsilon$. As $\int_a^x f$ is increasing, then $A - \int_a^x f < \epsilon$ for all $x \geq N$ and hence $\int_a^\infty f$ converges to A.

Let us look at the second item. If $\int_a^\infty f$ converges, then every sequence $\{x_n\}$ going to infinity works. The trick is proving the other direction. Suppose $\{x_n\}$ is such that $\lim x_n = \infty$ and

$$\lim_{n\to\infty} \int_a^{x_n} f = A$$

converges. Given $\epsilon > 0$, pick N such that for all $n \geq N$, we have $A - \epsilon < \int_a^{x_n} f < A + \epsilon$. Because $\int_a^x f$ is increasing as a function of x, we have that for all $x \geq x_N$

$$A - \epsilon < \int_a^{x_N} f \leq \int_a^x f.$$

As $\{x_n\}$ goes to ∞, then for any given x, there is an x_m such that $m \geq N$ and $x \leq x_m$. Then

$$\int_a^x f \leq \int_a^{x_m} f < A + \epsilon.$$

In particular, for all $x \geq x_N$, we have $\left|\int_a^x f - A\right| < \epsilon$. $\square$

Proposition 5.43 (Comparison test for improper integrals)**.** *Let $f\colon [a,\infty) \to \mathbb{R}$ and $g\colon [a,\infty) \to \mathbb{R}$ be functions that are Riemann integrable on $[a,b]$ for all $b > a$. Suppose that for all $x \geq a$,*

$$|f(x)| \leq g(x).$$

(i) If $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges, and in this case $\left|\int_a^\infty f\right| \leq \int_a^\infty g$.

(ii) If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Proof. We start with the first item. For every b and c, such that $a \leq b \leq c$, we have $-g(x) \leq f(x) \leq g(x)$, and so

$$\int_b^c -g \leq \int_b^c f \leq \int_b^c g.$$

In other words, $\left|\int_b^c f\right| \leq \int_b^c g$.

Let $\epsilon > 0$ be given. Because of Proposition 5.41,

$$\int_a^\infty g = \int_a^b g + \int_b^\infty g.$$

As $\int_a^b g$ goes to $\int_a^\infty g$ as b goes to infinity, $\int_b^\infty g$ goes to 0 as b goes to infinity. Choose B such that

$$\int_B^\infty g < \epsilon.$$

As g is nonnegative, if $B \leq b < c$, then $\int_b^c g < \epsilon$ as well. Let $\{x_n\}$ be a sequence going to infinity. Let M be such that $x_n \geq B$ for all $n \geq M$. Take $n, m \geq M$, with $x_n \leq x_m$,

$$\left|\int_a^{x_m} f - \int_a^{x_n} f\right| = \left|\int_{x_n}^{x_m} f\right| \leq \int_{x_n}^{x_m} g < \epsilon.$$

Therefore, the sequence $\{\int_a^{x_n} f\}_{n=1}^\infty$ is Cauchy and hence converges.

We need to show that the limit is unique. Suppose $\{x_n\}$ is a sequence converging to infinity such that $\{\int_a^{x_n} f\}$ converges to L_1, and $\{y_n\}$ is a sequence converging to infinity is such that $\{\int_a^{y_n} f\}$ converges to L_2. Then there must be some n such that $\left|\int_a^{x_n} f - L_1\right| < \epsilon$ and $\left|\int_a^{y_n} f - L_2\right| < \epsilon$. We can also suppose $x_n \geq B$ and $y_n \geq B$. Then

$$|L_1 - L_2| \leq \left|L_1 - \int_a^{x_n} f\right| + \left|\int_a^{x_n} f - \int_a^{y_n} f\right| + \left|\int_a^{y_n} f - L_2\right| < \epsilon + \left|\int_{x_n}^{y_n} f\right| + \epsilon < 3\epsilon.$$

As $\epsilon > 0$ was arbitrary, $L_1 = L_2$, and hence $\int_a^\infty f$ converges. Above we have shown that $\left|\int_a^c f\right| \leq \int_a^c g$ for all $c > a$. By taking the limit $c \to \infty$, the first item is proved.

The second item is simply a contrapositive of the first item. $\square$

Example 5.44: The improper integral

$$\int_0^\infty \frac{\sin(x^2)(x+2)}{x^3+1}\, dx$$

converges.

Proof: Observe we simply need to show that the integral converges when going from 1 to infinity. For $x \geq 1$ we obtain

$$\left| \frac{\sin(x^2)(x+2)}{x^3+1} \right| \leq \frac{x+2}{x^3+1} \leq \frac{x+2}{x^3} \leq \frac{x+2x}{x^3} \leq \frac{3}{x^2}.$$

Then

$$\int_1^\infty \frac{3}{x^2}\, dx = 3 \int_1^\infty \frac{1}{x^2}\, dx = 3.$$

So using the comparison test and the tail test, the original integral converges.

Example 5.45: You should be careful when doing formal manipulations with improper integrals. The integral

$$\int_2^\infty \frac{2}{x^2-1}\, dx$$

converges via the comparison test using $\frac{1}{x^2}$ again. However, if you succumb to the temptation to write

$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

and try to integrate each part separately, you will not succeed. It is *not* true that you can split the improper integral in two; you cannot split the limit.

$$\begin{aligned}
\int_2^\infty \frac{2}{x^2-1}\, dx &= \lim_{b\to\infty} \int_2^b \frac{2}{x^2-1}\, dx \\
&= \lim_{b\to\infty} \left(\int_2^b \frac{1}{x-1}\, dx - \int_2^b \frac{1}{x+1}\, dx \right) \\
&\neq \int_2^\infty \frac{1}{x-1}\, dx - \int_2^\infty \frac{1}{x+1}\, dx.
\end{aligned}$$

The last line in the computation does not even make sense. Both of the integrals there diverge to infinity, since we can apply the comparison test appropriately with $\frac{1}{x}$. We get $\infty - \infty$.

Now suppose we need to take limits at both endpoints.

Definition 5.46. Suppose $f\colon (a,b) \to \mathbb{R}$ is a function that is Riemann integrable on $[c,d]$ for all c, d such that $a < c < d < b$, then we define

$$\int_a^b f := \lim_{c\to a^+} \lim_{d\to b^-} \int_c^d f$$

if the limits exist.

Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that f is Riemann integrable on all bounded intervals $[a,b]$. Then we define

$$\int_{-\infty}^{\infty} f := \lim_{c \to -\infty} \lim_{d \to \infty} \int_c^d f$$

if the limits exist.

We similarly define improper integrals with one infinite and one finite improper endpoint, we leave this to the reader.

One ought to always be careful about double limits. The definition given above says that we first take the limit as d goes to b or ∞ for a fixed c, and then we take the limit in c. We will have to prove that in this case it does not matter which limit we compute first.

Example 5.47: Let us see an example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} \bigl(\arctan(b) - \arctan(a)\bigr) = \pi.$$

In the definition the order of the limits can always be switched if they exist. Let us prove this fact only for the infinite limits.

Proposition 5.48. *If $f\colon \mathbb{R} \to \mathbb{R}$ is a function integrable on every bounded interval $[a,b]$. Then*

$$\lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f \quad \textit{converges} \qquad \textit{if and only if} \qquad \lim_{b \to \infty} \lim_{a \to -\infty} \int_a^b f \quad \textit{converges,}$$

in which case the two expressions are equal. If either of the expressions converges, then the improper integral converges and

$$\lim_{a \to \infty} \int_{-a}^{a} f = \int_{-\infty}^{\infty} f.$$

Proof. Without loss of generality assume $a < 0$ and $b > 0$. Suppose the first expression converges. Then

$$\begin{aligned} \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f &= \lim_{a \to -\infty} \lim_{b \to \infty} \left(\int_a^0 f + \int_0^b f \right) = \left(\lim_{a \to -\infty} \int_a^0 f \right) + \left(\lim_{b \to \infty} \int_0^b f \right) \\ &= \lim_{b \to \infty} \left(\left(\lim_{a \to -\infty} \int_a^0 f \right) + \int_0^b f \right) = \lim_{b \to \infty} \lim_{a \to -\infty} \left(\int_a^0 f + \int_0^b f \right). \end{aligned}$$

Similar computation shows the other direction. Therefore, if either expression converges, then the improper integral converges and

$$\begin{aligned} \int_{-\infty}^{\infty} f = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f &= \left(\lim_{a \to -\infty} \int_a^0 f \right) + \left(\lim_{b \to \infty} \int_0^b f \right) \\ &= \left(\lim_{a \to \infty} \int_{-a}^0 f \right) + \left(\lim_{a \to \infty} \int_0^a f \right) = \lim_{a \to \infty} \left(\int_{-a}^0 f + \int_0^a f \right) = \lim_{a \to \infty} \int_{-a}^{a} f. \end{aligned}$$

□

Example 5.49: On the other hand, you must be careful to take the limits independently before you know convergence. Let $f(x) = \frac{x}{|x|}$ for $x \neq 0$ and $f(0) = 0$. If $a < 0$ and $b > 0$, then

$$\int_a^b f = \int_a^0 f + \int_0^b f = a + b.$$

For every fixed $a < 0$, the limit as $b \to \infty$ is infinite. So even the first limit does not exist, and the improper integral $\int_{-\infty}^{\infty} f$ does not converge. On the other hand, if $a > 0$, then

$$\int_{-a}^{a} f = (-a) + a = 0.$$

Therefore,

$$\lim_{a \to \infty} \int_{-a}^{a} f = 0.$$

Example 5.50: An example to keep in mind for improper integrals is the so-called *sinc function*[7]. This function comes up quite often in both pure and applied mathematics. Define

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

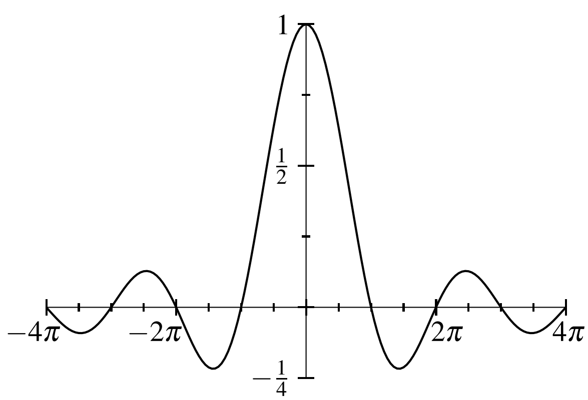

Figure 5.6: The sinc function.

It is not difficult to show that the sinc function is continuous at zero, but that is not important right now. What is important is that

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) \, dx = \pi, \qquad \text{while} \qquad \int_{-\infty}^{\infty} |\operatorname{sinc}(x)| \, dx = \infty.$$

The integral of the sinc function is a continuous analogue of the alternating harmonic series $\sum \frac{(-1)^n}{n}$, while the absolute value is like the regular harmonic series $\sum \frac{1}{n}$. In particular, the fact that the integral converges must be done directly rather than using comparison test.

[7] Shortened from Latin: *sinus cardinalis*

We will not prove the first statement exactly. Let us simply prove that the integral of the sinc function converges, but we will not worry about the exact limit. Because $\frac{\sin(-x)}{-x} = \frac{\sin(x)}{x}$, it is enough to show that

$$\int_{2\pi}^{\infty} \frac{\sin(x)}{x} \, dx$$

converges. We also avoid $x = 0$ this way to make our life simpler.

For every $n \in \mathbb{N}$, we have that for $x \in [\pi 2n, \pi(2n+1)]$,

$$\frac{\sin(x)}{\pi(2n+1)} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\pi 2n},$$

as $\sin(x) \geq 0$. For $x \in [\pi(2n+1), \pi(2n+2)]$,

$$\frac{\sin(x)}{\pi(2n+1)} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\pi(2n+2)},$$

as $\sin(x) \leq 0$.

Via the fundamental theorem of calculus,

$$\frac{2}{\pi(2n+1)} = \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{\pi(2n+1)} \, dx \leq \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{x} \, dx \leq \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{\pi 2n} \, dx = \frac{1}{\pi n}.$$

Similarly,

$$\frac{-2}{\pi(2n+1)} \leq \int_{\pi(2n+1)}^{\pi(2n+2)} \frac{\sin(x)}{x} \, dx \leq \frac{-1}{\pi(n+1)}.$$

Adding the two together we find

$$0 = \frac{2}{\pi(2n+1)} + \frac{-2}{\pi(2n+1)} \leq \int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x} \, dx \leq \frac{1}{\pi n} + \frac{-1}{\pi(n+1)} = \frac{1}{\pi n(n+1)}.$$

See Figure 5.7.

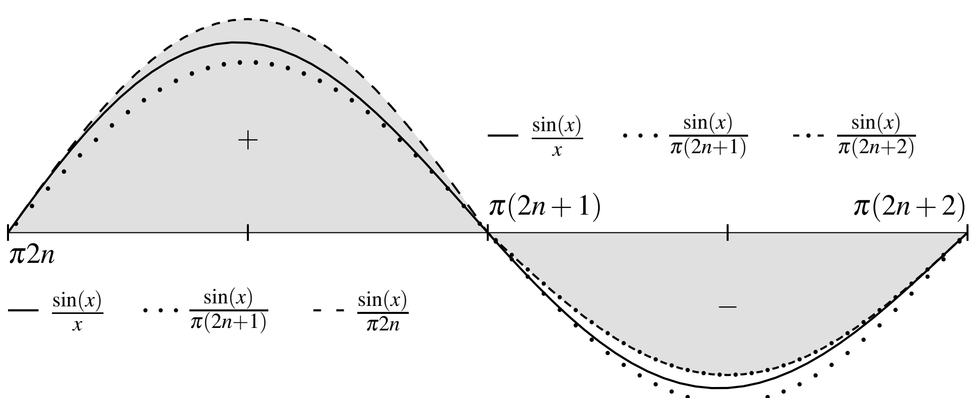

Figure 5.7: Bound of $\int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x} \, dx$ using the shaded integral (signed area $\frac{1}{\pi n} + \frac{-1}{\pi(n+1)}$).

For $k \in \mathbb{N}$,

$$\int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\, dx = \sum_{n=1}^{k-1} \int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x}\, dx \leq \sum_{n=1}^{k-1} \frac{1}{\pi n(n+1)}.$$

We find the partial sums of a series with positive terms. The series converges as $\sum \frac{1}{\pi n(n+1)}$ is a convergent series. Thus as a sequence,

$$\lim_{k\to\infty} \int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\, dx = L \leq \sum_{n=1}^{\infty} \frac{1}{\pi n(n+1)} < \infty.$$

Let $M > 2\pi$ be arbitrary, and let $k \in \mathbb{N}$ be the largest integer such that $2k\pi \leq M$. For $x \in [2k\pi, M]$, we have $\frac{-1}{2k\pi} \leq \frac{\sin(x)}{x} \leq \frac{1}{2k\pi}$, and so

$$\left| \int_{2k\pi}^{M} \frac{\sin(x)}{x}\, dx \right| \leq \frac{M - 2k\pi}{2k\pi} \leq \frac{1}{k}.$$

As k is the largest k such that $2k\pi \leq M$, then as $M \in \mathbb{R}$ goes to infinity, so does $k \in \mathbb{N}$.

Then

$$\int_{2\pi}^{M} \frac{\sin(x)}{x}\, dx = \int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\, dx + \int_{2k\pi}^{M} \frac{\sin(x)}{x}\, dx.$$

As M goes to infinity, the first term on the right-hand side goes to L, and the second term on the right-hand side goes to zero. Hence

$$\int_{2\pi}^{\infty} \frac{\sin(x)}{x}\, dx = L.$$

The double-sided integral of sinc also exists as noted above. We leave the other statement—that the integral of the absolute value of the sinc function diverges—as an exercise.

5.5.1 Integral test for series

The fundamental theorem of calculus can be used in proving a series is summable and to estimate its sum.

Proposition 5.51 (Integral test)**.** *Suppose $f\colon [k,\infty) \to \mathbb{R}$ is a decreasing nonnegative function where $k \in \mathbb{Z}$. Then*

$$\sum_{n=k}^{\infty} f(n) \quad \textit{converges} \qquad \textit{if and only if} \qquad \int_k^{\infty} f \quad \textit{converges.}$$

In this case

$$\int_k^{\infty} f \leq \sum_{n=k}^{\infty} f(n) \leq f(k) + \int_k^{\infty} f.$$

See Figure 5.8, for an illustration with $k = 1$. By Proposition 5.26, f is integrable on every interval $[k, b]$ for all $b > k$, so the statement of the theorem makes sense without additional hypotheses of integrability.

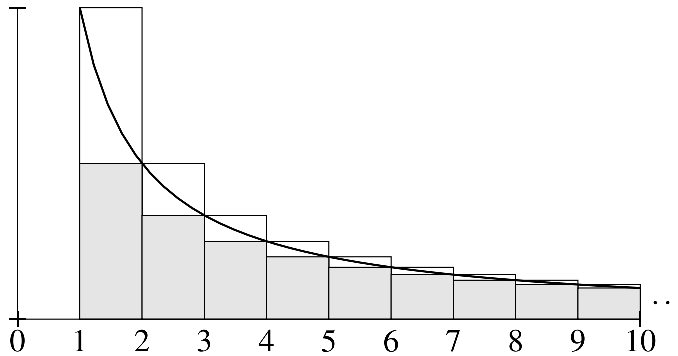

Figure 5.8: The area under the curve, $\int_1^\infty f$, is bounded below by the area of the shaded rectangles, $f(2)+f(3)+f(4)+\cdots$, and bounded above by the area entire rectangles, $f(1)+f(2)+f(3)+\cdots$.

Proof. Let $\ell, m \in \mathbb{Z}$ be such that $m > \ell \geq k$. Because f is decreasing, we have $\int_n^{n+1} f \leq f(n) \leq \int_{n-1}^n f$. Therefore,

$$\int_\ell^m f = \sum_{n=\ell}^{m-1} \int_n^{n+1} f \leq \sum_{n=\ell}^{m-1} f(n) \leq f(\ell) + \sum_{n=\ell+1}^{m-1} \int_{n-1}^{n} f \leq f(\ell) + \int_\ell^{m-1} f. \tag{5.3}$$

Suppose first that $\int_k^\infty f$ converges and let $\epsilon > 0$ be given. As before, since f is positive, then there exists an $L \in \mathbb{N}$ such that if $\ell \geq L$, then $\int_\ell^m f < \frac{\epsilon}{2}$ for all $m \geq \ell$. The function f must decrease to zero (why?), so make L large enough so that for $\ell \geq L$, we have $f(\ell) < \frac{\epsilon}{2}$. Thus, for $m > \ell \geq L$, we have via (5.3),

$$\sum_{n=\ell}^{m} f(n) \leq f(\ell) + \int_\ell^m f < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The series is therefore Cauchy and thus converges. The estimate in the proposition is obtained by letting m go to infinity in (5.3) with $\ell = k$.

Conversely, suppose $\int_k^\infty f$ diverges. As f is positive, then by Proposition 5.42, the sequence $\{\int_k^m f\}_{m=k}^\infty$ diverges to infinity. Using (5.3) with $\ell = k$, we find

$$\int_k^m f \leq \sum_{n=k}^{m-1} f(n).$$

As the left-hand side goes to infinity as $m \to \infty$, so does the right-hand side. □

Example 5.52: The integral test can be used not only to show that a series converges, but to estimate its sum to arbitrary precision. Let us show $\sum_{n=1}^\infty \frac{1}{n^2}$ exists and estimate its sum to within 0.01. As this series is the p-series for $p = 2$, we already proved it converges (let us pretend we do not know that), but we only roughly estimated its sum.

The fundamental theorem of calculus says that for all $k \in \mathbb{N}$,

$$\int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k}.$$

In particular, the series must converge. But we also have

$$\frac{1}{k} = \int_k^\infty \frac{1}{x^2}\,dx \leq \sum_{n=k}^\infty \frac{1}{n^2} \leq \frac{1}{k^2} + \int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k^2} + \frac{1}{k}.$$

Adding the partial sum up to $k-1$ we get

$$\frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2} \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq \frac{1}{k^2} + \frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2}.$$

In other words, $\frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2}$ is an estimate for the sum to within $\frac{1}{k^2}$. Therefore, if we wish to find the sum to within 0.01, we note $\frac{1}{10^2} = 0.01$. We obtain

$$1.6397\ldots \approx \frac{1}{10} + \sum_{n=1}^{9} \frac{1}{n^2} \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq \frac{1}{100} + \frac{1}{10} + \sum_{n=1}^{9} \frac{1}{n^2} \approx 1.6497\ldots.$$

The actual sum is $\frac{\pi^2}{6} \approx 1.6449\ldots$.

5.5.2 Exercises

***Exercise* 5.55:** *Finish the proof of Proposition 5.40.*

***Exercise* 5.56:** *Find out for which $a \in \mathbb{R}$ does $\sum\limits_{n=1}^\infty e^{an}$ converge. When the series converges, find an upper bound for the sum.*

***Exercise* 5.57:**

a) Estimate $\sum\limits_{n=1}^\infty \frac{1}{n(n+1)}$ correct to within 0.01 using the integral test.

b) Compute the limit of the series exactly and compare. Hint: The sum telescopes.

***Exercise* 5.58:** *Prove*

$$\int_{-\infty}^\infty \left\lvert \operatorname{sinc}(x) \right\rvert \,dx = \infty.$$

Hint: Again, it is enough to show this on just one side.

***Exercise* 5.59:** *Can you interpret*

$$\int_{-1}^1 \frac{1}{\sqrt{\lvert x \rvert}}\,dx$$

as an improper integral? If so, compute its value.

***Exercise* 5.60:** *Take $f \colon [0,\infty) \to \mathbb{R}$, Riemann integrable on every interval $[0,b]$, and such that there exist M, a, and T, such that $\lvert f(t) \rvert \leq M e^{at}$ for all $t \geq T$. Show that the* Laplace transform *of f exists. That is, for every $s > a$ the following integral converges:*

$$F(s) := \int_0^\infty f(t) e^{-st}\,dt.$$

***Exercise* 5.61:** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function on every interval $[a, b]$, and such that $\int_{-\infty}^{\infty} |f(x)| \ dx < \infty$. Show that the* Fourier sine and cosine transforms *exist. That is, for every $\omega \geq 0$ the following integrals converge*

$$F^s(\omega) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega t) \ dt, \qquad F^c(\omega) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t) \ dt.$$

Furthermore, show that F^s and F^c are bounded functions.

***Exercise* 5.62:** *Suppose $f\colon [0, \infty) \to \mathbb{R}$ is Riemann integrable on every interval $[0, b]$. Show that $\int_0^\infty f$ converges if and only if for every $\epsilon > 0$ there exists an M such that if $M \leq a < b$, then $\left|\int_a^b f\right| < \epsilon$.*

***Exercise* 5.63:** *Suppose $f\colon [0, \infty) \to \mathbb{R}$ is nonnegative and* decreasing. *Prove:*

a) If $\int_0^\infty f < \infty$, then $\lim\limits_{x \to \infty} f(x) = 0$.

b) The converse does not hold.

***Exercise* 5.64:** *Find an example of an* unbounded *continuous function $f\colon [0, \infty) \to \mathbb{R}$ that is nonnegative and such that $\int_0^\infty f < \infty$. Note that $\lim_{x \to \infty} f(x)$ will not exist; compare previous exercise. Hint: On each interval $[k, k + 1]$, $k \in \mathbb{N}$, define a function whose integral over this interval is less than say 2^{-k}.*

***Exercise* 5.65** (More challenging)**:** *Find an example of a function $f\colon [0, \infty) \to \mathbb{R}$ integrable on all intervals such that $\lim_{n \to \infty} \int_0^n f$ converges as a limit of a sequence (so $n \in \mathbb{N}$), but such that $\int_0^\infty f$ does not exist. Hint: For all $n \in \mathbb{N}$, divide $[n, n + 1]$ into two halves. On one half make the function negative, on the other make the function positive.*

***Exercise* 5.66:** *Suppose $f\colon [1, \infty) \to \mathbb{R}$ is such that $g(x) := x^2 f(x)$ is a bounded function. Prove that $\int_1^\infty f$ converges.*

It is sometimes desirable to assign a value to integrals that normally cannot be interpreted even as improper integrals, e.g. $\int_{-1}^{1} \frac{1}{x} \ dx$. Suppose $f\colon [a, b] \to \mathbb{R}$ is a function and $a < c < b$, where f is Riemann integrable on the intervals $[a, c - \epsilon]$ and $[c + \epsilon, b]$ for all $\epsilon > 0$. Define the *Cauchy principal value* of $\int_a^b f$ as

$$p.v.\!\int_a^b f := \lim_{\epsilon \to 0^+} \left(\int_a^{c-\epsilon} f + \int_{c+\epsilon}^{b} f \right),$$

if the limit exists.

***Exercise* 5.67:**

a) Compute $p.v.\!\int_{-1}^{1} \frac{1}{x} \ dx$.

b) Compute $\lim_{\epsilon \to 0^+} (\int_{-1}^{-\epsilon} \frac{1}{x} \ dx + \int_{2\epsilon}^{1} \frac{1}{x} \ dx)$ and show it is not equal to the principal value.

c) Show that if f is integrable on $[a, b]$, then $p.v.\!\int_a^b f = \int_a^b f$ (for an arbitrary $c \in (a, b)$).

d) Suppose $f\colon [-1, 1] \to \mathbb{R}$ is an odd function ($f(-x) = -f(x)$) that is integrable on $[-1, -\epsilon]$ and $[\epsilon, 1]$ for all $\epsilon > 0$. Prove that $p.v.\!\int_{-1}^{1} f = 0$

e) Suppose $f\colon [-1, 1] \to \mathbb{R}$ is continuous and differentiable at 0. Show that $p.v.\!\int_{-1}^{1} \frac{f(x)}{x} \ dx$ exists.

***Exercise* 5.68:** *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions, where $g(x) = 0$ for all $x \notin [a, b]$ for some interval $[a, b]$.*

a) Show that the convolution

$$(g * f)(x) := \int_{-\infty}^{\infty} f(t)g(x - t)\ dt$$

is well-defined for all $x \in \mathbb{R}$.

b) Suppose $\int_{-\infty}^{\infty} |f(x)|\ dx < \infty$. Prove that

$$\lim_{x\to-\infty} (g * f)(x) = 0, \qquad \textit{and} \qquad \lim_{x\to\infty} (g * f)(x) = 0.$$

Bibliography

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