Chapter 3

Continuous Functions

3.1 Limits of functions

Before we define continuity of functions, we must visit a somewhat more general notion of a limit. Given a function $f \colon S \to \mathbb{R}$, we want to see how $f(x)$ behaves as x tends to a certain point.

3.1.1 Cluster points

First, we return to a concept we have previously seen in an exercise. When moving within the set S we can only approach points that have elements of S arbitrarily near.

Definition 3.1. Let $S \subset \mathbb{R}$ be a set. A number $x \in \mathbb{R}$ is called a *cluster point* of S if for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$ is not empty.

That is, x is a cluster point of S if there are points of S arbitrarily close to x. Another way of phrasing the definition is to say that x is a cluster point of S if for every $\epsilon > 0$, there exists a $y \in S$ such that $y \neq x$ and $|x - y| < \epsilon$. Note that a cluster point of S need not lie in S.

Let us see some examples.

(i) The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ has a unique cluster point zero.

(ii) The cluster points of the open interval $(0, 1)$ are all points in the closed interval $[0, 1]$.

(iii) The set of cluster points of $\mathbb{Q}$ is the whole real line $\mathbb{R}$.

(iv) The set of cluster points of $[0, 1) \cup \{2\}$ is the interval $[0, 1]$.

(v) The set $\mathbb{N}$ has no cluster points in $\mathbb{R}$.

Proposition 3.2. *Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}$ such that $x_n \neq x$ and $x_n \in S$ for all n, and $\lim x_n = x$.*

Proof. First suppose x is a cluster point of S. For every $n \in \mathbb{N}$, pick x_n to be an arbitrary point of $(x - \frac{1}{n}, x + \frac{1}{n}) \cap S \setminus \{x\}$, which is nonempty because x is a cluster point of S. Then x_n is within $\frac{1}{n}$ of x, that is,

$$|x - x_n| < \frac{1}{n}.$$

As $\{\frac{1}{n}\}$ converges to zero, $\{x_n\}$ converges to x.

On the other hand, if we start with a sequence of numbers $\{x_n\}$ in S converging to x such that $x_n \neq x$ for all n, then for every $\epsilon > 0$ there is an M such that, in particular, $|x_M - x| < \epsilon$. That is, $x_M \in (x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$. □

3.1.2 Limits of functions

If a function f is defined on a set S and c is a cluster point of S, then we define the limit of $f(x)$ as x gets close to c. It is irrelevant for the definition whether f is defined at c or not. Even if the function is defined at c, the limit of the function as x goes to c can very well be different from $f(c)$.

Definition 3.3. Let $f\colon S \to \mathbb{R}$ be a function and c a cluster point of $S \subset \mathbb{R}$. Suppose there exists an $L \in \mathbb{R}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, we have

$$|f(x) - L| < \epsilon.$$

We then say $f(x)$ *converges* to L as x goes to c. We say L is the *limit* of $f(x)$ as x goes to c. We write

$$\lim_{x \to c} f(x) := L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to c.$$

If no such L exists, then we say that the limit does not exist or that f *diverges* at c.

Again the notation and language we are using above assumes the limit is unique even though we have not yet proved uniqueness. Let us do that now.

Proposition 3.4. *Let c be a cluster point of $S \subset \mathbb{R}$ and let $f\colon S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as x goes to c is unique.*

Proof. Let L_1 and L_2 be two numbers that both satisfy the definition. Take an $\epsilon > 0$ and find a $\delta_1 > 0$ such that $|f(x) - L_1| < \frac{\epsilon}{2}$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_1$. Also find $\delta_2 > 0$ such that $|f(x) - L_2| < \frac{\epsilon}{2}$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_2$. Put $\delta := \min\{\delta_1, \delta_2\}$. Suppose $x \in S$, $|x - c| < \delta$, and $x \neq c$. As $\delta > 0$ and c is a cluster point, such an x exists. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As $|L_1 - L_2| < \epsilon$ for arbitrary $\epsilon > 0$, then $L_1 = L_2$. □

Example 3.5: Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$. Then for any $c \in \mathbb{R}$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 = c^2.$$

Proof: Let $c \in \mathbb{R}$ be fixed, and suppose $\epsilon > 0$ is given. Write

$$\delta := \min \left\{ 1, \frac{\epsilon}{2|c|+1} \right\}.$$

Take $x \neq c$ such that $|x-c| < \delta$. In particular, $|x-c| < 1$. By reverse triangle inequality, we get

$$|x| - |c| \leq |x-c| < 1.$$

Adding $2|c|$ to both sides, we obtain $|x| + |c| < 2|c| + 1$. We compute

$$\begin{aligned} \left| f(x) - c^2 \right| &= \left| x^2 - c^2 \right| \\ &= |(x+c)(x-c)| \\ &= |x+c|\,|x-c| \\ &\leq (|x| + |c|)\,|x-c| \\ &< (2|c|+1)\,|x-c| \\ &< (2|c|+1)\frac{\epsilon}{2|c|+1} = \epsilon. \end{aligned}$$

Example 3.6: Define $f\colon [0,1) \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = 0,$$

even though $f(0) = 1$.

Proof: Let $\epsilon > 0$ be given. Let $\delta := \epsilon$. For $x \in [0,1)$, $x \neq 0$, and $|x-0| < \delta$, we get

$$|f(x) - 0| = |x| < \delta = \epsilon.$$

3.1.3 Sequential limits

Let us connect the limit as defined above with limits of sequences.

Lemma 3.7. *Let $S \subset \mathbb{R}$, let c be a cluster point of S, let $f\colon S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$.*

Then $f(x) \to L$ as $x \to c$ if and only if for every sequence $\{x_n\}$ of numbers such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim x_n = c$, we have that the sequence $\{f(x_n)\}$ converges to L.

Proof. Suppose $f(x) \to L$ as $x \to c$, and $\{x_n\}$ is a sequence such that $x_n \in S \setminus \{c\}$ and $\lim x_n = c$. We wish to show that $\{f(x_n)\}$ converges to L. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x-c| < \delta$, then $|f(x) - L| < \epsilon$. As $\{x_n\}$ converges to c, find an M such that for $n \geq M$, we have that $|x_n - c| < \delta$. Therefore, for $n \geq M$,

$$|f(x_n) - L| < \epsilon.$$

Thus $\{f(x_n)\}$ converges to L.

For the other direction, we use proof by contrapositive. Suppose it is not true that $f(x) \to L$ as $x \to c$. The negation of the definition is that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists an $x \in S \setminus \{c\}$, where $|x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

Let us use $\frac{1}{n}$ for δ in the statement above to construct a sequence $\{x_n\}$. We have that there exists an $\epsilon > 0$ such that for every n, there exists a point $x_n \in S \setminus \{c\}$, where $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$. The sequence $\{x_n\}$ just constructed converges to c, but the sequence $\{f(x_n)\}$ does not converge to L. And we are done. $\square$

It is possible to strengthen the reverse direction of the lemma by simply stating that $\{f(x_n)\}$ converges without requiring a specific limit. See Exercise 3.11.

Example 3.8: $\displaystyle\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist, but $\displaystyle\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$. See Figure 3.1.

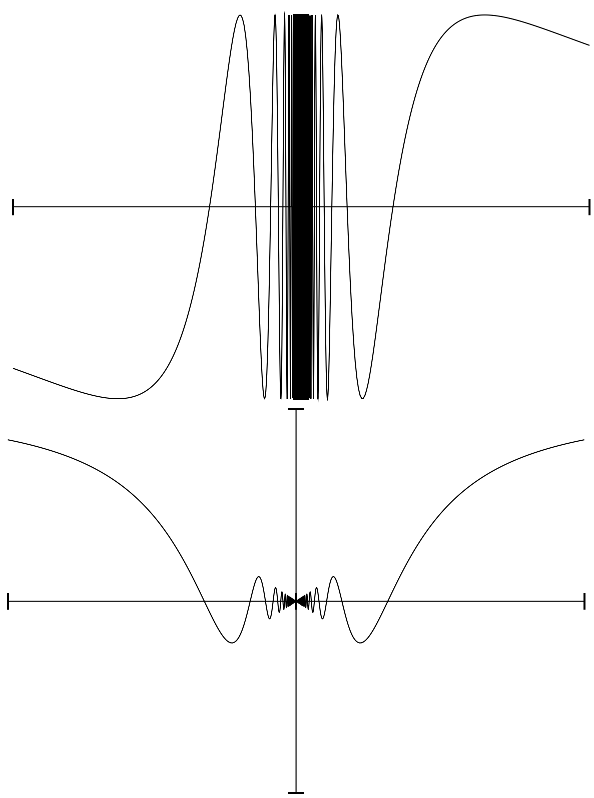

Figure 3.1: Graphs of $\sin(\frac{1}{x})$ and $x \sin(\frac{1}{x})$. Note that the computer cannot properly graph $\sin(\frac{1}{x})$ near zero as it oscillates too fast.

Proof: We start with $\sin(\frac{1}{x})$. Define a sequence by $x_n := \frac{1}{\pi n + \frac{\pi}{2}}$. It is not hard to see that $\lim x_n = 0$. Furthermore,

$$\sin(\frac{1}{x_n}) = \sin(\pi n + \frac{\pi}{2}) = (-1)^n.$$

Therefore, $\left\{\sin(\frac{1}{x_n})\right\}$ does not converge. By Lemma 3.7, $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

Now consider $x \sin(\frac{1}{x})$. Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n, and such that $\lim x_n = 0$. Notice that $\left|\sin(t)\right| \leq 1$ for all $t \in \mathbb{R}$. Therefore,

$$\left| x_n \sin(\frac{1}{x_n}) - 0 \right| = \left| x_n \right| \left| \sin(\frac{1}{x_n}) \right| \leq \left| x_n \right|.$$

As x_n goes to 0, then $\left|x_n\right|$ goes to zero, and hence $\left\{x_n \sin(\frac{1}{x_n})\right\}$ converges to zero. By Lemma 3.7, $\lim\limits_{x \to 0} x \sin(\frac{1}{x}) = 0$.

Keep in mind the phrase "for every sequence" in the lemma. For example, take $\sin(\frac{1}{x})$ and the sequence given by $x_n := \frac{1}{\pi n}$. Then $\left\{\sin(\frac{1}{x_n})\right\}$ is the constant zero sequence, and therefore converges to zero, but the limit of $\sin(\frac{1}{x})$ as $x \to 0$ does not exist.

Using Lemma 3.7, we can start applying everything we know about sequential limits to limits of functions. Let us give a few important examples.

Corollary 3.9. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f \colon S \to \mathbb{R}$ and $g \colon S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and*

$$f(x) \leq g(x) \qquad \text{for all } x \in S.$$

Then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Proof. Take $\{x_n\}$ be a sequence of numbers in $S \setminus \{c\}$ that converges to c. Let

$$L_1 := \lim_{x \to c} f(x), \qquad \text{and} \qquad L_2 := \lim_{x \to c} g(x).$$

Lemma 3.7 says that $\{f(x_n)\}$ converges to L_1 and $\{g(x_n)\}$ converges to L_2. We also have $f(x_n) \leq g(x_n)$. We obtain $L_1 \leq L_2$ using Lemma 2.21. □

By applying constant functions, we get the following corollary. The proof is left as an exercise.

Corollary 3.10. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f \colon S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that*

$$a \leq f(x) \leq b \qquad \text{for all } x \in S.$$

Then

$$a \leq \lim_{x \to c} f(x) \leq b.$$

Using Lemma 3.7 in the same way as above, we also get the following corollaries, whose proofs are again left as exercises.

Corollary 3.11. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f \colon S \to \mathbb{R}$, $g \colon S \to \mathbb{R}$, and $h \colon S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in S.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Corollary 3.12. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist. Then*

(i) $\lim_{x \to c} \bigl(f(x) + g(x)\bigr) = \left(\lim_{x \to c} f(x)\right) + \left(\lim_{x \to c} g(x)\right).$

(ii) $\lim_{x \to c} \bigl(f(x) - g(x)\bigr) = \left(\lim_{x \to c} f(x)\right) - \left(\lim_{x \to c} g(x)\right).$

(iii) $\lim_{x \to c} \bigl(f(x)g(x)\bigr) = \left(\lim_{x \to c} f(x)\right) \left(\lim_{x \to c} g(x)\right).$

(iv) If $\lim_{x \to c} g(x) \neq 0$, and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.$$

Corollary 3.13. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f\colon S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Then*

$$\lim_{x \to c} \lvert f(x) \rvert = \left\lvert \lim_{x \to c} f(x) \right\rvert.$$

3.1.4 Limits of restrictions and one-sided limits

Sometimes we work with the function defined on a subset.

Definition 3.14. Let $f\colon S \to \mathbb{R}$ be a function and $A \subset S$. Define the function $f|_A\colon A \to \mathbb{R}$ by

$$f|_A(x) := f(x) \qquad \text{for } x \in A.$$

The function $f|_A$ is called the *restriction* of f to A.

The function $f|_A$ is simply the function f taken on a smaller domain. The following proposition is the analogue of taking a tail of a sequence.

Proposition 3.15. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and let $f\colon S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ such that $(A \setminus \{c\}) \cap (c-\alpha, c+\alpha) = (S \setminus \{c\}) \cap (c-\alpha, c+\alpha)$.*

(i) The point c is a cluster point of A if and only if c is a cluster point of S.

(ii) Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A(x) \to L$ as $x \to c$.

Proof. First, let c be a cluster point of A. Since $A \subset S$, then if $(A \setminus \{c\}) \cap (c-\epsilon, c+\epsilon)$ is nonempty for every $\epsilon > 0$, then $(S \setminus \{c\}) \cap (c-\epsilon, c+\epsilon)$ is nonempty for every $\epsilon > 0$. Thus c is a cluster point of S. Second, suppose c is a cluster point of S. Then for $\epsilon > 0$ such that $\epsilon < \alpha$ we get that $(A \setminus \{c\}) \cap (c-\epsilon, c+\epsilon) = (S \setminus \{c\}) \cap (c-\epsilon, c+\epsilon)$, which is nonempty. This is true for all $\epsilon < \alpha$ and hence $(A \setminus \{c\}) \cap (c-\epsilon, c+\epsilon)$ must be nonempty for all $\epsilon > 0$. Thus c is a cluster point of A.

Now suppose c is a cluster point of S and $f(x) \to L$ as $x \to c$. That is, for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \epsilon$. Because $A \subset S$, if x is in $A \setminus \{c\}$, then x is in $S \setminus \{c\}$, and hence $f|_A(x) \to L$ as $x \to c$.

Finally suppose $f|_A(x) \to L$ as $x \to c$. For every $\epsilon > 0$ there is a $\delta' > 0$ such that if $x \in A \setminus \{c\}$ and $|x - c| < \delta'$, then $\bigl|f|_A(x) - L\bigr| < \epsilon$. Take $\delta := \min\{\delta', \alpha\}$. Now suppose $x \in S \setminus \{c\}$ and $|x - c| < \delta$. As $|x - c| < \alpha$, then $x \in A \setminus \{c\}$, and as $|x - c| < \delta'$, we have $|f(x) - L| = \bigl|f|_A(x) - L\bigr| < \epsilon$. □

The hypothesis of the proposition is necessary. For an arbitrary restriction we generally only get implication in only one direction, see Exercise 3.6.

The usual notation for the limit is

$$\lim_{\substack{x \to c \\ x \in A}} f(x) := \lim_{x \to c} f|_A(x).$$

The most common use of restriction with respect to limits are the *one-sided limits*[1].

Definition 3.16. Let $f \colon S \to \mathbb{R}$ be function and let c be a cluster point of $S \cap (c, \infty)$. Then if the limit of the restriction of f to $S \cap (c, \infty)$ as $x \to c$ exists, define

$$\lim_{x \to c^+} f(x) := \lim_{x \to c} f|_{S \cap (c,\infty)}(x).$$

Similarly, if c is a cluster point of $S \cap (-\infty, c)$ and the limit of the restriction as $x \to c$ exists, define

$$\lim_{x \to c^-} f(x) := \lim_{x \to c} f|_{S \cap (-\infty,c)}(x).$$

The proposition above does not apply to one-sided limits. It is possible to have one-sided limits, but no limit at a point. For example, define $f \colon \mathbb{R} \to \mathbb{R}$ by $f(x) := 1$ for $x < 0$ and $f(x) := 0$ for $x \geq 0$. We leave it to the reader to verify that

$$\lim_{x \to 0^-} f(x) = 1, \qquad \lim_{x \to 0^+} f(x) = 0, \qquad \lim_{x \to 0} f(x) \quad \text{does not exist.}$$

We have the following replacement.

Proposition 3.17. *Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f \colon S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and*

$$\lim_{x \to c} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

That is, a limit exists if both one-sided limits exist and are equal, and vice versa. The proof is a straightforward application of the definition of limit and is left as an exercise. The key point is that $\bigl(S \cap (-\infty, c)\bigr) \cup \bigl(S \cap (c, \infty)\bigr) = S \setminus \{c\}$.

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[1] There are a plethora of notations for one-sided limits. E.g. for $\lim\limits_{x \to c^-} f(x)$ one sees $\lim\limits_{\substack{x \to c \\ x < c}} f(x)$, $\lim\limits_{x \uparrow c} f(x)$, or $\lim\limits_{x \nearrow c} f(x)$.

3.1.5 Exercises

***Exercise* 3.1:** *Find the limit (and prove it of course) or prove that the limit does not exist*

a) $\lim\limits_{x\to c} \sqrt{x}$, *for* $c \geq 0$ b) $\lim\limits_{x\to c} x^2 + x + 1$, *for* $c \in \mathbb{R}$ c) $\lim\limits_{x\to 0} x^2 \cos(\frac{1}{x})$
d) $\lim\limits_{x\to 0} \sin(\frac{1}{x}) \cos(\frac{1}{x})$ e) $\lim\limits_{x\to 0} \sin(x) \cos(\frac{1}{x})$

***Exercise* 3.2:** *Prove Corollary 3.10.*

***Exercise* 3.3:** *Prove Corollary 3.11.*

***Exercise* 3.4:** *Prove Corollary 3.12.*

***Exercise* 3.5:** *Let* $A \subset S$*. Show that if* c *is a cluster point of* A*, then* c *is a cluster point of* S*. Note the difference from Proposition 3.15.*

***Exercise* 3.6:** *Let* $A \subset S$*. Suppose* c *is a cluster point of* A *and it is also a cluster point of* S*. Let* $f \colon S \to \mathbb{R}$ *be a function. Show that if* $f(x) \to L$ *as* $x \to c$*, then* $f|_A(x) \to L$ *as* $x \to c$*. Note the difference from Proposition 3.15.*

***Exercise* 3.7:** *Find an example of a function* $f \colon [-1,1] \to \mathbb{R}$*, where for* $A := [0,1]$*, we have* $f|_A(x) \to 0$ *as* $x \to 0$*, but the limit of* $f(x)$ *as* $x \to 0$ *does not exist. Note why you cannot apply Proposition 3.15.*

***Exercise* 3.8:** *Find example functions* f *and* g *such that the limit of neither* $f(x)$ *nor* $g(x)$ *exists as* $x \to 0$*, but such that the limit of* $f(x) + g(x)$ *exists as* $x \to 0$*.*

***Exercise* 3.9:** *Let* c_1 *be a cluster point of* $A \subset \mathbb{R}$ *and* c_2 *be a cluster point of* $B \subset \mathbb{R}$*. Suppose* $f \colon A \to B$ *and* $g \colon B \to \mathbb{R}$ *are functions such that* $f(x) \to c_2$ *as* $x \to c_1$ *and* $g(y) \to L$ *as* $y \to c_2$*. If* $c_2 \in B$*, also suppose that* $g(c_2) = L$*. Let* $h(x) := g\big(f(x)\big)$ *and show* $h(x) \to L$ *as* $x \to c_1$*. Hint: Note that* $f(x)$ *could equal* c_2 *for many* $x \in A$*, see also Exercise 3.14.*

***Exercise* 3.10** (note[2])**:** *Let* c *be a cluster point of* $A \subset \mathbb{R}$*, and* $f \colon A \to \mathbb{R}$ *be a function. Suppose for every sequence* $\{x_n\}$ *in* A*, such that* $\lim x_n = c$*, the sequence* $\{f(x_n)\}_{n=1}^{\infty}$ *is Cauchy. Prove that* $\lim_{x\to c} f(x)$ *exists.*

***Exercise* 3.11:** *Prove the following stronger version of one direction of Lemma 3.7: Let* $S \subset \mathbb{R}$*,* c *be a cluster point of* S*, and* $f \colon S \to \mathbb{R}$ *be a function. Suppose that for every sequence* $\{x_n\}$ *in* $S \setminus \{c\}$ *such that* $\lim x_n = c$ *the sequence* $\{f(x_n)\}$ *is convergent. Then show that the limit of* $f(x)$ *as* $x \to c$ *exists.*

***Exercise* 3.12:** *Prove Proposition 3.17.*

***Exercise* 3.13:** *Suppose* $S \subset \mathbb{R}$ *and* c *is a cluster point of* S*. Suppose* $f \colon S \to \mathbb{R}$ *is bounded. Show that there exists a sequence* $\{x_n\}$ *with* $x_n \in S \setminus \{c\}$ *and* $\lim x_n = c$ *such that* $\{f(x_n)\}$ *converges.*

***Exercise* 3.14** (Challenging)**:** *Show that the hypothesis that* $g(c_2) = L$ *in Exercise 3.9 is necessary. That is, find* f *and* g *such that* $f(x) \to c_2$ *as* $x \to c_1$ *and* $g(y) \to L$ *as* $y \to c_2$*, but* $g\big(f(x)\big)$ *does not go to* L *as* $x \to c_1$*.*

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[2] This exercise is almost identical to the next one. It will be replaced in the next major edition.

Exercise* 3.15*: *Show that the condition of being a cluster point is necessary to have a reasonable definition of a limit. That is, suppose c is not a cluster point of $S \subset \mathbb{R}$, and $f \colon S \to \mathbb{R}$ is a function. Show that every L would satisfy the definition of limit at c without the condition on c being a cluster point.*

Exercise* 3.16*:

a) Prove Corollary 3.13.

b) Find an example showing that the converse of the corollary does not hold.

3.2 Continuous functions

You undoubtedly heard of continuous functions in your schooling. A high-school criterion for this concept is that a function is continuous if we can draw its graph without lifting the pen from the paper. While that intuitive concept may be useful in simple situations, we require rigor. The following definition took three great mathematicians (Bolzano, Cauchy, and finally Weierstrass) to get correctly and its final form dates only to the late 1800s.

3.2.1 Definition and basic properties

Definition 3.18. Let $S \subset \mathbb{R}$, $c \in S$, and let $f : S \to \mathbb{R}$ be a function. We say that f is *continuous at* c if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

When $f : S \to \mathbb{R}$ is continuous at all $c \in S$, then we simply say f is a *continuous function*.

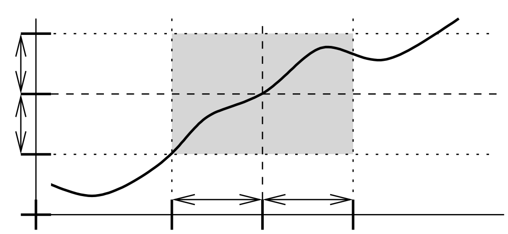

Figure 3.2: For $|x - c| < \delta$, the graph of $f(x)$ should be within the gray region.

If f is continuous for all $c \in A$, we say f is *continuous on* $A \subset S$. A straightforward exercise (Exercise 3.23) shows that this implies that $f|_A$ is continuous, although the converse does not hold.

Continuity may be the most important definition to understand in analysis, and it is not an easy one. See Figure 3.2. Note that δ not only depends on ϵ, but also on c; we need not pick one δ for all $c \in S$. It is no accident that the definition of continuity is similar to the definition of a limit of a function. The main feature of continuous functions is that these are precisely the functions that behave nicely with limits.

Proposition 3.19. *Consider a function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then:*

(i) If c is not a cluster point of S, then f is continuous at c.

(ii) If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and

$$\lim_{x \to c} f(x) = f(c).$$

(iii) The function f is continuous at c if and only if for every sequence $\{x_n\}$ where $x_n \in S$ and $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Proof. We start with the first item. Suppose c is not a cluster point of S. Then there exists a $\delta > 0$ such that $S \cap (c-\delta, c+\delta) = \{c\}$. For any $\epsilon > 0$, simply pick this given δ. The only $x \in S$ such that $|x-c| < \delta$ is $x = c$. Then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.

Let us move to the second item. Suppose c is a cluster point of S. Let us first suppose that $\lim_{x\to c} f(x) = f(c)$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Also $|f(c) - f(c)| = 0 < \epsilon$, so the definition of continuity at c is satisfied. On the other hand, suppose f is continuous at c. For every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x \in S$ where $|x-c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. Then the statement is, of course, still true if $x \in S \setminus \{c\} \subset S$. Therefore, $\lim_{x\to c} f(x) = f(c)$.

For the third item, first suppose f is continuous at c. Let $\{x_n\}$ be a sequence such that $x_n \in S$ and $\lim x_n = c$. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in S$ where $|x - c| < \delta$. Find an $M \in \mathbb{N}$ such that for $n \geq M$, we have $|x_n - c| < \delta$. Then for $n \geq M$, we have that $|f(x_n) - f(c)| < \epsilon$, so $\bigl\{f(x_n)\bigr\}$ converges to $f(c)$.

We prove the other direction of the third item by contrapositive. Suppose f is not continuous at c. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \in S$ such that $|x-c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$. Let us define a sequence $\{x_n\}$ as follows. Let $x_n \in S$ be such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \epsilon$. Now $\{x_n\}$ is a sequence of numbers in S such that $\lim x_n = c$ and such that $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ does not converge to $f(c)$. It may or may not converge, but it definitely does not converge to $f(c)$. □

The last item in the proposition is particularly powerful. It allows us to quickly apply what we know about limits of sequences to continuous functions and even to prove that certain functions are continuous. It can also be strengthened, see Exercise 3.29.

Example 3.20: The function $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is continuous.

Proof: Fix $c \in (0,\infty)$. Let $\{x_n\}$ be a sequence in $(0,\infty)$ such that $\lim x_n = c$. Then we know that

$$f(c) = \frac{1}{c} = \frac{1}{\lim x_n} = \lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} f(x_n).$$

Thus f is continuous at c. As f is continuous at all $c \in (0,\infty)$, f is continuous.

We have previously shown $\lim_{x\to c} x^2 = c^2$ directly. Therefore the function x^2 is continuous. We can use the continuity of algebraic operations with respect to limits of sequences, which we proved in the previous chapter, to prove a much more general result.

Proposition 3.21. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a* polynomial. *That is*

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

for some constants $a_0, a_1, \ldots, a_d$. Then f is continuous.

Proof. Fix $c \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $\lim x_n = c$. Then

$$\begin{aligned} f(c) &= a_d c^d + a_{d-1} c^{d-1} + \cdots + a_1 c + a_0 \\ &= a_d (\lim x_n)^d + a_{d-1} (\lim x_n)^{d-1} + \cdots + a_1 (\lim x_n) + a_0 \\ &= \lim_{n\to\infty} \left(a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_1 x_n + a_0 \right) = \lim_{n\to\infty} f(x_n). \end{aligned}$$

Thus f is continuous at c. As f is continuous at all $c \in \mathbb{R}$, f is continuous. □

By similar reasoning, or by appealing to Corollary 3.12, we can prove the following proposition. The proof is left as an exercise.

Proposition 3.22. *Let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be functions continuous at $c \in S$.*

(i) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continuous at c.

(ii) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x) - g(x)$ is continuous at c.

(iii) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c.

(iv) If $g(x) \neq 0$ for all $x \in S$, the function $h\colon S \to \mathbb{R}$ defined by $h(x) := \frac{f(x)}{g(x)}$ is continuous at c.

Example 3.23: The functions $\sin(x)$ and $\cos(x)$ are continuous. In the following computations we use the sum-to-product trigonometric identities. We also use the simple facts that $|\sin(x)| \leq |x|$, $|\cos(x)| \leq 1$, and $|\sin(x)| \leq 1$.

$$\begin{aligned}
|\sin(x) - \sin(c)| &= \left| 2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \\
&= 2\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\cos\left(\frac{x+c}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \\
&\leq 2\left|\frac{x-c}{2}\right| = |x-c|
\end{aligned}$$

$$\begin{aligned}
|\cos(x) - \cos(c)| &= \left| -2\sin\left(\frac{x-c}{2}\right)\sin\left(\frac{x+c}{2}\right)\right| \\
&= 2\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\sin\left(\frac{x+c}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \\
&\leq 2\left|\frac{x-c}{2}\right| = |x-c|
\end{aligned}$$

The claim that sin and cos are continuous follows by taking an arbitrary sequence $\{x_n\}$ converging to c, or by applying the definition of continuity directly. Details are left to the reader.

3.2.2 Composition of continuous functions

You probably already realized that one of the basic tools in constructing complicated functions out of simple ones is composition. Recall that for two functions f and g, the composition $f \circ g$ is defined by $(f \circ g)(x) := f\bigl(g(x)\bigr)$. A composition of continuous functions is again continuous.

Proposition 3.24. *Let $A, B \subset \mathbb{R}$ and $f\colon B \to \mathbb{R}$ and $g\colon A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g\colon A \to \mathbb{R}$ is continuous at c.*

Proof. Let $\{x_n\}$ be a sequence in A such that $\lim x_n = c$. As g is continuous at c, we have $\{g(x_n)\}$ converges to $g(c)$. As f is continuous at $g(c)$, we have $\{f(g(x_n))\}$ converges to $f(g(c))$. Thus $f \circ g$ is continuous at c. $\square$

Example 3.25: Claim: $\left(\sin(\frac{1}{x})\right)^2$ *is a continuous function on* $(0, \infty)$.

Proof: The function $\frac{1}{x}$ is continuous on $(0, \infty)$ and $\sin(x)$ is continuous on $(0, \infty)$ (actually on $\mathbb{R}$, but $(0, \infty)$ is the range for $\frac{1}{x}$). Hence the composition $\sin(\frac{1}{x})$ is continuous. Also, x^2 is continuous on the interval $(-1, 1)$ (the range of sin). Thus the composition $\left(\sin(\frac{1}{x})\right)^2$ is continuous on $(0, \infty)$.

3.2.3 Discontinuous functions

When f is not continuous at c, we say f is *discontinuous* at c, or that it has a *discontinuity* at c. The following proposition is a useful test and follows immediately from third item of Proposition 3.19.

Proposition 3.26. *Let* $f : S \to \mathbb{R}$ *be a function and* $c \in S$. *Suppose there exists a sequence* $\{x_n\}$, $x_n \in S$, *and* $\lim x_n = c$ *such that* $\{f(x_n)\}$ *does not converge to* $f(c)$. *Then* f *is discontinuous at* c.

Again, saying that $\{f(x_n)\}$ does not converge to $f(c)$ means that it either does not converge at all, or it converges to something other than $f(c)$.

Example 3.27: The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

is not continuous at 0.

Proof: Take the sequence $\{-\frac{1}{n}\}$, which converges to 0. Then $f(-\frac{1}{n}) = -1$ for every n, and so $\lim f(-\frac{1}{n}) = -1$, but $f(0) = 1$. Thus the function is not continuous at 0. See Figure 3.3.

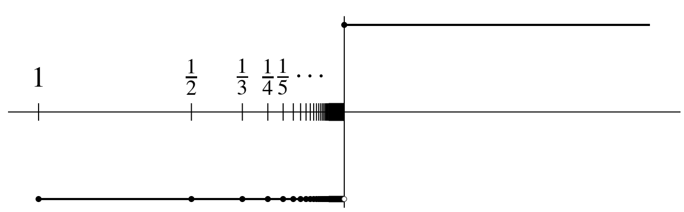

Figure 3.3: Graph of the jump discontinuity. The values of $f(-\frac{1}{n})$ and $f(0)$ are marked as black dots.

Notice that $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Hence, $\lim f(\frac{1}{n}) = f(0) = 1$. So $\{f(x_n)\}$ may converge to $f(0)$ for some specific sequence $\{x_n\}$ going to 0, despite the function being discontinuous at 0.

Finally, consider $f\left(\frac{(-1)^n}{n}\right) = (-1)^n$. This sequence diverges.

Example 3.28: For an extreme example, take the so-called *Dirichlet function*[3].

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

The function f is discontinuous at all $c \in \mathbb{R}$.

Proof: Suppose c is rational. Take a sequence $\{x_n\}$ of irrational numbers such that $\lim x_n = c$ (why can we?). Then $f(x_n) = 0$ and so $\lim f(x_n) = 0$, but $f(c) = 1$. If c is irrational, take a sequence of rational numbers $\{x_n\}$ that converges to c (why can we?). Then $\lim f(x_n) = 1$, but $f(c) = 0$.

Let us test the limits of our intuition. Can there exist a function continuous at all irrational numbers, but discontinuous at all rational numbers? There are rational numbers arbitrarily close to any irrational number. Perhaps strangely, the answer is yes, such a function exists. The following example is called the *Thomae function*[4] or the *popcorn function.*

Example 3.29: Define $f \colon (0,1) \to \mathbb{R}$ as

$$f(x) := \begin{cases} \frac{1}{k} & \text{if } x = \frac{m}{k}, \text{ where } m,k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

See the graph of f in Figure 3.4. We claim that f is continuous at all irrational c and discontinuous at all rational c.

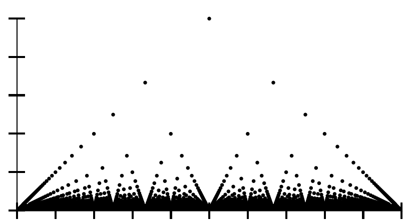

Figure 3.4: Graph of the "popcorn function."

Proof: Suppose $c = \frac{m}{k}$ is rational. Take a sequence of irrational numbers $\{x_n\}$ such that $\lim x_n = c$. Then $\lim f(x_n) = \lim 0 = 0$, but $f(c) = \frac{1}{k} \neq 0$. So f is discontinuous at c.

Now let c be irrational, so $f(c) = 0$. Take a sequence $\{x_n\}$ in $(0,1)$ such that $\lim x_n = c$. Given $\epsilon > 0$, find $K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$ by the Archimedean property. If $\frac{m}{k} \in (0,1)$ is in lowest terms (no common divisors), then $0 < m < k$. So there are only finitely many rational numbers in $(0,1)$ whose denominator k in lowest terms is less than K. As $\lim x_n = c$, every number not equal to c can appear at most finitely many times in $\{x_n\}$. Hence, there is an M such that for $n \geq M$, all the numbers x_n that are rational have a denominator larger than or equal to K. Thus for $n \geq M$,

$$|f(x_n) - 0| = f(x_n) \leq \frac{1}{K} < \epsilon.$$

Therefore, f is continuous at irrational c.

[3]Named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).
[4]Named after the German mathematician Carl Johannes Thomae (1840–1921).

Let us end on an easier example.

Example 3.30: Define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) := 0$ if $x \neq 0$ and $g(0) := 1$. Then g is not continuous at zero, but continuous everywhere else (why?). The point $x = 0$ is called a *removable discontinuity*. That is because if we would change the definition of g, by insisting that $g(0)$ be 0, we would obtain a continuous function. On the other hand, let f be the function of Example 3.27. Then f does not have a removable discontinuity at 0. No matter how we would define $f(0)$ the function would still fail to be continuous. The difference is that $\lim_{x\to 0} g(x)$ exists while $\lim_{x\to 0} f(x)$ does not.

We stay with this example to show another phenomenon. Let $A := \{0\}$, then $g|_A$ is continuous (why?), while g is not continuous on A. Similarly, if $B := \mathbb{R} \setminus \{0\}$, then $g|_B$ is also continuous, and g is in fact continuous on B.

3.2.4 Exercises

***Exercise* 3.17:** *Using the definition of continuity directly prove that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.*

***Exercise* 3.18:** *Using the definition of continuity directly prove that $f: (0,\infty) \to \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is continuous.*

***Exercise* 3.19:** *Let $f: \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

***Exercise* 3.20:** *Let $f: \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

***Exercise* 3.21:** *Let $f: \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) := \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

***Exercise* 3.22:** *Prove Proposition 3.22.*

***Exercise* 3.23:** *Prove the following statement. Let $S \subset \mathbb{R}$ and $A \subset S$. Let $f: S \to \mathbb{R}$ be a continuous function. Then the restriction $f|_A$ is continuous.*

***Exercise* 3.24:** *Suppose $S \subset \mathbb{R}$, such that $(c-\alpha, c+\alpha) \subset S$ for some $c \in \mathbb{R}$ and $\alpha > 0$. Let $f: S \to \mathbb{R}$ be a function and $A := (c-\alpha, c+\alpha)$. Prove that if $f|_A$ is continuous at c, then f is continuous at c.*

***Exercise* 3.25:** *Give an example of functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ such that the function h defined by $h(x) := f(x) + g(x)$ is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?*

Exercise* 3.26*: *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all x.*

Exercise* 3.27*: *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$, we have $f(x) > 0$.*

Exercise* 3.28*: *Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.*

Exercise* 3.29*: *Let $f\colon S \to \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}$ in S with $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges. Show that f is continuous at c.*

Exercise* 3.30*: *Suppose $f\colon [-1,0] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ are continuous and $f(0) = g(0)$. Define $h\colon [-1,1] \to \mathbb{R}$ by $h(x) := f(x)$ if $x \leq 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.*

Exercise* 3.31*: *Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f\colon \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x - y)$ for all x and y. Show that f is continuous.*

Exercise* 3.32** (Challenging):*** *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at 0 and such that $f(x + y) = f(x) + f(y)$ for every x and y. Show that $f(x) = ax$ for some $a \in \mathbb{R}$. Hint: Show that $f(nx) = nf(x)$, then show f is continuous on $\mathbb{R}$. Then show that $\frac{f(x)}{x} = f(1)$ for all rational x.*

Exercise* 3.33*: *Suppose $S \subset \mathbb{R}$ and let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be continuous functions. Define $p\colon S \to \mathbb{R}$ by $p(x) := \max\{f(x), g(x)\}$ and $q\colon S \to \mathbb{R}$ by $q(x) := \min\{f(x), g(x)\}$. Prove that p and q are continuous.*

Exercise* 3.34*: *Suppose $f\colon [-1,1] \to \mathbb{R}$ is a function continuous at all $x \in [-1,1] \setminus \{0\}$. Show that for every ϵ such that $0 < \epsilon < 1$, there exists a function $g\colon [-1,1] \to \mathbb{R}$ continuous on all of $[-1,1]$, such that $f(x) = g(x)$ for all $x \in [-1,-\epsilon] \cup [\epsilon, 1]$, and $|g(x)| \leq |f(x)|$ for all $x \in [-1,1]$.*

Exercise* 3.35** (Challenging):*** *A function $f\colon I \to \mathbb{R}$ is* convex *if whenever $a \leq x \leq b$ for a, x, b in I, we have $f(x) \leq f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}$. In other words, if the line drawn between $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$ is above the graph of f.*

a) Prove that if $I = (\alpha, \beta)$ an open interval and $f\colon I \to \mathbb{R}$ is convex, then f is continuous.

b) Find an example of a convex $f\colon [0,1] \to \mathbb{R}$ which is not continuous.

3.3 Min-max and intermediate value theorems

Continuous functions on closed and bounded intervals are quite well behaved.

3.3.1 Min-max or extreme value theorem

Recall a function $f: [a, b] \to \mathbb{R}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$. We have the following lemma.

Lemma 3.31. *A continuous function $f: [a, b] \to \mathbb{R}$ is bounded.*

Proof. Let us prove this claim by contrapositive. Suppose f is not bounded. Then for each $n \in \mathbb{N}$, there is an $x_n \in [a, b]$, such that

$$|f(x_n)| \geq n.$$

The sequence $\{x_n\}$ is bounded as $a \leq x_n \leq b$. By the Bolzano–Weierstrass theorem, there is a convergent subsequence $\{x_{n_i}\}$. Let $x := \lim x_{n_i}$. Since $a \leq x_{n_i} \leq b$ for all i, then $a \leq x \leq b$. The sequence $\{f(x_{n_i})\}$ is not bounded as $|f(x_{n_i})| \geq n_i \geq i$. Thus f is not continuous at x as

$$f(x) = f\Big(\lim_{i\to\infty} x_{n_i}\Big), \qquad \text{but} \qquad \lim_{i\to\infty} f(x_{n_i}) \text{ does not exist.} \qquad \square$$

Notice a key point of the proof. Boundedness of $[a, b]$ allows us to use Bolzano–Weierstrass, while the fact that it is closed gives us that the limit is back in $[a, b]$. The technique is a common one: Find a sequence with a certain property, then use Bolzano–Weierstrass to make such a sequence that also converges.

Recall from calculus that $f: S \to \mathbb{R}$ achieves an *absolute minimum* at $c \in S$ if

$$f(x) \geq f(c) \qquad \text{for all } x \in S.$$

On the other hand, f achieves an *absolute maximum* at $c \in S$ if

$$f(x) \leq f(c) \qquad \text{for all } x \in S.$$

If such a $c \in S$ exists, then f *achieves an absolute minimum (resp. absolute maximum) on* S.

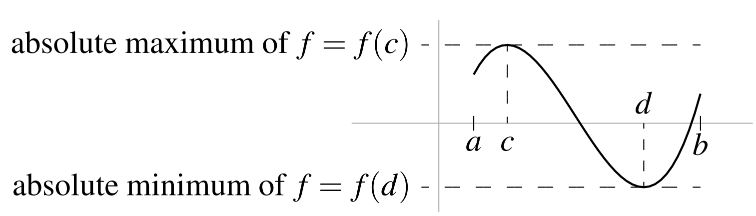

Figure 3.5: $f: [a, b] \to \mathbb{R}$ achieves an absolute maximum $f(c)$ at c, and an absolute minimum $f(d)$ at d.

If S is a closed and bounded interval, then a continuous f must achieve an absolute minimum and an absolute maximum on S.

Theorem 3.32 (Minimum-maximum theorem / Extreme value theorem)**.** *A continuous function $f\colon [a,b] \to \mathbb{R}$ on a closed and bounded interval $[a,b]$ achieves both an absolute minimum and an absolute maximum on $[a,b]$.*

Proof. The lemma says that f is bounded, so the set $f\big([a,b]\big) = \big\{ f(x) : x \in [a,b] \big\}$ has a supremum and an infimum. There exist sequences in the set $f\big([a,b]\big)$ that approach its supremum and its infimum. That is, there are sequences $\big\{ f(x_n) \big\}$ and $\big\{ f(y_n) \big\}$, where x_n and y_n are in $[a,b]$, such that

$$\lim_{n\to\infty} f(x_n) = \inf f\big([a,b]\big) \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = \sup f\big([a,b]\big).$$

We are not done yet; we need to find where the minima and the maxima are. The problem is that the sequences $\{x_n\}$ and $\{y_n\}$ need not converge. We know $\{x_n\}$ and $\{y_n\}$ are bounded (their elements belong to a bounded interval $[a,b]$). Apply the Bolzano–Weierstrass theorem, to find convergent subsequences $\{x_{n_i}\}$ and $\{y_{m_i}\}$. Let

$$x := \lim_{i\to\infty} x_{n_i} \qquad \text{and} \qquad y := \lim_{i\to\infty} y_{m_i}.$$

As $a \leq x_{n_i} \leq b$ for all i, we have $a \leq x \leq b$. Similarly, $a \leq y \leq b$. So x and y are in $[a,b]$. A limit of a subsequence is the same as the limit of the sequence, and we can take a limit past the continuous function f:

$$\inf f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i}) = f\Big(\lim_{i\to\infty} x_{n_i} \Big) = f(x).$$

Similarly,

$$\sup f([a,b]) = \lim_{n\to\infty} f(y_n) = \lim_{i\to\infty} f(y_{m_i}) = f\Big(\lim_{i\to\infty} y_{m_i} \Big) = f(y).$$

Therefore, f achieves an absolute minimum at x and f achieves an absolute maximum at y. ☐

Example 3.33: The function $f(x) := x^2 + 1$ defined on the interval $[-1,2]$ achieves a minimum at $x = 0$ when $f(0) = 1$. It achieves a maximum at $x = 2$ where $f(2) = 5$. Do note that the domain of definition matters. If we instead took the domain to be $[-10,10]$, then $x = 2$ would no longer be a maximum of f. Instead the maximum would be achieved at either $x = 10$ or $x = -10$.

We show by examples that the different hypotheses of the theorem are truly needed.

Example 3.34: The function $f(x) := x$, defined on the whole real line, achieves neither a minimum, nor a maximum. So it is important that we are looking at a bounded interval.

Example 3.35: The function $f(x) := \frac{1}{x}$, defined on $(0,1)$ achieves neither a minimum, nor a maximum. The values of the function are unbounded as we approach 0. Also as we approach $x = 1$, the values of the function approach 1, but $f(x) > 1$ for all $x \in (0,1)$. There is no $x \in (0,1)$ such that $f(x) = 1$. So it is important that we are looking at a closed interval.

Example 3.36: Continuity is important. Define $f\colon [0,1] \to \mathbb{R}$ by $f(x) := \frac{1}{x}$ for $x > 0$ and let $f(0) := 0$. The function does not achieve a maximum. The problem is that the function is not continuous at 0.

3.3.2 Bolzano's intermediate value theorem

Bolzano's intermediate value theorem is one of the cornerstones of analysis. It is sometimes only called the intermediate value theorem, or just Bolzano's theorem. To prove Bolzano's theorem we prove the following simpler lemma.

Lemma 3.37. *Let $f \colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a,b)$ such that $f(c) = 0$.*

Proof. We define two sequences $\{a_n\}$ and $\{b_n\}$ inductively:

(i) Let $a_1 := a$ and $b_1 := b$.

(ii) If $f\left(\frac{a_n+b_n}{2}\right) \geq 0$, let $a_{n+1} := a_n$ and $b_{n+1} := \frac{a_n+b_n}{2}$.

(iii) If $f\left(\frac{a_n+b_n}{2}\right) < 0$, let $a_{n+1} := \frac{a_n+b_n}{2}$ and $b_{n+1} := b_n$.

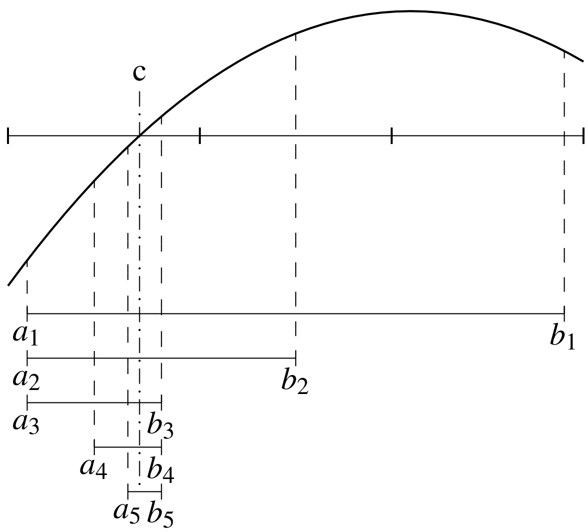

Figure 3.6: Finding roots (bisection method).

See Figure 3.6 for an example defining the first five steps. If $a_n < b_n$, then $a_n < \frac{a_n+b_n}{2} < b_n$. So $a_{n+1} < b_{n+1}$. Thus by induction $a_n < b_n$ for all n. Furthermore, $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n, that is, the sequences are monotone. As $a_n < b_n \leq b_1 = b$ and $b_n > a_n \geq a_1 = a$ for all n, the sequences are also bounded. Therefore, the sequences converge. Let $c := \lim\, a_n$ and $d := \lim\, b_n$, where also $a \leq c \leq d \leq b$. We need to show that $c = d$. Notice

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}.$$

By induction,

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = 2^{1-n}(b-a).$$

As $2^{1-n}(b-a)$ converges to zero, we take the limit as n goes to infinity to get

$$d - c = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} 2^{1-n}(b-a) = 0.$$

In other words, $d = c$.

By construction, for all n, we have

$$f(a_n) < 0 \qquad \text{and} \qquad f(b_n) \geq 0.$$

Since $\lim a_n = \lim b_n = c$ and as f is continuous, we may take limits in those inequalities:

$$f(c) = \lim f(a_n) \leq 0 \qquad \text{and} \qquad f(c) = \lim f(b_n) \geq 0.$$

As $f(c) \geq 0$ and $f(c) \leq 0$, we conclude $f(c) = 0$. Thus also $c \neq a$ and $c \neq b$, so $a < c < b$. ☐

Theorem 3.38 (Bolzano's intermediate value theorem)**.** *Let $f \colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists a $c \in (a,b)$ such that $f(c) = y$.*

The theorem says that a continuous function on a closed interval achieves all the values between the values at the endpoints.

Proof. If $f(a) < y < f(b)$, then define $g(x) := f(x) - y$. Then $g(a) < 0$ and $g(b) > 0$, and we apply Lemma 3.37 to g to find c. If $g(c) = 0$, then $f(c) = y$.

Similarly, if $f(a) > y > f(b)$, then define $g(x) := y - f(x)$. Again, $g(a) < 0$ and $g(b) > 0$, and we apply Lemma 3.37 to find c. As before, if $g(c) = 0$, then $f(c) = y$. ☐

If a function is continuous, then the restriction to a subset is continuous; if $f \colon S \to \mathbb{R}$ is continuous and $[a,b] \subset S$, then $f|_{[a,b]}$ is also continuous. We generally apply the theorem to a function continuous on some large set S, but we restrict our attention to an interval.

The proof of the lemma tells us how to find the root c. The proof is not only useful for us pure mathematicians, it is a useful idea in applied mathematics, where it is called the *bisection method*.

Example 3.39 (Bisection method)**:** The polynomial $f(x) := x^3 - 2x^2 + x - 1$ has a real root in $(1,2)$. We simply notice that $f(1) = -1$ and $f(2) = 1$. Hence there must exist a point $c \in (1,2)$ such that $f(c) = 0$. To find a better approximation of the root we follow the proof of Lemma 3.37. We look at 1.5 and find that $f(1.5) = -0.625$. Therefore, there is a root of the polynomial in $(1.5, 2)$. Next we look at 1.75 and note that $f(1.75) \approx -0.016$. Hence there is a root of f in $(1.75, 2)$. Next we look at 1.875 and find that $f(1.875) \approx 0.44$, thus there is a root in $(1.75, 1.875)$. We follow this procedure until we gain sufficient precision. In fact, the root is at $c \approx 1.7549$.

The technique above is the simplest method of finding roots of polynomials, which is perhaps the most common problem in applied mathematics. In general, finding roots is hard to do quickly, precisely, and automatically. There are other, faster methods of finding roots of polynomials, such as Newton's method. One advantage of the method above is its simplicity. The moment we find an initial interval where the intermediate value theorem applies, we are guaranteed to find a root up to a desired precision in finitely many steps. Furthermore, the bisection method finds roots of any continuous function, not just a polynomial.

The theorem guarantees at least one c such that $f(c) = y$, but there may be many different roots of the equation $f(c) = y$. If we follow the procedure of the proof, we are guaranteed to find approximations to one such root. We need to work harder to find any other roots.

Polynomials of even degree may not have any real roots. There is no real number x such that $x^2 + 1 = 0$. Odd polynomials, on the other hand, always have at least one real root.

Proposition 3.40. *Let $f(x)$ be a polynomial of odd degree. Then f has a real root.*

Proof. Suppose f is a polynomial of odd degree d. We write

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

where $a_d \neq 0$. We divide by a_d to obtain a *monic polynomial*[5]

$$g(x) := x^d + b_{d-1} x^{d-1} + \cdots + b_1 x + b_0,$$

where $b_k = \frac{a_k}{a_d}$. Let us show that $g(n)$ is positive for some large $n \in \mathbb{N}$. We first compare the highest order term with the rest:

$$\begin{aligned} \left| \frac{b_{d-1} n^{d-1} + \cdots + b_1 n + b_0}{n^d} \right| &= \frac{\left| b_{d-1} n^{d-1} + \cdots + b_1 n + b_0 \right|}{n^d} \\ &\leq \frac{\left| b_{d-1} \right| n^{d-1} + \cdots + \left| b_1 \right| n + \left| b_0 \right|}{n^d} \\ &\leq \frac{\left| b_{d-1} \right| n^{d-1} + \cdots + \left| b_1 \right| n^{d-1} + \left| b_0 \right| n^{d-1}}{n^d} \\ &= \frac{n^{d-1} \big(\left| b_{d-1} \right| + \cdots + \left| b_1 \right| + \left| b_0 \right| \big)}{n^d} \\ &= \frac{1}{n} \big(\left| b_{d-1} \right| + \cdots + \left| b_1 \right| + \left| b_0 \right| \big). \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} \frac{b_{d-1} n^{d-1} + \cdots + b_1 n + b_0}{n^d} = 0.$$

Thus there exists an $M \in \mathbb{N}$ such that

$$\left| \frac{b_{d-1} M^{d-1} + \cdots + b_1 M + b_0}{M^d} \right| < 1,$$

which implies

$$-(b_{d-1} M^{d-1} + \cdots + b_1 M + b_0) < M^d.$$

Therefore, $g(M) > 0$.

Next, consider $g(-n)$ for $n \in \mathbb{N}$. By a similar argument, there exists a $K \in \mathbb{N}$ such that $b_{d-1}(-K)^{d-1} + \cdots + b_1(-K) + b_0 < K^d$ and therefore $g(-K) < 0$ (see Exercise 3.40). In the proof, make sure you use the fact that d is odd. In particular, if d is odd, then $(-n)^d = -(n^d)$.

We appeal to the intermediate value theorem to find a $c \in [-K, M]$, such that $g(c) = 0$. As $g(x) = \frac{f(x)}{a_d}$, then $f(c) = 0$, and the proof is done. □

[5]The word *monic* means that the coefficient of x^d is 1.

Example 3.41: Interestingly, there exist discontinuous functions with the intermediate value property. The function

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0; however, f has the intermediate value property: Whenever $a < b$ and y is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, there exists a c such that $f(y) = c$. Proof is left as Exercise 3.39.

The intermediate value theorem says that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then $f\bigl([a,b]\bigr)$ contains all the values between $f(a)$ and $f(b)$. In fact, more is true. Combining all the results of this section one can prove the following useful corollary whose proof is left as an exercise.

Corollary 3.42. *If $f\colon [a,b] \to \mathbb{R}$ is continuous, then the direct image $f([a,b])$ is a closed and bounded interval or a single number.*

3.3.3 Exercises

Exercise* 3.36*: *Find an example of a discontinuous function $f\colon [0,1] \to \mathbb{R}$ where the conclusion of the intermediate value theorem fails.*

Exercise* 3.37*: *Find an example of a* bounded *discontinuous function $f\colon [0,1] \to \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.*

Exercise* 3.38*: *Let $f\colon (0,1) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).*

Exercise* 3.39*: *Let*

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

Show that f has the intermediate value property. That is, whenever $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there exists a $c \in (a,b)$ such that $f(c) = y$.

Exercise* 3.40*: *Suppose $g(x)$ is a monic polynomial of odd degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Show that there exists a $K \in \mathbb{N}$ such that $g(-K) < 0$. Hint: Make sure to use the fact that d is odd. You will have to use that $(-n)^d = -(n^d)$.

Exercise* 3.41*: *Suppose $g(x)$ is a monic polynomial of positive even degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Suppose $g(0) < 0$. Show that g has at least two distinct real roots.

Exercise* 3.42*: *Prove Corollary 3.42: Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function. Prove that the direct image $f([a,b])$ is a closed and bounded interval or a single number.*

Exercise* 3.43*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period $P > 0$. That is, $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. Show that f achieves an absolute minimum and an absolute maximum.*

Exercise* 3.44** (Challenging):*** *Suppose $f(x)$ is a bounded polynomial, in other words, there is an M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f must be a constant.*

Exercise* 3.45*: *Suppose $f\colon [0,1] \to [0,1]$ is continuous. Show that f has a fixed point, in other words, show that there exists an $x \in [0,1]$ such that $f(x) = x$.*

Exercise* 3.46*: *Find an example of a continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$ that does not achieve an absolute minimum nor an absolute maximum on $\mathbb{R}$.*

Exercise* 3.47*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $x \leq f(x) \leq x+1$ for all $x \in \mathbb{R}$. Find $f(\mathbb{R})$.*

Exercise* 3.48*: *True/False, prove or find a counterexample. If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f|_{\mathbb{Z}}$ is bounded, then f is bounded.*

Exercise* 3.49*: *Suppose $f\colon [0,1] \to (0,1)$ is a bijection. Prove that f is not continuous.*

Exercise* 3.50*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.*

a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

b) Find a continuous function f such that $f(\mathbb{R}) = \mathbb{R}$, but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

Exercise* 3.51*: *Suppose $g(x)$ is a monic polynomial of even degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Show that g achieves an absolute minimum on $\mathbb{R}$.

Exercise* 3.52*: *Suppose $f(x)$ is a polynomial of degree d and $f(\mathbb{R}) = \mathbb{R}$. Show that d is odd.*

3.4 Uniform continuity

3.4.1 Uniform continuity

We made a fuss of saying that the δ in the definition of continuity depended on the point c. There are situations when it is advantageous to have a δ independent of any point, and so we give a name to this concept.

Definition 3.43. Let $S \subset \mathbb{R}$, and let $f \colon S \to \mathbb{R}$ be a function. Suppose for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Then we say f is *uniformly continuous*.

A uniformly continuous function must be continuous. The only difference in the definitions is that in uniform continuity, for a given $\epsilon > 0$ we pick a $\delta > 0$ that works for all $c \in S$. That is, δ can no longer depend on c, it only depends on ϵ. The domain of definition of the function makes a difference now. A function that is not uniformly continuous on a larger set, may be uniformly continuous when restricted to a smaller set. We will say *uniformly continuous on* X to mean that f restricted to X is uniformly continuous, or perhaps to just emphasize the domain. Note that x and c are not treated any differently in this definition.

Example 3.44: $f \colon [0,1] \to \mathbb{R}$, defined by $f(x) := x^2$ is uniformly continuous.

Proof: Note that $0 \leq x, c \leq 1$. Then

$$\left|x^2 - c^2\right| = |x + c|\,|x - c| \leq (|x| + |c|)\,|x - c| \leq (1 + 1)\,|x - c|\,.$$

Therefore, given $\epsilon > 0$, let $\delta := \frac{\epsilon}{2}$. If $|x - c| < \delta$, then $|x^2 - c^2| < \epsilon$.

On the other hand, $g \colon \mathbb{R} \to \mathbb{R}$, defined by $g(x) := x^2$ is not uniformly continuous.

Proof: Suppose it is uniformly continuous, then for every $\epsilon > 0$, there would exist a $\delta > 0$ such that if $|x - c| < \delta$, then $|x^2 - c^2| < \epsilon$. Take $x > 0$ and let $c := x + \frac{\delta}{2}$. Write

$$\epsilon > \left|x^2 - c^2\right| = |x + c|\,|x - c| = (2x + \frac{\delta}{2})\frac{\delta}{2} \geq \delta x.$$

Therefore, $x < \frac{\epsilon}{\delta}$ for all $x > 0$, which is a contradiction.

Example 3.45: The function $f \colon (0,1) \to \mathbb{R}$, defined by $f(x) := \frac{1}{x}$ is not uniformly continuous.

Proof: Given $\epsilon > 0$, then $\epsilon > \left|\frac{1}{x} - \frac{1}{y}\right|$ holds if and only if

$$\epsilon > \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{|xy|} = \frac{|y - x|}{xy},$$

or

$$|x - y| < xy\epsilon.$$

Suppose $\epsilon < 1$, and we wish to see if a small $\delta > 0$ would work. If $x \in (0,1)$ and $y = x + \frac{\delta}{2} \in (0,1)$, then $|x - y| = \frac{\delta}{2} < \delta$. We plug y into the inequality above to get $\frac{\delta}{2} < x\bigl(x + \frac{\delta}{2}\bigr)\epsilon < x$. If the definition of uniform continuity is satisfied, then the inequality $\frac{\delta}{2} < x$ holds for all $x > 0$. But then $\delta \leq 0$. Therefore, there is no single $\delta > 0$ that works for all points.

The examples show that if f is defined on an interval that is either not closed or not bounded, then f can be continuous, but not uniformly continuous. For a closed and bounded interval $[a, b]$, we can, however, make the following statement.

Theorem 3.46. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. We prove the statement by contrapositive. Suppose f is not uniformly continuous. We will prove that there is some $c \in [a,b]$ where f is not continuous. Let us negate the definition of uniformly continuous. There exists an $\epsilon > 0$ such that for every $\delta > 0$, there exist points x, y in $[a,b]$ with $|x-y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

So for the $\epsilon > 0$ above, we find sequences $\{x_n\}$ and $\{y_n\}$ such that $|x_n - y_n| < \frac{1}{n}$ and such that $|f(x_n) - f(y_n)| \geq \epsilon$. By Bolzano–Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Let $c := \lim x_{n_k}$. As $a \leq x_{n_k} \leq b$ for all k, we have $a \leq c \leq b$. Estimate

$$|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \frac{1}{n_k} + |x_{n_k} - c| .$$

As $\frac{1}{n_k}$ and $|x_{n_k} - c|$ both go to zero when k goes to infinity, $\{y_{n_k}\}$ converges and the limit is c. We now show that f is not continuous at c. Estimate

$$\begin{aligned} |f(x_{n_k}) - f(c)| &= |f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c)| \\ &\geq |f(x_{n_k}) - f(y_{n_k})| - |f(y_{n_k}) - f(c)| \\ &\geq \epsilon - |f(y_{n_k}) - f(c)| . \end{aligned}$$

Or in other words,

$$|f(x_{n_k}) - f(c)| + |f(y_{n_k}) - f(c)| \geq \epsilon.$$

At least one of the sequences $\{f(x_{n_k})\}$ or $\{f(y_{n_k})\}$ cannot converge to $f(c)$, otherwise the left-hand side of the inequality would go to zero while the right-hand side is positive. Thus f cannot be continuous at c. $\square$

As before, note what is key in the proof: We can apply Bolzano–Weierstrass because the interval $[a,b]$ is bounded, and the limit of the subsequence is back in $[a,b]$ because the interval is closed.

3.4.2 Continuous extension

Before we get to continuous extension, we show the following useful lemma. It says that uniformly continuous functions behave nicely with respect to Cauchy sequences. The new issue here is that for a Cauchy sequence we no longer know where the limit ends up; it may not end up in the domain of the function.

Lemma 3.47. *Let $f\colon S \to \mathbb{R}$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence in S. Then $\{f(x_n)\}$ is Cauchy.*

Proof. Let $\epsilon > 0$ be given. There is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in S$ and $|x - y| < \delta$. Find an $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $|x_n - x_k| < \delta$. Then for all $n, k \geq M$, we have $|f(x_n) - f(x_k)| < \epsilon$. $\square$

An application of the lemma above is the following extension result. It says that a function on an open interval is uniformly continuous if and only if it can be extended to a continuous function on the closed interval.

Proposition 3.48. *A function $f\colon (a,b) \to \mathbb{R}$ is uniformly continuous if and only if the limits*

$$L_a := \lim_{x\to a} f(x) \qquad \textit{and} \qquad L_b := \lim_{x\to b} f(x)$$

exist and the function $\widetilde{f}\colon [a,b] \to \mathbb{R}$ defined by

$$\widetilde{f}(x) := \begin{cases} f(x) & \textit{if } x \in (a,b), \\ L_a & \textit{if } x = a, \\ L_b & \textit{if } x = b, \end{cases}$$

is continuous.

Proof. One direction is not difficult. If $\widetilde{f}$ is continuous, then it is uniformly continuous by Theorem 3.46. As f is the restriction of $\widetilde{f}$ to (a,b), then f is also uniformly continuous (easy exercise).

Now suppose f is uniformly continuous. We must first show that the limits L_a and L_b exist. Let us concentrate on L_a. Take a sequence $\{x_n\}$ in (a,b) such that $\lim x_n = a$. The sequence $\{x_n\}$ is Cauchy, so by Lemma 3.47 the sequence $\{f(x_n)\}$ is Cauchy and thus convergent. We have some number $L_1 := \lim f(x_n)$. Take another sequence $\{y_n\}$ in (a,b) such that $\lim y_n = a$. By the same reasoning we get $L_2 := \lim f(y_n)$. If we show that $L_1 = L_2$, then the limit $L_a = \lim_{x\to a} f(x)$ exists. Let $\epsilon > 0$ be given. Find $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \frac{\epsilon}{3}$. Find $M \in \mathbb{N}$ such that for $n \geq M$, we have $|a-x_n| < \frac{\delta}{2}$, $|a-y_n| < \frac{\delta}{2}$, $|f(x_n)-L_1| < \frac{\epsilon}{3}$, and $|f(y_n)-L_2| < \frac{\epsilon}{3}$. Then for $n \geq M$,

$$|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

So

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x_n) + f(x_n) - f(y_n) + f(y_n) - L_2| \\ &\leq |L_1 - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - L_2| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $L_1 = L_2$. Thus L_a exists. To show that L_b exists is left as an exercise.

Now that we know that the limits L_a and L_b exist, we are done. If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} \widetilde{f}(x)$ exists (see Proposition 3.15). Similarly with L_b. Hence $\widetilde{f}$ is continuous at a and b. And since f is continuous at $c \in (a,b)$, then $\widetilde{f}$ is continuous at $c \in (a,b)$. $\square$

A common application of this proposition (together with Proposition 3.17) is the following. Suppose $f\colon (-1,0) \cup (0,1) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 0} f(x)$ exists and the function has what is called an *removable singularity*, that is, we can extend the function to a continuous function on $(-1,1)$.

3.4.3 Lipschitz continuous functions

Definition 3.49. A function $f: S \to \mathbb{R}$ is *Lipschitz continuous*[6], if there exists a $K \in \mathbb{R}$, such that

$$|f(x) - f(y)| \leq K\,|x - y| \qquad \text{for all } x \text{ and } y \text{ in } S.$$

A large class of functions is Lipschitz continuous. Be careful, just as for uniformly continuous functions, the domain of definition of the function is important. See the examples below and the exercises. First, we justify the use of the word *continuous*.

Proposition 3.50. *A Lipschitz continuous function is uniformly continuous.*

Proof. Let $f: S \to \mathbb{R}$ be a function and let K be a constant such that $|f(x) - f(y)| \leq K\,|x - y|$ for all x, y in S. Let $\epsilon > 0$ be given. Take $\delta := \frac{\epsilon}{K}$. For all x and y in S such that $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq K\,|x - y| < K\delta = K\frac{\epsilon}{K} = \epsilon.$$

Therefore, f is uniformly continuous. □

We interpret Lipschitz continuity geometrically. Let f be a Lipschitz continuous function with some constant K. We rewrite the inequality to say that for $x \neq y$, we have

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq K.$$

The quantity $\frac{f(x)-f(y)}{x-y}$ is the slope of the line between the points $\bigl(x, f(x)\bigr)$ and $\bigl(y, f(y)\bigr)$, that is, a *secant line*. Therefore, f is Lipschitz continuous if and only if every line that intersects the graph of f in at least two distinct points has slope less than or equal to K. See Figure 3.7.

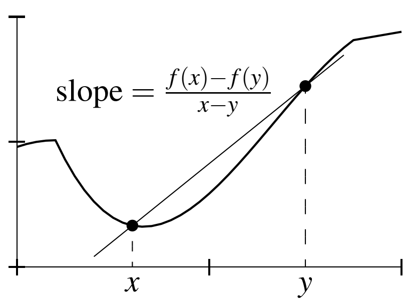

Figure 3.7: The slope of a secant line. A function is Lipschitz if $|\text{slope}| = \left|\frac{f(x)-f(y)}{x-y}\right| \leq K$ for all x and y.

Example 3.51: The functions $\sin(x)$ and $\cos(x)$ are Lipschitz continuous. In Example 3.23 we have seen the following two inequalities.

$$|\sin(x) - \sin(y)| \leq |x - y| \qquad \text{and} \qquad |\cos(x) - \cos(y)| \leq |x - y|\,.$$

Hence sine and cosine are Lipschitz continuous with $K = 1$.

[6]Named after the German mathematician Rudolf Otto Sigismund Lipschitz (1832–1903).

Example 3.52: The function $f\colon [1,\infty) \to \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is Lipschitz continuous. Proof:

$$\left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}.$$

As $x \geq 1$ and $y \geq 1$, we see that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$. Therefore,

$$\left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{1}{2}\left|x-y\right|.$$

On the other hand, $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is not Lipschitz continuous. Let us see why: Suppose

$$\left|\sqrt{x} - \sqrt{y}\right| \leq K\left|x-y\right|,$$

for some K. Set $y = 0$ to obtain $\sqrt{x} \leq Kx$. If $K > 0$, then for $x > 0$ we then get $\frac{1}{K} \leq \sqrt{x}$. This cannot possibly be true for all $x > 0$. Thus no such $K > 0$ exists and f is not Lipschitz continuous.

The last example is a function that is uniformly continuous but not Lipschitz continuous. To see that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$, note that it is uniformly continuous on $[0,1]$ by Theorem 3.46. It is also Lipschitz (and therefore uniformly continuous) on $[1,\infty)$. It is not hard (exercise) to show that this means that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

3.4.4 Exercises

Exercise* 3.53*: *Let $f\colon S \to \mathbb{R}$ be uniformly continuous. Let $A \subset S$. Then the restriction $f|_A$ is uniformly continuous.*

Exercise* 3.54*: *Let $f\colon (a,b) \to \mathbb{R}$ be a uniformly continuous function. Finish the proof of Proposition 3.48 by showing that the limit $\lim\limits_{x\to b} f(x)$ exists.*

Exercise* 3.55*: *Show that $f\colon (c,\infty) \to \mathbb{R}$ for some $c > 0$ and defined by $f(x) := \frac{1}{x}$ is Lipschitz continuous.*

Exercise* 3.56*: *Show that $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is not Lipschitz continuous.*

Exercise* 3.57*: *Let A, B be intervals. Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be uniformly continuous functions such that $f(x) = g(x)$ for $x \in A \cap B$. Define the function $h\colon A \cup B \to \mathbb{R}$ by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B \setminus A$.*

a) Prove that if $A \cap B \neq \emptyset$, then h is uniformly continuous.

b) Find an example where $A \cap B = \emptyset$ and h is not even continuous.

Exercise* 3.58** (Challenging):*** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $d \geq 2$. Show that f is not Lipschitz continuous.*

Exercise* 3.59*: *Let $f\colon (0,1) \to \mathbb{R}$ be a bounded continuous function. Show that the function $g(x) := x(1-x)f(x)$ is uniformly continuous.*

Exercise* 3.60*: *Show that $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := \sin(\frac{1}{x})$ is not uniformly continuous.*

Exercise* 3.61** (Challenging):*** *Let $f\colon \mathbb{Q} \to \mathbb{R}$ be a uniformly continuous function. Show that there exists a uniformly continuous function $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = \widetilde{f}(x)$ for all $x \in \mathbb{Q}$.*

Exercise* 3.62*:

a) Find a continuous $f\colon (0,1) \to \mathbb{R}$ and a sequence $\{x_n\}$ in $(0,1)$ that is Cauchy, but such that $\big\{f(x_n)\big\}$ is not Cauchy.

b) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, and $\{x_n\}$ is Cauchy, then $\big\{f(x_n)\big\}$ is Cauchy.

Exercise* 3.63*: *Prove:*

a) If $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are uniformly continuous, then $h\colon S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f\colon S \to \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h\colon S \to \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

Exercise* 3.64*: *Prove:*

a) If $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are Lipschitz, then $h\colon S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz.

b) If $f\colon S \to \mathbb{R}$ is Lipschitz and $a \in \mathbb{R}$, then $h\colon S \to \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz.

Exercise* 3.65*:

a) If $f\colon [0,1] \to \mathbb{R}$ is given by $f(x) := x^m$ for an integer $m \geq 0$, show f is Lipschitz and find the best (the smallest) Lipschitz constant K (depending on m of course). Hint: $(x-y)(x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \cdots + xy^{m-2} + y^{m-1}) = x^m - y^m$.

b) Using the previous exercise, show that if $f\colon [0,1] \to \mathbb{R}$ is a polynomial, that is, $f(x) := a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, then f is Lipschitz.

Exercise* 3.66*: *Suppose for $f\colon [0,1] \to \mathbb{R}$, we have $\lvert f(x) - f(y)\rvert \leq K\lvert x-y\rvert$ for all x, y in $[0,1]$, and $f(0) = f(1) = 0$. Prove that $\lvert f(x)\rvert \leq \frac{K}{2}$ for all $x \in [0,1]$. Further show by example that $\frac{K}{2}$ is the best possible, that is, there exists such a continuous function for which $\lvert f(x)\rvert = \frac{K}{2}$ for some $x \in [0,1]$.*

Exercise* 3.67*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period $P > 0$. That is, $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.*

Exercise* 3.68*: *Suppose $f\colon S \to \mathbb{R}$ and $g\colon [0,\infty) \to [0,\infty)$ are functions, g is continuous at 0, $g(0) = 0$, and whenever x and y are in S, we have $\lvert f(x) - f(y)\rvert \leq g\big(\lvert x-y\rvert\big)$. Prove that f is uniformly continuous.*

Exercise* 3.69*: *Suppose $f\colon [a,b] \to \mathbb{R}$ is a function such that for every $c \in [a,b]$ there is a $K_c > 0$ and an $\epsilon_c > 0$ for which $\lvert f(x) - f(y)\rvert \leq K_c\lvert x-y\rvert$ for all x and y in $(c-\epsilon_c, c+\epsilon_c) \cap [a,b]$. In other words, f is "locally Lipschitz."*

a) Prove that there exists a single $K > 0$ such that $\lvert f(x) - f(y)\rvert \leq K\lvert x-y\rvert$ for all x, y in $[a,b]$.

b) Find a counterexample to the above if the interval is open, that is, find an $f\colon (a,b) \to \mathbb{R}$ that is locally Lipschitz, but not Lipschitz.

3.5 Limits at infinity

3.5.1 Limits at infinity

As for sequences, a continuous variable can also approach infinity. Let us make this notion precise.

Definition 3.53. We say ∞ is a *cluster point* of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \geq M$. Similarly, $-\infty$ is a *cluster point* of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \leq M$.

Let $f \colon S \to \mathbb{R}$ be a function, where ∞ is a cluster point of S. If there exists an $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

whenever $x \in S$ and $x \geq M$, then we say $f(x)$ *converges* to L as x goes to ∞. We call L the *limit* and write

$$\lim_{x \to \infty} f(x) := L.$$

Alternatively we write $f(x) \to L$ as $x \to \infty$.

Similarly, if $-\infty$ is a cluster point of S and there exists an $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

whenever $x \in S$ and $x \leq M$, then we say $f(x)$ *converges* to L as x goes to $-\infty$. We call L the *limit* and write

$$\lim_{x \to -\infty} f(x) := L.$$

Alternatively we write $f(x) \to L$ as $x \to -\infty$.

We cheated a little bit again and said *the* limit. We leave it as an exercise for the reader to prove the following proposition.

Proposition 3.54. *The limit at ∞ or $-\infty$ as defined above is unique if it exists.*

Example 3.55: Let $f(x) := \frac{1}{|x|+1}$. Then

$$\lim_{x \to \infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 0.$$

Proof: Let $\epsilon > 0$ be given. Find $M > 0$ large enough so that $\frac{1}{M+1} < \epsilon$. If $x \geq M$, then $\frac{1}{x+1} \leq \frac{1}{M+1} < \epsilon$. Since $\frac{1}{|x|+1} > 0$ for all x the first limit is proved. The proof for $-\infty$ is left to the reader.

Example 3.56: Let $f(x) := \sin(\pi x)$. Then $\lim_{x \to \infty} f(x)$ does not exist. To prove this fact note that if $x = 2n + \frac{1}{2}$ for some $n \in \mathbb{N}$, then $f(x) = 1$, while if $x = 2n + \frac{3}{2}$, then $f(x) = -1$. So they cannot both be within a small ϵ of a single real number.

We must be careful not to confuse continuous limits with limits of sequences. We could say

$$\lim_{n \to \infty} \sin(\pi n) = 0, \qquad \text{but} \qquad \lim_{x \to \infty} \sin(\pi x) \text{ does not exist.}$$

Of course the notation is ambiguous: Are we thinking of the sequence $\{\sin(\pi n)\}_{n=1}^{\infty}$ or the function $\sin(\pi x)$ of a real variable? We are simply using the convention that $n \in \mathbb{N}$, while $x \in \mathbb{R}$. When the notation is not clear, it is good to explicitly mention where the variable lives, or what kind of limit are you using. If there is possibility of confusion, one can write, for example,

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \sin(\pi n).$$

There is a connection of continuous limits to limits of sequences, but we must take all sequences going to infinity, just as before in Lemma 3.7.

Lemma 3.57. *Suppose $f\colon S \to \mathbb{R}$ is a function, ∞ is a cluster point of $S \subset \mathbb{R}$, and $L \in \mathbb{R}$. Then*

$$\lim_{x\to\infty} f(x) = L$$

if and only if

$$\lim_{n\to\infty} f(x_n) = L$$

for all sequences $\{x_n\}$ in S such that $\lim\limits_{n\to\infty} x_n = \infty$.

The lemma holds for the limit as $x \to -\infty$. Its proof is almost identical and is left as an exercise.

Proof. First suppose $f(x) \to L$ as $x \to \infty$. Given an $\epsilon > 0$, there exists an M such that for all $x \geq M$, we have $|f(x) - L| < \epsilon$. Let $\{x_n\}$ be a sequence in S such that $\lim x_n = \infty$. Then there exists an N such that for all $n \geq N$, we have $x_n \geq M$. And thus $|f(x_n) - L| < \epsilon$.

We prove the converse by contrapositive. Suppose $f(x)$ does not go to L as $x \to \infty$. This means that there exists an $\epsilon > 0$, such that for every $n \in \mathbb{N}$, there exists an $x \in S$, $x \geq n$, let us call it x_n, such that $|f(x_n) - L| \geq \epsilon$. Consider the sequence $\{x_n\}$. Clearly $\{f(x_n)\}$ does not converge to L. It remains to note that $\lim x_n = \infty$, because $x_n \geq n$ for all n. ☐

Using the lemma, we again translate results about sequential limits into results about continuous limits as x goes to infinity. That is, we have almost immediate analogues of the corollaries in §3.1.3. We simply allow the cluster point c to be either ∞ or $-\infty$, in addition to a real number. We leave it to the student to verify these statements.

3.5.2 Infinite limit

Just as for sequences, it is often convenient to distinguish certain divergent sequences, and talk about limits being infinite almost as if the limits existed.

Definition 3.58. Let $f\colon S \to \mathbb{R}$ be a function and suppose S has ∞ as a cluster point. We say $f(x)$ *diverges to infinity* as x goes to ∞, if for every $N \in \mathbb{R}$ there exists an $M \in \mathbb{R}$ such that

$$f(x) > N$$

whenever $x \in S$ and $x \geq M$. We write

$$\lim_{x\to\infty} f(x) := \infty,$$

or we say that $f(x) \to \infty$ as $x \to \infty$.

A similar definition can be made for limits as $x \to -\infty$ or as $x \to c$ for a finite c. Also similar definitions can be made for limits being $-\infty$. Stating these definitions is left as an exercise. Note that sometimes *converges to infinity* is used. We can again use sequential limits, and an analogue of Lemma 3.7 is left as an exercise.

Example 3.59: Let us show that $\lim_{x\to\infty} \frac{1+x^2}{1+x} = \infty$.

Proof: For $x \geq 1$, we have

$$\frac{1+x^2}{1+x} \geq \frac{x^2}{x+x} = \frac{x}{2}.$$

Given $N \in \mathbb{R}$, take $M = \max\{2N+1, 1\}$. If $x \geq M$, then $x \geq 1$ and $\frac{x}{2} > N$. So

$$\frac{1+x^2}{1+x} \geq \frac{x}{2} > N.$$

3.5.3 Compositions

Finally, just as for limits at finite numbers we can compose functions easily.

Proposition 3.60. *Suppose $f \colon A \to B$, $g \colon B \to \mathbb{R}$, $A, B \subset \mathbb{R}$, $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of A, and $b \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of B. Suppose*

$$\lim_{x\to a} f(x) = b \qquad \text{and} \qquad \lim_{y\to b} g(y) = c$$

for some $c \in \mathbb{R} \cup \{-\infty, \infty\}$. If $b \in B$, then suppose $g(b) = c$. Then

$$\lim_{x\to a} g\bigl(f(x)\bigr) = c.$$

The proof is straightforward, and left as an exercise. We already know the proposition when $a, b, c \in \mathbb{R}$, see Exercises 3.9 and 3.14. Again the requirement that g is continuous at b, if $b \in B$, is necessary.

Example 3.61: Let $h(x) := e^{-x^2+x}$. Then

$$\lim_{x\to\infty} h(x) = 0.$$

Proof: The claim follows once we know

$$\lim_{x\to\infty} -x^2 + x = -\infty$$

and

$$\lim_{y\to-\infty} e^y = 0,$$

which is usually proved when the exponential function is defined.

3.5.4 Exercises

***Exercise* 3.70:** *Prove Proposition 3.54.*

***Exercise* 3.71:** *Let* $f : [1,\infty) \to \mathbb{R}$ *be a function. Define* $g : (0,1] \to \mathbb{R}$ *via* $g(x) := f(\frac{1}{x})$*. Using the definitions of limits directly, show that* $\lim_{x\to 0^+} g(x)$ *exists if and only if* $\lim_{x\to\infty} f(x)$ *exists, in which case they are equal.*

***Exercise* 3.72:** *Prove Proposition 3.60.*

***Exercise* 3.73:** *Let us justify terminology. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function such that* $\lim_{x\to\infty} f(x) = \infty$ *(diverges to infinity). Show that* $f(x)$ *diverges (i.e. does not converge) as* $x \to \infty$*.*

***Exercise* 3.74:** *Come up with the definitions for limits of* $f(x)$ *going to* $-\infty$ *as* $x \to \infty$*,* $x \to -\infty$*, and as* $x \to c$ *for a finite* $c \in \mathbb{R}$*. Then state the definitions for limits of* $f(x)$ *going to* ∞ *as* $x \to -\infty$*, and as* $x \to c$ *for a finite* $c \in \mathbb{R}$*.*

***Exercise* 3.75:** *Suppose* $P(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *is a* monic polynomial *of degree* $n \geq 1$ *(monic means that the coefficient of* x^n *is 1).*

a) Show that if n *is even, then* $\lim_{x\to\infty} P(x) = \lim_{x\to-\infty} P(x) = \infty$*.*

b) Show that if n *is odd, then* $\lim_{x\to\infty} P(x) = \infty$ *and* $\lim_{x\to-\infty} P(x) = -\infty$ *(see previous exercise).*

***Exercise* 3.76:** *Let* $\{x_n\}$ *be a sequence. Consider* $S := \mathbb{N} \subset \mathbb{R}$*, and* $f : S \to \mathbb{R}$ *defined by* $f(n) := x_n$*. Show that the two notions of limit,*

$$\lim_{n\to\infty} x_n \qquad \textit{and} \qquad \lim_{x\to\infty} f(x)$$

are equivalent. That is, show that if one exists so does the other one, and in this case they are equal.

***Exercise* 3.77:** *Extend Lemma 3.57 as follows. Suppose* $S \subset \mathbb{R}$ *has a cluster point* $c \in \mathbb{R}$*,* $c = \infty$*, or* $c = -\infty$*. Let* $f : S \to \mathbb{R}$ *be a function and suppose* $L = \infty$ *or* $L = -\infty$*. Show that*

$$\lim_{x\to c} f(x) = L \qquad \textit{if and only if} \qquad \lim_{n\to\infty} f(x_n) = L \ \textit{ for all sequences } \{x_n\} \textit{ such that } \lim x_n = c.$$

***Exercise* 3.78:** *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a 2-periodic function, that is* $f(x+2) = f(x)$ *for all* x*. Define* $g : \mathbb{R} \to \mathbb{R}$ *by*

$$g(x) := f\left(\frac{\sqrt{x^2+1}-1}{x}\right)$$

a) Find the function $\varphi : (-1,1) \to \mathbb{R}$ *such that* $g\bigl(\varphi(t)\bigr) = f(t)$*, that is* $\varphi^{-1}(x) = \frac{\sqrt{x^2+1}-1}{x}$*.*

b) Show that f *is continuous if and only if* g *is continuous and*

$$\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = f(1) = f(-1).$$

3.6 Monotone functions and continuity

Definition 3.62. Let $S \subset \mathbb{R}$. We say $f: S \to \mathbb{R}$ is *increasing* (resp. *strictly increasing*) if $x, y \in S$ with $x < y$ implies $f(x) \leq f(y)$ (resp. $f(x) < f(y)$). We define *decreasing* and *strictly decreasing* in the same way by switching the inequalities for f.

If a function is either increasing or decreasing, we say it is *monotone*. If it is strictly increasing or strictly decreasing, we say it is *strictly monotone*.

Sometimes *nondecreasing* (resp. *nonincreasing*) is used for increasing (resp. decreasing) function to emphasize it is not strictly increasing (resp. strictly decreasing).

If f is increasing, then $-f$ is decreasing and vice versa. Therefore, many results about monotone functions can just be proved for, say, increasing functions, and the results follow easily for decreasing functions.

3.6.1 Continuity of monotone functions

One-sided limits for monotone functions are computed by computing infima and suprema.

Proposition 3.63. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, $f: S \to \mathbb{R}$ be increasing, and $g: S \to \mathbb{R}$ be decreasing. If c is a cluster point of $S \cap (-\infty, c)$, then*

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c, x \in S\} \qquad \textit{and} \qquad \lim_{x \to c^-} g(x) = \inf\{g(x) : x < c, x \in S\}.$$

If c is a cluster point of $S \cap (c, \infty)$, then

$$\lim_{x \to c^+} f(x) = \inf\{f(x) : x > c, x \in S\} \qquad \textit{and} \qquad \lim_{x \to c^+} g(x) = \sup\{g(x) : x > c, x \in S\}.$$

If ∞ is a cluster point of S, then

$$\lim_{x \to \infty} f(x) = \sup\{f(x) : x \in S\} \qquad \textit{and} \qquad \lim_{x \to \infty} g(x) = \inf\{g(x) : x \in S\}.$$

If $-\infty$ is a cluster point of S, then

$$\lim_{x \to -\infty} f(x) = \inf\{f(x) : x \in S\} \qquad \textit{and} \qquad \lim_{x \to -\infty} g(x) = \sup\{g(x) : x \in S\}.$$

Namely, all the one-sided limits exist whenever they make sense. For monotone functions therefore, when we say the left-hand limit $x \to c^-$ exists, we mean that c is a cluster point of $S \cap (-\infty, c)$, and same for the right-hand limit.

Proof. Let us assume f is increasing, and we will show the first equality. The rest of the proof is very similar and is left as an exercise.

Let $a := \sup\{f(x) : x < c, x \in S\}$. If $a = \infty$, then given an $M \in \mathbb{R}$, there exists an $x_M \in S$, $x_M < c$, such that $f(x_M) > M$. As f is increasing, $f(x) \geq f(x_M) > M$ for all $x \in S$ with $x > x_M$. If we take $\delta := c - x_M > 0$, then we obtain the definition of the limit going to infinity.

Next suppose $a < \infty$. Let $\epsilon > 0$ be given. Because a is the supremum and $S \cap (-\infty, c)$ is nonempty, $a \in \mathbb{R}$ and there exists an $x_\epsilon \in S$, $x_\epsilon < c$, such that $f(x_\epsilon) > a - \epsilon$. As f is increasing, if $x \in S$ and $x_\epsilon < x < c$, we have $a - \epsilon < f(x_\epsilon) \leq f(x) \leq a$. Let $\delta := c - x_\epsilon$. Then for $x \in S \cap (-\infty, c)$ with $|x - c| < \delta$, we have $|f(x) - a| < \epsilon$. $\square$

Suppose $f\colon S \to \mathbb{R}$ is increasing, $c \in S$, and that both one-sided limits exist. Since $f(x) \leq f(c) \leq f(y)$ whenever $x < c < y$, taking the limits we obtain

$$\lim_{x \to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x).$$

Then f is continuous at c if and only if both limits are equal to each other (and hence equal to $f(c)$). See also Proposition 3.17. See Figure 3.8 to get an idea of a what a discontinuity looks like.

Corollary 3.64. *If $I \subset \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is monotone and not constant, then $f(I)$ is an interval if and only if f is continuous.*

Assuming f is not constant is to avoid the technicality that $f(I)$ is a single point: $f(I)$ is a single point if and only if f is constant. A constant function is continuous.

Proof. Without loss of generality, suppose f is increasing.

First suppose f is continuous. Take two points $f(x_1) < f(x_2)$ in $f(I)$. As f is increasing, then $x_1 < x_2$. By the intermediate value theorem, given y with $f(x_1) < y < f(x_2)$, we find a $c \in (x_1, x_2) \subset I$ such that $f(c) = y$, so $y \in f(I)$. Hence, $f(I)$ is an interval.

Let us prove the reverse direction by contrapositive. Suppose f is not continuous at $c \in I$, and that c is not an endpoint of I. Let

$$a := \lim_{x \to c^-} f(x) = \sup\bigl\{ f(x) : x \in I, x < c \bigr\}, \qquad b := \lim_{x \to c^+} f(x) = \inf\bigl\{ f(x) : x \in I, x > c \bigr\}.$$

As c is a discontinuity, $a < b$. If $x < c$, then $f(x) \leq a$, and if $x > c$, then $f(x) \geq b$. Therefore no point in $(a, b) \setminus \{f(c)\}$ is in $f(I)$. However there exists $x_1 \in I$, $x_1 < c$, so $f(x_1) \leq a$, and there exists $x_2 \in I$, $x_2 > c$, so $f(x_2) \geq b$. Both $f(x_1)$ and $f(x_2)$ are in $f(I)$, but there are points in between them that are not in $f(I)$. So $f(I)$ is not an interval. See Figure 3.8.

When $c \in I$ is an endpoint, the proof is similar and is left as an exercise. $\square$

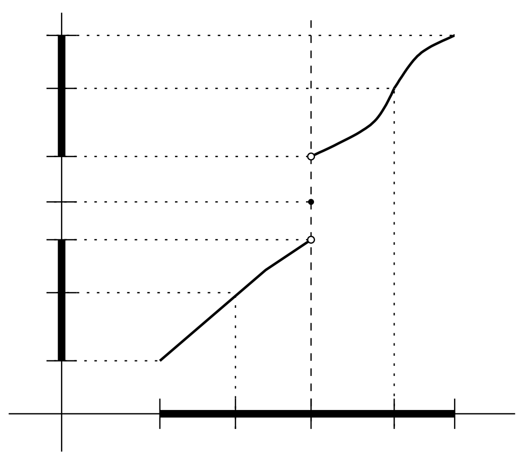

Figure 3.8: Increasing function $f\colon I \to \mathbb{R}$ discontinuity at c.

A striking property of monotone functions is that they cannot have too many discontinuities.

Corollary 3.65. *Let $I \subset \mathbb{R}$ be an interval and $f \colon I \to \mathbb{R}$ be monotone. Then f has at most countably many discontinuities.*

Proof. Let $E \subset I$ be the set of all discontinuities that are not endpoints of I. As there are only two endpoints, it is enough to show that E is countable. Without loss of generality, suppose f is increasing. We will define an injection $h \colon E \to \mathbb{Q}$. For each $c \in E$ the one-sided limits of f both exist as c is not an endpoint. Let

$$a := \lim_{x \to c^-} f(x) = \sup\bigl\{ f(x) : x \in I, x < c \bigr\}, \qquad b := \lim_{x \to c^+} f(x) = \inf\bigl\{ f(x) : x \in I, x > c \bigr\}.$$

As c is a discontinuity, we have $a < b$. There exists a rational number $q \in (a,b)$, so let $h(c) := q$. If $d \in E$ is another discontinuity, then if $d > c$, then there exist an $x \in I$ with $c < x < d$, and so $\lim_{x \to d^-} f(x) \geq b$. Hence the rational number we choose for $h(d)$ is different from q, since $q = h(c) < b$ and $h(d) > b$. Similarly if $d < c$. So after making such a choice for every $c \in E$, we have a one-to-one (injective) function into $\mathbb{Q}$. Therefore, E is countable. $\square$

Example 3.66: By $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Define $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) := x + \sum_{n=0}^{\lfloor 1/(1-x) \rfloor} 2^{-n},$$

for $x < 1$ and $f(1) := 3$. It is left as an exercise to show that f is strictly increasing, bounded, and has a discontinuity at all points $1 - \frac{1}{k}$ for $k \in \mathbb{N}$. In particular, there are countably many discontinuities, but the function is bounded and defined on a closed bounded interval. See Figure 3.9.

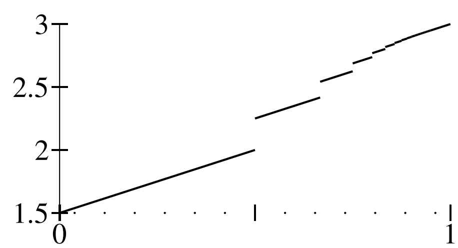

Figure 3.9: Increasing function with countably many discontinuities.

Similarly, one can find an example of a function discontinuous on a dense set such as the rational numbers. See the exercises.

3.6.2 Continuity of inverse functions

A strictly monotone function f is one-to-one (injective). To see this fact, notice that if $x \neq y$, then we can assume $x < y$. Either $f(x) < f(y)$ if f is strictly increasing or $f(x) > f(y)$ if f is strictly decreasing, so $f(x) \neq f(y)$. Hence, f must have an inverse f^{-1} defined on its range.

Proposition 3.67. *If $I \subset \mathbb{R}$ is an interval and $f \colon I \to \mathbb{R}$ is strictly monotone, then the inverse $f^{-1} \colon f(I) \to I$ is continuous.*

Proof. Let us suppose f is strictly increasing. The proof is almost identical for a strictly decreasing function. Since f is strictly increasing, so is f^{-1}. That is, if $f(x) < f(y)$, then we must have $x < y$ and therefore $f^{-1}\bigl(f(x)\bigr) < f^{-1}\bigl(f(y)\bigr)$.

Take $c \in f(I)$. If c is not a cluster point of $f(I)$, then f^{-1} is continuous at c automatically. So let c be a cluster point of $f(I)$. Suppose both of the following one-sided limits exist:

$$x_0 := \lim_{y \to c^-} f^{-1}(y) = \sup\bigl\{f^{-1}(y) : y < c, y \in f(I)\bigr\} = \sup\bigl\{x \in I : f(x) < c\bigr\},$$
$$x_1 := \lim_{y \to c^+} f^{-1}(y) = \inf\bigl\{f^{-1}(y) : y > c, y \in f(I)\bigr\} = \inf\bigl\{x \in I : f(x) > c\bigr\}.$$

We have $x_0 \leq x_1$ as f^{-1} is increasing. For all $x \in I$ where $x > x_0$, we have $f(x) \geq c$. As f is strictly increasing, we must have $f(x) > c$ for all $x \in I$ where $x > x_0$. Therefore,

$$\{x \in I : x > x_0\} \subset \bigl\{x \in I : f(x) > c\bigr\}.$$

The infimum of the left-hand set is x_0, and the infimum of the right-hand set is x_1, so we obtain $x_0 \geq x_1$. So $x_1 = x_0$, and f^{-1} is continuous at c.

If one of the one-sided limits does not exist, the argument is similar and is left as an exercise. □

Example 3.68: The proposition does not require f itself to be continuous. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x < 0, \\ x+1 & \text{if } x \geq 0. \end{cases}$$

The function f is not continuous at 0. The image of $I = \mathbb{R}$ is the set $(-\infty, 0) \cup [1, \infty)$, not an interval. Then $f^{-1}\colon (-\infty, 0) \cup [1, \infty) \to \mathbb{R}$ can be written as

$$f^{-1}(y) = \begin{cases} y & \text{if } y < 0, \\ y-1 & \text{if } y \geq 1. \end{cases}$$

It is not difficult to see that f^{-1} is a continuous function. See Figure 3.10 for the graphs.

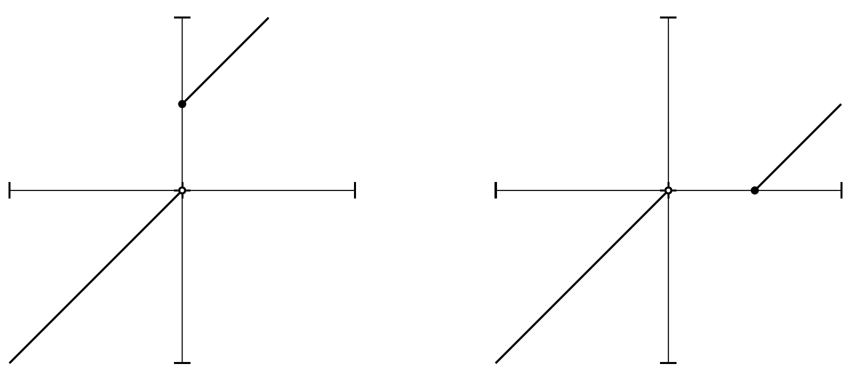

Figure 3.10: Graph of f on the left and f^{-1} on the right.

Notice what happens with the proposition if $f(I)$ is an interval. In that case, we could simply apply Corollary 3.64 to both f and f^{-1}. That is, if $f\colon I \to J$ is an onto strictly monotone function and I and J are intervals, then both f and f^{-1} are continuous. Furthermore, $f(I)$ is an interval precisely when f is continuous.

3.6.3 Exercises

Exercise* 3.79*: *Suppose $f\colon [0,1] \to \mathbb{R}$ is monotone. Prove f is bounded.*

Exercise* 3.80*: *Finish the proof of Proposition 3.63. Hint: You can halve your work by noticing that if g is decreasing, then $-g$ is increasing.*

Exercise* 3.81*: *Finish the proof of Corollary 3.64.*

Exercise* 3.82*: *Prove the claims in Example 3.66.*

Exercise* 3.83*: *Finish the proof of Proposition 3.67.*

Exercise* 3.84*: *Suppose $S \subset \mathbb{R}$, and $f\colon S \to \mathbb{R}$ is an increasing function. Prove:*

a) If c is a cluster point of $S \cap (c,\infty)$, then $\lim\limits_{x\to c^+} f(x) < \infty$.

b) If c is a cluster point of $S \cap (-\infty,c)$ and $\lim\limits_{x\to c^-} f(x) = \infty$, then $S \subset (-\infty,c)$.

Exercise* 3.85*: *Let $I \subset \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ a function. Suppose that for each $c \in I$, there exist $a, b \in \mathbb{R}$ with $a > 0$ such that $f(x) \geq ax + b$ for all $x \in I$ and $f(c) = ac + b$. Show that f is strictly increasing.*

Exercise* 3.86*: *Suppose I and J are intervals and $f\colon I \to J$ is a continuous, bijective (one-to-one and onto) function. Show that f is strictly monotone.*

Exercise* 3.87*: *Consider a monotone function $f\colon I \to \mathbb{R}$ on an interval I. Prove that there exists a function $g\colon I \to \mathbb{R}$ such that $\lim\limits_{x\to c^-} g(x) = g(c)$ for all c in I except the smaller (left) endpoint of I, and such that $g(x) = f(x)$ for all but countably many $x \in I$.*

Exercise* 3.88*:

a) Let $S \subset \mathbb{R}$ be a subset. If $f\colon S \to \mathbb{R}$ is increasing and bounded, then show that there exists an increasing $F\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x)$ for all $x \in S$.

b) Find an example of a strictly increasing bounded $f\colon S \to \mathbb{R}$ such that an increasing F as above is never strictly increasing.

Exercise* 3.89** (Challenging):*** *Find an example of an increasing function $f\colon [0,1] \to \mathbb{R}$ that has a discontinuity at each rational number. Then show that the image $f([0,1])$ contains no interval. Hint: Enumerate the rational numbers and define the function with a series.*

Exercise* 3.90*: *Suppose I is an interval and $f\colon I \to \mathbb{R}$ is monotone. Show that $\mathbb{R} \setminus f(I)$ is a countable union of disjoint intervals.*

Exercise* 3.91*: *Suppose $f\colon [0,1] \to (0,1)$ is increasing. Show that for every $\epsilon > 0$, there exists a strictly increasing $g\colon [0,1] \to (0,1)$ such that $g(0) = f(0)$, $f(x) \leq g(x)$ for all x, and $g(1) - f(1) < \epsilon$.*

Exercise* 3.92*: *Prove that the Dirichlet function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) := 1$ if x is rational and $f(x) := 0$ otherwise cannot be written as a difference of two increasing functions. That is, there do not exist increasing g and h such that, $f(x) = g(x) - h(x)$.*

Exercise* 3.93*: *Suppose $f\colon (a,b) \to (c,d)$ is a strictly increasing onto function. Prove that there exists a $g\colon (a,b) \to (c,d)$, which is also strictly increasing and onto, and $g(x) < f(x)$ for all $x \in (a,b)$.*