# Discrete Mathematics
# Mathematical reasoning

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# Outline


1. Direct proof
2. Proof by Induction
3. Proof by Contradiction
4. The Pigeonhole principle

# Direct proof

## Example

Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

## Proof.

- $A = 1 + 2 + \cdots + n$
- $2A = (1 + \cdots + n) + n + \cdots + 1)$
- $2A = (1 + n) + (2 + (n - 1)) + \cdots + (n + 1)$
- $2A = n \times (n + 1)$
- $\Rightarrow A = \frac{n(n+1)}{2}$

□

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$\square$

# Direct proof

**Example**

Prove that $n^3 - n$ is divisible by 3

**Proof.**

- $n^3 - n = n(n-1)(n+1)$
- If $n \bmod 3 = 0$, then $n(n-1)(n+1)$ is divisible by 3
- If $n \bmod 3 = 1$, then $n-1$ is divisible by 3, thus $n(n-1)(n+1)$ is divisible by 3
- If $n \bmod 3 = 2$, then $n+1$ is divisible by 3, thus $n(n-1)(n+1)$ is divisible by 3

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# Direct proof

## Example

Prove that $S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)\times n} = \frac{n-1}{n}$

## Proof.

- $\frac{1}{i\times(i+1)} = \frac{1}{i} - \frac{1}{i+1}$
- $S_n = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + \frac{1}{n-1} - \frac{1}{n}$
- $S_n = \frac{1}{1} = \frac{1}{n} - \frac{n-1}{n}$

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**Induction principle**

- Let $P(n)$ be a statement which involves a natural number $n$, i.e., $n = 1, 2, \ldots,$. $P(n)$ is true for all $n$
  - $P(1)$ is true
  - $P(k) \Rightarrow P(k+1)$ for all natural number $k$

# Proof by Induction

**Example**

Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

**Proof.**

- $n = 1, \sum_{i=1}^{1} i = 1 = \frac{1\times(1+1)}{2}$, this means $P(1)$ is true
- Suppose that $P(k)$ is true that $\sum_{i=1}^{k} i = \frac{k\times(k+1)}{2}$
- Now $n = k + 1$,
  $\sum_{i=1}^{k+1} = \sum_{i=1}^{k} i + (k+1) = \frac{k\times(k+1)}{2} + (k+1) = \frac{(k+1)\times(k+2)}{2}$
  $\Rightarrow P(k+1)$ is true
- $P(n)$ is true for all natural number $n$

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  $\Rightarrow P(k+1)$ is true
- $P(n)$ is true for all natural number $n$

$\square$

**Example**

Prove that $n^3 - n$ is divisible by 3

**Proof.**

- For $n = 1$, $1^3 - 1 = 0$ which is divisible by 3
- Suppose that $P(k)$ is true that $k^3 - k$ is divisible by 3
- For $n = k + 1$, $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = (k^3 - k) + 3(k^2 + k)$ which is divisible by 3 because $k^3 - k$ is divisible by 3. This means $P(k+1)$ is true
- $\Rightarrow n^3 - n$ is divisible by 3 for all natural number $n$

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- $\Rightarrow n^3 - n$ is divisible by 3 for all natural number $n$

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# Proof by Induction

## Example

Prove that $S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)\times n} = \frac{n-1}{n}$

## Proof.

- For $n = 1$, $S_1 = \frac{1}{1\times 2} = \frac{1}{2} = \frac{2-1}{2}$, $P(1)$ is true
- Suppose that $P(k)$ is true that $\sum_{i=1}^{k} \frac{1}{(i-1)\times i} = \frac{k-1}{k}$
- For $n = k+1$, $\sum_{i=1}^{k+1} \frac{1}{i\times(i+1)} = \sum_{i=1}^{k} \frac{1}{(i-1)\times i} + \frac{1}{k\times(k+1)} = \frac{k-1}{k} + \frac{1}{k\times(k+1)} = \frac{k^2-1+1}{k\times(k+1)} = \frac{k}{k+1}$. Thus $P(k+1)$ is true
- $\Rightarrow S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)\times n} = \frac{n-1}{n}$ for all natural number $n$

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- Suppose that $P(k)$ is true that $\sum_{i=1}^{k} \frac{1}{(i-1)\times i} = \frac{k-1}{k}$
- For $n = k+1$, $\sum_{i=1}^{k+1} \frac{1}{i\times(i+1)} = \sum_{i=1}^{k} \frac{1}{(i-1)\times i} + \frac{1}{k\times(k+1)} = \frac{k-1}{k} + \frac{1}{k\times(k+1)} = \frac{k^2-1+1}{k\times(k+1)} = \frac{k}{k+1}$. Thus $P(k+1)$ is true
- $\Rightarrow S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)\times n} = \frac{n-1}{n}$ for all natural number $n$

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# Proof by Contradiction

**Basic idea**

- Assume that the statement to be proved $\mathcal{P}$ is **false**
- Then, we show that this assumption leads to a contradiction
- $\Rightarrow$ So, the assumption above is **false**, thus $\mathcal{P}$ is **true**

# Proof by Contradiction

**Example**

Prove that if $a, b$ are integer, then $a^2 + 4b \neq 6$

**Proof.**

- If $a^2 + 4b = 6$, then $a$ is even: $a = 2k$
- Thus, $4k^2 + 4b = 6$
- Hence, $2(k^2 + b) = 3$ (contradiction)
- $\Rightarrow a^2 + 4b \neq 6$

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# Proof by Contradiction

## Example

Prove that if $\sqrt{3}$ is irrational

## Proof.

- Suppose $\exists a, b \in \mathbb{Z}$ such that $\sqrt{3} = \frac{a}{b}$ and $gcd(a, b) = 1$
- $\Rightarrow a^2 = 3b^2$
- $\Rightarrow a$ is divisible by 3: $\exists a_1 \in \mathbb{Z}$ such that $a = 3a_1$
- $\Rightarrow 9a_1^2 = 3b^2$
- $\Rightarrow b^2 = 3a_1^2$
- $\Rightarrow b$ is divisible by 3 (contradiction because $gcd(a, b) = 1$)
- $\Rightarrow \sqrt{3}$ is irrational

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- $\Rightarrow b$ is divisible by 3 (contradiction because $gcd(a, b) = 1$)
- $\Rightarrow \sqrt{3}$ is irrational

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# Proof by Contradiction

## Example

Given 7 segments $\{1, 2, \ldots, 7\}$. The segment $i$ has the length $a_i$ with $10 < a_i < 100, \forall i = 1 \ldots 7$. Prove that the proposition $\mathcal{P}$: "there exist 3 segments that can establish a triangle" is true.

## Proof.

- Assume that $\mathcal{P}$ is false
- Without loss of generality, assume that $a_1 \leq a_2 \leq \cdots \leq a_7$
- We have
  - $a_3 \geq a_1 + a_2 > 10 + 10 = 20$
  - $a_4 \geq a_2 + a_3 > 10 + 20 = 30$
  - $a_5 \geq a_3 + a_4 > 20 + 30 = 50$
  - $a_6 \geq a_4 + a_5 > 30 + 50 = 80$
  - $a_7 \geq a_5 + a_6 > 50 + 80 = 130$ (contradiction with the assumption that $a_7 < 100$)
- $\Rightarrow \mathcal{P}$ is true

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  - $a_7 \geq a_5 + a_6 > 50 + 80 = 130$ (contradiction with the assumption that $a_7 < 100$)
- $\Rightarrow \mathcal{P}$ is true

# Proof by Contradiction

## Example

10 integers $\{0, 1, \ldots, 9\}$ are arrange in a circle. Prove that the proposition $\mathcal{P}$: "There exist 3 consecutive integers having sum greater than 13" is true

## Proof.

- Assume that $\mathcal{P}$ is false
- We have
  - $k_1 = x_1 + x_2 + x_3 \leq 13$
  - $k_2 = x_2 + x_3 + x_4 \leq 13$
  - ...
  - $k_9 = x_0 + x_{10} + x_1 \leq 13$
  - $k_{10} = x_{10} + x_1 + x_2 \leq 13$
- $\Rightarrow k_1 + \cdots + k_{10} = 3(x_1 + \cdots + x_{10}) \leq 13 \times 10 = 130$
- $\Rightarrow 3(0 + 1 + \cdots + 9) = 135 \leq 130$ (contradiction)
- $\Rightarrow \mathcal{P}$ is true

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  - $k_{10} = x_{10} + x_1 + x_2 \leq 13$
- $\Rightarrow k_1 + \cdots + k_{10} = 3(x_1 + \cdots + x_{10}) \leq 13 \times 10 = 130$
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- $\Rightarrow \mathcal{P}$ is true

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# The Pigeonhole principle

## Theorem

- *If* $n+1$ *objects are put into* $n$ *boxes, then at least one box contains two or more objects*
- *If* $n$ *objects are put into* $k$ *boxes, then there exists one box containing at least* $\frac{n}{k}$ *objects*

## Example

- Among 13 people, there exist two people having their birthdays in the same month
- Among 100 people, there are at least 9 people having their brithdays in the same month

# The Pigeonhole principle

## Example

In a meeting room, there are $n$ people. Prove that there are at leats two people having the same number of friends (If $A$ is a friend of $B$ then $B$ is a friend of $A$)

## Proof.

- The number of friends of a person is an integer $k$ with $0 \leq k \leq n-1$
- If there is a person having $n-1$ friends, then all the other people in the room are his friends. This means that no one has 0 friend
- Hence, 0 and $n-1$ cannot be simultaneously the number of friends of some people in the room
- $\Rightarrow$ There are at least two people having the same number of friends

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$\square$

# The Pigeonhole principle

## Example

Given $n$ integers $a_1, \ldots, a_n$, not necessarily distinct. Prove that there exist indices $i$ and $j$ with $1 \leq i \leq j \leq n$ such that $a_i + \cdots + a_j$ is divisible by $n$

## Proof.

- Consider $n$ integers: $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$
- Dividing these integers by $n$, we have:
  $a_1 + a_2 + \cdots + a_i = q_i \times n + r_i, 0 \leq r_i \leq n - 1, \forall i = 1, \ldots, n$
- Two cases:
  - If there exists $r_k = 0$, then $a_1 + a_2 + \cdots + a_k$ is divisible by $n$
  - If none of $r_1, r_2, \ldots, r_n$ is zero, then there exist indices $i < j$ such that $r_i = r_j$. Then $(a_1 + \cdots + a_j) - (a_1 + \cdots + a_i)$ is divisible by $n$. Hence $a_{i+1} + a_{i+2} + \cdots + a_j$ is divisible by $n$

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  - If none of $r_1, r_2, \dots, r_n$ is zero, then there exist indices $i < j$ such that $r_i = r_j$. Then $(a_1 + \cdots + a_j) - (a_1 + \cdots + a_i)$ is divisible by $n$. Hence $a_{i+1} + a_{i+2} + \cdots + a_j$ is divisible by $n$

□

## Example

Given 51 integer from $1, 2, \ldots, 100$. Prove that there exist at least two integers such that one of them is divisible by the other

## Proof.

- Any integer can be written in the form $2^k \times a$, where $0 \leq k$ and $a$ is odd.
- The value $a$ can be one of the 50 numbers $1, 3, 5, \ldots, 99$. Thus, among 51 integers selected, there exist two integers $x$ and $y$ which have the same value of $a$ when they are written in the form $x = 2^r \times a$ and $y = 2^s \times a$.
- At that time, if $r \leq s$, then y is divisible by $x$, and $x$ is divisible by $y$, otherwise.

# The Pigeonhole principle

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- At that time, if $r \leq s$, then y is divisible by $x$, and $x$ is divisible by $y$, otherwise.

$\square$

# The Pigeonhole principle

## Example

There are 6 points $A, B, C, D, E, F$ on the plane. Every 2 points is connected by a segment (edge) of color red or blue. Every 3 points establishes a triangle. Prove that there is triangle with its 3 edges having the same color.

## Proof.

- Consider a point $A$. 5 edges $AB, AC, AD, AE, AF$ are painted with two colors red and blue. Thus, there are 3 edges having the same color, say, $AB, AC, AD$ are red
- If one of 3 edges $BC, CD, DB$ is red, say, $CD$, then the triangle $ACD$ is red
- Otherwise, 3 edges $BC, CD, DB$ are blue, then the triangle $BCD$ is blue

□

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# The Pigeonhole principle: strong form

## Theorem

*Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are put into $n$ boxes, then either the $1^{st}$ box contains at least $q_1$ objects, or the $2^{nd}$ box contains at least $q_2$ objects,$\ldots$, or the $n^{th}$ box contains at least $q_n$ objects*

**Proof** can easily be obtained by contradiction

## Example

A basket contains 21 fruits of 3 kinds: apples, bananas, and oranges. Then, there are either at leats 6 apples, or at least 10 bananas, or at least 7 oranges