Data structures and Algorithms
Searching

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Outline

1. Sequential searches
2. Binary searches
3. Binary search trees
4. AVL trees
5. String searching
6. Map and Hashing

Sequential searches

- Input: a list $A[1..n]$ and an item x
- Output: position i such that $A[i] = x$

```
int sSearch(int A[], int L, int R, int x){
  for(int i = L; i <= R; i++)
    if(A[i]==x)
      return i;
  return -1;
}
```

Outline

Binary search

- Input: a list $A[1..n]$ in non-decreasing order and an item x
- Output: position i such that $A[i] = x$

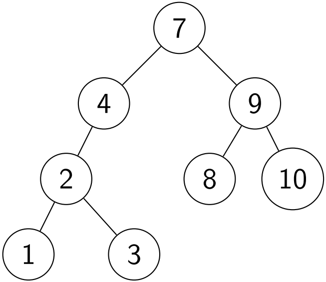
```
int bSearch(int A[], int L, int R, int x){
  int l = L;
  int r = R;
  while(l <= r){
    int mid = (l+r)/2;
    if(A[mid] == x)
      return mid;
    if(A[mid] > x)
      r = mid-1;
    else
      l = mid+1;
  }
  return -1;
}
```

Outline

1. Sequential searches
2. Binary searches
3. **Binary search trees**
4. AVL trees
5. String searching
6. Map and Hashing

Binary search trees (BST)

- Each node has unique key
- Keys of nodes of left subtree of a node v are less than the key of v
- Keys of nodes of right subtree of a node v are greater than or equal to the key of v



Binary search trees

```
struct Node{
  int key;
  Node* leftChild;
  Node* rightChild;
};

Node* root;
```

BST

- makeNode(int v): create a node with key = v, return the new created node
- insert(Node* ptr, int v): insert a new node with key = v to the BST pointed by ptr
- search(Node* ptr, int v): search a node having key = v, return this node if found
- findMin(Node* ptr): return the node having smallest key of the BST pointed by ptr
- del(Node* ptr, int v): remove a node having key = v

BST

```
Node* makeNode(int x){
  Node* p = new Node;
  p->key = x;
  p->leftChild = NULL;
  p->rightChild = NULL;
  return p;
}
```

BST

```
Node* insertNode(Node* r, int x){
  if(r == NULL){
    r = makeNode(x);
  }else if(r->key > x){
    r->leftChild = insertNode(r->leftChild,x);
  }else if(r->key <= x){
    r->rightChild = insertNode(r->rightChild,x);
  }
  return r;
}
```

BST

```
Node* search(Node* r, int x){
  if(r != NULL){
    if(r->key == x)
      return r;
    else if(r->key > x)
      return search(r->leftChild,x);
    else
      return search(r->rightChild,x);
  }
  return NULL;
}
```

BST

```
Node* findMin(Node* r){
  if(r == NULL) return NULL;
  Node* lmin = findMin(r->leftChild);
if(lmin == NULL) return r;
if(lmin->key < r->key)
  return lmin;
else
  return r;
}
```

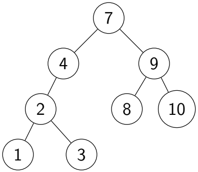
BST

```
Node* del(Node* r, int x){
  Node* tmp;
  if(r == NULL) printf("Not Found\n");
  else if(x < r->key) r->leftChild = del(r->leftChild,x);
  else if(x > r->key) r->rightChild = del(r->rightChild,x);
  else{// x = r->key, remove r means that make the smallest node of the right tree of r the root of the new tree
    if(r->leftChild != NULL && r->rightChild != NULL){
      tmp = findMin(r->rightChild);
      r->key = tmp->key;
      r->rightChild = del(r->rightChild,tmp->key);
    }else{
      tmp = r;// keep this node for freeing memory
      if(r->leftChild == NULL)        r = r->rightChild;
      else if(r->rightChild == NULL)        r = r->leftChild;
      delete tmp;
    }
  }
  return r;
}
```

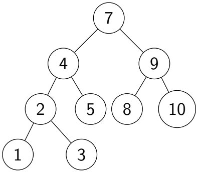
Outline

AVL trees

- AVL is a BST (G. M. Adelson-Velskii and E. M. Landis, 1962)
  - the height of the left child differs from the height of the right child by at most 1 (balance property)
  - left and right subtrees are both AVL
- Modification (insertion, deletion of nodes) on AVL must conserve the balance property
- Assumption: the keys of all nodes are different (e.g., cannot construct an AVL for 8, 8, 9)



a. BST (not AVL)



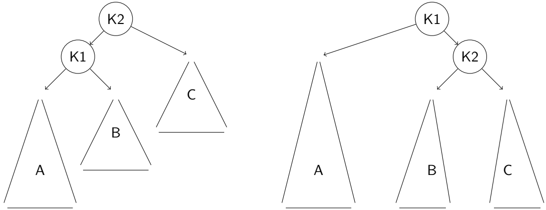
b. AVL

AVL tree - recovery the balance property

- A BST T whose left and right subtrees are AVL
- Perform rotation actions so that the resulting BST is an AVL
- After the insertion or deletion of a node on an AVL
  - balance property of the AVL may loss
  - the height of any subtree change at most 1
  - identify which subtrees loosing balance property

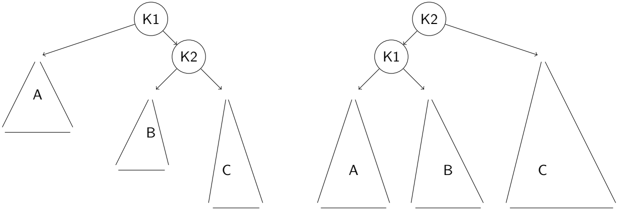
Recovery balance property

Case 1 (perform right rotation at K2)



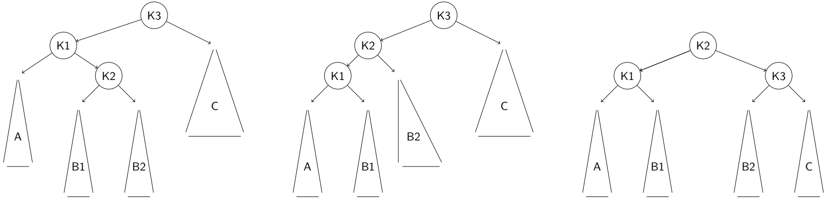
Recovery balance property

Case 2 (perform left rotation at K1)

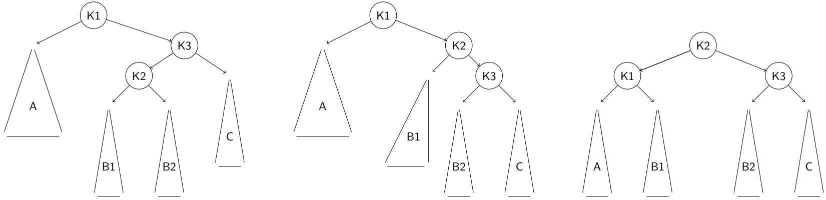


Recovery balance property

Case 3 (perform left rotation at K1 and then right rotation at K3)



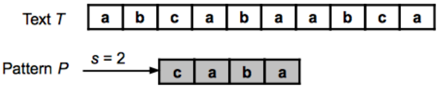
Case 4 (perform right rotation at K3 and then left rotation at K1)



Outline

String searching

- String matching problem: find one or all occurrences of a pattern in a given text
- Applications
  - information retrieval
  - Text editors
  - computational biology (DNA sequences)
- Formal formulation
  - A text is an array $T[1..n]$ and a pattern is an array $P[1..m]$ $(m \neq n)$
  - $T[i], P[j] \in$ a finite alphabet $\sum$ (e.g., $\sum = \{0, 1\}$ or $\sum = \{a, \ldots, z\}$)
  - We say that pattern P **occurs with shift s** in T if $0 \leq s \leq n - m$ and $T[s + 1..s + m]$=$P[1..m]$



String searching algorithms

- Naive
- Boyer-Moore
- Rabin-Karp
- Knuth-Morris-Pratt (KMP)

Naive algorithm

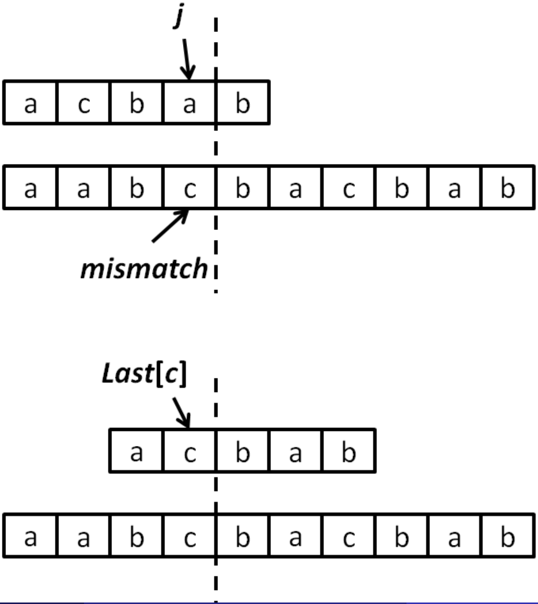
Algorithm 1: NaiveSM(P, T)

```
foreach s = 0..n-m do
    i ← 1;
    while i ≤ m and P[i] = T[i + s] do
        i ← i + 1;
    if i ≥ m then
        Output(s);
```

Boyer-Moore algorithm

- Left to right shift
- Right to left scan
- Use information gained by preprocessing P in order to skip as many alignment as possible
- Bad character shift rule
  - $last[c]$: the right-most occurrence of c in P
  - When mismatch: shift P right by $max\{j - last[c], 1\}$ where j is the position of mismatch character of P

Boyer-Moore algorithm



Boyer-Moore algorithm

```
void computeLast(){
  for(int c = 0; c < 256; c++)
   last[c] = 0;
  for(int i = m; i >= 1; i--){
    if(last[P[i]] == 0)
      last[P[i]] = i;
  }
}
void BoyerMoore(){
  int s = 0;
  while(s <= n-m){
    int j = m;
    while(j > 0 && T[j+s] == P[j])   j--;
    if(j == 0){
      Output(s);
      s = s + 1;
    }else{
      int k = last[T[j+s]];
      s = s + max(j-k,1);
    }
  }
```

Rabin-Karp algorithm

- Convert the pattern $P[1..m]$ to a number:

$$p = P[1] * d^{m-1} + P[2] * d^{m-2} + \cdots + P[m] * d^0$$

  where each character $P[i]$ is viewed as a nonnegative integer $< d$, and d is the size of the alphabet

- Using Horner's rule:

$$p = P[m] + d * (P[m-1] + d * (\cdots + d * P[1]) + \dots)$$

- Convert $T[s+1..s+m]$ to the integer

$$t_s = T[s+1] * d^{m-1} + \cdots + T[s+m]$$

- **Note**: t_{s+1} can easily be computed from t_s as follows: $t_{s+1} = (t_s - T[s+1] * d^{m-1}) * d + T[s+m+1]$

Rabin-Karp algorithm

- Drawback: when m is large, then the computation of p and t_s does not take constant time
- Solution: Compute p and t_s modulo a suitable number q
  - Still problem: $p \equiv t_s (mod\ q)$ does not mean that $p = t_s$, we have to check $P[1..m]$ and $T[s+1..s+m]$ character by character to see if they are really identical
- Worst-case time is $\mathcal{O}(mn)$ where $P = a^m$ and $T = a^n$

Knuth-Morris-Pratt (KMP) algorithm

- Comparison: from left to right
- Shift: more than one position
- Preprocessing the pattern
  - Pattern $P[1..m]$
  - $\pi[q]$ is the length of the longest prefix of $P[1..m]$ which is also the **strictly** suffix of $P[1..q]$

Example

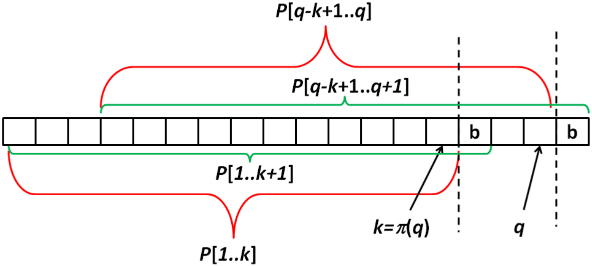
q	1	2	3	4	5	6	7	8	9	10
$P[q]$	a	b	a	b	a	b	a	b	c	a
$\pi[q]$	0	0	1	2	3	4	5	6	0	1

Knuth-Morris-Pratt (KMP) algorithm - preprocessing

```
void computePI(){
  pi[1] = 0;
  int k = 0;
  for(int q = 2; q <= m; q++){
    while(k > 0 && P[k+1] != P[q])
      k = pi[k];
    if(P[k+1] == P[q])
      k = k + 1;
    pi[q] = k;
  }
}
```

Denote $k = \pi[q]$

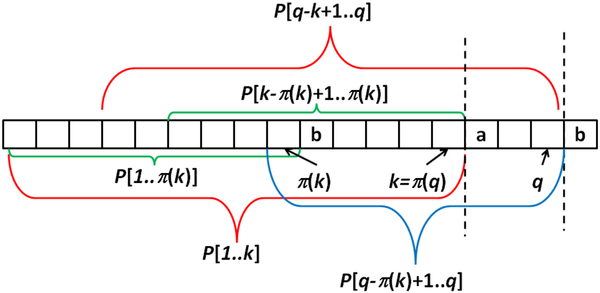
- If $P[q+1] = P[k+1]$, then $\pi[q+1] = \pi[q] + 1$



Knuth-Morris-Pratt (KMP) algorithm - preprocessing

Denote $k = \pi[q]$

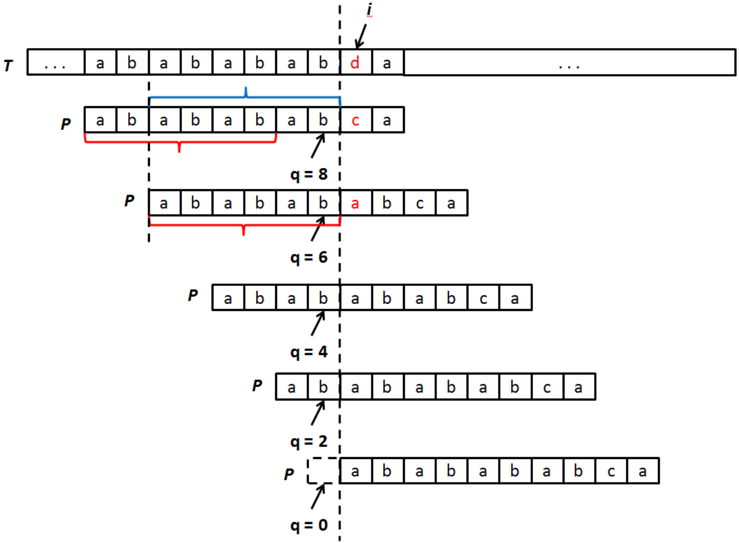
- if $P[q+1] \neq P[k+1]$ and $P[q+1] = P[\pi[k]+1] = b$:
  - $P[1..k] = P[q-k+1..q] \Rightarrow P[k-\pi[k]+1..k] = P[q-\pi[k]+1..q]$
  - Moreover, $P[k-\pi[k]+1] = P[1..\pi[k]$, so $P[1..\pi[k]] = P[q-\pi[k]+1..q]$,
  - Hence $P[1..\pi[k]+1] = P[q-\pi[k]+1..q+1]$, this means $\pi[q+1] = \pi[k]+1$



Knuth-Morris-Pratt (KMP) algorithm

```
void kmp(){
  int q = 0;
  for(int i = 1; i <= n; i++){
    while(q > 0 && P[q+1] != T[i]){
      q = pi[q];
    }
    if(P[q+1] == T[i])
      q++;
    if(q == m){
      cout << "match at position " << i-m+1 << endl;
      q = pi[q];
    }
  }
}
```

Knuth-Morris-Pratt (KMP) algorithm



Outline

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2. Binary searches
3. Binary search trees
4. AVL trees
5. String searching
6. Map and Hashing

Map ADT

- Table stores key-value pairs
- Keys cannot be duplicated
- Operations
  - size()
  - empty()
  - get(k): return the item having key k
  - put(k, v): put a key-value pair (k, v) into the table
  - remove(k): remove item having key k from the table
- Operations should be performed efficiently without sorting items of the table

Map implementation

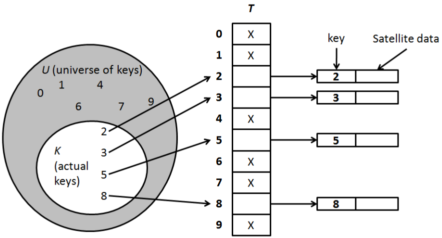
- Arrays
- Linked-lists
- Binary search trees
- **Hash tables** (focus of this topic)

Hashing

- A different approach to searching from the comparison-based methods
- Hashing tries to reference an item in the table directly based on its key without navigating through the table and comparing the search key with the keys of all items
- Hashing transform a key into a table address
- Two approaches
  - Direct-address tables
  - Hash tables

Direct-address tables

- Simple technique
- Work well when the universe U of keys is small
- Suppose each key is taken from $U = \{0, \ldots, m-1\}$ where m is not too large
- Use an array (**direct-address table**) $T[0..m-1]$
  - Each slot k corresponds to a key k

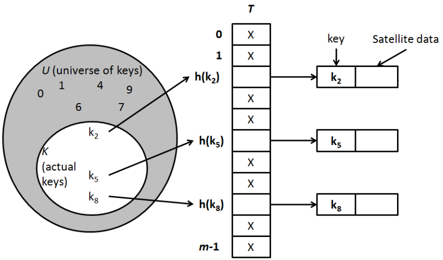


Hash table

- Each element with key k is stored in slot $h(k)$ of the hash table (called **hash function**)

$$h : U \to \{0, 1, \ldots, m-1\}$$

- The size m of hash table is typically much less than the size of U
- **Collision** when two keys k_1, k_2 hash the same slot: $h(k_1) = h(k_2)$
  - Solution by chaining
  - Solution by open addressing



Chaining

- Place all elements that hash to the same slot into the same linked list
- Slot i of the table stores a pointer to the head of the linked list

Algorithm 2: Put(k,v)

$x.key \leftarrow k$;
$x.value \leftarrow v$;
Insert x at the head of list $T[h(k)]$;

Algorithm 3: Get(k)

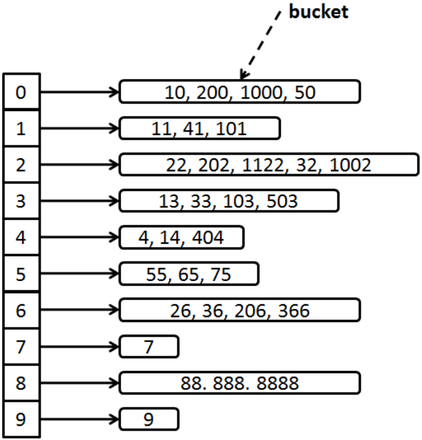
Search an element with key k in list $T[h(k)]$;
return x;

Algorithm 4: Remove(k)

$x \leftarrow \text{Get}(k)$;
Delete x from list $T[h(k)]$;

Hashing: example

- keys: integer
- m is chosen to be 10
- $h(k) = k \bmod 10$
- Collison: two keys that hash to the same value (e.g., 22, 202)



Analysis of hashing with chaining

- We define a **load factor** $\alpha = \frac{n}{m}$
- Assumption of **simple uniform hashing**
- Average running time of unsuccessful search is $\Theta(1 + \alpha)$ (proof is detailed in "Introduction to Algorithms" book)
- Average running time of successful search is $\Theta(1 + \alpha)$ (proof is detailed in "Introduction to Algorithms" book)
- Put and get actions take $\mathcal{O}(1)$ if the lists are implemented as doubly linked lists

Open addressing

- All elements are stored in table itself
- When searching for an element: examine tables slots
- No list and no elements stored outside the table, avoid pointers together
- The hash table can fill up, no further insertion can be made

Open addressing - insertion

- Sucessively examine (**probe**) the table until an empty slot is found to put the key
- Instead of fixing the order $0, 1, m-1$ ($\Theta(n)$ search time), the sequence of slots probed depends the key being inserted
- Extended hash function:
  $h : U \times \{0, 1, \ldots, m-1\} \rightarrow \{0, 1, \ldots, m-1\}$
- The probe sequence for key k is $\langle h(k,0), h(k,1), \ldots, h(k, m-1)\rangle$ which is a permutation of $0, 1, \ldots, m-1$

Open addressing - insertion

Algorithm 5: Put(k, v)

$x.key \leftarrow k$;
$x.value \leftarrow v$;
$i \leftarrow 0$;
while $i < m$ **do**
    $j \leftarrow h(k, i)$;
    if $T[j] = NULL$ **then**
        $T[j] \leftarrow x$;
        return j;
    $i \leftarrow i + 1$;
Error "hash table overflow";

Open addressing - search

Algorithm 6: Get(k)

```
i ← 0;
while i < m do
    j ← h(k, i);
    if T[j].key = k then
        return T[j];
    i ← i + 1;
return NULL;
```

Open addressing

- **Uniform hashing** assumption: Each key is equally likely to have any of $m!$ permutation of $\{0, 1, \ldots, m-1\}$ as its probe sequence
- Three common techniques for probe sequence computation
  - Linear probing
  - Quadratic probing
  - Double hashing
- All of three techniques guarantee that $h(k, 0), h(k, 1), \ldots, h(k, m-1)$ is a permutation of $0, 1, \ldots, m-1$ for each key k
- None of three techniques fulfills the assumption of uniform hashing

Open addressing

- Linear probing: $h(k, i) = (h'(k) + i)$ **mod** m where h' is an ordinary hash function
- Quadratic probing: $h(k, i) = (h'(k) + c_1 i + c_2 i^2)$ **mod** m where h' is ordinary hash function and $c_1, c_2 \neq 0$ are constants
- Double hashing (one of the best method for open addressing): $(h(k, i) = (h_1(k) + ih_2(k))$ **mod** m where h_1, h_2 are auxiliary hash functions
  - The value of $h_2(k)$ must be relatively prime to the hash table size m for the entire hash table to be searched (see "Introduction to Algorithms" for more detail)
  - Some approaches:
    - Let m be a power of 2 and design $h_2(k)$ such that it always returns an odd value
    - Let m be a prime and design $h_2(k)$ such that it always returns a positive integer less than m (e.g., $h_2(k) = 1 + k$ **mod** $(m - 1)$)

Open addressing - analysis

- Inserting an element into an open-address hash table with load factor $\alpha = \frac{n}{m}$ requires at most $\frac{1}{1-\alpha}$ probes on average, assuming uniform hashing

Universal hashing

- If the chosen hash function is fixed, then there might be n keys that hash to the same slot, yielding an average retrieval time of $\Theta(n)$
- Solution: universal hashing
  - Select the hash function randomly from a carefully designed class of functions at the beginning of the execution
  - Randomization guarantees that no single input will always evoke worst-case behavior
  - Good average running time

Definition

Let $\mathcal{H}$ be a finite collection of hash functions that map a given universe U of keys into the range $\{0, 1, \ldots, m-1\}$. Such collection is said to be **universal** if for each pair of distinct keys k, $l \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k) = h(l)$ is at most $\frac{\mathcal{H}}{m}$. In other words, for a given pair of distinct keys k, l, we pick a hash function from $\mathcal{H}$, the probability that $h(k) = h(l)$ is at most $\frac{1}{m}$.

Universal hashing

Theorem

Suppose that a hash function h is chosen randomly from a universal collection of hash functions and has been used to hash n keys into a table T of size m using chaining to resolve collisions. If key k is not in the table, then the expected length $E[n_{h(k)}]$ of the list that k hashes to is at most the load factor $\alpha = \frac{n}{m}$. If the key k is in the table, then the expected length $E[n_{h(k)}]$ of the list that k hashes to is at most $1 + \alpha$

Proof.

See "Introduction to Algorithms" book □

Universal hashing - example

- Choose a prime number p large enough so that every possible key k is in the range $0, \ldots, p-1$, inclusive. Let $\mathbb{Z}_p = \{0, \ldots, p-1\}$, and $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$
- $p > m$ by assumption that the size of universe U is greater than the number of slots of the hash table
- Define hash function $h_{a,b}(k) = ((ak + b) \textbf{ mod } p) \textbf{ mod } m, \forall a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p$
- $\mathcal{H}_{p,m} = \{h_{a,b} : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$

Theorem

$\mathcal{H}_{p,m}$ *defined above is* **universal**

Proof.

See "Introduction to Algorithms" book $\square$

Perfect hashing

- Hashing can provide excellent **worst-case** performance when the set of keys is **static**: once the keys are stored in the table, they never change
- **Perfect hashing**: $\mathcal{O}(1)$ memory access are required to perform a search in the worst-case
- Idea: Two-levels hashing with universal hashing at each level
  - First level: selected carefully from a family of universal hash functions
  - Second level uses hash tables instead of linked lists: Choose carefully hash function h_j for hash table S_j of slot j in order to guarantee that there are no collisions at the second level
    - Set the size m_j of hash table S_j to n_j^2 where n_j is the number of keys hashing to slot j

Perfect hashing