Discrete Mathematics

Counting Problem

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Hanoi, 2012

Outline

Introduction

Definition

How many objects with certain properties?

Example

- Q: How many binary strings of length 5?
- A: 2^5
- **Generalization**: The number of sequences of length n selected from a set of k objects is k^n

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- Q: How many ways to arrange 5 people A, B, C, D, E in a sequence such that A and B are not adjacent?
- A: 5! - 2$\times$4!

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- Q: How many ways to establish a jury of 3 members: a president, an examiner, and a secretary from 5 people A, B, C, D, E?
- A: $5 \times 4 \times 3 = \frac{5!}{(5-3)!}$
- **Generalization**: The number of ordered k-tuple $(a_1, \ldots, a_k)$ established from a set of n elements $\{1, 2, \ldots, n\}$ such that $a_i \neq a_j, \forall i \neq j \in \{1, 2, \ldots, k\}$ is $\frac{n!}{(n-k)!}$

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- Q: How many ways to establish a group of 3 people from 5 people A, B, C, D, E?
- A: $\frac{5\times4\times3}{3!} = \frac{5!}{3!\times2!}$
- **Generalization**: The number of ways to select k objects from a set of n objects is $C(k,n) = \frac{n!}{k!\times(n-k)!}$
  - $C(0,n) = C(n,n) = 1$
  - $C(k,n) = C(k-1,n-1) + C(k,n-1)$

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- Q: How many binary sequence of length $m + n$ in which there are n bits 0 and m bits 1?
- A: $C(m, m + n) = \frac{(m+n)!}{n! \times m!}$

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Basic counting rules

- Approach: Simplify the solution by decomposing the problem
- Two basic decomposition rules
  - Product rule: $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \ldots |A_n|$
  - Sum rule: A and B are independent: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion principle

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|$
- Generic case:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n}((-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} |\bigcap_{j=1}^{k} A_{i_j}|)$$

Example

- Question: How many numbers of $X = \{1, \ldots, 1000\}$ which are not divisible by 3, 4, 7 ?
- Answer: A_3, A_4, A_7 are respectively the sets of numbers of X divisible by 3, 4, and 7
  - Result $= |X| - |A_3 \cup A_4 \cup A_7| = 1000 - |A_3| - |A_4| - |A_7| + |A_3 \cap A_4| + |A_4 \cap A_7| + |A_7 \cap A_3| - |A_3 \cap A_4 \cap A_7| = 1000 - \lfloor \frac{1000}{3} \rfloor - \lfloor \frac{1000}{4} \rfloor - \lfloor \frac{1000}{7} \rfloor + \lfloor \frac{1000}{3 \times 4} \rfloor + \lfloor \frac{1000}{4 \times 7} \rfloor + \lfloor \frac{1000}{7 \times 3} \rfloor$ - $\lfloor \frac{1000}{3 \times 4 \times 7} \rfloor$

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Example

- Question: How many ordered $(X_1, X_2, \ldots, X_k)$ such that $X_1 + X_2 + \cdots + X_k = N, X_i \in \mathbb{Z}_+, \forall i = 1, \ldots, k$?
- Answer:
  - A solution is generated by putting $k-1$ delimiters at $n-1$ intervals $[1,2],[2,3],\ldots,[n-1,n]$
  - $C(k-1, n-1) = \frac{(n-1)!}{(k-1)!(n-k)!}$

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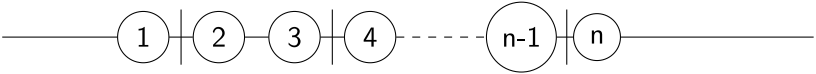

Example

- Question: How many ways to take k elements from a sequence of n elements such that no two adjacent elements are taken?
- Answer:
  - A solution is $(x_1, x_2, \dots, x_k)$ such that $1 \leq x_1 < x_2 - 1 < x_3 - 2 < \cdots < x_k - k + 1 \leq n - k + 1$ $(x_i \in \{1, \dots, n\})$
  - $C(k, n-k+1) = \frac{(n-k+1)!}{k!(n-2k+1)!}$

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Counting Problem - Basic counting problem

Example

- Question: How many ways to take k elements from n elements $\{1, 2, \ldots, n\}$ lying on a cycle such that no two adjacent elements are taken?
- Answer:
  - The set of solutions is divided into two sets
    1. S_1 contains solutions in which 1 is selected. In this case, 2 and n cannot be selected. Hence, we select $k-1$ elements from $n-3$ remaining elements lying on a line as above example
    2. $\overline{S_1}$ contains solutions in which 1 is not selected. In this case, we select k elements from $n-1$ elements lying on a line as above example
  - $|S_1| + |\overline{S_1}| = C(k-1, n-k-1) + C(k, n-k)$

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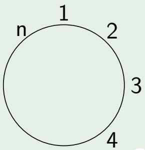


Outline

Definition

- $F_n = C_1 F_{n-1} + C_2 F_{n-2} + \cdots + C_k F_{n-k} + P(n)$
- $C_1, \ldots, C_k$ are constant
- $P(n)$ is a function of n

Linear Recurrence Relations - homogeneous

$$\begin{cases} F_n = C_1 F_{n-1} + C_2 F_{n-2} + \cdots + C_k F_{n-k} \\ F_0 = a_0, F_1 = a_1, \ldots, F_{k-1} = a_{k-1} \end{cases}$$

- Characteristic equation: $x^k - C_1 x^{k-1} - C_2 x^{k-2} - \cdots - C_k = 0$

Theorem

If the characteristic equation has k distinct real roots $x_1, \ldots, x_k$, then $F_n = \alpha_1 x_1^n + \alpha_2 x_2^n + \cdots + \alpha_k x_k^n$

Linear Recurrence Relations - homogeneous

$$\begin{cases} F_n = C_1 F_{n-1} + C_2 F_{n-2} \\ F_0 = a_0, F_1 = a_1 \end{cases}$$

- Characteristic equation: $x^2 - C_1 x - C_2 = 0$

Theorem

If the characteristic equation has exactly one real root x_0, then $F_n = \alpha_1 x_0^n + \alpha_2 n x_0^n$

Definition

$$\begin{cases} F_n = C_1F_{n-1} + C_2F_{n-2} + \cdots + C_kF_{n-k} + t^n \times P_q(n) \\ F_0 = a_0, F_1 = a_1, \ldots, F_{k-1} = a_{k-1} \end{cases}$$

$P_q(n)$ is a polynomial of degree q and t is a constant

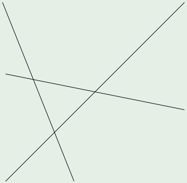
- Find F_n of the form $F_n = F_n^{(h)} + F_n^{(p)}$
  - $F_n^{(h)}$ is a generic solution to the associated homogeneous recurrence relation: $F_n = C_1F_{n-1} + C_2F_{n-2} + \cdots + C_kF_{n-k}$
  - $F_n^{(p)}$ is a particular solution to the original heterogeneous recurrence relation

Linear Recurrence Relations - heterogeneous

- If t is not a root of the characteristic equation of the homogeneous recurrence relation, then there is a particular solution of heterogeneous recurrence relation the form:
$F_n^{(p)} = t^n \times (A_q n^q + A_{q-1} n^{q-1} + \cdots + A_0)$
- If t is a root of the characteristic equation of multiplicity m, then there is a particular solution to the heterogeneous recurrence relation of the form: $F_n^{(p)} = t^n \times n^m \times (A_q n^q + A_{q-1} n^{q-1} + \cdots + A_0)$

Example

How many parts the plane is divided by n lines in which two any lines intersect, and no three lines intersect at one point?



$$\begin{cases} S_{n+1} = S_n + n + 1 & \forall n \geq 1 \\ S_1 = 2 \end{cases}$$

Example

How many binary sequences containing no 2 consecutive bits 0?

- S_n is the sets of binary sequences of length n containing no 2 consecutive bits 0
- S_n^1 is the subset of S_n in which each sequence starts with 1
- S_n^0 is the subset of S_n in which each sequence starts with 0
- $|S_n| = |S_n^1| + |S_n^0|$
- $|S_n^1| = |S_{n-1}|$ and $|S_n^0| = |S_{n-2}|$
- $\Rightarrow$

$$\begin{cases} |S_n| = |S_{n-1}| + |S_{n-2}|, n > 2 \\ |S_1| = 2, |S_2| = 3 \end{cases}$$

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- $\Rightarrow$
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Example

How many binary sequences containing no 3 consecutive bits 0?

- S_n is the sets of binary sequences of length n containing no 3 consecutive bits 0
- S_n^1 is the subset of S_n in which each sequence starts with 1
- S_n^{00} is the subset of S_n in which each sequence starts with 00
- S_n^{01} is the subset of S_n in which each sequence starts with 01
- We have $|S_n| = |S_n^1| + |S_n^{00}| + |S_n^{01}|$
- $|S_n^1| = |S_{n-1}|$, $|S_n^{00}| = |S_{n-3}|$, $|S_n^{01}| = |S_{n-2}|$
- $\Rightarrow F_n = F_{n-1} + F_{n-2} + F_{n-3}, \forall n \geq 4$ and $F_1 = 1, F_2 = 4, F_3 = 7$

Linear Recurrence Relations and Counting Problem

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- $\Rightarrow F_n = F_{n-1} + F_{n-2} + F_{n-3}, \forall n \geq 4$ and $F_1 = 1, F_2 = 4, F_3 = 7$

Example

How many configurations $(a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1, 2\}$ containing neither two consecutive 0s nor two consecutive 1s?

- S_n: set of configurations satisfying the given condition
- S_n^0: subset of S_n such that $a_1 = 0$
- S_n^1: subset of S_n such that $a_1 = 1$
- S_n^2: subset of S_n such that $a_1 = 2$
- $\Rightarrow |S_n| = |S_n^0| + |S_n^1| + S_n^2|$
- $|S_n^2| = |S_{n-1}|$
- $|S_n^0| = |S_n^1|$

Linear Recurrence Relations and Counting Problem

Example

How many configurations $(a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1, 2\}$ containing neither two consecutive 0s nor two consecutive 1s?

- S_n^{01}: subset of S_n^0 such that $a_2 = 1$
- S_n^{02}: subset of S_n^0 such that $a_2 = 2$
- $\Rightarrow |S_n^0| = |S_n^{01}| + |S_n^{02}|$
- $|S_n^{02}| = |S_{n-2}|$
- $|S_n^{01}| = |S_{n-1}^1| = |S_{n-1}^0|$
- Denote $Q_n = |S_n|$ and $P_n = |S_n^0|$
- We have

$$\begin{cases} Q_n = 2P_n + Q_{n-1} & (1) \\ P_n = P_{n-1} + Q_{n-2} & (2) \end{cases}$$

Example

How many configurations $(a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1, 2\}$ containing neither two consecutive 0s nor two consecutive 1s?

- $(1) \Rightarrow P_n = \frac{Q_n - Q_{n-1}}{2} (3)$
- (2) and $(3) \Rightarrow \frac{Q_n - Q_{n-1}}{2} = \frac{Q_{n-1} - Q_{n-2}}{2} + Q_{n-2}$
- $\Rightarrow$

$$\begin{cases} Q_n = 2Q_{n-1} + Q_{n-2} \\ Q_1 = 3, Q_2 = 7 \end{cases}$$

Example

How many ways to divide a rectangle of size $2 \times n$ into sub-rectangles of sizes $1 \times 2, 2 \times 1, 2 \times 2$ and the sides of these sub-rectangles are parallel with the sides of the original rectangle?

$$Answer \begin{cases} S_n = S_{n-1} + 2S_{n-2}, \forall n \geq 3 \\ S_1 = 1, S_2 = 3 \end{cases}$$

Outline

Definition

Generating function g(x) of the sequence $\{h_n \mid n = 1, 2, \dots\}$ is the formal power series

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots = \sum_{i=0}^{\infty} h_i x^i$$

If the sequence if finite, then there exists m such that $h_i = 0, i > m$: $g(x)$ is a polynomial of degree m

Example

$g(x) = (1 + x)^m$ is the generating function of the sequence $\{h_k = C(m, k) \mid k = 0, 1, \dots, m\}$:

$$(1 + x)^m = \sum_{k=0}^{m} C(m, k) x^k$$

Example

- $g(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots$ is the generating function of the sequence $1, 1, 1, \dots$
- $g(x) = \frac{1-x^k}{1-x} = 1 + x + x^2 + \dots + x^k$ is the generating function of the sequence of k 1 elements: $1, 1, \dots, 1$
- $g(x) = 1+3x+\frac{1}{1-x} = 1+3x+(1+x+x^2+x^3\dots) = 2+4x+x^2+\dots$ is the generating function of the sequence $2, 4, 1, 1, \dots$

Generating Functions

- $g(x) = \frac{1}{(1-x)^k} = (1 + x + x^2 + \dots)^k = \sum_{i=0}^{\infty} h_i x^i$ where h_n is the number of natural roots of the equation $t_1 + t_2 + \cdots + t_k = n$
- It is known that $h_k = C(n + k - 1, n)$
- Hence, $\frac{1}{(1-x)^k}$ is the generating function of the sequence $\{C(n + k - 1, n) \mid n = 0, 1, 2, \dots\}$

Generating Functions

- $\frac{x^k}{1-x} = x^k(1 + x + x^2 + \dots) = x^k + x^{k+1} + x^{k+2} + \dots$
- $\frac{1}{1-x^2} = 1 + x^2 + x^4 + \dots$
- $\frac{x}{1-x^2} = x + x^3 + x^5 + \dots$
- $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

- $g(x) = (1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)$
- The coefficient of x^n in $g(x)$ is the number of natural roots of the equation $a + b + c = n$ with $0 \le a \le 3$, $0 \le b \le 2$, and $0 \le c \le 4$
- In general, it is not easy to have explicit formula of the result of that counting problem.
- The value of such counting problem can be obtained by a computer program

- The number of natural roots of the equation $X_1 + X_2 + X_3 + X_4 = N$ where X_1 is even, X_2 is odd, $X_3 \leq 4$, and $X_4 \geq 2$
- The coefficient of x^n in the generating function $g(x) = (1 + x^2 + x^4 + x^6 + \dots)(x + x^3 + x^5 + \dots)(1 + x + x^2 + x^3 + x^4)(x^2 + x^3 + \dots) = \frac{1}{1-x^2} \times \frac{x}{1-x^2} \times \frac{1-x^5}{1-x} \times \frac{x^2}{1-x}$

- The number of natural roots of the equation $X_1 + \cdots + X_k = N$ where X_i is odd, $\forall i \in \{1, \ldots, k\}$
- The coefficient of x^n in the generating function $g(x) = (x + x^3 + x^5 \ldots)^k = (\frac{x}{1-x^2})^k$

Example

$$\begin{cases} h_n = 4h_{n-2} \\ h_0 = 0, h_1 = 1 \end{cases}$$

- $g(x) = h_0 + h_1 x + h_2 x^2 + \dots$
- $g(x) - 4x^2 g(x) = h_0 + h_1 x = x$
- $g(x) = \frac{x}{(1-2x)(1+2x)} = \frac{1}{4}(\frac{1}{1-2x} - \frac{1}{1+2x}) = \sum_{k=0}^{\infty} \frac{1}{4}(2^k - (-2)^k)x^k$
- $h_k = \frac{1}{4}(2^k - (-2)^k), \forall k = 0, 1, 2, \dots$

Example

$$\begin{cases} f_n = f_{n-1} + f_{n-2} \\ f_0 = 0, f_1 = 1 \end{cases}$$

- $g(x) = \sum_{k=0}^{\infty} f_k x^k = f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k = f_0 + f_1 x + \sum_{k=2}^{\infty} (f_{k-1} + f_{k-2}) x^k = f_0 + f_1 x + x(g(x) - 1) + x^2 g(x)$
- $\Rightarrow g(x) = \frac{1}{1-x-x^2}$
- $g(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$ in which $\alpha = \frac{1-\sqrt{5}}{2}, \beta = \frac{1+\sqrt{5}}{2}$, and $A = \frac{\alpha}{\alpha-\beta}$, $B = \frac{-\beta}{\alpha-\beta}$
- $\Rightarrow g(x) = \sum_{k=0}^{\infty} \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} x^k$
- $\Rightarrow f_k = \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}]$