Discrete Mathematics

Logic, Sets, Functions

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Hanoi, 2012

Outline

Propositional Logic

Definition

A proposition is a statement that can be either true or false

Example

- Hanoi is the capital of Vietnam (true)
- $1+2 = 5$ (false)

Definition

Let p be a proposition. The statement "It is not the case that p" is called the negation of p, denoted by $\neg p$

Definition

- Let p and q be propositions. The proposition "p and q", denoted by $p \wedge q$, is the proposition that is true when both p and q are true and is false otherwise.
- $p \wedge q$ is called conjunction of p and q

Definition

- Let p and q be propositions. The proposition "p or q", denoted by $p \vee q$, is the proposition that is false when both p and q are false and is true otherwise.
- $p \vee q$ is called disjunction of p and q

Definition

Let p and q be propositions

- The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.
- The implication $p \to q$ is the proposition that is false when p is true and q is false and is true otherwise
- The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the sam truth value and is false otherwise

Propositional Equivalences

Definition

The propositions p and q are called logically equivalent ($p \Leftrightarrow q$) if $p \leftrightarrow q$ is always true

Example

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ (see truth table)

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

- $p \rightarrow q \Leftrightarrow \neg p \vee q$

Propositional Equivalences

- $p \wedge \mathrm{T} \Leftrightarrow p$
- $p \vee \mathrm{F} \Leftrightarrow p$
- $p \vee \mathrm{T} \Leftrightarrow \mathrm{T}$
- $p \wedge \mathrm{F} \Leftrightarrow \mathrm{F}$
- $p \wedge p \Leftrightarrow p$
- $p \vee p \Leftrightarrow p$
- $\neg(\neg p) \Leftrightarrow p$
- $p \vee q \Leftrightarrow q \vee p$
- $p \wedge q \Leftrightarrow q \wedge p$
- $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
- $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
- $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

- $\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$
- $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$
- $p \wedge \neg p \Leftrightarrow \mathrm{F}$
- $p \vee \neg p \Leftrightarrow \mathrm{T}$

Exercise

- Show that $(p \wedge q) \rightarrow (p \vee q) \Leftrightarrow \mathrm{T}$

Predicates and Quantifiers

- The propositional logic is not powerful enough:
- Example: the assertion "x is greater than 5", where x is a variable, is not a proposition because we cannot tell whether it is true or false unless you know the value of x

Example

- Q: Let $P(x)$ denote the statement "$x > 5$". What are the truth values of $P(1)$ and $P(7)$?
- A: $P(1)$ is false and $P(7)$ is true

Definition

Propositional function: $P(x_1, \ldots, x_n)$

Quantifiers

- When all the variables in a propositional function are assigned values, the resulting statement has a truth value
- Two types of quantification
  - Universal quantification $\forall$
  - Existential quantfication $\exists$

Definition

- The universal quantification of $P(x)$ is the proposition "$P(x)$ is true for all values of x" (denoted by $\forall x P(x)$)
- The existential quantification of $P(x)$ is the proposition "There exists a value of x such that $P(x)$ is true" (denoted by $\exists x P(x)$)

Example

- Let $Q(x, y)$ denote "$x + y = 0$". What are truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$?
- Let $Q(x, y, z)$ denote "$x + y = z$". What are truth values of the quantifications $\exists z \forall y \forall x Q(x, y, z)$ and $\forall x \forall y \exists z Q(x, y, z)$?

NEGATION

- $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$
- $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Outline

Sets

- Sets are used to group objects having similar properties
- The objects in a set are also called the **elements**, or **members** of the set
- A set is said to contain its elements

Example

- Set of even positive integers less than 8 can be expressed by $\{2, 4, 6\}$
- Set of positive integers divisible by 5 less than 20 is $\{5, 10, 15\}$

Sets

Definition

- Two sets are equal if and only if they have the same elements
- The set A is called to be a subset of another set B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B: $\forall x(x \in A \to x \in B)$
- Let S be a set. If there are exactly n distinct elements in S ($n \geq 0$), we say that S is a **finite set** and n is **cardinality** of S, denoted by $|S|$: $|S| = n$

Cartesian product

Definition

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 is the first element, a_2 is the second element, $\ldots$, and a_n is its n^{th} element.
- $a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two ordered tuples. $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ iff $a_i = b_i, \forall i = 1, \ldots, n$.
- Let A and B be two sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
- $A_1 \times A_2 \times \ldots A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \forall i = 1, 2, \ldots, n\}$

Set operations

Definition

- $A \cup B = \{x \mid x \in A \vee x \in B\}$
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- $A - B$ (or $A \backslash B$) $= \{x \mid x \in A \wedge x \notin B\}$
- $\overline{A} = \{x \mid x \notin A\}$

Properties

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C))$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Outline

Functions

Definition

- Let A and B be two sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write $f : A \to B$

Functions can be specified in different ways:

- Explicitly state the assignment
- Use formula, for exapmle $f(x) = x^2 + 2x$
- Write a computer program to specify a function

Functions

Definition

- A function f is said to be **one-to-one**, or **injective** iff $f(x) = f(y)$ implies that $x = y$
- A function f is said to be **surjective** iff for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$
- A function f is said to be bijective if it is both injective and surjective

Example

- The function $f(x) = x^2$ from the set of integers to the set of integers is neither injective nor surjective
- The function $f(x) = x - 4$ from the set of integers to the set of integers is both injective and surjective