$$\max_{u_t \in \mathcal{U}} \sum_{t=0}^{T} f(t, x_t, u_t)$$

$$x_{t+1} = g(t, x_t, u_t)$$

$$x_0 \in \mathbb{R}$$

Eks: $$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{3} (1 + x_t - u_t^2)$$

$$x_{t+1} = x_t + u_t, \qquad t = 0, 1, 2, 3$$

$$u_t \in \mathbb{R}$$

$$x_0 = 0$$

$$f(t, x_t, u_t) = 1 + x_t - u_t^2$$

$$g(t, x_t, u_t) = x_t + u_t$$

- I periode $T = 3$ er $J_3(x_3)$ den maksimale værdi af $1 + x_3 - u_3^2$ og derfor

$$u_3^* = 0 \implies J_3(x_3) = 1 + x_3$$

- I periode $s = T - 1 = 2$

$$\max_{u_2} \left(1 + x_2 - u_2^2 + J_3(\overbrace{x_2 + u_2}^{= x_3})\right)$$

$$\Leftrightarrow \max_{u_2} \left(1 + x_2 - u_2^2 + 1 + x_2 + u_2\right)$$

$$\Leftrightarrow \max_{u_2} (\underbrace{2 + 2x_2 - u_2^2 + u_2}_{=: h_2(u_2)}) \quad \text{konkav i } u_2$$

$$h_2'(u_2) = -2u_2 + 1 = 0 \quad \Rightarrow \quad u_2^* = \frac{1}{2}$$

$$h_2\left(\frac{1}{2}\right) = 2 + 2x_2 - \frac{1}{4} + \frac{1}{2} = \frac{9}{4} + 2x_2$$

$$\Rightarrow \quad J_2(x_2) = \frac{9}{4} + 2x_2$$

- I periode $s = 1$:

$$\max_{u_1} \left(1 + x_1 - u_1^2 + J_2(x_1 + u_1)\right)$$

$$\Leftrightarrow \quad \max_{u_1} \left(1 + x_1 - u_1^2 + \frac{9}{4} + 2(x_1 + u_1)\right)$$

$$\Leftrightarrow \quad \max_{u_1} \Big(\underbrace{\frac{13}{4} + 3x_1 + 2u_1 - u_1^2}_{=:\, h_1(u_1) \text{ konkav i } u_1}\Big)$$

$$h_1'(u_1) = -2u_1 + 2 = 0 \quad \Rightarrow \quad u_1^* = 1$$

$$h_1(1) = 3x_1 + \frac{17}{4}$$

$$\Rightarrow \quad J_1(x_1) = \frac{17}{4} + 3x_1$$

- I periode $s = 0$:

$$\max_{u_0} \left(1 + x_0 - u_0^2 + J_1(x_0 + u_0)\right)$$

$$\Leftrightarrow \quad \max_{u_0} \left(1 + x_0 - u_0^2 + \frac{17}{4} + 3(x_0 + u_0)\right)$$

$$\Leftrightarrow \quad \max_{u_0} \Big(\underbrace{\frac{21}{4} + 4x_0 + 3u_0 - u_0^2}_{=:\, h_0(u_0) \text{ konkav i } u_0}\Big)$$

$$h_0'(u_0) = -2u_0 + 3 \quad \Rightarrow \quad u_0^* = \frac{3}{2}$$

$$\Rightarrow \quad (u_0^*, u_1^*, u_2^*, u_3^*) = \left(\tfrac{3}{2}, 1, \tfrac{1}{2}, 0\right)$$

$$\Rightarrow \quad x_1^* = x_0 + u_0^* = \tfrac{3}{2}$$

$$x_2^* = x_1^* + u_1^* = \tfrac{3}{2} + 1 = \tfrac{5}{2}$$

$$x_3^* = x_2^* + u_2^* = \tfrac{5}{2} + \tfrac{1}{2} = 3$$

Alternativ løsningsmetode:

$$x_{t+1} = x_t + u_t$$

$$x_1 = x_0 + u_0$$

$$x_2 = x_1 + u_1 = x_0 + u_0 + u_1$$

$$x_3 = x_2 + u_2 = x_0 + u_0 + u_1 + u_2$$

$$I = \sum_{t=0}^{3} (1 + x_t - u_t^2)$$

$$= (1 - u_0^2) + (1 + x_1 - u_1^2) + (1 + x_2 - u_2^2) + (1 + x_3 - u_3^2)$$

$$= 1 - u_0^2 + 1 + x_0 + u_0 - u_1^2 + 1 + x_0 + u_0 + u_1 - u_2^2 + 1 + x_0 + u_0 + u_1 + u_2 - u_3^2$$

$$= 4 + 3u_0 + 2u_1 + u_2 - u_0^2 - u_1^2 - u_2^2 - u_3^2$$

$$\frac{\partial I}{\partial u_0} = 3 - 2u_0 \quad \Rightarrow \quad u_0^* = \frac{3}{2}$$

$$\frac{\partial I}{\partial u_1} = 2 - 2u_1 \quad \Rightarrow \quad u_1^* = 1$$

$$\frac{\partial I}{\partial u_2} = 1 - 2u_2 \quad \Rightarrow \quad u_2^* = \frac{1}{2}$$

$$\frac{\partial I}{\partial u_3} = -2u_3 = 0 \Rightarrow u_3^* = 0$$

Eks :

$$\begin{cases} \max_{u_t \geq 0} \sum_{t=0}^{T-1} \left(-\frac{2}{3} u_t x_t\right) + \log x_T \\ x_{t+1} = x_t(1 + u_t x_t), \quad x_0 > 0 \text{ givet} \end{cases}$$

- $x_0 > 0$, $u_t \geq 0$ for alle $t \Rightarrow x_t > 0$ for alle t
- $f(T, x_T, u_T) = \log x_T$

$$\Rightarrow J_T(x_T) = \log x_T$$

uafh. af u_T, derved optimal for alle u_T

- $s = T-1$:

$$J_{T-1}(x_{T-1}) = \max_{u_{T-1} \geq 0} \left\{ -\frac{2}{3} u_{T-1} x_{T-1} + J_T\left(x_{T-1}(1 + u_{T-1} x_{T-1})\right) \right\}$$

$$= \max_{u_{T-1} \geq 0} \left\{ \underbrace{-\frac{2}{3} u_{T-1} x_{T-1} + \log x_{T-1} + \log(1 + u_{T-1} x_{T-1})}_{=: h_{T-1}(u_{T-1}) \text{ konkav i } u_{T-1}} \right\}$$

$$h'_{T-1}(u_{T-1}) = -\frac{2}{3} x_{T-1} + \frac{x_{T-1}}{1 + u_{T-1} x_{T-1}} = 0$$

$$\Rightarrow u_{T-1}^* = \frac{1}{2 x_{T-1}^*}$$

$$h_{T-1}(u_{T-1}^*) = h_{T-1}\left(\frac{1}{2 x_{T-1}}\right) = -\frac{1}{3} + \log x_{T-1} + \log \frac{3}{2}$$

$$\Rightarrow J_{T-1}(x_{T-1}) = \log x_{T-1} + C, \quad C = \log \frac{3}{2} - \frac{1}{3}$$

- $s = T-2$:

$$J_{T-2}(x_{T-2}) = \max_{u_{T-2} \geq 0} \left\{ -\frac{2}{3} u_{T-2} x_{T-2} + J_{T-1}\left(x_{T-2}(1 + u_{T-2} x_{T-2})\right) \right\}$$

$$= \max_{u_{T-2} \geq 0} \Big\{ \underbrace{-\frac{2}{3} u_{T-2} x_{T-2} + \log x_{T-2} + \log(1 + u_{T-2} x_{T-2}) + C}_{=: \, h_{T-2}(u_{T-2}) \text{ konkav i } u_{T-2}} \Big\}$$

$$h'_{T-2}(u_{T-2}) = -\frac{2}{3} x_{T-2} + \frac{x_{T-2}}{1 + x_{T-2} u_{T-2}} = 0$$

$$\Rightarrow \quad u^*_{T-2} = \frac{1}{2 x^*_{T-2}}$$

$$h_{T-2}\left(\frac{1}{2 x_{T-2}}\right) = -\frac{1}{3} + \log x_{T-2} + C + \log \frac{3}{2}$$

$$= \log x_{T-2} + 2C$$

- Mønstret fortsætter for $k = 0, 1, 2, \ldots, T$:

$$J_{T-k}(x_{T-k}) = \log x_{T-k} + kC, \quad C = \log \frac{3}{2} - \frac{1}{3}$$

$$u^*_{T-k} = \frac{1}{2 x^*_{T-k}}$$

- Lad $t = T - k$, $k = 0, 1, \ldots, T$:

$$J_t(x_t) = \log x_t + (T - t) C$$

$$u^*_t = \frac{1}{2 x^*_t}$$

- $$x^*_{t+1} = x^*_t \left(1 + \frac{1}{2 \cancel{x^*_t}} \cancel{x^*_t}\right) = \frac{3}{2} x^*_t$$

$$\Rightarrow \quad x^*_t = \left(\frac{3}{2}\right)^t x_0$$

$$u^*_t = \frac{1}{2 x^*_t} = \left(\frac{2}{3}\right)^t \frac{1}{2 x_0} \qquad \square$$