Kædereglen:

$$\begin{array}{ccccc} & \mathbb{R}^n & & \mathbb{R}^m & \\ & \cup & & \cup & \\ f: & X & \to & Y & \end{array} \qquad f(x) = y$$

$$g : Y \to \mathbb{R}^k$$

$$g \circ f(x) = g(f(x))$$

$$\underset{k\times n}{D(g\circ f)(x)} = \underset{k\times m}{Dg(y)}\ \underset{m\times n}{Df(x)}$$

Bevis: Lad $A = Df(x)$, $B = Dg(y)$

Vi vise, at $D(g\circ f)(x) = BA$.

f, g differentiable:

$$\underset{m}{f(x+h)} = \underset{m}{f(x)} + \underset{m\times n}{A}\underset{n}{h} + \underset{m}{e_f(h)}$$

$$g(y+p) = g(y) + Bp + e_g(p)$$

$$\lim_{h\to 0} \frac{e_f(h)}{\|h\|} = 0 \qquad \lim_{p\to 0} \frac{e_g(p)}{\|p\|} = 0$$

Derved,

$$\begin{aligned}(g\circ f)(x+h) &= g(f(x+h)) \\ &= g(\underbrace{f(x)}_{y} + \underbrace{Ah + e_f(h)}_{p(h)}) \\ &= g(f(x)) + B(Ah + e_f(h)) + e_g(Ah + e_f(h)) \\ &= \underbrace{g(f(x))}_{g\circ f(x)} + BAh + \underbrace{Be_f(h) + e_g(Ah + e_f(h))}_{e(h)}\end{aligned}$$

Vi skal vise, at

$$\lim_{h\to 0} \frac{e(h)}{\|h\|} = 0$$

Vi har: $\lim_{h\to 0} (Ah + e_f(h)) = 0$

$$\lim_{h\to 0} \frac{Be_f(h)}{\|h\|} = B \lim_{h\to 0} \frac{e_f(h)}{\|h\|} = 0$$

Udsagnet $\lim_{p\to 0} \frac{e_g(p)}{\|p\|} = 0$

ensbetydende med, at for ethvert lille tal $\varepsilon_1 > 0$ og p tilstrækkeligt lille:

$$\|e_g(p)\| < \varepsilon_1 \|p\|$$

På samme måde, for ethvert lille $\varepsilon_2 > 0$ og h tilstrækkeligt lille

$$\|e_f(h)\| < \varepsilon_2 \|h\|$$

Derved,

$$\begin{aligned}\|e_g(Ah + e_f(h))\| &< \varepsilon_1 \|Ah + e_f(h)\| \\ &\leq \varepsilon_1 \|Ah\| + \varepsilon_1 \|e_f(h)\| \\ &< \varepsilon_1 \|Ah\| + \varepsilon_1\varepsilon_2 \|h\|\end{aligned}$$

eller $\lim_{h\to 0} \frac{e_g(Ah + e_f(h))}{\|h\|} = 0$

Vi får

$$\begin{aligned}\|e(h)\| &= \|Be_f(h) + e_g(Ah + e_f(h))\| \\ &\leq \|Be_f(h)\| + \|e_g(Ah + e_f(h))\| \\ &\leq k\|h\|\end{aligned}$$

for et $k > 0$ der afhænger af $\varepsilon_1, \varepsilon_2$. Og

$$\lim_{h\to 0} \frac{e(h)}{\|h\|} = 0 \qquad \square$$