$$f: \mathbb{R}^n \to \mathbb{R}$$
$$x \mapsto f(x)$$

$f \in C^k$ i en åben omegn X af x_0.

Lad $h = (h_1, h_2, \dots, h_n)'$ og

$$x_0 + th \in X, \quad t \in [0,1]$$

Lad

$$g(t) = f(x_0 + th)$$

Taylor (en-dimensional): Der findes et $c \in [0,1]$ således, at

$$\underset{0+1}{g(1)} = g(0) + g'(0) + \tfrac{1}{2} g''(c)$$

Vi har

$$g'(t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x_0 + th) h_i$$
$$= \langle \nabla f(x_0 + th), h \rangle$$
$$g''(t) = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial}{\partial x_i} f(x_0 + th) h_i \right)$$
$$\frac{d}{dt}\left(\frac{\partial}{\partial x_i} f(x_0 + th) \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f(x_0 + th) h_j$$

$$\begin{bmatrix} f''_{11}(x) & f''_{12}(x) \\ f''_{12}(x) & f''_{22}(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(x) & \frac{\partial^2}{\partial x_1 \partial x_2} f(x) \\ \frac{\partial^2}{\partial x_1 \partial x_2} f(x) & \frac{\partial^2}{\partial x_2^2} f(x) \end{bmatrix}$$

$$\Rightarrow g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(x_0 + th) h_i h_j$$
$$= h^T \operatorname{Hess} f(x_0 + th)\, h$$
$$= \langle h, \operatorname{Hess} f(x_0 + th) h \rangle$$

$$g(1) = g(0) + g'(0) + \frac{g''(c)}{2}$$
$$f(x_0 + h) = f(x_0) + \langle \nabla f(x_0), h \rangle + \tfrac{1}{2} h^T \operatorname{Hess} f(x_0 + ch)\, h$$

Til anden orden:

$$f(x+h) = f(x) + \langle \nabla f(x), h \rangle + \tfrac{1}{2} h^T \operatorname{Hess} f(x_0) h + R_3$$

<u>Eks</u>:

$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto x_1^2 + 2x_1x_2 + x_2^2$$
$$\nabla f(x) = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\operatorname{Hess} f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \qquad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$
$$f(x+h) = f(x) + \langle \nabla f(x), h \rangle + h^T \operatorname{Hess} f(x) h + R_3$$
$$= x_1^2 + 2x_1x_2 + x_2^2 + (2x_1 + 2x_2, 2x_1 + 2x_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$
$$+ \tfrac{1}{2} \underbrace{(h_1, h_2) \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}_{2h_1^2 + 2h_1h_2 + 2h_1h_2 + 2h_2^2}$$
$$\Rightarrow R_3 = 0$$

$$x_1^2 + 2x_1x_2 + x_2^2 = (x_1, x_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

<u>Eks</u>: (Afledede af en kvadratisk form)

$A \in \mathbb{R}^{n \times n}$ symmetrisk.

$$q_A(x) = x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$
$$= \langle x, Ax \rangle$$

Evaluér i $x + h$:

$$q_A(x+h) = \langle x+h, A(x+h) \rangle$$
$$= \langle x+h, Ax + Ah \rangle$$
$$= \langle x, Ax \rangle + \langle x, Ah \rangle + \langle h, Ax \rangle + \langle h, Ah \rangle$$
$$\langle x, Ah \rangle = x^T A h = (x^T A h)^T$$
$$= h^T A x = \langle h, Ax \rangle$$
$$= \underbrace{\langle x, Ax \rangle}_{q_A(x)} + \underbrace{2\langle Ax, h \rangle}_{\langle 2Ax, h \rangle} + \underbrace{\langle h, Ah \rangle}_{q_A(h) = R(h)}$$
$$\lim_{h \to 0} \frac{R(h)}{\|h\|} = \lim_{h \to 0} \frac{h^T A h}{\|h\|} = 0$$
$$\Rightarrow C = Dq_A(x) = 2Ax$$
$$\operatorname{Hess} q_A(x) = 2A \qquad \square$$

$$f(x) = ax^2 \qquad f'(x) = 2ax$$

A ikke symmetrisk: $(x^T A x)' = (A + A^T)x$

<u>Eks</u>:

$$f: \mathbb{R}_+^2 \to \mathbb{R}$$
$$(x, y) \mapsto \frac{x - y}{x + y}$$
$$\nabla f(x,y) = \begin{bmatrix} \frac{2y}{(x+y)^2} \\ -\frac{2x}{(x+y)^2} \end{bmatrix}$$
$$\operatorname{Hess} f(x,y) = \frac{1}{(x+y)^3} \begin{bmatrix} -4y & 2(x-y) \\ 2(x-y) & 4x \end{bmatrix}$$

Taylor til anden orden i $(\frac{1}{2}, \frac{1}{2})$:

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = 0$$
$$\nabla f(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\operatorname{Hess} f(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow f(x_0 + x, y_0 + y)$$
$$= f(x_0, y_0) + \left\langle \nabla f(x_0, y_0), \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle + \tfrac{1}{2}(x, y) \operatorname{Hess} f(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix} + R_3$$
$$= 0 + \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle + \tfrac{1}{2}(x, y) \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + R_3$$
$$= x - y - x^2 + y^2 + R_3$$

Man kan vise, at

$$R_3 = 2\, \frac{(x+y)^2 (x-y)}{(2 + c(x+y))^4}, \quad c \in [0,1]$$

$\square$