Konstante koefficienter

$$x_{t+2} + a x_{t+1} + b x_t = 0$$

Antag $\quad x_t = m^t \quad , m \in \mathbb{R} \,, \ m \neq 0$

(Kapitel 6: $\ x(t) = e^{rt}$)

$$m^{t+2} + a m^{t+1} + b m^t = 0 \quad | : m^t$$

$$m^2 + a m + b = 0$$

$$\Rightarrow \quad m_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$$

1) $\frac{a^2}{4} - b > 0 \quad \Rightarrow \quad m_1 \neq m_2 \,, \ m_{1,2} \in \mathbb{R}$

$$\Rightarrow \quad u_t^{(1)} = m_1^t \qquad u_t^{(2)} = m_2^t$$

2) $\frac{a^2}{4} - b = 0 \quad \Rightarrow \quad m = -\frac{a}{2}$ dobbelt rod

$$\Rightarrow \quad u_t^{(1)} = m^t$$

$$u_t^{(2)} = t m^t$$

$$0 \overset{?}{=} (t+2) m^{t+2} + a(t+1) m^{t+1} + b t m^t \quad / : m^t$$

$$\overset{?}{=} (t+2) m^2 + a(t+1) m + bt$$

$$= t(\underbrace{m^2 + am + b}_{=0}) + \underbrace{2m^2 + am}_{= m(2m+a)}$$

$$\overset{!}{=} 0$$

$$2m + a = 2\left(-\frac{a}{2}\right) + a = -a + a = 0$$

3) $\frac{a^2}{4} - b < 0$

$$m_{1,2} = \underbrace{-\frac{a}{2}}_{\alpha} \pm i \underbrace{\sqrt{b - \frac{a^2}{4}}}_{\beta}$$

$$= r(\cos\theta \pm i \sin\theta)$$

$$\tilde{u}_t^1 = m_1^t = (\alpha + i\beta)^t$$

$$= r^t(\cos\theta + i\sin\theta)^t$$

$$= r^t\left(\cos(\theta t) + i\sin(\theta t)\right)$$

$$\tilde{u}_t^2 = m_2^t = (\alpha - i\beta)^t$$

$$= r^t\left(\cos(\theta t) - i\sin(\theta t)\right)$$

Linearkombinationene:

$$u_t^1 = \frac{\tilde{u}_t^1 + \tilde{u}_t^2}{2} = r^t \cos(\theta t)$$

$$u_t^2 = \frac{\tilde{u}_t^1 - \tilde{u}_t^2}{2i} = r^t \sin(\theta t)$$

er reelle løsninger.

Eks : (1) $x_{t+2} - 5x_{t+1} + 6x_t = 0$

$$m^2 - 5m + 6 = 0$$

$$m_{1,2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$$

$$= \frac{5}{2} \pm \frac{1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

$$\Rightarrow \quad u_t^1 = 3^t, \quad u_t^2 = 2^t$$

$$\Rightarrow \quad x(t) = A3^t + B2^t$$

(2) $x_{t+2} - 6x_{t+1} + 9x_t = 0$

$$m^2 - 6m + 9 = 0$$

$$m_{1,2} = 3 \pm \sqrt{0} \quad \Rightarrow \quad m = 3 \text{ dobbelt rod}$$

$$\Rightarrow \quad u_t^1 = 3^t, \quad u_t^2 = t3^t$$

$$x(t) = A3^t + Bt3^t$$

(3) $x_{t+2} - x_{t+1} + x_t = 0$

$$m^2 - m + 1 = 0$$

$$\Rightarrow m_{1,2} = \frac{1}{2} \pm i\sqrt{\frac{3}{4}} = \underbrace{\frac{1}{2}}_{\alpha} \pm i \underbrace{\frac{\sqrt{3}}{2}}_{\beta}$$

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\cos\theta = \frac{\operatorname{Re} m}{r} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

$$\Rightarrow \quad u^1_t = \cos\left(\frac{\pi}{3}t\right), \quad u^2_t = \sin\left(\frac{\pi}{3}t\right)$$

$$x_t = A\cos\left(\frac{\pi}{3}t\right) + B\sin\left(\frac{\pi}{3}t\right)$$

<u>Eks</u>: $\quad x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3$

$$u^*_t = C4^t + Dt^2 + Et + F$$

$$\begin{array}{lllll}
x_{t+2}: & C4^{(t+2)} & + D(t+2)^2 & + E(t+2) & + F \\
-5x_{t+1}: & -5C4^{(t+1)} & -5D(t+1)^2 & -5E(t+1) & -5F \\
+6x_t: & +6C4^t & +6Dt^2 & +6Et & +6F
\end{array}$$

$$\stackrel{!}{=} 4^t + t^2 + 3$$

Ryd op på leddene:

$$2C4^t + 2Dt^2 + (-6D+2E)t + (-D-3E+2F) \stackrel{!}{=} 4^t + t^2 + 3$$

$$\Rightarrow \quad \begin{aligned} 2C &= 1 \\ 2D &= 1 \\ -6D+2E &= 0 \\ -D-3E+2F &= 3 \end{aligned}$$

$$\Rightarrow \quad C = \frac{1}{2},\ D = \frac{1}{2},\ E = \frac{3}{2},\ F = 4$$

$$\Rightarrow \quad x_t = A2^t + B3^t + \frac{4^t}{2} + \frac{t^2}{2} + \frac{3}{2}t + 4 \qquad \square$$