$t = 0, 1, 2, 3, \dots$

Eks : $x_{t+1} = a\, x_t \qquad a \in \mathbb{R}$

$$x_1 = a\, x_0$$
$$x_2 = a\, x_1 = a(a x_0) = a^2 x_0$$
$$x_3 = a\, x_2 = a(a^2 x_0) = a^3 x_0$$
$$\dots$$
$$x_t = a^t x_0$$
$$|a| < 1 \Rightarrow a^t x_0 \xrightarrow[t \to \infty]{} 0$$
$$|a| > 1 \Rightarrow a^t x_0 \longrightarrow \infty$$

Eks : AR(1) $\quad x_t = \phi\, x_{t-1} + \varepsilon_t$

$$\varepsilon_t \sim N(0, \sigma^2)$$

$$x_t = \phi(\phi\, x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$
$$= \phi(\phi(\phi\, x_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t$$
$$= \dots$$
$$= \phi^t x_0 + \sum_{j=1}^{t-1} \phi^j \varepsilon_{t-j}$$

Eks $\quad x_t = a\, x_{t-1} + b, \qquad a, b \in \mathbb{R}$

$$= a(a x_{t-2} + b) + b$$
$$= a(a(a x_{t-3} + b) + b) + b$$
$$= \dots$$

$$= a^t x_0 + b + ba + ba^2 + \ldots + ba^{t-1}$$
$$= a^t x_0 + b\left(1 + a + a^2 + \ldots + a^{t-1}\right)$$

$$\begin{array}{rl} S_{t-1} &= 1 + a + a^2 + \ldots + a^{t-1} \\ a S_{t-1} &= \quad a + a^2 + \ldots + a^{t-1} + a^t \end{array} \Big| -$$

$$(1-a) S_{t-1} = 1 - a^t$$

$$\Rightarrow S_{t-1} = \frac{1-a^t}{1-a}$$

$$\Rightarrow x_t = a^t x_0 + b \frac{1-a^t}{1-a}$$
$$= \frac{b}{1-a} + a^t \left(x_0 - \frac{b}{1-a}\right)$$

$$\left.\begin{array}{l} |a| < 1 \\ t \to \infty : \quad a^t \to 0 \end{array}\right\} \Rightarrow \lim_{t \to \infty} x_t = \frac{b}{1-a}$$

$$x_{t+1} = a x_t + b_t$$
$$x_t = a x_{t-1} + b_{t-1}$$
$$= a\left(a x_{t-2} + b_{t-2}\right) + b_{t-1}$$
$$= a\left(a\left(a x_{t-3} + b_{t-3}\right) + b_{t-2}\right) + b_{t-1}$$
$$= \ldots$$
$$= a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}$$

Eks: Du låner et beløb G_0 i begyndelsen af totalperioden $t = 0, \ldots, n$ år.

Du betales beløbet A tilbage ved enden af hvert år.

Rentesatsen er r. Gælden i tidspunkt t følger differensligningen

$$G_{t+1} = (1+r)\,G_t - A$$

Løs:

$$x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

$$G_t = (1+r)^t\left(G_0 - \left(\frac{-A}{1-1-r}\right)\right) + \frac{(-A)}{1-1-r}$$

$$= (1+r)^t\left(G_0 - \frac{A}{r}\right) + \frac{A}{r}$$

$$\boxed{= G_0 q^t - A\,\frac{q^t - 1}{q-1}}$$

$q = 1+r$ Annuitetsformel

Eks: Formueakkumulation i Kapitel 5, 5.4

I diskret tid:

$$w_{t+1} = (1+s)\,w_t + y_{t+1} - c_{t+1}$$

w_t formue

y_t indtægt

c_t forbrug

$q = 1+r$

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}$$

$$w_t = q^t w_0 + \sum_{k=1}^{t} q^{t-k}(y_k - c_k)$$

I nutidsværdi:

$$w_t q^{-t} = w_0 + \sum_{k=1}^{t} q^{-k}(y_k - c_k)$$

I kontinuert tid:

$$w(t)e^{-rt} = w(0) + \int_0^t [y(s) - c(s)]e^{-rs}\,ds$$

II