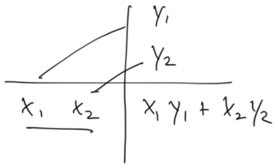
$$\|x+y\|^2 = \|x-y\|^2$$
$$(x_1+y_1)^2 + (x_2+y_2)^2 = (x_1-y_1)^2 + (x_2-y_2)^2$$
$$\cancel{x_1^2} + \underline{2x_1y_1} + \cancel{y_1^2} + \cancel{x_2^2} + 2x_2y_2 + \cancel{y_2^2}$$
$$= \cancel{x_1^2} - 2x_1y_1 + \cancel{y_1^2} + \cancel{x_2^2} - 2x_2y_2 + \cancel{y_2^2}$$
$$4x_1y_1 + 4x_2y_2 = 0$$
$$x_1y_1 + x_2y_2 = 0$$



$$\langle x, y\rangle = x^T y$$

$$\langle \cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$c^2 = \|x+y\|^2$$
$$= \langle x+y, x+y\rangle$$
$$\underset{\text{bil.}}{=} \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$\underset{\text{sym.}}{=} \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$$
$$\left\langle \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \right\rangle = a\cdot 0 + 0b = 0$$
$$= \underbrace{\|x\|^2}_{a^2} + \underbrace{\|y\|^2}_{b^2}$$

C-S $\quad |\langle x,y\rangle| \leq \|x\| \, \|y\|$

- x eller $y = 0$: $\quad 0 = 0$
- x, y lin. afhængige : $\quad y = rx, \quad r \neq 0$

$$\langle x,y\rangle = \langle x, rx\rangle = r\langle x,x\rangle$$
$$= r\,\|x\|\,\|x\|$$
$$= \underbrace{\|rx\|}_{y}\,\|x\|$$

- x, y lin. uafhængige, $\quad x + ry \neq 0$ for alle $r \in \mathbb{R}$

$$0 < \langle x+ry, x+ry\rangle$$
$$= \langle x,x\rangle + r\langle y,x\rangle + r\langle x,y\rangle + r^2\langle y,y\rangle$$
$$= \langle x,x\rangle + 2r\langle y,x\rangle + r^2\langle y,y\rangle$$
$$= \|x\|^2 + 2r\langle x,y\rangle + r^2\|y\|^2$$

Lad $r = -\dfrac{\langle x,y\rangle}{\|y\|^2}$, så gælder:

$$0 < \|x\|^2 - 2\frac{\langle x,y\rangle^2}{\|y\|^2} + \frac{\langle x,y\rangle^2}{\|y\|^2}$$
$$0 < \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2}$$
$$\langle x,y\rangle^2 < \|x\|^2\,\|y\|^2$$
$$|\langle x,y\rangle| < \|x\|\,\|y\| \qquad \square$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

[Ortogonalsæt er lineært uafhængige.

Bevis [ved modstrid]:

Lad x, y $(\neq 0)$ være ortogonale $(x^T y = 0)$, men lineært afhængige.

$\Rightarrow \quad y = rx, \quad r \neq 0.$

$\Rightarrow \quad x^T y = 0 = x^T(rx)$
$$= \langle x, rx\rangle$$
$$= r\|x\|^2$$

Den sidste ligning kan kun holde, hvis enten $r = 0$ eller $x = 0$. Modstrid ↯ $\square$