Eks:
$$\dot{x} = y$$
$$\dot{y} = \frac{y^2}{x} \qquad x, y > 0$$
$$x(1) = 1, \quad y(1) = 2$$

Nu er
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} = \frac{y}{x}$$
$$\int \frac{dy}{y} = \int \frac{dx}{x} + \tilde{c}, \quad \tilde{c} \in \mathbb{R}$$
$$\log y = \log x + \tilde{c}$$
eller $y = cx, \quad c = e^{\tilde{c}}$
$$\Rightarrow \quad \dot{x} = cx$$
$$\Rightarrow x = de^{ct}, \quad d \in \mathbb{R}$$
$$\Rightarrow \quad y = cde^{ct}$$
$$x(1) = de^{c} = 1$$
$$y(1) = cde^{c} = 2 \Rightarrow c = 2$$
$$d = e^{-2}$$
□

Differentiér den første ligning ift. t:
$$\ddot{x} = a_{11}\dot{x} + a_{12}\dot{y} + \dot{b}_1$$
$$= a_{11}\dot{x} + a_{12}(a_{21}x + a_{22}y + b_2) + \dot{b}_1$$
Første ligning:
$$y = \frac{1}{a_{12}}(\dot{x} - a_{11}x - b_1)$$
$$\Rightarrow \ddot{x} = a_{11}\dot{x} + a_{12}a_{21}x + a_{22}(\dot{x} - a_{11}x - b_1) + a_{12}b_2 + \dot{b}_1$$
$$\ddot{x} - (a_{11} + a_{22})\dot{x} + (a_{11}a_{22} - a_{12}a_{21})x = -a_{22}b_1 + a_{12}b_2 + \dot{b}_1$$
$$\Rightarrow \ddot{x} - \operatorname{tr}(A)\dot{x} + \det(A)x = -a_{22}b_1 + a_{12}b_2 + \dot{b}_1$$
Analogt:
$$\ddot{y} - \operatorname{tr}(A)\dot{y} + \det(A)y = a_{21}b_1 - a_{11}b_2 + \dot{b}_2$$
Den homogne ligning
$$\ddot{x} - \operatorname{tr}(A)\dot{x} + \det(A)x = 0$$
har karakteristisk ligning
$$r^2 - \operatorname{tr}(A)r + \det A = 0$$
$$\det\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = ve^{\lambda t} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \lambda v e^{\lambda t}$$
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\lambda v e^{\lambda t} = A v e^{\lambda t} \quad / \, e^{-\lambda t}$$
$$\boxed{\lambda v = Av}$$

Eks:
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2y \\ x+y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$
Kar. poly.:
$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2$$
$$= (\lambda + 1)(\lambda - 2)$$
$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 2$$
Egenvektorene:
$$v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
Løsningen:
$$\begin{bmatrix} x \\ y \end{bmatrix} = ce^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + de^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$c, d \in \mathbb{R}$$
$$= \begin{bmatrix} -2ce^{-t} + de^{2t} \\ ce^{-t} + de^{2t} \end{bmatrix}$$

Inhomogen:
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2y + 6 \\ x + y - 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$
Ligevægtspunkt: $\dot{x} = \dot{y} = 0$
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$
$$\Rightarrow \quad \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$
$$A \quad x = b$$
løsning: $x = 6, \; y = -3$

Substituér:
$$\left.\begin{aligned} z &= x - 6 \\ w &= y + 3 \end{aligned}\right\} \text{afstand fra ligevægt}$$
$$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2(w-3) + 6 \\ (z+6) + (w-3) - 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2w \\ z + w \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}$$
homogen!

Løsning:
$$\begin{bmatrix} z \\ w \end{bmatrix} = ce^{-1t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + de^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2ce^{-1t} + de^{2t} \\ ce^{-1t} + de^{2t} \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2ce^{-1t} + de^{2t} + 6 \\ ce^{-t} + de^{2t} - 3 \end{bmatrix}$$
□