$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix} + b = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \quad A\begin{bmatrix}x\\ y\end{bmatrix} = -b$$

$$\Rightarrow \quad \begin{bmatrix}x^*\\ y^*\end{bmatrix} = A^{-1}(-b)$$

$$x^* = \frac{\det A_1(-b)}{\det A}$$

$$= \frac{\det\begin{bmatrix}-b_1 & a_{12}\\ -b_2 & a_{22}\end{bmatrix}}{\det A} = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$y^* = \frac{\det A_2(-b)}{\det A}$$

$$= \frac{\det\begin{bmatrix}a_{11} & -b_1\\ a_{21} & -b_2\end{bmatrix}}{\det A} = \frac{a_{21}b_1 - a_{11}b_2}{\det A}$$

Eks:

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \underbrace{\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix}}_{A}\begin{bmatrix}x\\ y\end{bmatrix} + \underbrace{\begin{bmatrix}2\\ 8\end{bmatrix}}_{b}$$

Ligevægtspunkt:

$$Ax = -b$$

$$x = A^{-1}(-b)$$

$$= \begin{bmatrix}-\frac{1}{2} & -\frac{1}{4}\\ 0 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-2\\ -8\end{bmatrix} = \begin{bmatrix}3\\ 4\end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix}-2-\lambda & 1\\ 0 & -2-\lambda\end{bmatrix}$$

$$= (-2-\lambda)^2 = \lambda^2 + 4\lambda + 4$$

$$\lambda_{1,2} = -2 \pm \sqrt{4-4} \Rightarrow \lambda = -2 \text{ dobbelt}$$

$$(A - \lambda I)x = 0$$

x	y	
0	1	0
0	0	0

$$\Rightarrow v = \begin{pmatrix}1\\0\end{pmatrix} \text{ egenvektor}$$

Lad $w = y - 4$

så kan den anden ligning skrives som

$$\dot w = -2w$$

med løsning

$$w(t) = e^{-2t}, \quad w(0) = 1.$$

Substituér $z = x - 3$

$$\dot z = -2z + w = -2z + e^{-2t}$$

med løsning

$$z(t) = te^{-2t}$$

$$\Rightarrow \dot z = e^{-2t} - 2te^{-2t}$$

$$= -2z + e^{-2t} \quad ✓$$

$$\Rightarrow \begin{bmatrix}z\\ w\end{bmatrix} = \begin{bmatrix}te^{-2t}\\ e^{-2t}\end{bmatrix} = u_2(t)$$

En anden løsning er:

$$u_1(t) = \begin{bmatrix}z\\ w\end{bmatrix} = e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} = e^{\lambda t}v$$

Så får vi den fuldstændige løsning:

$$\begin{bmatrix}z\\ w\end{bmatrix} = cu_1 + du_2, \quad c, d \in \mathbb{R}$$

$$= \begin{pmatrix}ce^{-2t}\\ 0\end{pmatrix} + d\begin{pmatrix}te^{-2t}\\ e^{-2t}\end{pmatrix}$$

$$\begin{pmatrix}\dot z\\ \dot w\end{pmatrix} = \begin{pmatrix}-2 & 1\\ 0 & -2\end{pmatrix}\begin{bmatrix}z\\ w\end{bmatrix}$$

I (x, y)-koordinater:

$$\begin{bmatrix}x\\ y\end{bmatrix} = cu_1 + du_2 + \begin{bmatrix}3\\ 4\end{bmatrix}$$

$$= \begin{bmatrix}ce^{-2t} + dte^{-2t} + 3\\ de^{-2t} + 4\end{bmatrix}$$

□

Eks:

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}0 & 2\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Ligevægtspunktet er $\begin{pmatrix}0\\0\end{pmatrix}$.

Egenværdi / -vektor parrene er givne ved

$$\lambda_1 = 1 + i, \quad v_1 = \begin{bmatrix}1\\ \frac{1}{2} + \frac{1}{2}i\end{bmatrix}$$

$$\lambda_2 = 1 - i, \quad v_2 = \begin{bmatrix}1\\ \frac{1}{2} - \frac{1}{2}i\end{bmatrix}$$

De komplekse løsninger er

$$e^{(1+i)t} = e^t\begin{bmatrix}e^{it}\\ \frac{1}{2}e^{it} + \frac{i}{2}e^{it}\end{bmatrix}$$

$$= e^t\begin{bmatrix}\cos t + i\sin t\\ \frac{1}{2}[\cos t - \sin t + i(\cos t + \sin t)]\end{bmatrix}$$

$$e^{(1-i)t} = e^t\begin{bmatrix}e^{-it}\\ \frac{1}{2}e^{-it} - \frac{i}{2}e^{-it}\end{bmatrix}$$

$$= e^t\begin{bmatrix}\cos t - i\sin t\\ \frac{1}{2}[\cos t - \sin t - i(\cos t + \sin t)]\end{bmatrix}$$

konstruér reelle løsninger

$$\frac{1}{2}\left(e^{(1+i)t}v_1 + e^{(1-i)t}v_2\right)$$

$$= \frac{e^t}{2}\begin{bmatrix}\cos t\\ \cos t - \sin t\end{bmatrix}$$

$$\frac{1}{2i}\left(e^{(1+i)t}v_1 - e^{(1-i)t}v_2\right)$$

$$= \frac{e^t}{2}\begin{bmatrix}\sin t\\ \cos t + \sin t\end{bmatrix}$$

Tilpas begyndelsesbetingelser

$$\begin{pmatrix}x\\ y\end{pmatrix} = \frac{ce^t}{2}\begin{bmatrix}\cos t\\ \cos t - \sin t\end{bmatrix}$$

$$+ \frac{de^t}{2}\begin{bmatrix}\sin t\\ \cos t + \sin t\end{bmatrix}, \quad c, d \in \mathbb{R}$$

□

Eks:

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Ligevægtspunkt er $(0, 0)$.

Egenværdi / -vektor parrene er

$$\lambda_1 = i, \quad v_1 = \begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\lambda_2 = -i, \quad v_2 = \begin{pmatrix}1\\ i\end{pmatrix}$$

Komplekse løsninger

$$e^{it}v_1 = \begin{pmatrix}e^{it}\\ -ie^{it}\end{pmatrix} = \begin{pmatrix}\cos t + i\sin t\\ -i\cos t + \sin t\end{pmatrix}$$

$$e^{-it}v_2 = \begin{pmatrix}e^{-it}\\ ie^{-it}\end{pmatrix} = \begin{pmatrix}\cos t - i\sin t\\ i\cos t + \sin t\end{pmatrix}$$

Reelle løsninger:

$$\frac{1}{2}\left(e^{it}v_1 + e^{-it}v_2\right) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

$$\frac{1}{2i}\left(e^{it}v_1 - e^{-it}v_2\right) = \begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}$$

Tilpas begyndelsesbetingelser

$$\begin{pmatrix}x\\ y\end{pmatrix} = c\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + d\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix},$$

$$c, d \in \mathbb{R}$$

For hver t gælder at

$$x^2(t) + y^2(t) = c^2 + d^2$$

$\Rightarrow$ løsninger er kredser med radius

$$\sqrt{c^2 + d^2}$$

□

$$\dot x = y$$
$$\dot y = -2x - y$$

isokliner til nulniveauet

$$\dot x = 0 = y$$

$$\Rightarrow \quad \dot y = -2x \Rightarrow \begin{matrix}x > 0 \Rightarrow y \downarrow\\ x < 0 \Rightarrow y \uparrow\end{matrix}$$

$$\dot y = 0$$

$$y = -2x$$

$$\dot x = y \qquad \begin{matrix}y > 0 \Rightarrow x \rightarrow\\ y < 0 \Rightarrow x \leftarrow\end{matrix}$$