$$\ddot{x} + a\dot{x} + bx = f(t)$$

$$x(t) = A u_1(t) + B u_2(t) + u^*(t)$$

$u^* \in \mathbb{R}$

$\dot{u}^* = 0$

$\ddot{u}^* = 0$

DL : $\underbrace{\ddot{u}^*}_{0} + a \underbrace{\dot{u}^*}_{0} + b u^* = A$

$b u^* = A$

$\Rightarrow u^* = \frac{A}{b}$

Eks : $\underbrace{\ddot{x} - 4\dot{x} + 4x}_{u_1 = e^{2t},\ u_2 = te^{2t}} = t^2 + 2$

$u^* = At^2 + Bt + C$

$\dot{u}^* = 2At + B$

$\ddot{u}^* = 2A$

$\ddot{u}^* + \overset{-4}{a} \dot{u}^* + \overset{4}{b} u^* = t^2 + 2$

$2A - 4(2At + B) + 4(At^2 + Bt + C)$

$= 4At^2 + (-8A + 4B)t$

$+ 2A - 4B + 4C$

$$\Rightarrow \begin{aligned} 4A &= 1 &\Rightarrow A = \tfrac{1}{4} \\ -8A + 4B &= 0 &\Rightarrow B = \tfrac{1}{2} \\ 2A - 4B + 4C &= 2 &\Rightarrow C = \tfrac{7}{8} \end{aligned}$$

$\frac{1}{2} \quad -2 \qquad \frac{3.5}{4}$

$\Rightarrow u^* = \frac{t^2}{4} + \frac{t}{2} + \frac{7}{8}$

$\Rightarrow x(t) = Ae^{2t} + Bte^{2t} + \frac{t^2}{4} + \frac{t}{2} + \frac{7}{8}$

$f(t) = pe^{qt}$

$u^* = Ae^{qt} \Rightarrow \dot{u}^* = qAe^{qt}$

$\ddot{u}^* = q^2 Ae^{qt}$

DL : $\ddot{u}^* + a\dot{u}^* + bu^* = pe^{qt}$

$q^2 Ae^{qt} + aqAe^{qt} + bAe^{qt} = pe^{qt}$

$= Ae^{qt}(q^2 + aq + b) = pe^{qt} \ / e^{-qt}$

$A(q^2 + aq + b) = p$

$\Rightarrow A = \frac{p}{q^2 + aq + b}$

$q^2 + aq + b \neq 0$

Eks : $\ddot{x} - 4\dot{x} + 4x = 2\cos(2t)$

Antag : $u^* = A\sin(2t) + B\cos(2t)$

$\Rightarrow \dot{u}^* = 2A\cos(2t) - 2B\sin(2t)$

$\ddot{u}^* = -4A\sin(2t) - 4B\cos(2t)$

DL : $\ddot{u}^* - 4\dot{u}^* + 4u^* = 2\cos(2t)$

$-4A\cancel{\sin(2t)} - \cancel{4}B\cos(2t)$

$-4(2A\cos(2t) - 2B\sin(2t))$

$+4(\cancel{A\sin}(2t) + \cancel{B}\cos(2t))$

$= -8A\cos(2t) - 8B\sin(2t)$

$\overset{!}{=} 2\cos(2t)$

$\Rightarrow -8A = 2 \Rightarrow A = -\frac{1}{4}$

$-8B = 0 \Rightarrow B = 0$

$\Rightarrow u^* = -\frac{1}{4}\sin(2t)$

$x(t) = e^{2t}(A + Bt) - \frac{1}{4}\sin(2t)$

Eulers DL :

$t^2\ddot{x} + at\dot{x} + bx = 0, \quad t > 0$

$s := \log t$

$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{1}{t}\frac{dx}{ds}$

$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\frac{dx}{ds}\right) = -\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t}\frac{d^2x}{ds^2}\frac{ds}{dt}$

$= -\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t^2}\frac{d^2x}{ds^2}$

DL : $t^2\frac{d^2x}{dt^2} + at\frac{dx}{dt} + bx$

$= t^2\left(-\frac{1}{t^2}\frac{dx}{ds} + \frac{1}{t^2}\frac{d^2x}{ds^2}\right)$

$+ at\left(\frac{1}{t}\frac{dx}{ds}\right) + bx$

$= \frac{d^2x}{ds^2} + (a-1)\frac{dx}{ds} + bx = 0$

$r_{1,2} = -\frac{a-1}{2} \pm \sqrt{\frac{(a-1)^2}{4} - b}$

1) $\frac{(a-1)^2}{4} - b > 0$

$x(t) = Ae^{r_1 s} + Be^{r_2 s}$

$= A\underbrace{e^{r_1 \log t}}_{\underbrace{(e^{\log t})^{r_1}}_{t^{r_1}}} + Be^{r_2 \log t}$

$= At^{r_1} + Bt^{r_2}$

2) $\frac{(a-1)^2}{4} - b = 0 \Rightarrow r \in \mathbb{R}$

$x(t) = e^{rs}(A + Bs)$

$= e^{r\log t}(A + B\log t)$

$= t^r(A + B\log t)$

3) $\frac{(a-1)^2}{4} - b < 0$

$r_{1,2} = \underbrace{-\frac{a-1}{2}}_{\alpha} \pm i\underbrace{\sqrt{b - \frac{(a-1)^2}{4}}}_{\beta}$

$x(t) = e^{\alpha s}(A\cos(\beta s) + B\sin(\beta s))$

$= t^{\alpha}(A\cos(\beta\log t) + B\sin(\beta\log t))$

$r^2 + ar + b = (r - r_1)(r - r_2)$

$= r^2 - (r_1 + r_2)r + r_1 r_2$

$\Rightarrow a = -r_1 - r_2$

$b = r_1 r_2$

$r_1 < 0,\ r_2 < 0 \iff a > 0,\ b > 0$

Hvis $r_{1,2} = \alpha \pm i\beta$

$\alpha = -\frac{a}{2} < 0 \iff a > 0$

$\frac{a^2}{4} - b < 0 \iff b > \frac{a^2}{4} > 0$