$$x_t = \underset{\mathbb{R}}{\lambda^t}\ \underset{\mathbb{R}^n}{v},$$

$$x_{t+1} = A x_t$$

$$\lambda^{t+1} v = A \lambda^t v \qquad / : \lambda^t, \ \lambda \neq 0$$

$$\boxed{\lambda v = A v}$$

Eks:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

Find egenværdiene og egenvektorene:

$$(\lambda_1, v_1) = \left(1, \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}\right)$$

$$(\lambda_2, v_2) = \left(\frac{1}{6}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$\Rightarrow x(t) = \underset{\mathbb{R}}{\alpha}\ \lambda_1^t v_1 + \beta \lambda_2^t v_2$$

$$= \alpha \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} + \beta \left(\frac{1}{6}\right)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_{t+1} = A x_t + b_t$$

$$x_t \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}, \quad b_t \in \mathbb{R}^n$$

$$\begin{aligned}
x_t &= A x_{t-1} + b_{t-1} \\
&= A(A x_{t-2} + b_{t-2}) + b_{t-1} \\
&= A\big(A(A x_{t-3} + b_{t-3}) + b_{t-2}\big) + b_{t-1} \\
&= \dots \\
&= A^t x_0 + \sum_{k=1}^{t} A^{k-1} b_{t-k}, \quad A^0 = I
\end{aligned}$$

Hvis $b_t \equiv b$ for alle t:

$$x_t = A^t x_0 + \underbrace{(I + A + A^2 + \dots + A^{t-1})}_{S_t} b$$

$$\begin{aligned}
S_t &= I + A + A^2 + \dots + A^{t-1} \\
A S_t &= \phantom{I + {}} A + A^2 + \dots + A^{t-1} + A^t \quad \Big| -
\end{aligned}$$

$$(I - A) S_t = I - A^t$$

$$\Rightarrow S_t = (I-A)^{-1}(I - A^t)$$

hvis $I - A$ has fuld rang

$$x_t = A^t x_0 + (I-A)^{-1}(I-A^t) b$$

Hvis A diagonaliserbar

$$A = P \Lambda P^{-1}, \quad P \text{ fuld rang}$$

$$\Rightarrow I - A = P I P^{-1} - P \Lambda P^{-1}$$

$$= P(I - \Lambda)P^{-1}$$

Hvis $I - \Lambda$ har fuld rang, findes $(I-A)^{-1}$

Hvis egenværdierne i Λ har absolutværdi mindre end 1, så gælder, at

$$\lim_{t\to\infty} A^t = \lim_{t\to\infty} (P\Lambda P^{-1})^t$$

$$= \lim_{t\to\infty} \underbrace{P\Lambda \underbrace{P^{-1}P}_{I}\Lambda \underbrace{P^{-1}P}_{I}\Lambda P^{-1} \cdots P\Lambda P^{-1}}_{t\text{-gange}}$$

$$= \lim_{t\to\infty} P\Lambda^t P^{-1}$$

$$= P\left(\underbrace{\lim_{t\to\infty}\Lambda^t}\right)P^{-1}$$

$$\lim_{t\to\infty} \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)^t$$

$$= \lim_{t\to\infty} \operatorname{diag}(\lambda_1^t, \lambda_2^t, \ldots, \lambda_n^t)$$

$$= 0 \in \mathbb{R}^{n\times n}$$

$$\Rightarrow \underbrace{\sum_{k=1}^{\infty} A^{k-1} b}_{\lim_{t\to\infty} \sum_{k=1}^{t} A^{k-1} b} = (I-A)^{-1} b$$

$$\lim_{t\to\infty} x_t = (I-A)^{-1} b$$

(første afsnit : $\frac{b}{1-a}$)