Kontante koefficienter

$$\dot{x} + ax = b$$

Integrerende faktor: e^{at}

$$\underbrace{\dot{x}e^{at} + ae^{at}x}_{\frac{d}{dt}(xe^{at})} = be^{at}$$

$$\Rightarrow \quad xe^{at} = \int be^{at}\,dt + C$$

$$\Rightarrow \quad x(t) = Ce^{-at} + e^{-at} \underbrace{\int^t be^{as}\,ds}_{\frac{b}{a}e^{at}}$$

$$= \frac{b}{a} + Ce^{-at}$$

fuldstændig løsning, $C \in \mathbb{R}$

Begyndelsesbetingelse $x(0) = x_0 \in \mathbb{R}$

$$x(0) = C + \frac{b}{a} \overset{!}{=} x_0$$

$$\Rightarrow \quad C = x_0 - \frac{b}{a}$$

$$\Rightarrow \quad x(t) = \frac{b}{a} + \underbrace{e^{-at}\left(x_0 - \frac{b}{a}\right)}_{\xrightarrow[t\to\infty]{} 0}$$ partikulær løsning

$a > 0$

Eks: $\dot{x} + 3x = 2$

$$\underbrace{\dot{x}e^{3t} + 3e^{3t}x}_{\frac{d}{dt}(xe^{3t})} = 2e^{3t}$$

$$xe^{3t} = 2\int e^{3t}\,dt + C$$

$$= \frac{2}{3}e^{3t} + C$$

$$x(t) = \frac{2}{3} + e^{-3t}C$$

$x_0 = 1$

$$x(0) = \frac{2}{3} + C = 1 \quad \Rightarrow \quad C = \frac{1}{3}$$

$$x(t) = \frac{2}{3} + \frac{1}{3}e^{-3t}$$

$$\dot{x} + ax = b(t)$$

$$\dot{x}e^{at} + ae^{at}x = b(t)e^{at}$$

$$xe^{at} = \int b(s)e^{as}\,ds + C$$

$$\Rightarrow \quad x(t) = Ce^{-at} + e^{-at}\int b(s)e^{as}\,ds$$

Eks $\dot{x} - x = t$

$$\dot{x}e^{-t} - e^{-t}x = te^{-t}$$

$$\rightarrow \quad xe^{-t} = \int se^{-s}\,ds + C$$

$$\int \underset{F}{t}\,\underset{g}{e^{-t}}\,dt = \underset{FG}{-te^{-t}} + \underset{-\,fG}{\int e^{-t}\,dt}$$

$$= -te^{-t} + (-e^{-t})$$

$$= -e^{-t}(t+1)$$

$$\Rightarrow \quad x = Ce^{t} - t - 1$$ □

$$\dot{x} + a(t)x = b(t)$$

$e^{A(t)}$ således, at $\dot{A}(t) = a(t)$

$$\underbrace{\dot{x}e^{A(t)} + a(t)e^{A(t)}x}_{\frac{d}{dt}(xe^{A(t)})} = b(t)e^{A(t)}$$

$$xe^{A(t)} = \int b(s)e^{A(s)}\,ds + C$$

$$\Rightarrow \quad x(t) = Ce^{-A(t)} + e^{-A(t)}\int b(s)e^{A(s)}\,ds$$

$$A(t) = \int a(s)\,ds$$

Eks: $\dot{x} + 2tx = 4t$

$$A(t) = t^2$$

$$\underline{\dot{x}e^{t^2} + 2te^{t^2}x} = 4te^{t^2}$$

$$\Rightarrow \quad xe^{t^2} = 4\int se^{s^2}\,ds + C$$

$$x(t) = Ce^{-t^2} + 4e^{-t^2}\int se^{s^2}\,ds$$

$$4\int te^{t^2}\,dt = 2e^{t^2} + C$$

$$\Rightarrow \quad xe^{t^2} = 2e^{t^2} + C$$

$$\Rightarrow \quad x(t) = Ce^{-t^2} + 2$$

$$x_0 = 2 \quad \Rightarrow \quad C = -4$$ □

$$\dot{w} - r(t)w = y(t) - c(t)$$

Integrerende faktor: $e^{-\int r(s)\,ds}$

$$\underbrace{\dot{w}e^{-\int r(s)\,ds} - r(t)e^{-\int r(s)\,ds}}_{\big\downarrow} = (y(t) - c(t))e^{-\int r(s)\,ds}$$

$$\Rightarrow \quad we^{-\int r(s)\,ds} = \int (y(s) - c(s))e^{-\int r(\tau)\,d\tau}\,ds + C$$

$$\Rightarrow \quad w(t) = Ce^{\int_0^t r(s)\,ds} + e^{\int_0^t r(s)\,ds}\int_0^t (y(s) - c(s))e^{-\int_0^s r(\tau)\,d\tau}\,ds$$

Bestemte integraler og $t_0 \neq 0$. Vi betragter intervallet $[t_0, t]$:

$$\dot{x} + a(t)x = b(t)$$

$$\dot{x}e^{A(t)} + a(t)e^{A(t)}x = b(t)e^{A(t)}$$

$$xe^{A(t)} = \int_{t_0}^{t} e^{A(s)}b(s)\,ds + C$$

$$A(t) = \int_{t_0}^{t} a(s)\,ds$$

$$\Rightarrow \quad x(t) = Ce^{-\int_{t_0}^t a(s)\,ds} + e^{-\int_{t_0}^t a(s)\,ds}\int_{t_0}^{t} b(s)e^{\int_{t_0}^s a(\tau)\,d\tau}\,ds$$ □