Eks: $\begin{cases} \max\limits_{u_t \in \mathbb{R}} \sum\limits_{t=0}^{\infty} \beta^t \underbrace{\left(-\frac{2x_t^2}{3} - u_t^2\right)}_{\leq 0}, \quad \beta \in (0,1) \\ x_{t+1} = x_t + u_t, \quad x_0 \in \mathbb{R}, \quad t = 0, 1, 2, \dots \end{cases}$

Antag, at

$$J(x) = -\alpha x^2, \quad \alpha > 0$$

løser Bellman-ligningen. Ligningen bliver til

$$J(x) = \max_{u \in \mathbb{R}} \left\{ -\frac{2}{3}x^2 - u^2 + \beta J(x+u) \right\}$$

$$-\alpha x^2 = \max_{u \in \mathbb{R}} \left\{ \underbrace{-\frac{2}{3}x^2 - u^2 - \alpha\beta(x+u)^2}_{=: \phi(u) \text{ konkav i } u} \right\}$$

$$\phi'(u) = -2u - 2\alpha\beta(x+u) = 0$$

$$\Rightarrow \quad u^*(x) = -\frac{\alpha\beta x}{1+\alpha\beta}$$

og

$$x + u^*(x) = \frac{x}{1+\alpha\beta}$$

Bellman ligningen bliver til

$$-\alpha x^2 = \phi(u^*(x))$$

$$= -\frac{2}{3}x^2 - \frac{\alpha^2\beta^2}{(1+\alpha\beta)^2}x^2 - \frac{\alpha\beta}{(1+\alpha\beta)^2}x^2$$

$$= -\frac{2}{3}x^2 - \frac{\alpha^2\beta^2 + \alpha\beta}{(1+\alpha\beta)^2}x^2$$

$$\Rightarrow \quad \alpha = \frac{2}{3} + \frac{\alpha^2\beta^2 + \alpha\beta}{(1+\alpha\beta)^2}$$

$$\Rightarrow \quad 3\beta\alpha^2 + (3-5\beta)\alpha - 2 = 0$$

Kun den positive løsning for α er tilladt:

$$\alpha = \frac{5\beta - 3 + \sqrt{(5\beta-3)^2 + 24\beta}}{6\beta}$$

$$\Rightarrow \quad u^*(x) = -\frac{\alpha\beta}{1-\alpha\beta}\, x$$

$$= -\underbrace{\frac{\beta - 3 + \sqrt{(5\beta-3)^2 + 24\beta}}{6\beta}}_{=:\,\gamma}\, x$$

For x_0 givet, får vi

$$u_0^* = -\gamma x_0$$

$$x_1 = x_0 + u_0^* = x_0 - \gamma x_0 = (1-\gamma)\, x_0$$

$$u_1^* = -\gamma x_1$$
$$= -\gamma(1-\gamma)x_0$$

$$x_2 = x_1 + u_1^*$$
$$= (1-\gamma)x_0 - \gamma(1-\gamma)x_0$$
$$= (1-\gamma)(1-\gamma)x_0$$

$$u_2^* = -\gamma x_2 = -\gamma(1-\gamma)^2 x_0$$

$$x_3 = x_2 + u_2^*$$
$$= (1-\gamma)^2 x_0 - \gamma(1-\gamma)^2 x_0$$
$$= (1-\gamma)^3 x_0$$

$$\dots$$

$$\Rightarrow \quad x_t = (1-\gamma)^t x_0 \qquad \square$$

Eks:

$$\begin{cases} \max\limits_{u_t \in \mathbb{R}} \sum\limits_{t=0}^{3} (1 + x_t - u_t^2) \\ x_{t+1} = x_t + u_t, \quad t = 0,1,2,3 \\ x_0 = 0 \end{cases}$$

For $t<3$ er Hamilton funktionen

$$H(t,x,u,p) = 1 + x_t - u_t^2 + p_t(x_t + u_t)$$

$$H'_u = -2u_t + p_t = 0$$

$$\Rightarrow u_t^* = \frac{p_t}{2}, \quad t = 0,1,2$$

For $t=3$:

$$H(t,x,u,p) = 1 + x_t - u_t^2$$

$$H'_u = -2u_t = 0 \Rightarrow u_t^* = u_3^* = 0$$

$$\Rightarrow u_0^* = \frac{1}{2}p_0, \quad u_1^* = \frac{1}{2}p_1, \quad u_2^* = \frac{1}{2}p_2, \quad u_3^* = 0$$

Differensligningen for p er

$$p_{t-1} = H'_x(t,x,u,p)$$

$$= \begin{cases} 1 + p_t, & t = 0,1,2 \\ 1, & t = 3 \end{cases}$$

x_T fri $\Rightarrow p_T = 0 = p_3$

$$\Rightarrow p_2 = 1,$$

$$p_1 = 1 + p_2 = 2$$

$$p_0 = 1 + p_1 = 3$$

$$\Rightarrow \; u_0^* = \frac{3}{2}, \quad u_1^* = 1, \quad u_2^* = \frac{1}{2}, \quad u_3^* = 0$$

$$x_0 = 0$$

$$x_1 = x_0 + u_0 = \frac{3}{2}$$

$$x_2 = x_1 + u_1 = \frac{3}{2} + 1 = \frac{5}{2}$$

$$x_3 = x_2 + u_2 = \frac{5}{2} + \frac{1}{2} = 3$$

$$\left(x_4 = x_3 + u_3 = 3 + 0 = 3 \right)$$

$\square$