Def.: Den imaginære enhed i er løsningen på ligningen

$$x^2 + 1 = 0$$

Eller $\quad i^2 = -1$

I $\mathbb{R}^{2\times 2}$: Findes en matrix A, således at

$$A^2 + I = 0$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ \hline A^2 \quad \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & \underbrace{\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}}_{-I} \end{array}$$

$$A^2 = -I$$

$$A^2 + I = 0$$

Generelle form

$$z = a + ib \quad \in \mathbb{C}$$

$a = \mathrm{Re}\, z \in \mathbb{R}$ realdel

$b = \mathrm{Im}\, z \in \mathbb{R}$ imaginærdel

Addition:

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

Multiplikation:

$$(a+ib)(c+id) = \underbrace{(ac-bd)}_{\mathrm{Re}} + i\underbrace{(ad+bc)}_{\mathrm{Im}}$$

Def.: $z = a + ib$

(i) $\bar{z} = a - ib$

kompleks konjugerede tal til z

(ii) $|z| = \sqrt{a^2 + b^2}$

absolutværdi

$$|z| = \sqrt{z\bar{z}} = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2 + i(ab-ab) + b^2}$$

(iii) Reciprokke værdi / inverse

$$z^{-1} = \frac{\bar{z}}{|z|^2} \in \mathbb{C}$$

$$zz^{-1} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

Def.: Eksponentielle form:

$$z = a + ib = r e^{i\theta}$$

$r = |z|$

θ vinklen af diagonalen af rektanglet udgjort af $(a,b) \in \mathbb{R}^2$

(r, θ) polære koordinater

(a,b) Kartesiske koordinater

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler formlen

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i\sin(x+y) \\ &= e^{ix} e^{iy} \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x \cos y - \sin x \sin y \\ &\quad + i(\cos x \sin y + \sin x \cos y) \end{aligned}$$

$$\Rightarrow \cos(x+y) = \cos x \cos y - \sin x \sin y = \mathrm{Re}\, e^{i(x+y)}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y = \mathrm{Im}\, e^{i(x+y)}$$

$$\begin{aligned} e^{i(x-y)} &= \cos(x-y) + i\sin(x-y) \\ &= e^{ix} e^{-iy} \\ &= (\cos x + i\sin x)(\cos y - i\sin y) \\ &= \cos x \cos y + \sin x \sin y \\ &\quad + i(\sin x \cos y - \cos x \sin y) \end{aligned}$$

$$\Rightarrow \cos(x-y) = \mathrm{Re}\, e^{i(x-y)} = \cos x \cos y + \sin x \sin y$$

$$\sin(x-y) = \mathrm{Im}\, e^{i(x-y)} = \sin x \cos y - \cos x \sin y$$