Eks:

$$\begin{cases} \max\limits_{u} \int_0^\infty -u^2 e^{-rt}\,dt, & r > 0 \\ u \in \mathbb{R} \\ \dot{x} = u e^{-at}, & a > \frac{r}{2} \\ x(0) = 0 \\ \lim\limits_{t\to\infty} x(t) \geq K, & K > 0. \end{cases}$$

Current-value Hamiltonian,

$$H^c(t, x, u, \lambda) = -u^2 + \lambda u e^{-at}$$

Konkav i (x, u).

$$\frac{\partial H^c}{\partial u} = -2u + \lambda e^{-at} = 0 \Rightarrow u = \frac{\lambda}{2} e^{-at}$$

$$\dot{\lambda} - r\lambda = -\frac{\partial H^c}{\partial x} = 0$$

$$\dot{\lambda} = r\lambda \Rightarrow \lambda(t) = ce^{rt}$$

$$\Rightarrow \quad u = \frac{c}{2} e^{(r-a)t}$$

$$\Rightarrow \quad \dot{x} = \frac{c}{2} e^{(r-2a)t}, \quad x(0) = 0$$

$$\Rightarrow x(t) = \frac{c}{2} \int_0^t e^{(r-2a)s}\,ds$$

$$= \frac{c}{2} \left[\frac{1}{r-2a} e^{(r-2a)s}\right]_0^t$$

$$= \frac{c}{2(r-2a)}\left(e^{\underbrace{(r-2a)}_{<0}t} - 1\right)$$

$$\lim_{t\to\infty} x(t) = \frac{c}{2(2a-r)}$$

$$\Rightarrow \quad \frac{c}{2(2a-r)} \geq K$$

$$c \geq 2\underbrace{K}_{>0}\underbrace{(2a-r)}_{>0} > 0$$

$\Rightarrow$ Sl. 7 (c) : Der findes et tal t' således, at $\lambda(t) = ce^{rt} \geq 0$ for alle $t > t'$.
Gælder her for alle $t' \in [0, \infty)$.

Nu skal vi tjekke (A) :

$$\lim_{t\to\infty} \lambda(t)e^{-rt}(x_1 - x^*(t)) \geq 0$$

$$\lim_{t\to\infty} \underbrace{\lambda(t)e^{-rt}}_{c}\left[K - \frac{c}{2(2a-r)}\left(1 - e^{\underbrace{(r-2a)}_{<0}t}\right)\right]$$

$$= c\left(\underbrace{K - \frac{c}{2(2a-r)}}_{=0}\right), \quad c = 2K(2a-r)$$

og alle betingelser i Teorem 9.1.1 er opfyldte.

Bemærk: Den adjungerede funktion

$$p(t) = \lambda(t)e^{-rt} = 2K(2a-r) > 0$$

$\square$