$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

$$f(x+h) = f(x) + f'(x)h + R(h)$$

$$R(h) = f(x+h) - f(x) - f'(x)h$$

$$\lim_{h\to 0} \frac{R(h)}{h} = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} - f'(x)\right)$$

$$= 0$$

$$f'(x) = c \in \mathbb{R}$$

f differentiabel $\iff$ Der findes et tal c således, at

$$\lim_{h\to 0} \frac{f(x+h)-f(x)-ch}{h} = 0.$$

Eks: $f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

$$= 1 + x + R(x)$$

$$f(x+h) = e^{x+h}$$

$$\lim_{h\to 0}\frac{R(h)}{h} = \lim_{h\to 0}\left(\frac{e^{x+h}-e^x}{h} - c\right) = 0$$

$$= \lim_{h\to 0} \frac{e^x(e^h-1)}{h} - c = 0$$

$$= e^x \lim_{h\to} \frac{e^h-1}{h} - c = 0$$

$$\lim_{h\to 0} \frac{e^h-1}{h} = ?$$

$$e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots$$

$$\frac{e^h-1}{h} = 1 + \frac{h}{2} + \frac{h^2}{6} + \dots$$

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

$$\Rightarrow \lim_{h\to 0}\frac{R(h)}{h} = e^x - c$$

$$= e^x - f'(x) = 0$$

$$\Rightarrow f'(x) = e^x \qquad \square$$

$$\underset{\mathbb{R}^m}{f(x+h)} = \underset{\mathbb{R}^m}{f(x)} + \underset{\mathbb{R}^{m\times n}}{C}\,\underset{\mathbb{R}^n}{h} + \underset{\mathbb{R}^m}{R(h)}, \quad x, h \in \mathbb{R}^n$$

Eks: $f : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1^2 + 2x_1x_2 + x_2^2$$

$$f(x+h) = (x_1+h_1)^2 + 2(x_1+h_1)(x_2+h_2) + (x_2+h_2)^2$$

$$= x_1^2 + 2x_1h_1 + h_1^2 + 2x_1x_2 + 2x_1h_2$$

$$+ 2x_2h_1 + 2h_1h_2 + x_2^2 + 2x_2h_2 + h_2^2$$

$$f(x+h) - f(x) = 2x_1h_1 + 2x_1h_2 + 2x_2h_1$$

$$+ 2x_2h_2$$

$$+ h_1^2 + 2h_1h_2 + h_2^2$$

$$\underset{\mathbb{R}}{f(x+h)} = \underset{\mathbb{R}}{f(x)} + \underset{\mathbb{R}^{1\times 2}}{C}\,\underset{\mathbb{R}^2}{h} + \underset{\mathbb{R}}{R(h)}$$

$$C = \nabla f(x) \in \mathbb{R}^2$$

$$(2x_1 + 2x_2,\ 2x_1 + 2x_2)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= 2x_1h_1 + 2x_2h_1 + 2x_1h_2 + 2x_2h_2$$

$$\Rightarrow R(h) = h_1^2 + 2h_1h_2 + h_2^2$$

$$\lim_{h\to 0} \frac{h_1^2 + 2h_1h_2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} = 0$$

$$\Rightarrow C = [2x_1 + 2x_2 \quad 2x_1 + 2x_2] \qquad \square$$

(i) Fordi $\lim_{h\to 0} Ch = 0$

og $\lim_{h\to 0} \frac{R(h)}{\|h\|} = 0$

følger, at

$$\lim_{h\to 0} f(x+h) = f(x)$$

(ii) Vi har, at

$$f_i(x+h) = f_i(x) + \sum_{j=1}^{n} c_{ij}h_j + R_i(h)$$

$i = 1, \dots, m$

med $\lim_{h\to 0} \frac{R_i(h)}{\|h\|} = 0$

Lad $\delta > 0$ og e_j j-te standard basisvektor

$$f_i(x + \underbrace{\delta e_j}_{h}) = f_i(x) + \delta c_{ij} + R_i(\delta e_j)$$

Vi får:

$$\frac{\partial f_i}{\partial x_j} = \lim_{\delta\to 0} \frac{f_i(x+\delta e_j) - f_i(x)}{\delta}$$

$$= c_{ij} + \lim_{\delta\to 0}\frac{R_i(\delta e_j)}{\delta}$$

$$= c_{ij} \qquad \square$$