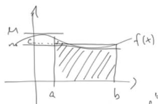
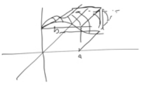


$$c(b-a) = \int_a^b f(x)\,dx$$

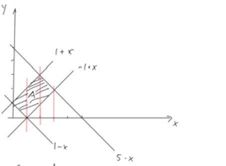


<u>Eks</u>: Bestem

$$I = \int_A (x^2+y^2-1)\,dx\,dy$$

A er begrænset af linjerne

$$\begin{aligned} x+y &= 1 & y &= 1-x \\ x+y &= 5 & y &= 5-x \\ x-y &= -1 & y &= x+1 \\ x-y &= 1 & y &= x-1 \end{aligned}$$



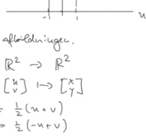
$$A = \{(x,y) \mid \begin{array}{l} x\in[0,1): 1-x \le y \le x+1, \\ x\in[1,2): x-1 \le y \le x+1, \\ x\in[2,3): x-1 \le y \le 5-x \end{array}\}$$

$$\begin{aligned} I &= \int_0^1 \left(\int_{1-x}^{x+1} (x^2+y^2-1)\,dy \right) dx \\ &+ \int_1^2 \left(\int_{-1+x}^{x+1} (x^2+y^2-1)\,dy \right) dx \\ &+ \int_2^3 \left(\int_{-1+x}^{5-x} (x^2+y^2-1)\,dy \right) dx \\ &= \frac{52}{3} \end{aligned}$$

Substitution:

$$u := x-y$$
$$v := x+y$$

$$\begin{aligned} x+y &= 1 & &\Rightarrow & v &= 1 \\ x+y &= 5 & &\Rightarrow & v &= 5 \\ x-y &= -1 & &\Rightarrow & u &= -1 \\ x-y &= 1 & &\Rightarrow & u &= 1 \end{aligned}$$



Koordinatafbildningen.

$$\begin{pmatrix} g \\ h \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = \tfrac{1}{2}(u+v)$$
$$y = \tfrac{1}{2}(-u+v)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{J\text{(acobi)}} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}$$

Integranden

$$\begin{aligned} x^2+y^2-1 &= \tfrac{1}{4}(u+v)^2 + \tfrac{1}{4}(-u+v)^2 - 1 \\ &= \frac{u^2}{2} + \frac{v^2}{2} - 1 \end{aligned}$$

$$\begin{aligned} I &= \int_A (x^2+y^2-1)\,dx\,dy \\ &= \int_{A'} \left(\frac{u^2}{2} + \frac{v^2}{2} - 1 \right) \frac{1}{2}\,du\,dv \\ &= \frac{1}{2} \int_{-1}^{1} \left(\int_1^5 \left(\frac{u^2}{2} + \frac{v^2}{2} - 1 \right) dv \right) du \\ &= \frac{52}{3} \end{aligned}$$

$\square$

<u>Eks</u>: (Polære koordinater)

$$\begin{bmatrix} g \\ h \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} x = g(r,\theta) = r\cos\theta \\ y = h(r,\theta) = r\sin\theta \end{bmatrix}$$

Jacobi-determinanten

$$\det \begin{bmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} \end{bmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta$$
$$= r\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

Vi får:

$$\int_A f(x,y)\,dx\,dy = \int_{A'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Bestem:

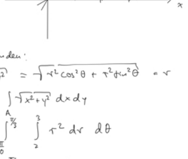
$$I = \int_A \sqrt{x^2+y^2}\,dx\,dy$$

hvor

$$A = \{(x,y) \mid 4 \le x^2+y^2 \le 9,\ \frac{x}{\sqrt{3}} \le y \le \sqrt{3}x \}$$

A er begrænset

$$\begin{aligned} x^2+y^2 &\ge 4 & &\Rightarrow & r &\ge 2 \\ x^2+y^2 &\le 9 & &\Rightarrow & r &\le 3 \\ y &\ge \frac{x}{\sqrt{3}} & &\Rightarrow & \theta &\ge \frac{\pi}{6} & &(30°) \\ y &\le \sqrt{3}x & &\Rightarrow & \theta &\le \frac{\pi}{3} & &(60°) \end{aligned}$$



Integranden:

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

$$\begin{aligned} \Rightarrow I &= \int_A \sqrt{x^2+y^2}\,dx\,dy \\ &= \int_{\frac{\pi}{6}}^{\pi/3} \int_2^3 r^2\,dr\,d\theta \\ &= \int_{\frac{\pi}{6}}^{\pi/3} \underbrace{\left[\tfrac{1}{3} r^3 \right]_2^3}_{\left(\frac{27}{3} - \frac{8}{3}\right)} d\theta \\ &= \int_{\frac{\pi}{6}}^{\pi/3} \frac{19}{3}\,d\theta = \frac{19}{3}\left(\frac{\pi}{3} - \frac{\pi}{6} \right) \end{aligned}$$