$$u : \mathbb{R} \to \mathcal{U}$$
$$t \mapsto u(t)$$

Bevis idé for Teorem 9.4.1 tilfælde (a), (a')
(c), (c')

Og $u$ indre punkt af $\mathcal{U}$.

Definér det følgende integral:

$$J := \int_0^T \left[ f(t,x,u) + p(t)\left(g(t,x,u) - \dot{x}\right)\right] dt$$

$$\int_0^T p(t)\,\dot{x}(t)\,dt \underset{\text{partiel int.}}{=} p(t)\,x(t)\Big|_0^T - \int_0^T \dot{p}(t)\,x(t)\,dt$$

$F \quad g \quad\quad F \quad G \quad\quad f \quad G$

$$= p(T)\,x(T) - p(0)\,x(0) - \int_0^T \dot{p}(t)\,x(t)\,dt$$

$$\Rightarrow J = \int_0^T \left[ f(t,x,u) + pg(t,x,u) + \dot{p}x\right] dt$$
$$- p(T)\,x(T) + p(0)\,x_0$$
$$= \int_0^T \left[ H(t,x,u,p) + \dot{p}x\right] dt$$
$$- p(T)\,x(T) + p(0)\,x_0$$

Lad $u^*$ være den optimale kontrolfunktion der maksimerer $J$ og lad $x^*$ være den tilsvarende funktion for tilstandsvariablen.

Den maksimale værdi af $J$ er

$$J^* = \int_0^T [H(t, x^*, u^*, p) + \dot{p} x^*] dt$$

$$- p(T) \underset{\text{(a): } x_T}{x^*(T)} + p(0) x_0$$

Betragt en perturberet kontrolfunktion

$$u = u^* + \varepsilon v \in \mathcal{U}$$

$u^*, v$ funktioner, $\varepsilon > 0$ tal, og tilsvarende tilstand $x(t, \varepsilon)$, således, at

$$x(t, 0) = x^*(t),$$

$$x(0, \varepsilon) = x_0 .$$

Nødvendig betingelse for et maksimum af $J(u^* + \varepsilon v)$ i $u^*$ er, at

$$\frac{d}{d\varepsilon} J(u^* + \varepsilon v) \Big|_{\varepsilon = 0} = 0 ,$$

eller

$$\frac{d}{d\varepsilon} J(u^* + \varepsilon v) \Big|_{\varepsilon = 0} \overset{\text{Leibniz-formlen}}{=} \int_0^T \Big[ \frac{\partial H}{\partial x} \frac{\partial x(t,\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon = 0} + \frac{\partial H}{\partial u} v$$

$$+ \dot{p} \frac{\partial x(t,\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon = 0} \Big] dt$$

$$-p(T)\left.\frac{\partial x(T,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

$$\stackrel{!}{=} 0$$

Omskriv til:

$$\int_0^T \left[ \frac{\partial H}{\partial u} v + \left( \frac{\partial H}{\partial x} + \dot{p} \right) \left.\frac{\partial x(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \right] dt$$

$$-p(T)\left.\frac{\partial x(T,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

For hver funktion $v$ således, at $u^* + \varepsilon v \in \mathcal{U}$, må gælde, at

$$\frac{\partial H}{\partial u} = 0$$

$$\frac{\partial H}{\partial x} + \dot{p} = 0$$

$$p(T) = 0 \qquad \square$$

Fleming & Rishel (1982)