$f: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $g \circ f(x) = g(f(x))$

$g: \mathbb{R}^m \to \mathbb{R}^k$

$h \in \mathbb{R}^n$

$Ah \in \mathbb{R}^m$

$e_f(h) \in \mathbb{R}^m$

$p \in \mathbb{R}^m$

$e_g(p) \in \mathbb{R}^k$

$$\boxed{\begin{aligned} e(h) &= Be_f(h) \\ &\quad + \underbrace{e_g(Ah + e_f(h))}_{\in \mathbb{R}^k} \\ \lim_{\|h\| \to 0} \frac{e(h)}{\|h\|} &= 0 \in \mathbb{R}^k \end{aligned}}$$

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ε-δ <u>Definitionen</u> af grænseværdien

Tallet A kaldes for limit (grænseværdi) af funktionen $f(x)$ når x går mod $a \in \mathbb{R}$, hvis for hvert tal $\varepsilon > 0$, der findes et tal $\delta > 0$ således, at

$$0 < |x-a| < \delta \Rightarrow |f(x) - A| < \varepsilon$$

$$\lim_{\|p\| \to 0} \frac{e_g(p)}{\|p\|} = 0 \in \mathbb{R}^k, \quad p \in \mathbb{R}^m$$

$\Leftrightarrow$ For alle $\tilde{\varepsilon} > 0$ findes et $\delta_i > 0$, $i = 1, \dots, k$, således, at

$$\|p\| < \delta_i \Rightarrow \frac{\{e_g(p)\}_i}{\|p\|} < \tilde{\varepsilon}$$

Lad $\delta = \min\{\delta_1, \delta_2, \dots, \delta_k\}$

$$\Longleftrightarrow \ \|p\| < \delta \quad \Longrightarrow \quad \frac{\{e_g(p)\}_i}{\|p\|} < \tilde{\varepsilon} \quad \text{for alle } i$$

$$\frac{\|e_g(p)\|}{\|p\|} < \sqrt{\underbrace{\tilde{\varepsilon}^2 + \tilde{\varepsilon}^2 + \dots + \tilde{\varepsilon}^2}_{k\text{-gange}}} = \sqrt{k\tilde{\varepsilon}^2} = \sqrt{k}\,\tilde{\varepsilon}$$

For $\varepsilon_1 > 0$, vælg $\tilde{\varepsilon}_1 = \frac{\varepsilon_1}{\sqrt{k}}$, så følger, at der findes $\delta_1 > 0$ således, at

$$\|p\| < \delta_1 \quad \Longrightarrow \quad \frac{\|e_g(p)\|}{\|p\|} < \sqrt{k}\,\tilde{\varepsilon}_1 = \varepsilon_1$$

eller $\quad \|e_g(p)\| < \varepsilon_1 \|p\| \qquad (*)$

På samme måde, for $f: \mathbb{R}^n \to \mathbb{R}^m$,

$$\lim_{\|h\| \to 0} \frac{e_f(h)}{\|h\|} = 0$$

$$\Longleftrightarrow \quad \|h\| < \delta_2 \ \Longrightarrow \ \|e_f(h)\| < \varepsilon_2 \|h\|$$

Derved,

$$\|e_g(\underbrace{Ah + e_f(h)}_{=:p})\| < \varepsilon_1 \|Ah + e_f(h)\|$$

$$< \varepsilon_1 \|Ah\| + \varepsilon_1 \|e_f(h)\|$$

trekantsulighed ($\leq$)

og $(*)$ $(<)$

$$< \varepsilon_1 \|Ah\| + \varepsilon_1 \varepsilon_2 \|h\| \qquad (**)$$

$$\underset{m\times n\ n\times 1}{Ah} = \begin{bmatrix} a_{11}h_1 + a_{12}h_2 + \cdots + a_{1n}h_n \\ a_{21}h_1 + a_{22}h_2 + \cdots + a_{2n}h_n \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ a_{m1}h_1 + a_{m2}h_2 + \cdots + a_{mn}h_n \end{bmatrix} \in \mathbb{R}^m$$

Lad $\bar{a}$ være den største indgang i $A \in \mathbb{R}^{m\times n}$:

$$\underset{i=1,\dots,m}{\{Ah\}_i} < \bar{a}(h_1 + h_2 + \cdots + h_n)$$

$$\|Ah\| < \left\| \begin{matrix} \bar{a}\sum^n h_i \\ \bar{a}\sum^n h_i \\ \vdots \\ \bar{a}\sum^n h_i \end{matrix} \right\| = \sqrt{m}\,\bar{a}\sum^n h_i$$

$$\underset{m\times 1}{\left\| \begin{matrix} c \\ c \\ \vdots \\ c \end{matrix} \right\|} = \sqrt{mc^2} = \sqrt{m}\,c$$

$$\sum^n h_i = h_1 + h_2 + \cdots + h_n = \langle h, 1 \rangle, \quad 1 \in \mathbb{R}^n$$

Cauchy-Schwarz:

$$|\langle h, 1 \rangle| \leq \|h\|\,\|1\| = \sqrt{n}\,\|h\|$$

$$\Rightarrow \sqrt{m}\,\bar{a}\sum^n h_i \leq \sqrt{m}\sqrt{n}\,\bar{a}\,\|h\|$$

$$\Rightarrow \|Ah\| \leq \sqrt{nm}\,\bar{a}\,\|h\|$$

$$\Rightarrow \|e(h)\| = \|Be_f(h) + e_g(Ah + e_f(h))\|$$

$$\leq \|Be_f(h)\| + \underbrace{\|e_g(Ah + e_f(h))\|}_{\underset{(**)}{<} \varepsilon_1 \sqrt{nm}\,\bar{a}\,\|h\| + \varepsilon_1\varepsilon_2\|h\|}$$

$$\lim_{\|h\|\to 0} \frac{Be_f(h)}{\|h\|} = \underset{k\times m}{B} \lim_{\|h\|\to 0} \frac{e_f(h)}{\|h\|} = B\underset{\mathbb{R}^m}{\hat{0}} = \underset{\mathbb{R}^k}{\hat{0}}$$

eller $\|Be_f(h)\| < \varepsilon_3 \|h\|$ for $\|h\| < \delta_3$.

$$\Rightarrow \|e(h)\| \leq (\varepsilon_3 + \sqrt{nm}\,\bar{a}\,\varepsilon_1 + \varepsilon_1\varepsilon_2)\|h\|$$

$\Rightarrow$ For $K > 0$, vælg $\varepsilon_1, \varepsilon_2, \varepsilon_3$ således, at

$$K \geq \varepsilon_3 + (\sqrt{nm}\,\bar{a} + \varepsilon_2)\varepsilon_1$$

Vælg $\|h\| \leq \min\{\delta_1, \delta_2, \delta_3\} =: \delta$

$$\Rightarrow \|e(h)\| \leq K\|h\|$$

$$\Rightarrow \lim_{\|h\|\to 0} \frac{e(h)}{\|h\|} = 0$$

$\square$