Eks:

$$\begin{cases} \max\limits_{u_t \in \mathbb{R}} \displaystyle\sum_{t=0}^{3} \left(1 + x_t - u_t^2\right) \\ x_{t+1} = x_t + u_t, \qquad t = 0, 1, 2, 3 \\ x_0 = 0 \end{cases}$$

$$x_{t+1} = x_t + u_t$$

$$\Rightarrow \quad u_t = x_{t+1} - x_t$$

$$\Rightarrow \quad \text{maks} \sum_{t=0}^{3} \left(1 + x_t - (x_{t+1} - x_t)^2\right)$$

$$F(t, x_t, x_{t+1}) = 1 + x_t - x_t^2 - x_{t+1}^2 + 2x_t x_{t+1}$$

- $t = T = 3$:

$$x_{T+1}^*(x_T) = \text{argmaks}\left(1 + x_T - x_T^2 - x_{T+1}^2 + 2x_T x_{T+1}\right)$$

$$D_3 F(T, x_T, x_{T+1}) = -2x_{T+1} + 2x_T = 0$$

$$\Rightarrow \quad x_{T+1}^*(x_T) = x_T$$

$$x_4^*(x_3) = x_3$$

- $t = T-1 = 2$:

$$x_T^*(x_{T-1}) = \text{argmaks}\left\{F(T-1, x_{T-1}, x_T) + F(T, x_T, x_{T+1}^*(x_T))\right\}$$

$$= \text{argmaks}\left\{F(2, x_2, x_3) + F(3, x_3, x_4^*(x_3))\right\}$$

$$D_3 F(2, x_2, x_3) + D_2 F(3, x_3, x_4)$$

$$= \frac{\partial}{\partial x_3}\left(1 + x_2 - x_2^2 - x_3^2 + 2x_2 x_3\right) + \frac{\partial}{\partial x_3}\left(1 + x_3 - x_3^2 - x_4^2 + 2x_3 x_4\right)$$

$$= \frac{\partial}{\partial x_3}\left(2 + x_2 + x_3 - x_2^2 - 2x_3^2 - x_4^2 + 2x_2 x_3 + 2x_3 x_4\right)$$

$$= 1 - 4x_3 + 2x_2 + 2x_4 \stackrel{!}{=} 0$$

$$\underset{x_4^*=x_3^*}{=} 1-4x_3^* + 2x_2 + 2x_3^*$$

$$= 1-2x_3^* + 2x_2 \overset{!}{=} 0 \quad \Rightarrow \quad x_3^*(x_2) = \frac{1}{2} + x_2$$

- $t = T-2 = 1$:

$$x_2^*(x_1) = \operatorname{argmax}\{F(T-2, x_{T-2}, x_{T-1}) + F(T-1, x_{T-1}, x_T)\}$$

$$= \operatorname{argmax}\{F(1, x_1, x_2) + F(2, x_2, x_3)\}$$

$$D_3 F(1, x_1, x_3) + D_2 F(2, x_2, x_3)$$

$$= \frac{\partial}{\partial x_2}(1+x_1-x_1^2-x_2^2+2x_1x_2) + \frac{\partial}{\partial x_2}(1+x_2-x_2^2-x_3^2+2x_2x_3)$$

$$= \frac{\partial}{\partial x_2}(2+x_1+x_2-x_1^2-2x_2^2-x_3^2+2x_1x_2+2x_2x_3)$$

$$= 1-4x_2+2x_1+2x_3 \overset{!}{=} 0$$

$$= 1-4x_2+2x_1+2\left(\frac{1}{2}+x_2\right)$$

$$= 2-2x_2+2x_1 = 0$$

$$\Rightarrow \quad x_2^*(x_1) = 1+x_1$$

- $t = T-3 = 0$:

$$x_1^*(x_0) = \operatorname{argmax}\{F(0, x_0, x_1) + F(1, x_1, x_2)\}$$

$$D_3 F(0, x_0, x_1) + D_2 F(1, x_1, x_2)$$

$$= \frac{\partial}{\partial x_1}(1+x_0-x_0^2-x_1^2+2x_0x_1) + \frac{\partial}{\partial x_1}(1+x_1-x_1^2-x_2^2-2x_1x_2)$$

$$= \frac{\partial}{\partial x_1}(2+x_0+x_1-x_0^2-2x_1^2-x_2^2+2x_0x_1+2x_1x_2)$$

$$= 1-4x_1+2x_0+2x_2 \overset{!}{=} 0$$

$$\overset{x_2^* = 1+x_1}{\underset{x_0=0}{=}} 1 - 4x_1 + 2(1+x_1)$$

$$= 3 - 2x_1 = 0 \implies x_1^* = \frac{3}{2}$$

$$x_1^* = \frac{3}{2} \qquad\qquad u_0^* = x_1^* - x_0 = \frac{3}{2}$$

$$x_2^* = 1 + x_1^* = \frac{5}{2} \qquad\qquad u_1^* = x_2^* - x_1^* = 1$$

$$x_3^* = \frac{1}{2} + x_2^* = 3 \qquad\qquad u_2^* = x_3^* - x_2^* = \frac{1}{2}$$

$$x_4^* = x_3^* = 3 \qquad\qquad u_3^* = x_4^* - x_3^* = 0$$

$$u_t = x_{t+1} - x_t$$

<u>Eks</u>:
$$\begin{cases} \max\limits_{c_t} \sum\limits_{t=0}^{T-1} \log c_t + \log x_T \\ x_{t+1} = \alpha(x_t - c_t), \quad \alpha > 1 \\ t = 0, 1, \ldots, T-1 \end{cases}$$

- x_t formue i periode t (tilstand)
- c_t forbruk —"— (kontroll)
- $x_t - c_t$ opsparing, vokser om α hver periode

$$\implies c_t = \frac{\alpha x_t - x_{t+1}}{\alpha} = x_t - \underbrace{\frac{1}{\alpha}}_{=:\beta} x_{t+1}$$

$$\implies \max_{\{x_t\}} \left\{ \sum_{t=0}^{T-1} \log(x_t - \beta x_{t+1}) + \log x_T \right\}$$

- I periode $t = T$:

$$D_2 F(T, x_T, x_{T+1}) + D_3 F(T-1, x_{T-1}, x_T)$$
$$= D_2 \log x_T + D_3 \log(x_{T-1} - \beta x_T)$$
$$= \frac{1}{x_T} - \frac{\beta}{x_{T-1} - \beta x_T} = 0$$
$$\Rightarrow \quad x_T = \frac{x_{T-1} - \beta x_T}{\beta}$$
$$\Rightarrow \quad x_{T-1} = 2\beta x_T$$

- I periode $t = T-1$:

$$D_2 F(T-1, x_{T-1}, x_T) + D_3 F(T-2, x_{T-2}, x_{T-1})$$
$$= D_2 \log(x_{T-1} - \beta x_T) + D_3 \log(x_{T-2} - \beta x_{T-1})$$
$$= \frac{1}{x_{T-1} - \beta x_T} - \frac{\beta}{x_{T-2} - \beta x_{T-1}} = 0$$
$$x_{T-1} - \beta x_T = \frac{x_{T-2} - \beta x_{T-1}}{\beta}$$
$$\Rightarrow \quad x_{T-2} = 2\beta x_{T-1} - \beta^2 x_T$$

Dette mønster fortsætter for $t = T-2, T-3, \ldots, 0$:

$$x_{t-1} = 2\beta x_t - \beta^2 x_{t+1}$$

anden-ordens differensligning med konstante koefficienter

$$x_{T-2} = 2\beta x_{T-1} - \beta^2 x_T$$

$$\underset{x_{T-1} = 2\beta x_T}{=} 4\beta^2 x_T - \beta^2 x_T = 3\beta^2 x_T$$

$$x_{T-3} = 2\beta x_{T-2} - \beta^2 x_{T-1}$$

$$= 6\beta^3 x_T - 2\beta^3 x_T = 4\beta^3 x_T$$

$$\cdots$$

$$x_t = (T+1-t)\,\beta^{T-t} x_T$$

$$\cdots$$

$$x_0 = (T+1)\,\beta^T x_T$$

$$\Rightarrow \left[\begin{array}{l} x_T^* = \dfrac{x_0}{\beta^T (T+1)} \\[2ex] x_t^* = \dfrac{T+1-t}{T+1}\,\beta^{-t} x_0 \\[2ex] c_t^* = x_t^* - \beta x_{t+1}^* = \dfrac{\beta^{-t} x_0}{T+1} \end{array}\right.$$

$$\square$$