$$\max_{x,y} \; f(x,y) = 2x^2 + 3xy$$

$$\text{s.a.} \quad g_1(x,y) = \tfrac{1}{2}x^2 + y \le 4$$
$$g_2(x,y) = -y \le -2 \quad (y \ge 2)$$

$$\mathcal{L}(x,y) = 2x^2 + 3xy - \lambda_1\left(\tfrac{1}{2}x^2 + y - 4\right) - \lambda_2(-y+2)$$

$$\lambda_{1,2} \ge 0$$

(1) $\frac{\partial \mathcal{L}}{\partial x} = 4x + 3y - \lambda_1 x = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = 3x - \lambda_1 + \lambda_2 = 0$

(3) $\lambda_1\left(\tfrac{1}{2}x^2 + y - 4\right) = 0$

(4) $\lambda_2(-y+2) = 0$

4 Tilfælde:

1. Begge bibetingelser aktive
2. g_1 aktiv, g_2 inaktiv
3. g_1 inaktiv, g_2 aktiv
4. Begge bibetingelser inaktive

1. $\tfrac{1}{2}x^2 + y = 4, \quad \lambda_1 > 0$

$y = 2, \quad \lambda_2 > 0$

$\Rightarrow \tfrac{1}{2}x^2 + 2 = 4 \Rightarrow x = \pm 2$

(a) $x = 2$:

(1) $\Rightarrow 8 + 6 - 2\lambda_1 = 0 \Rightarrow \lambda_1 = 7 > 0$

(2) $\Rightarrow 6 - \lambda_1 + 7 = 0 \Rightarrow \lambda_2 = 1 > 0$

$\Rightarrow$ Kandidatpunkt:

$(x,y) = (2,2) \qquad (\lambda_1, \lambda_2) = (7,1)$

(b) $x = -2$

(1) $\Rightarrow \underset{-2}{4x} + 3\underset{2}{y} - \lambda_1 \underset{-2}{x} = 0$

$\Rightarrow \lambda_1 = 1 > 0$

(2) $\Rightarrow 3\underset{-2}{x} - \underset{1}{\lambda_1} + \lambda_2 = 0$

$\Rightarrow \lambda_2 = 7 > 0$

$\Rightarrow$ Kandidatpunkt:

$(x,y) = (-2,2), \quad (\lambda_1, \lambda_2) = (1,7)$

2. g_1 aktiv, g_2 inaktiv

$\tfrac{1}{2}x^2 + y = 4, \quad \lambda_1 > 0$

$y > 2, \quad \lambda_2 = 0$

$\Rightarrow y = 4 - \tfrac{1}{2}x^2$

(2) $\Rightarrow 3x - \lambda_1 = 0 \Rightarrow \lambda_1 = 3x$

(1) $\Rightarrow 4x + 3y - \lambda_1 x = 0$

$4x + 3\left(4 - \tfrac{1}{2}x^2\right) - 3x^2 = 0$

$-\tfrac{9}{2}x^2 + 4x + 12 = 0 \quad \Big| \cdot -\tfrac{2}{9}$

$x^2 - \tfrac{8}{9}x - \tfrac{8}{3} = 0$

$$x_{1,2} = \frac{4}{9} \pm \sqrt{\underbrace{\frac{1}{4}\frac{64}{81}}_{\frac{16}{81}} + \frac{8}{3}}$$

$$= \frac{4}{9} \pm \sqrt{\frac{16 + 216}{81}}$$

$$= \frac{4}{9} \pm \frac{\sqrt{232}}{9}$$

(a) $x = \frac{4}{9} + \frac{\sqrt{232}}{9}$

$$y = 4 - \tfrac{1}{2}x^2 = 4 - \tfrac{1}{2}\left(\frac{4}{9} + \frac{\sqrt{232}}{9}\right)^2$$
$$= 4 - \tfrac{1}{2}\left(\frac{16}{81} + \frac{8\sqrt{232}}{81} + \frac{232}{81}\right)$$
$$= \frac{200 - 4\sqrt{232}}{81}$$
$$\hat{=}\; 1.717 < 2 \qquad ↯$$

(b) $x = \frac{4}{9} - \frac{\sqrt{232}}{9}$

$\lambda_1 = 3x = 3\left(\frac{4}{9} - \frac{\sqrt{232}}{9}\right) < 0 \qquad ↯$

3. g_1 inaktiv, g_2 aktiv

$\tfrac{1}{2}x^2 + y < 4, \quad \lambda_1 = 0$

$y = 2, \quad \lambda_2 > 0$

(2) $3x - \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -3x$

(1) $4x + 3y - \lambda_1 x = 0$

$4x + 6 = 0 \Rightarrow x = -\tfrac{3}{2}$

$\Rightarrow \lambda_2 = \tfrac{9}{2} > 0$

Kandidatpunkt:

$(x,y) = \left(-\tfrac{3}{2}, 2\right) \qquad (\lambda_1, \lambda_2) = \left(0, \tfrac{9}{2}\right)$

4. Begge bibetingelser inaktive

$\tfrac{1}{2}x^2 + y < 4, \quad \lambda_1 = 0$

$y > 2, \quad \lambda_2 = 0$

(2) $3x = 0 \Rightarrow x = 0$

(1) $3y = 0 \Rightarrow y = 0$

$y > 2 \qquad ↯$

$f(x,y) = 2x^2 + 3xy$

$(x,y) = (2,2) \xrightarrow{(\lambda_1,\lambda_2) = (7,1)} f(x,y) = 20$

$(x,y) = (-2,2) \Rightarrow f(x,y) = -4$

$(x,y) = \left(-\tfrac{3}{2}, 2\right) \Rightarrow f(x,y) = -\tfrac{9}{2}$

Anden-ordens betingelse

$$\mathcal{L}(x,y)\Big|_{(\lambda_1,\lambda_2) = (7,1)}$$
$$= 2x^2 + 3xy - \tfrac{7}{2}x^2 - 7y + 28 + y - 2$$
$$= -\tfrac{3}{2}x^2 + 3xy - 6y + 26$$

konkav i (x,y)

$\Rightarrow (2,2)$ er maksimum

Constraint qualification

$$\nabla g_1 = \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$\nabla g_1, \nabla g_2$ lin. afh. i $x = 0$

$\tfrac{1}{2}x^2 + y \le 4 \Rightarrow y \le 4$

$y \ge 2 \Rightarrow y \in [2,4]$

$f(x,y) = 2x^2 + 3xy = 0 < 20$

$\Rightarrow (x,y) = (2,2)$ er maksimumspunkt.

$\square$