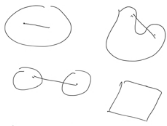
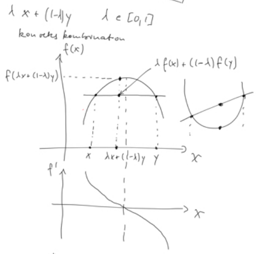


$\lambda x + (1-\lambda) y \quad \lambda \in [0,1]$

konveks kombination



$f(\lambda x + (1-\lambda) y) \underset{\leq}{\geq} \lambda f(x) + (1-\lambda) f(y)$ konkav / konveks

Bevis:

$\underbrace{f: I \to \mathbb{R},\ f \in C^2,\ f \text{ konkav}}_{A} \iff \underbrace{f''(x) \leq 0 \ \forall x \in I}_{B}$

(a) "$\Leftarrow$"

$f''(x) \leq 0 \ \forall x \Rightarrow f'(x)$ monotont ikke-stigende

$\Rightarrow [x_1, x_2 \in I : x_1 < x_2 \Rightarrow f'(x_1) \geq f'(x_2)]$

Lad $\lambda \in (0,1)$ og

$x := \lambda x_1 + (1-\lambda) x_2 \in (x_1, x_2)$

Middelværdisætning: Der findes

$c_1 \in (x_1, x)$ og $c_2 \in (x, x_2)$

således, at

$f'(c_1) = \frac{f(x) - f(x_1)}{x - x_1}$

$f'(c_2) = \frac{f(x_2) - f(x)}{x_2 - x}$

og $f'(c_1) \geq f'(c_2)$ fordi f' ikke-stigende

$x - x_1 = (1-\lambda)(x_2 - x_1)$

$x_2 - x = \lambda (x_2 - x_1)$

$\Rightarrow \frac{f(x) - f(x_1)}{1-\lambda} \geq \frac{f(x_2) - f(x)}{\lambda}$

$\lambda (f(x) - f(x_1)) \geq (1-\lambda)(f(x_2) - f(x))$

$\underbrace{\lambda f(x) + (1-\lambda) f(x)}_{f(x)} \geq \lambda f(x_1) + (1-\lambda) f(x_2)$ $\boxed{a}$

(b) f konkav $\Rightarrow f''(x) \leq 0$

A $\Rightarrow$ B

Kontraposition: $\neg B \Rightarrow \neg A$

$f''(x_0) > 0 \Rightarrow f$ ikke konkav

Lad $x_0 \in I$ således, at

$f''(x_0) > 0$

Definér $F(x) = f(x) - f'(x_0)(x - x_0)$.

F er to gange diff. i x_0 med

$F'(x_0) = f'(x_0) - f'(x_0) = 0$

$F''(x_0) = f''(x_0) > 0$

$\Rightarrow F$ har et minimum i x_0.

$\Rightarrow$ Der findes $\varepsilon > 0$ således, at

$[x_0 - \varepsilon, x_0 + \varepsilon] \subset I$

$F(x_0) = f(x_0) < F(x_0 - \varepsilon)$

$< F(x_0 + \varepsilon)$

$\Rightarrow$ For alle $\lambda \in [0,1]$:

$F(x_0) = f(x_0) < \lambda F(x_0 + \varepsilon) + (1-\lambda) F(x_0 - \varepsilon)$

$\Rightarrow$ Især $\lambda = 1 - \lambda = \frac{1}{2}$

$\frac{1}{2} F(x_0 + \varepsilon) + \frac{1}{2} F(x_0 - \varepsilon)$

$= \frac{1}{2}[f(x_0 + \varepsilon) - f'(x_0)(x_0 + \varepsilon - x_0)]$

$+ \frac{1}{2}[f(x_0 - \varepsilon) - f'(x_0)(x_0 - \varepsilon - x_0)]$

$= \frac{1}{2} f(x_0 + \varepsilon) + \frac{1}{2} f(x_0 - \varepsilon)$

Lad $x_1 = x_0 + \varepsilon$, $x_2 = x_0 - \varepsilon$

$\Rightarrow x_0 = \frac{1}{2} x_1 + \frac{1}{2} x_2 = \lambda x_1 + (1-\lambda) x_2$

$\Rightarrow f(x_0) = f(\lambda x_1 + (1-\lambda) x_2)$

$< \lambda f(x_1) + (1-\lambda) f(x_2)$

$\Rightarrow f$ ikke konkav. $\boxed{b}$

Bevis: (2) $\underset{A}{f \text{ konkav}} \iff \underset{B}{\text{Hess} f \preceq 0}$

(a) "$\Leftarrow$": Lad $x, y \in X$ og $t \in [0,1]$

Lad $g(t) = f(y + t(x-y))$

$= f(tx + (1-t)y)$

Kæderegel:

$g'(t) = \sum_{i=1}^{n} f'_i(y + t(x-y))(x_i - y_i)$

$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{ij}(y + t(x-y))(x_i - y_i)(x_j - y_j)$

$= (x-y)^T \text{Hess} f(y + t(x-y))(x-y) \underset{\text{Hess} f \leq 0}{\leq} 0$

$\Rightarrow g$ konkav på $[0,1]$

$\Rightarrow g(t) = g(t \cdot 1 + (1-t) 0)$

$\geq t g(1) + (1-t) g(0)$

$f(tx + (1-t)y) \geq t f(x) + (1-t) f(y)$ $\boxed{a}$

(b) "$\Rightarrow$": Vi skal vise: f konkav $\Rightarrow$

$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{ij} h_i h_j \leq 0$

for alle $x \in X$, $h = (h_1, h_2, \ldots, h_n)' \in \mathbb{R}^n$.

X er en åben mængde $\Rightarrow$ For hvert (x, h) findes et tal $a > 0$ således, at

$x + th \in X$ for alle $|t| < a$.

Lad $I = (-a, a)$ og $p: I \to \mathbb{R}$

$p(t) = f(x + th)$

Siden f er konkav, er p konkav i I (FMEA s. 60).

$\Rightarrow p''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{ij}(x + th) h_i h_j$

$= h^T \text{Hess} f(x + th) h \leq 0$

for alle $t \in I$ og $x \in X$. $\boxed{b}$

(a) Lad $F = (f_1, \ldots, f_m)$

og $G(x) = \langle F(x), a \rangle$

$a = (a_1, \ldots, a_m)'$

F konkav $\Rightarrow$

$G(\lambda x + (1-\lambda) y) = \langle F(\lambda x + (1-\lambda) y), a \rangle$

$\geq \langle \lambda F(x) + (1-\lambda) F(y), a \rangle$

$= \lambda \langle F(x), a \rangle + (1-\lambda) \langle F(y), a \rangle$

$= \lambda G(x) + (1-\lambda) G(y)$ $\square$

(c) Lad $x, y \in X$ og $\lambda \in [0,1]$

Lad $\tilde{u} = -F \circ f$

F konveks, aftagende $\Rightarrow -F$ konkav, voksende

f konkav $\Rightarrow f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y)$

$-F$ konkav, voksende $\Rightarrow$

$-F(f(\lambda x + (1-\lambda) y)) \underset{\text{voks.}}{\geq} -F(\lambda f(x) + (1-\lambda) f(y))$

$\underset{\text{konk}}{\geq} -\lambda F(f(x)) - (1-\lambda) F(f(y))$

$\tilde{u}(\lambda x + (1-\lambda) y) \geq \lambda \tilde{u}(x) + (1-\lambda) \hat{u}(y)$

$\Rightarrow \hat{u}$ konkav $\Rightarrow -\hat{u} = F \circ f$ konveks