$$\|x\|^2 = x_1^2 + x_2^2$$

$$\|x+y\|^2 = \|x-y\|^2$$

$$(x_1+y_1)^2 + (x_2+y_2)^2 = (x_1-y_1)^2 + (x_2-y_2)^2$$

$$\cancel{x_1^2} + \underline{2x_1y_1} + \cancel{y_1^2} + \cancel{x_2^2} + \underline{2x_2y_2} + \cancel{y_2^2}$$

$$= \cancel{x_1^2} - 2x_1y_1 + \cancel{y_1^2} + \cancel{x_2^2} - 2x_2y_2 + \cancel{y_2^2}$$

$$4x_1y_1 + 4x_2y_2 = 0 \quad / :4$$

$$x_1y_1 + x_2y_2 = 0$$

$$x \perp y \;:\Longleftrightarrow\; \underline{x_1y_1 + x_2y_2} = 0 \qquad x, y \in \mathbb{R}^2$$

$$\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\left\langle \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \right\rangle = a \cdot 0 + 0 \cdot b = 0 \quad ✓$$

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1^2 + x_2^2$$

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1y_1 + x_2y_2$$

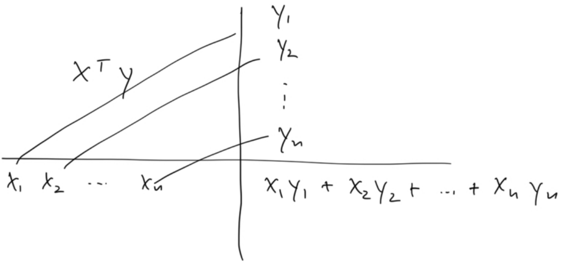
$$c^2 = \|x+y\|^2$$

$$= \langle x+y, x+y \rangle$$

$$\overset{\text{lin.}}{=} \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$$

$$\underset{\text{sym.}}{=} \langle x,x \rangle + 2\langle x,y \rangle + \langle y,y \rangle$$

$$= \underbrace{\|x\|^2}_{a^2} + \underbrace{\|y\|^2}_{b^2} + \underbrace{2\langle x,y \rangle}_{=0}$$



$$|\langle x,y \rangle| \leq \|x\|\,\|y\|$$

x eller y lig med nul : $\quad 0 = 0 \quad \leftarrow$

x, y lin. afhængige : $\quad y = rx, \quad r \neq 0$

$$\langle x,y \rangle = \langle x, rx \rangle \underset{\text{lin.}}{=} r\langle x,x \rangle$$

$$= r\|x\|\,\|x\|$$

$$\underset{\text{lin.}}{=} \|\underbrace{rx}_{y}\|\,\|x\|$$

x, y lin. uafhængige: $\quad x + ry \neq 0, \quad r \in \mathbb{R}$

$$0 \overset{\text{pos.def.}}{<} \langle x+ry, x+ry \rangle$$

$$\underset{\text{lin.}}{=} \langle x,x \rangle + r\langle x,y \rangle + r\langle y,x \rangle + r^2\langle y,y \rangle$$

$$\underset{\text{sym.}}{=} \langle x,x \rangle + 2r\langle x,y \rangle + r^2\langle y,y \rangle$$

$$= \|x\|^2 + 2r\langle x,y \rangle + r^2\|y\|^2$$

Lad $r = -\dfrac{\langle x,y \rangle}{\|y\|^2}$, så gælder

$$0 < \|x\|^2 - \frac{2\langle x,y \rangle^2}{\|y\|^2} + \frac{\langle x,y \rangle^2}{\|y\|^2}$$

$$0 < \|x\|^2 - \frac{\langle x,y \rangle^2}{\|y\|^2}$$

$$\langle x,y \rangle^2 < \|x\|^2\,\|y\|^2$$

$$|\langle x,y \rangle| < \|x\|\,\|y\| \qquad \square$$