u^* maksimerer H $=: A$

$\dot{p}(t) = -H'_x = -\frac{\partial H}{\partial x} =: B$

H konkav i (x, u) $=: C$

(x^*, u^*) optimalt par $=: Z$

nødv. betingelser:

$Z \Rightarrow A \wedge B$

tilstr. betingelser

$A \wedge B \wedge C \Rightarrow Z$

Eks: $\max\limits_{u} \int_0^T (1 - tx - u^2)\, dt$

lineær, neg. parabel

$\Rightarrow f$ konkav

$\dot{x} = u \in \mathbb{R}$

$x(0) = x_0$, $x(T)$ fri

$x_0, T > 0$

Hamilton-funksjon:

$H(t, x, u, p) = 1 - tx - u^2 + pu$

$H'_u = \frac{\partial H}{\partial u} = -2u + p \overset{!}{=} 0$

$$\Rightarrow \quad u^* = \frac{p}{2}$$

$$\dot{p} = -H'_x = t \quad , \quad p(T) = 0$$

transversalitets betingelse

$$\Rightarrow \quad p = \frac{t^2}{2} + C$$

$$p(T) = \frac{T^2}{2} + C = 0 \quad \Rightarrow \quad C = -\frac{T^2}{2}$$

$$\Rightarrow \quad u^* = \frac{p}{2} = \frac{1}{4}(t^2 - T^2)$$

$$\Rightarrow \quad \dot{x} = u = \frac{1}{4}(t^2 - T^2)$$

$$\Rightarrow \quad x^*(t) = \frac{1}{12}t^3 - \frac{1}{4}T^2 t + x_0$$

□

Eks: $\max_u \int_0^T \left(-x^2 - \frac{1}{2}u^2\right) dt$

Typ II

$$\dot{x} = x + u \quad , \quad u \in (-\infty, \infty)$$

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$$x(0) = 1 \quad , \quad x(T) \text{ fri } \Rightarrow p(T) = 0$$

Hamilton-funktion:

$$H(t, x, u, p) = -x^2 - \frac{1}{2}u^2 + p(x + u)$$

$$H'_u = -u + p = 0 \quad \Rightarrow \quad u = p$$

$$\dot{p} = -H'_x = 2x - p$$

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$$\Rightarrow \quad \begin{aligned} \dot{x} &= x + p \\ \dot{p} &= 2x - p \end{aligned}$$

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

1 $$p(\lambda) = (1-\lambda)(-1-\lambda) - 2$$
$$= \lambda^2 - 3$$
$$\Rightarrow \quad \lambda_{1,2} = \pm\sqrt{3}$$

$$\left.\begin{matrix} 1-\sqrt{3} & 1 \\ 2 & -1-\sqrt{3} \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix}$$

$$\rightsquigarrow \left.\begin{matrix} 1 & \frac{1}{1-\sqrt{3}} \\ 0 & \underbrace{-1-\sqrt{3} - \frac{2}{1-\sqrt{3}}}_{=0} \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \qquad \Rightarrow \quad v_1 = \begin{pmatrix} -\frac{1}{1-\sqrt{3}} \\ 1 \end{pmatrix}$$

$$\left.\begin{matrix} 1+\sqrt{3} & 1 \\ 2 & -1+\sqrt{3} \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix}$$

$$\left.\begin{matrix} 1 & \frac{1}{1+\sqrt{3}} \\ 0 & \underbrace{-1+\sqrt{3} - \frac{2}{1+\sqrt{3}}}_{=0} \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \qquad \Rightarrow \quad v_2 = \begin{pmatrix} -\frac{1}{1+\sqrt{3}} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ p \end{bmatrix} = A \begin{pmatrix} -\frac{1}{1-\sqrt{3}} \\ 1 \end{pmatrix} e^{\sqrt{3}t} + B \begin{pmatrix} -\frac{1}{1+\sqrt{3}} \\ 1 \end{pmatrix} e^{-\sqrt{3}t}$$

$$x(0) = -\frac{1}{1-\sqrt{3}} A - \frac{1}{1+\sqrt{3}} B = 1$$
$$p(T) = e^{\sqrt{3}T} A + e^{-\sqrt{3}T} B = 0$$

1 $$\begin{bmatrix} -\frac{1}{1-\sqrt{3}} & -\frac{1}{1+\sqrt{3}} \\ e^{\sqrt{3}T} & e^{-\sqrt{3}T} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -\frac{1}{1-\sqrt{3}} & -\frac{1}{1+\sqrt{3}} & 1 \\ e^{\sqrt{3}T} & e^{-\sqrt{3}T} & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|l} 1 & 0 & \frac{-(1+\sqrt{3})(1-\sqrt{3})}{1+\sqrt{3}-(1-\sqrt{3})e^{2\sqrt{3}T}} = A \\ 0 & 1 & \frac{(1-\sqrt{3})(1+\sqrt{3})}{e^{-2\sqrt{3}T}(1+\sqrt{3})-(1-\sqrt{3})} = B \end{array}\right]$$

$$\Rightarrow x^*(t) = \frac{1+\sqrt{3}}{1+\sqrt{3}-(1-\sqrt{3})e^{2\sqrt{3}T}} e^{\sqrt{3}t} - \frac{1-\sqrt{3}}{e^{-2\sqrt{3}T}(1+\sqrt{3})-(1-\sqrt{3})} e^{-\sqrt{3}t}$$

$$p(t) = A e^{\sqrt{3}t} + B e^{-\sqrt{3}t}$$

$$u^*(t) = p(t) = A e^{\sqrt{3}t} + B e^{-\sqrt{3}t}$$