Solving Linear Second-Order PDE with the Finite Element Method

Based on Hermes2D (`http://hpfem.org/hermes`)

1 Module Description

This module is designed to solve general linear second-order partial differential equations (PDE) of the form

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f$$

where $d = 2$ is the spatial dimension, and a_{ij}, b_i, c and f are constants, which can be different in subdomains. One can prescribe Dirichlet, Neumann, and Newton (Robin) boundary conditions. The boundary conditions can be combined arbitrarily.

2 Interesting Special Cases

With $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = b_1 = b_2 = c = 0$ we obtain the *Poisson equation*

$$-\Delta u = f.$$

This equation has several important applications in physics. It is used to model electrostatics (u being the electric potential and f the electric charge density divided by the electric permittivity), stationary heat transfer equation (u being the temperature and f the heat sources or losses), and other diffusive processes.

Setting moreover $f = 0$, one obtains the *Laplace equation*

$$-\Delta u = 0$$

which describes linear magnetostatics (u being the scalar magnetic potential), stationary wave equation (u being the amplitude), and it can also be used to compute the shape of an elastic membrane (smallest surface) that spans a closed curve.

With $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = b_1 = b_2 = f = 0$ and $c < 0$ one obtains the *Helmholtz equation*

$$-\Delta u - k^2 u = 0$$

where $k^2 = -c$ is the square of the wave number.

As the last example, with $a_{11} = a_{22} = D$, $a_{12} = a_{21} = 0$, $b_1 = v_1$, $b_2 = v_2$, and $f = c = 0$ one obtains the *stationary advection-diffusion equation*

$$-\mathrm{div}(D\nabla u) + \mathbf{v} \cdot \nabla u = 0.$$

Here u is the advected quantity, $D > 0$ is the diffusion coefficient, and $\mathbf{v} = (v_1, v_2)$ is the velocity of the flowing medium that is advecting the quantity u.

3 Boundary Conditions

One can split the boundary into subsets where different boundary conditions are prescribed:

- *Dirichlet conditions*: $u = u^*$ where u^* is a constant.
- *Neumann conditions*:

  $$\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \nu_i = c$$

  where (ν_1, ν_2) is the unit outer normal vector to the domain's boundary, and c is a constant.
- *Newton (Robin) conditions*:

  $$\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \nu_i + c_1 u = c_2$$

  where (ν_1, ν_2) is the unit outer normal vector to the domain's boundary, and c_1, c_2 are constants.