Linear Elasticity Module

Based on Hermes2D (`http://hpfem.org/hermes`)

1 Module Description

The linear elasticity module calculates displacements and stresses caused by deforming[1] solid objects. The objects can consist of one or more material subdomains with generally different values of the Young modulus E and Poisson ratio ν. In these subdomains, the material is assumed homogeneous.

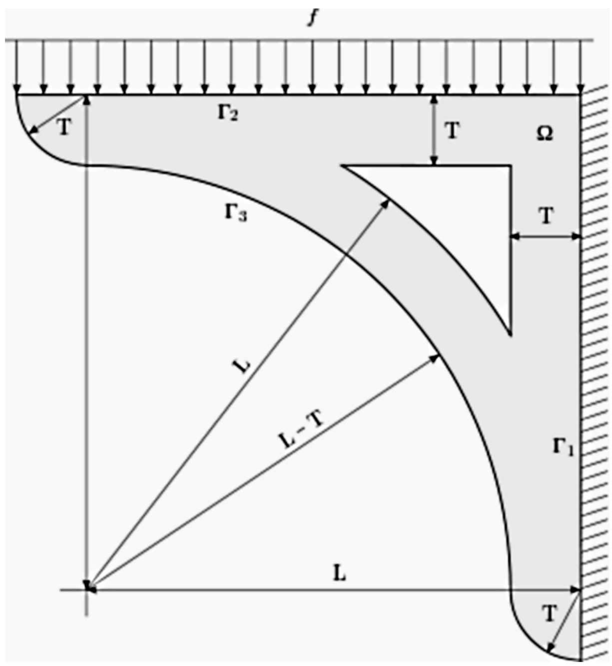

Figure 1: Steel bookshelf bracket.

On the boundary, one can prescribe *displacements* (right vertical edge in Fig. 1), *surface force* (top edge), or nothing (curved parts and the inner triangle). Prescribing nothing is equivalent to prescribing zero surface force. Other boundary conditions one can prescribe are *normal force*, or *pressure*. The model accounts for volumetric forces caused by the gravity or another acceleration. Calculated are displacements and the Von Mises stress.

[1]Linear elasticity assumes that stress is a linear function of strain, and thus is applies to small deformations only.

2 Underlying Equations

Equations of linear elasticity consist of *equilibrium equations*, *strain-displacement equations* and *constitutive equations*. Omitting their derivation (which can be found in many textbooks as well as on Wikipedia), the equations have the form

$$\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} + F_i = 0, \quad i = 1, 2$$

(Einstein's summation over repeated indices is used). Here u_1, u_2 are unknown displacements, $F = (F_1, F_2)$ the vector of volumetric forces, and λ, μ are Lamé constants. If just gravitational acceleration is present, then we have

$$(F_1, F_2) = (0, -\varrho g)$$

where ϱ is the material density and g the gravitational acceleration. The Lamé constants λ and μ are related to E and ν as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

3 Boundary Conditions

There are four types of boundary conditions:

- *Fixed displacement*: $u_1 = u_1^*$ and $u_2 = u_2^*$ where u_1^*, u_2^* are constants. Setting these constants to zero means a fixed boundary that does not move, but nonzero boundary displacement can be also prescribed if deformation of the boundary is known. It is possible to prescribe different displacements on different parts of the boundary.
- *Surface force*: $f_1 = f_1^*$ and $f_2 = f_2^*$ where f_1^*, f_2^* are constants. It is possible to prescribe different surface forces on different parts of the boundary.
- *Normal force*: Similar to *surface force*, but the components are calculated from the value, according to the normal direction to the boundary. It is possible to prescribe different normal forces on different parts of the boundary.
- *Pressure*: Constant pressure in Pa on the part of the boundary. It is possible to prescribe different pressures on different parts of the boundary.
- *Free boundary*: Where neither displacement nor surface force is prescribed, the boundary remains free. This is equivalent to prescribing $f_1 = 0$ N and $f_2 = 0$ N.

4 Sample Results

The following results correspond to the geometry from Fig. 1 with $E = 200$ GPa, $\nu = 0.3$, $\varrho = 8000$ kg/m^3, $g = 9.81$ ms^{-2}, $F_1 = 0$ N, and $F_2 = -1000$ N.

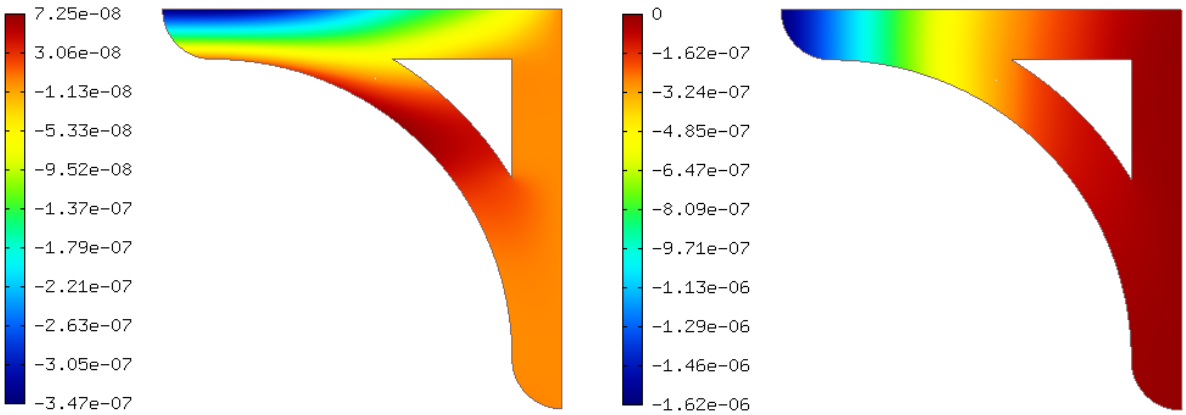

Figure 2: Displacements u_1 and u_2.

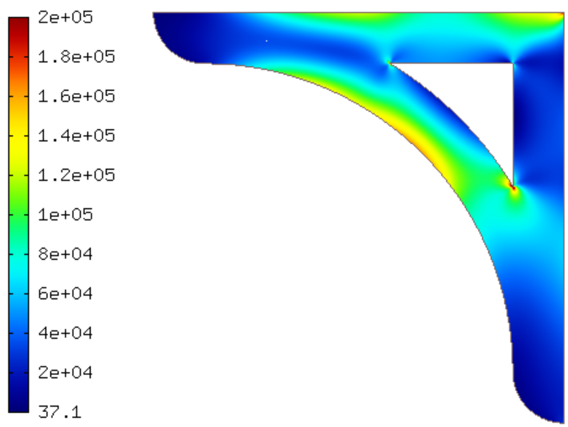

Figure 3: Von Mises stress.

In the last figure, notice singularities (pointwise infinite values) in the Von Mises stress at all three re-entrant corners. Singularities are a known difficulty of linear elasticity models. They are easily underresolved if a coarse mesh is used. In order to resolve singularities accurately, adaptive finite element methods must be used. Adaptive *hp*-FEM methods that are provided by the Hermes library are among the best adaptive finite element methods available.