MTH 101A- 2021 -2022
Quiz -II: Tentative Marking Scheme

1. Discuss the differentiability of the function $f(x,y) = \sin(x)\sqrt{|xy|}$ at all points on the x-axis.

[7]

2. (a) Is the improper integral $\int\limits_{\pi}^{\infty} \sin^2 x \; dx$ convergent? Justify your answer.

[2]

   (b) Determine all values of $p \in \mathbb{R}$ for which the improper integral $\int\limits_{\pi}^{\infty} \frac{\sin^2 x}{x^p} dx$ is convergent.

[5]

3. Let C be the cone generated by revolving the line $y = \frac{x}{\sqrt{8}}$ starting from $(0,0)$ around y-axis. Let S be the solid sphere of radius 3 i.e., $x^2 + y^2 + z^2 \leq 9$. Find the volume of the portion of S that lies inside C.

[6]

Question 1 solution:

- Let $(A, 0)$ be any arbitrary point on the x-axis.

$$f_x(A,0) = \lim_{h\to 0} \frac{f(A+h,0) - f(0,0)}{h} = 0.$$

[1]

$$f_y(A,0) = \lim_{k\to 0} \frac{f(A,k) - f(0,0)}{k} = \lim_{k\to 0} \sin(A)\frac{\sqrt{|A||k|}}{k}.$$

- The partial derivative $f_y(A,0) = 0$ if $A = n\pi$. [1]
- The partial derivative $f_y(A,0)$ does not exist if $A \neq n\pi$. [1]
- Since $f_y(A,0)$ does not exist if $A \neq n\pi$, therefore total derivative cannot exist at these points.

[1]

- Now for the points $A = n\pi$, let us look at the error function

$$|e(h,k)| = \left|\frac{f(A+h,k) - f(A,0) - (h,k)\cdot\nabla f(A,0)}{\sqrt{h^2+k^2}}\right|$$
$$= |\sin(n\pi + h)|\frac{\sqrt{|n\pi + h||k|}}{\sqrt{h^2+k^2}} = |\sin(h)|\frac{\sqrt{|n\pi + h||k|}}{\sqrt{h^2+k^2}}.$$

[1]

- Now using the inequality for $|h|$ small enough, we know $|\sin h| \leq |h| \leq \sqrt{h^2+k^2}$.

[1]

- Therefore we get $|e(h,k)| \leq \sqrt{|n\pi + h||k|} \to 0$ as $(h,k) \to (0,0)$. Hence f is differentiable at $(n\pi, 0)$. [1]

Question 2 solution:

- By definition $\int\limits_{\pi}^{M} \sin^2 x \; dx = \frac{1}{2} \int\limits_{\pi}^{M} (1 - \cos(2x)) \; dx = \frac{M}{2} - \frac{\pi}{2} - \frac{\sin 2M}{4}$ [1]
- Thus $\lim\limits_{M \to \infty} \int\limits_{\pi}^{M} \sin^2 x \; dx = \infty$. Hence the intergal is divergent. [1]
- $p \leq 0$. In this case $\frac{\sin^2 x}{x^p} \geq \sin^2 x$ since $x \geq \pi$. Hence by comparison test $\int\limits_{\pi}^{\infty} \frac{\sin^2 x}{x^p} dx$ is divergent. [1]
- $p > 1$. We use the inequality $0 \leq \frac{\sin^2 x}{x^p} \leq \frac{1}{x^p}$. Hence by comparison test $\int\limits_{\pi}^{\infty} \frac{\sin^2 x}{x^p} dx$ is convergent. [1]
- $0 < p \leq 1$. Now $\frac{\sin^2 x}{x^p} = \frac{1}{2x^p} - \frac{\cos 2x}{2x^p}$. In this case, $\int\limits_{\pi}^{\infty} \frac{1}{x^p} dx$ is divergent. [1]
- In this case, $\int\limits_{\pi}^{\infty} \frac{\cos 2x}{x^p} dx$ is convergent by Dirichlet test. [1]
- Therefore $\int\limits_{\pi}^{\infty} \frac{\sin^2 x}{x^p} dx$ is divergent for $0 < p \leq 1$.. [1]

Question 3 solution:

- Volume of the upper hemisphere $= \frac{4}{6}\pi 3^3 = 18\pi$. [1]
- Intersection point of the circle on xy-plane $x^2 + y^2 = 9$ and and line $x = \sqrt{8}y$ is $= (\sqrt{8}, 1)$.

  [2]
- Volume of the portion outside the cone and inside the sphere is $= \int\limits_0^1 \pi(9 - y^2 - 8y^2)\ dy = 6\pi$. [2]
- The required volume $= 18\pi - 6\pi = 12\pi$.

  [1]

Alternative solution:

- Intersection point of the circle on xy-plane $x^2 + y^2 = 9$ and and line $x = \sqrt{8}y$ is $= (\sqrt{8}, 1)$.

  [2]
- Volume of the cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi.8.1 = 8\pi/3$. [1]
- Volume of the upper curved portion $= \int\limits_1^3 \pi x^2\ dy = \int\limits_1^3 \pi(9 - y^2)\ dy = \frac{28}{3}\pi$.

  [2]
- The required volume. $=\frac{8}{3}\pi + \frac{28}{3}\pi = 12\pi$

  [1]