MTH 101A 2021-2022
Mid-Sem: Solutions

Q1. **(a)** Let $A = \{x \in \mathbb{R} : x^3 + x < 1\}$. Show that the set A is bounded above. If $a = \sup A$, then show that $a^3 + a = 1$. **[7 marks]**

(b) Let $a_1 = 1$ and $a_{n+1} = \left(1 + \frac{(-1)^n}{2^n}\right) a_n$, for $n \in \mathbb{N}$. Show that the sequence (a_n) is convergent. **[8 marks]**

Solutions:

(a) We first show that 1 is an upper bound for A. If $x > 1$ for some x then $x^3 + x > 2$, a contradiction.

Consider the sequence $x_n = a + 1/n$. Since a is an upper bound for A we have $x_n \notin A$. So $x_n^3 + x_n \geq 1$. Further, notice that $\lim\limits_{n\to\infty} x_n = z$, so $\lim\limits_{n\to\infty} (x_n^3 + x_n) = a^3 + a$. But since $x_n^3 + x_n \geq 1$, it follows that $a^3 + a \geq 1$.

Similarly, consider the sequence $y_n = a - 1/n$. Then y_n is not an upper bound of A. So there exists $x \in A$ such that $y_n < x$. Then $y_n^3 + y_n < x^3 + x < 1$ as 1 is an upper bound. Then $\lim\limits_{n\to\infty} y_n^3 + y_n = a^3 + a \leq 1$. So we conclude that $a^3 + a = 1$.

Alternate Solution:

Let $f(x) = x + x^3$. Note that f is continuous.

Note $f(0) = 0$ and $f(1) = 2$. Then by IVP (Intermediate Value Property) there exists $c \in (0, 1)$ such that $f(c) = 1$

Note f is increasing.

So $f(x) < 1$ for $x < c$ and $f(x) > 1$ for $x > c$.

So we can identify $A = (-\infty, c)$.

So $a = \sup A = c$. Hence $f(a) = a + a^3 = 1$.

(b)

$$
\begin{aligned}
|a_{n+1} - a_n| &= \frac{1}{2^n}|a_n| \\
|a_n - a_{n-1}| &= \frac{1}{2^{n-1}}|a_{n-1}| \\
\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}|} &= \frac{1}{2}\left|\frac{a_n}{a_{n-1}}\right| \\
&= \frac{1}{2}\left|1 + \frac{(-1)^{n-1}}{2^{n-1}}\right| \\
&\leq \frac{1}{2}(1 + \frac{1}{2}) \\
&\leq \frac{3}{4} < 1
\end{aligned}
$$

Hence, (a_n) is Cauchy. So converges.

Alternate Solution:

$|a_{n+1}| \leq (1+\frac{1}{2^n})(1+\frac{1}{2^{n-1}})\cdots(1+\frac{1}{2}) \leq (\frac{n+1}{n})^n < 3$. (Using AM-GM Inequality.)

$$|a_{n+1}-a_n| = \frac{|a_n|}{2^n} < \frac{3}{2^n}$$

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&< \frac{3}{2^{m-1}} + \frac{3}{2^{m-2}} + \cdots + \frac{3}{2^n} \\
&= \sum_{k=1}^{m} \frac{3}{2^{k-1}} - \sum_{k=1}^{n} \frac{3}{2^{k-1}}
\end{aligned}$$

Since, $\sum\limits_{k=1}^{\infty} \frac{3}{2^{k-1}}$ is convergent series, we see that the sequence (a_n) is Cauchy Sequence.

Hence (a_n) is convergent.

Q2. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Using ϵ-δ definition of continuity, show that the function $f(x)$ is not continuous at 0. Find a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence. **[8 marks]**

(b) Let $f : (a,b) \to \mathbb{R}$ be an infinitely differentiable function, and let $x_0 \in (a,b)$. Suppose there exist $k \geq 1$ such that $f''(x_0) = f^{(3)}(x_0) = \cdots = f^{(2k)} = 0$. If $f^{(2k+1)}(x_0) \neq 0$ then show that x_0 is a point of inflection for f. **[7 marks]**

Solutions:

(a) If the function is continous at 0, then by definition, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \epsilon$.

We choose $\epsilon = 1/2$ and let $\delta > 0$ be given.

Note that $\cos(\frac{1}{x}) = 1$ if $x = 1/(2k\pi)$.

Choose k large enough so that $x = 1/(2k\pi)$ and $|x| \leq \delta$. Then $|1 - 0| < \epsilon = 1/2$, a contradiction.

Let $(x_n) = 2/n\pi$, then $f(x_n) = 0$ if n is odd and $f(x_n) = (-1)^k$ if n is even. So $(f(x_n))$ is not Cauchy.

(b) Since $f^{(2k+1)}(x_0) \neq 0$, by the continuity of $f^{(2k+1)}$ there exists a $\delta > 0$ such that $f^{(2k+1)}(x)$ has same sign as $f^{(2k+1)}(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

By Taylor's theorem, for any $x \in (x_0 - \delta, x_0 + \delta)$ there exists a point c between x and x_0 such that

$$f''(x) = \frac{f^{(2k+1)}(c)}{(2k-1)!}(x - x_0)^{2k-1}.$$

If $f^{(2k+1)}(x_0) > 0$ then $f''(x) \leq 0$ for $x \in (x_0 - \delta, x_0)$ and $f''(x_0) \geq 0$ for $x \in (x_0, x_0 + \delta)$.

Similary, if $f^{(2k+1)}(x_0) < 0$ then $f''(x) \geq 0$ for $x \in (x_0 - \delta, x_0)$ and $f''(x_0) \leq 0$ for $x \in (x_0, x_0 + \delta)$.

For both the case we conclude that x_0 is a point of inflection for f.

Q3. **(a)** Does the following series converge? Does it converge absolutely?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{5n + (\log_e n)^3}.$$

[6 marks]

(b) Find the Maclaurin series for $f(x) = \frac{1}{1+2x^2}$. Find all $x \in \mathbb{R}$ such that the Maclaurin series is convergent at x. **[6 marks]**

(c) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^3}{n^4 + 1} x^{3n}.$$

[3 marks]

Solutions:

(a) Since $\frac{1}{5n+(\log_e n)^3}$ is a sequence which is positive, decreasing, and tending to 0, by the alternating series test, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{5n+(\log_e n)^3}$ converges.

Note that

$$\lim_{n\to\infty} \frac{\frac{1}{5n+(\log_e n)^3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n + (\log_e n)^3} = \lim_{n\to\infty} \frac{1}{5 + \frac{(\log_e n)^3}{n}} = \frac{1}{5}.$$

Since $0 < \frac{1}{5} < \infty$, by limit comparison, the series $\sum\limits_{n=1}^{\infty} \frac{1}{5n+(\log_e n)^3}$ behaves the same as the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. Since the latter is a divergent p-series, $\sum\limits_{n=1}^{\infty} \frac{1}{5n+(\log_e n)^3}$ diverges.

(b) Writing $\frac{1}{1+2x^2}$ as

$$\frac{1}{1 - (-2x^2)},$$

we can use the geometric series to see that

$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}.$$

Since the equality $\frac{1}{1-r} = \sum r^n$ is only valid when $|r| < 1$, we see that this series converges for $|-2x^2| < 1$, meaning that $|x| < 1/\sqrt{2}$ (we could also have seen this by using the Ratio Test), so the interval of convergence is

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

We have to check convergence at the endpoints:

when $x = 1/\sqrt{2}$, the series is

$$\sum_{n=0}^{\infty} (-1)^n 2^n \left(\frac{1}{\sqrt{2}}\right)^{2n} = \sum_{n=0}^{\infty} (-1)^n,$$

which diverges.

When $x = -1/\sqrt{2}$, the series is

$$\sum_{n=0}^{\infty}(-1)^n 2^n \left(\frac{-1}{\sqrt{2}}\right)^{2n} = \sum_{n=0}^{\infty} 1,$$

which also diverges. Therefore, the series diverges at both endpoints and the interval of convergence is as stated above.

(c) Using the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{(n+1)^3x^{3n+3}}{(n+1)^4+1}}{\frac{n^3x^{3n}}{n^4+1}}\right| &= \lim_{n\to\infty}\left|\frac{(n^4+1)(n+1)^3x^3}{n^3((n+1)^4+1)}\right| \\ &= |x|^3\lim_{n\to\infty}\frac{(n^4+1)(n+1)^3}{n^3((n+1)^4+1)} \\ &= |x|^3\lim_{n\to\infty}\frac{n^7+\cdots}{n^7+\cdots} \\ &= |x|^3.\end{aligned}$$

The limit is less than 1 when $|x| < 1$, so the radius of convergence is 1.

Q4. **(a)** Define a function f on $[0,1]$ by $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational.}\end{cases}$.

Is f Riemann integrable on $[0,1]$? If yes, then find $\int_0^1 f(x)dx$. (You may use the formula: $\sum_{k=0}^{n} k^3 = (\frac{n(n+1)}{2})^2$). **[8 marks]**

(b) Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2}$ using definite integrals. **[7 marks]**

Solutions:

(a) Take a partition P of $[0,1]$ where $P = \{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < \frac{n}{n} = 1\}$

Since $x^2 > x^3$ in each subinterval and $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $M_i = \dfrac{i^2}{n^2}; m_i = \dfrac{(i-1)^3}{n^3}$, where $M_i = \sup f(x)$ in subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ and $m_i = \inf f(x)$ in this subinterval.

Then $U(P,f) = \sum_{i=1}^{n} M_i\Delta_i = \dfrac{1}{n}\{\dfrac{1}{n^2} + \dfrac{4}{n^2} + \ldots + \dfrac{n^2}{n^2}\} = \dfrac{n(n+1)(2n+1)}{6n^3}$.

Again $L(P,f) = \sum_{i=1}^{n} m_i\Delta_i = \dfrac{1}{n}\{0 + \dfrac{1}{n^3} + \dfrac{8}{n^3} + \ldots + \dfrac{(n-1)^3}{n^3}\} = \dfrac{n^2(n-1)^2}{4n^4}$.

Then as $n \to \infty$;$U(P,f) - L(P,f) = \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{12}$, which does not go to 0. Hence f is not Riemann integrable.

(b) We apply the formula

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f(k/n)}{n}$$

to the function $f(x) = \frac{x}{1+x^2}$.

A simple calculation shows that

$$\sum_{k=1}^{\infty} \frac{f(k/n)}{n} = \sum_{k=1}^{n} \frac{k}{k^2+n^2}.$$

Also, $\int\limits_0^1 f(x)dx = \int\limits_0^1 \frac{d}{dx}\frac{\log(1+x^2)}{2}dx$. By fundamental Theorem of Calculus, $\int\limits_0^1 f(x)dx = \frac{\log 2}{2} = \log\sqrt{2}$. This gives $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k}{k^2+n^2} = \log\sqrt{2}$.