MTH 101-Calculus
2021-2022
Quiz -1: Solutions

Q1. Let $x_1 = 4$ and $x_{n+1} = 8 + \sqrt{x_n}$ for all n. Show that the sequence (x_n) is convergent and find $\lim_{n\to\infty} x_n$. **(7 marks)**

Solution.

Method 1:

Step 1. To prove that the sequence (x_n) is increasing, i.e, $x_{n+1} - x_n \geq 0$ for all n.

Proof by induction: For $n = 1$ we have $(x_2 - x_1) = 10 - 4 > 0$.

Note that $x_{n+2} - x_{n+1} = \sqrt{x_{n+1}} - \sqrt{x_n} = \frac{1}{\sqrt{x_{n+1}}+\sqrt{x_n}}(x_{n+1} - x_n)$

By induction hypothesis, $x_{n+1} - x_n \geq 0$. Hence, $x_{n+2} - x_{n+1} \geq 0$ as $x_n > 0$ for all n.

The sequence (x_n) is increasing.

Step 2. The sequence (x_n) is bounded above.

Proof by induction: $x_n \leq 12$ for all n.

As the sequence (x_n) is increasing and bounded above, (x_n) is convergent.

Step 2. If $\lim_{n\to\infty} x_n = l$ then $l^2 - 17l + 64 = 0$.

$\implies l = \frac{17+\sqrt{33}}{2}$ or $\frac{17-\sqrt{33}}{2}$.

As $\frac{17-\sqrt{33}}{2} < 6$ and $x_n \geq 8$ for $n \geq 2$, we have $\lim_{n\to\infty} x_n = \frac{17+\sqrt{33}}{2}$.

Method 2:

Step 1. To show (x_n) is Cauchy sequence.

$x_{n+2} - x_{n+1} = \sqrt{x_{n+1}} - \sqrt{x_n} = \frac{1}{\sqrt{x_{n+1}}+\sqrt{x_n}}(x_{n+1} - x_n)$

As $x_n \geq 4$ for all n, $(x_{n+2} - x_{n+1}) \leq \frac{1}{4}(x_{n+1} - x_n)$

$\implies$ (x_n) satisfy the contractive condition.

$\implies$ (x_n) is Cauchy sequence.

$\implies$ (x_n) is convergent.

Step 2. If $\lim_{n\to\infty} x_n = l$ then $l^2 - 17l + 64 = 0$.

$\implies l = \frac{17+\sqrt{33}}{2}$ or $\frac{17-\sqrt{33}}{2}$.

As $\frac{17-\sqrt{33}}{2} < 6$ and $x_n \geq 8$ for $n \geq 2$, we have $\lim_{n\to\infty} x_n = \frac{17+\sqrt{33}}{2}$.

Q2. Let $p > 1$ be a real number. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = -f(px)$ for all $x \in \mathbb{R}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$. **(7 marks)**

Solution.

Given that f is continuous and $f(x) = -f(px)$ for all $x \in \mathbb{R}$.

So, $f(0) = 0$.

Using induction it is easy to show that $f(x) = (-1)^n f(\frac{x}{p^n})$ for any positive integer n.

Since the sequence $\frac{x}{p^n}$ converges to 0, by continuity of f, we have : $f(\frac{x}{p^n}) \to f(0) = 0$ when $n \to \infty$ and for any real number x.

Hence for every real number x, we have : $|f(x)| = \lim_{n\to\infty} |f(x)| = \lim_{n\to\infty} |(-1)^n f(\frac{x}{p^n})| = |f(0)| = 0$. So f is the zero function.

Q3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x) < 0$ for all $x \in \mathbb{R}$ and $f(2) = 3$ and $f(3) = 1$. Then prove that $f'(2) > -2$. **(6 marks)**

Solution.

Method 1:

Given that $f(2) = 3,\ f(3) = 1$.

So by mean value theorem there exists a $c \in (2, 3)$ such that $f'(c) = (f(3)-f(2))/(3-2) = -2$.

Now since $f''(x) < 0$, $f'(x)$ is strictly decreasing.

(One mark to be deducted if "strictly decreasing" is not realised.)

As $f'(c) = -2$ for some $c \in (2, 3)$, we have $f'(2) > -2$.

Method 2:

Given that $f(2) = 3,\ f(3) = 1$.

So by mean value theorem there exists a $c \in (2, 3)$ such that $f'(c) = (f(3)-f(2))/(3-2) = -2$.

By Mean Value Thm, there exists $c_1 \in (2, c)$ such that $f''(c_1) = (f'(c) - f'(2))/(c - 2)$

Since $f''(c_1) < 0$ we have $f'(2) > f'(c) = -2$.

Method 3:

By Taylor's Theorem, $f(3) = f(2) + f'(2) + \frac{1}{2}f''(c)$ for some $c \in (2, 3)$. So $1 < 3 + f'(2) \Rightarrow f'(2) > -2$.