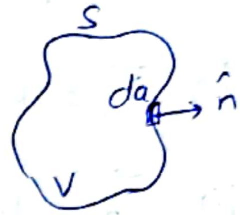
Solution of the mid-sem exam (14-01-2022)

Q-1 (a) $A = \int s\,d\phi\,dz = 3\int_0^5 dz \int_{\pi/6}^{\pi} d\phi = 3.5.\left(\pi - \frac{\pi}{6}\right)$

$$= \frac{25\pi}{2}\ m^2$$

(b) Divergence theorem:



$$\int_V (\vec\nabla\cdot\vec B)\,d\tau = \oint_S \vec B\cdot d\vec a = \oint (\vec B\cdot\hat n)\,da$$

substitute $\vec B = \vec A\times\vec C$, where $\vec C$ is a constant vector,

$$\int_V \vec\nabla\cdot(\vec A\times\vec C)\,d\tau = \oint_S (\vec A\times\vec C)\cdot\hat n\,da$$

$$\Rightarrow \int_V \vec C\cdot(\vec\nabla\times\vec A)\,d\tau = \oint_S \vec C\cdot(\hat n\times\vec A)\,da$$

$$\Rightarrow \vec C\cdot\int_V (\vec\nabla\times\vec A)\,d\tau = \vec C\cdot\oint_S (\hat n\times\vec A)\,da$$

since $\vec C$ is a constant vector,

$$\int_V (\vec\nabla\times\vec A)\,d\tau = \oint_S (\hat n\times\vec A)\,da$$

$$\vec\nabla\cdot(\vec A\times\vec C) = \vec C\cdot(\vec\nabla\times\vec A) - \underbrace{\vec A\cdot(\vec\nabla\times\vec C)}_{0}$$

$$(\vec A\times\vec C)\cdot\hat n = \vec A\cdot(\vec C\times\hat n) = \vec C\cdot(\hat n\times\vec A)$$

(c) $f(\theta) = \cos\theta \Rightarrow \nabla f = \frac{1}{r}\frac{\partial f}{\partial\theta}\hat\theta = -\frac{\sin\theta}{r}\hat\theta$

$$f(x,y,z) = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \nabla f = \frac{\partial f}{\partial x}\hat x + \frac{\partial f}{\partial y}\hat y + \frac{\partial f}{\partial z}\hat z$$

$$= -\frac{xz}{r^3}\hat x - \frac{yz}{r^3}\hat y + \left[\frac{1}{r} - \frac{z^2}{r^3}\right]\hat z$$

Thus, $$\frac{\sin\theta}{r}\hat\theta = \frac{r\sin\theta\cos\phi . r\cos\theta}{r^3}\hat x + \frac{r\sin\theta\sin\phi . r\cos\theta}{r^3}\hat y - \left[\frac{r^2(1-\cos^2\theta)}{r^3}\right]\hat z$$

$$\Rightarrow \frac{\sin\theta}{r}\hat\theta = \frac{\sin\theta\cos\theta\cos\phi}{r}\hat x + \frac{\sin\theta\cos\theta\sin\phi}{r}\hat y - \frac{\sin^2\theta}{r}\hat z$$

So, $$\hat\theta\cdot\hat x = \cos\phi\cos\theta = \frac{x}{r\sin\theta}\cdot\frac{z}{r} = \frac{xz}{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2+z^2})}$$

$$\hat\theta\cdot\hat y = \sin\phi\cos\theta = \frac{y}{r\sin\theta}\cdot\frac{z}{r} = \frac{yz}{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2+z^2})}$$

$$\hat\theta\cdot\hat z = -\sin\theta = \frac{\sqrt{x^2+y^2}}{r} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

[D] For a conservative vector field, its line integral does not depend on the path. Therefore, its closed loop line integral is always zero. Therefore, its Curl is also zero.

$$\vec{\nabla} \times \vec{F} = 0.$$

(i) $\vec{F_1} = K \exp\left(-\frac{r^2}{R^2}\right)\hat{r} \Rightarrow \vec{\nabla} \times \vec{F_1} = 0$

(ii) $\vec{F_2} = K(x^3\hat{y} - y^3\hat{x}) \Rightarrow \vec{\nabla} \times \vec{F_2} \neq 0$

(iii) $\vec{F_3} = K(x^3\hat{x} + y^3\hat{y}) \Rightarrow \vec{\nabla} \times \vec{F_3} = 0$

(iv) $\vec{F_4} = K\frac{\hat{\phi}}{r} \Rightarrow \vec{\nabla} \times \vec{F_4} \neq 0.$

Therefore, $\vec{F_1}$ & $\vec{F_3}$ represent conservative vector field.

Q.2(a) Given, $V(\vec{r}) = q\frac{e^{-\lambda r}}{r} + q\frac{\lambda e^{-\lambda r}}{2}$

So, $\vec{E} = -\vec{\nabla}V = -q\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r} + \frac{\lambda e^{-\lambda r}}{2}\right)\hat{r}$

$$= -q\left[\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2} - \frac{\lambda^2}{2}e^{-\lambda r}\right]\hat{r} = qe^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\frac{\hat{r}}{r^2}$$

Now, $\rho = \epsilon_0(\vec{\nabla}\cdot\vec{E})$ and $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$$\Rightarrow \rho = \underbrace{\epsilon_0 q e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)}_{\Downarrow \atop \mathrm{I}} + \underbrace{\epsilon_0 q\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left[e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\right]}_{\Downarrow \atop \mathrm{II}}$$

$$\mathrm{I} \Rightarrow \epsilon_0 q e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \epsilon_0 q e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)4\pi\delta^3(\vec{r})$$
$$= \epsilon_0 q\, 4\pi\,\delta^3(\vec{r})$$

$$\mathrm{II} \Rightarrow \epsilon_0 q\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left[e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\right] = \epsilon_0 q\frac{\hat{r}}{r^2}\cdot\frac{\partial}{\partial r}\left[e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right)\right]\hat{r}$$
$$= \epsilon_0 q\frac{\hat{r}}{r^2}\cdot\left[-\lambda e^{-\lambda r}\left(1+\lambda r+\frac{\lambda^2}{2}r^2\right) + e^{-\lambda r}(\lambda+\lambda^2 r)\right]\hat{r}$$
$$= -\epsilon_0 q\frac{\lambda^3}{2}e^{-\lambda r}$$

Therefore, $\rho = \epsilon_0 q\left[4\pi\delta^3(\vec{r}) - \frac{\lambda^3}{2}e^{-\lambda r}\right]$.

(b) The electric field at radius s outside the given cylinder is $\frac{\lambda}{2\pi\epsilon_0 s}$, where $\lambda = \pi R^2\rho$. Consider a length l of the cylinder. The energy stored in the external field within this length, out to a radius B, equals,

$$U_{ext} = \frac{\epsilon_0}{2}\int_V E^2 d\tau = \frac{\epsilon_0}{2}\int_R^B\left(\frac{\pi R^2\rho}{2\pi\epsilon_0 s}\right)^2(2\pi s\,ds)\cdot l$$

$$= \frac{\pi\rho^2R^4 l}{4\epsilon_0}\int_R^B\frac{ds}{s} = \frac{\pi\rho^2R^4 l}{4\epsilon_0}\ln\left(\frac{B}{R}\right)$$

The field at radius s inside the cylinder is due to only the charge inside radius s. This charge has line charge density $\lambda = \pi s^2 \rho$. So the electric field will be $\frac{\lambda}{2\pi\epsilon_0 s} = \frac{\rho s}{2\epsilon_0}$.

The energy stored in the electric field, within a length l, is then

$$U_{int} = \frac{\epsilon_0}{2} \int_0^R \left(\frac{\rho s}{2\epsilon_0}\right)^2 (2\pi s\,ds).l = \frac{\pi \rho^2 l}{4\epsilon_0} \int_0^R s^3 ds$$

$$= \frac{\pi \rho^2 R^4 l}{16\epsilon_0}.$$

So, total energy,

$$U = \frac{\pi \rho^2 R^4 l}{4\epsilon_0} \left[\ln\left(\frac{B}{R}\right) + \frac{1}{4} \right]$$

And, energy per unit length, $u = \frac{\lambda^2}{4\pi\epsilon_0} \left[\ln\left(\frac{B}{R}\right) + \frac{1}{4} \right]$.

<u>NOTE</u>: The energy diverges as $B \to \infty$, and $R \to 0$.

<u>(c)</u> We ignore the z component.

Along path (i) $\int_{(0,0)}^{(x_1,y_1)} \vec{E}.d\vec{l} = \int_0^{x_1} E_x(x,0)\,dx + \int_0^{y_1} E_y(x_1,y)\,dy$

$$= 0 + \int_0^{y_1} (3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3$$

Along path (ii) $\int_{(0,0)}^{(x_1,y_1)} \vec{E}.d\vec{l} = \int_0^{y_1} E_y(0,y)\,dy + \int_0^{x_1} E_x(x,y_1)\,dx$

$$= \int_0^{y_1} (0 - 3y^2)\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1$$

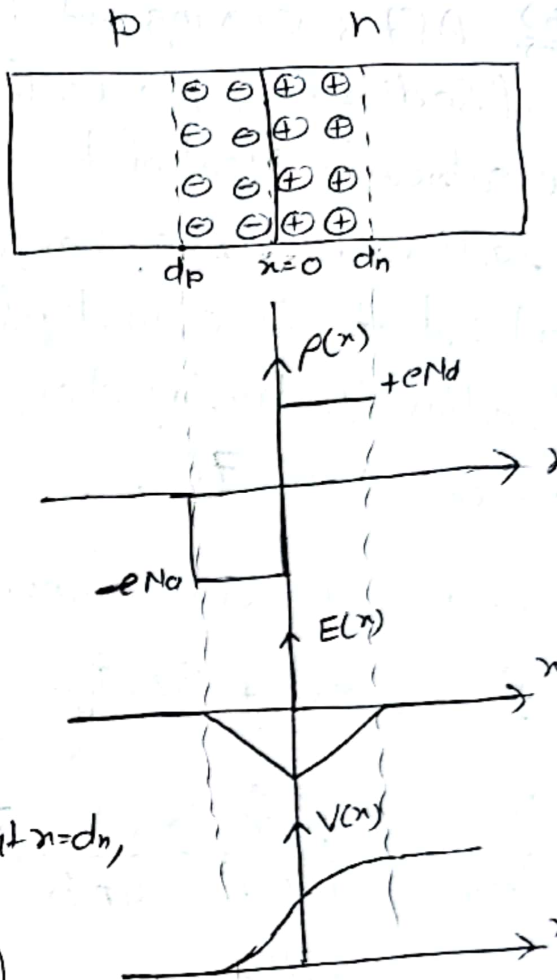
So, both are equal.

The electric potential, if taken to be zero at (0,0), is just the negative of the result because $\vec{E} = -\vec{\nabla}\phi$, or $V = -\int \vec{E}.d\vec{l}$

So, $V(x,y) = y^3 - 3x^2 y$

Q-3(a) The charge density varies in only x-direction,

So, $$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0} \qquad (\epsilon = \epsilon_0 \epsilon_r)$$



$$\Rightarrow \frac{d^2V}{dx^2} = +\frac{eN_a}{\epsilon_0} \qquad -d_p < x < 0$$
$$= -\frac{eN_d}{\epsilon_0} \qquad 0 < x < d_n$$

Integrating above eqn.

$$\frac{dV}{dx} = \frac{eN_a}{\epsilon_0}x + C_1 \qquad -d_p < x < 0$$
$$= -\frac{eN_d}{\epsilon_0}x + C_2 \qquad 0 < x < d_n$$

Using B.C. that $E\left(=-\frac{dV}{dx}\right) = 0$ at $x = -d_p$ & at $x = d_n$,

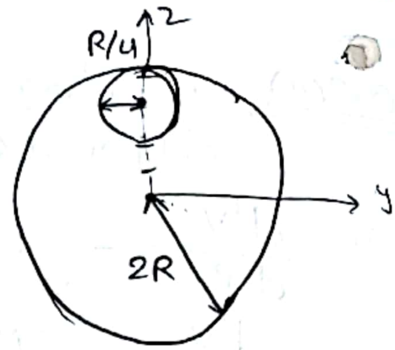
$$\Rightarrow \frac{dV}{dx} = \frac{eN_a}{\epsilon}x + \frac{eN_a d_p}{\epsilon} \qquad -d_p < x < 0$$
$$= -\frac{eN_d}{\epsilon}x + \frac{eN_d d_n}{\epsilon} \qquad 0 < x < d_n$$
$$= -E(x)$$

Integrating again & using B.C. that potential is continuous at $x=0$.

$$\Rightarrow V = \frac{eN_a}{2\epsilon}(x+d_p)^2 \qquad -d_p < x < 0$$
$$= -\frac{eN_d}{2\epsilon}(x-d_n)^2 + \frac{e}{2\epsilon}(N_a d_p^2 + N_d d_n^2) \qquad 0 < x < d_n$$

Potential in the depletion region is the potential difference between the points $x = -d_p$ and $x = d_n$,

$$V_{(x=d_n)} - V_{(x=-d_p)} = \frac{e}{2\epsilon}\left[N_a d_p^2 + N_d d_n^2\right]$$



(b) After carving out the smaller sphere of radius $R/4$, the configuration is electrically identical to the origin sphere of radius $2R$ having charge $q_1 \left(= \frac{4}{3}\pi (2R)^3 \rho\right)$ centered at the origin plus another sphere of radius $R/4$ having charge $q_2 \left(= -\frac{4}{3}\pi \left(\frac{R}{4}\right)^3 \rho\right)$ centered at $z = \frac{7R}{4}$.

Therefore the dipole moment of the system is,

$$\vec{p} = q_1 \times 0 + q_2 \cdot \frac{7R}{4}\hat{z} = \frac{7q_2 R}{4}\hat{z}$$

So,

$$V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0}\frac{7q_2 R}{4r^2}\cdot\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{7R\cos\theta}{4r^2}\cdot\frac{4}{3}\pi\frac{R^3}{64}\cdot\rho$$

$$= \frac{7R^4\rho\cos\theta}{768 \cdot \epsilon_0 \cdot r^2}$$

If Someone used R/2 then,

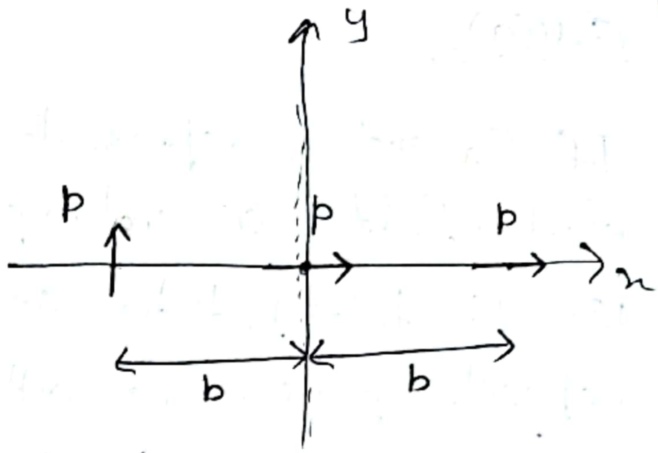
$$q_2 = -\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\rho.$$

$$\vec{p} = q_1 \cdot 0 + q_2 \cdot \frac{3R}{2}\hat{z}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{3q_2 R}{2r^2}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{3R\cos\theta}{2r^2}\cdot\frac{4}{3}\pi\frac{R^3}{8}\cdot\rho$$

$$= \frac{R^4\rho\cos\theta}{16\epsilon_0 r^2}$$

(c) Let the middle dipole be located at the origin. So, the downward field at the origin due to the left dipole is slightly stronger at the (negative) left end of the middle dipole than at its (positive) right end.



The left end, therefore, feels a larger force upward than the right end feels downward. So, there will be a net upward force due to the field of the left dipole.

Similar reasoning shows that there will be a net rightward force due to the field of the right dipole. So, the net force on the middle dipole is upward & rightword.

Let's see this.

~~component of~~ force on a dipole, $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

Let's calculate the y ~~field~~ force due to the field from left dipole,

$$E_y = \frac{-p}{4\pi\epsilon_0 (b+x)^3} \quad , \quad \left.\frac{\partial E_y}{\partial x}\right|_{x=0} = \frac{3p}{4\pi\epsilon_0 b^4}$$

$$\text{So,} \quad F_y = (\vec{p}\cdot\vec{\nabla})E_y = p_x(\nabla E_y)_x = p\cdot\frac{3p}{4\pi\epsilon_0 b^4} = \frac{3p^2}{4\pi\epsilon_0 b^4}$$

Now, x force due to the field from right dipole,

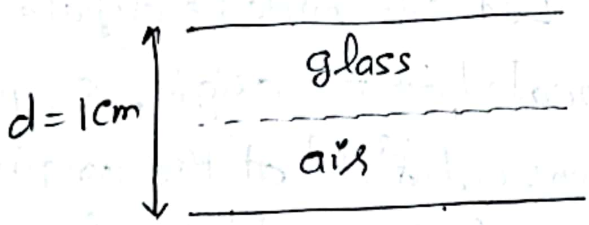
$$E_x = \frac{2p}{4\pi\epsilon_0 (b-x)^3} \quad , \quad \left.\frac{\partial E_x}{\partial x}\right|_{x=0} = \frac{6p}{4\pi\epsilon_0 b^4}$$

$$\text{So,} \quad F_x = (\vec{p}\cdot\vec{\nabla})E_x = p_x(\nabla E_x)_x = \frac{6p^2}{4\pi\epsilon_0 b^4}$$

$$\Rightarrow \frac{F_y}{F_x} = 1/2 = \tan\theta \Rightarrow \theta \simeq 27^\circ$$

$$\& \quad F_{net} = \frac{3\sqrt{5}\,p^2}{4\pi\epsilon_0 b^4}$$

Q.4(a)



If E_a and E_g represents the electric field in air and in glass, then the total potential on the capacitor plate can be written as;

$$V = E_a \frac{d}{2} + E_g \frac{d}{2}$$

, but $E_g = \frac{E_a}{\epsilon_r}$

where ϵ_r = dielectric constant of glass.

$$= E_a \frac{d}{2} + \frac{E_a}{\epsilon_r} \frac{d}{2}$$

$$\Rightarrow V = \frac{(\epsilon_r + 1)}{\epsilon_r} E_a \cdot \frac{d}{2}$$

Given that breakdown in air will occur when $E_a = 3\,V/\mu m$.

which gives

$$V = \frac{(\epsilon_r+1)}{\epsilon_r} \cdot 3\times10^{6} \times \frac{10^{-2}}{2} = \left(15 + \frac{15}{\epsilon_r}\right) kV.$$

$\left[(15+\ldots)kV\right.$

For this applied field, the electric field in the glass will be,

$$E_g = \frac{E_a}{\epsilon_r} = \frac{3\times10^{6}}{\epsilon_r} V/m$$

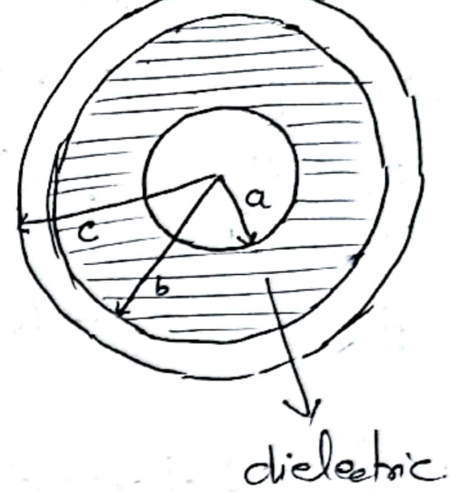
and this will be always less than the breakdown field in the glass i.e 30×10^{6} V/m.

So, the maximum permissible potential on the capacitor should be slightly less than $\left(15 + \frac{15}{\epsilon_r}\right)$ kV.

(b) Given that $K = \alpha r$.

so, $\epsilon = \epsilon_0 K = \epsilon_0 \alpha r$

Displacement vector can be obtained using Gauss's law:



$$\oint \vec{D} \cdot d\vec{a} = Q_{f,enc} \Rightarrow \vec{D} = \frac{Q_{f,enc}}{4\pi r^2} \hat{r}$$

Since regions $r<a$ and $b<r<c$, are inside conductors, so

$$\vec{D} = 0, \vec{E} = 0 \quad \text{for } r<a \text{ \& } b<r<c$$

For the region, $a<r<b$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} \quad , \quad \vec{E} = \frac{Q}{4\pi \epsilon_0 \alpha r^3} \hat{r}$$

For the region, $r>c$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} \quad , \quad \vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}$$

So, the total energy associated with the electric field confined in the system,

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau$$

$$= \frac{1}{2} \left[\int_a^b \frac{Q}{4\pi r^2} \cdot \frac{Q}{4\pi \epsilon_0 \alpha r^3} \cdot 4\pi r^2 dr + \int_c^\infty \frac{Q}{4\pi r^2} \cdot \frac{Q}{4\pi \epsilon_0 r^2} 4\pi r^2 dr \right]$$

$$= \frac{Q^2}{8\pi \epsilon_0 \alpha} \left[\int_a^b \frac{dr}{r^3} + \alpha \int_c^\infty \frac{dr}{r^2} \right]$$

$$W = \frac{Q^2}{8\pi \epsilon_0 \alpha} \left[\frac{b^2 - a^2}{2a^2 b^2} + \frac{\alpha}{c} \right].$$

(c) similar to the last problem,

$\vec{E} = \vec{D} = 0$ for $r<a$ and $b<r<c$.

For the region, $a<r<b$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \quad , \quad \vec{E} = \frac{Q}{4\pi\epsilon_0\alpha r^3}\hat{r}$$

(i) $$\vec{P} = \vec{D} - \epsilon_0\vec{E} = (\epsilon-\epsilon_0)\vec{E} = \frac{(\alpha r - 1)Q}{4\pi\alpha r^3}\hat{r}$$

volume bound charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$$\Rightarrow \rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \cdot \frac{(\alpha r-1)Q}{4\pi\alpha r^3}\right] = -\frac{Q}{4\pi\alpha r^4}$$

surface bound charge density:

$$\sigma_b(r=a) = \vec{P}\cdot\hat{n}|_{r=a} = \frac{(1-\alpha a)Q}{4\pi\alpha a^3}$$

$$\sigma_b(r=b) = \vec{P}\cdot\hat{n}|_{r=b} = \frac{(\alpha b-1)Q}{4\pi\alpha b^3}$$

(ii) Potential difference,

$$V = -\int_b^a \vec{E}\cdot d\vec{r} = -\int_b^a \frac{Q\,dr}{4\pi\epsilon_0\alpha r^3}$$

$$\Rightarrow V = \frac{Q(b^2-a^2)}{8\pi\epsilon_0\alpha a^2 b^2}.$$