MTH 101-Calculus
Spring-2021

1. (a) Let (a_n) be a Cauchy sequence with $a_n \geq a > 0$ for all n. Using the definition of Cauchy sequence decide whether the sequence $(\frac{1}{a_n^2})$ is a Cauchy sequence or not.

[5]

Solution:
Since (a_n) is Cauchy, it is bounded. So there exists $M > 0$ such that $|a_n| < M$ for all n.

[1]

$|\frac{1}{a_n^2} - \frac{1}{a_k^2}| = \frac{|a_n - a_k||a_n + a_k|}{|a_n^2 a_k^2|} \leq 2Ma^{-4}|a_n - a_k|$.

[2]

Now, given, any $\epsilon > 0$, as (a_n) is Cauchy, there is an m, so that if $m < n, k$, then $|a_n - a_k| < \frac{1}{2}M^{-1}a^4\epsilon$. Then $|\frac{1}{a_n^2} - \frac{1}{a_k^2}| < \epsilon$. Hence $(\frac{1}{a_n^2})$ is a Cauchy sequence.

[2]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = x^6 + 2e^{-x} + 8(x+1)^2 - 13\cos x$, for all $x \in \mathbb{R}$. Find the number of real roots of $g(x)$.

[6]

Solution.
Since $g''(x) = 30x^4 + 2e^{-x} + 16 + 13\cos x > 0$ for all x, by Rolle's theorem g' has at most one root and so $g(x) = 0$ has at most two real roots, again by Rolle's theorem.

[2]

Observe that $f(0) < 0, f(2) > 0$ and $f(-2) > 0$.

[2]

By IVP, $f(x) = 0$ has at least two real roots. Therefore $f(x) = 0$ has exactly two real roots.

[2]

(c) Find all values of $x \in \mathbb{R}$ for which the following series converge:

$$\sum_{n=1}^{\infty} \frac{(2x-3)^{2n+1}}{n^{5/2}}$$

[6]

Solution
For each x, comparing with series $\sum u_n$, we apply ratio test of series:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-3)^{2n+3}}{(n+1)^{5/2}} \frac{n^{5/2}}{(2x-3)^{2n+1}}\right| = (2x-3)^2.$$

[2]

Thus by Ratio test, the given series is convergent for $|2x - 3| < 1$ and divergent for $|2x - 3| > 1$. This the series is convergent for $x \in (1, 2)$ and divergent for $x > 2$ or $x < 1$.

[2]

For the endpoint $x = 2$ the series becomes $\sum \frac{1}{n^{5/2}}$, which is convergent.

[1]

For the endpoint $x = 1$ the series becomes $\sum -\frac{1}{n^{5/2}}$, which is also convergent.

[1]

Hence the series converges if and only if $x \in [1, 2]$.

2. (a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function with $f(x) \geq 0$ for all $x \in [a,b]$. Show that

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}} = \sup\{f(x) : x \in [a,b]\}.$$

[10]

Solution
Let $M = \sup\{f(x) : x \in [a,b]\}$. Then

$$\left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}} \leq \left(\int_a^b M^n dx\right)^{\frac{1}{n}} = M(b-a)^{1/n}.$$

[1]

Taking limit as $n \to \infty$

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}} \leq M.$$

[1]

Since f is continuous on a closed interval, there exists x_0 such that $f(x_0) = M$. For any $\epsilon > 0$, there exists $\delta > 0$ such that
$x_0 - \delta < x < x_0 + \delta \implies M - \epsilon < f(x) < M + \epsilon.$ [2]
Since $f \leq 0$, we have,

$$(M-\epsilon)(2\delta)^{1/n} \leq (\int\limits_{x_0-\delta}^{x_0+\delta} f^n dx)^{1/n} \leq (\int\limits_a^b f^n dx)^{1/n}$$

[2]

Taking limit as $n \to \infty$, we have $M - \epsilon \leq \lim\limits_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}}$. [2]

This is true for any $\epsilon > 0$. Hence $M \leq \lim\limits_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}}$.

[2]

(b) Let $f : (-1,1) \to \mathbb{R}$ be a differentiable function and $f'(x)$ is continuous. If $f(0) = 0$ and $f'(x) \geq 2|x|$ for all $x \in (-1,1)$ then show that $|f(x)| \geq x^2$ for all $x \in (-1,1)$.

[7]

Solution: Since $f(0) = 0$ the statement holds for $x = 0$.

[1]

We treat this in two cases: $x \in (0,1)$ and $x \in (-1,0)$.
Let $x \in (0,1)$. Since f is differentiable and $f'(x)$ is continuous, by fundamental theorem of calculus we have
$|f(x)| \geq f(x) = f(x) - f(0) = \int_0^x f'(t)dt \geq \int_0^x 2t dt = x^2$

[3]

Let $x \in (-1,0)$. By FTC again
$|f(x)| \geq -f(x) = f(0) - f(x) = \int_x^0 f'(t)dt \geq \int_x^0 -2t dt = x^2$.

[3]

3. (a) A farmer wants to build a rectangular box without a top, with a volume of 500 cubic meters. Find the dimension of the box such that the amount of material required is minimum.

[10]

Solution: Let x, y, z be the dimension of the box. Then the problem can be formulated as , minimizing the function

$$s(x, y, z) = xy + 2z(x + y),$$

constrained to

$$xyz = 500 \quad (A)$$

where x, y, z denotes the length, breath and height of the box respectively.

[2]

Applying Lagrange multiplier method, we get For some $\lambda \in \mathbb{R}$,

$$y + 2z = \lambda yz \quad (B)$$
$$x + 2z = \lambda xz \quad (C)$$
$$2x + 2y = \lambda xy \quad (D)$$

[3]

Using (C),(B) to get either $x = y$ or $\lambda z = 1$
For ruling out $\lambda z = 1$ as then from (B) $z = 0$.
Using (D) to get, $\lambda x = \lambda y = 4$. Using (A) -to get $4z = 125\lambda^2$
To get $\lambda = \frac{2}{5}$

[2].

To get the only critical point $(10, 10, 5)$.

[3]

(b) Consider the function

$$f(x, y) = (y - 4x^2)(y - x^2).$$

Show that $(0, 0)$ is a critical point of f and find out whether it is a local maximum or local minimum or a saddle point for the function. [6]

Solution: For calculating $\nabla f(0, 0) = (0, 0)$.

[2]

For choosing the sequence $x_\epsilon = (\epsilon, 0) \to (0, 0)$ (or any other sequence) , such that

$$f(x_\epsilon) = 4\epsilon^4 > f(0, 0) = 0.$$

[2]

For choosing the sequence $y_\epsilon = (\epsilon, 3\epsilon^2) \to (0, 0)$ (or any other sequence) , such that

$$f(y_\epsilon) = -2\epsilon^4 < f(0, 0) = 0.$$

[2]

4. (a) Let γ be the line segment joining from $(0,0,0)$ to $(1,1,1)$. Compute the following line integral:

$$\int_{\gamma} 3x^2yzdx + x^3zdy + x^3ydz.$$

[4]

Solution : Notice if $g(x,y,z) = x^3yz$, then

$$\nabla g = (3x^2yz, x^3z, x^3y).$$

[2]

Using second fundamental theorem for line integral,

$$\int_{\gamma} 3x^2yzdx + x^3zdy + x^3ydz = g(1,1,1) - g(0,0,0) = 1.$$

[2].

Alternative Solution : Parametrising the line by

$$r(t) = (t,t,t), \ t \in (0,1).$$

[2]

By using directly the definition,

$$\int_{\gamma} 3x^2yzdx + x^3zdy + x^3ydz = \int_0^1 5t^4dt = 1.$$

[2]

(b) Apply Stokes theorem to evaluate the following integral

$$\oint_C x^2y^3dx + dy + zdz,$$

where C is the closed curve parameterized by $(2\cos t, 2\sin t, \sqrt{12})$, $\quad 0 \le t \le 2\pi$. [12]

Solution: For writing a surface S, whose boundary is C. For example, one can take

$$S = \{(x,y,z) \mid x^2 + y^2 \le 2, z = \sqrt{12}\}.$$

[2]

To ensure that all conditions of Stoke's theorem are satisfied: that is ,

$$F(x,y,z) = (x^2y^3, 1, z)$$

has continious Partial Derivatives. [1]

Calculating Curl$F \cdot N = -3x^2y^2$, where $N = (0,0,1)$ is the *correct* normal to the surface. [2]

Applying the Stoke's theorem,

$$I = \int_C x^2y^3dx + dy + zdz = -\int_S 3x^2y^2d\sigma$$

[2]

For changing to double integral by using definition,

$$I = \int_{\{x^2+y^2\leq 1\}} -3x^2y^2dxdy.$$

[2]

For using polar coordianates (or Fubini)

$$I = -3\int_0^{2\pi}\int_0^2 r^5 \sin^2 t \cos^2 tdt$$

[2]

For getting the final answer that $I = -8\pi$.

[1].

5. (a) Let T be the region in $\mathbb{R}^2$ bounded by the curves $y = 4 - x^2$, $x = 0$ and $y = 0$. Draw the region T by indicating the coordinates of intersecting points of the three curves above, taken two at a time. Evaluate the following double integral:

$$\iint_T \frac{xe^{2y}}{4-y}\, dxdy.$$

[7]

Solution: Draw the picture please. [1-marks]

For writing the following line, after using Fubini

$$\iint_T \frac{xe^{2y}}{4-y}\, dxdy = \int_0^4 \left(\int_{x=0}^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx \right) dy.$$

[4-mark]

The above 4-mark is subdivided in to 2+2, for putting the 2-limits correctly.

$$\iint_T \frac{xe^{2y}}{4-y}\, dxdy = \frac{1}{2}\int_0^4 e^{2y} dy.$$

[1-mark]

To get the final answer correctly $\frac{1}{4}(e^8 - 1)$. [1-mark].

Remark:

Please note that if a student uses Fubini by fixing points in x-axis first, i.e.

$$\iint_T \frac{xe^{2y}}{4-y}\, dxdy = \int_0^2 \left(\int_{y=0}^{4-x^2} \frac{xe^{2y}}{4-y} dy \right) dx.$$

[4-mark]

They will not be proceed after that.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y\sqrt{|y|}}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0,0)$? Do the directional derivatives of f at $(0,0)$ exist along any direction? Is f differentiable at $(0,0)$? Justfy your answers.

[10]

Solution: Let $\epsilon > 0$. Then $|f(x,y) - f(0,0)| = |\frac{x^2y\sqrt{|y|}}{x^4+y^2}| \leq |x^2y\sqrt{|y|}|\frac{1}{2x^2|y|} = \frac{\sqrt{|y|}}{2}$ as by $AM \geq GM$ we have $\frac{x^4+y^2}{2} \geq x^2|y|$.

Again we know that $|y| \leq \sqrt{x^2+y^2}$ and so $\frac{\sqrt{|y|}}{2} \leq \frac{||(x,y)||^{1/4}}{2}$. Heence $\frac{\sqrt{|y|}}{2} < \epsilon$ if $||(x,y) - (0,0)|| \leq \delta$ where $\delta = \epsilon^4/16$. So f is continuous at $(0,0)$.

[4]

Let $U = (u_1, u_2) \in \mathbb{R}^2$ such that $||U|| = 1$.

Then $\lim\limits_{t\to 0} \frac{f((0,0)+t(u_1,u_2))-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{u_1^2u_2\sqrt{|tu_2|}}{t^2u_1^4+u_2^2} = 0$ if $u_2 \neq 0$. For $u_2 = 0, u_1 = 1$ we have the directional derivative $f_x(0,0) = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$. So all the directional derivatives exists at the origin and are equal to zero.

[3]

So $\nabla f(0,0) = (0,0)$.

For differentiability, we look at

$$E(x) = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0).(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2 y\sqrt{|y|}}{(x^4+y^2)\sqrt{x^2+y^2}}$$

We will show that this function does not have a limit at $(0,0)$. For that we take $y = mx^2$. Then $E(x) \to \frac{m\sqrt{|m|}}{(1+m^2)}$ as $x \to 0$ for any m.

[3]

6. (a) Determine the convergence/ divergence of the improper integral

$$\int_1^{10} \frac{t^3 \sin t}{e^t \ln t} dt.$$

[6]

Solution

Limit comparison test with $\frac{1}{t-1}$ we get

[2]

$$\lim_{t \to 1} \frac{t^3 \sin t}{e^t \ln t}.(t-1) = (\sin 1)/e > 0.$$

But $\int_1^{10} \frac{1}{t-1} dt$ is divergent. Hence the given integral is divergent.

[4]

(b) Consider the region bounded by the curves $y = \sqrt{16 - x^2}$, $y = \sqrt{9 - x^2}$ and $y = 0$. Using Pappus theorem, find the volume of the solid obtained by revolving the above region around the line $y = -2$.

[5]

Solution: ρ is the distance of the centroid of the surface from x-axis.

$$\rho = \frac{148}{21\pi}$$

[1 marks]

By applying second Pappus theorem, we get , by revolving the surface arounf x-axis,

$$\frac{4}{3}\pi(4^3 - 3^3) = \pi^2(4^2 - 3^2)\rho,$$

[2-mark]

Therefore the required volume is

$$V = 2\pi(\rho + 2)7\frac{\pi}{2}$$

[2-mark]

(c) Prove that

$$\left|\sin\left(||X||^2\right) - \sin\left(||Y||^2\right)\right| \leq 2||X - Y|| \text{ for all } X, Y \in \mathbb{R}^2 \text{ such that } ||X||, ||Y|| \leq 1.$$

[6]

Solution Proving the following inequality

$$|\sin(a) - \sin(b)| \leq |a - b|$$

[2-marks]

Then using it with $a = ||X||^2, b = ||Y||^2$ to get

$$|\sin(||X||^2) - \sin(||Y||^2)| \leq \left|||X||^2 - ||Y||^2\right| = |||X|| - ||Y||| \, (|||X|| + ||Y|||)$$

[1-mark]

Now using triangle inequality again to get

$$|\sin(||X||^2) - \sin(||Y||^2)| \leq ||X - Y||(|||X|| + ||Y|||)$$

[1-mark]

Then finally using the condition on X, Y to get the result. [1-mark]

Alternative Solution:
Applying MVT for function of several variable on the function $f(X) = \sin(||X||^2)$ to get

$$f(X) - f(Y) = \sin(||X||^2) - \sin(||Y||^2) = \nabla f(\xi) \cdot (X - Y)$$

where ξ is a point on line joining X and Y. [2-mark]

For calculating

$$|\nabla f(\xi)| = 2\cos(||\xi||^2)||\xi||$$

[1-mark]

Using this and Cauchy Scwartz to get

$$|\sin(||X||^2) - \sin(||Y||^2)| \leq 2||\xi|||X - Y|||$$

[2 mark]

Finally noticing that $||\xi|| \leq 1$ as it lies on the line joining X and Y , where $||X||, ||Y|| \leq 1$.
[1-mark]