Differential Calculus and Sage

(revised edition)

William Anthony Granville,
with extra material added by David Joyner
and updates by Charles Ross

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For further information on William granville, please see the Wikipedia article,[1] which has a short biography and links for further information. Granville's book "Elements of the Differential and Integral Calculus" fell into the public domain and then much of it (but not all, at the time of this writing) was scanned[2] primarily by P. J. Hall. This wikisource document uses MathML and LaTeX and some Greek letter fonts. The current LaTeX document is due to David Joyner, who is responsible for the formatting, editing for readability, the correction of any typos in the scanned version, and any extra material added (for example, the hyperlinked cross references, and the Sage material). Please email corrections to `wdjoyner@gmail.com`. In particular, the existence of this document owes itself primarily to three great open source projects: TeX/LaTeX Wikipedia, and Sage. More information on Sage can be found at the Sage website[3]. Some material from Sean Mauch's public domain text on Applied Mathematics,[4] was also included.

Acknowledgements: I thank the following readers for reporting typos: Mario Pernici, Jacob Hicks.

DAVID JOYNER
2007

[1] `http://en.wikipedia.org/wiki/William_Anthony_Granville`
[2] `http://en.wikisource.org/wiki/Elements_of_the_Differential_and_Integral_Calculus`
[3] `http://www.sagemath.org`
[4] `http://www.its.caltech.edu/~sean/book.html`
[5] `http://www.gnu.org/copyleft/fdl.html`

Chapter 0

Preface

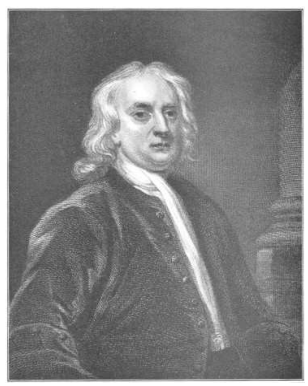

Figure 1: Sir Isaac Newton.

That teachers and students of the Calculus have shown such a generous appreciation of Granville's "Elements of the Differential and Integral Calculus" has been very gratifying to the author. In the last few years considerable progress has been made in the teaching of the elements of the Calculus, and in this revised edition of Granville's "Calculus" the latest and best methods are exhibited,—methods that have stood the test of actual classroom work. Those features of the first edition which contributed so much to its usefulness and popularity have been retained. The introductory matter has been cut down somewhat in order to get down to the real business of the Calculus sooner. As this is designed essentially for a drill book, the pedagogic principle that each result should be made intuitionally as well as analytically evident to the student has been kept constantly in mind. The object is not to teach the student to rely on his intuition, but, in some cases, to use this faculty in advance of analytical investigation. Graphical illustration has been drawn on very liberally.

This Calculus is based on the method of limits and is divided into two main parts,—Differential Calculus and Integral Calculus. As special features, attention may be called to the effort to make perfectly clear the nature and extent of each new theorem, the large number of carefully graded exercises, and the summarizing into working rules of the methods of solving problems. In the Integral Calculus the notion of integration over a plane area has been much enlarged upon, and integration as the limit of a summation is constantly emphasized. The existence of the limit e has been assumed and its approximate value calculated from its graph. A large number of new examples have been added, both with and without answers. At the end of almost every chapter will be found a collection of miscellaneous examples. Among the new topics added are approximate integration, trapezoidal rule, parabolic rule, orthogonal trajectories, centers of area and volume, pressure of liquids, work done, etc. Simple practical problems have been added throughout; problems that illustrate the theory and at the same time are of interest to the student. These problems do not presuppose an extended knowledge in any particular branch of science, but are based on knowledge that all students of the Calculus are supposed to have in common.

The author has tried to write a textbook that is thoroughly modern and teachable, and the capacity and needs of the student pursuing a first course in the Calculus have been kept constantly in mind. The book contains more material than is necessary for the usual course of one hundred lessons given in our colleges and engineering schools; but this gives teachers an opportunity to choose such subjects as best suit the needs of their classes. It is believed that the volume contains all topics from which a selection naturally would be made in preparing students either for elementary work in applied science or for more advanced work in pure mathematics.

WILLIAM A. GRANVILLE
Gettysburg, Pa.

Figure 2: Gottfried Wilhelm Leibnitz.

Chapter 1

Collection of Formulas

1.1 Formulas for Reference

For the convenience of the student we give the following list of elementary formulas from Algebra, Geometry, Trigonometry, and Analytic Geometry.

1. Binomial Theorem (n being a postive integer):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1} + \cdots$$

2. $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)\,n$.

3. In the quadratic equation $ax^2 + bx + c = 0$,

when $b^2 - 4ac > 0$, the roots are real and unequal;
when $b^2 - 4ac = 0$, the roots are real and equal;
when $b^2 - 4ac < 0$, the roots are imaginary.

4. When a quadratic equation is reduced to the form $x^2 + px + q = 0$,

$p =$sum of roots with sign changed, and
$q =$product of roots.

5. In an arithmetical series, a, $a+d$, $a+2d$, $\ldots$,

$$s = \sum_{i=0}^{n-1} a + id = \frac{n}{2}\left[2a + (n-1)\,d\right].$$

6. In a geometrical series, a, ar, ar^2, ...,

$$s = \sum_{i=0}^{n-1} ar^i = \frac{a\left(r^n - 1\right)}{r - 1}.$$

7. $\log ab = \log a + \log b$.

8. $\log \frac{a}{b} = \log a - \log b$.

9. $\log a^n = n \log a$.

10. $\log \sqrt[n]{a} = \frac{1}{n} \log a$.

11. $\log 1 = 0$.

12. $\log e = 1$.

13. $\log \frac{1}{a} = -\log a$.

14. [1] Circumference of circle $= 2\pi r$.

15. Area of circle $= \pi r^2$.

16. Volume of prism $= Ba$.

17. Volume of pyramid $= \frac{1}{3}Ba$.

18. Volume of right circular cylinder $= \pi r^2 a$.

19. Lateral surface of right circular cylinder $= 2\pi ra$.

20. Total surface of right circular cylinder $= 2\pi r\left(r + a\right)$.

21. Volume of right circular cone $= 2\pi r\left(r + a\right)$.

22. Lateral surface of right circular cone $= \pi rs$.

23. Total surface of right circular cone $= \pi r\left(r + s\right)$.

24. Volume of sphere $= \frac{4}{3}\pi r^3$.

25. Surface of sphere $= 4\pi r^2$.

26. $\sin x = \dfrac{1}{\csc x}$, $\cos x = \dfrac{1}{\sec x}$, $\tan x = \dfrac{1}{\cot x}$.

27. $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$,

28. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

29. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$; $\cos x = \sin\left(\frac{\pi}{2} - x\right)$; $\tan x = \cot\left(\frac{\pi}{2} - x\right)$.

[1] In formulas 14-25, r denotes radius, a altitude, B area of base, and s slant height.

30. $\sin(\pi - x) = \sin x$; $\cos(\pi - x) = -\cos x$; $\tan(\pi - x) = -\tan x$.

31. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

32. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

33. $\cos(x \pm y) = \cos x \cos y + \mp \sin x \sin y$.

34. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

35. $\tan(x - y) = \dfrac{\tan x \tan y}{1 + \tan x \tan y}$.

36. $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

37. $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$; $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$; $\tan x = \dfrac{2 \tan \frac{1}{2} x}{1 - \tan^2 \frac{1}{2} x}$.

38. $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$; $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$.

39. $1 + \cos x = 2 \cos^2 \frac{x}{2}$; $1 - \cos x = 2 \sin^2 \frac{x}{2}$.

40. $\sin \frac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$; $\cos \frac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$; $\tan \frac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$.

41. $\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$.

42. $\sin x - \sin y = 2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$.

43. $\cos x + \cos y = -2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$.

44. $\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$.

45. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$; Law of Sines.

46. $a^2 = b^2 + c^2 - 2bc \cos A$; Law of Cosines.

47. $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; distance between points (x_1, y_1) and (x_2, y_2).

48. $d = \dfrac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$; distance from line $Ax + By + C = 0$ to (x_1, y_1).

49. $x = \dfrac{x_1 + x_2}{2}$, $y = \dfrac{y_1 + y_2}{2}$; coördinates of middle point.

50. $x = x_0 + x'$, $y = y_0 + y'$; ransforming to new origin (x_0, y_0).

51. $x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$; transforming to new axes making the angle θ with old.

52. $x = \rho\cos\theta$, $y = \rho\sin\theta$; transforming from rectangular to polar coördinates.

53. $\rho = \sqrt{x^2+y^2}$, $\theta = \arctan\frac{y}{x}$; transforming from polarto rectangular coördinates.

54. Different forms of equation of a straight line:

    (a) $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$, two-point form;

    (b) $\frac{x}{a}+\frac{y}{b}=1$; intercept form;

    (c) $y-y_1 = m(x-x_1)$; slope-point form;

    (d) $y = mx+b$, slope-intercept form;

    (e) $x\cos\alpha + y\sin\alpha = p$, normal form;

    (f) $Ax+By+C=0$, general form.

55. $\tan\theta = \dfrac{m_1-m_2}{1+m_1m_2}$, angle between two lines whose slopes are m_1 and m_2.

$$m_1 = m_2 \text{ when lines are parallel, and}$$
$$m_1 = -\frac{1}{m_2} \text{ when lines are perpendicular.}$$

56. $(x=\alpha)^2 + (y-\beta)^2 = r^2$, equation of circle with center (α,β) and radius r.

Many of these facts are already know to Sage:

Sage

```
sage: a,b = var("a,b")
sage: log(sqrt(a))
log(a)/2
sage: log(a/b).simplify_log()
log(a) - log(b)
sage: sin(a+b).simplify_trig()
cos(a)*sin(b) + sin(a)*cos(b)
sage: cos(a+b).simplify_trig()
cos(a)*cos(b) - sin(a)*sin(b)
sage: (a+b)^5
(b + a)^5
sage: expand((a+b)^5)
b^5 + 5*a*b^4 + 10*a^2*b^3 + 10*a^3*b^2 + 5*a^4*b + a^5
```

"Under the hood" Sage used Maxima to do this simplification.

1.2 Greek Alphabet

Letters	**Names**	**Letters**	**Names**	**Letters**	**Names**
A, α	alpha	I, ι	iota	P, ρ	rho
B, β	beta	K, κ	kappa	Σ, σ	sigma
Γ, γ	gamma	Λ, λ	lambda	T, τ	tau
Δ, δ	delta	M, μ	mu	Υ, υ	upsilon
E, ϵ	epsilon	N, ν	nu	Φ, ϕ	phi
Z, ζ	zeta	Ξ, ξ	xi	X, χ	chi
H, η	eta	O, o	omicron	Ψ, ψ	psi
Θ, θ	theta	Π, π	pi	Ω, ω	omega

1.3 Rules for Sings of the Trigonometric Functions

Quadrant	Sin	Cos	Tan	Cot	Sec	Csc
First	+	+	+	+	+	+
Second	+	-	-	-	-	+
Third	-	-	+	+	-	-
Fourth	-	+	-	-	+	-

1.4 Natural Values of the Trigonometric Functions

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot	Sec	Csc
0	0	0	1	0	∞	1	∞
$\pi/6$	30	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
$\pi/4$	45	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	60	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
$\pi/2$	90	1	0	∞	0	∞	1
π	180	0	-1	0	∞	-1	∞
$3\pi/2$	270	-1	0	∞	0	∞	-1
2π	360	0	1	0	∞	1	∞

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot		
.0000	0	.0000	1.0000	.0000	Inf.	90	1.5708
.0175	1	.0175	.9998	.0175	57.290	89	1.5533
.0349	2	.0349	.9994	.0349	28.636	88	1.5359
.0524	3	.0523	.9986	.0524	19.081	87	1.5184
.0698	4	.0698	.9976	.0699	14.300	86	1.5010
.0873	5	.0872	.9962	.0875	11.430	85	1.4835
.1745	10	.1736	.9848	.1763	5.671	80	1.3963
.2618	15	.2588	.9659	.2679	3.732	75	1.3090
.3491	20	.3420	.9397	.3640	2.747	70	1.2217
.4863	25	.4226	.9063	.4663	2.145	65	1.1345
.5236	30	.5000	.8660	.5774	1.732	60	1.0472
.6109	35	.5736	.8192	.7002	1.428	55	.9599
.6981	40	.6428	.7660	.8391	1.192	50	.8727
.7854	45	.7071	.7071	1.0000	1.000	45	.7854
		Cos	Sin	Cot	Tan	Angle in Degrees	Angle in Radians

You can create a table like this in Sage:

Sage

```
sage: RR15 = RealField(15)
sage: rads1 = [n*0.0175 for n in range(1,6)]
sage: rads2 = [0.0875+n*0.0875 for n in range(1,9)]
sage: rads = rads1+rads2
sage: trigs = ["sin", "cos", "tan", "cot"]
sage: tbl = [[eval(x+"(%s)"%y) for x in trigs] for y in rads]
sage: tbl = [[RR15(eval(x+"(%s)"%y)) for x in trigs] for y in rads]
sage: print Matrix(tbl)
[0.01750  0.9998 0.01750   57.14]
[0.03499  0.9994 0.03502   28.56]
[0.05247  0.9986 0.05255   19.03]
[0.06994  0.9976 0.07011   14.26]
[0.08739  0.9962 0.08772   11.40]
[ 0.1741  0.9847  0.1768   5.656]
[ 0.2595  0.9658  0.2687   3.722]
[ 0.3429  0.9394  0.3650   2.740]
[ 0.4237  0.9058  0.4677   2.138]
[ 0.5012  0.8653  0.5792   1.726]
[ 0.5749  0.8182  0.7026   1.423]
[ 0.6442  0.7648  0.8423   1.187]
[ 0.7086  0.7056   1.004  0.9958]
```

The first column are the values of $\sin(x)$ at $x \in \{0.01750, 0.03500, ...0.7875\}$ (measured in radians). The second, third and fourth rows are the corresponding values for cos, tan and cot, resp.

1.5 Logarithms of Numbers and Trigonometric Functions

The common logarithm is the logarithm with base 10. The fractional part of the logarithm of x is known as the *mantissa* of the common logarithm of x. For example, if $x = 120$, then

$$\log_{10} 120 = \log_{10} \left(10^2 \times 1.2\right) = 2 + \log_{10} 1.2 \approx 2 + 0.079181,$$

so the very last number $(0.079181\ldots)$ is the mantissa. In the table below, this is given simply as 0792.

Table of mantissas of the common logarithms of numbers:

No.	0	1	2	3	4	5	6	7	8	9
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	07f9	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989

The table of logarithms of the trigonometric functions (given in Granville's original text) is omitted as they can be computed using a hand calculator or Sage.

Chapter 2

Variables and Functions

2.1 Variables and Constants

A *variable* is a quantity to which an unlimited number of values can be assigned. Variables are denoted by the later letters of the alphabet. Thus, in the equation of a straight line,

$$\frac{x}{a} + \frac{y}{b} = 1$$

x and y may be considered as the variable coördinates of a point moving along the line. A quantity whose value remains unchanged is called a *constant*.

Numerical or absolute constants retain the same values in all problems, as 2, 5, $\sqrt{7}$, π, etc.

Arbitrary constants, or *parameters*, are constants to which any one of an unlimited set of numerical values may be assigned, and they are supposed to have these assigned values throughout the investigation. They are usually denoted by the earlier letters of the alphabet. Thus, for every pair of values arbitrarily assigned to a and b, the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

represents some particular straight line.

2.2 Interval of a Variable

Very often we confine ourselves to a portion only of the number system. For example, we may restrict our variable so that it shall take on only such values as lie between a and b, where a and b may be included, or either or both excluded. We shall employ the symbol $[a,\ b]$, a being less than b, to represent the numbers $a,\ b$, and all the numbers between them, unless otherwise stated. This symbol $[a,\ b]$ is read the interval from a to b.

2.3 Continuous Variation

A variable x is said to vary continuously through an interval $[a, b]$, when x starts with the value a and increases until it takes on the value b in such a manner as to assume the value of every number between a and b in the order of their magnitudes. This may be illustrated geometrically as follows:

Figure 2.1: Interval from A to B

The origin being at O, layoff on the straight line the points A and B corresponding to the numbers a and b. Also let the point P correspond to a particular value of the variable x. Evidently the interval $[a, b]$ is represented by the segment AB. Now as x varies continuously from a to b inclusive, i.e. through the interval $[a, b]$, the point P generates the segment AB.

2.4 Functions

When two variables are so related that the value of the first variable depends on the value of the second variable, then the first variable is said to be a *function* of the second variable.

Nearly all scientific problems deal with quantities and relations of this sort, and in the experiences of everyday life we are continually meeting conditions illustrating the dependence of one quantity on another. For instance, the weight a man is able to lift depends on his strength, other things being equal. Similarly, the distance a boy can run may be considered as depending on the time. Or, we may say that the area of a square is a function of the length of a side, and the volume of a sphere is a function of its diameter.

Sage

```
sage: I1 = interval(1,3)
sage: I2 = interval(2,6)
sage: I3 = interval(min(I2),max(I1)) # the intersection
sage: P1 = plot(0, xmin=min(I1), xmax = max(I1), thickness=10, rgbcolor=(1,0,0),linestyle="--")
sage: P2 = plot(0, xmin=min(I2), xmax = max(I2), thickness=10, rgbcolor=(0,1,0),linestyle=":")
sage: P3 = plot(0, xmin=min(I3), xmax = max(I3), thickness=10, rgbcolor=(1,1,0))
sage: show(P1+P2+P3)
```

2.5 Independent and Dependent Variables

The second variable, to which values may be assigned at pleasure within limits depending on the particular problem, is called the *independent variable*, or *argument*; and the first variable, whose value is determined as soon as the value of the independent variable is fixed, is called the *dependent variable*, or *function*.

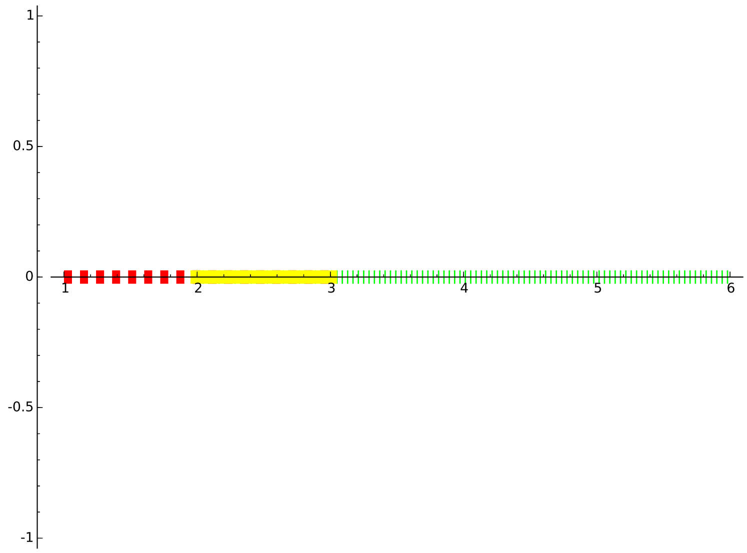

Figure 2.2: Interval $[1, 3]$ is dashed, $[2, 5]$ is dotted, $[1, 3][\in, \bigtriangledown] = [\in, \ni]$ is solid.

Frequently, when we are considering two related variables, it is in our power to fix upon whichever we please as the independent variable; but having once made the choice, no change of independent variable is allowed without certain precautions and transformations.

One quantity (the dependent variable) may be a function of two or more other quantities (the independent variables, or arguments). For example, the cost of cloth is a function of both the quality and quantity; the area of a triangle is a function of the base and altitude; the volume of a rectangular parallelepiped is a function of its three dimensions.

In the Sage example below, t is the independent variable and f is the dependent variable. Though this example uses differentiation (Sage's `f.diff` notation), you don't need to understand how to differentiate to appreciate the syntax Sage uses to express the command.

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: f(t).diff(t)
diff(f(t), t, 1)
sage: (2*f(t)+3).diff(t)
2*diff(f(t), t, 1)
sage: (f(t)^2).diff(t)
2*f(t)*diff(f(t), t, 1)
sage: f = cos
sage: f(pi/2)
0
sage: f(t).diff(t)
-sin(t)
```

2.6 Notation of Functions

The symbol $f(x)$ is used to denote a function of x, and is read "f of x". In order to distinguish between different functions, the prefixed letter is changed, as $F(x)$, $\phi(x)$, $f'(x)$, etc.

During any investigation the same functional symbol always indicates the same law of dependence of the function upon the variable. In the simpler cases this law takes the form of a series of analytical operations upon that variable. Hence, in such a case, the same functional symbol will indicate the same operations or series of operations, even though applied to different quantities. Thus, if

$$f(x) = x^2 - 9x + 14,$$

then

$$f(y) = y^2 - 9y + 14.$$

Also

$$f(a) = a^2 - 9a + 14,$$

$$f(b+1) = (b+1)^2 - 9(b+1) + 14 = b^2 - 7b + 6,$$

$$f(0) = 0^2 - 9 \cdot 0 + 14 = 14,$$

$$f(-1) = (-1)^2 - 9(-1) + 14 = 24,$$

$$f(3) = 3^2 - 9 \cdot 3 + 14 = -4,$$

$$f(7) = 7^2 - 9 \cdot 7 + 14 = 0,$$

etc. Similarly, $\phi(x,\ y)$ denotes a function of x and y, and is read "ϕ of x and y". If

$$\phi(x,\ y) = \sin(x+y),$$

then

$$\phi(a,\ b) = \sin(a+b),$$

and

$$\phi\left(\frac{\pi}{2}, 0\right) = \sin\frac{\pi}{2} = 1.$$

Again, if

$$F(x,\ y,\ z) = 2x + 3y - 12z,$$

then

$$F(m,\ -m,\ m) = 2m - 3m - 12m = -13m.$$

and

$$F(3,\ 2,\ 1) = 2 \cdot 3 + 3 \cdot 2 - 12 \cdot 1 = 0.$$

Evidently this system of notation may be extended indefinitely.

You can define a function in Sage in several ways:

Sage

```
sage: x,y = var("x,y")
sage: f = log(sqrt(x))
sage: f(4)
log(4)/2
sage: f(4).simplify_log()
log(2)
sage: f = lambda x: (x^2+1)/2
sage: f(x)
(x^2 + 1)/2
sage: f(1)
1
sage: f = lambda x,y: x^2+y^2
sage: f(3,4)
25
sage: R.<x> = PolynomialRing(CC,"x")
sage: f = x^2+2
sage: f.roots()
[(1.41421356237309*I, 1), (2.77555756156289e-17 - 1.41421356237309*I, 1)]
```

2.7 Values of the Independent Variable for which a Function is Defined

Consider the functions

$$x^2 - 2x + 5, \qquad \sin x, \qquad \arctan x$$

of the independent variable x. Denoting the dependent variable in each case by y, we may write

$$y = x^2 - 2x + 5, y = \sin x,\ \ y = \arctan x.$$

In each case y (the value of the function) is known, or, as we say, defined, for all values of x. This is not by any means true of all functions, as the following examples illustrating the more common exceptions will show.

$$y = \frac{a}{x-b} \tag{2.1}$$

Here the value of y (i.e. the function) is defined for all values of x except $x = b$. When $x = b$ the divisor becomes zero and the value of y cannot be computed from (2.1). Any value might be assigned to the function for this value of the argument.

$$y = \sqrt{x}. \tag{2.2}$$

In this case the function is defined only for positive values of x. Negative values of x give imaginary values for y, and these must be excluded here, where we are confining ourselves to real numbers only.

$$y = \log_a x. \qquad a > 0 \tag{2.3}$$

Here y is defined only for positive values of x. For negative values of x this function does not exist (see **??**).

$$y = \arcsin x,\ y = \arccos x. \tag{2.4}$$

Since sines, and cosines cannot become greater than $+1$ nor less than -1, it follows that the above functions are defined for all values of x ranging from -1 to $+1$ inclusive, but for no other values.

Sage
```
sage: t = var("t’’)
sage: f = function(’f’, t)
sage: g = function(’g’, t)
sage: f = sin
sage: g = asin
sage: f(g(t))
t
sage: g(f(t))
t
sage: g(f(0.2))
0.200000000000000
```

2.8 Exercises

1. Given $f(x) = x^3 - 10x^2 + 31x - 30$; show that

$$\begin{gathered} f(0) = -30, \quad f(y) = y^3 - 10y^2 + 31y - 30, \\ f(2) = 0, \quad f(a) = a^3 - 10a^2 + 31a - 30, \\ f(3) = f(5), \quad f(yz) = y^3z^3 - 10y^2z^2 + 31yz - 30, \\ f(1) > f(-3), \quad f(x-2) = x^3 - 16x^2 + 83x - 140, \\ f(-1) = 6f(6). \end{gathered}$$

2. If $f(x) = x^3 - 3x + 2$, find $f(0)$, $f(1)$, $f(-1)$, $f\left(-\frac{1}{2}\right)$, $f\left(\frac{4}{3}\right)$.

3. If $f(x) = x^3 - 10x^2 + 31x - 30$, and $\phi(x) = x^4 - 55x^2 - 210x - 216$, show that $f(2) = \phi(-2)$, $f(3) = \phi(-3)$, $f(5) = \phi(-4)$, $f(0) + \phi(0) + 246 = 0$.

4. If $F(x) = 2x$, find $F(0)$, $F(-3)$, $F\left(\frac{1}{3}\right)$, $F(-1)$.

5. Given $F(x) = x(x-1)(x+6)\left(x - \frac{1}{2}\right)\left(x + \frac{5}{4}\right)$, show that $F(0) = F(1) = F(-6) = F\left(\frac{1}{2}\right) = F\left(-\frac{5}{4}\right) = 0$.

6. If $f(m_1) = \frac{m_1 - 1}{m_1 + 1}$, show that $\frac{f(m_1) - f(m_2)}{1 + f(m_1)f(m_2)} = \frac{m_1 - m_2}{1 + m_1 m_2}$.

7. If $\phi(x) = a^x$, show that $\phi(y) \cdot \phi(z) = \phi(y+z)$.

8. Given $\phi(x) = \log \frac{1-x}{1+x}$, show that $\phi(x) + \phi(y) = \phi\left(\frac{x+y}{1+xy}\right)$.

9. If $f(\phi) = \cos\phi$, show that $f(\phi) = f(-\phi) = -f(\pi - \phi) = -f(\pi + \phi)$.

10. If $F(\theta) = \tan\theta$, show that $F(2\theta) = \frac{2F(\theta)}{1 - [F(\theta)]^2}$.

11. Given $\psi(x) = x^{2n} + x^{2m} + 1$, show that $\psi(1) = 3$, $\psi(0) = 1$, and $\psi(a) = \psi(-a)$.

12. If $f(x) = \frac{2x-3}{x+7}$, find $f\left(\sqrt{2}\right)$.

Here's how to verify the double angle identity for tan in Exercise 10 above:

Sage

```
sage: theta = var("theta")
sage: tan(2*theta).expand_trig()
2*tan(theta)/(1 - tan(theta)^2)
```

Chapter 3

Theory of Limits

Chapter 4

Differentiation

Chapter 5

Rules for Differentiating Standard Elementary Forms

5.1 Importance of General Rule

5.2 Differentiation of a Constant

5.3 Differentiation of a Variable with Respect to Itself

5.4 Differentiation of a Sum

5.5 Differentiation of the Product of a Constant and a Function

5.6 Differentiation of the Product of Two Functions

5.7 Differentiation of the Product of Any Finite Number of Functions

5.8 Differentiation of a Function with a Constant Exponent

5.9 Differentiation of a Quotient

5.10 Examples

5.11 Differentiation of a Function of a Function

5.12 Differentiation of Inverse Functions

5.13 Differentiation of a Logarithm

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Simple Applications of the Derivative

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Successive Differentiation

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7.2 Notation

7.3 The n-th Derivative

7.4 Leibnitz's Formula for the n-th Derivative of a Product

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Maxima, Minima and Inflection Points

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Curvature; Radius of Curvature

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Theorem of Mean Value; Indeterminant Forms

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13.8 Evaluation of a Function Taking on an Indeterminate Form

13.9 Evaluation of the Indeterminate Form $\frac{0}{0}$

13.9.1 Rule for Evaluating the Indeterminate Form $\frac{0}{0}$

13.9.2 Exercises

13.10 Evaluation of the Indeterminate Form $\frac{\infty}{\infty}$

13.11 Evaluation of the Indeterminate Form $0 \cdot \infty$

13.12 Evaluation of the Indeterminate Form $\infty - \infty$

13.12.1 Exercises

Chapter 14

Circle of Curvature; Center of Curvature

Chapter 15

References

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