Linear Time Invariant and Causal (LTIC) Systems

Digital Signal Processing

Notes

Contents

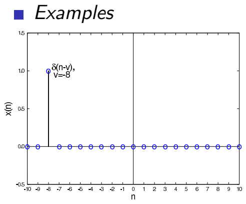
Notes

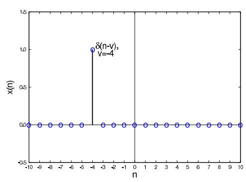
Unit Impulse Function

- Unit impulse function is a fundamental function in Digital Signal Processing (DSP)
- Symbol of Unit impulse function is the *Greek* delta: δ
- $\delta(n) = 1$ if $n = 0$, so that,

$$\delta(n-v) = \begin{cases} 1 & \text{if} \quad (n-v) = 0, \\ 0 & \quad \text{otherwise.} \end{cases}$$

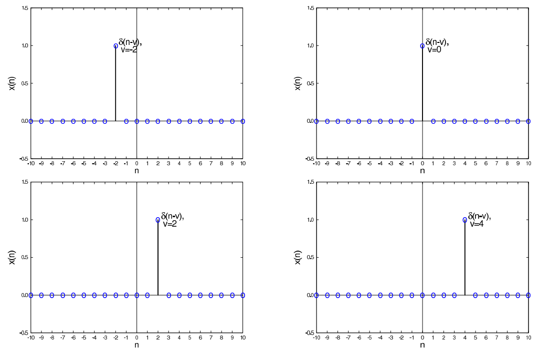
- *Examples*





Notes

Unit Impulse Function *cont'd.*



Notes

Scaling Unit Impulse Function

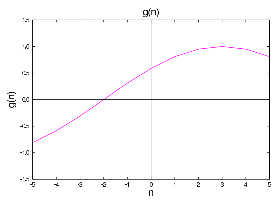
- Can scale unit implulse with any value, *i.e.*

$$g \times \delta(n-v) = \begin{cases} g & \text{if} \quad n-v=0, \\ 0 & \text{otherwise.} \end{cases}$$

- So if g is a function, such as $g(n)$ then

$$g(n)\delta(n-v) = \begin{cases} g(n) & \text{if} \quad n-v=0, \\ 0 & \text{otherwise.} \end{cases}$$

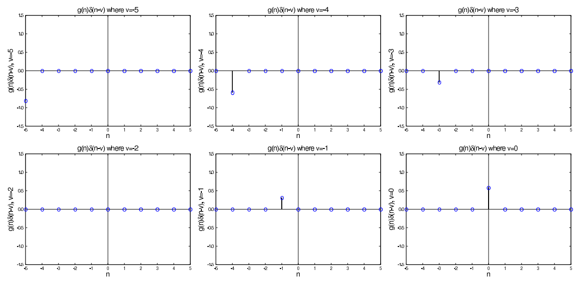
- This is useful for something called *sifting*
- Given a signal $g(n)$:



Notes

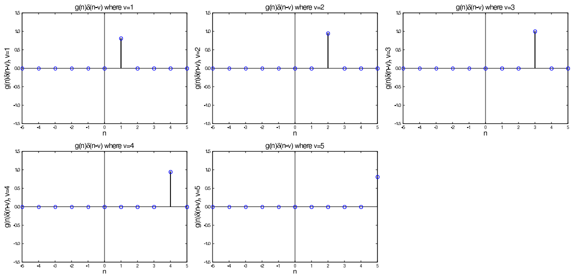
Sifting

- Calculate $g(n)\delta(n-v)$ for all values of v, *i.e.*



Notes

Sifting *cont'd.*

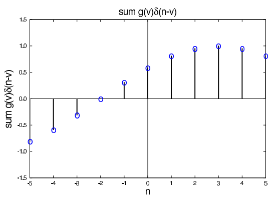


- We can now add all these together...

Notes

Sifting *cont'd.*

- Adding all the delta values together we get



- which is a discrete (*sifted*) representation of the original signal, $g(n)$.
- This process can be represented by

$$x[n] = ... + g(-5)\delta(n+5) + g(-4)\delta(n+4) + ... + g(4)\delta(n-4) + g(5)\delta(n-5) + ...$$

- where $[\cdot]$ signifies a discrete formulation. This can be shortened to $x[n] = \sum_{k=-\infty}^{\infty} g(k)\delta(n-k)$. For our case $x[n] = \sum_{k=-5}^{5} g(k)\delta(n-k)$.

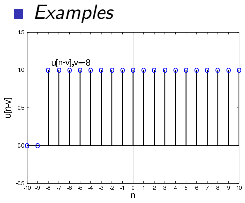
Notes

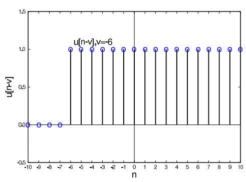
Unit Step Function

- The unit step function:

$$u[n-v] = \begin{cases} 1 & \text{if} \quad n-v \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

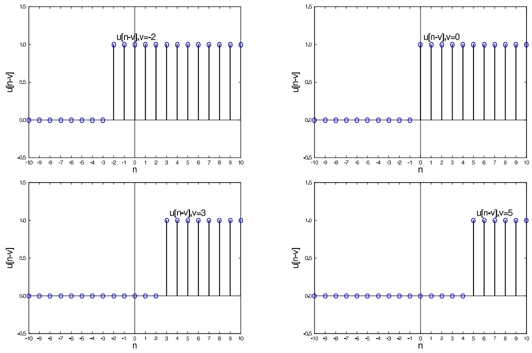
- switches from zero to unit value.
- *Examples*





Notes

Unit Step Function *cont'd.*



Notes

Unit Step Function

- It can be defined using the unit impulse function ($\delta[n-v]$):

$$u[n-v] = \sum_{m=v}^{\infty} \delta[n-m]$$

- Also

$$\delta[n-v] = u[n-v] - u[n-1-v].$$

- These are known as *recurrence* formula, where the current signal value is dependent on previous signal values:

  "to recur"

  Meaning: to repeat.

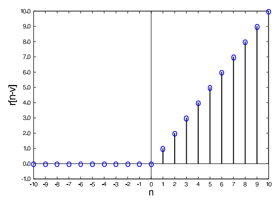
Notes

Ramp Function

- Another interesting function type is the ramp function.
- Given by

$$r[n-v] = (n-v)u[n].$$

Example



Notes

Digital Sinusoidal Functions

- Digital sine wave:

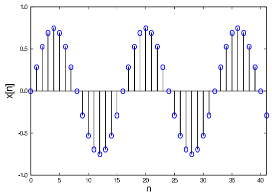
$$x[n] = a\sin(n\Omega + \theta)$$

- Digital cosine wave:

$$x[n] = a\cos(n\Omega + \theta)$$

- Ω is the *digital "frequency"* measured in *radians*
- 1 cycle every N samples. Also $\Omega = 2\pi/N$ so that $N = 2\pi/\Omega$

- *Example* $a = 0.75$, $\theta = 0$ and $\Omega = \pi/8$, therefore $N = 2 \times 8 = 16$ *i.e.* $x[n] = 0.75\sin(n\pi/8)$:



Notes

Comparison with Analog Sine Function

- Compare to a continuous analog sine wave:

$$x(t) = a\sin(t\omega + \theta)$$

where t could be time in seconds and $\omega = 2\pi f$ is the angular frequency, therefore in *radians per second*.

- The interval between each sample n is T_s seconds, so there is a sample at every $t = nT_s$ seconds
- The continuous sine wave can then be written as

$$x(n) = a\sin(nT_s 2\pi f + \theta)$$

- If we equate the continuous and digital versions, then

$$x[n] = x(n)$$

$$a\sin(n\Omega + \theta) = a\sin(nT_s 2\pi f + \theta)$$

- Therefore $\Omega = T_s 2\pi f$ *or* if sampling frequency is $f_s = 1/T_s$ then $\Omega = 2\pi f/f_s$.

Notes

Linear Time Invariant Systems

$$\text{Input} \longrightarrow \boxed{\text{Linear Time Invariant System}} \longrightarrow \text{Output}$$

- **Time Invariance**:
  - The same response to the same input at any time.

  If $q[n - v_1] = q[n - v_2] = x[n]$ for constants v_1 and v_2 then

$$F(q[n - v_1]) = F(q[n - v_2])$$

- **Linear System**:
  - *Principle of Superposition*:
    - If the input consists of a sum of signals then the output is the sum of the responses to those signals, *e.g.*

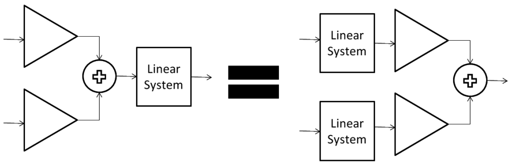
If the output of a system is $y_1[n]$ and $y_2[n]$ in response to two different inputs $x_1[n]$ and $x_2[n]$ respectively then the output of the same system for the two inputs weighted and combined *i.e.* $ax_1[n] + bx_2[n]$ will be $ay_1[n] + by_2[n]$ where a and b are constants.

Notes

Linear Time Invariant Systems

For a linear system $y[n] = F(x[n])$

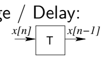
$$F(ax_1[n] + bx_2[n]) = aF(x_1[n]) + bF(x_2[n])$$



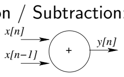
Notes

Linear Time Invariant Systems

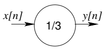
- A Linear Time Invariant (LTI) system consists of:
  - Storage / Delay:



  - Addition / Subtraction: *e.g.* $y[n] = x[n] + x[n-1]$



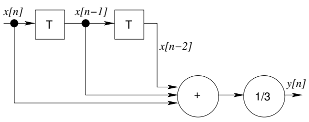
  - Multiplication by Constants: *e.g.* $y[n] = \frac{1}{3}x[n]$



Notes

Simple LTI System Example

- *Example*
  Moving average filter, $y[n] = \frac{x[n]+x[n-1]+x[n-2]}{3}$



Notes

Other System Properties

Notes

An LTI system is

- Associative, where a system can be broken down into simpler subsystems for analysis or synthesis
- Commutative, where if a system is composed of a series of subsystems then the subsystems can be arranged in any order

LTI systems may also have

- Causality: output does not depend on future input values
- Stability: output is bounded for a bounded input (see Lecture 04)
- Invertibility: input can be uniquely found from the input (*e.g.* the square of a number is not invertible)
- Memory: output depends on past input values

Examples of Linear Mathematical Operations

Notes

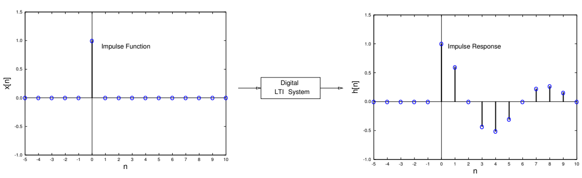
- Scaling (i.e. idealised gain or attenuation)
- Differentiation
- Integration
- The Laplace transform
- The Fourier transform
- The z-transform

Impulse Response

Notes

An LTI system possesses an Impulse Response which characterizes the system's output if an impulse function is applied to the input.

Example Impulse Response



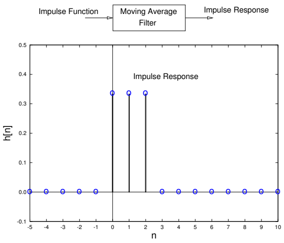
Impulse Response *Example*

Remember the moving average filter:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

If the input is the impulse function: $x[n=0] = \delta(0)$, then $y[n]$ is the output in response to an impulse function, *i.e.* the **impulse response** hence $h[n] = y[n]$...

$$\begin{array}{lll} \ldots & & \\ n=-2: & x[n=-2] = & \delta[-2]=0 \\ n=-1: & x[n=-1] = & \delta[-1]=0 \\ n=0: & x[n=\ \ 0] = & \delta[0]=1 \\ n=+1: & x[n=+1] = & \delta[+1]=0 \\ n=+2: & x[n=+2] = & \delta[+2]=0 \\ \ldots & & \end{array}$$



Notes

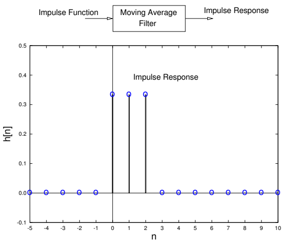
Impulse Response *Example*

Remember the moving average filter:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

If the input is the impulse function: $x[n=0] = \delta(0)$, then $y[n]$ is the output in response to an impulse function, *i.e.* the **impulse response** hence $h[n] = y[n]$...

$$\begin{array}{lll} \ldots & & \\ n=-2: & x[n-1=-3] = & \delta[-3]=0 \\ n=-1: & x[n-1=-2] = & \delta[-2]=0 \\ n=0: & x[n-1=-1] = & \delta[-1]=0 \\ n=+1: & x[n-1=\ \ 0] = & \delta[0]=1 \\ n=+2: & x[n-1=+1] = & \delta[+1]=0 \\ \ldots & & \end{array}$$



Notes

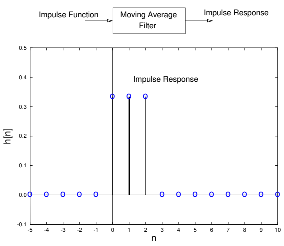
Impulse Response *Example*

Remember the moving average filter:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

If the input is the impulse function: $x[n=0] = \delta(0)$, then $y[n]$ is the output in response to an impulse function, *i.e.* the **impulse response** hence $h[n] = y[n]$...

$$\begin{array}{lll} \ldots & & \\ n=-2: & x[n-2=-4] = & \delta[-4]=0 \\ n=-1: & x[n-2=-3] = & \delta[-3]=0 \\ n=0: & x[n-2=-2] = & \delta[-2]=0 \\ n=+1: & x[n-2=-1] = & \delta[-1]=0 \\ n=+2: & x[n-2=\ \ 0] = & \delta[0]=1 \\ \ldots & & \end{array}$$



Notes

Impulse Response *Example*

Moving Average Filter:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

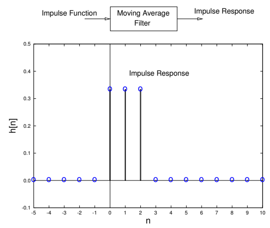
Example, Input Signal = Impulse Function:

$$x[n] = \delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore

- $h[n<0] = y[n<0] = 0$
- $h[0] = y[0] = \frac{1}{3}(\delta[0] + \delta[-1] + \delta[-2]) = \frac{1}{3}$
- $h[1] = y[1] = \frac{1}{3}(\delta[+1] + \delta[0] + \delta[-1]) = \frac{1}{3}$
- $h[2] = y[2] = \frac{1}{3}(\delta[+2] + \delta[+1] + \delta[0]) = \frac{1}{3}$
- $h[n>2] = y[n>2] = 0$

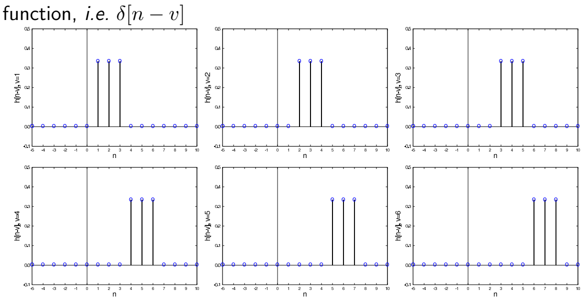
Response $h[n]$ is known as the **Impulse Response**.



Notes

Impulse Response
Examples - shifting

The impulse response can also be determined for a shifted impulse function, *i.e.* $\delta[n-v]$



What will the system output ($y[n]$) be if the input consists of more than one impulse function shifted by different amounts?

Notes

System Response to Multiple Shifted Impulse Responses

What will the system output ($y[n]$) be if the input consists of more than one impulse function shifted by different amounts?
Remember that all LTI systems obey the *"Principle of Superposition"*...
So, for the inputs

$$x_1[n] = a\delta[n] \text{ and } x_2[n] = b\delta[n-1],$$

where a and b are constants, the corresponding outputs will be

$$y_1[n] = ah[n] \text{ and } y_2[n] = bh[n-1],$$

i.e. impulse responses. Therefore if $x[n] = x_1[n] + x_2[n] = a\delta[n] + b\delta[n-1]$ then

$$y[n] = ah[n] + bh[n-1].$$

Notes

Other Signals: Step Function

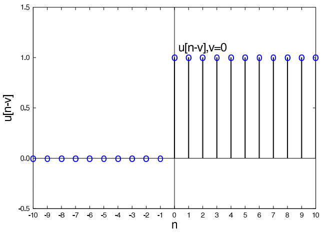
- The discrete step function can be thought of as a series of impulse functions (remember sifting).
- Each impulse function creates an impulse response.
- The output is then the joint response of all the impulse responses scaled by the inputs.

- A discretely sampled step input (starting at $n = 0$) is given by:

$$x[n] = \sum_{k=0}^{\infty} \delta(n-k).$$

- Therefore, using the *Principle of Superposition* we get

$$y[n] = \sum_{k=0}^{\infty} h(n-k).$$



Notes

Step Function Moving Average

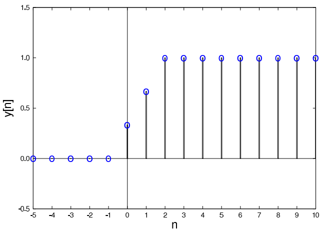
Moving average (with $k = 3$) has an impulse response:

- $h[n < 0] = y[n < 0] = 0$
- $h[0] = y[0] = \frac{1}{3}(\delta[0] + \delta[-1] + \delta[-2]) = \frac{1}{3}$
- $h[1] = y[1] = \frac{1}{3}(\delta[+1] + \delta[0] + \delta[-1]) = \frac{1}{3}$
- $h[2] = y[2] = \frac{1}{3}(\delta[+2] + \delta[+1] + \delta[0]) = \frac{1}{3}$
- $h[n > 2] = y[n > 2] = 0$

Moving average of a step function is then:

$$y[n] = \sum_{k=0}^{\infty} h(n-k)$$

$$= \begin{cases} 0 & \text{if} \quad n \leq 0 \\ 1/3 & \text{if} \quad n = 0 \\ 2/3 & \text{if} \quad n = 1 \\ 1 & \text{if} \quad n \geq 2 \end{cases}$$



Notes

Scaled Impulse Function Inputs

What happens when the step function is given by:

$$u[n-v] = \begin{cases} a & \text{if} \quad n - v \geq 0, \\ 0 & \text{otherwise} \end{cases} ?$$

The discrete impulse function version is

$$x[n] = \sum_{k=0}^{\infty} a\delta[n-k].$$

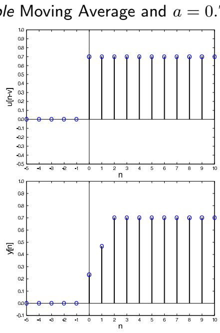
Using the *Principle of Superposition*:

$$y[n] = \sum_{k=0}^{\infty} ah[n-k].$$

Example Moving average filter, $k = 3$

$$y[n] = \begin{cases} 0 & \text{if} \quad n \leq 0 \\ a/3 & \text{if} \quad n = 0 \\ 2a/3 & \text{if} \quad n = 1 \\ a & \text{if} \quad n \geq 2 \end{cases}$$

Example Moving Average and $a = 0.7$



Notes

Digital Convolution

What happens if the scale of the input impulse functions (a) varies with n? *i.e.*

$$x[n] = a[n]\delta[n-k].$$

Using the *Principle of Superposition* we get

$$y[n] = \sum_{k=-\infty}^{\infty} a[k]h[n-k].$$

This is known as the **Convolution Sum**.

Example

$$x[n] = \begin{cases} 0 & \text{if} \quad n < 0 \\ a[0] & \text{if} \quad n = 0 \\ a[1] & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2 \end{cases},$$

which is the same as $x[n] = a[0]\delta[n] + a[1]\delta[n-1]$. Then

$$y[n] = a[0]h[n] + a[1]h[n-1].$$

Notes

Digital Convolution *Example*

Q. Find $y[n]$ if $a[0] = 1$ and $a[1] = 2$ using the impulse response of the moving average filter, $k = 3$.

A.

$$\begin{aligned} y[n] &= a[0]h[n] + a[1]h[n-1] = h[n] + 2h[n-1] \\ y[-1] &= h[-1] + 2h[-2] = 0 + 0 = 0 \\ y[0] &= h[0] + 2h[-1] = 1/3 + 0 = 1/3 \\ y[1] &= h[1] + 2h[0] = 1/3 + 2/3 = 1 \\ y[2] &= h[2] + 2h[1] = 1/3 + 2/3 = 1 \\ y[3] &= h[3] + 2h[2] = 0 + 2/3 = 2/3 \\ y[4] &= h[4] + 2h[3] = 0 + 0 = 0 \end{aligned}$$

Notes

Digital Convolution Trivia

Convolution is often represented by an asterik:

$$y[n] = \sum_{k=-\infty}^{\infty} a[k]h[n-k] = a[n] * h[n]$$

Convolution is commutative:

$$\begin{aligned} y[n] &= a[n] * h[n] = h[n] * a[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]a[n-k]. \end{aligned}$$

Convolution is associative: *cascaded systems*

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

Convolution is distributive: *systems in parallel*

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

Notes

Digital Convolution *Example*

Original signal: (1 cycle every 104 samples)

$$x_1[n] = \sin(n\pi/52)$$

Input signal:

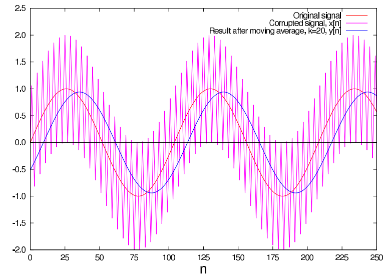
$$x[n] = x_1[n] + x_2[n].$$

Noise signal: (1 cycle every 4 samples)

$$x_2[n] = \sin(n\pi/2).$$

Moving average filter, $k = 20$:

$$y[n] = \frac{1}{20}\sum_{k=0}^{k=19} x[n-k].$$



Notes

Digital Cross-Correlation

Cross-correlation can be used to compare 2 signals.

- If $x_1[n]$ and $x_2[n]$ are two signals then digital cross-correlation is defined:

$$y[n] = \sum_{m=-\infty}^{\infty} x_1^*[m]x_2[n+m]$$

  where $x_1^*[n]$ is the complex conjugate of $x_1[n]$.
- For a real signal $x_1^*[n] = x_1[n]$.
- l is the *lag*.
- If $x_1[n]$ and $x_2[n]$ are the same signal but with a delay between them, then $y[l]$ is at a maximum when l is equal to this delay.

Notes

Digital Cross-Correlation *Example*

Q. Given $x_1 = (0\ 0\ 0.5\ 0.7\ 0)^{\mathrm{T}}$ and $x_2 = (0\ 0.5\ 0.7\ 0\ 0)^{\mathrm{T}}$.
Calculate the cross-correlation for these two real signals.
A. Cross correlation for a real signal is:

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n+m].$$

There are 5 elements in these vectors so (changing the limits):

$$y[n] = \sum_{m=0}^{4} x_1[m]x_2[n+m].$$

We can then calculate the results. Some example calculations:

$$\begin{aligned}
y[l=0] =& \overbrace{x_1[0]\times x_2[0]}^{l=0,m=0} + \overbrace{x_1[1]\times x_2[1]}^{l=0,m=1} + x_1[2]\times x_2[2] + x_1[3]\times x_2[3] + \overbrace{x_1[4]\times x_2[4]}^{l=0,m=4} \\
=& 0\times 0 + 0\times 0.5 + \underline{0.5\times 0.7} + 0.7\times 0 + 0\times 0 = \underline{0.5\times 0.7} = 0.35 \\
y[l=1] =& \overbrace{x_1[0]\times x_2[0+1]}^{l=1,m=0} + \overbrace{x_1[1]\times x_2[1+1]}^{l=1,m=1} + x_1[2]\times x_2[2+1] + \\
& x_1[3]\times x_2[3+1] + \overbrace{x_1[4]\times x_2[4+1]}^{l=1,m=4} \\
=& 0\times 0.5 + 0\times 0.7 + 0\times 0 + 0\times 0 + 0\times 0 = 0
\end{aligned}$$

Notes

Digital Cross-Correlation *cont'd.*

Here are the results for each combination of l and m values:

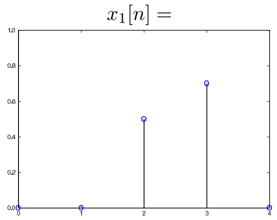
	m					
l	**0**	**1**	**2**	**3**	**4**	$y[l]$
-5	0	0	0	0	0	0
-4	0	0	0	0	0	0
-3	0	0	0	0	0	0
-2	0	0	0	0.35	0	0.35
-1	0	0	0.25	0.49	0	$\underbrace{0.74}_{PEAK}$
0	0	0	0.35	0	0	0.35
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0

- A peak at $l = -1$.
- l is the *lag*, so there is a lag of -1.
- This means x_1 has some similar signal as x_2 but lagged by 1 step.
- We can also see from the signal definitions $x_1 = (0\ 0\ 0.5\ 0.7\ 0)^{\mathrm{T}}$ and $x_2 = (0\ 0.5\ 0.7\ 0\ 0)^{\mathrm{T}}$ that $x_1[n-1] = x_2[n]$.

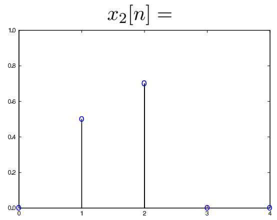
Notes

Digital Cross-Correlation *cont'd.*

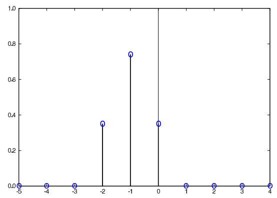
$x_1[n] =$



$x_2[n] =$



The result of the digital cross-correlation, $y[l] =$



Notes

Difference Equations

Difference equations are the name given to the equations that describe the digital signals and systems. For example the equation for the moving average filter with $k = 3$:

$$y[n] = \frac{x[n] + x[n-1] + x[n-2]}{3}$$

is known as a difference equation.
Difference equations for LTI systems can always be put in the form:

$$\sum_{m=0}^{N} a[m]y[n-m] = \sum_{m=0}^{M} b[m]x[n-m].$$

So for our moving average filter:

- $M = 2$ and $N = 0$.
- $a[m]$ and $b[m]$ are known as coefficients.
- For the moving average output y there is only one coefficient, $a[0] = 1$.
- For the moving average input x, there are three coefficients $b[0] = b[1] = b[2] = \frac{1}{3}$.

Notes

Summary

Today we have covered

- Types of digital signal, *e.g.* unit impulse function
- Sifting
- Digital sine and cosine functions
- Linear time invariant (LTI) systems
- Impulse response
- Moving average of a step function
- Digital convolution
- Digital cross-correlation
- Generalized difference equation for LTI systems

Notes

Notes

Notes