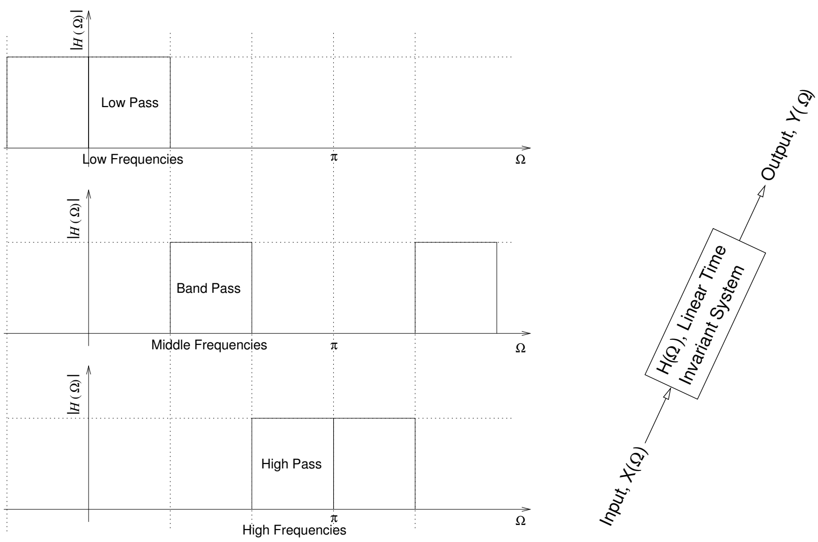
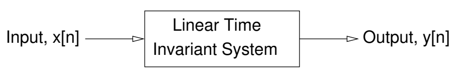
Infinite Impulse Response Filters

Contents

1 Introduction

Often used to remove some frequencies from a signal $X(\Omega)$ and to allow other frequencies to pass through to the output $Y(\Omega)$.





What is a *Recursive digital filter*?

- "*Recursive*" comes from the word "to recur"
  Meaning: to repeat

A recursive filter uses past output values ($y[n-i]$) for the calculation of the current output $y[n]$:

- *Recursive Filter Example*

$$y[n] = 0.5y[n-1] + 0.5x[n].$$

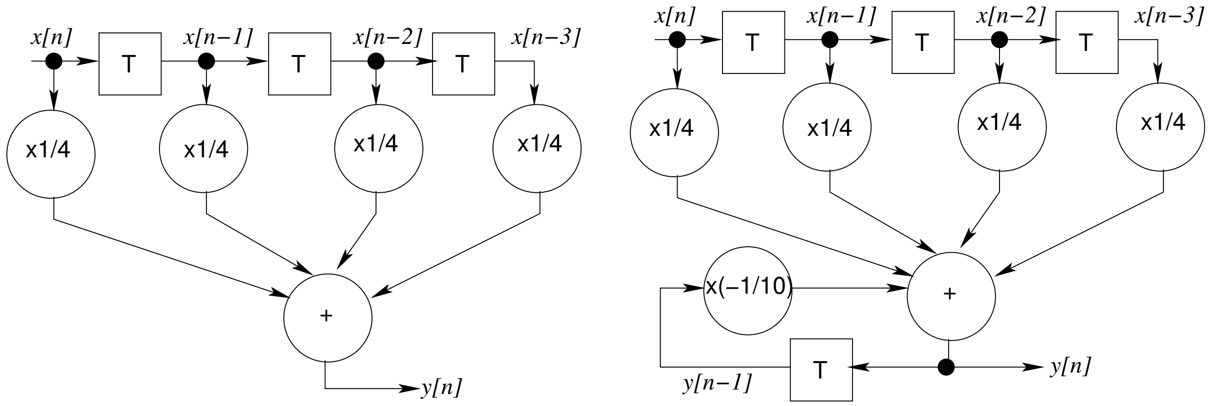


Figure 1: Examples of an FIR filter structure (left) and an IIR filter structure (right).

A non-recursive filter only uses input values $x[n-i]$:

- *Non-recursive Filter Example*

$$y[n] = 0.5x[n-1] + 0.5x[n].$$

Recall the generalised difference equation for causal LTI systems:

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k]$$

If $a[0] = 1$, this can then be changed to:

$$y[n] = \sum_{k=0}^{M} b[k]x[n-k] - \sum_{k=1}^{N} a[k]y[n-k].$$

Also recall the frequency response of such a system can be described by:

$$H(\Omega) = \frac{\sum\limits_{k=0}^{M} b[k]\exp(-jk\Omega)}{\exp(0) + \sum\limits_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{\sum\limits_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum\limits_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

This is the Fourier based frequency representation. There can also be a representation in the z-domain (z-transform):

$$H(z) = \frac{\sum\limits_{k=0}^{M} b[k]z^{-k}}{1 + \sum\limits_{k=1}^{N} a[k]z^{-k}}.$$

Both the z-domain and the Fourier representations describe types of frequency response of the system. The system structure of an IIR filter demonstrates the feedback of the output into the input again.

An example of an FIR filter structure is

$$y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right). \tag{1}$$

Notice there is only a single output term $y[n]$ on the left and no past output terms on the right. An example of an IIR filter structure is

$$y[n] = (x[n] + x[n-1] + x[n-2] + x[n-3]) - \frac{1}{10}y[n-1]. \tag{2}$$

Both examples can be seen illustrated in Fig. 1. In these systems a single z^{-1} is the same as a unit delay, “T” in a system diagram. Recursive digital filters are often known as Infinite Impulse Response (IIR) filters as the impulse response of an IIR filter often has an infinite number of coefficients:

- Require fewer calculations than FIR filters.

Table 1: Comparison of FIR and IIR filter characteristics. *Adapted from "Understanding digital signal processing" by R. G. Lyons.*

Characteristic	IIR	FIR
Multiplications	least	most
Coefficient quantification sensitivity	can be high	very low
Overflow errors	can be high	very low
Stability	by design	always
Linear phase	no	always
Simulate analog filter	yes	no
Coefficient memory	least	most
Design complexity	moderate	simple

- This can mean that they can have a faster response to the input signal,
- and this can also mean that they have a shorter frequency response i.e. *transition width*.

However, IIR filters can become unstable. Therefore there is a need to think carefully about stability when designing IIR Filters.

1.1 Comparison of IIR and FIR filters

A comparative summary of the characteristics of FIR and IIR filters can be seen in Table 1.1.

2 IIR Filter Design

Filter design can follow a number of different methodologies however most, if not all, will require the following steps:

1. Filter specification
2. Coefficient calculations
3. Convert transfer function to suitable filter structure
4. Error analysis
5. Implementation

2.1 Frequency Domain Parameters

An important aspect of filter design are the frequency domain parameters. These can include:

- δ_{p} passband ripple
- δ_{s} stopband ripple
- Ω_{s1} lower stop band edge frequency
- Ω_{p1} lower pass band edge frequency
- Ω_{p2} upper pass band edge frequency
- Ω_{s2} upper stop band edge frequency

The stop band is where there is high attenuation and the pass band is where there is low attenuation (gain$\approx$1). These parameters are illustrated in Fig. 2.

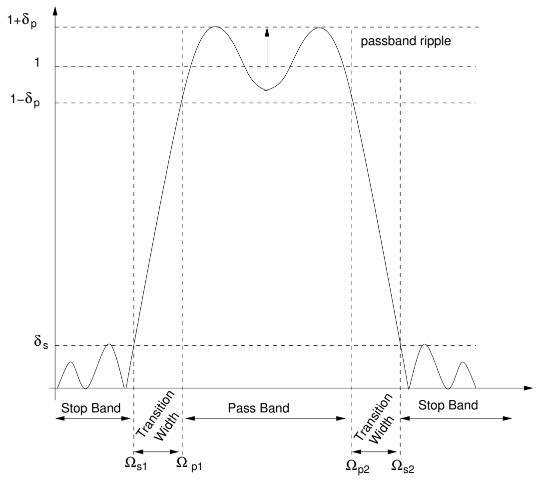


Figure 2: Illustration of some commonly found frequency domain parameters used in filter design.

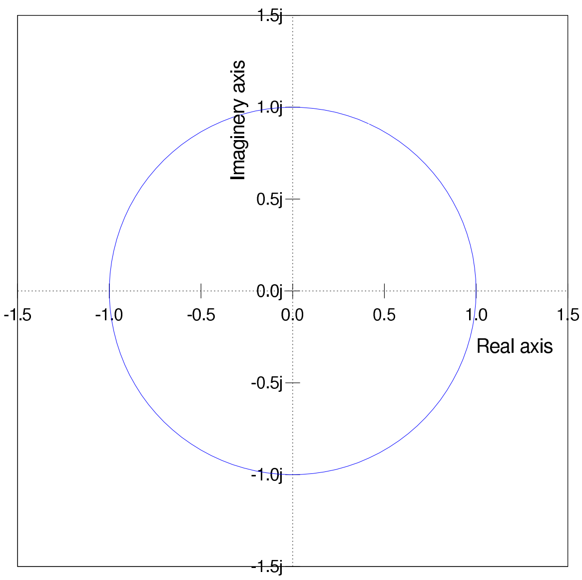


Figure 3: Illustration of the z-plane along with the unit circle.

3 Pole-Zero Placement Method

A filter can be described in the z-plane with poles and zeros. The poles are the roots of the denominator of the transfer function and the zeros are the roots of the numerator, *i.e.*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...} = \frac{\text{zeros}}{\text{poles}}$$

Here, you should be able to observe:

- Poles *located* at: z_1, z_2, z_3, ...
- Zeros *located* at: p_1, p_2, p_3, ...

An example z-plane is illustrated in Fig. 3. It is interesting to observe that:

- Poles (X) close to unit circle *make large peaks*;
- Zeros (O) close to unit circle *make troughs or minima*.

Angle of poles and zeros on z-plane correspond to frequencies that can be used for filter specification.

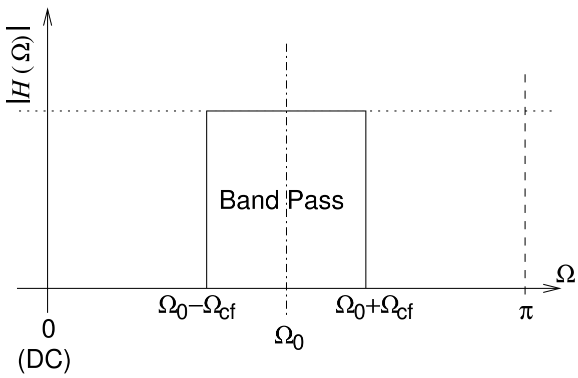


Figure 4: Illustration of a bandpass filter's frequency response.

3.1 Bandpass Filter Pole-Zero Placement Filter Design

A bandpass filter has a frequency response where a band of frequencies are passed but lower and higher frequencies are attenuated. This is illustrated in Fig. fig:bandpassfreqresp. A pole-zero placement design method can utilise the following observations:

- A bandpass filter, with centre frequency Ω_0 radians can have two poles at $\pm\Omega_0$ radians in the z-plane[1].
- Complete attenuation at two frequencies, $\Omega_{\mathrm{r1}} = 0$ radians and $\Omega_{\mathrm{r2}} = \pi$ radians can have two zeros at 0 and π radians.
- The radius of the poles can be calculated with:

  $$r \cong 1 - \Omega_{\mathrm{cf}}$$

  or

  $$r \cong 1 - \frac{\Omega_{\mathrm{bw}}}{2}$$

  where $\Omega_{\mathrm{bw}} = 2\Omega_{\mathrm{cf}}$ is the -3dB bandwidth of the filter.

Example Design a bandpass filter using the Pole-zero placement method with:

- centre frequency at $\Omega_0 = \pi/2$;
- a bandwidth of $\Omega_{\mathrm{bw}} = \pi/8$;
- complete attenuation at $\Omega_{\mathrm{r1}} = 0$ and $\Omega_{\mathrm{r2}} = \pi$;
- and peak unity pass band gain.

Solution Bandpass filter has x2 poles at $\pm\Omega_0 = \pm\pi/2$ radians.

$$\therefore H(z) = K\frac{\mathrm{zeros}}{(z - r\exp(j\pi/2))(z - r\exp(-j\pi/2))}$$

The radii of the poles are given by:

$$r \cong 1 - \frac{\Omega_{\mathrm{bw}}}{2} = 1 - \frac{\pi/8}{2} = 0.80365;$$

and the zeros are at $\Omega_{\mathrm{r1}} = 0$ and $\Omega_{\mathrm{r2}} = \pi$, so that

$$H(z) = K\frac{(z - \exp(j\Omega_{\mathrm{r1}}))(z - \exp(j\Omega_{\mathrm{r2}}))}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

As

- $\exp(\Omega_{\mathrm{r1}}) = \exp(j0) = \cos(0) + j\sin(0) = 1 - j0 = 1$

[1] Complex conjugate pair to make real filter coefficients, when $\Omega_0 \neq 0$ or $\Omega_0 \neq \pi$ radians (on the real line).

- $\exp(\Omega_{\mathrm{r2}}) = \exp(j\pi) = \cos(\pi) + j\sin(\pi) = -1 + j0 = -1,$

then the transfer function becomes:

$$H(z) = K\frac{(z-1)(z+1)}{(z-0.80365\exp(j\pi/2))(z-0.80365\exp(-j\pi/2))}.$$

Using Euler's identity,

- $\exp(j\pi/2) = \cos(\pi/2) + j\sin(\pi/2) = +j$
- and $\exp(j\pi/2) = \cos(\pi/2) - j\sin(\pi/2) = -j,$

so that

$$H(z) = K\frac{(z-1)(z+1)}{(z-0.80365j)(z+0.80365j)}. \tag{3}$$

the pole zero diagram can then be plotted which is shown in Fig. 5

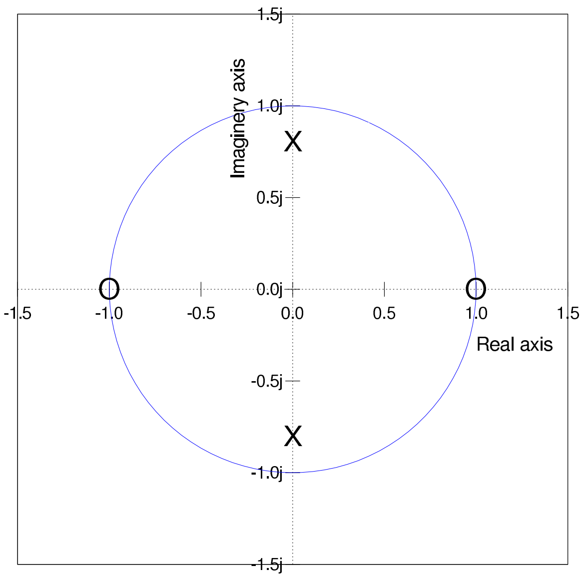


Figure 5: z-plane plot for the poles and zeros specified by the transfer function in (3).

Recall that $H(z) = \frac{Y(z)}{X(z)}$,

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{(z-1)(z+1)}{(z-0.80365j)(z+0.80365j)} = K\frac{z^2-1}{z^2+0.64585}.$$

Then

$$Y(z)(z^2+0.64585) = X(z)K(z^2-1).$$

Remembering that each z^{-1} is a unit delay, so that each z is a unit advance, then the difference equation is:

$$y[n+2] + 0.64585y[n] = K(x[n+2] - x[n])$$

which can be made causal by making $n = n-2$ so that

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]).$$

K is not known, but can be used to make the peak pass band gain to be unity.

The frequency response of the filter can be determined from the difference equation:

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]),$$

in combination with:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

So that (using Euler's identity):

$$H(\Omega) = \frac{K(1 - \cos(2\Omega) + j\sin(2\Omega))}{1 + 0.64585(\cos(2\Omega) - j\sin(2\Omega))}$$

which has magnitude frequency response:

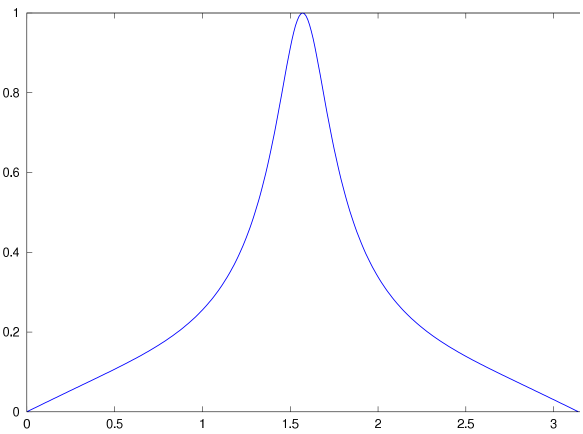


Figure 6: Magnitude Frequency Response

$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1-\cos(2\Omega))^2 + \sin^2(2\Omega))}{(1+0.64585\cos(2\Omega))^2 + \sin^2(2\Omega)}. \tag{4}$$

where $K = 0.17708$.

Relating the digital frequencies for previous example to actual frequencies...
If the sampling frequency is $f_s = 500$Hz, the sampling frequency corresponds to $\Omega = 2\pi$, therefore the filter parameters become:

- centre frequency at $\Omega_0 = \pi/2$, so actual centre frequency $f_0 = \frac{\pi/2}{2\pi} f_s = 125$Hz;
- a bandwidth of $\Omega_{\mathrm{bw}} = \pi/8$, so actual bandwidth $f_{\mathrm{bw}} = 31.25$Hz;
- complete attenuation at $\Omega_{\mathrm{r1}} = 0$ and $\Omega_{\mathrm{r2}} = \pi$, with actual frequencies $f_{\mathrm{r1}} = 0$Hz and $f_{\mathrm{r2}} = \frac{\pi}{2\pi}500 = 250$Hz.

3.2 Bandstop Filter Pole-Zero Placement Filter Design

A bandstop filter, sometimes referred to as a notch filter (usually for narrow bands) remove or significantly attenuate a finite band of frequencies so that low frequencies and high frequencies are not attenuated. An ideal bandstop filter frequency response is illustrated in Fig. 7.

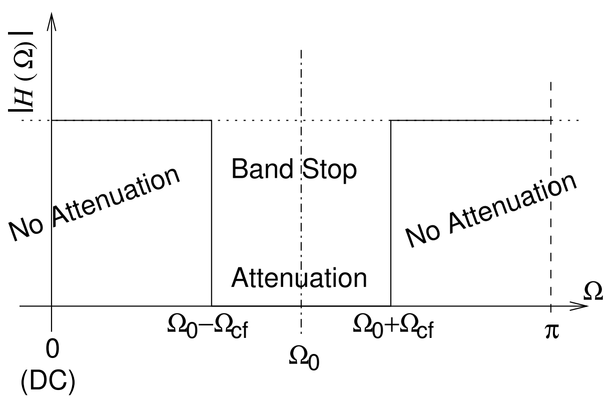


Figure 7: Illustration of an ideal bandstop filter's frequency response.

A bandstop filter can be broadly specified by the frequencies at which the band of frequencies are to be stopped, at the beginning and end.

Example Design a digital bandstop filter using pole-zero placement method with the following parameters:

- Centre frequency, $\Omega_0 = \pi/10$ radians (complete attenuation)
- Band stop width, $\Omega_\mathrm{w} = 2\Omega_\mathrm{cf} = \pi/20$ radians

Solution Complete attenuation at $\Omega_0 = \pi/10$, $\therefore$ x2 zeros (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{\mathrm{poles}}$$

The centre frequency is at $\Omega_0 = \pi/10$ radians, therefore there should be two poles (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$, which gives:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{(z-r\exp(j\pi/10))(z-r\exp(-j\pi/10))}$$

The poles are scaled with radius r to control the width of the band stop,

$$r \cong 1-\frac{\Omega_\mathrm{w}}{2} = 1-\frac{\pi/20}{2} = 0.92146.$$

This results in:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{(z-0.92146\exp(j\pi/10))(z-0.92146\exp(-j\pi/10))}$$

The transfer function is then (using Euler's identity like before):

$$H(z) = K\frac{z^2-1.9021z+1}{z^2-1.7527z+0.84909}$$

As before, each z is a unit advance, so

$$y[n+2]-1.7527y[n+1]+0.84909y[n] = K(x[n+2]-1.9021x[n+1]+x[n]).$$

Letting $n = n-2$, makes it causal:

$$y[n]-1.7527y[n-1]+0.84909y[n-2] = K(x[n]-1.9021x[n-1]+x[n-2]).$$

The frequency response is thus

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1+\sum_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{K(1-1.9021\exp(-j\Omega)+\exp(-j2\Omega))}{1-1.7527\exp(-j\Omega)+0.84909\exp(-j2\Omega)}. \tag{5}$$

Using Euler's identity:

$$H(\Omega) = \frac{K(1-1.9021(\cos\Omega-j\sin\Omega)+\cos 2\Omega-j\sin 2\Omega)}{1-1.7527(\cos\Omega-j\sin\Omega)+0.84909(\cos 2\Omega-j\sin 2\Omega)}.$$

Magnitude frequency response is then:

$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1-1.9021\cos\Omega+\cos 2\Omega)^2+(1.9021\sin\Omega-\sin 2\Omega)^2)}{(1-1.7527\cos\Omega+0.84909\cos 2\Omega)^2+(1.7527\sin\Omega-0.84909\sin 2\Omega)^2)}. \tag{6}$$

Magnitude frequency response of the notch or bandstop filter is shown in Fig. 8.

4 IIR Filter Design from Analogue Filters

This is a very common approach for IIR filter design because it means that it can use some very well-established analogue filter specifications to design the actual digital IIR filters. There are two common approaches:

- Impulse invariant method

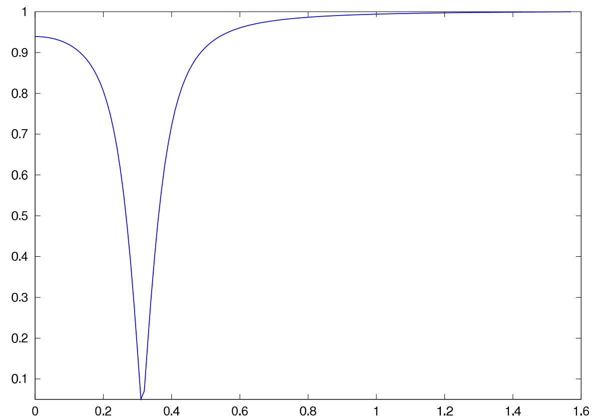
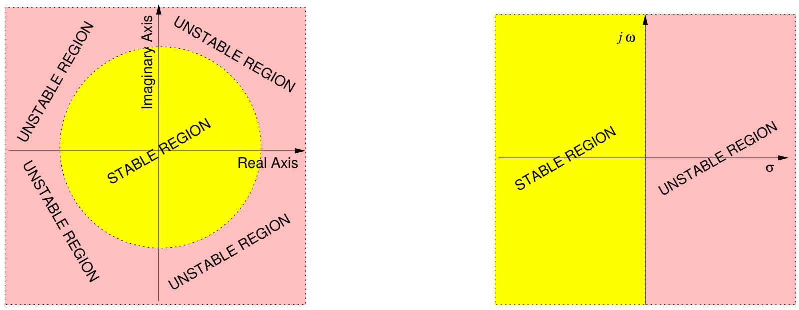


Figure 8: Magnitude frequency response of the derived bandstop filter using the pole zero placement method with frequency response given by the magnitude transfer function in (6).

Discrete (z-plane) $\Longleftrightarrow$ Analogue (s-plane)



z-transform $\mathcal{Z}(h[n])$ $\Longleftrightarrow$ Laplace transform $\mathcal{L}(h(t))$
z-plane $H[z]$ $\Longleftrightarrow$ s-plane $H(s)$
Difference equation, $h[n]$ $\Longleftrightarrow$ Differential equation $h(t)$

Figure 9: Comparison between digital z-plane and analogue s-plane.

- Bilinear transformation

Analogue filter transfer function $h(t)$ can be specified in the s-plane with the Laplace transform $\mathcal{L}(h(t)) = H(s)$. Hence, the s-plane is for analogue instead of the z-plane (for digital). The s-plane can be used to analyse stability of analogue filters, whilst the z-plane can be used to analyse for stability for digital filters. The s-plane and z-plane are illustrated and compared in Fig. 9.

There is a potential difficulty with the mapping between analogue and digital frequencies:

- Analogue frequencies range over $\omega = 0...\infty$.
- Whilst digital frequencies range over $\Omega = 0...2\pi$.

So how to convert analogue frequency to digital? When designing a digital filter with an analogue prototype, there is a need to swap analogue frequencies with digital frequencies...

- If $\Omega \to 2\pi$ then the corresponding analogue frequencies should also be very high analogue frequencies, *i.e.* ($\omega \to \infty$)
- If $\Omega \to 0$ then the corresponding analogue frequencies should also be very low analogue frequencies ($\omega \to 0$).

The Bilinear transformation (or BLT) method replaces *analogue frequency* s or $j\omega$ with *digital frequencies* Ω using:

$$s = j\omega = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right). \tag{7}$$

where ω is analogue frequency, Ω is digital *frequency* and $T_s = 1/f_s$ is the sampling period. This is often referred to as the frequency warping formula. It is used in conjunction with the actual Bilinear Transform as a pre-warping formula to convert digital frequencies to their analogue equivalent and then to return them back to the digital domain via the Bilinear Transform to find the z-transform $H(z)$:

$$s = j\omega = 2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right). \tag{8}$$

The application of the Bilinear Transform to an analogue prototype warps the frequencies specified. This means that application of (8) firstly requires pre-warping using (7).

Example Given an analog filter with:

$$H(\omega) = H(s)|_{s=j\omega} = \frac{K(j\omega - z_1)(j\omega - z_2)...}{(j\omega - p_1)(j\omega - p_2)...}$$

Then bilinear transformation gives

$$H(\Omega) = H(s)|_{s=j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)} = \frac{K(j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right) - z_1)(j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right) - z_2)...}{(j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right) - p_1)(j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right) - p_2)...} \tag{9}$$

or...

$$H(z) = H(s)|_{s=2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{K(2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_1)(2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_2)...}{(2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_1)(2f_\mathrm{s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_2)...} \tag{10}$$

Example Convert the single pole low pass analog filter:

$$H(s) = \frac{\omega_\mathrm{cf}}{s + \omega_\mathrm{cf}}$$

into a digital filter (z-plane form) with digital cut-off frequency $\Omega_\mathrm{cf} = 0.2\pi$ using the bilinear transformation.

Solution

1. Calculate analogue cut-off frequency ω_cf from digital cut-off frequency $\Omega_\mathrm{cf} = 0.2\pi$:

$$\omega_\mathrm{cf} = 2f_\mathrm{s}\tan(\Omega_\mathrm{cf}/2) = 2f_\mathrm{s}\tan(0.1\pi) = 2f_\mathrm{s}A$$

2. Therefore analogue transfer function:

$$H(s) = \frac{2f_\mathrm{s}A}{s + 2f_\mathrm{s}A}$$

3. Apply bilinear transformation: $s = 2f_\mathrm{s}\frac{1-z^{-1}}{1+z^{-1}}$:

$$H(z) = \frac{2f_\mathrm{s}A}{2f_\mathrm{s}\frac{1-z^{-1}}{1+z^{-1}} + 2f_\mathrm{s}A} = \left(\frac{2f_\mathrm{s}}{2f_\mathrm{s}}\right)\frac{A(1+z^{-1})}{(1-z^{-1}) + A(1+z^{-1})}$$

The z-transform transfer function of the filter is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A + Az^{-1}}{1 + A + (A-1)z^{-1}} \tag{11}$$

Stability analysis can now be performed. Rearranging to determine the poles for stability analysis gives:

$$H(z) = \frac{A}{1+A}\frac{z+1}{z + \frac{A-1}{1+A}}.$$

- So there is 1 pole at $z + \frac{A-1}{1+A} = 0$ or $z = -\frac{A-1}{1+A}$.
- Remember $A = \tan(0.1\pi)$, so the pole is: $z = -0.50953$,
- *the magnitude is less than 1, so the filter is stable.*

The difference equation can also be derived. The difference equation can now be found. Multiplying both sides by both denominators of equation (11) results in

$$Y(z)\left\{1+A+(A-1)z^{-1}\right\} = X(z)\left\{A+Az^{-1}\right\}$$

Remembering that each z^{-1} is a unit delay, so that

$$(1+A)y[n]+(A-1)y[n-1] = Ax[n]+Ax[n-1]$$

Dividing through by $(1+A)$ and rearranging gives

$$y[n] = \frac{A}{1+A}\left(x[n]+x[n-1]\right) - \frac{A-1}{1+A}y[n-1],$$

where $A = \tan(0.1\pi)$. This is now a difference equation we can use to filter a signal.

The frequency response can be found directly using the bilinear transformation or from the z-transform transfer function. We will compare both approaches. For the bilinear transformation, the following analogue transfer function from an earlier step is needed:

$$H(s) = \frac{2f_\mathrm{s}A}{s+2f_\mathrm{s}A}$$

The s-plane variable s can be replaced by the Fourier complex frequency variable $j\omega$,

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{2f_\mathrm{s}A}{j\omega+2f_\mathrm{s}A}.$$

The Fourier frequency can then be converted to the digital frequency Ω using $\omega = 2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)$:

$$H(\Omega) = H(\omega)\Big|_{\omega=2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)} = \frac{2f_\mathrm{s}A}{j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)+2f_\mathrm{s}A} = \frac{A}{j\tan\left(\frac{\Omega}{2}\right)+A}$$

So the magnitude frequency response calculated directly from the Bilinear transformation is:

$$|H(\Omega)| = \sqrt{\frac{A^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2+A^2}} = \sqrt{\frac{(\tan(0.1\pi))^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2+(\tan(0.1\pi))^2}} \tag{12}$$

The frequency response is illustrated in Fig. 10. The designed cut-off frequency $\Omega_\mathrm{cf} = 0.2\pi$ is confirmed by

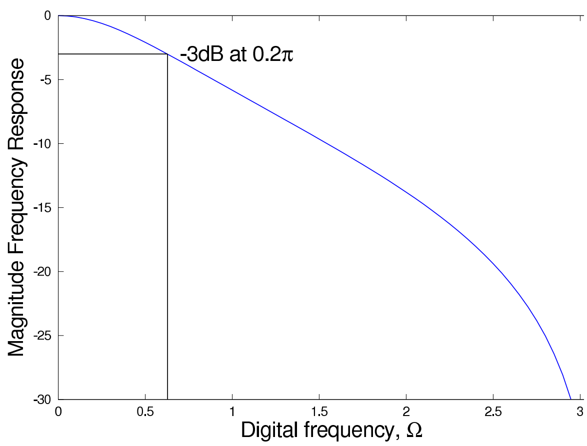


Figure 10: Magnitude frequeny response found using the bilinear transformation method with the result shown in (12).

this plot.

The frequency response can also be determined from the z-Transform transfer function. Remember the z-transform transfer function calculated earlier (equation (11)):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A+Az^{-1}}{1+A+(A-1)z^{-1}}.$$

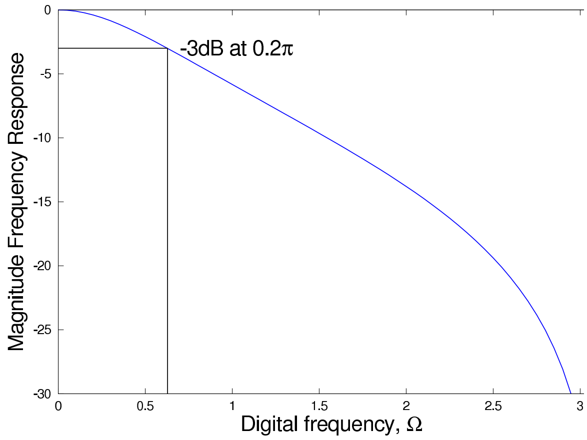


Figure 11: The frequency response obtained using the z-domain transfer function in (13).

This can be converted to the frequency response using

$$H(\Omega) = H(z)\Big|_{z=\exp(-jk\Omega)} = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{\sum_{k=0}^{N} a[k]\exp(-jk\Omega)}$$

So that the coefficients become:

- $b[0] = b[1] = A$,
- $a[0] = 1 + A$,
- $a[1] = A - 1$.

Thus, the frequency response is given by (using Euler's identity):

$$H(\Omega) = \frac{A + A(\cos(\Omega) - j\sin(\Omega))}{(1+A) + (A-1)(\cos(\Omega) - j\sin(\Omega)).}$$

The magnitude is then given by:

$$|H(\Omega)| = \sqrt{\frac{(A + A\cos(\Omega))^2 + (A\sin(\Omega))^2}{((1+A) + (A-1)\cos(\Omega))^2 + ((A-1)\sin(\Omega)))^2.}} \tag{13}$$

This frequency response result is illustrated in Fig. 11. This is the same as the frequency response calculated directly from the bilinear transformation. The Bilinear transformation is quicker here.

4.1 Digital Filter Design from Analogue Prototypes Recipe

The recipe to create a digital filter from an analogue specification can be stated as follows:

1. Determine the filter frequencies and gains (constants as outlined above);
2. Determine the digital frequency representations of the frequencies in radians per second;
3. Warp the frequencies using the digital to analogue frequency warping equations (as described in the IIR filters lectures) which are:
$$\omega_a = 2f_s \tan(\omega_d T_s/2)$$
4. Determine the analogue normalised frequencies;
5. Determine the required filter order (depending on Butterworth or Chebyshev filter equations) and rounding n up;
6. Apply a lowpass to lowpass transformation (or similarly to another filter type if desired in the analogue domain) as a function of s or $j\omega$;
7. Convert to digital form using the bilinear transformation (BLT);
8. Convert to appropriate filter type in the digital domain if in 6, this was not done (i.e. this can be done in the analogue or digital domain).

5 Frequency Transformation

Frequency transformation can be used to convert a low pass filter into:

- Another type of lowpass
- Highpass
- Bandpass
- or Bandstop.

Frequency transformations are summarised in Table 2.

Table 2: Prototype frequency transformations in the analogue domain.

Type	Transformation	Parameters
Lowpass	$s = \frac{s}{\omega_a}$	ω_a cut-off frequency (rad/s)
Highpass	$s = \frac{\omega_a}{s}$	
Bandpass	$s = \frac{s^2+\omega_o^2}{sW}$	$\omega_o^2 = \omega_{\mathrm{al}}\omega_{\mathrm{ah}}$, $W = \omega_{\mathrm{ah}} - \omega_{\mathrm{al}}$
Bandstop	$s = \frac{sW}{s^2+\omega_o^2}$	

It is important to note, as shown in Table 2, that the centre frequency is given by the geometric mean for the bandstop and bandpass filters, *i.e.*

$$\omega_o = \sqrt{\omega_{\mathrm{al}}\omega_{\mathrm{ah}}}. \tag{14}$$

Example Determine a 2nd order analogue Butterworth notch filter specification with bandwidth W and centre frequency ω_o.

Answer Starting with a first order Butterworth filter low pass prototype:

$$H(s) = \frac{1}{s+1}. \tag{15}$$

This can then be converted to a 2nd order notch filter via the appropriate transformation as shown in Table 2, where $s = sW/(s^2+\omega_o^2)$, to give

$$H_{\mathrm{BS}}(s) = H_{\mathrm{LP}}(s)|_{s=sW/(s^2+\omega_o^2)} = \frac{1}{\frac{sW}{s^2+\omega_o^2}+1}. \tag{16}$$

Multiplying top and bottom by $s^2+\omega_o^2$ yields

$$H_{\mathrm{BS}}(s) = \frac{s^2+\omega_o^2}{s^2+sW+\omega_o^2}. \tag{17}$$

5.1 A Summary of Frequency Transformation Results

Here is a summary of some commonly found frequency transformation results:

1st order standard LP to 1st order LP	$\frac{1}{s+1}$	$\frac{1}{\frac{s}{\omega_o}+1} = \frac{\omega_o}{s+\omega_o}$
1st order standard LP to 1st order HP	$\frac{1}{s+1}$	$\frac{1}{\frac{\omega_o}{s}+1} = \frac{s}{s+\omega_o}$
1st order standard LP to 2nd order BP	$\frac{1}{s+1}$	$\frac{1}{\frac{s^2+\omega_o^2}{sW}+1} = \frac{sW}{s^2+sW+\omega_o^2}$
1st order standard LP to 2nd order notch	$\frac{1}{s+1}$	$\frac{1}{\frac{sW}{s^2+\omega_o^2}+1} = \frac{s^2+\omega_o^2}{s^2+sW+\omega_o^2}$
2nd order standard LP to 2nd order LP	$\frac{1}{s^2+\frac{s}{Q}+1}$	$\frac{1}{\left(\frac{s}{\omega_o}\right)^2+\frac{s}{\omega_o Q}+1}$
2nd order standard LP to 2nd order HP	$\frac{1}{s^2+\frac{s}{Q}+1}$	$\frac{1}{\left(\frac{\omega_o}{s}\right)^2+\frac{\omega_o}{sQ}+1} = \frac{\left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)^2+\left(\frac{s}{Q\omega_o}\right)+1}$

Here, we have the bandwidth, $W = \omega_h - \omega_l$ where $\omega_h = 2\pi f_h$ rad/s and $\omega_l = 2\pi f_l$ rad/s and f_h and f_l are the higher and lower -3dB frequencies. The critical frequency, $\omega_o = 2\pi f_o$ rad/s which for the notch and bandpass filters is the geometric mean, *i.e.* $\omega_o = \sqrt{\omega_l \omega_h}$ of the upper and lower -3dB frequencies. Also, the quality factor or q-factor, $Q = \omega_o/W$.

The above summary is in terms of the s-domain variable $s = j\omega + \sigma$. However, the same expressions can be considered in terms of the radial frequency, assuming steady state ac-analysis, $s = j\omega$ where $\sigma = 0$ as follows.

1st order standard LP to 1st order LP	$\frac{1}{j\omega+1}$	$\frac{1}{\frac{j\omega}{\omega_o}+1} = \frac{\omega_o}{j\omega+\omega_o}$
1st order standard LP to 1st order HP	$\frac{1}{j\omega+1}$	$\frac{1}{\frac{\omega_o}{j\omega}+1} = \frac{j\omega}{j\omega+\omega_o}$
1st order standard LP to 2nd order BP	$\frac{1}{j\omega+1}$	$\frac{1}{\frac{(j\omega)^2+\omega_o^2}{j\omega W}+1} = \frac{j\omega W}{(j\omega)^2+j\omega W+\omega_o^2}$
1st order standard LP to 2nd order notch	$\frac{1}{j\omega+1}$	$\frac{1}{\frac{j\omega W}{(j\omega)^2+\omega_o^2}+1} = \frac{(j\omega)^2+\omega_o^2}{(j\omega)^2+j\omega W+\omega_o^2}$
2nd order standard LP to 2nd order LP	$\frac{1}{(j\omega)^2+\frac{j\omega}{Q}+1}$	$\frac{1}{\left(\frac{j\omega}{\omega_o}\right)^2+\frac{j\omega}{\omega_o Q}+1}$
2nd order standard LP to 2nd order HP	$\frac{1}{(j\omega)^2+\frac{j\omega}{Q}+1}$	$\frac{1}{\left(\frac{\omega_o}{j\omega}\right)^2+\frac{\omega_o}{j\omega Q}+1} = \frac{\left(\frac{j\omega}{\omega_o}\right)^2}{\left(\frac{j\omega}{\omega_o}\right)^2+\left(\frac{j\omega}{Q\omega_o}\right)+1}$

5.2 Summary of Common Digital Filter Results from Analogue Prototypes

Here is a summary of some commonly found frequency transformation results and their associated z-domain bilinear transformations:

1st order LP	$H(s)=\frac{\omega_o}{s+\omega_o}$	$H(z)=\frac{1}{1+\alpha}\times\frac{z+1}{z+\frac{1-\alpha}{1+\alpha}}$
where		$\omega_o=2f_s\tan\left(\frac{\Omega_o}{2}\right),\ \ \Omega_o=\frac{2\pi f_o}{f_s}$ and $\alpha=\frac{2f_s}{\omega_o}$
1st order HP	$H(s)=\frac{s}{s+\omega_o}$	$H(z)=\frac{\alpha}{1+\alpha}\times\frac{z-1}{z+\frac{1-\alpha}{1+\alpha}}$
2nd order BP	$H(s)=\frac{sW}{s^2+sW+\omega_o^2}$	$H(z)=\frac{\frac{\alpha}{Q}}{\alpha^2+\frac{\alpha}{Q}+1}\times\frac{(z-1)(z+1)}{z^2+z\frac{2(1-\alpha^2)}{\alpha^2+\frac{\alpha}{Q}+1}+\frac{\alpha^2-\frac{\alpha}{Q}+1}{\alpha^2+\frac{\alpha}{Q}+1}}$
where	$W=\omega_h-\omega_l$ $\omega_o=\sqrt{\omega_h\omega_l}$	$\omega_{h/l}=2f_s\tan\left(\frac{\Omega_{h/l}}{2}\right),\ \ \Omega_{h/l}=\frac{2\pi f_{h/l}}{f_s}$ and $Q=\frac{\sqrt{\omega_h\omega_l}}{\omega_h-\omega_l}$.
2nd order notch	$H(s)=\frac{s^2+\omega_o^2}{s^2+sW+\omega_o^2}$	$H(z)=\frac{\alpha^2+1}{\alpha^2+\frac{\alpha}{Q}+1}\times\frac{z^2+z\frac{2(1-\alpha^2)}{(1+\alpha^2)}+1}{z^2+z\frac{2(1-\alpha^2)}{\alpha^2+\frac{\alpha}{Q}+1}+\frac{\alpha^2-\frac{\alpha}{Q}+1}{\alpha^2+\frac{\alpha}{Q}+1}}$
2nd order LP	$H(s)=\frac{1}{\left(\frac{s}{\omega_o}\right)^2+\frac{s}{\omega_o Q}+1}$	$H(z)=\frac{1}{\alpha^2+\frac{\alpha}{Q}+1}\times\frac{(z+1)^2}{z^2+z\frac{2(1-\alpha^2)}{\alpha^2+\frac{\alpha}{Q}+1}+\frac{\alpha^2-\frac{\alpha}{Q}+1}{\alpha^2+\frac{\alpha}{Q}+1}}$
where	$\lvert H(\omega=\omega_o)\rvert=Q.$	$\omega_o=2f_s\tan\left(\frac{\Omega_o}{2}\right),\ \ \Omega_o=\frac{2\pi f_o}{f_s}$
2nd order HP	$H(s)=\frac{\left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)^2+\left(\frac{s}{Q\omega_o}\right)+1}$	$H(z)=\frac{\alpha^2}{\alpha^2+\frac{\alpha}{Q}+1}\times\frac{(z-1)^2}{z^2+z\frac{2(1-\alpha^2)}{\alpha^2+\frac{\alpha}{Q}+1}+\frac{\alpha^2-\frac{\alpha}{Q}+1}{\alpha^2+\frac{\alpha}{Q}+1}}$

Note that for the bandpass and bandstop (*i.e.* notch) filters, the critical frequency ω_o is defined in terms of the geometric mean of the bilinear transformed side frequencies. Hence, also the quality factor is determined after bilinear transformation of the side frequencies and subsequent calculation of the bandwidth W in terms of the transformed side frequencies.

5.3 Frequency Transformation in Digital Form

Frequency transformation can be performed in the analogue or digital domains, *i.e.* in the Laplace or z-domains. The frequency transformations in digital form are summarised in Table 5.3.

Table 3: Frequency transformation in digital form, *from Proakis and Manolakis, "Digital Signal Processing, Principles, Algorithms and Applications"*

Type	Transformation	Parameters
Lowpass	$z^{-1}\rightarrow\frac{z^{-1}-a}{1-az^{-1}}$	$a=\frac{\sin((\Omega_p-\Omega_p')/2)}{\sin((\Omega_p+\Omega_p')/2)}$
Highpass	$z^{-1}\rightarrow\frac{z^{-1}-a}{1+az^{-1}}$	$a=-\frac{\cos((\Omega_p+\Omega_p')/2)}{\cos((\Omega_p-\Omega_p')/2)}$
Bandpass	$z^{-1}\rightarrow-\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1=\frac{2\alpha K}{K+1},\ a_2=\frac{K-1}{K+1},\ \alpha=\frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)},\ K=\cot\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}.$
Bandstop	$z^{-1}\rightarrow-\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1=\frac{2\alpha}{K+1},\ a_2=\frac{(1-K)}{(1+K)},\ \alpha=\frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)},\ K=\tan\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}.$

6 Summary

These notes have covered a range of different topics including:

- Introduction to IIR filters.
- Frequency domain parameters.
- Pole-zero placement method for IIR filter design.
- Band stop filter design.
- IIR filter design from analog filters using bilinear transformation method.