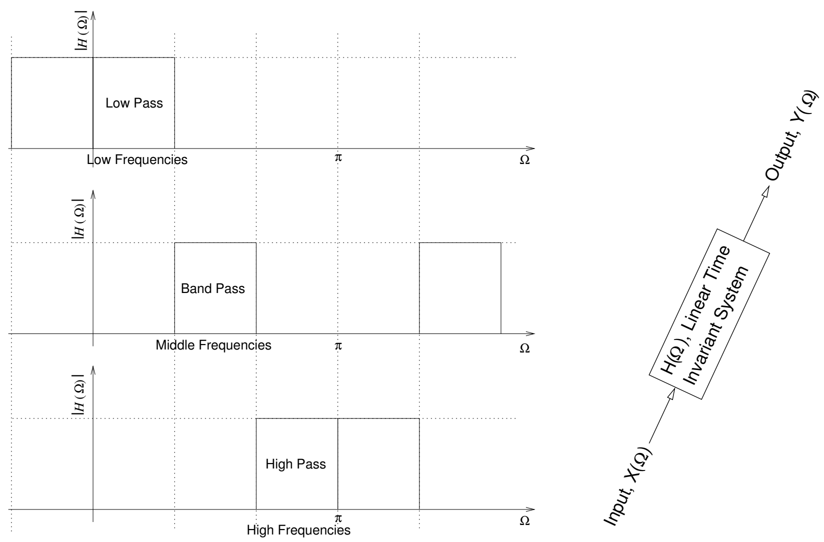
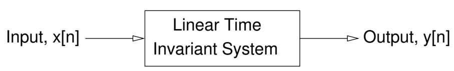
Infinite Impulse Response Filters

Contents

1 Introduction

Often used to remove some frequencies from a signal $X(\Omega)$ and to allow other frequencies to pass through to the output $Y(\Omega)$.





What is a *Recursive digital filter*?

- "*Recursive*" comes from the word "to recur"
  Meaning: to repeat

A recursive filter uses past output values ($y[n-i]$) for the calculation of the current output $y[n]$:

- *Recursive Filter Example*
$$y[n] = 0.5y[n-1] + 0.5x[n].$$

A non-recursive filter only uses input values $x[n-i]$:

- *Non-recursive Filter Example*
$$y[n] = 0.5x[n-1] + 0.5x[n].$$

Recall the generalised difference equation for causal LTI systems:

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k]$$

If $a[0] = 1$, this can then be changed to:

$$y[n] = \sum_{k=0}^{M} b[k]x[n-k] - \sum_{k=1}^{N} a[k]y[n-k].$$

Also recall the frequency response of such a system can be described by:

$$H(\Omega) = \frac{\sum\limits_{k=0}^{M} b[k]\exp(-jk\Omega)}{\exp(0) + \sum\limits_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{\sum\limits_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum\limits_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

This is the Fourier based frequency representation. There can also be a representation in the z-domain (z-transform):

$$H(z) = \frac{\sum\limits_{k=0}^{M} b[k]z^{-k}}{1 + \sum\limits_{k=1}^{N} a[k]z^{-k}}.$$

Both the z-domain and the Fourier representations describe types of frequency response of the system. The system structure of an IIR filter demonstrates the feedback of the output into the input again.

An example of an FIR filter structure is

$$y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right). \tag{1}$$

Notice there is only a single output term $y[n]$ on the left and no past output terms on the right. An example of an IIR filter structure is

$$y[n] = \left(x[n] + x[n-1] + x[n-2] + x[n-3]\right) - \frac{1}{10}y[n-1]. \tag{2}$$

Both examples can be seen illustrated in Fig. 1. In these systems a single z^{-1} is the same as a unit delay, "T" in a system diagram. Recursive digital filters are often known as Infinite Impulse Response (IIR) filters as the impulse response of an IIR filter often has an infinite number of coefficients:

- Require fewer calculations than FIR filters.
- This can mean that they can have a faster response to the input signal,
- and this can also mean that they have a shorter frequency response i.e. *transition width*.

However, IIR filters can become unstable. Therefore there is a need to think carefully about stability when designing IIR Filters.

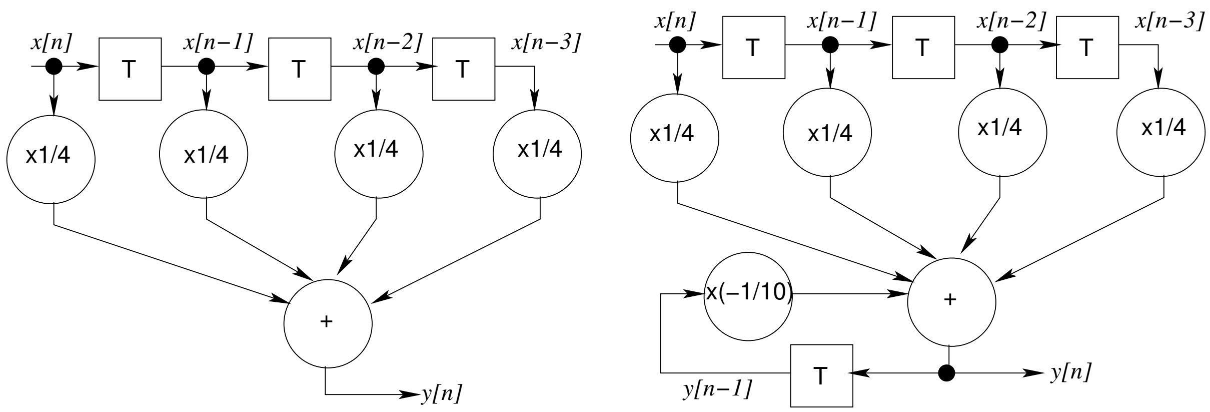


Figure 1: Examples of an FIR filter structure (left) and an IIR filter structure (right).

Table 1: Comparison of FIR and IIR filter characteristics. *Adapted from "Understanding digital signal processing" by R. G. Lyons.*

Characteristic	IIR	FIR
Multiplications	least	most
Coefficient quantification sensitivity	can be high	very low
Overflow errors	can be high	very low
Stability	by design	always
Linear phase	no	always
Simulate analog filter	yes	no
Coefficient memory	least	most
Design complexity	moderate	simple

1.1 Comparison of IIR and FIR filters

A comparative summary of the characteristics of FIR and IIR filters can be seen in Table 1.1.

2 IIR Filter Design

Filter design can follow a number of different methodologies however most, if not all, will require the following steps:

1. Filter specification
2. Coefficient calculations
3. Convert transfer function to suitable filter structure
4. Error analysis
5. Implementation

2.1 Frequency Domain Parameters

An important aspect of filter design are the frequency domain parameters. These can include:

- δ_p passband ripple
- δ_s stopband ripple
- $\Omega_{\mathrm{s}1}$ lower stop band edge frequency
- $\Omega_{\mathrm{p}1}$ lower pass band edge frequency
- $\Omega_{\mathrm{p}2}$ upper pass band edge frequency

- Ω_{s2} upper stop band edge frequency

The stop band is where there is high attenuation and the pass band is where there is low attenuation (gain≈1). These parameters are illustrated in Fig. 2.

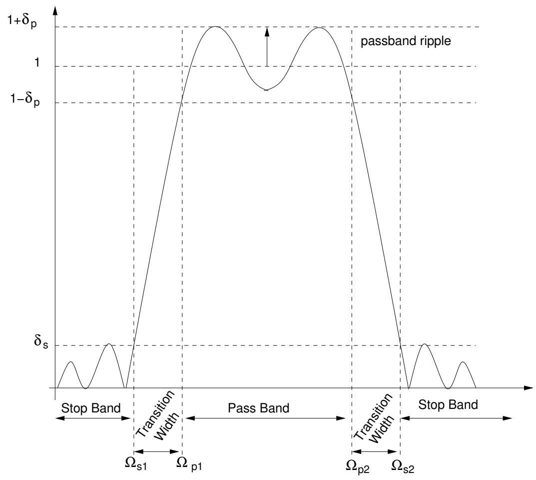


Figure 2: Illustration of some commonly found frequency domain parameters used in filter design.

3 Pole-Zero Placement Method

A filter can be described in the z-plane with poles and zeros. The poles are the roots of the denominator of the transfer function and the zeros are the roots of the numerator, *i.e.*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...} = \frac{\text{zeros}}{\text{poles}}$$

Here, you should be able to observe:

- Poles *located* at: z_1, z_2, z_3, ...
- Zeros *located* at: p_1, p_2, p_3, ...

An example z-plane is illustrated in Fig. 3. It is interesting to observe that:

- Poles (X) close to unit circle *make large peaks*;
- Zeros (O) close to unit circle *make troughs or minima*.

Angle of poles and zeros on z-plane correspond to frequencies that can be used for filter specification.

3.1 Bandpass Filter Pole-Zero Placement Filter Design

A bandpass filter has a frequency response where a band of frequencies are passed but lower and higher frequencies are attenuated. This is illustrated in Fig. fig:bandpassfreqresp. A pole-zero placement design method can utilise the following observations:

- A bandpass filter, with centre frequency Ω_0 radians can have two poles at $\pm\Omega_0$ radians in the z-plane[1].
- Complete attenuation at two frequencies, $\Omega_{r1} = 0$ radians and $\Omega_{r2} = \pi$ radians can have two zeros at 0 and π radians.
- The radius of the poles can be calculated with:

  $$r \cong 1 - \Omega_{cf}$$

  or

  $$r \cong 1 - \frac{\Omega_{bw}}{2}$$

  where $\Omega_{bw} = 2\Omega_{cf}$ is the -3dB bandwidth of the filter.

[1] Complex conjugate pair to make real filter coefficients, when $\Omega_0 \neq 0$ or $\Omega_0 \neq \pi$ radians (on the real line).

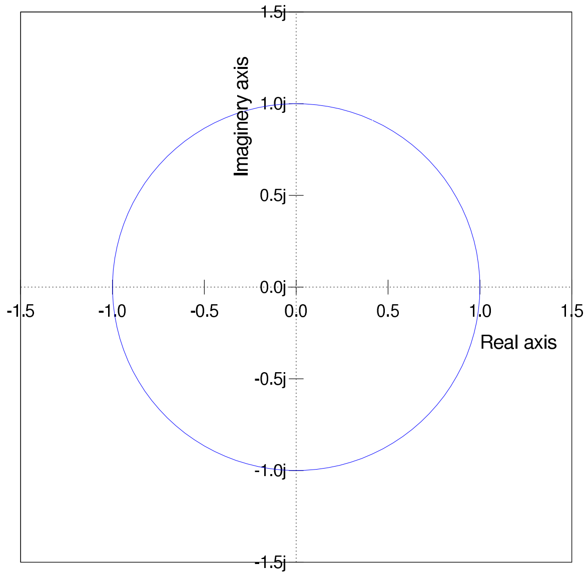


Figure 3: Illustration of the z-plane along with the unit circle.

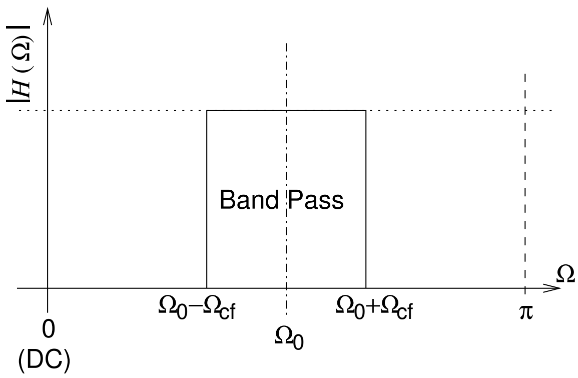


Figure 4: Illustration of a bandpass filter's frequency response.

Example Design a bandpass filter using the Pole-zero placement method with:

- centre frequency at $\Omega_0 = \pi/2$;
- a bandwidth of $\Omega_{\rm bw} = \pi/8$;
- complete attenuation at $\Omega_{\rm r1} = 0$ and $\Omega_{\rm r2} = \pi$;
- and peak unity pass band gain.

Solution Bandpass filter has x2 poles at $\pm\Omega_0 = \pm\pi/2$ radians.

$$\therefore H(z) = K \frac{\text{zeros}}{(z - r\exp(j\pi/2))(z - r\exp(-j\pi/2))}$$

The radii of the poles are given by:

$$r \cong 1 - \frac{\Omega_{\rm bw}}{2} = 1 - \frac{\pi/8}{2} = 0.80365;$$

and the zeros are at $\Omega_{\rm r1} = 0$ and $\Omega_{\rm r2} = \pi$, so that

$$H(z) = K \frac{(z - \exp(j\Omega_{\rm r1}))(z - \exp(j\Omega_{\rm r2}))}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

As

- $\exp(\Omega_{\rm r1}) = \exp(j0) = \cos(0) + j\sin(0) = 1 - j0 = 1$

- $\exp(\Omega_{r2}) = \exp(j\pi) = \cos(\pi) + j\sin(\pi) = -1 + j0 = -1$,

then the transfer function becomes:

$$H(z) = K\frac{(z-1)(z+1)}{(z-0.80365\exp(j\pi/2))(z-0.80365\exp(-j\pi/2))}.$$

Using Euler's identity,

- $\exp(j\pi/2) = \cos(\pi/2) + j\sin(\pi/2) = +j$
- and $\exp(j\pi/2) = \cos(\pi/2) - j\sin(\pi/2) = -j$,

so that

$$H(z) = K\frac{(z-1)(z+1)}{(z-0.80365j)(z+0.80365j)}. \tag{3}$$

the pole zero diagram can then be plotted which is shown in Fig. 5

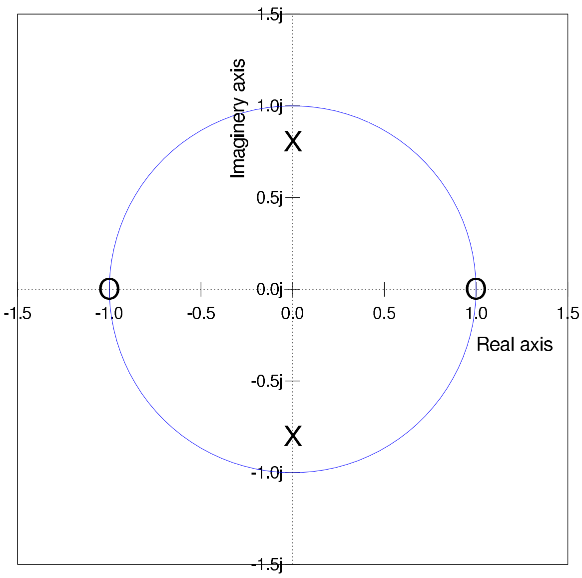


Figure 5: z-plane plot for the poles and zeros specified by the transfer function in (3).

Recall that $H(z) = \frac{Y(z)}{X(z)}$,

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{(z-1)(z+1)}{(z-0.80365j)(z+0.80365j)} = K\frac{z^2-1}{z^2+0.64585}.$$

Then

$$Y(z)(z^2+0.64585) = X(z)K(z^2-1).$$

Remembering that each z^{-1} is a unit delay, so that each z is a unit advance, then the difference equation is:

$$y[n+2] + 0.64585y[n] = K(x[n+2] - x[n])$$

which can be made causal by making $n = n-2$ so that

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]).$$

K is not known, but can be used to make the peak pass band gain to be unity.

The frequency response of the filter can be determined from the difference equation:

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]),$$

in combination with:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

So that (using Euler's identity):

$$H(\Omega) = \frac{K(1 - \cos(2\Omega) + j\sin(2\Omega))}{1 + 0.64585(\cos(2\Omega) - j\sin(2\Omega))}$$

which has magnitude frequency response:

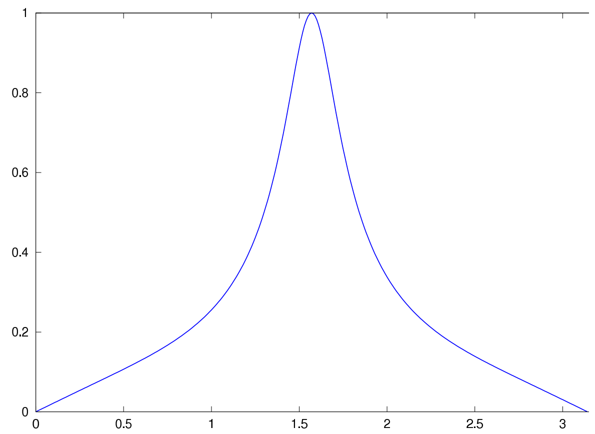


Figure 6: Magnitude Frequency Response

$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1 - \cos(2\Omega))^2 + \sin^2(2\Omega))}{(1 + 0.64585\cos(2\Omega))^2 + \sin^2(2\Omega)}. \quad (4)$$

where $K = 0.17708$.

Relating the digital frequencies for previous example to actual frequencies...
If the sampling frequency is $f_s = 500$Hz, the sampling frequency corresponds to $\Omega = 2\pi$, therefore the filter parameters become:

- centre frequency at $\Omega_0 = \pi/2$, so actual centre frequency $f_0 = \frac{\pi/2}{2\pi} f_s = 125$Hz;
- a bandwidth of $\Omega_{\text{bw}} = \pi/8$, so actual bandwidth $f_{\text{bw}} = 31.25$Hz;
- complete attenuation at $\Omega_{\text{r1}} = 0$ and $\Omega_{\text{r2}} = \pi$, with actual frequencies $f_{\text{r1}} = 0$Hz and $f_{\text{r2}} = \frac{\pi}{2\pi} 500 = 250$Hz.

3.2 Bandstop Filter Pole-Zero Placement Filter Design

A bandstop filter, sometimes referred to as a notch filter (usually for narrow bands) remove or significantly attenuate a finite band of frequencies so that low frequencies and high frequencies are not attenuated. An ideal bandstop filter frequency response is illustrated in Fig. 7.

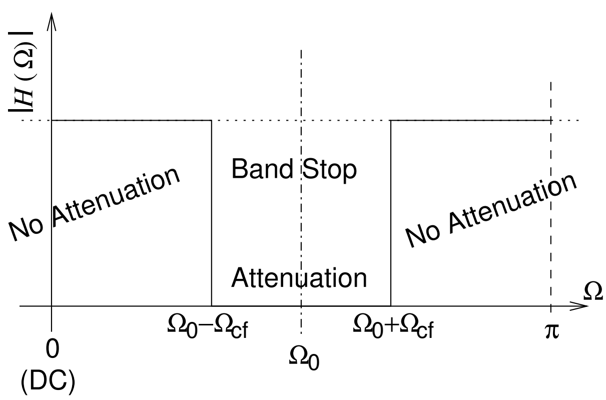


Figure 7: Illustration of an ideal bandstop filter's frequency response.

A bandstop filter can be broadly specified by the frequencies at which the band of frequencies are to be stopped, at the beginning and end.

Example Design a digital bandstop filter using pole-zero placement method with the following parameters:

- Centre frequency, $\Omega_0 = \pi/10$ radians (complete attenuation)
- Band stop width, $\Omega_\text{w} = 2\Omega_\text{cf} = \pi/20$ radians

Solution Complete attenuation at $\Omega_0 = \pi/10$, $\therefore$ x2 zeros (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{\text{poles}}$$

The centre frequency is at $\Omega_0 = \pi/10$ radians, therefore there should be two poles (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$, which gives:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{(z-r\exp(j\pi/10))(z-r\exp(-j\pi/10))}$$

The poles are scaled with radius r to control the width of the band stop,

$$r \cong 1-\frac{\Omega_\text{w}}{2} = 1-\frac{\pi/20}{2} = 0.92146.$$

This results in:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{(z-0.92146\exp(j\pi/10))(z-0.92146\exp(-j\pi/10))}$$

The transfer function is then (using Euler's identity like before):

$$H(z) = K\frac{z^2-1.9021z+1}{z^2-1.7527z+0.84909}$$

As before, each z is a unit advance, so

$$y[n+2]-1.7527y[n+1]+0.84909y[n] = K(x[n+2]-1.9021x[n+1]+x[n]).$$

Letting $n=n-2$, makes it causal:

$$y[n]-1.7527y[n-1]+0.84909y[n-2] = K(x[n]-1.9021x[n-1]+x[n-2]).$$

The frequency response is thus

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1+\sum_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{K(1-1.9021\exp(-j\Omega)+\exp(-j2\Omega))}{1-1.7527\exp(-j\Omega)+0.84909\exp(-j2\Omega)}. \tag{5}$$

Using Euler's identity:

$$H(\Omega) = \frac{K(1-1.9021(\cos\Omega-j\sin\Omega)+\cos2\Omega-j\sin2\Omega)}{1-1.7527(\cos\Omega-j\sin\Omega)+0.84909(\cos2\Omega-j\sin2\Omega)}.$$

Magnitude frequency response is then:

$$\text{Mag}(H(\Omega))^2 = \frac{K((1-1.9021\cos\Omega+\cos2\Omega)^2+(1.9021\sin\Omega-\sin2\Omega)^2)}{(1-1.7527\cos\Omega+0.84909\cos2\Omega)^2+(1.7527\sin\Omega-0.84909\sin2\Omega)^2)}. \tag{6}$$

Magnitude frequency response of the notch or bandstop filter is shown in Fig. 8.

4 IIR Filter Design from Analogue Filters

This is a very common approach for IIR filter design because it means that it can use some very well-established analogue filter specifications to design the actual digital IIR filters. There are two common approaches:

- Impulse invariant method

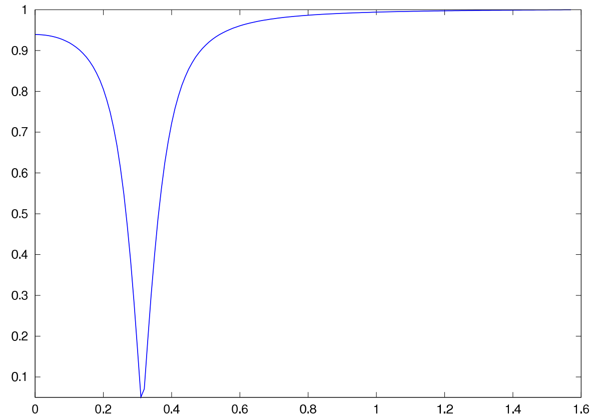
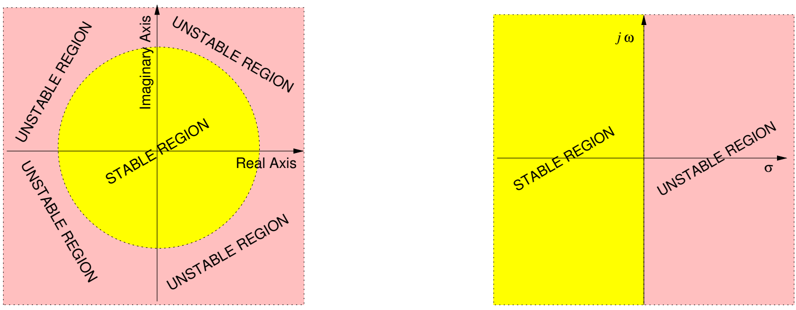


Figure 8: Magnitude frequency response of the derived bandstop filter using the pole zero placement method with frequency response given by the magnitude transfer function in (6).

Discrete (z-plane) $\Longleftrightarrow$ Analogue (s-plane)



z-transform $\mathcal{Z}(h[n])$ $\Longleftrightarrow$ Laplace transform $\mathcal{L}(h(t))$
z-plane $H[z]$ $\Longleftrightarrow$ s-plane $H(s)$
Difference equation, $h[n]$ $\Longleftrightarrow$ Differential equation $h(t)$

Figure 9: Comparison between digital z-plane and analogue s-plane.

- Bilinear transformation

Analogue filter transfer function $h(t)$ can be specified in the s-plane with the Laplace transform $\mathcal{L}(h(t)) = H(s)$. Hence, the s-plane is for analogue instead of the z-plane (for digital). The s-plane can be used to analyse stability of analogue filters, whilst the z-plane can be used to analyse for stability for digital filters. The s-plane and z-plane are illustrated and compared in Fig. 9.

There is a potential difficulty with the mapping between analogue and digital frequencies:

- Analogue frequencies range over $\omega = 0...\infty$.
- Whilst digital frequencies range over $\Omega = 0...2\pi$.

So how to convert analogue frequency to digital? When designing a digital filter with an analogue prototype, there is a need to swap analogue frequencies with digital frequencies...

- If $\Omega \to 2\pi$ then the corresponding analogue frequencies should also be very high analogue frequencies, *i.e.* ($\omega \to \infty$)
- If $\Omega \to 0$ then the corresponding analogue frequencies should also be very low analogue frequencies ($\omega \to 0$).

The Bilinear transformation (or BLT) method replaces *analogue frequency* s or $j\omega$ with *digital frequencies* Ω using:

$$s = j\omega = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right). \tag{7}$$

where ω is analogue frequency, Ω is digital *frequency* and $T_s = 1/f_s$ is the sampling period. This is often referred to as the frequency warping formula. It is used in conjunction with the actual Bilinear Transform as a pre-warping formula to convert digital frequencies to their analogue equivalent and then to return them back to the digital domain via the Bilinear Transform to find the z-transform $H(z)$:

$$s = j\omega = 2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right). \tag{8}$$

The application of the Bilinear Transform to an analogue prototype warps the frequencies specified. This means that application of (8) firstly requires pre-warping using (7).

Example Given an analog filter with:

$$H(\omega) = H(s)|_{s=j\omega} = \frac{K(j\omega - z_1)(j\omega - z_2)...}{(j\omega - p_1)(j\omega - p_2)...}$$

Then bilinear transformation gives

$$H(\Omega) = H(s)|_{s=j2f_{\mathrm{s}}\tan\left(\frac{\Omega}{2}\right)} = \frac{K(j2f_{\mathrm{s}}\tan\left(\frac{\Omega}{2}\right) - z_1)(j2f_{\mathrm{s}}\tan\left(\frac{\Omega}{2}\right) - z_2)...}{(j2f_{\mathrm{s}}\tan\left(\frac{\Omega}{2}\right) - p_1)(j2f_{\mathrm{s}}\tan\left(\frac{\Omega}{2}\right) - p_2)...} \tag{9}$$

or...

$$H(z) = H(s)|_{s=2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{K(2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_1)(2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_2)...}{(2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_1)(2f_{\mathrm{s}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_2)...} \tag{10}$$

Example Convert the single pole low pass analog filter:

$$H(s) = \frac{\omega_{\mathrm{cf}}}{s + \omega_{\mathrm{cf}}}$$

into a digital filter (z-plane form) with digital cut-off frequency $\Omega_{\mathrm{cf}} = 0.2\pi$ using the bilinear transformation.

Solution

1. Calculate analogue cut-off frequency ω_{cf} from digital cut-off frequency $\Omega_{\mathrm{cf}} = 0.2\pi$:

$$\omega_{\mathrm{cf}} = 2f_{\mathrm{s}}\tan(\Omega_{\mathrm{cf}}/2) = 2f_{\mathrm{s}}\tan(0.1\pi) = 2f_{\mathrm{s}}A$$

2. Therefore analogue transfer function:

$$H(s) = \frac{2f_{\mathrm{s}}A}{s + 2f_{\mathrm{s}}A}$$

3. Apply bilinear transformation: $s = 2f_{\mathrm{s}}\frac{1-z^{-1}}{1+z^{-1}}$:

$$H(z) = \frac{2f_{\mathrm{s}}A}{2f_{\mathrm{s}}\frac{1-z^{-1}}{1+z^{-1}} + 2f_{\mathrm{s}}A} = \left(\frac{2f_{\mathrm{s}}}{2f_{\mathrm{s}}}\right)\frac{A(1+z^{-1})}{(1-z^{-1}) + A(1+z^{-1})}$$

The z-transform transfer function of the filter is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A + Az^{-1}}{1 + A + (A-1)z^{-1}} \tag{11}$$

Stability analysis can now be performed. Rearranging to determine the poles for stability analysis gives:

$$H(z) = \frac{A}{1+A}\frac{z+1}{z + \frac{A-1}{1+A}}.$$

- So there is 1 pole at $z + \frac{A-1}{1+A} = 0$ or $z = -\frac{A-1}{1+A}$.
- Remember $A = \tan(0.1\pi)$, so the pole is: $z = -0.50953$,
- *the magnitude is less than 1, so the filter is stable.*

The difference equation can also be derived. The difference equation can now be found. Multiplying both sides by both denominators of equation (11) results in

$$Y(z)\left\{1+A+(A-1)z^{-1}\right\} = X(z)\left\{A+Az^{-1}\right\}$$

Remembering that each z^{-1} is a unit delay, so that

$$(1+A)y[n]+(A-1)y[n-1] = Ax[n]+Ax[n-1]$$

Dividing through by $(1+A)$ and rearranging gives

$$y[n] = \frac{A}{1+A}\left(x[n]+x[n-1]\right) - \frac{A-1}{1+A}y[n-1],$$

where $A = \tan(0.1\pi)$. This is now a difference equation we can use to filter a signal.

The frequency response can be found directly using the bilinear transformation or from the z-transform transfer function. We will compare both approaches. For the bilinear transformation, the following analogue transfer function from an earlier step is needed:

$$H(s) = \frac{2f_\mathrm{s}A}{s+2f_\mathrm{s}A}$$

The s-plane variable s can be replaced by the Fourier complex frequency variable $j\omega$,

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{2f_\mathrm{s}A}{j\omega+2f_\mathrm{s}A}.$$

The Fourier frequency can then be converted to the digital frequency Ω using $\omega = 2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)$:

$$H(\Omega) = H(\omega)\Big|_{\omega=2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)} = \frac{2f_\mathrm{s}A}{j2f_\mathrm{s}\tan\left(\frac{\Omega}{2}\right)+2f_\mathrm{s}A} = \frac{A}{j\tan\left(\frac{\Omega}{2}\right)+A}$$

So the magnitude frequency response calculated directly from the Bilinear transformation is:

$$|H(\Omega)| = \sqrt{\frac{A^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2+A^2}} = \sqrt{\frac{(\tan(0.1\pi))^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2+(\tan(0.1\pi))^2}} \tag{12}$$

The frequency response is illustrated in Fig. 10. The designed cut-off frequency $\Omega_\mathrm{cf} = 0.2\pi$ is confirmed by

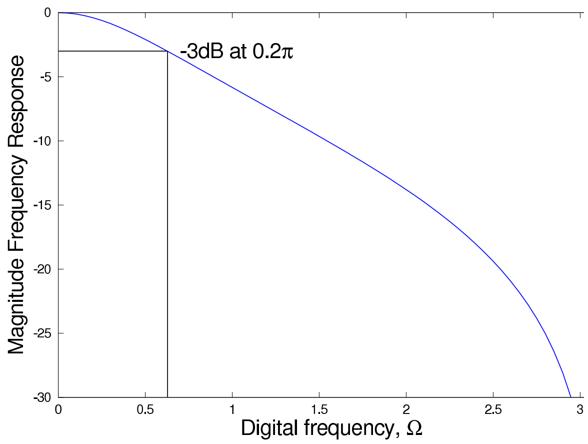


Figure 10: Magnitude frequeny response found using the bilinear transformation method with the result shown in (12).

this plot.

The frequency response can also be determined from the z-Transform transfer function. Remember the z-transform transfer function calculated earlier (equation (11)):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A+Az^{-1}}{1+A+(A-1)z^{-1}}.$$

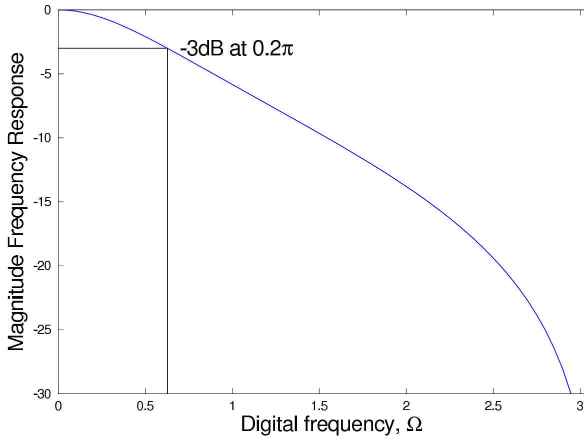


Figure 11: The frequency response obtained using the z-domain transfer function in (13).

This can be converted to the frequency response using

$$H(\Omega) = H(z)\Big|_{z=\exp(-jk\Omega)} = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{\sum_{k=0}^{N} a[k]\exp(-jk\Omega)}$$

So that the coefficients become:

- $b[0] = b[1] = A$,
- $a[0] = 1 + A$,
- $a[1] = A - 1$.

Thus, the frequency response is given by (using Euler's identity):

$$H(\Omega) = \frac{A + A(\cos(\Omega) - j\sin(\Omega))}{(1 + A) + (A - 1)(\cos(\Omega) - j\sin(\Omega)).}$$

The magnitude is then given by:

$$|H(\Omega)| = \sqrt{\frac{(A + A\cos(\Omega))^2 + (A\sin(\Omega))^2}{((1 + A) + (A - 1)\cos(\Omega))^2 + ((A - 1)\sin(\Omega)))^2.}} \tag{13}$$

This frequency response result is illustrated in Fig. 11. This is the same as the frequency response calculated directly from the bilinear transformation. The Bilinear transformation is quicker here.

4.1 Some Famous Analogue Filters

There are numerous different analogue filters. Two of the more commonly used filters are the Butterworth and Chebyshev analogue prototype filters.

The Butterworth magnitude frequency response is given by:

$$|H(\omega)| = \frac{1}{\left\{1 + \left(\frac{\omega}{\omega_{\rm cf}}\right)^{2n}\right\}^{1/2}}, \tag{14}$$

where n is the filter order and $\omega_{\rm cf}$ is the analogue radial cut-off frequency. The Chebyshev magnitude frequency response is given by:

$$|H(\omega)| = \frac{1}{\left\{1 + \varepsilon^2 C_n^2\left(\frac{\omega}{\omega_{\rm cf}}\right)\right\}^{1/2}} \tag{15}$$

where ε is the maximum passband attenuation and C_n is the Chebyshev function that takes the normalised frequency $\omega/\omega_{\rm cf}$ as an argument. These parameters will be discussed more shortly. The Butterworth and

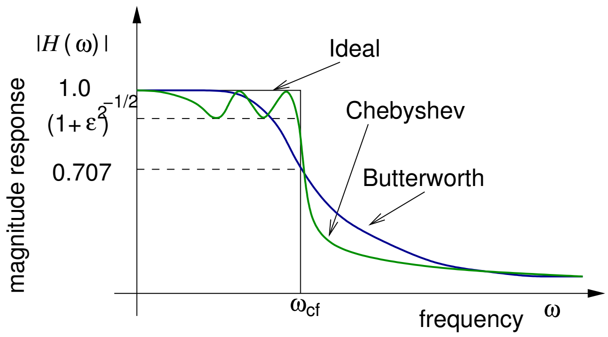


Figure 12: Illustrative comparison of Butterworth and Chebyshev analogue filter responses.

Table 2: 3dB Butterworth prototypes with $\varepsilon = 1$, *taken from Tan, Digital Signal Processing and Applications, 2007.*

n	prototype
1	$1/(s+1)$
2	$1/(s^2+\sqrt{2}s+1)$
3	$1/(s^3+2s^2+2s+1)$
4	$1/(s^4+2.6131s^3+3.4142s^2+2.6131s+1)$
5	$1/(s^5+3.2361s^4+5.2361s^3+5.2361s^2+3.2361s+1)$
6	$1/(s^6+3.8637s^5+7.4641s^4+9.1416s^3+7.4641s^2+3.8637s+1)$

Chebyshev two frequency responses are sketched for illustrative purposes and shown in Fig. 12 for comparison. The Chebyshev is closer to the ideal filter response but is not smooth which is not the case for the Butterworth filter response.

Table 4.1 lists the Butterworth low pass prototypes up to $n = 6$.

Table 4.1 lists the filter prototypes for the Chebyshev type 1 prototypes with 0.5dB of ripple.

5 Frequency Transformation

So far we have looked at low pass IIR filters only.
Frequency transformation can be used to convert a low pass filter into:

- Another type of lowpass
- Highpass
- Bandpass
- or Bandstop

Frequency transformation can be performed in the:

- Analogue form
- Or the digital form.

Table 3: Chebyshev Type 1 prototypes with 0.5dB ripple ($\varepsilon = 0.3493$), *taken from Tan, Digital Signal Processing and Applications, 2007.*

n	prototype
1	$2.8628/(s+2.8628)$
2	$1.4314/(s^2+1.4256s+1.5162)$
3	$0.7157/(s^3+1.2529s^2+1.5349s+0.7157)$
4	$0.3579/(s^4+1.1974s^3+1.7169s^2+1.0255s+0.3791)$
5	$0.1789/(s^5+1.1725s^4+1.9374s^3+1.3096s^2+0.7525s+0.1789)$
6	$0.0895/(s^6+1.1592s^5+2.1718s^4+1.5898s^3+1.1719s^2+0.4324s+0.0948)$

5.1 Frequency Transformation in Analogue Form

The frequency transformations in analogue form are summarised in Table 4.

Table 4: Prototype frequency transformations in the analogue domain.

Type	Transformation	Parameters
Lowpass	$s = \frac{s}{\omega_a}$	ω_a cut-off frequency (rad/s)
Highpass	$s = \frac{\omega_a}{s}$	
Bandpass	$s = \frac{s^2+\omega_o^2}{sW}$	$\omega_o^2 = \omega_{\rm al}\omega_{\rm ah}$, $W = \omega_{\rm ah} - \omega_{\rm al}$
Bandstop	$s = \frac{sW}{s^2+\omega_o^2}$	

It is important to note, as shown in Table 4, that the centre frequency is given by the geometric mean for the bandstop and bandpass filters, *i.e.*

$$\omega_o = \sqrt{\omega_{\rm al}\omega_{\rm ah}}. \tag{16}$$

Example Determine a 2nd order analogue Butterworth notch filter specification with bandwidth W and centre frequency ω_o.

Answer Starting with a first order Butterworth filter low pass prototype:

$$H(s) = \frac{1}{s+1}. \tag{17}$$

This can then be converted to a 2nd order notch filter via the appropriate transformation as shown in Table 4, where $s = sW/(s^2 + \omega_o^2)$, to give

$$H_{\rm BS}(s) = H_{\rm LP}(s)|_{s=sW/(s^2+\omega_o^2)} = \frac{1}{\frac{sW}{s^2+\omega_o^2} + 1}. \tag{18}$$

Multiplying top and bottom by $s^2 + \omega_o^2$ yields

$$H_{\rm BS}(s) = \frac{s^2 + \omega_o^2}{s^2 + sW + \omega_o^2}. \tag{19}$$

5.2 Frequency Transformation in Digital Form

The frequency transformations in digital form are summarised in Table 5.2.

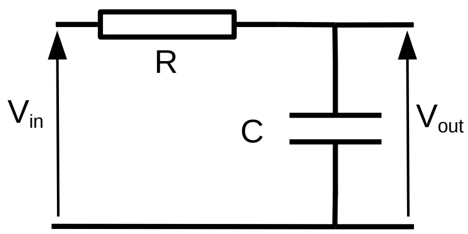
Table 5: Frequency transformation in digital form, *from Proakis and Manolakis, "Digital Signal Processing, Principles, Algorithms and Applications"*

Type	Transformation	Parameters
Lowpass	$z^{-1} \to \frac{z^{-1}-a}{1-az^{-1}}$	$a = \frac{\sin((\Omega_p - \Omega_p')/2)}{\sin((\Omega_p + \Omega_p')/2)}$
Highpass	$z^{-1} \to \frac{z^{-1}-a}{1+az^{-1}}$	$a = -\frac{\cos((\Omega_p + \Omega_p')/2)}{\cos((\Omega_p - \Omega_p')/2)}$
Bandpass	$z^{-1} \to -\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha K}{K+1}$, $a_2 = \frac{K-1}{K+1}$, $\alpha = \frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)}$, $K = \cot\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}$.
Bandstop	$z^{-1} \to -\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha}{K+1}$, $a_2 = \frac{(1-K)}{(1+K)}$, $\alpha = \frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)}$, $K = \tan\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}$.

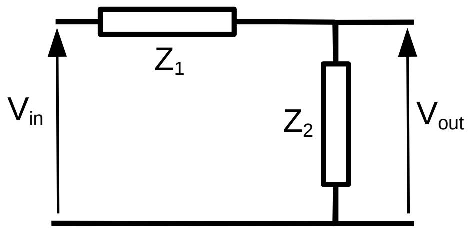
6 Analogue Filters Background

6.1 First Order Response

A simple first order low pass filter consists of a resistor and a capacitor. This is illustrated below:



This is, what is often known as a first order passive low pass filter. It is a very simple configuration of a single resistor and capacitor. The derivation for the transfer function for this type of filter can follow what should be a relatively familiar theme. Firstly, consider the resistor and capacitor as forming something similar to a voltage divider network, i.e.



Here the output voltage can be seen to be given by the following expression,

$$V_{\text{out}} = V_{\text{in}} \frac{Z_2}{Z_1 + Z_2} \tag{20}$$

where the impedances are given by $Z_1 = R$ and $Z_2 = X_c$. The second impedance is the reactance of the capacitance, which of course is given by $X_c = \frac{1}{j\omega C}$. Here C is the capacitance in Farads and $\omega = 2\pi f$ is the radial frequency in radians per second and f is the frequency in Hertz. Substituting these values into the equation we get

$$V_{\text{out}} = V_{\text{in}} \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}. \tag{21}$$

If we multiply the top and bottom of this equation by $j\omega C$ we get

$$V_{\text{out}} = V_{\text{in}} \frac{1}{j\omega CR + 1} \tag{22}$$

which is an interesting function as it combines together the capacitor and resistance values next to the radial frequency. Rearranging slightly and dividing through by the input voltage we can equate the resulting expression as a frequency dependent gain, *i.e.*

$$G(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j\omega CR}. \tag{23}$$

If we let $\omega_c = 1/(CR)$ and substituting we get

$$G(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j\omega/\omega_c}. \tag{24}$$

where ω_c is the cut-off frequency. For this particular circuit, you can say $\omega_c = 1/(RC)$. However it is important to note that this transfer function does not take into account the source impedance or the load impedance. So more in depth analysis could also be performed.

The magnitude response for this filter is given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}} \tag{25}$$

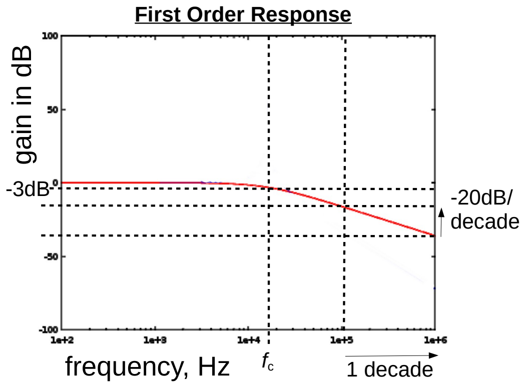
and the phase is given by

$$\phi = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right). \tag{26}$$

The slope for this first order type filter, after the cut-off frequency should be -20dB per decade of frequency. Also, for this circuit, when $\omega = \omega_c$ we can observe that

$$|H(\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707. \tag{27}$$

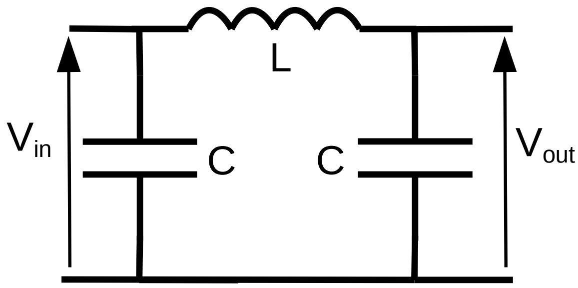
This indicates that the gain will fall to 0.707 at the point at which $\omega = \omega_c$. These things are illustrated below



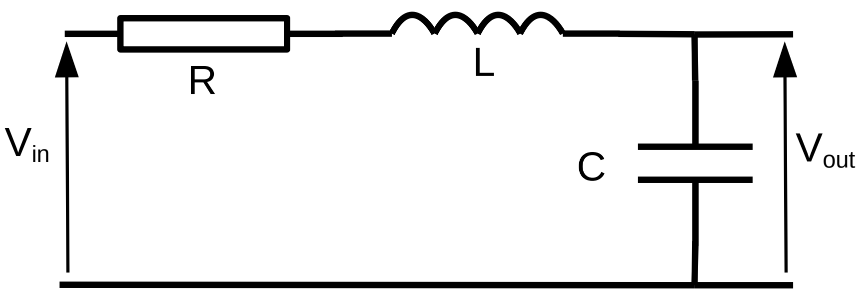
This is a useful result because it coincides with the half power point or the point at which the gain in decibels falls to -3dB, which is why it might be considered to be a point at which the filter is no longer able to provide a useful passband response.

6.2 Second Order Response

The first order low pass filter response can be compared with a second order low pass filter response. A second order filter is typically a circuit consisting of two energy storage elements, which for passive filters will often be a capacitor and an inductor. Examples of second order filters can be seen below



Pi LC Low Pass Filter



RLC Low Pass Filter

A second order transfer function can sometimes be expressed as follows

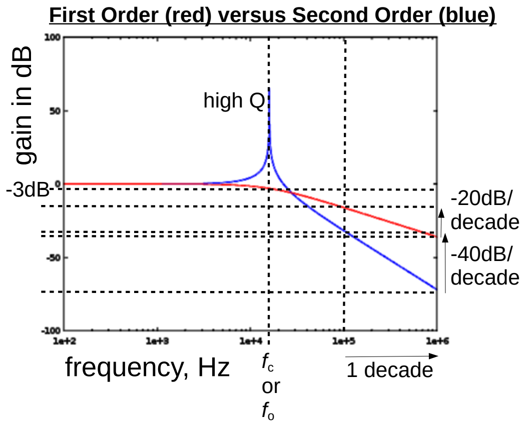
$$H(\omega) = \frac{1}{1 - \omega^2/\omega_o^2 + j\omega/(Q\omega_o)}. \tag{28}$$

Here ω_o can be referred to as a critical frequency and Q is the Q-factor. The Q-factor has some influence over the response of the system. For a particular single stage of an LC Pi network the Q factor is, in theory infinite, although that ignores the source and load impedances and also the non-zero internal resistance that an inductor is usually considered to possess (i.e. being lossy). For infinite Q the transfer function simplifies to

$$H(\omega) = \frac{1}{1-\omega^2/\omega_o^2}. \tag{29}$$

The critical frequency is given by $\omega_o = 1/\sqrt{LC}$. The response should peak at this frequency. Due to the lack of an imaginary component in the transfer function for infinite Q, we can observe that the magnitude response is equal to the absolute value of the transfer function, i.e. $|H(\omega)| = |\frac{1}{1-\omega^2/\omega_o^2}|$. Plotting the magnitude response will help to reveal that the slope is -40dB per decade at frequencies way past the critical frequency.

A comparison of the first order and second order magnitude responses should make this a bit more obvious. For instance:



The phase response is given by

$$\phi = -\tan^{-1}\left(\frac{\omega/\omega_o}{Q[1-(\omega/\omega_o)^2]}\right). \tag{30}$$

It is important to realise where this comes from. Looking back at equation (28), we can see that the numerator contributes zero phase shift but the denominator consists of an imaginary and a real component. In this case, the phase will depend on these values and depend on the quadrant in which the complex number resides. *i.e.* the argument (angle) of a complex number is calculated using

Rectangular		**Polar arg**
$z = x + jy$	$\rightarrow$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = -x + jy$	$\rightarrow$	$\phi = \pi - \tan^{-1}\left(\frac{y}{x}\right)$
$z = -x - jy$	$\rightarrow$	$\phi = \pi + \tan^{-1}\left(\frac{y}{x}\right)$
$z = x - jy$	$\rightarrow$	$\phi = 2\pi - \tan^{-1}\left(\frac{y}{x}\right)$

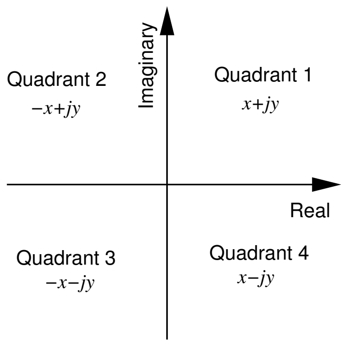
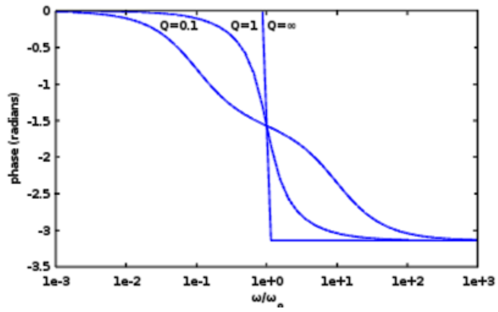


Illustration of the four quadrants in the complex domain.
The position of the complex vector affects the way in which the phase angle is computed.

The phase is plotted as a function of the normalised frequency ratio ω/ω_o for a number of different values of Q below.



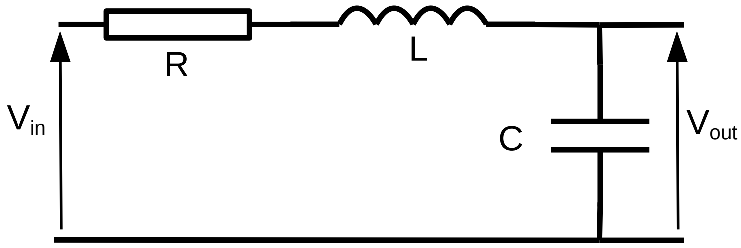
For the case of when $Q = \infty$ there is no imaginary component. The phase must then exist purely on the real line. However, this does not mean that it is fixed to one value. For infinite Q we must observe that we have $1 - (\omega/\omega_o)^2$ on the denominator which will affect the sign of the real term depending on the ratio $(\omega/\omega_o)^2$. We can therefore establish the two values of the phase in this case:

$$\phi = \begin{cases} \tan^{-1}(0) - \tan^{-1}(0) = 0 & \text{when} \quad (\omega/\omega_o)^2 < 1 \\ \tan^{-1}(0) - (\pi - \tan^{-1}(0)) = -\pi & \text{when} \quad (\omega/\omega_o)^2 > 1 \end{cases} \tag{31}$$

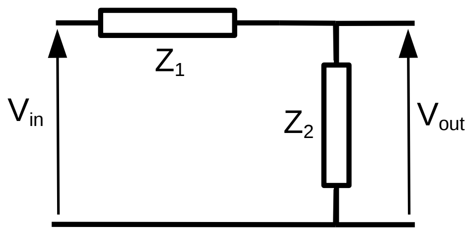
This does not include the case for when the real part is 0. In this case the value is not defined but one would normally plot a line going from 0 to $-\pi$ radians.

Example of Derivation of a Second Order Filter Response

Derive the transfer function for the following second order circuit RLC passive low pass filter:



A similar concept can be used here again as was used for the first order low pass filter analysis, *i.e.* in terms of a voltage divider equation in conjunction with some generic impedances. Recall the impedance based version of the circuit, we have



If we express the function of the circuit in terms of a frequency dependent gain equation, we have

$$G(\omega) = \frac{Z_2}{Z_1 + Z_2} \tag{32}$$

where the impedances are given by $Z_1 = R + X_l$ and $Z_2 = X_c$. The reactance of the inductor is given by $X_l = j\omega L$. Substituting in the expressions for the impedances we have

$$G(\omega) = \frac{X_c}{R + X_l + X_c} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}. \tag{33}$$

Multiplying top and bottom by $j\omega C$ results in

$$G(\omega) = \frac{1}{j\omega CR + (j\omega)^2 CL + 1} = \frac{1}{j\omega CR - \omega^2 CL + 1}. \tag{34}$$

Standard Second Order Response

If we let $\omega_o = 1/\sqrt{LC}$ and substitute it into the middle term in the denominator, to get

$$G(\omega) = \frac{1}{j\omega CR - \omega^2/\omega_o^2 + 1}. \tag{35}$$

This can be compared with equation (28). To determine the expression that we can substitute into the left hand term of the denominator, equate the similar term in equations (28) and (35)

$$CR = \frac{1}{Q\omega_o}$$

we have defined $\omega_0 = 1/\sqrt{LC}$ so that

$$CR = \frac{\sqrt{LC}}{Q}$$

and solving for Q we get

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

This shows that the *Q* factor is infinite for zero resistance. However, in practical circuits, even without resistance, an inductor will possess a finite amount of resistance.

Damping Factor

An alternative way to represent a frequency response yet useful is in terms of a damping factor $\zeta = \frac{R}{L}$ or *zeta*. The smaller the damping factor the greater the *Q* factor. We can see this if we rearrange the general second order response with

$$H(\omega) = \frac{\omega_o^2}{\omega_o^2 + (j\omega)^2 + j\omega\omega_o/Q}. \tag{36}$$

Substituting in $2\zeta = \frac{1}{Q}$ we obtain

$$H(\omega) = \frac{\omega_o^2}{\omega_o^2 + (j\omega)^2 + j\omega 2\zeta\omega_o}. \tag{37}$$

The magnitude response is then

$$|H(\omega)| = \frac{\omega_o^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\zeta\omega_o\omega)^2}} \tag{38}$$

and the phase is given by

$$\phi = -\tan^{-1}\left(\frac{2\zeta\omega_o\omega}{\omega_o^2 - \omega^2}\right). \tag{39}$$

This means when $\omega = \omega_o$ we have

$$|H(\omega = \omega_c)| = \frac{1}{2\zeta} \text{ and } \phi(\omega = \omega_o) = -\frac{\pi}{2}\text{radians.} \tag{40}$$

7 Butterworth and Chebyshev Filters

Butterworth and Chebyshev filters are two types of filters that have been found to have favourable properties in the analogue domain but also in the digital domain.

7.1 Butterworth Filter Analogue Specification

Butterworth filters can be specified with the following constants:

- v is the normalized frequency, $v = \Omega/\Omega_p$;
- v_s is a frequency in the stop band at which the maximum gain is specified;
- v_p is a frequency at the end of the passband at which the maximum attenuation is specified;
- ε is the maximum passband attenuation or ripple;

- A_s is the maximum gain in the stopband in decibels;
- A_p is the maximum attenuation in the passband.

A Butterworth filter has the following specification:

$$|H(v)| = \frac{1}{\sqrt{1+\varepsilon^2 v^{2n}}} \tag{41}$$

where v is a normalized frequency. The absolute ripple specification is ε and the filter order is n.

The meaning of the normalized frequency is such that $v = \Omega/\Omega_p$ where Ω_p is a frequency of the filter at which the maximum amount of attenation is specified. When $\Omega = \Omega_p$ we have $v = 1$.

The absolute ripple specification is the maximum amount of attenuation permissible in the passband and can be determined from

$$\varepsilon^2 = 10^{0.1A_p} - 1 \tag{42}$$

where A_p is the attenuation in decibels.

After ε is determined, we can determine the filter order. This is based on the maximum amount of attenuation in the passband but also the maximum ripple (or gain) allowed in the stop band A_s at frequency v_s

$$n \geq \frac{1}{2\log_{10}(v_s)} \log_{10}\left(\frac{10^{0.1A_s}-1}{\varepsilon^2}\right). \tag{43}$$

7.2 Chebyshev Analogue Filter Specification

Chebyshev filters can be specified with the following constants:

- v is the normalized frequency, $v = \Omega/\Omega_p$;
- v_s is a frequency in the stop band at which the maximum gain is specified;
- v_p is a frequency at the end of the passband at which the maximum attenuation is specified (there can be other points with this attenuation in the passband but not less than for the Chebyshev filter);
- ε is the maximum passband attenuation or ripple;
- A_s is the maximum gain in the stopband in decibels;
- A_p is the maximum attenuation in the passband.

Chebyshev filters are specified with the following function:

$$|H(v)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(v)}} \tag{44}$$

where ε is a minimum attenuation constant, similar for the Butterworth filter and C_n is the Chebyshev function which is given by

$$C_n(v) = \begin{cases} \cos\left(n\cos^{-1}(v)\right) & \text{for} \quad 0 \leq v \leq 1; \quad \approx \textit{passband} \\ \cosh\left(n\cosh^{-1}(v)\right) & \text{for} \quad v > 1. \quad \approx \textit{stopband} \end{cases} \tag{45}$$

Similar as for the Butterworth filter, the maximum passband attenuation is specified in decibels, i.e. A_p so that

$$\varepsilon^2 = 10^{0.1A_p} - 1. \tag{46}$$

For the Chebyshev filter order we have

$$n \geq \frac{1}{\cosh^{-1}(v_s)} \cosh^{-1}\left(\sqrt{\left(\frac{10^{0.1A_s}-1}{\varepsilon^2}\right)}\right). \tag{47}$$

7.3 Digital Filter Design from Analogue Prototypes Recipe

The recipe to create a digital filter from an analogue specification can be stated as follows:

1. Determine the filter frequencies and gains (constants as outlined above);
2. Determine the digital frequency representations of the frequencies in radians per second;
3. Warp the frequencies using the digital to analogue frequency warping equations (as described in the IIR filters lectures) which are:
$$\omega_a = 2f_s \tan\left(\omega_d T_s/2\right)$$
4. Determine the analogue normalised frequencies;
5. Determine the required filter order (depending on Butterworth or Chebyshev filter equations) and rounding n up;
6. Apply a lowpass to lowpass transformation (or similarly to another filter type if desired in the analogue domain) as a function of s or $j\omega$;
7. Convert to digital form using the bilinear transformation (BLT);
8. Convert to appropriate filter type in the digital domain if in 5. this was not done (i.e. this can be done in the analogue or digital domain).

8 Summary

- Introduction to IIR filters.
- Frequency domain parameters.
- Pole-zero placement method for IIR filter design.
- Band stop filter design.
- IIR filter design from analog filters using bilinear transformation method.