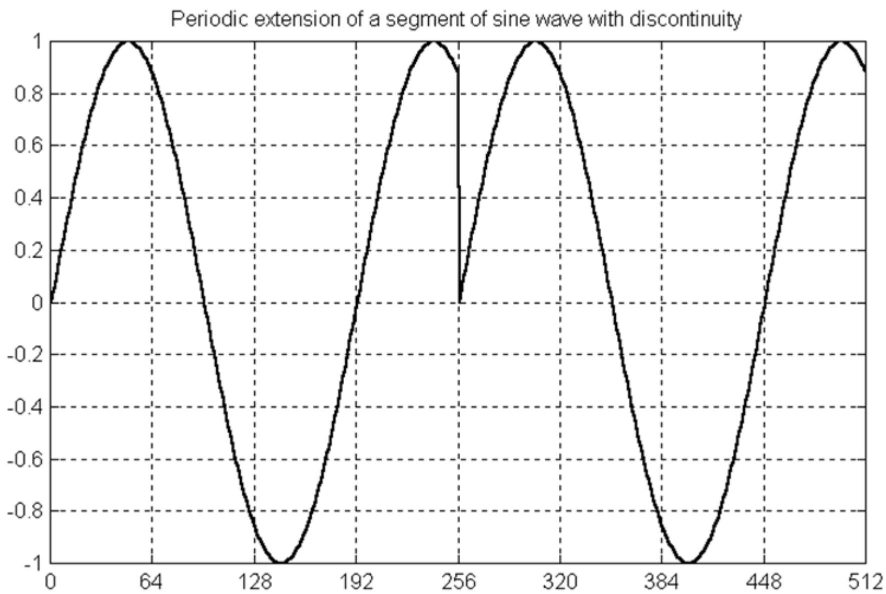
Spectral Leakage and Windowing

Contents

1 Implications of processing a finite block of samples

If a signal which we wish to analyse using the DFT (whether or not we are using the FFT version) was actually periodic with the period of the length of the block of samples being processed then the periodic extension which is implicit in the use of the DFT (discussed when we derived the DFT) causes no problems. However, if this is not the case then periodic extension creates a discontinuity at the point where the original data ends and the extension starts (as in the Figure below).



To illustrate the effect this has in the frequency domain, consider a single complex sinusoid, frequency f_0, whose waveform $x(t) = e^{j2\pi f_0 t}$ exists for all time and whose spectrum is $\delta(f - f_0)$ i.e. it is only non-zero at one frequency, f_0. If we take just a finite chunk of it, length T_c, then the resulting continuous-time waveform has a spectrum which is defined and in general non-zero at all frequencies:

$$X_{T_c}(f) = \int_{-T_c/2}^{+T_c/2} e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \int_{-T_c/2}^{+T_c/2} e^{-j2\pi(f-f_0)t} dt$$

Because of the symmetry of the limits of integration, only the even part of the integrand needs to be considered (integrating the odd part over symmetric limits will give a result of zero):

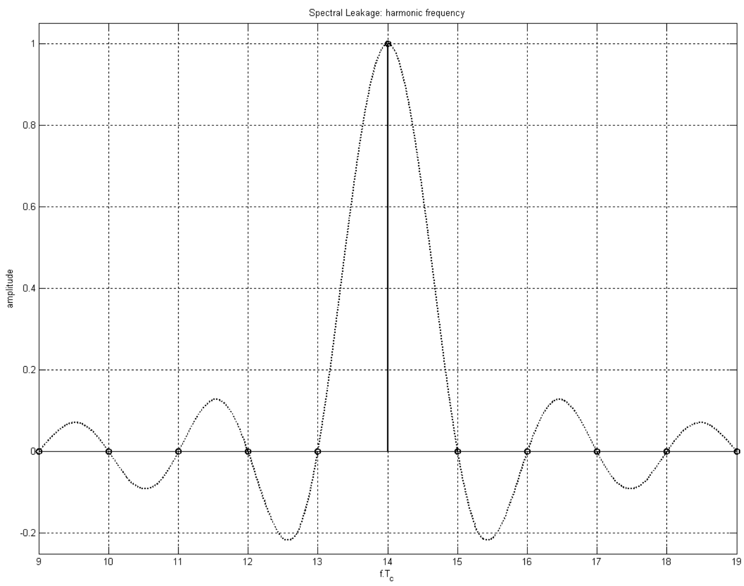
$$\begin{aligned} X_{T_c}(f) &= \int_{-T_c/2}^{+T_c/2} cos(2\pi(f-f_0)t) dt \\ &= \left[\frac{\sin(2\pi(f-f_0)t)}{2\pi(f-f_0)} \right]_{-T_c/2}^{+T_c/2} \\ &= \frac{2\sin(\pi(f-f_0)T_c)}{2\pi(f-f_0)} \\ &= T_c \frac{\sin(\pi(f-f_0)T_c)}{\pi(f-f_0)T_c} = T_c \mathrm{sinc}((f-f_0)T_c) \end{aligned}$$

If we now sample the spectrum at frequencies $f = k/T_c$, thereby performing an implicit time-domain periodic extension (with period Tc i.e. an immediate repetition, no zero-padding), the resulting discrete spectrum is given by

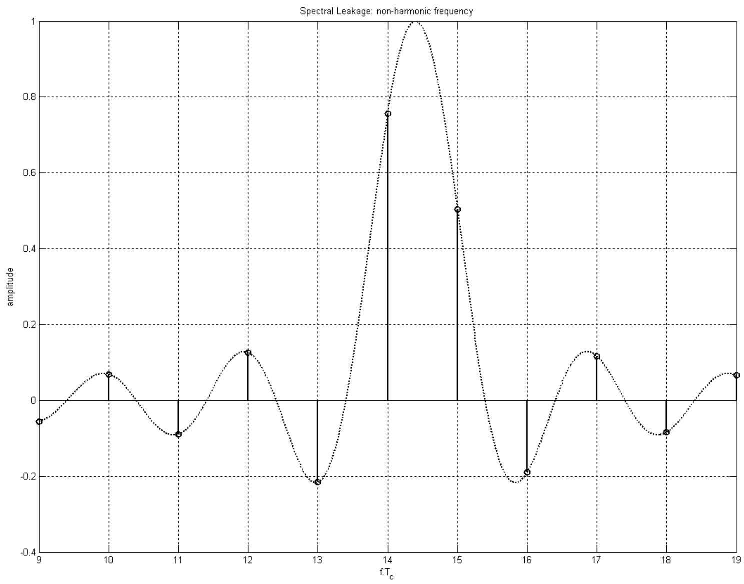
$$X[k] = T_c.\mathrm{sinc}(k - f_0 T_c)$$

The pattern that results depends on the value of f_0. There are two cases:

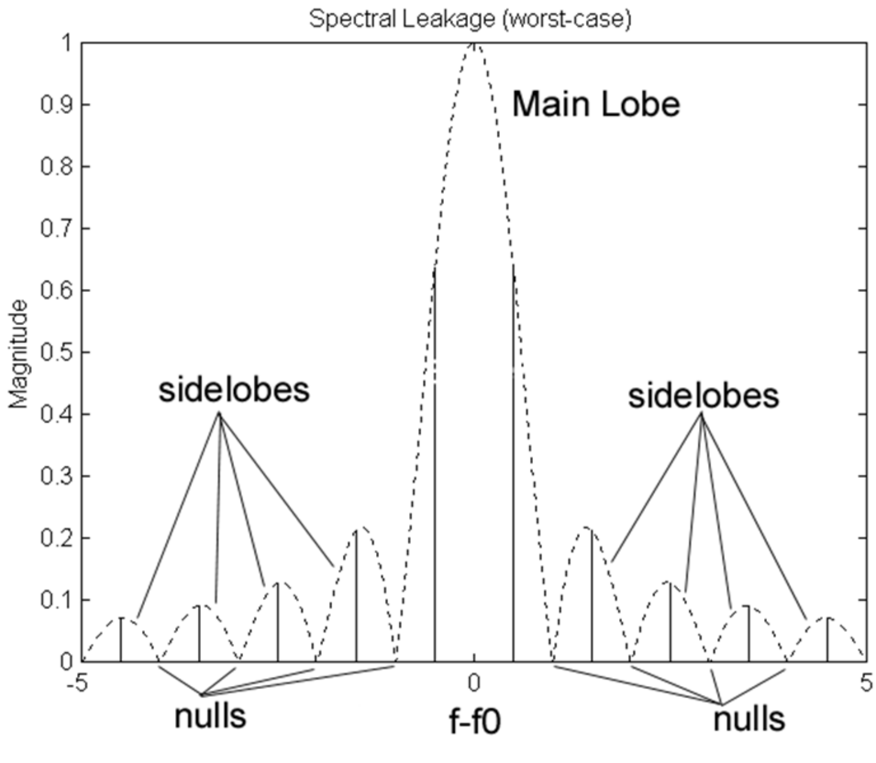
- harmonic frequencies: if $f_0 T_c$ happens to equal an integer, say n, then the properties of the sinc function dictate that X[k]=0 except for $k = n$ i.e. the spectrum we get is “right” because the periodic extension matches what the original sinusoid would have done: there is no discontinuity.



- nonharmonic frequencies: if $f_0 T_c$ is not an integer (which is usually the case) then $X[k]$ is generally nonzero for all k. In other words, taking only a finite length of data and periodically extending it has the effect of replacing an individual frequency by a sinc function which is defined over all frequencies (because the periodic extension is “wrong” and a discontinuity is created by it).



The problem in the second case is called spectral leakage because the single frequency “leaks” into all the other frequencies (values of k), mostly into those nearby but some into the whole of the frequency spectrum! To see this, consider the modulus of $X[k]$. This consists of a main peak (the main lobe) and a number of subsidiary peaks (sidelobes) separated by zeros (nulls) as illustrated below.



The implications of this for Signal Processing are:

- *main lobe*
  - ◇ The height of the largest frequency sample is $T_c\text{sinc}(k^* - f_0T_c)$, where k^* is the integer closest to f_0T_c. For a harmonic frequency, the height is $T_c\text{sinc}(0)$. For a non-harmonic frequency, the worst case (lowest possible) height is $T_c\text{sinc}(0.5)$ which is a reduction of about 4 dB (Why is this the worst case?). This loss of sensitivity is known as Scalloping Loss.
  - ◇ The single frequency has become “blurred” into a region of frequencies which reduces the Resolution of a spectral analysis of the signal. In particular, if there are two frequencies close together they may appear as a single peak: resolution is the ability to “resolve” which in this case means

the ability to see two peaks in the spectrum when two frequencies of equal amplitude are present. The minimum spacing necessary to guarantee two peaks are visible is equal to the 6dB-width of the main lobe (why the 6dB width rather than the 3dB width?)

- *sidelobes*
  - ◇ A peak observed in an analysis may not be a true signal, it may be an artifact of the interaction of the sidelobes of other signals (seeing something that isn't really there)
  - ◇ It may be impossible to identify a small signal in the presence of a large one because it is swamped by the larger signal's sidelobes (not seeing something which really is there). This is a problem of Dynamic Range, which is generally defined as the ratio of the largest value to the smallest value a system can handle. In this context it is the ratio of the amplitudes of the largest and smallest signals which can be detected simultaneously. The dynamic range (usually measured in dB) must therefore be taken as the height of the main lobe above the highest sidelobe (since that sidelobe has the greatest possibility of concealing a smaller signal).
  - ◇ There is another closely related issue: the "far-field dynamic range" is a qualitative property of the sidelobes far from the main lobe and depends on the sidelobe fall-off rate as well as the height of the highest sidelobe. If the sidelobes fall off rapidly then a very small signal can be detected in the presence of a large one, provided its frequency is far enough away from the frequency of the large signal.
  - ◇ The interaction of the sidelobes or the main lobe of one signal with the main lobe of another can result in a main lobe peak which no longer coincides with the signal's location, i.e. using DFT spectrum peaks to identify the frequencies present in a signal can yield biased estimates of the frequency locations.

One possible way to alleviate these problems is to take more data, though that may not always be possible.

Given all these problems one might wonder if the DFT is worth the trouble. It is! We just have to work within its limitations and not take the results it gives automatically at face value.

2 Taking Control of Spectral Leakage

There is no way to avoid spectral leakage: it is an inevitable consequence of only having a finite set of samples to process with our DFT. However, there are things that can be done to make its impact better-controlled. In particular we will look at two:

- zero-padding which fills in the dotted-lines in the main-lobe+sidelobes diagram in the previous section, thereby allowing us to see the spectral leakage clearly
- windowing which can be used to make the sidelobes lower, though at the cost of making the main-lobe wider

2.1 Zero Padding

In deriving the DFT, we assumed that the periodic extension is taking place immediately at the end of the available data. This need not be the case, however. It would be perfectly possible to take a longer period thus including a certain amount of null data (i.e. zeros) tagged onto the end of the actual data. This is known as zero-padding and it has the effect of sampling the spectrum at a finer frequency spacing:

$$\frac{1}{\text{transform-length}} < \frac{1}{\text{data-length}}$$

Note: We need to be cautious in the use of the term "frequency bin" (or simply "bin") in a DFT with zero padding:

- some authors use it to mean the spacing on the frequency axis which is the reciprocal of the transform length and changes with the amount of zero padding;
- others use it to mean the reciprocal of the data length (which is not affected by zero-padding).

We will use this second definition.

Although zero padding gives us more points on the frequency axis, it does not add any new information: it simply makes use of spectral leakage to interpolate the spectrum (i.e. it turns a bug into a feature!). The benefit is in getting a better "picture" which enables us (or even a computer) to locate, for example, the peaks in a spectrum more accurately. We are in effect attempting to get closer to a continuous spectrum by periodically extending after some suitable multiple M of the data length i.e. taking frequencies which are multiples of $\frac{1}{M \times T_c}$ rather than the $\frac{1}{T_c}$ of the un-padded transform.

Note: It is, of course, possible to do a zero-padding which isn't an integer multiple of the original data length. Under these conditions, the spectrum peaks generally won't coincide with those which would have arisen if a transform equal to the data length had been used. This is often not a problem with the DFT since the original signal wasn't actually periodic anyway (i.e. $f_0.T_c$ is not an integer)! A particular case which frequently arises is where we are using an FFT (whose length is a power of 2) but the original data length isn't a power of 2.

Note: The term "resolution" is sometimes used in two conflicting ways by different authors:

- the ability to distinguish two close frequencies of equal amplitude by detecting two peaks in the transform;
- the sample spacing along the frequency axis.

Zero padding does not improve resolution in sense 1 though it obviously improves it in sense 2! We will restrict the usage of the term "resolution" to sense 1.

Exercises:

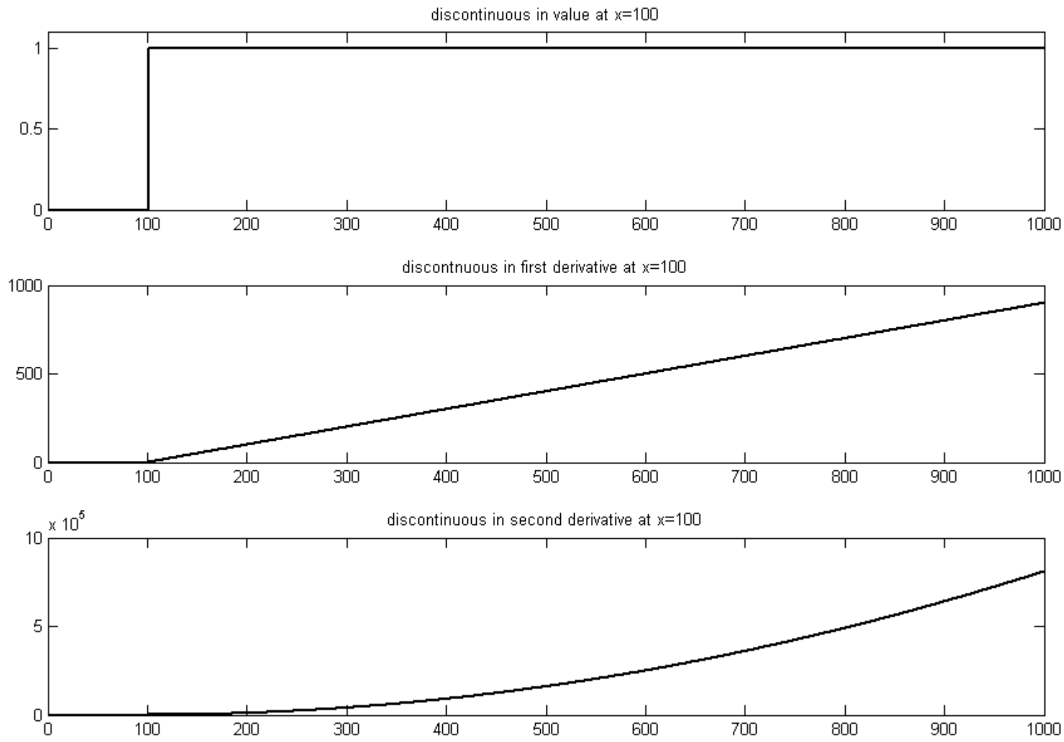
- What effect does having a longer data length have on the spectral leakage of a frequency, on both the main lobe and the sidelobes?
- What effect will zero padding by a factor M have on the Scalloping Loss?

2.2 Windowing

The problems associated with spectral leakage in a DFT can be quite severe. It is therefore of interest to attempt to ameliorate them. The starting point of this is the recognition that taking only a finite chunk of a waveform can be viewed as multiplying it by a rectangular window which equals unity over the data length and is zero elsewhere. The step transition at the ends of the rectangular window give rise to the discontinuity which can be held responsible for the spectral leakage.

An alternative approach involves multiplying the data segment by some other carefully chosen function w(t) which tapers to zero at its ends, thereby eliminating the discontinuity in value (but still having discontinuity somewhere in the derivatives). Windowing a signal has the effect of replacing each individual frequency component in the original spectrum by a $W(f)$ (the Fourier transform of $w(t)$) which we can control to a certain extent by choosing $w(t)$. In particular we can moderate the effect of the discontinuity at the point where the original data ends and then re-starts (or where zero-padding starts if it is being used).

The modified discontinuity has the following impact on the spectrum sidelobes: the order of the discontinuity determines how rapidly the sidelobes fall off as you move away from the main lobe. If the data is discontinuous in value (like the rectangular window), so that the ends do not join up, then the sidelobes fall off at 6dB per octave. If, instead of rectangular windowing (i.e. doing nothing to compensate for the discontinuity), we arrange for the periodically extended data to join up at the ends (by forcing it to zero) whilst being discontinuous in its first derivative, then the sidelobes fall off as $1/f_2$ i.e. 12 dB per octave, and so on for higher orders of discontinuity. The following figure illustrates these types of discontinuity (all the discontinuities are at $x = 100$).



This process of forcing the data to join (by forcing it towards zero at both ends) is the essence of windowing. It aims to reduce the sidelobes and hence to improve the dynamic range of the transform (i.e. its ability to see small signals in the presence of large ones). Unfortunately the law of unintended consequences applies:

- improving the dynamic range (by reducing the sidelobe levels) degrades the resolution (by making the main lobe wider)

Exactly what trade-off between resolution and dynamic range we choose will depend on our application. This why there is such a bewildering variety of windows available: they all represent different compromises.

To see how sidelobe fall-off arises we need to consider the envelope of the spectrum of the window because this is what determines the amplitudes of the sidelobes.

2.3 The Rectangular Window

A rectangular window of length T has a (continuous) Fourier transform which can be written in the form:

$$W_{rect}(f) = T\text{sinc}(fT) = [T] \times [\sin(\pi fT)] \times \left[\frac{1}{\pi fT}\right]$$

This is an example of a general situation where the spectrum of a symmetric window can be written as

$$W(f) = [T] \times [g] \times [\sin(\pi fT)] \times [e(fT)]$$

where

- g is the “coherent gain” of the window ()
- $|e(fT)|$ is the “envelope” of the window’s spectrum.

Note: The term “coherent gain” refers to the average value of the window w(t) and indicates the scaling that will be applied to the amplitude of a frequency which is exactly equal to one of the spectral lines which are harmonics of the original data length i.e. one of the frequencies which would not experience any spectral leakage provided no zero-padding were used

Note: The “envelope” of a window’s spectrum is analogous to the (time-domain) envelope of an amplitude-modulated carrier in radio communications.

In the case of a rectangular window, $g = 1$ and if we work in bins rather than Hz (using a variable $x = fT$), the envelope for the spectrum of the rectangular window is given by:

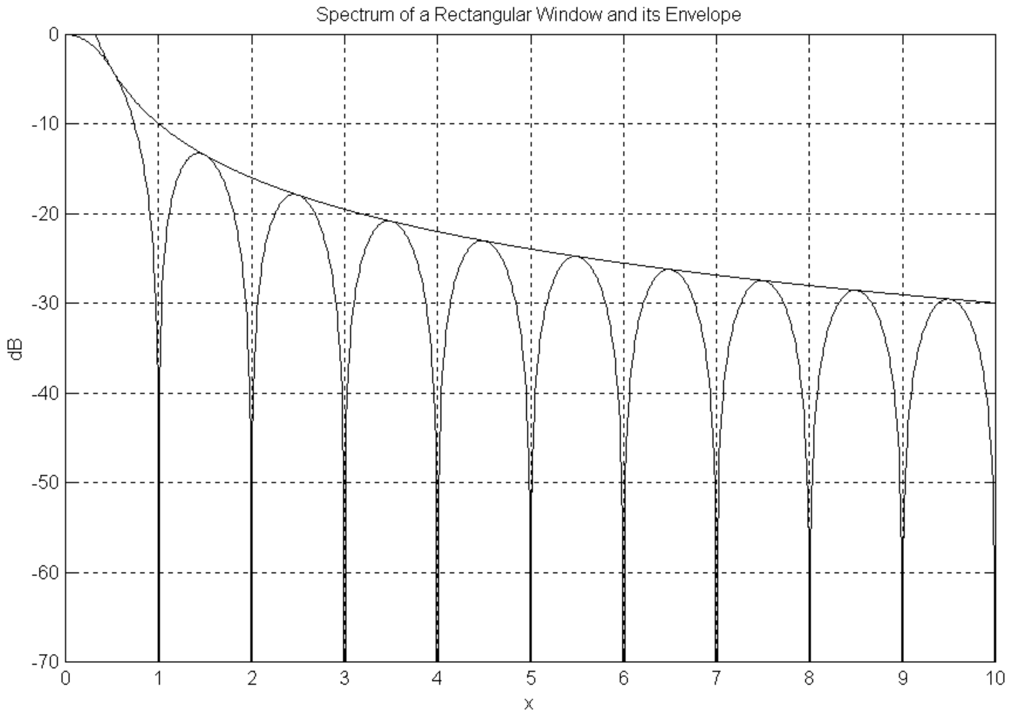
$$env_{rect}(x) = \frac{1}{\pi|x|}$$

and it is just this $1/x$ shape which leads to the 6dB/octave slope for the spectrum of this window.

Exercise

- Show that $1/x$ is indeed the same as 6 dB per octave.

The rectangular window spectrum and its envelope are shown below.



3 Some Common Windows

We have previously analysed the properties of the rectangular window. This has high sidelobes and a slow fall-off rate of the sidelobes against frequency. Other windows do better on these parameters, albeit at the expense of a wider main lobe.

3.1 Bartlett

If the data being analysed using the DFT arrives sequentially and is to be processed as it arrives (e.g. a spectrum analyser or some forms of filtering) then we may want to be able to overlap our DFTs so as not to lose any information. A simple example of a window which allows this is the Bartlett (triangular) window which overlaps exactly with a shift of 50% of the window length. The Bartlett window has the advantage of being very simple to calculate:

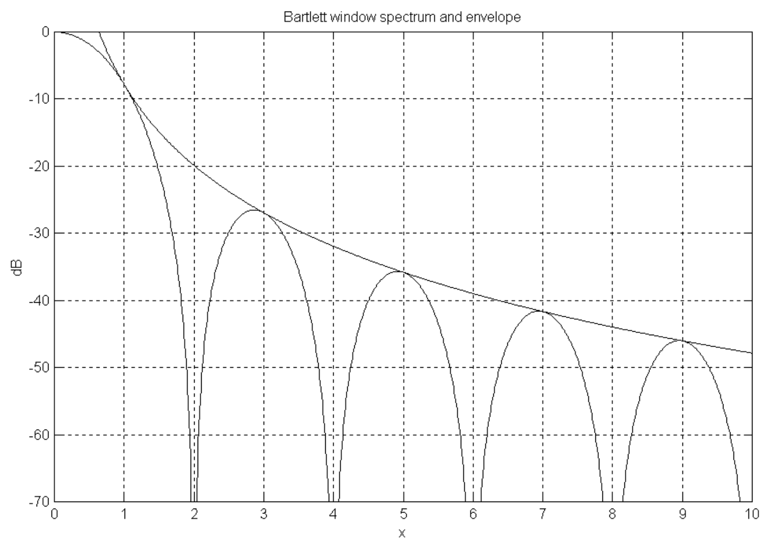
$$w[n] = 1 - \frac{2n - N}{N} : n = 0, ..., N - 1$$

The *spectrum* of the Bartlett window is best viewed as a sine-squared wave multiplied by an envelope of the form:

$$env_{Bart}(x) = \frac{4}{\pi^2 x^2}$$

where $x = fT$.

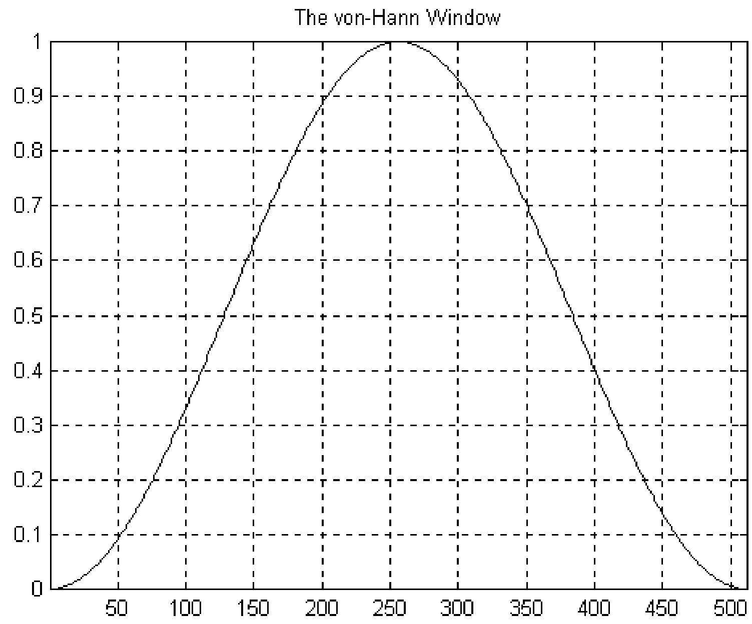
This $1/x^2$ asymptotic trend for the Bartlett window gives it a fall-off rate of 12 dB/octave. Unlike the rectangular window, it is continuous in value at its ends but discontinuous in its first derivative. Its spectrum is illustrated below:



3.2 Von Hann

Another window which shares the 50% overlap property with the Bartlett but whose sidelobes fall off at 18dB per octave is the von-Hann (sin2) window (also known as the "Hann" window and, confusingly, the "Hanning" window). This is continuous in its first derivative but discontinuous in its second. Needless to say, its main lobe is wider than the Bartlett (though not by much). Its formula is:

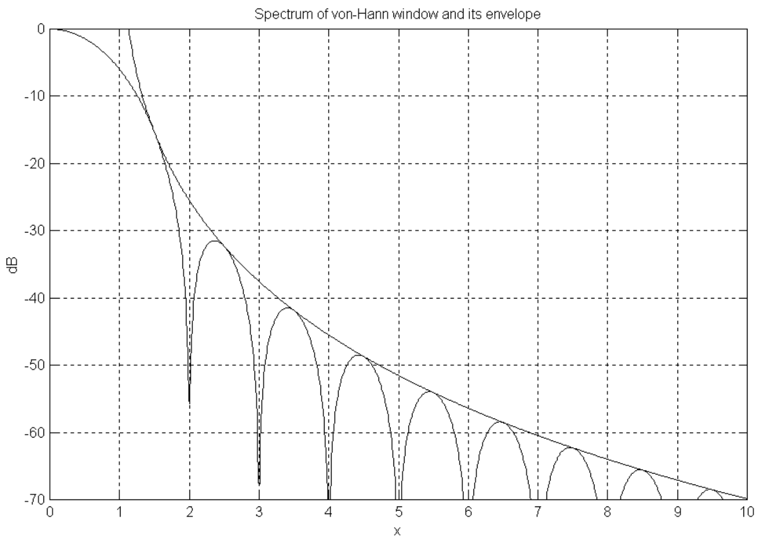
$$w[n] = 0.5 \left(1 - cos \left(\frac{2\pi n}{N}\right)\right) : n = 0, ..., N - 1$$



The envelope of the spectrum of the von-Hann window is:

$$env_{Von-Hann}(x) = \frac{1}{\pi|x(x^2-1)|}$$

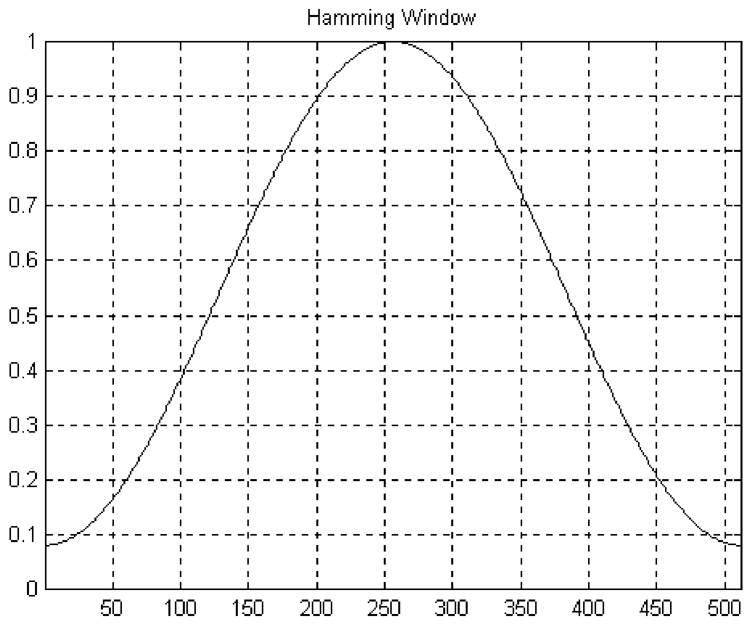
which tends asymptotically to $\frac{1}{\pi x^3}$ for large x ($>> 1$), giving rise to the 18dB/octave slope:



3.3 Hamming

Although the von-Hann window's rapid fall-off of sidelobes is desirable because it reduces the problems of long-distance spectral leakage, for some applications it may be more important to kill off the largest sidelobe, which is generally but not always the first, i.e. the one closest to the main lobe. The Hamming window does this. It adds a 8% of the rectangular window to 92% of the von-Hann window, which has the effect of cancelling out the first sidelobe. This makes it once more discontinuous in value (i.e. the sidelobes have only a 6dB/octave fall off) but its biggest sidelobe is lower than the von-Hann's. Again it is a different compromise. Its formula is:

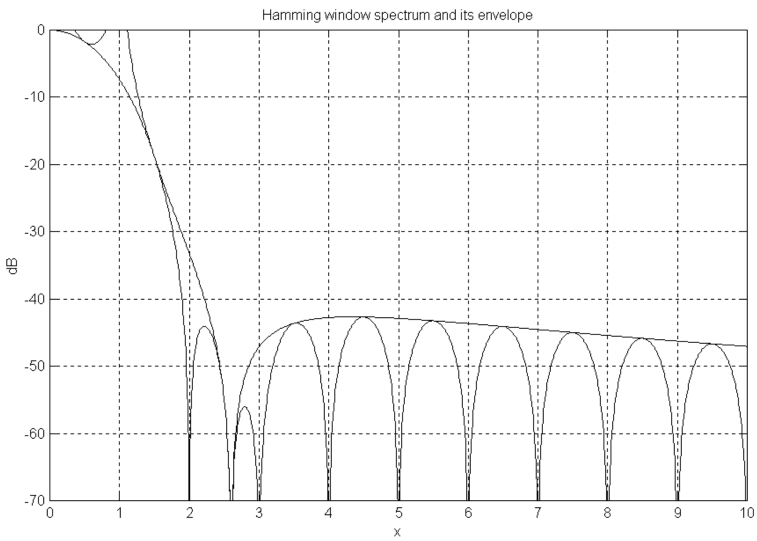
$$w[n] = 0.54 - 0.46cos\left(\frac{2\pi n}{N}\right) : n = 0, ..., N-1$$



The envelope of its spectrum is:

$$env_{Hamming}(x) = \frac{|\left(\frac{0.08}{0.54}\right)x^2 - 1|}{\pi|x(x^2-1)|}$$

The envelope for the Hamming window tends asymptotically to $\frac{0.148...}{\pi|x|}$ for large x ($>> 1$), which gives rise to only 6dB/octave sidelobe fall-off, as with the rectangular window, but the distant sidelobes are 16.6dB lower than for the rectangular because of the 0.148 factor. Also, the expression in its numerator gives rise to additional nulls at $x = \pm 2.598$ which is what suppresses the first sidelobe:

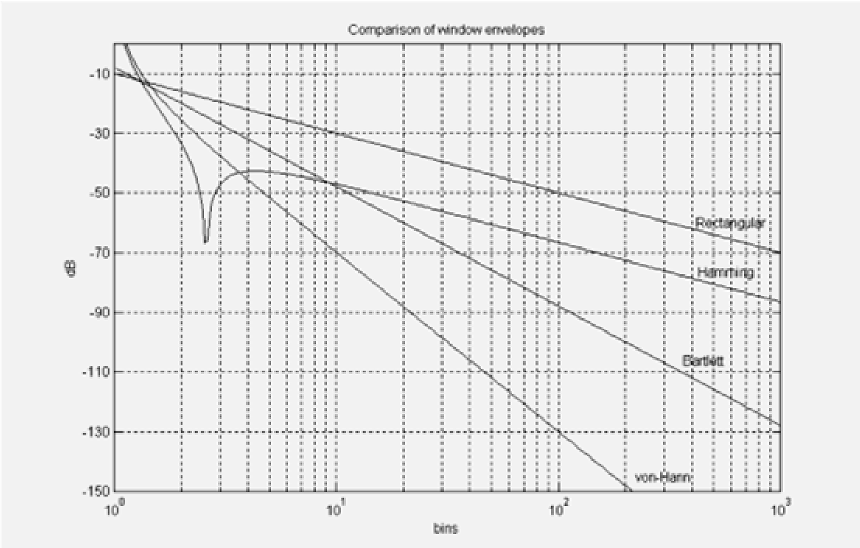


4 Windows' Asymptotes

The envelopes of DFT/FFT windows' spectra (defined previously) control the behaviour of their sidelobes. If we plot them on a log/log graph we can see their asymptotic behaviour (see figure below).

- The Rectangular & Hamming windows both fall off asymptotically at 20dB per decade, which is 6dB per octave, but the Hamming is 16.6dB lower and has an additional null at $x = 2.598$ which helps suppress the first sidelobe.

- The Bartlett falls off at 40dB per decade, which is 12dB per octave.
- The Von-Hann falls off at 60dB per decade, which is 18dB per octave.



This is reflected in their formulas:

- Rectangular

$$e_R(x) = \frac{1}{\pi|x|} \approx \frac{0.3183}{x}$$

  for positive x

- Hamming

$$e_H(x) = \frac{\frac{0.08}{0.54}x^2 - 1}{\pi|x(x^2-1)|} \approx \frac{0.04716}{x}$$

  for large positive x

- Bartlett

$$e_B(x) = \frac{4}{\pi^2 x^2} \approx \frac{0.40528}{x^2}$$

  for large x

- Von-Hann

$$e_{V-H}(x) = \frac{1}{\pi|x(x^2-1)|} \approx \frac{0.3183}{x^3}$$

  for large positive x

Exercises

- Check from these formulas that the Hamming window's asymptotic level is indeed 16.6 dB lower than that of the rectangular window.
- Check that these four asymptotic formulas do indeed yield the fall off rates in dB/decade and dB/octave which are stated above.

5 Choice of Window

Every DFT window is a compromise between main lobe width and side lobe level i.e. between resolution and dynamic range. The requirements in a particular situation (together with the actual number of data points available!) will determine the choice of window. If a finer frequency sampling is desired then a zero-padded transform can be adopted, in which case the time available to carry out the transform (in real-time applications) may also become a constraining factor. The following table of window parameters will help us summarise the situation:

Window	Highest side-lobe (dB)	side lobe fall off (dB/oct)	coherent gain	3dB B/W (bins)	6dB B/W (bins)	Scalloping loss (dB)
Rectangular	-13	-6	1	0.89	1.21	3.92
Bartlett	-27	-12	0.5	1.28	1.78	1.82
von Hann	-32	-18	0.5	1.44	2	1.42
Hamming	-43	-6	0.54	1.3	1.81	1.78
Dolph-Chebyshev ($\alpha = 2.5$)	-50	0	0.53	1.33	1.85	1.7

Note: The Chebyshev (or Dolph-Tchebychev or Dolph-Chebyshev or ...) window is an asymmetrical window originally invented for use in radar as a way of minimising the sidelobes of an antenna's beam-shape. The derivation of this is very complex but the result is simple: its sidelobes don't fall off! This may seem like bad news but the good news is that they are all equally low. In other words we choose to tolerate a certain level of sidelobes (everywhere) and we regain some resolution since this is the window with the narrowest main lobe for a given sidelobe level. It is actually defined by its shape in the frequency domain and its time-domain form has to be calculated using the inverse DFT.

Exercises:

- Which window would you use for the following:
  - ◇ finding the amplitude & frequency of a single sine wave
  - ◇ resolving two equal-amplitude sine waves
  - ◇ identifying a number of frequencies known to be widely spaced but with unknown different amplitudes
  - ◇ processing a completely general signal for display on a digital spectrum analyser
- Which of the above would benefit from zero-padding?

6 Derivations of Windows' Spectra

6.1 Rectangular Window

The rectangular window is defined as

$$w_R(t) = \begin{cases} 1 & |t| \leq \frac{T}{2} \\ 0 & else \end{cases}$$

Substituting this into the Fourier Transform we get

$$W_R(f) = \int_{-\infty}^{+\infty} w_R(t) e^{-j2\pi ft} dt = \int_{-\frac{T}{2}}^{+\frac{T}{2}} 1 \times e^{-j2\pi ft} dt$$

Since the limits of the integral are symmetric, we need only consider the even part of the integrand (the odd part will cancel itself out)

$$\begin{aligned} W_R(f) &= \int_{-\frac{T}{2}}^{+\frac{T}{2}} \cos(2\pi ft) dt = \left[\frac{\sin(2\pi ft}{2\pi f}\right]_{-\frac{T}{2}}^{+\frac{T}{2}} \\ &= \frac{2\sin(\pi fT)}{2\pi f} = T\frac{\sin(\pi fT)}{\pi fT} = T\mathrm{sinc}(fT) \end{aligned}$$

Note: This is the spectrum of the continuous rectangular window. If we wanted to find the spectrum of the discrete rectangular window of length N, we would need to z-transform a signal consisting of N ones followed by infinitely many zeros.

If you're interested, this is

$$W_R(z) = \sum_{k=0}^{k=N-1} 1 \times z^{-k} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

From this we could find a $W(f)$ by replacing z by $e^{j2\pi fT}$.

Note: Because the discrete window started at zero rather than being symmetric about zero, the result will be complex rather than real.

6.2 Bartlett Window

The continuous-time Bartlett window is defined as

$$w_B(t) = \begin{cases} 1 - \frac{|2t|}{T} & |t| \leq \frac{T}{2} \\ 0 & otherwise \end{cases}$$

Substituting this into the Fourier transform we get

$$W_B(f) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left(1 - \frac{|2t|}{T}\right) e^{-j2\pi ft} dt$$

As it stands, this integral is impossible because of the modulus sign inside the integral. However, because the limits are symmetric we can again omit the odd-part of the integrand. Now $\left(1 - \frac{|2t|}{T}\right)$, the part from $w_B(t)$, has even symmetry so the odd part of the integrand will be $\left(1 - \frac{|2t|}{T}\right)$ multiplied by the odd part of $e^{-j2\pi ft}$, which is $-j\sin(2\pi ft)$. This is the bit which will vanish when integrated between symmetric limits, which means that we can write our integral as

$$W_B(f) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left(1 - \frac{|2t|}{T}\right) \cos(2\pi ft) dt$$

We still have the modulus sign in the integral but now we have an integrand which is symmetric about zero so that the part of the integral to the left of zero is the same as the part to the right of zero, which means we can write

$$W_B(f) = 2\int_{0}^{+\frac{T}{2}} \left(1 - \frac{|2t|}{T}\right) \cos(2\pi ft) dt$$

and now that we are only dealing with non-negative values of t, we don't need the modulus sign any more:

$$W_B(f) = 2\int_{0}^{+\frac{T}{2}} \left(1 - \frac{2t}{T}\right) \cos(2\pi ft) dt$$

We now have an integral which is the product of two functions, so we need to make use of the method of integration by parts:

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du$$

For u we need to pick the element which gets simpler when differentiated, i.e.

$$u = \left(1 - \frac{2t}{T}\right) \implies du = -\frac{2}{T} dt$$

Therefore

$$dv = \cos(2\pi ft)dt \implies v = \frac{\sin(2\pi ft)}{2\pi f}$$

From these we get

$$W_B(f) = 2\left[\left(1 - \frac{2t}{T}\right) \frac{\sin(2\pi ft)}{2\pi f}\right]_0^{+\frac{T}{2}} - 2\int_0^{+\frac{T}{2}} \frac{\sin(2\pi ft)}{2\pi f} \left(-\frac{2}{T}\right) dt$$

The first term will vanish because the $\sin()$ term is zero at the lower limit and the other part is zero at the upper limit. Thus our expression reduces to

$$W_B(f) = \frac{2}{\pi fT} \int_0^{+\frac{T}{2}} \sin(2\pi ft) dt = \frac{2}{\pi fT} \left[\frac{-\cos(2\pi ft)}{2\pi f}\right]_0^{+\frac{T}{2}} = \frac{1}{\pi^2 f^2 T} \left(1 - \cos(\pi fT)\right)$$

We can apply a compound angle formula to this

$$1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$$

So

$$W_B(f) = \frac{2}{\pi^2 f^2 T} \sin^2\left(\frac{fT}{2}\right) = \frac{T}{2} \frac{\sin^2\left(\frac{\pi f T}{2}\right)}{\left(\frac{\pi f T}{2}\right)^2}$$

i.e.

$$W_B(f) = \frac{T}{2} \mathrm{sinc}^2\left(\frac{fT}{2}\right)$$

Note: Looking at the penultimate line of this derivation we can see that the Bartlett window has an envelope of

$$\frac{1}{\left(\frac{\pi f T}{2}\right)^2} = \frac{4}{\pi^2 f^2 T^2}$$

or, in terms of bins (x=fT):

$$env_B(x) = \frac{4}{\pi^2 x^2}$$

6.3 Von-Hann Window

The continuous version of the Von-Hann window is:

$$w_{VH}(t) = \begin{cases} 0.5 + 0.5cos\left(\frac{2\pi t}{T}\right) & |t| \leq \frac{T}{2} \\ 0 & otherwise \end{cases}$$

Substituting this into the Fourier transform yields

$$W_{VH}(f) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left(0.5 + 0.5cos\left(\frac{2\pi t}{T}\right)\right) e^{-j2\pi f t} dt$$

One way to solve this integral would be to use the argument based on symmetric limits to replace the exponential by a cosine and then use compound angle formulas. However, we will, instead, use the Euler formula

$$\begin{aligned} W_{VH}(f) &= \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left(0.5 + 0.25e^{j\frac{2\pi t}{T}} + 0.25e^{-j\frac{2\pi t}{T}}\right) e^{-j2\pi f t} dt \\ &= 0.5 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j2\pi f t} dt + 0.25 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{j\frac{2\pi t}{T}} e^{-j2\pi f t} dt + 0.25 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j\frac{2\pi t}{T}} e^{-j2\pi f t} dt \\ &= 0.5 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j2\pi f t} dt + 0.25 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j2\pi\left(f - \frac{1}{T}\right)t} dt + 0.25 \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j2\pi\left(f + \frac{1}{T}\right)t} dt \end{aligned}$$

Comparing this with the spectrum of the rectangular window we can immediately write down

$$W_{VH}(f) = 0.5T\mathrm{sinc}(fT) + 0.25T\mathrm{sinc}(fT - 1) + 0.25T\mathrm{sinc}(fT + 1)$$

since all three integrals are in the same form except where the first has f the others have $f \pm \frac{1}{T}$.

Exercise Work through the first integral to confirm this result.

6.4 Hamming Window

The continuous version of the Hamming window is:

$$w_H(t) = \begin{cases} 0.54 + 0.46cos\left(\frac{2\pi t}{T}\right) & |t| \leq \frac{T}{2} \\ 0 & otherwise \end{cases}$$

Comparing this with the spectrum of the Von-Hann window we can immediately write down

$$W_H(f) = 0.54T\mathrm{sinc}(fT) + 0.23T\mathrm{sinc}(fT - 1) + 0.23T\mathrm{sinc}(fT + 1)$$

Exercise Work through the formulas in the same way as for the Von-Hann window