Introduction to (Discrete) Fourier Analysis

Digital Signal Processing

Notes

Contents

Notes

Continuous vs Discrete
Periodic vs Aperiodic

Notes

- A continuous signal is a signal that has not been sampled.
- A discrete signal is a signal that has been sampled.
- A periodic signal is a signal that repeats.
- An aperiodic signal does not repeat and continues forever ($-\infty$ to ∞).

Discrete-time Fourier Analysis

Notes

$\times 4$ **Fourier techniques**:

- Fourier Transform: continuous and aperiodic (CA)
- Fourier Series: continuous and periodic (CP)
- Discrete-time Fourier Transform: discrete and aperiodic (DA)
- Discrete-time Fourier Series: discrete and periodic (DP)

Discrete-time Fourier Analysis

Notes

- Fourier Transform: continuous and aperiodic (CA)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\mathrm{d}t \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\mathrm{d}\omega$$

- Fourier Series: continuous and periodic (CP)

$$x_k = \frac{1}{T}\int_{a}^{a+T} x(t)\exp(-j2\pi kt/T)\mathrm{d}t \qquad x(t) = \sum_{k=-\infty}^{\infty} x_k \exp(j2\pi kt/T)$$

- Discrete-time Fourier Transform: discrete and aperiodic (DA)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n) \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\exp(j\Omega n)\mathrm{d}\Omega.$$

- **DFT**: Discrete-time Fourier Series: discrete and periodic (DP)

$$X[k] = \sum_{n=0}^{N-1} x[n]\exp(-j2\pi kn/N) \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\exp(j2\pi kn/N)$$

Periodic Digital Signals Spectra

Notes

- Periodic digital signal $x[n]$ can be represented by Fourier Series
- Line spectrum coefficients can be found using the *analysis equation*:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right)$$

where $X[k]$ is the k spectral component or harmonic and N is the number of sample values in each period of the signal.

- Regeneration of the original signal $x[n]$ can be found using the *synthesis equation*:

$$x[n] = \sum_{k=0}^{N-1} X[k]\exp\left(\frac{j2\pi kn}{N}\right).$$

In fact the $\frac{1}{N}$ can be used on the regeneration of the signal instead, which is what we will normally do except for this lecture.

Finding Line Spectra

- Remember Euler's identity:

$$\exp\left(\frac{-j2\pi kn}{N}\right) = \underbrace{\overbrace{\cos\left(\frac{2\pi kn}{N}\right)}^{\text{real part}}}_{\mathrm{Re}(\cdot)} \underbrace{\overbrace{-j\sin\left(\frac{2\pi kn}{N}\right)}^{\text{imaginary part}}}_{\mathrm{Im}(\cdot)}$$

$$\mathrm{Re}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = \cos\left(\frac{2\pi kn}{N}\right)$$

$$\mathrm{Im}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = -j\sin\left(\frac{2\pi kn}{N}\right)$$

- The real $\mathrm{Re}(X[k])$ and imaginary $\mathrm{Im}(X[k])$ components can be calculated individually

Notes

Finding Line Spectra *with a computer*

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right)$$

Two loops:

- A loop over n
- A loop over k

Pseudo-code (*assumes $X[n]$ is real only, no imaginary components*):

1. Given $x[n]$ and N:
2. Create $X[k]$ size N and set all elements to zero $X[k] = 0$
3. For k=0 to N-1 % *outer loop*
   1. For n=0 to N-1 % *inner loop*
      1. $\mathrm{Re}(X[k]) = X[k] + x[n] \times \cos(2\pi kn/N)$ % *real part*
      2. $\mathrm{Im}(X[k]) = X[k] - x[n] \times \sin(2\pi kn/N)$ % *imaginary part*
   2. End For
   3. Let $\mathrm{Re}(X[k]) = \mathrm{Re}(X[k])/N$ and Let $\mathrm{Im}(X[k]) = \mathrm{Im}(X[k])/N$
4. End For

Notes

Finding Line Spectra *Example 1*

Let $N = 2$ and $x[0] = 7$, $x[1] = 1$. Find $X[k]$. To start find the real part

- *To start* Let $n = 0$ and $k = 0$, then

  $\mathrm{Re}(X[0]) = \mathrm{Re}(X[0]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 0 \times 0/2) = 7$

- Increment n, so $n = 1$, then

  $\mathrm{Re}(X[0]) = \mathrm{Re}(X[0]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 0 \times 1/2) = 8$

- Divide by N: $\mathrm{Re}(X[0]) = \mathrm{Re}(X[0])/N = 4$
- *If we increment n any more then it will be larger than $N - 1$ so* Let $n = 0$ and increment k so that $k = 1$, then

  $\mathrm{Re}(X[1]) = \mathrm{Re}(X[1]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 1 \times 0/2) = 7$

- Increment n, so $n = 1$, then

  $\mathrm{Re}(X[1]) = \mathrm{Re}(X[1]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 1 \times 1/2) = 6$

- Divide by N: $\mathrm{Re}(X[1]) = \mathrm{Re}(X[1])/N = 3$

So the real part of $X[k]$ is given by $\mathrm{Re}(X[0]) = 4$ and $\mathrm{Re}(X[1]) = 3$.

Notes

Finding Line Spectra *Example 1*

Finding the imaginary part of $X[k]$...

- *To start* Let $n = 0$ and $k = 0$, then

  $$\mathrm{Im}(X[0]) = \mathrm{Im}(X[0]) - x[0] \times \sin(2\pi kn/2) = 0 - 7 \times \sin(2\pi 0 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

  $$\mathrm{Im}(X[0]) = \mathrm{Im}(X[0]) - x[1] \times \sin(2\pi kn/2) = 0 - 1 \times \sin(2\pi 0 \times 1/2) = 0$$

- Divide by N: $\mathrm{Im}(X[0]) = \mathrm{Im}(X[0])/N = 0$
- *If we increment n any more then it will be larger than $N - 1$ so* Let $n = 0$ and increment k so that $k = 1$, then

  $$\mathrm{Im}(X[1]) = \mathrm{Im}(X[1]) - x[0] \times \sin(2\pi kn/2) = 0 - 7 \times \sin(2\pi 1 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

  $$\mathrm{Im}(X[1]) = \mathrm{Im}(X[1]) - x[1] \times \sin(2\pi kn/2) = 0 - 1 \times \sin(2\pi 1 \times 1/2) = 0$$

- Divide by N: $\mathrm{Im}(X[0]) = \mathrm{Im}(X[0])/N = 0$

So the imaginary part of $X[k]$ is given by $\mathrm{Im}(X[0]) = 0$ and $\mathrm{Im}(X[1]) = 0$.

Notes

Finding Line Spectra *Example 1*

So the line spectra for signal:

$$x = (7 \;\; 1)$$

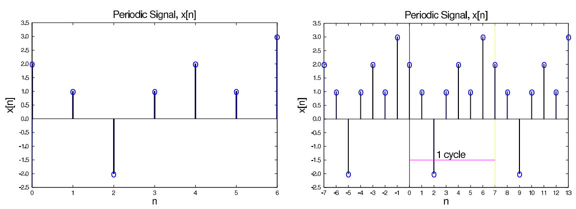
are

$$a = (4 \;\; 3).$$

Notes

Finding Line Spectra *Example 2 - by computer*

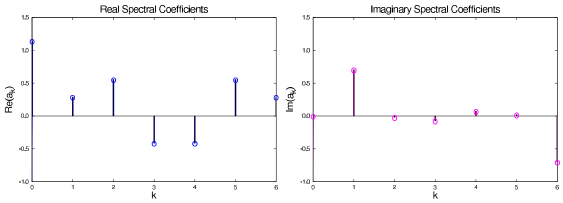
This time let $N = 7$ so that $x = (+2 \;\; +1 \;\; -2 \;\; +1 \;\; +2 \;\; +1 \;\; +3)^{\mathrm{T}}$ or $x[0] = +2$, $x[1] = +1$, ..., $x[6] = +3$. A plot of this *periodic* signal is given by



Notes

Finding Line Spectra
Example 2 - by computer

The line spectra in this case are found by a computer program using the algorithm described earlier.

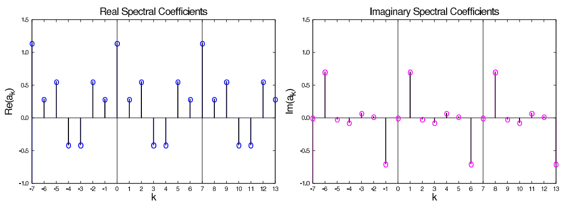


Notice: (for the real coefficients) the mirror image, where $X[1] = X[6]$ and $X[2] = X[5]$ etc. For the imaginary coefficients a similar effect, except for a change of sign, *e.g.* $X[1] = -X[6]$.

Notes

Finding Line Spectra
Example 2 - by computer

The line spectra are also periodic. The line spectra have the same periodicity as the original signal.



Notes

Magnitude and Phase

- The magnitude can be calculated with:

$$\mathrm{Mag}(X[k]) = \sqrt{\mathrm{Re}(X[k])^2 + \mathrm{Im}(X[k])^2}$$

- and the phase with:

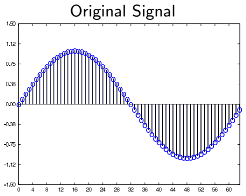
$$\phi(X[k]) = \tan^{-1}\left(\frac{\mathrm{Im}(X[k])}{\mathrm{Re}(X[k])}\right).$$

- The magnitude indicates the relative strength of the signal at different frequencies;
- The phase indicates the phase angle of the signal at different frequencies.

Notes

Magnitude and Phase *Example*

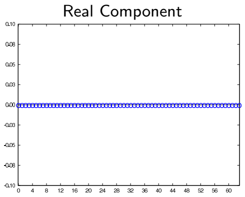
Original Signal



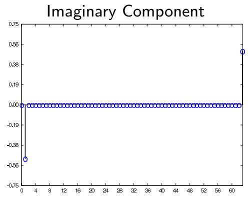
$$x[n] = \sin(2\pi n/64)$$

The real and imaginary components contain separate information about the frequency content of the signal. The sine function is not present in the real part.

Real Component



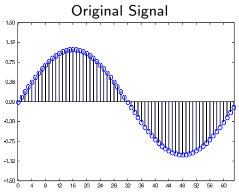
Imaginary Component



Notes

Magnitude and Phase *Example*

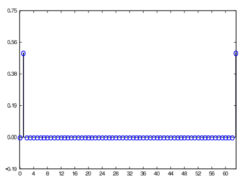
Original Signal



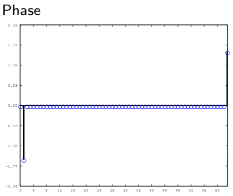
$$x[n] = \sin(2\pi n/64)$$

The magnitude provides a convenient overview of the frequency content of the signal.

Magnitude



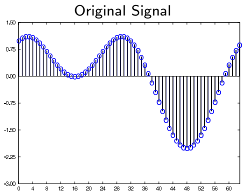
Phase



Notes

Magnitude and Phase *Example*

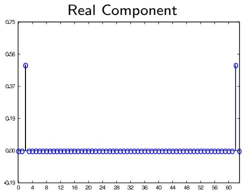
Original Signal



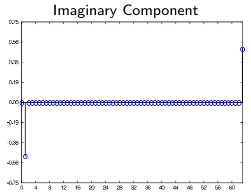
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine part of the signal is present in the real part.

Real Component



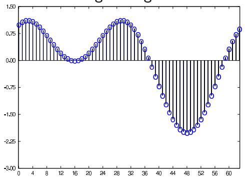
Imaginary Component



Notes

Magnitude and Phase *Example*

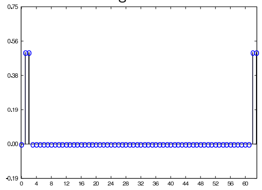
Original Signal



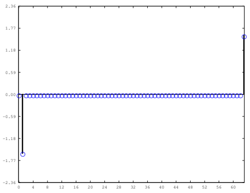
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine frequency is not present in the phase as it has zero phase. The sine part is present as it has $\pm\pi/2$ phase.

Magnitude



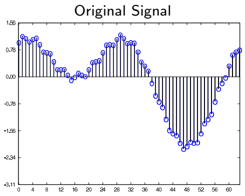
Phase



Notes

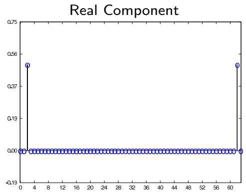
Magnitude and Phase *Example*

Original Signal

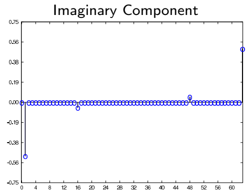


$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$
$$+0.1\sin(2\pi n/4)$$

Real Component



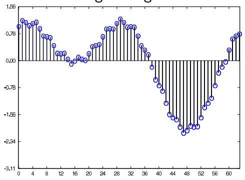
Imaginary Component



Notes

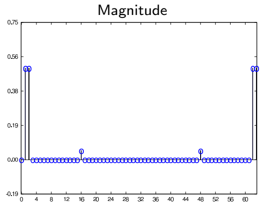
Magnitude and Phase *Example*

Original Signal

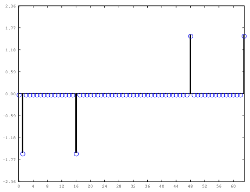


$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$
$$+0.1\sin(2\pi n/4)$$

Magnitude



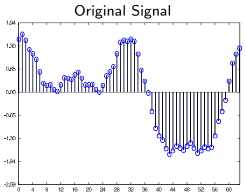
Phase



Notes

Magnitude and Phase *Example*

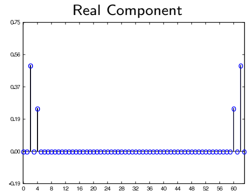
Original Signal



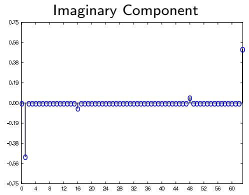
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

$$+0.1\sin(2\pi n/4) + 0.5\cos(2\pi n/16)$$

Real Component



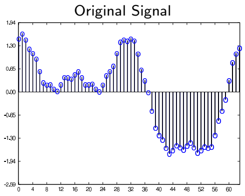
Imaginary Component



Notes

Magnitude and Phase *Example*

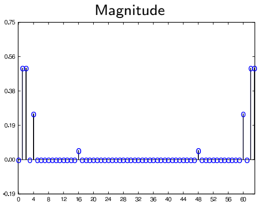
Original Signal



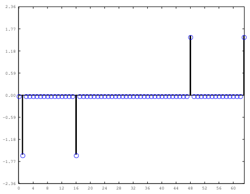
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

$$+0.1\sin(2\pi n/4) + 0.5\cos(2\pi n/16)$$

Magnitude



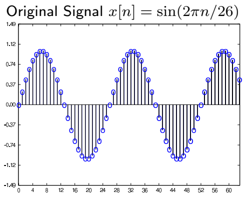
Phase



Notes

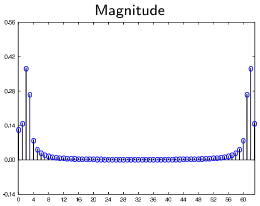
Discontinuities

Original Signal $x[n] = \sin(2\pi n/26)$

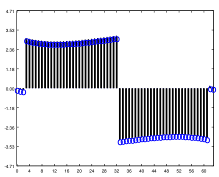


- This signal is not periodic as the sine function is analyzed over $\sim 2\frac{2}{5}$ periods. The end of the signal does not join up with the beginning, resulting in a discontinuity.
- The discontinuity has many frequency components with different phases.

Magnitude



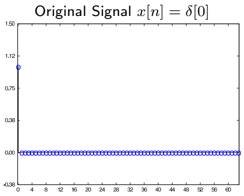
Phase



Notes

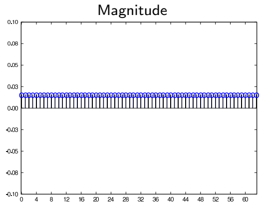
Impulse Function

Original Signal $x[n] = \delta[0]$

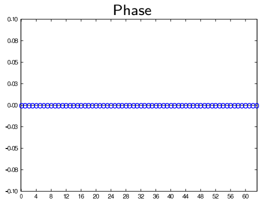


- The line spectra of a (periodic) impulse function is composed of all frequencies
- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Zero phase because the function is even, *i.e.* $x[n] = x[-n]$, frequency response composed cosine functions only.

Magnitude



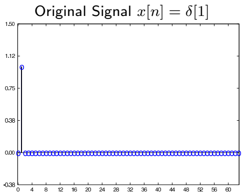
Phase



Notes

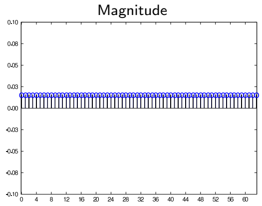
Impulse Function

Original Signal $x[n] = \delta[1]$

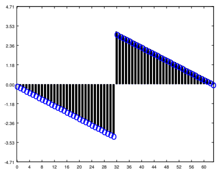


- The line spectra of a (periodic) impulse function is composed of all frequencies
- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Phase components present when odd function, *i.e.* $x[n] = -x[-n]$, composed of sine functions only.

Magnitude



Phase



Notes

Useful Properties

Parseval's theorem

Equates the total power of a signal in the time and frequency domains:

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \sum_{k=0}^{N-1}(\mathrm{Mag}(X[k]))^2$$

Example

Impulse function, $\delta[0] = 1$

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \frac{1}{N}$$

and

$$\sum_{k=0}^{N-1}(\mathrm{Mag}(X[k]))^2 = N \times \left(\frac{1}{N}\right)^2 = \frac{1}{N}$$

which are equal.

Notes

Other Useful Properties

$x[n] \leftrightarrow X[k]$ symbolizes $X[k]$ is the discrete Fourier Series of $x[n]$.

- **Linearity:**
  If $x_1[n] \leftrightarrow a_1[k]$ and $x_2[n] \leftrightarrow a_2[k]$ then

$$w_1 x_1[n] + w_2 x_2[n] \leftrightarrow w_1 a_1[k] + w_2 a_2[k]$$

- **Time-shifting (invariance):**
  If $x[n] \leftrightarrow X[k]$ then

$$x[n - n_0] \leftrightarrow X[k] \exp(-j2\pi k n_0/N),$$

  i.e. The shift is just a phase shift and does not affect the magnitude.

Notes

Aperiodic Digital Sequences

- Different analysis and synthesis equations are necessary for aperiodic sequences, known as the **Discrete Time Fourier Transform** (for aperiodic digital sequences)

$$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n)$$

- and the **inverse Fourier Transform** for aperiodic digital sequences

$$x[n] = \mathcal{F}^{-1}(X(\Omega)) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \exp(j\Omega n) \mathrm{d}\Omega.$$

- *Note: $X(\Omega)$ is a continuous function. It is also periodic which is a result of the ambiguities in discretely sampled signals.*

Notes

Aperiodic Digital Sequences

Comparing the Fourier Transform:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n), \qquad (1)$$

with the Fourier Series analysis equations:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-j2\pi k n}{N}\right). \qquad (2)$$

We can see that $\Omega = \frac{2\pi k}{N}$. n has also been taken to $\pm\infty$ and because of this the Fourier Transform is no longer divided by N (otherwise $X(\Omega)$ would be zero) so that $X(\Omega)$ can in some way be equated with $NX[k]$.

Notes

Fourier Transform *Boxcar Example*

The Fourier Transform of the impulse function $\delta[0]$:

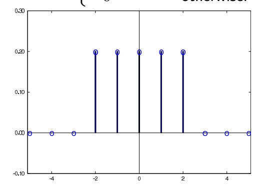
$$
\begin{aligned}
x[n] &= \delta[0] \\
\therefore\ X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[0] \exp(-j\Omega n) \\
&= \exp(-j\Omega \times 0) \\
&= 1.
\end{aligned}
$$

In other words, the Fourier Transform of an impulse function consists of all frequencies. Similar to the Fourier Series representation of a periodic impulse function, calculated earlier.

Notes

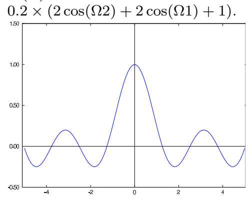
Fourier Transform *Example*

If $x[n] = \begin{cases} 0.2 & \text{if} \quad -2 \le n \le 2, \\ 0 & \text{otherwise.} \end{cases}$



$\mathcal{F}$
$\Rightarrow$

$X(\Omega) =$
$0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).$



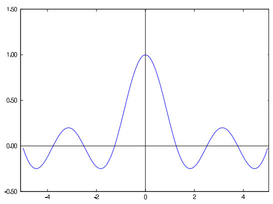
Then

$$
\begin{aligned}
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n) = \sum_{n=-2}^{2} 0.2 \exp(-j\Omega n) \\
=& 0.2 \times (\exp(j\Omega 2) + \exp(j\Omega 1) + \exp(-j\Omega 0) + \exp(-j\Omega 1) + \exp(-j\Omega 2)) \\
=& 0.2 \times (\cos(\Omega 2) + j\sin(\Omega 2) + \cos(\Omega 1) + j\sin(\Omega 1) + 1 \\
&+ \cos(\Omega 1) - j\sin(\Omega 1) + \cos(\Omega 2) - j\sin(\Omega 2)) \\
=& 0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).
\end{aligned}
$$

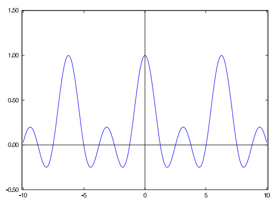
Notes

Periodicity of Fourier Transform

Also note that the Fourier Transform of an aperiodic signal is periodic.



$\Rightarrow$

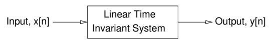


The periodicity is every 2π periods, a result of the sampling in the digitisation process.

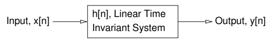
Notes

Frequency Response of LTI Systems

An LTI system has an input $x[n]$ and an output $y[n]$:



Recall that an LTI system has an impulse response, $h[n]$:



which describes the response of the system when an impulse function is given as the input. The impulse response is useful as it can be used to calculate the output signal for a given input signal:

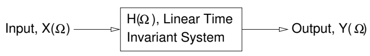
$$y[n] = x[n] * h[n]$$

where $*$ is convolution *NOT* multiplication.

Notes

Frequency Response of LTI Systems

An LTI system can also be described in the frequency domain:

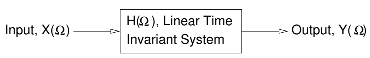


where

- The input frequency domain signal is $X(\Omega) = \mathcal{F}(x[n])$,
- The output frequency domain signal is $Y(\Omega) = \mathcal{F}(y[n])$
- The LTI system is described by $H(\Omega) = \mathcal{F}(h[n])$ which is known as the *frequency response* of the system and is the Fourier Transform of the impulse response.

Notes

Frequency Response of LTI Systems



In the frequency domain, the output can be calculated more easily:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

where multiplication *IS* used here. In other words, *convolution* is performed by multiplication in the frequency domain.

Notes

Frequency Response of LTI Systems

The frequency domain convolution (multiplication) equation:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

can be re-arranged so that:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

so if we want to find the frequency response of a system then we can find it via this equation or via the Fourier transform of the time domain representation $h[n]$.

Notes

Frequency Response of LTI Systems

Recall the general form of LTI difference equations:

$$\sum_{m=0}^{N} a[m]y[n-m] = \sum_{m=0}^{M} b[m]x[n-m].$$

Using the linearity and time-shifting properties of Fourier transforms we can convert it to an expression using frequency domain terms:

$$\sum_{m=0}^{N} a[m]\exp(-jk\Omega)Y(\Omega) = \sum_{m=0}^{M} b[m]\exp(-jk\Omega)X(\Omega).$$

Notes

Frequency Response of LTI Systems

Therefore the frequency response of a system can also be described by

$$H(\Omega) = \frac{\sum_{m=0}^{M} b[m]\exp(-jm\Omega)}{\sum_{m=0}^{N} a[m]\exp(-jm\Omega)}.$$

This equation can be used to directly find the frequency response of a system even if only the coefficients $a[m]$ and $b[m]$ are known.

Notes

Frequency Response *Example*

Q. A moving average filter has $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$. Find the frequency response of this filter.

A. We can find the frequency response by using the coefficients:

- There is only 1 output coefficient, $a[0] = 1$.
- There are 3 input coefficients, $b[0] = b[1] = b[2] = \frac{1}{3}$
- Therefore

$$H(\Omega) = \frac{\frac{1}{3}\sum_{m=0}^{M} \exp(-jm\Omega)}{\exp(-j0\Omega)} = \frac{1}{3}(1 + \exp(-j\Omega) + \exp(-j2\Omega))$$
$$= \frac{1}{3}(1 + \cos(\Omega) - j\sin(\Omega) + \cos(2\Omega) - j\sin(2\Omega))$$
$$= \frac{1}{3}(1 + \cos(\Omega) + \cos(2\Omega) - j(\sin(\Omega) + \sin(2\Omega)))$$

Notes

Frequency Response *Example*

- Magnitude:

$$\mathrm{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}\left((1 + \cos(\Omega) + \cos(2\Omega))^2 + (\sin(\Omega) + \sin(2\Omega))^2\right)}$$

- Phase:

$$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega) + \sin(2\Omega))}{(1 + \cos(\Omega) + \cos(2\Omega))}\right)$$

The magnitude can be simplified using:

- $2\sin(\Omega)\sin(2\Omega) = \cos(\Omega) - \cos(3\Omega)$
- $2\cos(\Omega)\cos(2\Omega) = \cos(\Omega) + \cos(3\Omega)$
- $\sin^2(\Omega) = \frac{1-\cos(2\Omega)}{2}$
- $\sin^2(2\Omega) = \frac{1-\cos(4\Omega)}{2}$
- $\cos^2(\Omega) = \frac{1+\cos(2\Omega)}{2}$
- $\cos^2(2\Omega) = \frac{1+\cos(4\Omega)}{2}$

Resulting in:

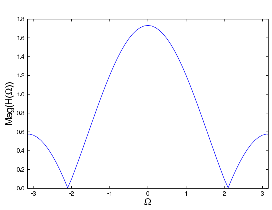
- $\mathrm{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3 + 2(2\cos(\Omega) + \cos(2\Omega)))}$

Notes

Frequency Response *Example*

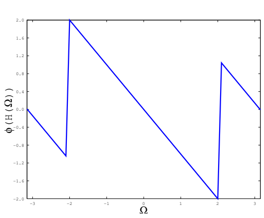
Moving Average Filter (k=3) Frequency Response

Magnitude



$\mathrm{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3 + 2(2\cos(\Omega) + \cos(2\Omega)))}$

Phase



$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega)+\sin(2\Omega))}{(1+\cos(\Omega)+\cos(2\Omega))}\right)$

Notes

Derivation of the DFT

A discretely sampled signal $x[n] : n = 0, ..., N-1$ is:

$$x_s(t) = \sum_{n=0}^{N-1} x[n]\delta(t - nT)$$

where T is the sampling interval. Fourier transform is:

$$X_s(f) = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi fnT).$$

This is periodic, i.e. $X_s(f) = X_s(f + 1/T)$.

Notes

Derivation of the DFT *cont.*

Discretizing $X_s(f)$ requires looking at regularly spaced values of f, i.e. $f_k = k \times f_0$ for some f_0:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f_k nT) = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi k f_0 nT).$$

Discrete sampling in frequency domain implies periodicity in time domain, i.e. $x[n+N] = x[n]$, $\therefore$ assume $f_0 = \frac{1}{NT}$, then:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi kn}{N}\right)$$

This is the DFT. The inverse DFT is given by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \exp\left(+j\frac{2\pi kn}{N}\right).$$

Notes

Summary

- Introduction to frequency domain analysis
- Discrete Fourier Series
- Spectra of Periodic Digital Signals
- Magnitude and Phase of Line Spectra
- The Fourier Transform for aperiodic digital sequences

Notes