Solutions for: Recursive Digital Filters Tutorial

1. What is the condition on α for stability? **Solution** For stability of a continuous-time system we require that the poles of the system be in the left half-plane, i.e. $\Re(p) < 0$. In this example, the poles are at $s = \alpha \pm j\omega_0$, so for both poles we have $\Re(p) = \alpha$ so we must have that $\alpha < 0$ for the analogue notch filter to be stable.

2. What procedure would you use to find $h(t)$ from $H(s)$? **Solution** I would use a partial-fraction expansion:

$$H(s) = \frac{A}{s - (\alpha + j\omega_0)} + \frac{B}{s - (\alpha - j\omega_0)}$$

   This is not an analogue electronics unit so you are not required to actually do it.

   However, if you're interested, here are the first few steps:

$$H(s) = \frac{s^2 + \omega_0^2}{(s-\alpha)^2 + \omega_0^2} = \frac{A(s - (\alpha - j\omega_0) + B(s - (\alpha + j\omega_0))}{(s-\alpha)^2 + \omega_0^2}$$
$$s^2 + \omega_0^2 = A(s - (\alpha - j\omega_0)) + B(s - (\alpha + j\omega_0))$$

   Setting $s = \alpha + j\omega_0$ tells us A

$$(\alpha + j\omega_0)^2 + \omega_0^2 = \alpha^2 + j2\alpha\omega_0 = A(\alpha + j\omega_0 - (\alpha - j\omega_0)) = A(j2\omega_0)$$
$$A = \frac{\alpha^2 + j2\alpha\omega_0}{j2\omega_0} = \alpha - j\frac{\alpha^2}{2\omega_0}$$

   Similarly, setting $s = \alpha - j\omega_0$ tells us B.

   Once we have A and B we can substitute for them and get an expression for H(s) which we can turn back into h(t) using tables.

3. Use the structure of the AR part of the filter to explain why its impulse response never quite dies away and thus why digital filters with feedback (and potentially feed-forward as well) are generally known as Infinite Impulse Response (IIR) filters. **Solution** The presence of a nonzero value at the output at some time instant $y[k] \neq 0$ means that the output at subsequent instants will contain contributions from $y[k]$, scaled by the feedback coefficients, so they won't be zero either. This means that even if the coefficients are such that the later outputs are smaller than $y[k]$, they won't settle down to exactly zero, i.e. the output sequence will run on indefinitely, which makes the filter an IIR.

   Note: in practice, finite precision effects mean that output values might fall to smaller than one least-significant-bit which could result in the output locking at zero. However, this is depending on a nonlinear process, which can also result in endless oscillation of the LSB in an effect called a limit cycle. This is too advanced a topic to cover in an introductory unit but you are welcome to look it up if you are interested.

4. A first order filter has a transfer function:

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

   Determine the zeros and poles for this filter. **Solution** First multiply through by z^{+1}:

$$H(z) = \frac{a_0 z + a_1}{z + b_1}$$

   The poles are the roots of the denominator, where $\frac{1}{H(z)} = 0$. We can see that if we let $z = -b_1$ we then have:

$$\frac{1}{H(z)} = 0.$$

   The zeros can be found as the roots of the numerator, where $H(z) = 0$. First manipulate the numerator:

$$H(z) = a_0 \frac{z + \frac{a_1}{a_0}}{z + b_1}.$$

   If we now let $z = -\frac{a_1}{a_0}$ we get:

$$H(z) = 0.$$

5. Explain why the set of values $z = e^{j\Omega}$ is the unit circle. **Solution** Eulers identity gives us:

$$z = \cos(\Omega) + j\sin(\Omega).$$

We can see that the magnitude of z is always 1 because:

$$\begin{aligned} r &= \sqrt{\cos^2(\Omega) + \sin^2(\Omega)} \\ &= 1. \end{aligned}$$

which is always true for all values of Ω (i.e. all normalised radian frequencies).

6. A first-order digital filter has impulse response:

$$h[n] = \frac{a_1}{b_1}\delta[n] + \left(a_0 - \frac{a_1}{b_1}\right)(-b_1)^n.$$

Explain why this must be an IIR filter rather than an FIR filter. **Solution** IIR impulse response continues forever. Here:

$$h[n] = \underbrace{\frac{a_1}{b_1}\delta[n]}_{\text{finite}} + \underbrace{\left(a_0 - \frac{a_1}{b_1}\right)(-b_1)^n}_{\text{infinite}}.$$

So it must be an IIR filter because it contains at least one term that will continue forever.

7. Find the z-plane poles and zeros of the transfer function of the biquadratic filter with

$$H(z) = \frac{a_0 z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}.$$

Solution The poles and zeros are the roots of the denominator and numerator, respectively. We can therefore find the roots by using a standard formula to find the roots of a quadratic polynomial:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the quadratic is assumed to have the form:

$$ax^2 + bx + c.$$

For the numerator we have:

- $a = a_0$, $b = a_1$ and $c = a_2$.

Therefore there are roots (zeros) at:

$$z_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}.$$

For the denominator we have:

- $a = 1$, $b = b_1$ and $c = b_2$.

Therefore there are roots (poles) at:

$$\begin{aligned} p_{1,2} =& \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} \\ =& -\frac{b_1}{2} \pm \sqrt{\left(\frac{b_1}{2}\right)^2 - b_2}. \end{aligned}$$

8. Find the previous question filter's impulse response $h[n]$ by using the partial fraction method. Use it to explain why this filter must be IIR rather than FIR. **Solution** It will be quite time consuming if we have to re-write the full general expression for each of the poles and each of the zeros. Therefore we will use the form:

$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}.$$

Partial fraction expansion:

$$H(z) = \frac{A}{z-p_1} + \frac{B}{z-p_2}.$$

To find A, using cover up rule:

$$\begin{aligned} A &= \left.\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}\right|_{z=p_1} \\ &= \frac{(p_1-z_1)(p_1-z_2)}{p_1-p_2}. \end{aligned}$$

To find B, using cover up rule:

$$\begin{aligned} B &= \left.\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}\right|_{z=p_2} \\ &= \frac{(p_2-z_1)(p_2-z_2)}{(p_2-p_1)}. \end{aligned}$$

Partial fraction expansion then given by:

$$H(z) = \frac{\frac{(p_1-z_1)(p_1-z_2)}{p_1-p_2}}{z-p_1} + \frac{\frac{(p_2-z_1)(p_2-z_2)}{(p_2-p_1)}}{z-p_2}.$$

Again, we have some quite complicated expressions here, that could be simplified, but perhaps it is easier for this exercise just to replace the numerators of both fractions with the constants A and B, i.e.

$$H(z) = \frac{A}{z-p_1} + \frac{B}{z-p_2}.$$

If we have actual values for the variables p_1, p_2, z_1 and z_2 then we could calculate A and B and use those instead, however this is not the case here. Nevertheless we can also determine the inverse z-transform of $H(z)$ which will be the impulse response of the biquadratic.

The inverse z-transform of $\frac{z}{z-a}$ is (from z-transform tables):

$$\mathcal{Z}^{-1}\left(\frac{z}{z-a}\right) = a^n \mathcal{H}[n]$$

where $\mathcal{H}[n]$ is the Heaviside (unit step) function. Furthermore, if we multiply by z^{-1} in the z-domain, then this is equivalent to a single time delay, i.e.

$$\mathcal{Z}^{-1}\left(z^{-1}\frac{z}{z-a}\right) = \mathcal{Z}^{-1}\left(\frac{1}{z-a}\right) = a^{n-1}\mathcal{H}[n-1]$$

The two fractions in $H(z)$ have the above form but also multiplied by a constant which, the z-transform being a linear operator:

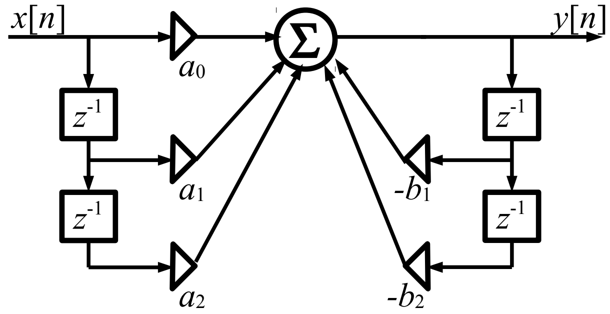
$$\mathcal{Z}^{-1}(af(z)) = a\mathcal{Z}^{-1}(f(z))$$

where $f(z)$ is any function of z. Thus

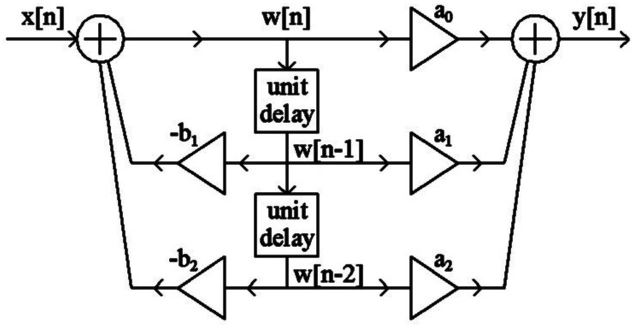
$$\begin{aligned} \mathcal{Z}^{-1}(H(z)) = \mathcal{Z}^{-1}\left(\frac{A}{z-p_1} + \frac{B}{z-p_2}\right) &= \mathcal{Z}^{-1}\left(\frac{A}{z-p_1}\right) + \mathcal{Z}^{-1}\left(\frac{B}{z-p_2}\right) \\ &= Ap_1^{n-1}\mathcal{H}[n-1] + Bp_2^{n-1}\mathcal{H}[n-1]. \end{aligned}$$

This is the impulse response. Here both terms in the impulse response continue forever after $n \geq 0$, thereby making the filter's response to an impulse function continue forever. This means that it is an *Infinite Impulse Response* type filter.

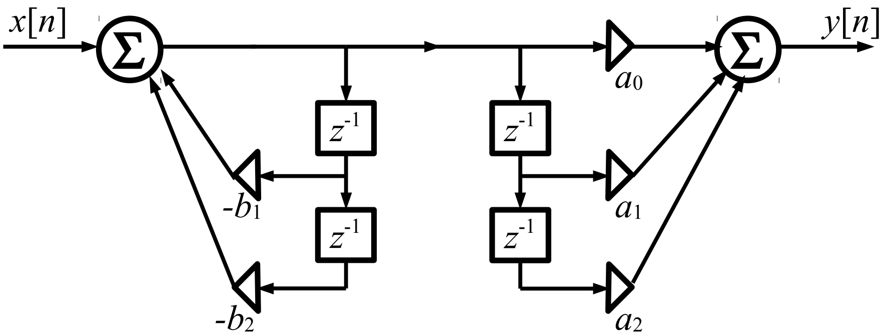
9. Draw a block diagram which represents the difference equation $y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] - b_1 y[n-1] - b_2 y[n-2]$ and explain the similarities and differences between it and the diagram of the biquadratic filter. **Solution**



Compare this with the diagram of the biquadratic filter:



we can see that the Biquad computes things in a different order. The filters illustrated in the two diagrams will produce the same result however the second one is more efficient because it has fewer time delays. They can be seen to be equivalent if we change the order of the calculations, which is possible because of the commutative property of linear systems.



We can show that they are equivalent by deriving the difference equation for the Biquadratic and comparing it with the difference equation as stated in the question. The derivation of the difference equation for the Biquadratic is best done via the z-domain.

- First split the biquadratic filter into 2 stages, 1) a feedback part with input $x[n]$ and output $w[n]$; and 2) a feedforward part with input $w[n]$ and output $y[n]$.
- Then obtain an expression for the feedback part:

$$w[n] = x[n] - b_1 w[n-1] - b_2 w[n-2]$$

with z-transform:

$$\begin{aligned} W(z) &= X(z) - z^{-1}b_1 W(z) - z^{-2}b_2 W(z) \\ &= X(z) - W(z)(z^{-1}b_1 + z^{-2}b_2) \\ \therefore W(z)\left(1 + z^{-1}b_1 + z^{-2}b_2\right) &= X(z) \\ \therefore W(z) &= \frac{X(z)}{(1 + z^{-1}b_1 + z^{-2}b_2)} \end{aligned}$$

leading to a transfer function:

$$H_{\text{back}}(z) = \frac{W(z)}{X(z)} = \frac{1}{(1 + z^{-1}b_1 + z^{-2}b_2)}$$

- Similarly for the feedforward part:

$$y[n] = a_0 w[n] + a_1 w[n-1] + a_2 w[n-2]$$

with z-transform

$$\begin{aligned} Y(z) &= a_0 W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) \\ &= W(z)(a_0 + a_1 z^{-1} + a_2 z^{-2}) \end{aligned}$$

leading to a transfer function:

$$H_{\text{fwd}}(z) = \frac{Y(z)}{W(z)} = a_0 + a_1 z^{-1} + a_2 z^{-2}.$$

- We can then obtain the combined transfer function:

$$H(z) = H_{\text{back}}(z) \times H_{\text{fwd}}(z) = \frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)} = \frac{Y(z)}{X(z)},$$

leading to

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + z^{-1}b_1 + z^{-2}b_2} \\ \therefore Y(z)(1 + z^{-1}b_1 + z^{-2}b_2) &= X(z)(a_0 + a_1 z^{-1} + a_2 z^{-2}) \end{aligned}$$

Multiplying out $Y(z)$ and $X(z)$ and performing an inverse z-transform yields

$$\begin{aligned} y[n] + b_1 y[n-1] + b_2 y[n-2] &= a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] \\ \therefore y[n] &= a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] - b_1 y[n-1] - b_2 y[n-2] \end{aligned}$$

This is a difference equation for the complete Biquadratic.

- We can compare the above different equation for the difference equation that was stated as part of this question and we can see that they are the same. This shows that the two structures will compute the same result however, as already stated, the Biquadratic is more efficient because it has fewer delays.

10. Find the frequency-domain response of this filter from the difference equation by assuming its input is a complex sinusoid of frequency f and its output is the same frequency but multiplied by complex $H(f)$. Compare the result with the z-domain transfer-function H(z). **Solution** The question requests:

$$x[n] = e^{j2\pi f nT} \quad \text{and} \quad y[n] = H(f) \times e^{j2\pi f nT}.$$

These expressions for $x[n]$ and $y[n]$ can be substituted into the difference equation:

$$\begin{aligned} y[n] + b_1 y[n-1] + b_2 y[n-2] &= a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] \\ H(f)e^{j2\pi f nT} + b_1 H(f) e^{j2\pi f(n-1)T} + b_2 H(f) e^{j2\pi f(n-2)T} &= a_0 e^{j2\pi f nT} + a_1 e^{j2\pi f(n-1)T} + a_2 e^{j2\pi f(n-2)T} \end{aligned}$$

Dividing through by the common factor and then rearranging:

$$\begin{aligned} H(f) + b_1 H(f) e^{-j2\pi fT} + b_2 H(f) e^{-2j2\pi fT} &= a_0 + a_1 e^{-j2\pi fT} + a_2 e^{-2j2\pi fT} \\ \therefore H(f) &= \frac{a_0 + a_1 e^{-j2\pi fT} + a_2 e^{-2j2\pi fT}}{1 + b_1 e^{-j2\pi fT} + b_2 e^{-2j2\pi fT}} \end{aligned}$$

Digital filters often use normalised radian frequency, or Ω, therefore performing a change of variables $\Omega = 2\pi \frac{f}{f_s}$ yields:

$$H(\Omega) = \frac{a_0 + a_1 e^{-j\Omega} + a_2 e^{-2j\Omega}}{1 + b_1 e^{-j\Omega} + b_2 e^{-2j\Omega}}.$$

An alternative approach to deriving the frequency response can be via the z-domain transfer function $H(z)$, substituting $z = e^{j\Omega}$. Previously we had

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + z^{-1} b_1 + z^{-2} b_2}$$

and performing the substitution we get

$$H(z) = \frac{a_0 + a_1 e^{-j\Omega} + a_2 e^{-2j\Omega}}{1 + e^{-j\Omega} b_1 + e^{-2j\Omega} b_2}$$

which is the same. This is often simpler and quicker if the z-domain transfer function is already known.

11. Determine (i) a cascade and (ii) a parallel realisation for the following transfer function using only first-order structures:

$$H(z) = \frac{z(z-1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)}.$$

Sketch the block diagrams which result and compare them with the block diagram you would get from implementing the above as a single biquadratic structure. **Solution**

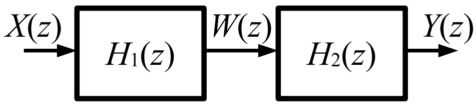
(a) A cascade realisation is one in which the filter is split into a series of consecutive structures, so that in the z-domain the overall transfer function $H(z)$ is given by the product of the transfer functions:

$$H(z) = H_1(z) \times H_2(z) \times ...$$

We already know $H(z)$ and it is already factorized into individual polynomials of degree 1, so that:

$$H(z) = \underbrace{\left(\frac{z}{z - \frac{1}{2}}\right)}_{H_1(z)} \times \underbrace{\left(\frac{z-1}{z - \frac{1}{8}}\right)}_{H_2(z)}.$$

Both $H_1(z)$ and $H_2(z)$ are first order. We can therefore create a cascade realisation from each of these transfer functions:



which will involve the determination of a difference equation for each section using an intermediate variable $W(z)$ resulting in a difference variable $w[n]$ to describe the output of the first stage and the input to the second stage. To determine the difference equations we will need to find $w[n]$ and the output of the second stage (entire system) $y[n]$.

i. Creating an intermediate output $W(z)$ for $H_1(z)$ we can say:

$$H_1(z) = \frac{W(z)}{X(z)} = \frac{z}{z - \frac{1}{2}}$$

Manipulating this we get

$$W(z)\left(z - \frac{1}{2}\right) = X(z)z$$

$$zW(z) - \frac{1}{2}W(z) = zX(z).$$

Inverse z-transform results in:

$$w[n+1] - \frac{1}{2}w[n] = x[n+1].$$

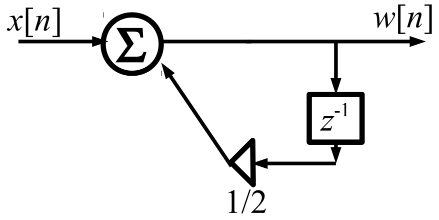
This is not causal, we can make it causal by substituting $n - 1 = n$

$$w[n] - \frac{1}{2}w[n-1] = x[n].$$

Then solving for $w[n]$ we get

$$w[n] = x[n] + \frac{1}{2}w[n-1].$$

This first stage of the cascade is illustrated below.



ii. We can say that:

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{z-1}{z-\frac{1}{8}}.$$

Following a similar series of steps to obtain an expression for $y[n]$:

$$Y(z)\left(z-\frac{1}{8}\right) = W(z)(z-1)$$
$$zY(z) - \frac{1}{8}Y(z) = zW(z) - W(z)$$

Inverse z-transform yields:

$$y[n+1] - \frac{1}{8}y[n] = w[n+1] - w[n].$$

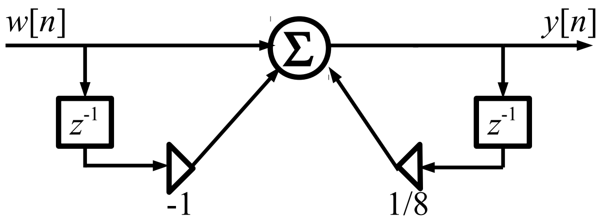
Making it causal by substituting $n-1=n$:

$$y[n] - \frac{1}{8}y[n-1] = w[n] - w[n-1].$$

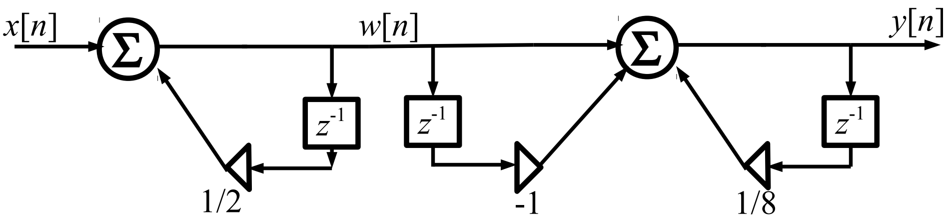
Rearranging for $y[n]$:

$$y[n] = w[n] - w[n-1] + \frac{1}{8}y[n-1].$$

This second stage of the cascade is illustrated below.

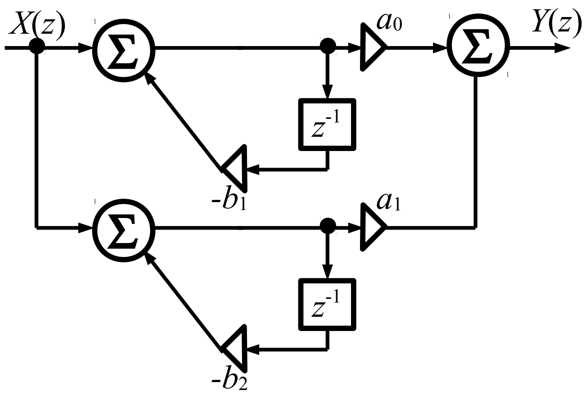


Combining the two cascade realizations together to implement the full transfer function $H(z)$ we get:



The above structure could be further simplified by merging the parallel path consisting of two equivalent delays, reducing the requirements of the overall system.

(b) A parallel realization splits the filter into parallel stages, e.g.



This is done by finding the partial fraction expansion, e.g. for the above structure we have

$$H(z) = \frac{z + a_0}{z + b_1} + \frac{z + a_1}{z + b_2}$$

The above diagram is a parallel form 1 realization of the above transfer function $H(z)$ consisting of two partial fractions. Note: the parallel form 1 realization is similar to the direct form 2 structure that we saw earlier.

Therefore we need to perform a partial fraction expansion on the transfer function $H(z)$. We therefore need to find the constants A and B where:

$$\frac{H(z)}{z} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{1}{8}}.$$

where we divide both sides by z to enable us to find the partial fractions. Using the cover up rule,

i.

$$\begin{aligned} A &= \left.\frac{(z-1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)}\right|_{z=\frac{1}{2}} = \frac{\frac{1}{2} - 1}{\frac{1}{2} - \frac{1}{8}} \\ &= \frac{-\frac{1}{2}}{\frac{1}{2}\left(1 - \frac{1}{4}\right)} \\ &= -\frac{1}{\frac{3}{4}} = -\frac{4}{3}. \end{aligned}$$

ii.

$$\begin{aligned} B &= \left.\frac{z-1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)}\right|_{z=\frac{1}{8}} = \frac{\frac{1}{8} - 1}{\frac{1}{8} - \frac{1}{2}} \\ &= \frac{-\frac{7}{8}}{\frac{1}{8}(1-4)} = \frac{7}{3}. \end{aligned}$$

We now have a partial fraction for $H(z)$. We should also check to make sure we have not made any mistakes:

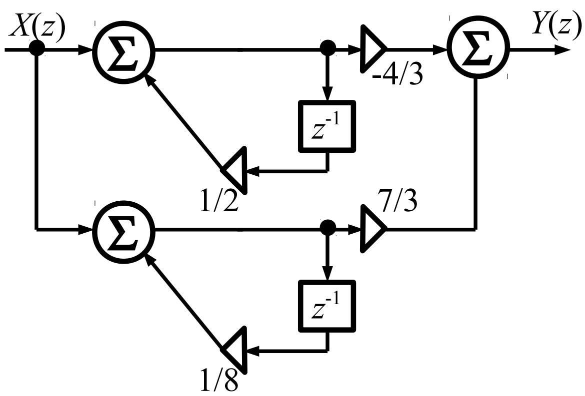
$$\begin{aligned} 3\frac{H(z)}{z} &= -\frac{4}{z - \frac{1}{2}} + \frac{7}{z - \frac{1}{8}} \\ &= \frac{-4\left(z - \frac{1}{8}\right) + 7\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)} = \frac{-4z + \frac{1}{2} + 7z - \frac{7}{2}}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)} \\ &= \frac{3z - \frac{6}{2}}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)} = \frac{3z - 3}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{8}\right)} \end{aligned}$$

Dividing through by $3/z$ confirms the original form for $H(z)$ so the values of A and B are correct.

The paralle form 1 realization for the filter using the transfer function consisting of two partial fractions:

$$H(z) = -\frac{z4/3}{z - \frac{1}{2}} + \frac{z7/3}{z - \frac{1}{8}}$$

is shown below:



(c) The biquadratic structure can be found by finding the difference equation for the full system, described by:

$$H(z) = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}$$

drawing out the direct form 1 structure then converting it into the direct form 2 structure.
Thus, manipulating the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 - z}{z^2 - \frac{1}{2}z - \frac{1}{8}z + \frac{1}{16}} = \frac{z^2 - z}{z^2 - \frac{3}{8}z + \frac{1}{16}}.$$

Multiplying both sides with the denominators:

$$Y(z)\left(z^2 - \frac{3}{8}z + \frac{1}{16}\right) = X(z)\left(z^2 + z\right)$$

Performing inverse z-transform:

$$y[n+2] - \frac{3}{8}y[n+1] + \frac{1}{16}y[n] = x[n+2] - x[n+1]$$

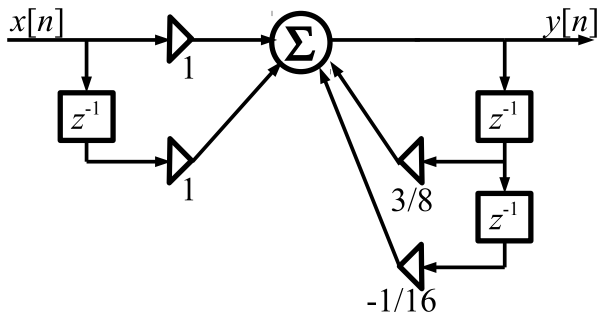
and making it causal:

$$y[n] - \frac{3}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - x[n-1].$$

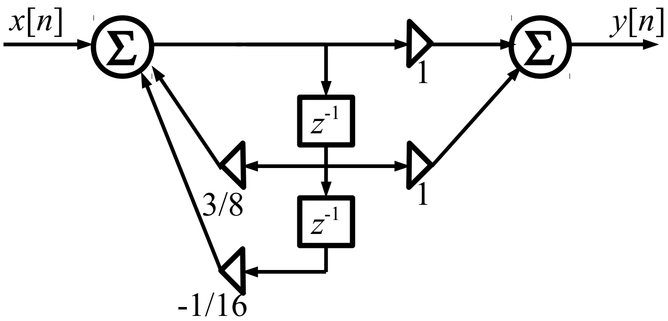
Rearranging for $y[n]$:

$$y[n] = x[n] - x[n-1] + \frac{3}{8}y[n-1] - \frac{1}{16}y[n-2].$$

We can now draw a direct form 1 realisation:



which can be converted to direct form 2 realisation by swapping the order in which the calculations are performed resulting in:



Some of the above filter structures have advantages over other structures. The principal advantage of the cascade and parallel filter structures is to reduce the effect of round off errors or limited precision which can affect the filter response. Splitting the system into smaller structures can reduce the effect of round off errors, helping to make sure that the filter retains the properties that were intended in the original filter design. This can include making sure the filter remains stable. Splitting the system into 1st or perhaps 2nd order substructures, whether parallel or in cascade form can help provide control on the overall affect of individual rounding errors.

The biquadratic structure, i.e. the direct form 2 structure shown above is better than the direct form 1 structure because it has fewer time delays. You can also see that the cascade structure will often require fewer multiplications, except here we have a number of multipliers equal to just 1 or -1.

12. Follow the steps below to design an IIR digital low-pass filter $G(z)$ with sampling frequency 44.1kHz and 3dB frequency of 11.025kHz using a second-order Butterworth filter analogue prototype:

$$G(s) = \frac{1}{\left(\frac{s}{\alpha}\right)^2 + \sqrt{2}\left(\frac{s}{\alpha}\right) + 1}.$$

where α is the analogue 3dB angular frequency.

(a) Find the value of α given the above sampling and cut-off frequency. **Solution** The normalized radian 3dB frequency is:

$$\Omega_{3dB} = 2\pi\frac{f_{3dB}}{f_s} = \frac{\pi}{2}$$

The analogue angular frequency is therefore

$$\alpha = \frac{2}{T}\tan\left(\frac{\Omega_{3dB}}{2}\right) = 88200.$$

(b) Find the z-domain transfer function of the digital filter. **Solution**

$$\begin{aligned} G(z) &= G(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}} \\ &= \frac{1}{\left(\frac{88200\frac{z-1}{z+1}}{88200}\right)^2 + \sqrt{2}\left(\frac{88200\frac{z-1}{z+1}}{88200}\right) + 1} \\ &= \frac{(z+1)^2}{(z-1)^2 + \sqrt{2}(z-1)(z+1) + (z+1)^2}. \end{aligned}$$

(c) Find the poles and zeros of $G(z)$ **Solution**

$$\begin{aligned} G(z) &= \frac{(z+1)^2}{(z-1)^2 + \sqrt{2}(z-1)(z+1) + (z+1)^2} \\ &= \frac{(z+1)^2}{(2+\sqrt{2})z^2 + (2-\sqrt{2})}. \end{aligned}$$

Therefore there are zeros at $z = -1$ and poles at:

$$\frac{1}{G(z)} = 0 \Rightarrow z^2 = -\frac{2-\sqrt{2}}{2+\sqrt{2}} \Rightarrow z = \pm j\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \pm j0.41421...$$

(d) Find a difference equation that can approximate the transfer function then draw a direct form 1 realisation for the difference equation. **Solution**

$$\begin{aligned} G(z) &= \frac{z^2+2z+1}{z^2-2z+1+\sqrt{2}z^2-\sqrt{2}+z^2+2z+1} \\ &= \frac{z^2+2z+1}{(2+\sqrt{2})z^2+(2-\sqrt{2})} \\ &= \frac{1}{2+\sqrt{2}}\left(\frac{z^2+2z+1}{z^2+\frac{(2-\sqrt{2})}{(2+\sqrt{2})}}\right) \\ &\approx 0.29289\left(\frac{1+2z^{-1}+z^{-2}}{1+0.17157z^{-2}}\right). \end{aligned}$$

The transfer function is given by:

$$G(z) = \frac{Y(z)}{X(z)}$$

where $X(z)$ is the input and $Y(z)$ is the output frequency response. We can therefore find a difference equation by multiplying both sides by the denominators then determining the inverse z transforms of each of the individual summed terms.

$$\frac{Y(z)}{X(z)} \approx 0.29289 \left(\frac{1 + 2z^{-1} + z^{-2}}{1 + 0.17157z^{-2}} \right)$$
$$\Rightarrow Y(z)(1 + 0.17157z^{-2}) = 0.29289X(z)(1 + 2z^{-1} + z^{-2})$$
$$\Rightarrow Y(z) + Y(z)0.17157z^{-2} = 0.29289X(z) + 0.58578X(z)z^{-1} + 0.29289X(z)z^{-2}$$

Inverse z transforms yields

$$y[n] + 0.17157y[n-2] = 0.29289x[n] + 0.58578x[n-1] + 0.29289x[n-2]$$

manipulating both sides to obtain an equation for $y[n]$ gives

$$y[n] = 0.29289x[n] + 0.58578x[n-1] + 0.29289x[n-2] - 0.17157y[n-2].$$

This can be easily drawn as a direct form 1 realisation. This is left as an exercise.