Infinite Impulse Response (IIR) Filters

Digital Signal Processing

Notes

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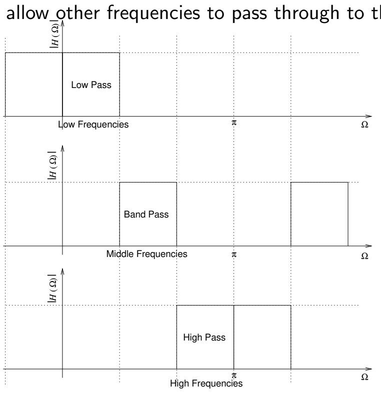
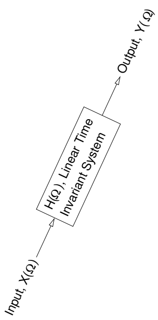
Notes

What is a Digital Filter?

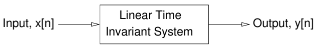
Often used to remove some frequencies from a signal $X(\Omega)$ and to allow other frequencies to pass through to the output $Y(\Omega)$.





Notes

Recursive digital filters



What is a *Recursive digital filter*?

- "*Recursive*" comes from the word "to recur"
  Meaning: to repeat

A recursive filter uses past output values ($y[n-i]$) for the calculation of the current output $y[n]$:

- *Recursive Filter Example*

$$y[n] = 0.5y[n-1] + 0.5x[n].$$

A non-recursive filter only uses input values $x[n-i]$:

- *Non-recursive Filter Example*

$$y[n] = 0.5x[n-1] + 0.5x[n].$$

Notes

Generalised Difference Equation

Recall the generalised difference equation for causal LTI systems:

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k]$$

If $a[0] = 1$, this can then be changed to:

$$y[n] = \sum_{k=0}^{M} b[k]x[n-k] - \sum_{k=1}^{N} a[k]y[n-k].$$

(Recall) The Frequency Response of such a system can be described by:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{\exp(0) + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

Notes

z-transform Representation

Fourier based frequency representation:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

Can also be represented in the z-domain (z-transform):

$$H(z) = \frac{\sum_{k=0}^{M} b[k]z^{-k}}{1 + \sum_{k=1}^{N} a[k]z^{-k}}.$$

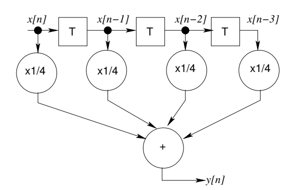
Both describe a type of **frequency response** of the system.

Notes

Comparison of IIR and FIR System Structures

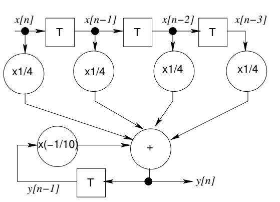
The system structure of an IIR filter demonstrates the feedback of the output into the input again.

FIR Filter



$$y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$$

IIR Filter

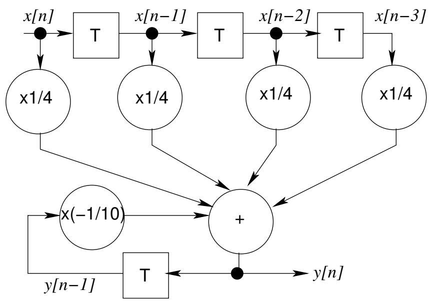


$$y[n] = \left(x[n] + x[n-1] + x[n-2] + x[n-3]\right) - \frac{1}{10}y[n-1]$$

Notes

Unit Delay in the z-plane

A single z^{-1} is the same as a unit delay, "T" in a system diagram.



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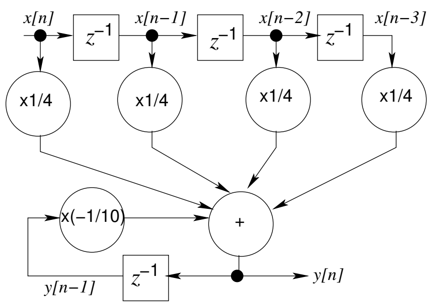


Notes

Recursive Digital Filters

Recursive digital filters are often known as

- **Infinite Impulse Response (IIR)** Filters

as the impulse response of an IIR filter often has an infinite number of coefficients.

IIR Filters

- Require fewer calculations than FIR filters.
- ∴ Faster response to the input signal,
- and ∴ shorter frequency response *transition width*.

However!

- IIR filters can become unstable.
- ∴ Need to think carefully about **stability** when designing IIR Filters.

Notes

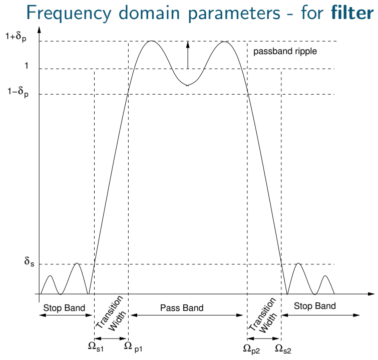
IIR Filter Design Overview

1. Filter specification
2. Coefficient calculations
3. Convert transfer function to suitable filter structure
4. Error analysis
5. Implementation

Notes

Filter Specification

Frequency domain parameters - for **filter description**.

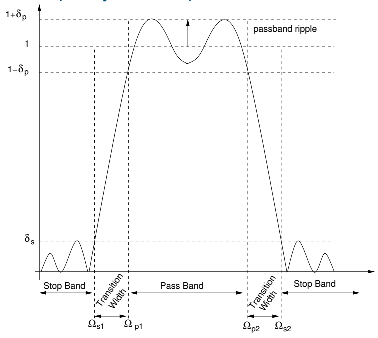


- Stop band - high attenuation
- Pass band - low attenuation (gain≈1)

Notes

Filter Specification

Frequency domain parameters - for **filter description**.



- $\delta_{\rm p}$ passband ripple
- $\delta_{\rm s}$ stopband ripple
- $\Omega_{\rm s1}$ lower stop band edge frequency
- $\Omega_{\rm p1}$ lower pass band edge frequency
- $\Omega_{\rm p2}$ upper pass band edge frequency
- $\Omega_{\rm s2}$ upper stop band edge frequency

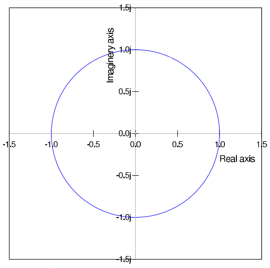
Notes

Pole-Zero Placement Method

A filter can be described in the z-plane with Poles and Zeros:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...} = \frac{\text{zeros}}{\text{poles}}$$

- Zeros *located* at: $z_1, z_2, z_3, \ldots$
- Poles *located* at: $p_1, p_2, p_3, \ldots$



- Poles (X) close to unit circle
  - **make large peaks**
- Zeros (O) close to unit circle
  - **make troughs or minima**

Notes

Pole-Zero Placement Method

Angle of poles and zeros on z-plane correspond to frequencies that can be used for filter specification.

- A bandpass filter, with centre frequency Ω_0 radians can have two poles at $\pm\Omega_0$ radians in the z-plane[1].
- Complete attenuation at two frequencies, $\Omega_{\rm r1} = 0$ radians and $\Omega_{\rm r2} = \pi$ radians can have two zeros at 0 and π radians.

[1]Complex conjugate pair to make real filter coefficients, when $\Omega_0 \neq 0$ or $\Omega_0 \neq \pi$ radians (on the real line).

Notes

Pole-Zero Placement Method

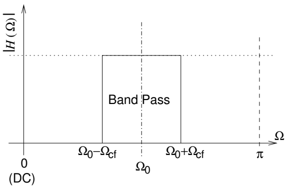
- The radius of the poles can be calculated with:

$$r \cong 1 - \Omega_{\mathrm{cf}}$$

or

$$r \cong 1 - \frac{\Omega_{\mathrm{bw}}}{2}$$

where $\Omega_{\mathrm{bw}} = 2\Omega_{\mathrm{cf}}$ is the -3dB bandwidth of the filter.



Notes

Pole-Zero Placement Method: *Example*

Q. Design a bandpass filter using the Pole-zero placement method with:

- centre frequency at $\Omega_0 = \pi/2$;
- a bandwidth of $\Omega_{\mathrm{bw}} = \pi/8$;
- complete attenuation at $\Omega_{\mathrm{r1}} = 0$ and $\Omega_{\mathrm{r2}} = \pi$;
- and peak unity pass band gain.

Notes

Pole-Zero Placement Method: *Example*

A. Bandpass filter has x2 poles at $\pm\Omega_0 = \pm\pi/2$ radians.

$$\therefore H(z) = K \frac{\text{zeros}}{(z - r\exp(j\pi/2))(z - r\exp(-j\pi/2))}$$

The radii of the poles are given by:

$$r \cong 1 - \frac{\Omega_{\mathrm{bw}}}{2} = 1 - \frac{\pi/8}{2} = 0.80365;$$

and the zeros are at $\Omega_{\mathrm{r1}} = 0$ and $\Omega_{\mathrm{r2}} = \pi$, so that

$$H(z) = K \frac{(z - \exp(j\Omega_{\mathrm{r1}}))(z - \exp(j\Omega_{\mathrm{r2}}))}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

Notes

Pole-Zero Placement Method:
Example cont'd.

As
- $\exp(\Omega_{\mathrm{r}1}) = \exp(j0) = \cos(0) + j\sin(0) = 1 - j0 = 1$
- $\exp(\Omega_{\mathrm{r}2}) = \exp(j\pi) = \cos(\pi) + j\sin(\pi) = -1 + j0 = -1$,

then the transfer function becomes:

$$H(z) = K\frac{(z-1)(z+1)}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

Notes

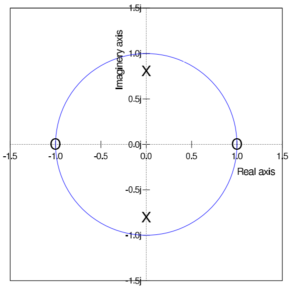
Pole-Zero Placement Method:
Example cont'd.

Using Euler's identity,
- $\exp(j\pi/2) = \cos(\pi/2) + j\sin(\pi/2) = +j$
- and $\exp(j\pi/2) = \cos(\pi/2) - j\sin(\pi/2) = -j$,

so that

$$H(z) = K\frac{(z-1)(z+1)}{(z - 0.80365j)(z + 0.80365j)}.$$

the **pole zero diagram** can then be plotted.



Notes

Pole-Zero Placement Method:
Example cont'd.

Recall that $H(z) = \frac{Y(z)}{X(z)}$,

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{(z-1)(z+1)}{(z - 0.80365j)(z + 0.80365j)} = K\frac{z^2 - 1}{z^2 + 0.64585}.$$

Then

$$Y(z)(z^2 + 0.64585) = X(z)K(z^2 - 1).$$

Notes

Pole-Zero Placement Method:
Example cont'd.

Remembering that each z^{-1} is a unit **delay**, so that each z is a unit **advance**, then the difference equation is:

$$y[n+2] + 0.64585y[n] = K(x[n+2] - x[n])$$

which can be made causal by making $n = n - 2$ so that

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]).$$

K is not known, but can be used to make the peak pass band gain to be **unity**.

Notes

Pole-Zero Placement Method:
Example cont'd.

The **frequency response** of the filter can be determined from the difference equation:

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]),$$

in combination with:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

Notes

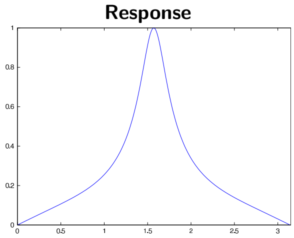
Pole-Zero Placement Method:
Example cont'd.

So that (using Euler's identity):

$$H(\Omega) = \frac{K(1 - \cos(2\Omega) + j\sin(2\Omega))}{1 + 0.64585(\cos(2\Omega) - j\sin(2\Omega))}$$

which has magnitude frequency response:

Magnitude Frequency Response



$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1 - \cos(2\Omega))^2 + \sin^2(2\Omega))}{(1 + 0.64585\cos(2\Omega))^2 + (0.64585\sin(2\Omega))^2}.$$

where $K = 0.17708$.

Notes

Pole-Zero Placement Method:
Example cont'd.

Relating the digital frequencies for previous example to actual frequencies...

If the sampling frequency is $f_s = 500$Hz, the sampling frequency corresponds to $\Omega = 2\pi$, therefore the filter parameters become:

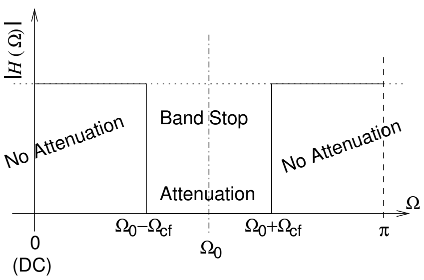
- centre frequency at $\Omega_0 = \pi/2$, so actual centre frequency $f_0 = \frac{\pi/2}{2\pi} f_s = 125$Hz;
- a bandwidth of $\Omega_{\text{bw}} = \pi/8$, so actual bandwidth $f_{\text{bw}} = 31.25$Hz;
- complete attenuation at $\Omega_{\text{r}1} = 0$ and $\Omega_{\text{r}2} = \pi$, with actual frequencies $f_{\text{r}1} = 0$Hz and $f_{\text{r}2} = \frac{\pi}{2\pi} 500 = 250$Hz.

Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter

Q. Design digital **bandstop** filter using pole-zero placement method with the following parameters:

- Centre frequency, $\Omega_0 = \pi/10$ radians (complete attenuation)
- Band stop width, $\Omega_{\text{w}} = 2\Omega_{\text{cf}} = \pi/20$ radians



Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter

- Complete attenuation at $\Omega_0 = \pi/10$, $\therefore$ x2 zeros (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$:

$$H(z) = K \frac{(z - \exp(j\pi/10))(z - \exp(-j\pi/10))}{\text{poles}}$$

- Centre frequency at $\Omega_0 = \pi/10$ radians, $\therefore$ x2 poles (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$,

$$H(z) = K \frac{(z - \exp(j\pi/10))(z - \exp(-j\pi/10))}{(z - r\exp(j\pi/10))(z - r\exp(-j\pi/10))}$$

- The poles are **scaled** with radius r to control **the width of the band stop**,

$$r \cong 1 - \frac{\Omega_{\text{w}}}{2} = 1 - \frac{\pi/20}{2} = 0.92146$$

Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter cont'd.

- resulting in:

$$H(z) = K\frac{(z-\exp(j\pi/10))(z-\exp(-j\pi/10))}{(z-0.92146\exp(j\pi/10))(z-0.92146\exp(-j\pi/10))}$$

- Transfer function is then (using Euler's identity like before):

$$H(z) = K\frac{z^2-1.9021z+1}{z^2-1.7527z+0.84909}$$

- As before, each z is a **unit advance**, so

$$y[n+2]-1.7527y[n+1]+0.84909y[n] = K(x[n+2]-1.9021x[n+1]+x[n])$$

Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter cont'd.

- letting $n=n-2$, making it causal:

$$y[n]-1.7527y[n-1]+0.84909y[n-2] = K(x[n]-1.9021x[n-1]+x[n-2]).$$

- With **frequency response**:

$$H(\Omega) = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{1+\sum_{k=1}^{N} a[k]\exp(-jk\Omega)} = \frac{K(1-1.9021\exp(-j\Omega)+\exp(-j2\Omega))}{1-1.7527\exp(-j\Omega)+0.84909\exp(-j2\Omega)}.$$

Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter cont'd.

Using Euler's identity:

$$H(\Omega) = \frac{K(1-1.9021(\cos\Omega-j\sin\Omega)+\cos 2\Omega-j\sin 2\Omega)}{1-1.7527(\cos\Omega-j\sin\Omega)+0.84909(\cos 2\Omega-j\sin 2\Omega)}.$$

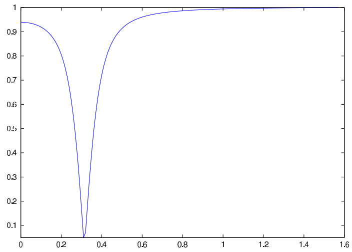
Magnitude Frequency response is then:

$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1-1.9021\cos\Omega+\cos 2\Omega)^2+(1.9021\sin\Omega-\sin 2\Omega)^2)}{(1-1.7527\cos\Omega+0.84909\cos 2\Omega)^2+(1.7527\sin\Omega-0.84909\sin 2\Omega)^2}.$$

Notes

Pole-Zero Placement Method,
Example 2: Band Stop Filter cont'd.

Magnitude frequency response of the notch or bandstop filter:



Notes

Converting Analogue Filters to Digital Filters

Most common approach for **IIR filter design**.

- *Use well-established analogue filter specifications to design digital IIR filters*

Two common approaches include:

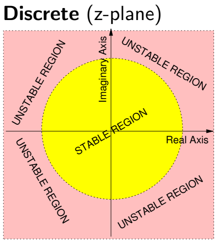
- **Impulse invariant method**
- **Bilinear transformation** *As Discussed Here*

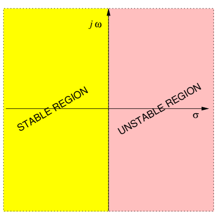
Notes

Laplace Transform

- **Analogue filter** transfer function $h(t)$ can be specified in the s-plane with the Laplace transform $\mathcal{L}(h(t)) = H(s)$
- Hence, the s-plane is for analogue instead of the z-plane (for digital).
- Can be used to analyse **stability** of analogue filters,
  - *Similar to* the z-transform for digital filters.

Discrete (z-plane) $\Longleftrightarrow$ **Analogue** (s-plane)





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- Can be used to analyse **stability** of analogue filters,
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Discrete	$\Longleftrightarrow$	**Analogue**
z-transform $\mathcal{Z}(h[n])$	$\Longleftrightarrow$	Laplace transform $\mathcal{L}(h(t))$
z-plane $H[z]$	$\Longleftrightarrow$	s-plane $H(s)$
Difference equation, $h[n]$	$\Longleftrightarrow$	Differential equation $h(t)$

Notes

How to Convert Analogue Frequency to Digital?

Problem!

- Analogue frequency, $\omega = 0 ... \infty$.
- But digital frequency, $\Omega = 0 ... 2\pi$.

So how to convert analogue frequency to digital?

Need to swap analogue frequencies with digital frequencies...

- If $\Omega \to 2\pi$ then **Very high** analogue frequencies ($\omega \to \infty$)
- If $\Omega \to 0$ then **Very low** analogue frequencies ($\omega \to 0$).

Notes

Bilinear Transformation IIR Filter Design

Bilinear Transformation method replaces *analog frequency* s or $j\omega$ with *digital frequency* Ω using **frequency warping formula**:

$$s = j\omega = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right).$$

where ω is analogue frequency, Ω is digital *frequency* and $T_s = 1/f_s$ is the sampling period.
Bilinear transformation can be applied to find the z-transform $H(z)$:

$$s = j\omega = 2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

Notes

Bilinear Transformation
IIR Filter Design

Example:

Given an analog filter with:

$$H(\omega) = H(s)|_{s=j\omega} = \frac{K(j\omega - z_1)(j\omega - z_2)...}{(j\omega - p_1)(j\omega - p_2)...}$$

Then bilinear transformation gives

$$H(\Omega) = H(s)|_{s=j2f_s \tan\left(\frac{\Omega}{2}\right)}$$
$$= \frac{K(j2f_s \tan\left(\frac{\Omega}{2}\right) - z_1)(j2f_s \tan\left(\frac{\Omega}{2}\right) - z_2)...}{(j2f_s \tan\left(\frac{\Omega}{2}\right) - p_1)(j2f_s \tan\left(\frac{\Omega}{2}\right) - p_2)...}$$

or...

$$H(z) =$$
$$H(s)|_{s=2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{K(2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_1)(2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - z_2)...}{(2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_1)(2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - p_2)...}$$

Notes

Design Procedure Summary

- Identify critical frequencies of the final digital filter response, typically:
  - dc and "corner frequency" for a low pass;
  - folding frequency and "corner frequency" for a high pass;
  - the upper and lower band edges for a band-pass or band-stop filter.
- Translate into Ω values using $\Omega = 2\pi f/f_s$ and apply bilinear frequency warping $\omega \leftarrow \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)$
- Design the s-domain analogue filter to have the required response at these frequencies.
- Apply the bilinear transformation $s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$ to this analogue filter to obtain the required z-domain formula.

Notes

Bilinear Transformation *Example*

Q. Convert the **single pole low pass analog filter**:

$$H(s) = \frac{\omega_{\text{cf}}}{s + \omega_{\text{cf}}}$$

into a **digital filter** (z-plane form) with **digital cut-off frequency** $\Omega_{\text{cf}} = 0.2\pi$ using the **bilinear transformation**.

Notes

Bilinear Transformation *Example*

A.

1. Calculate **analogue cut-off frequency** ω_{cf} from **digital cut-off frequency** $\Omega_{\text{cf}} = 0.2\pi$:

$$\omega_{\text{cf}} = 2f_{\text{s}} \tan(\Omega_{\text{cf}}/2) = 2f_{\text{s}} \tan(0.1\pi) = 2f_{\text{s}} A$$

2. Therefore **analogue transfer function**:

$$H(s) = \frac{2f_{\text{s}}A}{s + 2f_{\text{s}}A}$$

3. Apply **bilinear transformation**: $s = 2f_{\text{s}}\frac{1-z^{-1}}{1+z^{-1}}$:

$$H(z) = \frac{2f_{\text{s}}A}{2f_{\text{s}}\frac{1-z^{-1}}{1+z^{-1}} + 2f_{\text{s}}A} = \left(\frac{2f_{\text{s}}}{2f_{\text{s}}}\right)\frac{A(1+z^{-1})}{(1-z^{-1}) + A(1+z^{-1})}$$

Notes

Bilinear Transformation

Example cont'd.

The z-transform transfer function of the filter is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A + Az^{-1}}{1 + A + (A-1)z^{-1}} \qquad (1)$$

Stability Analysis

Rearranging to determine **the poles** for **stability analysis** gives:

$$H(z) = \frac{A}{1+A}\frac{z+1}{z + \frac{A-1}{1+A}}.$$

- So there is 1 pole at $z + \frac{A-1}{1+A} = 0$ or $z = -\frac{A-1}{1+A}$.
- Remember $A = \tan(0.1\pi)$, so the pole is: $z = -0.50953$,
- *the magnitude is less than 1, so the filter is stable.*

Notes

Bilinear Transformation

Example cont'd.

Difference Equation

The **difference equation** can now be found.
Multiplying both sides by both denominators of equation (1) results in

$$Y(z)\left\{1 + A + (A-1)z^{-1}\right\} = X(z)\left\{A + Az^{-1}\right\}$$

Remembering that each z^{-1} is a unit **delay**, so that

$$(1+A)y[n] + (A-1)y[n-1] = Ax[n] + Ax[n-1]$$

Dividing through by $(1+A)$ and rearranging gives

$$y[n] = \frac{A}{1+A}\left(x[n] + x[n-1]\right) - \frac{A-1}{1+A}y[n-1],$$

where $A = \tan(0.1\pi)$.
This is now a difference equation we can use to filter a signal.

Notes

Bilinear Transformation

Example cont'd.

Frequency Response

The frequency response can be found directly using the bilinear transformation or from the z-transform transfer function. We will compare both approaches.

Notes

Bilinear Transformation

Example cont'd.

Bilinear Transformation

The analogue transfer function from step 2 in earlier slide was:

$$H(s) = \frac{2f_s A}{s + 2f_s A}$$

The s-plane variable s can be replaced by the Fourier complex frequency variable $j\omega$,

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{2f_s A}{j\omega + 2f_s A}.$$

The Fourier frequency can then be converted to the digitial frequency Ω using $\omega = 2f_s \tan\left(\frac{\Omega}{2}\right)$ (see earlier slide):

$$H(\Omega) = H(\omega)\Big|_{\omega=2f_s\tan\left(\frac{\Omega}{2}\right)} = \frac{2f_s A}{j2f_s\tan\left(\frac{\Omega}{2}\right) + 2f_s A} = \frac{A}{j\tan\left(\frac{\Omega}{2}\right) + A}$$

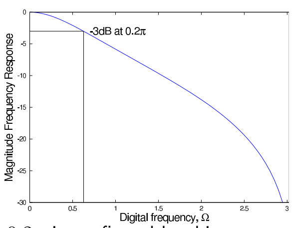
Notes

Bilinear Transformation

Example cont'd.

So the magnitude frequency response calculated directly from the Bilinear transformation is:

$$|H(\Omega)| = \sqrt{\frac{A^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2 + A^2}}$$
$$= \sqrt{\frac{(\tan(0.1\pi))^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2 + (\tan(0.1\pi))^2}}$$



The designed cut-off frequency $\Omega_{cf} = 0.2\pi$ is confirmed by this plot.

Notes

Bilinear Transformation

Example cont'd.

Frequency Response from z-Transform Transfer Function

Remember the z-transform transfer function calculated earlier (equation (1)):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A + Az^{-1}}{1 + A + (A-1)z^{-1}}.$$

This can be converted to the frequency response using

$$H(\Omega) = H(z)\Big|_{z=\exp(-jk\Omega)} = \frac{\sum_{k=0}^{M} b[k]\exp(-jk\Omega)}{\sum_{k=0}^{N} a[k]\exp(-jk\Omega)}$$

Notes

Bilinear Transformation

Example cont'd.

So:

- $b[0] = b[1] = A$,
- $a[0] = 1 + A$,
- $a[1] = A - 1$.

So the frequency response is given by (using Euler's identity):

$$H(\Omega) = \frac{A + A(\cos(\Omega) - j\sin(\Omega))}{(1+A) + (A-1)(\cos(\Omega) - j\sin(\Omega))}.$$

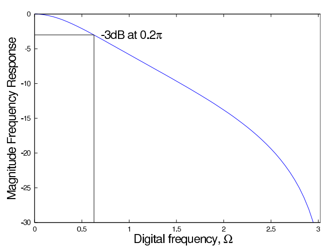
Notes

Bilinear Transformation

Example cont'd.

So the magnitude is given by:

$$|H(\Omega)| = \sqrt{\frac{(A + A\cos(\Omega))^2 + (A\sin(\Omega))^2}{((1+A) + (A-1)\cos(\Omega))^2 + ((A-1)\sin(\Omega))^2}}.$$



which is the same as the frequency response calculated directly from the bilinear transformation. The Bilinear transformation is quicker here.

Notes

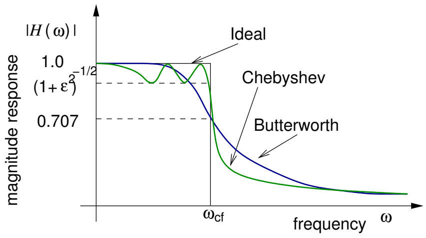
Famous Analogue Filters

Butterworth

- Magnitude frequency response: $|H(\omega)| = \frac{1}{\left\{1+\left(\frac{\omega}{\omega_{\rm cf}}\right)^{2n}\right\}^{1/2}}$

Chebyshev

- Magnitude frequency response: $|H(\omega)| = \frac{1}{\left\{1+\epsilon^2 C_n^2\left(\frac{\omega}{\omega_{\rm cf}}\right)\right\}^{1/2}}$



Notes

Butterworth and Chebyshev Analogue Filters

Bilinear transformation: $j\omega = j2f_s\tan(\Omega/2)$, therefore:

	Magnitude Frequency Response (*Analogue*) $\lvert H(\omega)\rvert$	(*Digital*) $\lvert H(\Omega)\rvert$
Butterworth	$\frac{1}{\left[1+\left(\frac{\omega}{\omega_{\rm cf}}\right)^{2n}\right]^{1/2}}$	$\frac{1}{\left[1+\left(\frac{\tan(\Omega/2)}{\tan(\Omega_{\rm cf}/2)}\right)^{2n}\right]^{1/2}}$
Chebyshev	$\frac{1}{\left[1+\epsilon^2 C_n^2\left(\frac{\omega}{\omega_{\rm cf}}\right)\right]^{1/2}}$	$\frac{1}{\left[1+\epsilon^2 C_n^2\left(\frac{\tan(\Omega/2)}{\tan(\Omega_{\rm cf}/2)}\right)\right]^{1/2}}$

Notes

Comparison of IIR and FIR filters

Characteristic	IIR	FIR
Multiplications	least	most
Coefficient quantification sensitivity	can be high	very low
Overflow errors	can be high	very low
Stability	by design	always
Linear phase	no	always
Simulate analog filter	yes	no
Coefficient memory	least	most
Design complexity	moderate	simple

adapted from "Understanding digital signal processing" by R. G. Lyons

Notes

Frequency Transformation

So far we have looked at **low pass IIR filters** only.
Frequency transformation can be used to convert a low pass filter into:

- Another type of lowpass
- Highpass
- Bandpass
- or Bandstop

Frequency transformation can be performed in the:

- Analogue form
- Or the digital form.

Notes

Frequency Transformation of Digital Filters

Type	**Transformation**	**Parameters**
Lowpass	$z^{-1} \rightarrow \frac{z^{-1}-a}{1-az^{-1}}$	$a = \frac{\sin((\Omega_p-\Omega_p')/2)}{\sin((\Omega_p+\Omega_p')/2)}$
Highpass	$z^{-1} \rightarrow \frac{z^{-1}-a}{1+az^{-1}}$	$a = -\frac{\cos((\Omega_p+\Omega_p')/2)}{\cos((\Omega_p-\Omega_p')/2)}$
Bandpass	$z^{-1} \rightarrow -\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha K}{K+1}$. $a_2 = \frac{K-1}{K+1}$. $\alpha = \frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)}$. $K = \cot\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}$.
Bandstop	$z^{-1} \rightarrow -\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha}{K+1}$ $a_2 = \frac{(1-K)}{(1+K)}$ $\alpha = \frac{\cos((\Omega_u+\Omega_l)/2)}{\cos((\Omega_u-\Omega_l)/2)}$ $K = \tan\frac{\Omega_u-\Omega_l}{2}\tan\frac{\Omega_p}{2}$

from Proakis and Manolakis, "Digital Signal Processing, Principles, Algorithms and Applications"

Notes

Summary

- Introduction to IIR filters.
- Frequency domain parameters.
- Pole-zero placement method for IIR filter design
- Band stop filter design
- IIR filter design from analog filters using bilinear transformation method

Notes