Solutions for: Fourier Series Tutorial

1. **Discrete Periodic Fourier Series**

   (a) The discrete Fourier series can be calculated with

$$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \exp\left(\frac{-j2\pi kn}{N}\right)$$

where N is the number of samples in one period of the signal $x[n]$ and $a[k]$ are the discrete Fourier series. Also recall Euler's identity:

$$\exp\left(\frac{-j2\pi kn}{N}\right) = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right). \tag{1}$$

Assuming $x[n]$ is composed of just a real part (no imaginary part) then the real part of the discrete Fourier series can therefore be calculated with

$$\mathrm{Re}(a[k]) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi kn}{N}\right)$$

and the imaginary part:

$$\mathrm{Im}(a[k]) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] j\sin\left(\frac{2\pi kn}{N}\right).$$

For the signal $x[n] = \sin(2\pi n/N)$ where $N = 8$:

i. Calculate the real part of the discrete Fourier series, showing all essential working and steps. **Solution** For $k = 0$ for the real part we have:

$$\begin{aligned}
\mathrm{Re}(a[k=0]) = &\frac{1}{8}(\sin(2\pi 0/8)\cos(0) + \sin(2\pi 1/8)\cos(0) + \sin(2\pi 2/8)\cos(0) + \sin(2\pi 3/8)\cos(0) \\
&+ \sin(2\pi 4/8)\cos(0) + \sin(2\pi 5/8)\cos(0) + \sin(2\pi 6/8)\cos(0) + \sin(2\pi 7/8)\cos(0)) \\
&= \frac{1}{8}(\sin(0) + \sin(\pi 2/8) + \sin(\pi 4/8) + \sin(\pi 6/8) \\
&+ \sin(\pi) + \sin(\pi 10/8) + \sin(\pi 12/8) + \sin(\pi 14/8) \\
&= \frac{1}{8}(0.707 + 1 + 0.707 + 0 - 0.707 - 1 - 0.707) = 0.
\end{aligned}$$

As you can see there are quite a few calculations and it is probably easier to put it in table form like so for the real part:

		n: 0	1	2	3	4	5	6	7	
k	$\cos(2\pi kn/N)$	$\sin(0)$ $=0$	$\sin(\pi/4)$ $=1/\sqrt{2}$	$\sin(\pi/2)$ $=1$	$\sin(3\pi/4)$ $=1/\sqrt{2}$	$\sin(\pi)$ $=0$	$\sin(5\pi/4)$ $=-1/\sqrt{2}$	$\sin(3\pi/2)$ $=-1$	$\sin(7\pi/4)$ $=-1/\sqrt{2}$	$\mathrm{Re}(a[k])$ $\frac{1}{N}\sum$
0	1	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0
1	$\cos(\pi n/4)$	0	0.5	0	-0.5	0	0.5	0	-0.5	0
2	$\cos(\pi n/2)$	0	0	-1	0	0	0	1	0	0
3	$\cos(\pi n3/4)$	0	-0.5	0	0.5	0	-0.5	0	0.5	0
4	$\cos(\pi n)$	0	$-1/\sqrt{2}$	1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	-1	$1/\sqrt{2}$	0
5	$\cos(\pi n5/4)$	0	-0.5	0	0.5	0	-0.5	0	0.5	0
6	$\cos(\pi n3/2)$	0	0	-1	0	0	0	1	0	0
7	$\cos(\pi n7/4)$	0	0.5	0	-0.5	0	0.5	0	-0.5	0

ii. Calculate the imaginary part of the discrete Fourier series, showing all essential working and steps. **Solution** For the imaginary part:

		n: 0	1	2	3	4	5	6	7	
k	$\cos(2\pi kn/N)$	$\sin(0)$ $=0$	$\sin(\pi/4)$ $=1/\sqrt{2}$	$\sin(\pi/2)$ $=1$	$\sin(3\pi/4)$ $=1/\sqrt{2}$	$\sin(\pi)$ $=0$	$\sin(5\pi/4)$ $=-1/\sqrt{2}$	$\sin(3\pi/2)$ $=-1$	$\sin(7\pi/4)$ $=-1/\sqrt{2}$	$\mathrm{Im}(a[k])$ $\frac{1}{N}\sum$
0	$\sin(0)$	0	0	0	0	0	0	0	0	0
1	$\sin(\pi n/4)$	0	0.5	1	0.5	0	0.5	1	0.5	0.5
2	$\sin(\pi n/2)$	0	0.70711	0	-0.70711	0	-0.70711	0	0.70711	0
3	$\sin(\pi n3/4)$	0	0.5	-1	0.5	0	0.5	-1	0.5	0
4	$\sin(\pi n)$	0	0	0	0	0	0	0	0	0
5	$\sin(\pi n5/4)$	0	-0.5	1	-0.5	0	-0.5	1	-0.5	0
6	$\sin(\pi n3/2)$	0	-0.70711	0	0.70711	0	0.70711	0	-0.70711	0
7	$\sin(\pi n7/4)$	0	-0.5	-1	-0.5	0	-0.5	-1	-0.5	-0.5

(b) The real and imaginary parts both contain useful information that is often best viewed combined using the magnitude. Calculate the magnitude of this discrete Fourier series. **Solution** The magnitude is given by:

$$\mathrm{Mag}(a[k]) = \sqrt{\mathrm{Re}(a[k])^2 + \mathrm{Im}(a[k])^2}.$$

The magnitude values for the signal can be found below:

	k							
	0	1	2	3	4	5	6	7
$\mathrm{Re}(a[k])$	0	0	0	0	0	0	0	0
$\mathrm{Im}(a[k])$	0	-0.5	0	0	0	0	0	0.5
$\mathrm{Mag}(a[k])$	0	0.5	0	0	0	0	0	0.5

(c) Explain the magnitude signal in two sentences. **Solution** The magnitude signal contains two line spectra, one corresponding to the frequency at which the the $\sin(2\pi n/8)$ occurs and another at the equivalent negative frequency. The negative frequencies can be translated like so:

$$\mathrm{Mag}(a[k]) = (0, 0.5, 0, 0, \underbrace{0, 0, 0, 0.5})^{\mathrm{T}}$$

so that we have:

$$\mathrm{Mag}(a[k]) = (\underbrace{0, 0, 0, 0.5}, 0, 0.5, 0, 0)^{\mathrm{T}}$$

(d) Total power: Show that the total power in the time and frequency domains are equal and name the relevant *theorem*. **Solution** Total power in the time domain $\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2$ is

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \frac{1}{8}\sum_{n=0}^{7}(x[n])^2 = 0.5.$$

is equal to the total power in the frequency domain $\sum_{k=0}^{N-1}\mathrm{Mag}(X[k])^2$:

$$\sum_{k=0}^{N-1}(\mathrm{Mag}(a[k]))^2 = 0 + 0.5^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0.5^2 = 0.5.$$

This shows that the power is equal irrespective of the domain. This is known as Parseval's theorem.

2. **Fourier Transform for Discrete Aperiodic Sequences**

(a) These questions are about the Fourier Transform for discrete aperiodic sequences which are digital signals that do not repeat or are not periodic. This Fourier transform for discrete aperiodic sequences is defined by

$$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n)$$

where $x[n]$ is the discrete aperiodic signal or sequence.

(b) The Dirac delta function is defined as

$$\delta[n-v] = \begin{cases} 1 & \text{for} \quad n = v, \\ 0 & \text{elsewhere.} \end{cases}$$

i. Calculate the Fourier transform magnitude and phase of a dirac delta function $\delta[n-v]$ centered at $v = 1$. Show all working and essential steps. **Solution**

A. For $x[n] = \delta[n-1]$,

$$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=1}^{1}\exp(-j\Omega n) = \exp(-j\Omega)$$

and using Euler's formula

$$X(\Omega) = \cos(\Omega) - j\sin(\Omega)$$

- The magnitude is given by:

$$\text{Mag}(X(\Omega)) = \sqrt{\cos(\Omega)^2 + \sin(\Omega)^2}.$$

This corresponds to a trigonometric identity,

$$\sin(a)^2 + \cos(a)^2 = 1.$$

Therefore

$$\text{Mag}(X(\Omega)) = 1.$$

The phase is given by

$$\Phi(X(\Omega)) = \tan^{-1}\left(-\frac{\sin(\Omega)}{\cos(\Omega)}\right).$$

ii. Calculate the Fourier transform magnitude and phase of a dirac delta function $\delta[n-v]$ centered at $v=0$. Show all working and essential steps. **Solution** For $x[n]=\delta[n]$, $X(\Omega) = \sum_{n=0}^{0} \exp(-j\Omega n) = \exp(0) = 1.$
- The magnitude is: $\text{Mag}(X(\Omega)) = 1.$
- The phase is: $\Phi(X(\Omega)) = 0.$

iii. Explain the difference between the two responses with $v=1$ and $v=0$. Consider the role of odd and even functions. **Solution** The two responses are different in the phase because they are the same signal but with a phase shift between them. Also when $v=0$, $\delta[n-v]$ becomes an even function (where $x[n]=x[-n]$). The real or cosine part of the complex number is also even and even functions can be represented purely by the real part of the Fourier spectrum. This is in comparison to when $v \neq 0$, so the delta function is no longer even and it is not an odd ($x[n]=-x[-n]$) function. The resulting Fourier spectrum is complex.

iv. What is the significance of the magnitude of the Fourier transform for the Dirac delta function? **Solution** The Dirac delta function contains all frequencies and it requires all frequencies in the frequency domain to construct a Dirac delta function in the time domain. This is particularly useful for the characterization of linear time invariant systems where we may want to know a systems response to any frequency.

v. Calculate the *inverse* discrete Fourier transform of a Dirac delta function centred at $v=0$. **Solution** Inverse aperiodic discrete Fourier transform is given by:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\exp(j\Omega n)\mathrm{d}\Omega$$

and for $X(\Omega)=\delta(\Omega)$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)\exp(j\Omega n)\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\delta(\Omega=0)\exp(0) \\ &= \frac{1}{2\pi}. \end{aligned}$$

(c) A square wave can be defined as

$$x[n] = \begin{cases} a & \text{for} \quad v_1 \leq n \leq v_2, \\ 0 & \quad \text{elsewhere.} \end{cases}$$

i. Calculate the Fourier transform magnitude of a square wave for $v_1=-1$, $v_2=1$ and $a=1/3$ and sketch the response.

ii. Calculate the Fourier transform magnitude of a square wave for $v_1=-2$, $v_2=2$ and $a=1/5$ and sketch the response.

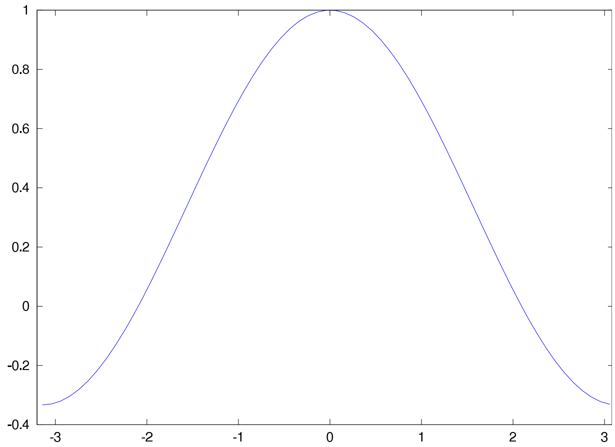

Figure 1: Aperiodic discrete Fourier transform for a square wave with $v_1 = -1$, $v_2 = 1$ and $a = \frac{1}{3}$.

iii. These square waves can be considered as non-causal moving average filter coefficients, so that
   A. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$ and
   B. $y[n] = \frac{1}{5}(x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2])$.
   Convert causal 3 term and 5 term moving average filters into the generalized difference equation forms (see question 1(b)) and calculate the magnitude frequency response for both filters.
iv. Sketch the response to these filters for $\Omega = \pm\pi$. Are these filters high pass, band pass, notch or low pass filters?

Solution

i. For a square wave $v_1 = -1$, $v_2 = 1$ and $a = 1/3$, the Fourier transform is:

$$\begin{aligned}
X(\Omega) &= \sum_{n=v_1}^{v_2} a \exp(-j\Omega n) \\
&= \sum_{-1}^{1} \frac{1}{3} \exp(-j\Omega n) \\
&= \frac{1}{3}\left(\exp(j\Omega) + \exp(0) + \exp(-j\Omega)\right).
\end{aligned}$$

Using Euler's formula (see equation (1)):

$$\begin{aligned}
X(\Omega) &= \frac{1}{3}\left(\cos(\Omega) + j\sin(\Omega) + 1 + \cos(\Omega) - j\sin(\Omega)\right) \\
&= \frac{1}{3}\left(1 + 2\cos(\Omega)\right).
\end{aligned}$$

The magnitude is then:

$$\mathrm{Mag}(X(\Omega)) = \frac{1}{3}\left(1 + 2\cos(\Omega)\right).$$

The response was calculated in octave using (but you can sketch the response by hand using a calculator for a few values of $-\pi \leq \Omega \leq \pi$):

- omega=-pi():0.1:pi();
- plot(omega,(1+2*cos(omega))/3)
- print('-djpg','sqResponse1.jpg');

and shown in figure 1.

ii. For a square wave $v_1 = -2$, $v_2 = 2$ and $a = 1/5$, the Fourier transform is:

$$\begin{aligned}
X(\Omega) &= \frac{1}{5}\sum_{-2}^{2} \exp(-j\Omega n) \\
&= \frac{1}{5}\left(\exp(2j\Omega) + \exp(j\Omega) + \exp(0) + \exp(-j\Omega) + \exp(-2j\Omega)\right).
\end{aligned}$$

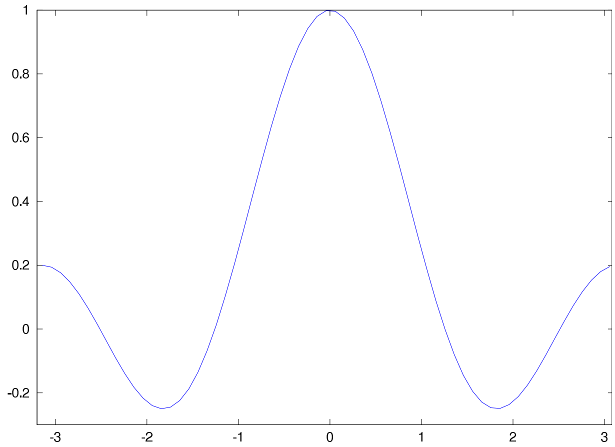

Figure 2: Aperiodic discrete Fourier transform for a square wave with $v_1 = -2$, $v_2 = 2$ and $a = \frac{1}{5}$.

Using Euler's formula (see equation (1)):

$$\begin{aligned} X(\Omega) &= \frac{1}{5}\left(\cos(2\Omega) + j\sin(2\Omega) + \cos(\Omega) + j\sin(\Omega) + 1 + \cos(\Omega) - j\sin(\Omega) + \cos(2\Omega) - j\sin(2\Omega)\right) \\ &= \frac{1}{5}\left(1 + 2(\cos(\Omega) + \cos(2\Omega))\right). \end{aligned}$$

The magnitude is then:

$$\mathrm{Mag}(X(\Omega)) = \frac{1}{5}\left(1 + 2(\cos(\Omega) + \cos(2\Omega))\right).$$

Response calculated in octave using (but sketch response by hand using a few values of $-\pi \leq \Omega \leq \pi$):

- omega=-pi():0.1:pi();
- plot(omega,(1+2*(cos(omega)+cos(2*omega))/5)
- print('-djpg','sqResponse2.jpg');

and shown in figure 2.

iii. Two causal moving average filters:

- 3 term:

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right). \tag{2}$$

The generalized difference equation is of the form

$$\sum_{m=0}^{M} a[m]y[n-m] = \sum_{v=0}^{V} b[v]x[n-v]. \tag{3}$$

Hence, converting (2) to generalized form gives us,

$$y[n] = \sum_{m=0}^{2} \frac{1}{3}x[n-m] \tag{4}$$

- 5 term:

$$y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right). \tag{5}$$

In generalized form:

$$y[n] = \sum_{m=0}^{4} \frac{1}{5}x[n-m] \tag{6}$$

Frequency response from generalized form:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^{M} b[m]\exp(-jm\Omega)}{\sum_{m=0}^{N} a[m]\exp(-jm\Omega)}.$$

- For the 3 term filter:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\frac{1}{3}(\exp(0) + \exp(-j\Omega) + \exp(-j2\Omega))}{\exp(0)}.$$

  Using Euler's formula:

$$H(\Omega) = \frac{1}{3}(1 + \cos(\Omega) - j\sin(\Omega) + \cos(2\Omega) - j\sin(2\Omega))$$

  Magnitude is:

$$\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{9}\left((1 + \cos(\Omega) + \cos(2\Omega))^2 + (\sin(\Omega) + \sin(2\Omega))^2\right)}$$

  The magnitude can be simplified using the trigonometric identities:

$$\begin{array}{ll} 2\sin(\Omega)\sin(2\Omega) = \cos(\Omega) - \cos(3\Omega) & \sin^2(2\Omega) = \frac{1-\cos(4\Omega)}{2} \\ 2\cos(\Omega)\cos(2\Omega) = \cos(\Omega) + \cos(3\Omega) & \cos^2(\Omega) = \frac{1+\cos(2\Omega)}{2} \\ \sin^2(\Omega) = \frac{1-\cos(2\Omega)}{2} & \cos^2(2\Omega) = \frac{1+\cos(4\Omega)}{2} \end{array}$$

  So that

$$\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{9}(3 + 2(2\cos(\Omega) + \cos(2\Omega)))}.$$

  Phase:

$$\Phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega) + \sin(2\Omega))}{(1 + \cos(\Omega) + \cos(2\Omega))}\right)$$

- For the 5 term filter:

$$H(\Omega) = \frac{1}{5}(1 + \cos(\Omega) - j\sin(\Omega) + \cos(2\Omega) - j\sin(2\Omega) + \cos(3\Omega) - j\sin(3\Omega) + \cos(4\Omega) - j\sin(4\Omega))$$

  Magnitude is:

$$\begin{aligned}\text{Mag}(H(\Omega)) = \Big(\frac{1}{25}&((1+\cos(\Omega) + \cos(2\Omega) + \cos(3\Omega) + \cos(4\Omega))^2 \\ &+ (\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega))^2\Big)^{1/2},\end{aligned}$$

  and Phase

$$\Phi(H(\Omega)) = \tan^{-1}\Big(-\frac{\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega)}{1 + \cos(\Omega) + \cos(2\Omega) + \cos(3\Omega) + \cos(4\Omega)}\Big).$$

iv. The magnitude frequency response for the 3 term filter is shown in figure 3 and the 5 term filter is shown in figure 4.
These filters are non-ideal low pass filters.

3. Replace t by $t+T$ in the expression for the input signal: $e^{j2\pi ft}$. By solving $x(t+T) = x(t)$, find the period of the signal, i.e. the smallest nonzero value of T for which the equation works. **Solution** Ans:

$$e^{j2\pi f(t+T)} = e^{j2\pi ft}$$

$$e^{j2\pi ft}e^{j2\pi fT} = e^{j2\pi ft}$$

$$e^{j2\pi fT} = 1$$

i.e.

$$cos(2\pi fT) + j.sin(2\pi fT) = 1$$

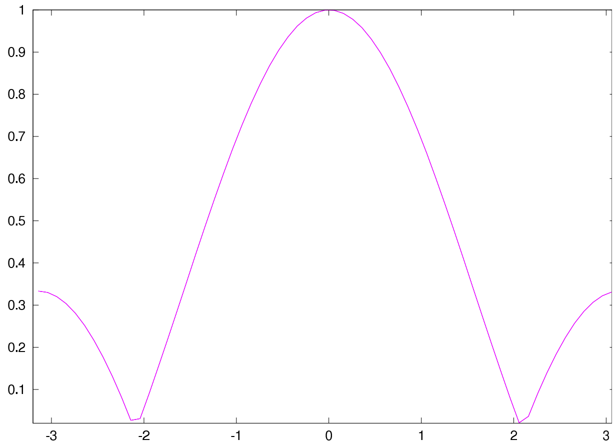

Figure 3: Magnitude frequency response for the causal 3 term moving average filter.

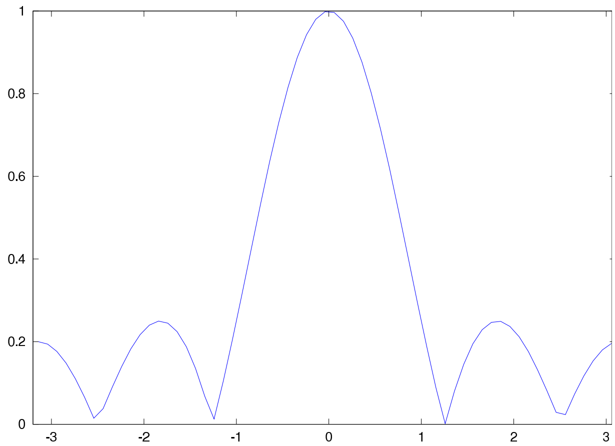

Figure 4: Magnitude frequency response for the causal 5 term moving average filter.

Obviously $T = 0$ works but that's simply saying a signal equals itself, which must be true for all signals! The functions $sin(x)$ and $cos(x)$ are periodic with period 2π, so the next value of T for which we get equality is

$$2\pi fT = 2\pi$$

From which we get the standard result that

$$T = \frac{1}{f}$$

4. Using compound angle formulae, do the same for sine and cosine functions:

$$cos\left(2\pi f(t+T)\right) = cos\left(2\pi ft\right)$$

$$sin\left(2\pi f(t+T)\right) = sin\left(2\pi ft\right)$$

Solution Ans:

$$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$$

$$cos\left(2\pi f(t+T)\right) = cos\left(2\pi ft\right)cos\left(2\pi fT\right) - sin\left(2\pi ft\right)sin\left(2\pi fT\right)$$

If we want this to equal what we started with for all values of t then we must have that

$$cos\left(2\pi fT\right) = 1$$

and

$$sin\left(2\pi fT\right) = 0$$

As with $e^{j2\pi ft}$, we use the fact that cos is periodic with period 2π to conclude that

$$T = \frac{1}{f}$$

The $sin()$ part is left as an exercise for the student

5. Write down the Fourier series for $x(t+T)$ based on the expression for $x(t)$:

$$x(t) = \sum_{k=-\infty}^{+\infty} x_k e^{j2\pi(kf_0)t}$$

Use algebra to show that $x(t+T)$ is the same as $x(t)$ (i.e. that $x(t)$ is a periodic signal with period T). You will need to use the fact that $e^{j2\pi} = 1$, which is a consequence of $e^{j\theta} = cos(\theta) + jsin(\theta)$. **Solution** Ans:

$$\begin{aligned} x(t+T) &= \sum_{k=-\infty}^{+\infty} x_k e^{j2\pi kf_0(t+T)} \\ &= \sum_{k=-\infty}^{+\infty} x_k e^{j2\pi kf_0t} e^{j2\pi kf_0T} \\ &= e^{j2\pi kf_0T} \sum_{k=-\infty}^{+\infty} x_k e^{j2\pi kf_0t} \\ &= e^{j2\pi kf_0T} x(t) \end{aligned}$$

This will equal $x(t)$ provided

$$e^{j2\pi kf_0T} = 1$$

We can rewrite this as

$$\left(e^{j2\pi f_0T}\right)^k$$

and using our previous result that

$$T = \frac{1}{f_0}$$

the expression becomes

$$\left(e^{j2\pi}\right)^k$$

and we have previously shown that

$$e^{j2\pi} = 1$$

and we know that

$$1^k = 1$$

which means we have

$$x(t+T) = 1 \times x(t)$$

which is the required result.

6. Remembering a previously emphasised point about variables of summation, we can write our Fourier series as

$$x(t) = \sum_{m=-\infty}^{+\infty} x_m e^{j2\pi mt/T}$$

Substitute this into the formula for x_k and see what you get. Hint: you may need to handle the k=m case separately from the other cases; you will also need to use the fact that integration is a linear operation (so the integral of a sum is a sum of integrals). **Solution** Ans:

$$\begin{aligned}\frac{1}{T}\int_a^{a+T} x(t)e^{-j2\pi kt/T}dt &= \frac{1}{T}\int_a^{a+T} \sum_{m=-\infty}^{+\infty} x_m e^{j2\pi mt/T} e^{-j2\pi kt/T} dt \\ &= \frac{1}{T}\sum_{m=-\infty}^{+\infty} x_m \int_a^{a+T} e^{j2\pi mt/T} e^{-j2\pi kt/T} dt \\ &= \frac{1}{T}\sum_{m=-\infty}^{+\infty} x_m \int_a^{a+T} e^{j2\pi(m-k)t/T} dt\end{aligned}$$

In the case where $m \neq k$ the integral is of a complex sinusoid of period $\frac{T}{|m-k|}$ over a time T, i.e. over $|m-k|$ whole periods. If we integrate a sinusoid over a whole period the result is zero, so in the above expression, all the terms for which $m \neq k$ contribute nothing to the sum so only the $m = k$ term remains. Therefore the expression can be written as

$$= \frac{x_k}{T}\int_a^{a+T} e^0 dt = \frac{x_k}{T}\int_a^{a+T} 1 dt = \frac{x_k}{T} \times T = x_k$$

In other words the formulas do work out as they should.

7. Find the Fourier series expansion of a rectangular pulse train where the period of the train is T

$$x(t+T) = x(t)$$

and the width of the pulses is τ ($< T$ of course):

$$x(t) = \begin{cases} \frac{1}{\tau} & |t| \leq \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$

Hint: do the integral from -T/2 to +T/2
Note: The height of the pulses has been chosen to be useful in later sections dealing with sampling.
Solution Ans:

$$\begin{aligned} x_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ &= \frac{1}{T}\int_{-\tau/2}^{\tau/2} \frac{1}{\tau} e^{-j2\pi kt/T}dt \\ &= \frac{1}{\tau}\frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-j2\pi kt/T}dt \end{aligned}$$

Since the limits of the integral are symmetric about zero, only the even part of the integrand will contribute to the result (the odd part will cancel out), so we can write this as

$$
\begin{aligned}
x_k &= \frac{1}{\tau}\frac{1}{T}\int_{-\tau/2}^{\tau/2} cos(2\pi kt/T)dt \\
&= \frac{1}{\tau}\frac{1}{T}\left[\frac{sin(2\pi kt/T)}{2\pi k/T}\right]_{-\tau/2}^{\tau/2} \\
&= \frac{1}{\tau}\frac{1}{2\pi k}\left[sin(2\pi kt/T)\right]_{-\tau/2}^{\tau/2} \\
&= \frac{1}{\tau}\frac{1}{2\pi k}\left(sin(\pi k\tau/T) - sin(-\pi k\tau/T)\right) \\
&= \frac{1}{\tau}\frac{1}{2\pi k}2sin(\pi k\tau/T) \\
&= \frac{1}{\tau}\frac{sin(\pi k\tau/T)}{\pi k} \\
&= \frac{1}{T}\frac{sin(\pi k\tau/T)}{\pi k\tau/T} = \frac{1}{T}sinc\left(\frac{k\tau}{T}\right)
\end{aligned}
$$

8. Check that the coefficients have conjugate symmetry **Solution** Ans
   The coefficients are actually real in this case so we only need to check that they have even symmetry and the sinc function has even symmetry because it is the ratio of two functions both having the same symmetry (odd).

9. What happens to the coefficients as $\tau \to 0$? **Solution** Ans

   Later in the unit we will meet the Dirac comb, a theoretical pulse train where the pulses have zero width but have an area of 1, forcing them to be of infinite height. We can use the above formula by setting $\tau = 0$, which gives us the result that

   $$x_k = T, (k = -\infty ... + \infty)$$

   i.e. all the harmonics have the same amplitude.

10. Use the Parseval's theorem formula to explain why $|X(f)|^2$ is sometimes known as the Energy Spectral Density of the signal. **Solution** Ans:

    Parseval's theorem says that you can find the signal energy either by integrating $|x(t)|^2$ with respect to time or by integrating $|X(f)|^2$ with respect to frequency.

    This is referred to as "energy"" because if $x(t)$ is in volts then $|x(t)|^2$ is in volts-squared, which is proportional to power $\left(\frac{V^2}{R}\right)$ and power is energy per unit time, so integrating power gives us energy. Effectively, if we assume a 1Ω load, it is the energy.

    This means that (again assuming a 1Ω load) $|X(f)|^2$ must have the units of Joules-per-Hertz, since we integrate it against frequency to get the energy. A "density" is "something-per-unit-something-else," eg mass density kg/m^3 or heat flux density W/m^{-2}. So J/Hz is an energy density and because it is per unit frequency it is a spectral density, hence "Energy Spectral Density".