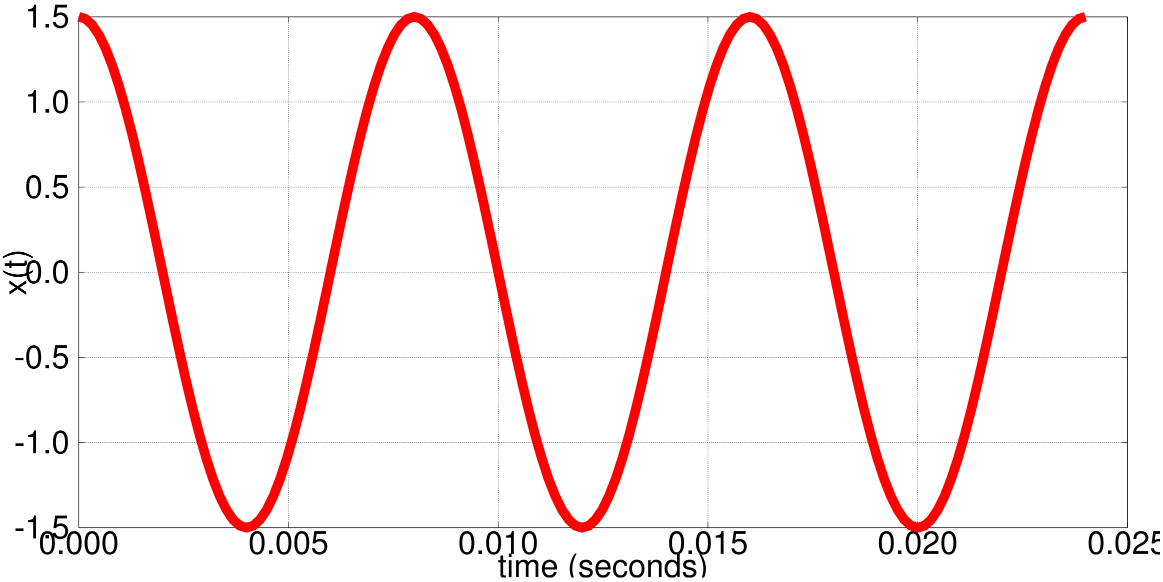
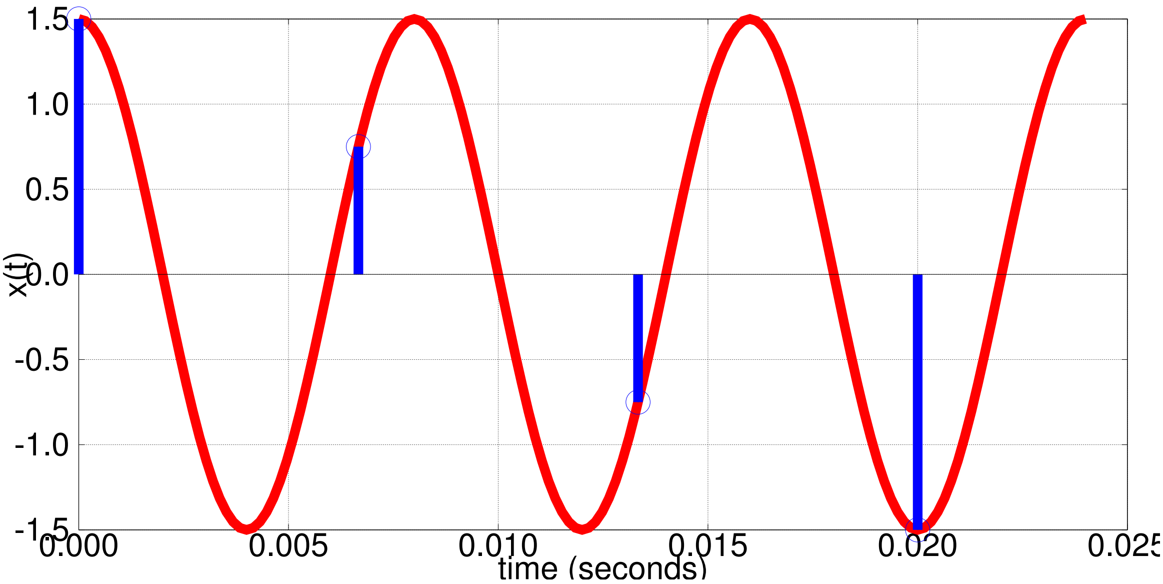
Solutions for: Sampled Signals Tutorial

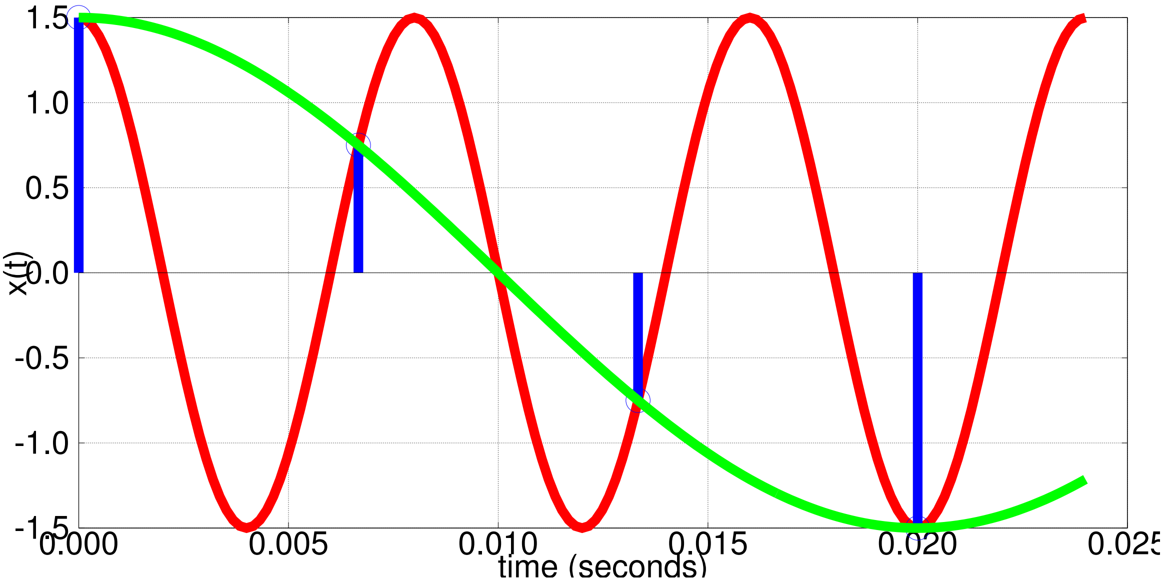
1. Sketch 3 cycles of a cosine wave with frequency 125Hz and amplitude $A = 1.5$ with zero phase.
   Hint: A cosine wave has equation $x(t) = A\cos(2\pi f t + \phi)$.
   Solution



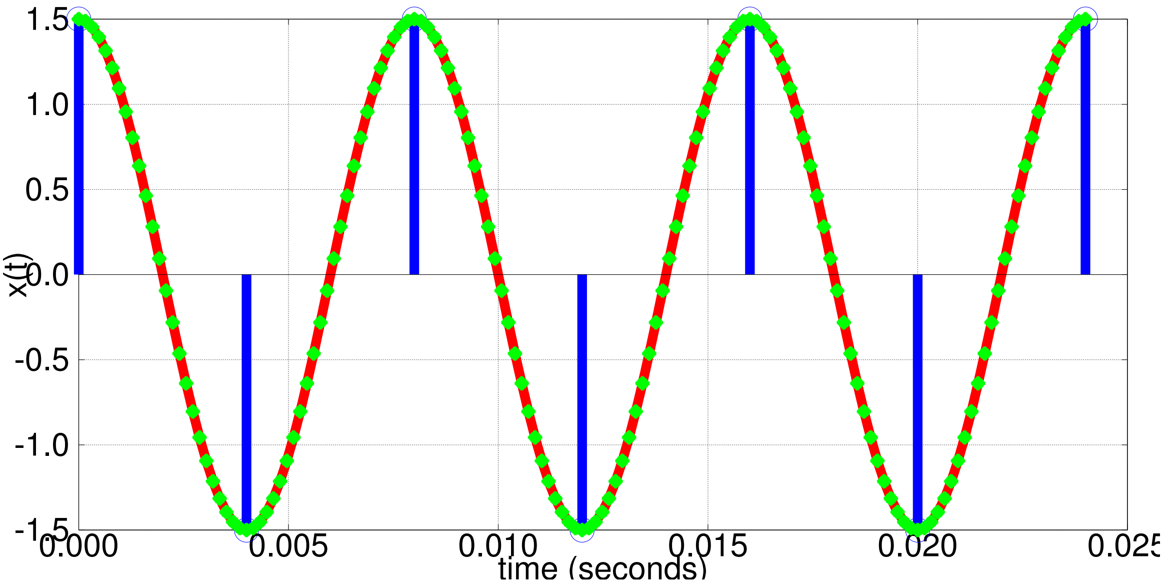
2. Sketch the above waveform again but include discrete points at which sampling occurs for a sampling frequency of 150Hz, assuming the first sample occurs at $t = 0$. **Solution**



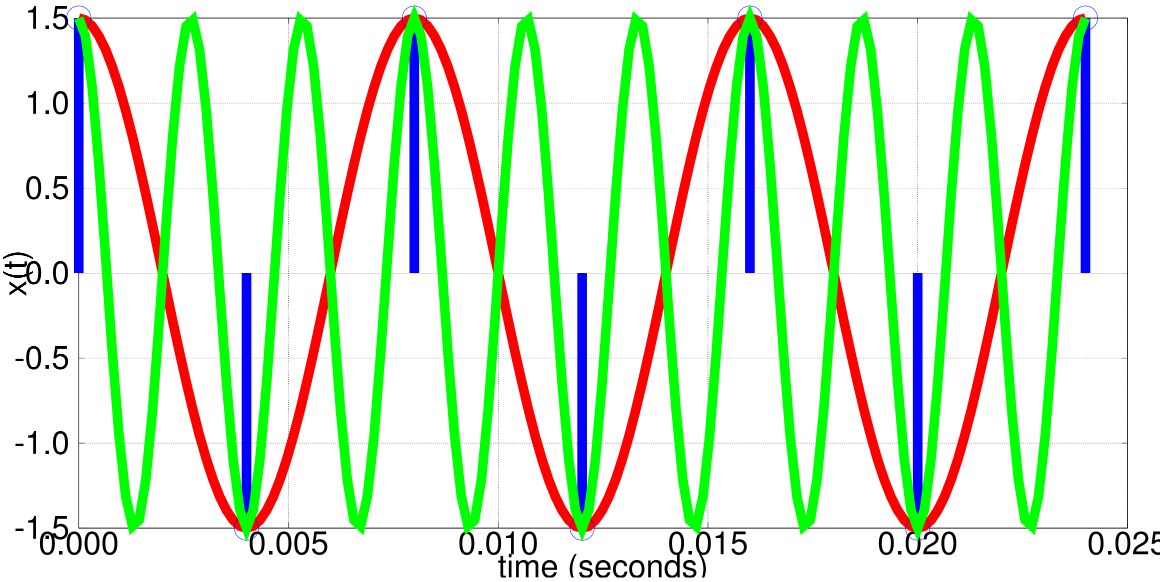
3. Sketch the above waveform again but include the aliasing frequency.
   Hint: An aliasing frequency occurs at $f_{\text{alias}} = f - f_s$. **Solution**



4. The above waveform is now sampled at twice the signal frequency, i.e. $f_s = 2 \times f = 2 \times 125 = 250$Hz. Resketch the waveform along with signal that could be assumed from the discrete samples upon conversion back to an analogue form. **Solution**



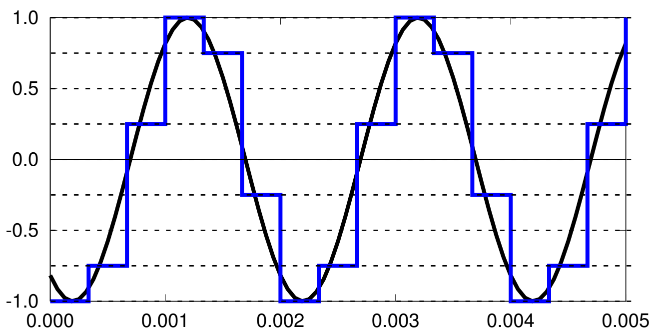
5. Draw the signal that originates from a signal component of the sampled signal that was drawn above, sampled at the Nyquist rate. Explain what you have drawn.
Hint: Signal components occur at $n \times f_s - f$ and $n \times f_s + f$ where n is an integer. **Solution** The digitisation process means that the signal is stored as a series of discrete samples. These discrete samples represent the original analogue signal. Even if the signal has been sampled at twice the maximum frequency, the discrete representation can still be interpreted as representing analogue signals with frequencies that are not the original sampled signal. An example folded frequency occurs at $f + f_s$:



6. What is the purpose of antialiasing filtering? **Solution** Anti-aliasing filter is a low pass filter that is used to remove frequencies above the maximum frequency content of interest in a signal. This helps an Analogue to Digital Conversion (ADC) process to adequately sample the signal in digital form. Signals will often contain component frequencies above the maximum of interest and so it is important to remove those components. If they are not then the digitised version will contain subtantial errors in the representation and prevent accurate reconstruction of the analogue signal at some later date.

7. Reconstruction Filtering:

   (a) What is the purpose of a reconstruction filter? **Solution** A reconstruction filter filters the signal after Digital to Analogue Conversion (DAC). After the DAC, the signal will typically consist of a series of discrete levels or stairs, e.g.

   The reconstruction filter will remove the stairs by smoothing the signal and eliminating the high frequencies, by low pass filtering the DAC signal.
   Another way of looking at the reconstruction filter is as an anti-imaging filter. An anti-imaging filter is designed to remove the high frequency components that are present in the digital signal often referred to as images of the actual sampled signal. These images are at different frequencies higher than the original sampled signal. A Digital to Analog Converter (DAC) will usually hold the digital output value until the next digital sample is to be output. This holding helps to attenuate the high frequency components in a sinc like function, i.e.

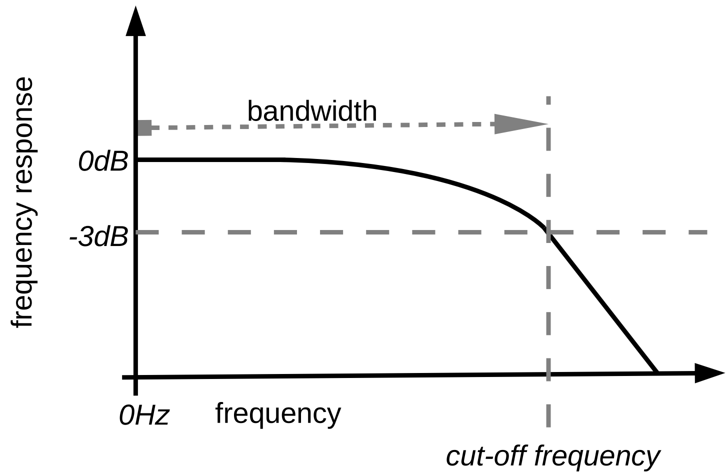
$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}.$$

   This sinc function arises because of the square like appearance of each digital held sample value. A square wave has a sinc frequency response so if we look at the frequency content of the DAC output, before anti-image filtering, the frequencies present in the signal should be the original signal and frequencies corresponding multiples of the original signal but attenuated by the sinc response.

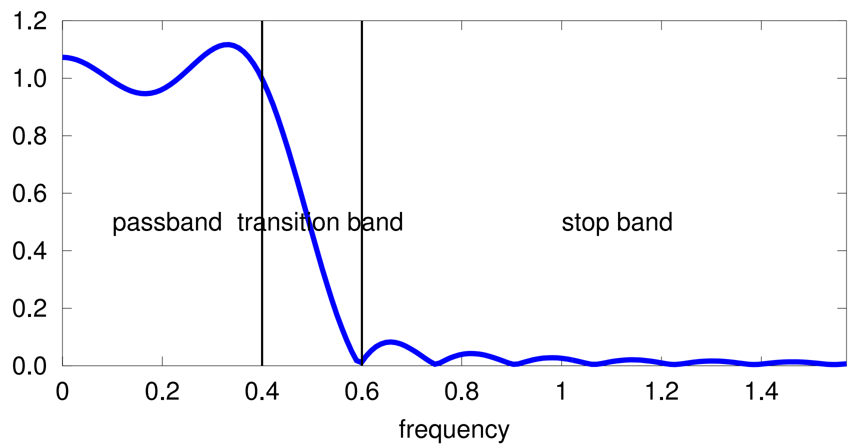
   (b) State the theoretical relationship between f_s and the bandwidth B of an ideal reconstruction filter. **Solution** The reconstruction filter is a low pass filter which should have a cut off frequency equal to the highest frequency that was present in the original analogue signal that was digitised at some point earlier. The signal may also have been sampled at a rate higher than twice the highest frequency, i.e. $f_s > 2f_{\max}$. Therefore, in the absence of direct knowledge regarding $f_{\max}$, the bandwidth of an ideal reconstruction filter should follow:

$$B_{\text{recon}} = \frac{f_s}{2}.$$

(c) Real world filters are not ideal. What does this mean when you want to use a real world filter for reconstruction of an analogue signal? **Solution** An ideal filter will be able to completely stop any frequency outside the filter bandwidth. However, real world filters are not ideal which means that frequencies outside of the bandwidth of the filter may only be attenuated, not usually completely stopped. The bandwidth for a filter is usually defined to be between the lowest and highest frequencies that can be attenuated in terms of their power by 50% or by -3dB:

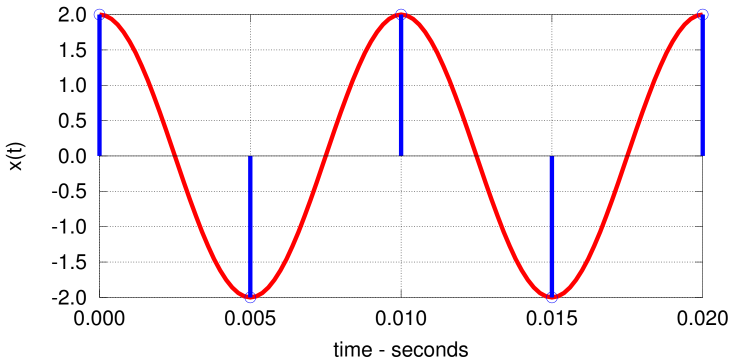


This means that if we want to reconstruct an analogue signal with a real world reconstruction filter then the response of the filter will need to be taken into account. Increasing the sampling rate can help because the quantisation noise will be reduced, particularly for any signal that the filter cannot completely attenuate because of a long transition band. Transition band is illustrated here:

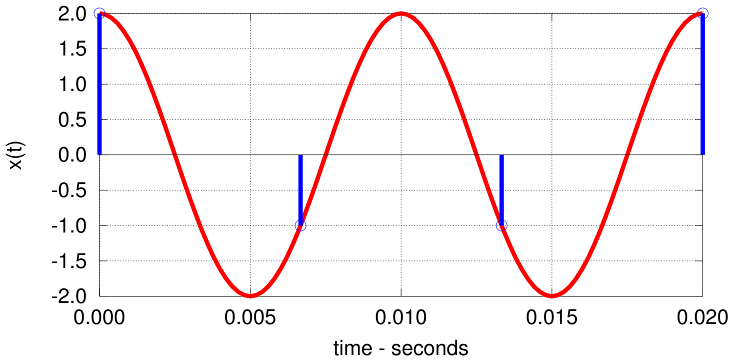


In theory, the stop band is the region that can be considered to completely stop any signal. However as illustrated above, the stop band still does not completely attenuate signals at all frequencies. To match the quality of the Analogue to Digital Converter (ADC) the smallest quantizable level of signal should be greater than the signal attenuation in the stop band. Furthermore, the stop band frequency of the antialiasing filter should be taken to be the maximum frequency component in the calculation of the sampling frequency to limit the effects of aliasing.

8. What are all the frequency components below $(3.5 \times f_s)$Hz that theory predicts will be present in sampled waveforms. **Solution** Nyquist's sampling theorem states that a signal with maximum frequency $f_{\max}$ must be sampled at $2 \times f_{\max}$. To see why this is true consider a simple Sine or Sinusoidal signal:

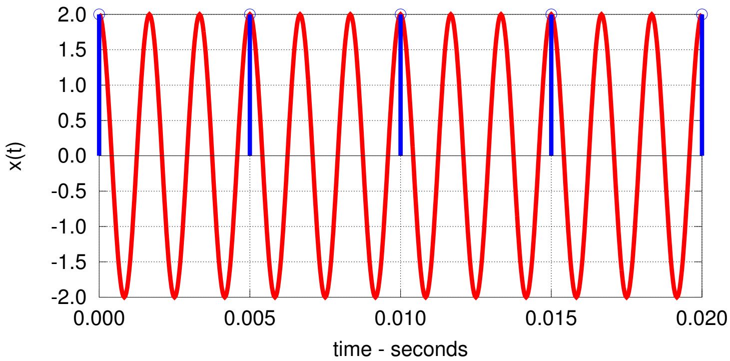


You can see here that the discrete samples of the continuous cosine waveform capture the signal twice in a single cycle. This is necessary to enable the digital information to record the amplitude of the signal at this frequency. If the frequency of the signal increases then the cycle time of the sine wave will decrease and the number of samples will be less than twice in a single cycle. E.g.

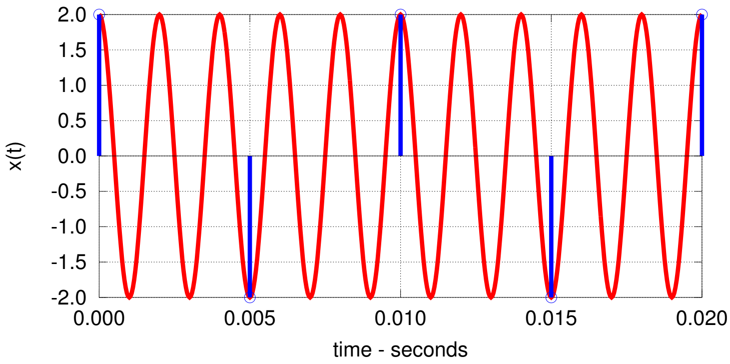


This signal has not been sampled enough times in a single cycle. This means that the digital information recorded about this signal is not enough to be able to reconstruct the signal if and when the signal is to be converted back from digital to the analogue form.
Something else that we can notice is the information recorded about a signal if the sampling frequency is kept constant but the frequency of the analogue signal is varied.
Here you will see some examples for varying the frequency of the analogue frequency f while f_s remains constant:

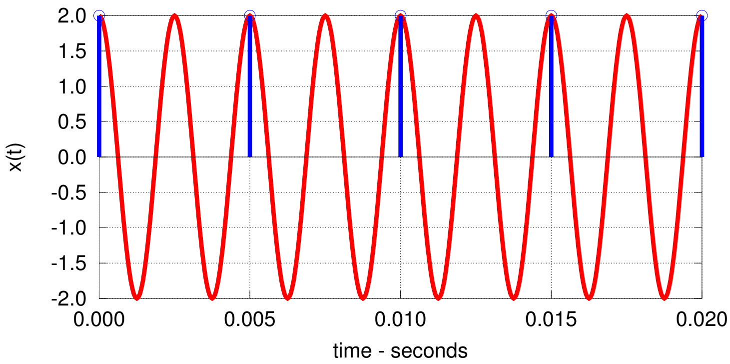
- $f/f_s = 3$



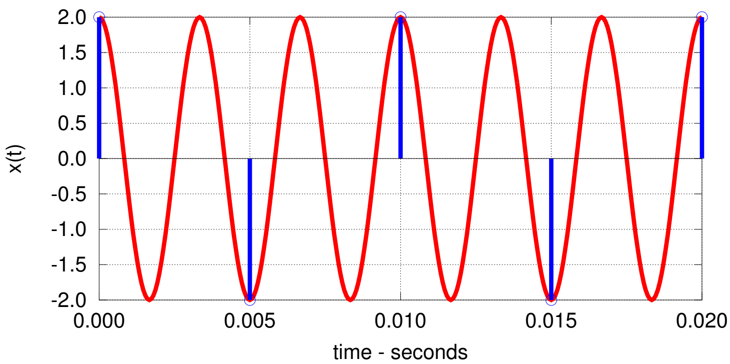
- $f/f_s = 2.5$



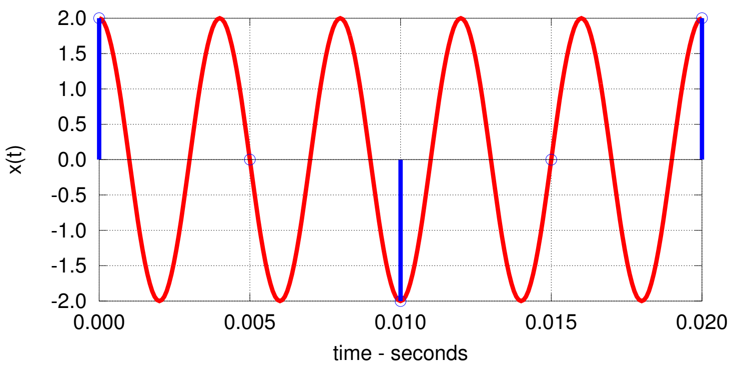
- $f/f_s = 2$



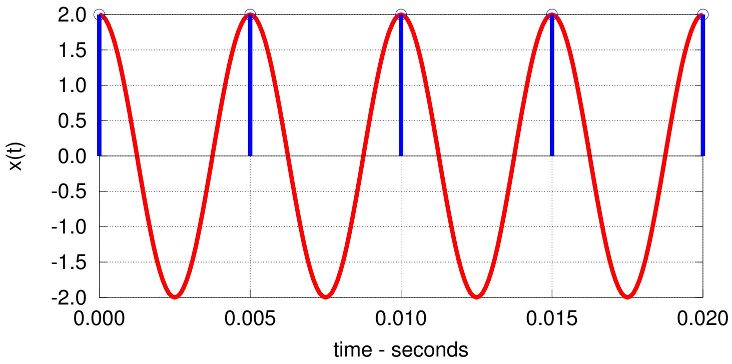
- $f/f_s = 1.5$



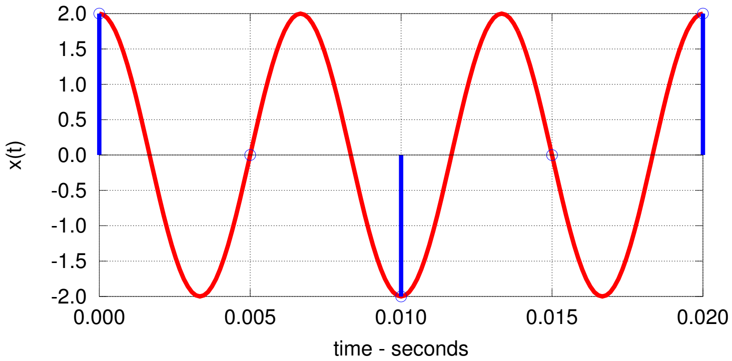
- $f/f_s = 1.25$



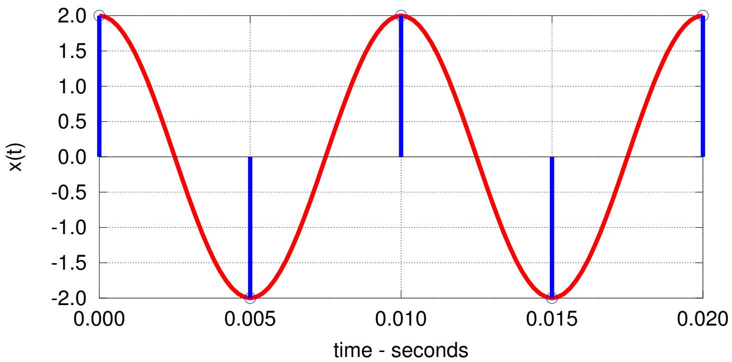
- $f/f_s = 1$



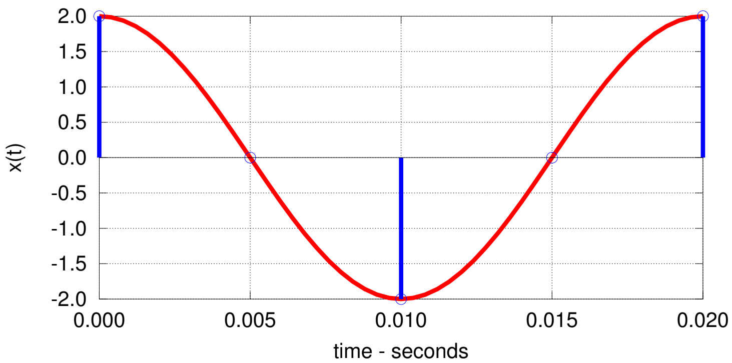
- $f/f_s = 0.75$



- $f/f_s = 0.5$

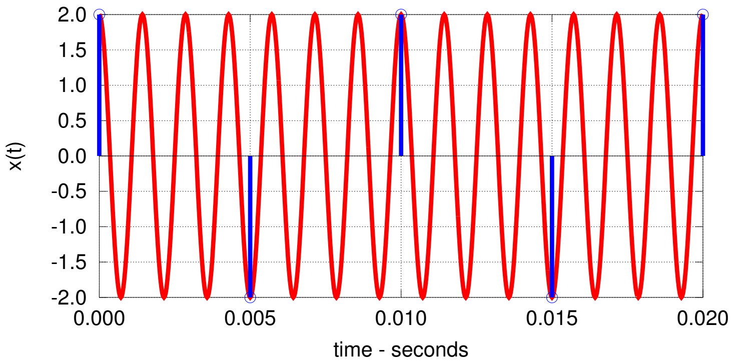


- $f/f_s = 0.25$

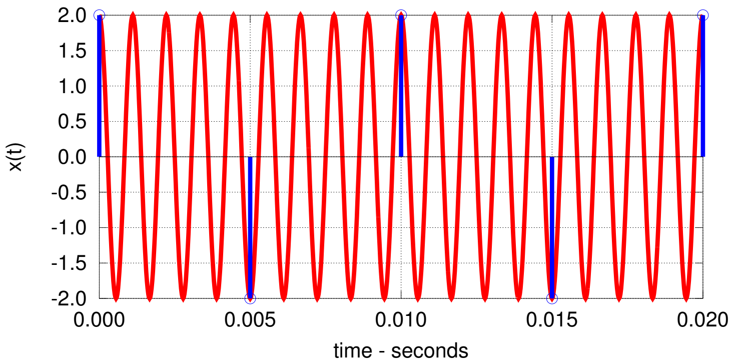


Here you should notice that if $f/f_s \leq 0.5$ then the analogue waveform is sampled frequently enough, *i.e.* Nyquist's sampling frequency of $f_s > 2f_{\max}$ is obeyed. However there is also another interesting observation to be made. Notice how at $f/f_s = 1.5$ and $f/f_s = 2.5$ the analogue signal has been sampled at the peaks. We can see this happening for larger similar values of f/f_s as well:

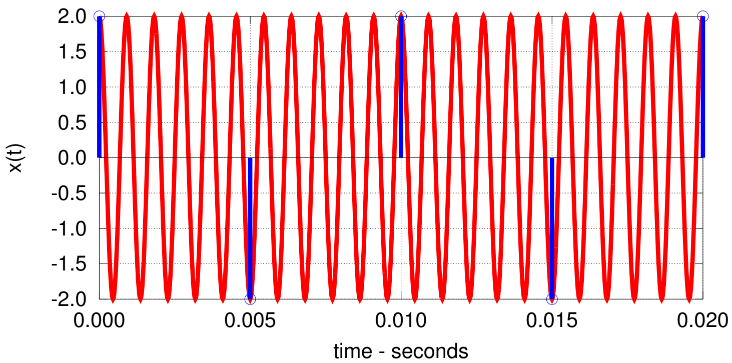
- $f/f_s = 3.5$



- $f/f_s = 4.5$



- $f/f_s = 5.5$



This shows the digitized signal contains a peak in energy because it captures the signal at the peaks of the signals only. These peaks in the digitized frequency spectrum correspond to frequencies that are odd multiples of the highest frequency. To understand this better we have to consider that the analogue signal has been correctly sampled with $f_s = 2f_{\max}$. The digitized signal however is ambiguous because it represents the analogue domain with much less information, discretely. This discrete information *could represent* signals with different frequencies at higher than $f_{\max}$ but very poorly. These potentially poorly sampled signals mirror the actual sampled signal with peaks in energy corresponding to odd multiples of $f_{\max}$. i.e. at $1 \times f_{\max}$, $3 \times f_{\max}$, $5 \times f_{\max}$, $7 \times f_{\max}$, ...
If an ADC samples at a frequency of f_s and there is an input frequency f_i then the digitized signal will contain components at:

$$(f_i), (f_s - f_i), (f_s + f_i), (2f_s - f_i), (2f_s + f_i), (3f_s - f_i), (3f_s + f_i), (4f_s - f_i), (4f_s + f_i), ...$$

So if $f_i = f_{\max}$ and $f_s = 2f_{\max}$ then the digitized signal will contain frequency components at:

$$(1 \times f_i), (3 \times f_i), (5 \times f_i), (7 \times f_i), ...$$

This is not the answer to the question. However it should help you understand how to answer it.

9. List the frequency components below 375kHz that theory predicts would be present in the sampled signal when $f_s = 150$kHz and $f_{\max} = 140$kHz. **Solution** See the information for the previous question.