Tutorial Questions:
Fourier Series

1. **Discrete Periodic Fourier Series**

   (a) The discrete Fourier series can be calculated with

   $$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right)$$

   where N is the number of samples in one period of the signal $x[n]$ and $a[k]$ are the discrete Fourier series. Also recall Euler's identity:

   $$\exp\left(\frac{-j2\pi kn}{N}\right) = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right).$$

   Assuming $x[n]$ is composed of just a real part (no imaginary part) then the real part of the discrete Fourier series can therefore be calculated with

   $$\mathrm{Re}(a[k]) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi kn}{N}\right)$$

   and the imaginary part:

   $$\mathrm{Im}(a[k]) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]j\sin\left(\frac{2\pi kn}{N}\right).$$

   For the signal $x[n] = \sin(2\pi n/N)$ where $N = 8$:

   i. Calculate the real part of the discrete Fourier series, showing all essential working and steps.
   ii. Calculate the imaginary part of the discrete Fourier series, showing all essential working and steps.

   (b) The real and imaginary parts both contain useful information that is often best viewed combined using the magnitude. Calculate the magnitude of this discrete Fourier series.

   (c) Explain the magnitude signal in two sentences.

   (d) Total power: Show that the total power in the time and frequency domains are equal and name the relevant *theorem*.

2. **Fourier Transform for Discrete Aperiodic Sequences**

   (a) These questions are about the Fourier Transform for discrete aperiodic sequences which are digital signals that do not repeat or are not periodic. This Fourier transform for discrete aperiodic sequences is defined by

   $$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n)$$

   where $x[n]$ is the discrete aperiodic signal or sequence.

   (b) The Dirac delta function is defined as

   $$\delta[n-v] = \begin{cases} 1 & \text{for} \quad n = v, \\ 0 & \text{elsewhere.} \end{cases}$$

   i. Calculate the Fourier transform magnitude and phase of a dirac delta function $\delta[n-v]$ centered at $v = 1$. Show all working and essential steps.
   ii. Calculate the Fourier transform magnitude and phase of a dirac delta function $\delta[n-v]$ centered at $v = 0$. Show all working and essential steps.
   iii. Explain the difference between the two responses with $v = 1$ and $v = 0$. Consider the role of odd and even functions.
   iv. What is the significance of the magnitude of the Fourier transform for the Dirac delta function?
   v. Calculate the *inverse* discrete Fourier transform of a Dirac delta function centred at $v = 0$.

(c) A square wave can be defined as

$$x[n] = \begin{cases} a & \text{for} \quad v_1 \le n \le v_2, \\ 0 & \text{elsewhere.} \end{cases}$$

i. Calculate the Fourier transform magnitude of a square wave for $v_1 = -1$, $v_2 = 1$ and $a = 1/3$ and sketch the response.
ii. Calculate the Fourier transform magnitude of a square wave for $v_1 = -2$, $v_2 = 2$ and $a = 1/5$ and sketch the response.
iii. These square waves can be considered as non-causal moving average filter coefficients, so that
A. $y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$ and
B. $y[n] = \frac{1}{5}\left(x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2]\right)$.
Convert causal 3 term and 5 term moving average filters into the generalized difference equation forms (see question 1(b)) and calculate the magnitude frequency response for both filters.
iv. Sketch the response to these filters for $\Omega = \pm\pi$. Are these filters high pass, band pass, notch or low pass filters?

3. Replace t by $t+T$ in the expression for the input signal: $e^{j2\pi ft}$. By solving $x(t+T) = x(t)$, find the period of the signal, i.e. the smallest nonzero value of T for which the equation works.

4. Using compound angle formulae, do the same for sine and cosine functions:

$$cos\left(2\pi f(t+T)\right) = cos\left(2\pi ft\right)$$

$$sin\left(2\pi f(t+T)\right) = sin\left(2\pi ft\right)$$

5. Write down the Fourier series for $x(t+T)$ based on the expression for $x(t)$:

$$x(t) = \sum_{k=-\infty}^{+\infty} x_k e^{j2\pi(kf_0)t}$$

Use algebra to show that $x(t+T)$ is the same as $x(t)$ (i.e. that $x(t)$ is a periodic signal with period T). You will need to use the fact that $e^{j2\pi} = 1$, which is a consequence of $e^{j\theta} = cos(\theta) + jsin(\theta)$.

6. Remembering a previously emphasised point about variables of summation, we can write our Fourier series as

$$x(t) = \sum_{m=-\infty}^{+\infty} x_m e^{j2\pi mt/T}$$

Substitute this into the formula for x_k and see what you get. Hint: you may need to handle the k=m case separately from the other cases; you will also need to use the fact that integration is a linear operation (so the integral of a sum is a sum of integrals).

7. Find the Fourier series expansion of a rectangular pulse train where the period of the train is T

$$x(t+T) = x(t)$$

and the width of the pulses is τ ($< T$ of course):

$$x(t) = \begin{cases} \frac{1}{\tau} & |t| \le \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$

Hint: do the integral from -T/2 to +T/2
Note: The height of the pulses has been chosen to be useful in later sections dealing with sampling.

8. Check that the coefficients have conjugate symmetry

9. What happens to the coefficients as $\tau \to 0$?

10. Use the Parseval's theorem formula to explain why $|X(f)|^2$ is sometimes known as the Energy Spectral Density of the signal.