Solutions for: Windowing and Zero Padding Tutorial

1. A single complex sinusoid with frequency f_0 exists for all time and has a spectrum $\delta(f-f_0)$. Show that this is the case. **Solution** We can work back using the inverse Fourier transform because this will enable us to use the standard result from the sifting property of the delta function $\int \delta(x-y)g(x)\mathrm{d}x = g(y)$. So the inverse Fourier transform of $\delta(f-f_0)$ should, according to the above have a complex sinusoid form like so:
$$x(t) = \exp(j2\pi f_0 t).$$
So the inverse Fourier transform of $\delta(f-f_0)$ can be determined with:
$$x(t) = \mathcal{F}^{-1}(\delta(f-f_0)) = \int_{-\infty}^{\infty} \delta(f-f_0)\exp(j2\pi f t)\mathrm{d}f.$$
From the sifting property of the delta function we know that the above must be true because:
$$\begin{aligned}\int_{-\infty}^{\infty} \delta(f-f_0)\exp(j2\pi f t)\mathrm{d}f &= \begin{cases} \exp(j2\pi f t) & \text{for} \quad f = f_0 \\ 0 & \text{everywhere else} \end{cases} \\ &= \exp(j2\pi f_0 t).\end{aligned}$$

2. Determine the spectrum for the above single complex sinusoid but now multiplied by a rectangular window with width T_c and centered at $t=0$ *i.e.* from $-T_c/2$ to $+T_c/2$. **Solution** We can observe the spectrum by taking the Fourier transform for this period:
$$\begin{aligned}X(f) &= \int_{-T_c/2}^{T_c/2} \exp(j2\pi f_0 t)\exp(-j2\pi f t)\mathrm{d}t \\ &= \int_{-T_c/2}^{T_c/2} \exp(-j2\pi(f-f_0)t)\mathrm{d}t\end{aligned}$$
Using Euler's identity and then the sine (odd) term is zero after integrating symmetrically around zero:
$$\begin{aligned}X(f) &= \int_{-T_c/2}^{T_c/2} \left(\cos(2\pi(f-f_0)t) - j\sin(2\pi(f-f_0)t)\right)\mathrm{d}t \\ &= \int_{-T_c/2}^{T_c/2} \cos(2\pi(f-f_0)t)\mathrm{d}t - j\underbrace{\int_{-T_c/2}^{T_c/2} \sin(2\pi(f-f_0)t)\mathrm{d}t}_{=0} \\ &= \int_{-T_c/2}^{T_c/2} \cos(2\pi(f-f_0)t)\mathrm{d}t\end{aligned}$$
Performing the integration we get:
$$\begin{aligned}X(f) &= \left[\frac{\sin(2\pi(f-f_0)t)}{2\pi(f-f_0)}\right]_{-T_c/2}^{+T_c/2} \\ &= \frac{\sin(2\pi(f-f_0)T_c/2)}{2\pi(f-f_0)} - \frac{\sin(2\pi(f-f_0)(-T_c/2))}{2\pi(f-f_0)} \\ &= \frac{2\sin(2\pi(f-f_0)T_c/2)}{2\pi(f-f_0)} = \frac{\sin(\pi(f-f_0)T_c)}{\pi(f-f_0)} \\ &= \frac{T_c}{T_c}\frac{\sin(\pi(f-f_0)T_c)}{\pi(f-f_0)}\end{aligned}$$

Therefore the resulting spectrum after multiplication by a window (width T_c) is:

$$X(f) = T_c \mathrm{sinc}((f - f_0)T_c).$$

3. A sinc function results in the above case. Sketch the magnitude spectrum of the determined sinc function and determine the locations of the nulls under two scenarios:

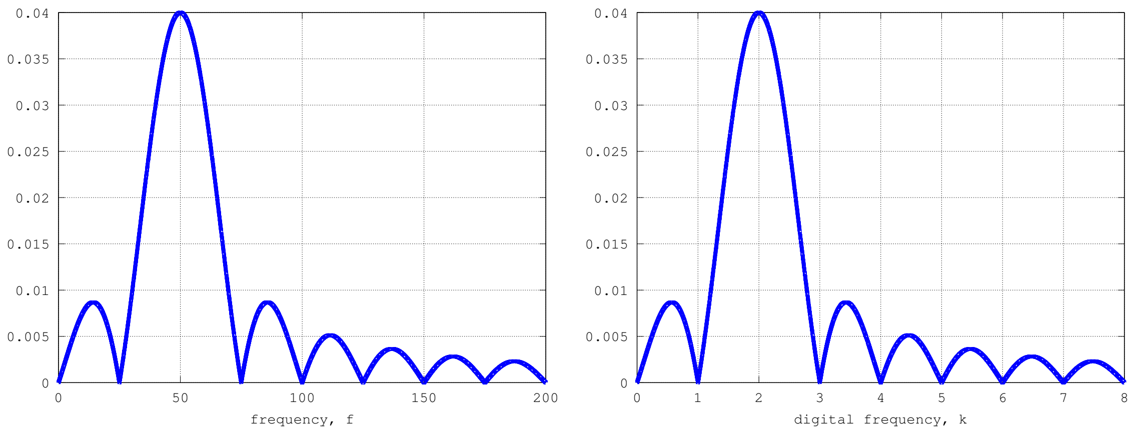
(a) When $f_0 T_c = n$ where n is an integer; **Solution** Here, the sinc function should have nulls at integer multiples of $1/T_c$, *i.e.* where

$$f = \frac{k}{T_c}$$

where k is another integer. k is like a digital frequency and so can vary but n is fixed, dependent on the frequency of the complex exponential and also the period over which the Fourier transform has been calculated. So nulls should occur at locations satisfied by:

$$k = f \times T_c.$$

For example, if we have a complex sinusoid with frequency $f_0 = 50$ Hz, and perform Fourier transform from time $-\frac{2}{50}/2 \leq f \leq \frac{2}{50}/2$ then $n = 2$ so that $X(f) = T_c$ when $f = 50$ Hz and nulls will occur when $f = k \times 50/2 = k \times 25$ where k is any integer. This can be seen below for this particular example:



(b) When $f_0 T_c$ is not equal to an integer. **Solution** The sinc function is zero if the argument is an integer, *i.e.* $\mathrm{sinc}(1) = 0$, $\mathrm{sinc}(2) = 0$, $\mathrm{sinc}(3) = 0$ *etc.* . This means that the argument above should be an integer to find the points at which there are the nulls. *i.e.*

$$\mathrm{sinc}((f - f_0)T_c) = 0 \quad \text{when} \quad (f - f_0)T_c = j, \text{ an integer.}$$

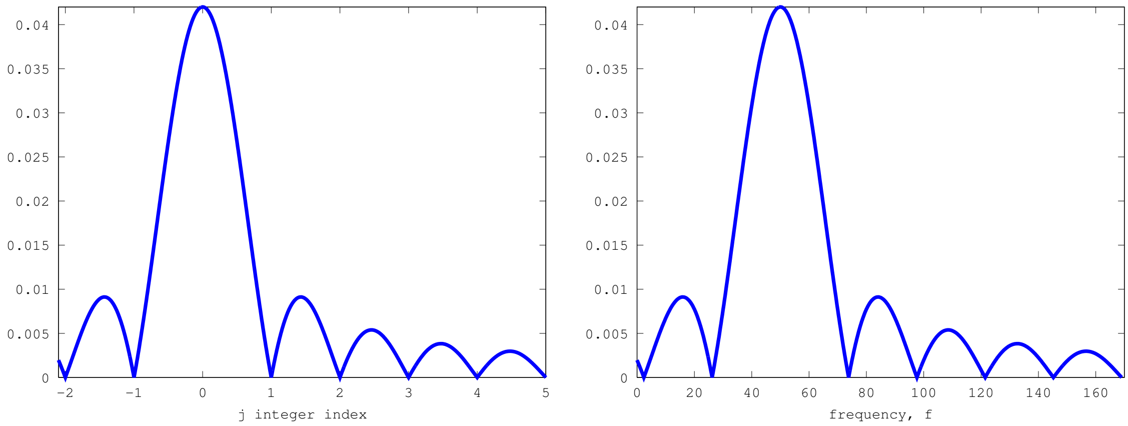
Thus:

$$(f - f_0)T_c = j$$

T_c and f_0 are given therefore just need to find f:

$$f - f_0 = \frac{j}{T_c}$$
$$\therefore f = \frac{j}{T_c} + f_0 = \frac{j + f_0 T_c}{T_c}.$$

For example, if again $f_0 = 50$Hz, but this time we let $T_c = \frac{2.1}{50}$ s, then nulls will occur when $f = k \times 50/2.1 = k \times 23.81$.



4. Sketch the magnitude of a sinc function and label the main lobe, nulls and side lobes.

5. Explain clearly, in your own words, the following terms, in the context of the DFT and windowing giving equations where possible:

   (a) Coherent gain; **Solution** Coherent gain is the gain of the window at DC frequency. Alternatively it can be calculated from the time domain window coefficients[1]:

   $$\text{CG} = \frac{1}{N}\sum_{n=0}^{N-1} w[n] = W(0).$$

   (b) Side lobe fall off rate in:

      i. Decibels per decade;

      ii. Decibels per octave.

   Solution Side lobe fall off rate is rate at which the peaks of the side lobes decrease in magnitude. Decibels per decade is the number of decibels the side lobes reduce for every 10-times an increase in frequency. Decibels per octave is the number of decibels it reduces over an octave which is a doubling in frequency.

   (c) Spectral leakage; **Solution** Spectral leakage is where a signal composed of e.g. a single frequency is actually represented as a spread-out range of frequencies extending to all frequencies in a calculated frequeny domain.

   (d) Scalloping loss; **Solution** Scalloping is where the magnitude of an DFT estimated frequency component has been reduced because the frequency component does not exactly coincide with one of the DFT's discrete frequency bins. Formula given by Poularikas:

   $$\begin{aligned}\text{scalloping loss} &= \frac{\left|\sum_n w(nT)\exp\left(-j\pi n/N\right)\right|}{\sum_n w(nT)} \\ &= \frac{\text{coherent gain for a tone located half a bin from DFT sample point}}{\text{coherent gain for a tone located at a DFT sample point}} \\ &= \text{maximum reduction in PG due to signal frequency}\end{aligned}$$

   where

   $$PG = \frac{1}{ENBW} = \frac{\text{output signal-to-noise ratio}}{\text{input signal-to-noise ratio}}.$$

   (e) Absolute dynamic range; **Solution** Dynamic range is the ability of a system to differentiate between values. For the DFT, it is important to be able to distinguish between a frequency component with large magnitude and a neighbouring frequency component, possibly of lower magnitude. It is generally defined as:

   $$\text{dynamic range} = \left.\frac{\text{largest detectable signal}}{\text{smallest detectable signal}}\right\}\text{that are simultaneously detectable}$$

[1] Given by A. Poularikis "Windows" in "The Handbook of Formulas and Tables for Signal Processing", CRC Press 1999

(f) Far-field dynamic range; **Solution** Far-field dynamic range is a qualitative property of the sidelobes far from the main lobe and depends on the sidelobe fall-off rate as well as the height of the highest sidelobe. If the sidelobes fall off rapidly then a very small signal can be detected in the presence of a large on, provided its frequency is far enough away from the frequency of the large signal.

(g) Resolution. **Solution** Resolution is the ability to resolve two peaks in the spectrum when two frequencies of equal amplitude are present.

6. If a signal has a magnitude A and a measurement of the signal yields a value B instead, write down, in simple text form (i.e. no special formatting; use _ for subscript and ˆ for superscript), the formulas for:

   (a) The measurement error; **Solution**
   $$B - A$$

   (b) The fractional error; **Solution**
   $$\frac{(B - A)}{|A|}$$

   (c) The percentage error; **Solution**
   $$\frac{(B - A)}{|A|} \times 100\%$$

   (d) The dB error. **Solution** Decibels are a measure of ratio and amplitudes are voltage rather than power therefore:
   $$20 \times \log_{10}\left(\frac{B}{A}\right).$$

7. When carrying out an FFT of a cosine wave whose frequency is 0.1Hz and whose amplitude is unity, explain why the peak amplitude measurement obtained in the range 0.09Hz to 0.11Hz should ideally have a value of 0.5? **Solution** Please see question 1 of the Fourier series tutorial. Here you will see that if you take the DFT of a sine wave then the imaginary component has two values. A signal of any single frequency will have two values whether it is a sine wave or a cosine wave. This can also be related to Parseval's theorem where the total energy in the time domain is equal to the total energy in frequency domain. A single frequency in the time domain corresponds to a positive and negative frequency in the Fourier domain. Therefore for Parseval's theorem to be true, for a signal with single frequency of unity amplitude then that signal has energy 0.5. In the Fourier domain the signal, consisting of a positive and negative frequency must have energy values that sum to 0.5, which is the case if the peak amplitude measurement is a 0.5.

8. Explain the concept of linear interpolation in your own words, without using any mathematical formulae. **Solution** Linear interpolation is estimating a value between other known values where the estimated value is determined by a point on a straight line that connects the known values together.

9. Explain the reasons for performing zero padding? Describe the process in the time domain and the effect on the resulting frequency domain. **Solution** Please see section 2.1 of the notes on Windowing.