Notes

Sampled Signals

Digital Signal Processing

Contents

Notes

Digital Signals

Sampling, Aliasing and Nyquist

Spectrum

Filters

Spectra of Sampled Signals

Fourier Series

Negative Frequencies

Sampling Theorem

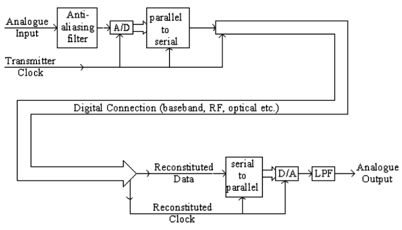
Fourier Transform

Sampling in Frequency Domain

Digitization Example

Notes

- Signal sampling is performed by an Analogue to Digital Converter or ADC to create a digital signal from an analogue signal

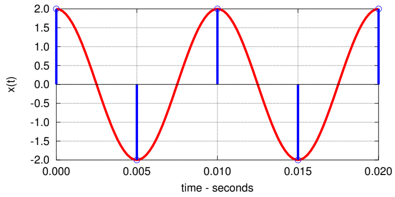


PCM System

- The analogue signal is reconstructed from the digital signal by a Digital to Analogue Converter (DAC).

Minimum Sampling Frequency

- An analogue signal at frequency f_i should be sampled at least $f_s = 2 \times f_i$
- Otherwise the digital signal will not contain sufficient information to enable the original analogue signal to be reconstructed



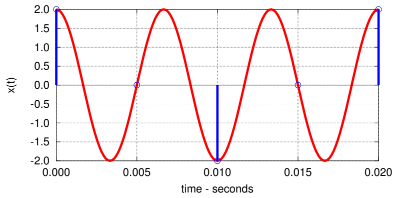
This analogue signal (cosine wave) with frequency f_i has been sampled at

$$f_s = 2f_i.$$

Notes

Under Sampling

- If an analogue signal is undersampled where $f_i > f_s/2$ then the digital signal will not contain sufficient information to reconstruct the analogue signal

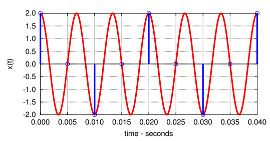


The discrete samples here are insufficient to recreate the original signal.

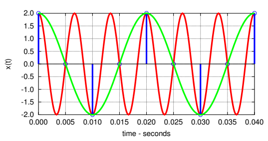
Notes

Aliasing (Errors)

- Aliasing occurs when the sampling frequency is not high enough $f_s < 2f_i$ or
- (Equivalently) the sampled analogue signal has too high frequency $f_i > f_s/2$



$f_i = 150\text{Hz}$



$f_{\text{alias}} = 50\text{Hz}$

$f_s = 200\text{Hz}$

Aliasing (error) frequencies occur:

f_{alias} is one of $\{(f_i + n \times f_s)$ or $(n \times f_s - f_i)$: for $-\infty \le n \le \infty\}$

Notes

Sampling and Nyquist

A digital signal is sampled f_s times a second.

- The **Nyquist frequency**, also known as the **folding frequency** is

$$\frac{f_s}{2}.$$

- The **Nyquist rate** which is:

$$2 \times B$$

  where B is the bandwidth of the signal.
- The **sampling frequency** should be greater than the Nyquist rate:

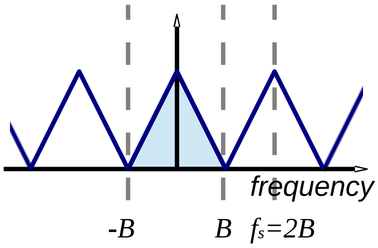
$$f_s > 2B$$

  to prevent **aliasing** errors.

Notes

Spectrum: no aliasing

If $f_s = 2B$ frequency spectrum of digital signal will be:



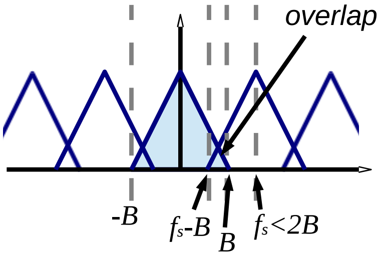
$f_s/2$ is known as folding frequency because all frequencies from 0Hz to BHz fold over to BHz to $2B$Hz:

Creating a mirror image of the frequencies.

Notes

Spectrum: with aliasing

If $f_s < 2B$ frequency spectrum of digital signal will be:

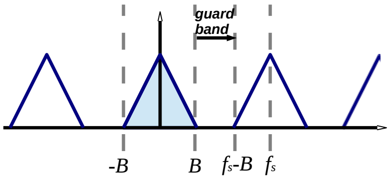


Aliasing error frequencies occur in the overlap region between $(f_s - B)$Hz and BHz.

Notes

Spectrum: $f_s > 2B$

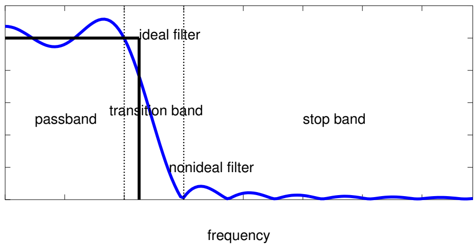
Here there is a gap known as a guard band between B Hz and $(f_s - B)$Hz.



Notes

Non-Ideal Filters

This guard band can be useful because the antialiasing filter and reconstruction filters will not be ideal filters and will have finite width transition bands between the pass and stop bands:

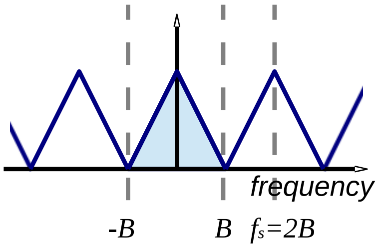


Notes

Spectra of Sampled Signals

The spectra of sampled signals contains:

- Negative frequencies from $-f_s/2$ to 0 Hertz
- Original signal frequencies from 0 Hertz to $f_s/2$ Hertz
- Images of the negative and original signal frequencies



Notes

Fourier Series

Discrete Fourier series approximation of periodic digital signal $x[n]$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(-\frac{j2\pi kn}{N}\right).$$

with $0 \leq n < N$.
a_k represents the spectral components or harmonics for $0 \leq k < N$.
The original signal can be reconstructed with:

$$x[n] = \sum_{k=0}^{N-1} a_k \exp\left(-\frac{j2\pi kn}{N}\right).$$

Notes

Euler's Formula

Euler's formula is often used to expand the complex exponentials:

$$\exp\left(-\frac{j2\pi kn}{N}\right) = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right).$$

So the signal $x[n]$ is approximated by a combination of cosine and complex sinusoidal functions.

Notes

Negative Frequencies

- Negative frequencies are the mirror image of the frequencies in the signal.
- How to represent a real signal (i.e. no $j = \sqrt{-1}$) with a combination of cosine and complex sinusoidal functions?
- **Only use the real part of the complex exponential.**

Example
Represent $\cos(\Omega n + \theta)$ with complex exponentials where $\Omega = \frac{2\pi k}{N}$.

Notes

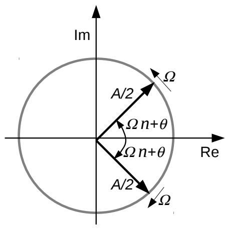
Negative Frequencies

Example Representing $\cos(\Omega n + \theta)$ with complex exponentials:

$$A\cos(\Omega n + \theta) = \frac{A}{2}\exp(j\Omega n + \theta) + \frac{A}{2}\exp(-j\Omega n + \theta)$$

- The **positive frequencies** travel anti-clockwise
- The **negative frequencies** travel clockwise
- Positive and negative frequencies project same value onto **real (Re) axis**.

Complex Plane:



where $\Omega = \frac{2\pi k}{N}$.

Notes

Sampling Theorem

An analogue signal $x(t)$ with bandwidth B is uniquely specified by its samples $x[n] = x(mT)$, $m = 0, \pm1, \pm2, \ldots$ with sampling period $T = 1/f_s$ and sampling frequency f_s where

$$f_s > 2B.$$

Notes

Sampled Signal

The sampled signal can be described by a train of unit impulse functions:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

here we are using the unit impulse function:

$$\delta(t - nT) = \begin{cases} 1 & \text{when} \quad t = nT \\ 0 & \text{elsewhere.} \end{cases}$$

The analogue signal $x(t)$ can be multiplied by $x_\delta(t)$ to get

$$x[n] = x(t)x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT),$$

using sifting property of delta function $x(t)\delta(t - nT) = x(nt)\delta(t - nT)$.

Notes

Fourier Transform

Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi f t)\,\mathrm{d}t.$$

Inverse Fourier transform:

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(-j2\pi f t)\,\mathrm{d}f.$$

Notes

Sampling in Frequency Domain

Fourier Transform of impulse train:

$$X_\delta(f) = \frac{1}{T}\sum_n \delta(f - nf_s)$$

Fourier (or inverse) Transform of multiplication is convolution,
$\therefore$ Fourier transform of $x(t) \times x_\delta(t)$ is

$$X(f) * \delta(f - nf_s)$$

Convolution of any function e.g. $G(f)$ with impulse function shifts $G(f)$:

$$G(f) * \delta(f - nf_s) = G(f - nf_s).$$

Using above we get:

$$\begin{aligned} X(f) * X_\delta(f) &= X(f) * \left[\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f - nf_s)\right] \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f - nf_s) \end{aligned}$$

Notes

Sampling in Frequency Domain

$\therefore$ Fourier Transform of sampling process $x(t)x_\delta(t)$:

$$X(f) * X_\delta(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f - nf_s)$$

shows spectrum of $x(t)$ is repeated at intervals spaced by nf_s.

For analogue signal with frequency f_i, the sampled signal's frequency content for the positive frequency will be

$$\ldots, f_i - 2f_s, f_i - f_s, f_i, f_i + f_s, f_i + 2f_s, \ldots$$

Notes

Sampling in Frequency Domain

Sampling images of negative frequencies of signal:

$$..., -2f_s - f_i, -f_s - f_i, -f_i, f_s - f_i, 2f_s - f_i, ...$$

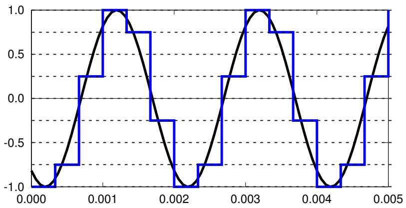
Combining positive and negative frequencies:

$$((f_i + n \times f_s), (n \times f_s - f_i) : \text{ for } -\infty \leq n \leq \infty)$$

Notes

Reconstruction

The frequency images of the original signal have to be removed when the analogue signal is reconstructed.
A reconstruction filter can help remove the noisy components that may otherwise manifest as a series of steps.



Notes

Sampled Signals: Summary

Digital signal processing systems perform:

- filtering
- analogue to digital conversion
- signal processing function
- digital to analogue conversion
- filtering

Analogue to Digital Conversion (ADC) and Digital to Analogue Conversion (DAC) are important steps that convert and reconstruct an analogue signal to and from the digital domain.
Aliasing errors may occur if the signal is not sampled frequently enough.
Furthermore the digital representation is ambiguous with images of the original signal that need to be removed upon reconstruction to an analogue form.

Notes