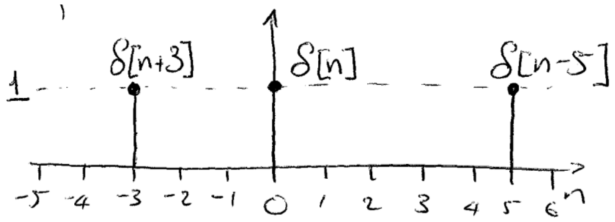
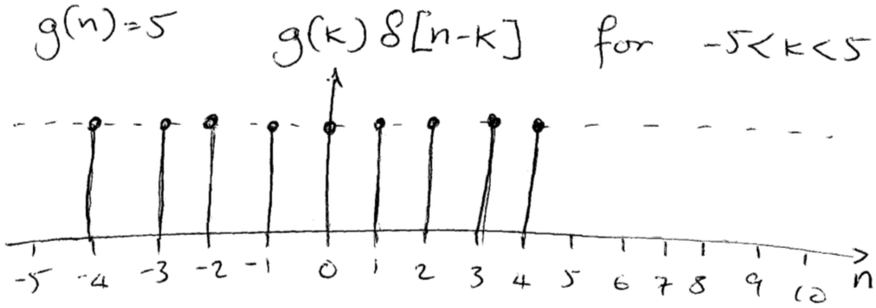
Solutions for: LTIC Systems Tutorial

1. **Impulse function and Digital Signals**

   (a) Draw three impulse function, $\delta[n-v]$ for values of $v = 0$, $v = 5$ and $v = -3$.
   Solution Three impulse functions can be seen below:



   (b) Given a function $g(n) = 5$ draw $g(k)\delta[n-k]$ for values of $-5 < k < 5$:
   Solution The train of impulse functions can be seen below:



   The dotted line should show a value of 5 on the vertical axis.

   (c) If a continuous sine wave is defined as $x(t) = a\sin(t2\pi f)$ define the digital version using n for the samples and T_s seconds for the sample interval. **Solution** As we have $x(t) = a\sin(2\pi ft)$ a digital version is often quoted in the form:

$$x[n] = a\sin(n\Omega)$$

   where Ω is normalized (digital) frequency calculated with:

$$\Omega = 2\pi f T_s$$

   where f_s is the sampling frequency. The amount of time between samples or sampling interval T_s is then

$$T_s = \frac{1}{f_s}.$$

   (d) State the sample frequency of this digitized sine wave (remember to include the units): **Solution** The sample frequency is f_s which can be calculated from:

$$f_s = \frac{1}{T_s}.$$

   (e) If the frequency of the sine wave is 8000 Hertz what is the period (remember to include the units)?
   Solution Frequency $f = 8000$ Hertz the period is then:

$$T = \frac{1}{f} = 125\mu s.$$

   or $0.125ms$.

2. **Linear Time Invariant Systems and Digital Convolution**

   (a) What is the *Principle of Superposition* and why is it useful?
   Solution The Principle of Superposition can be quoted as:

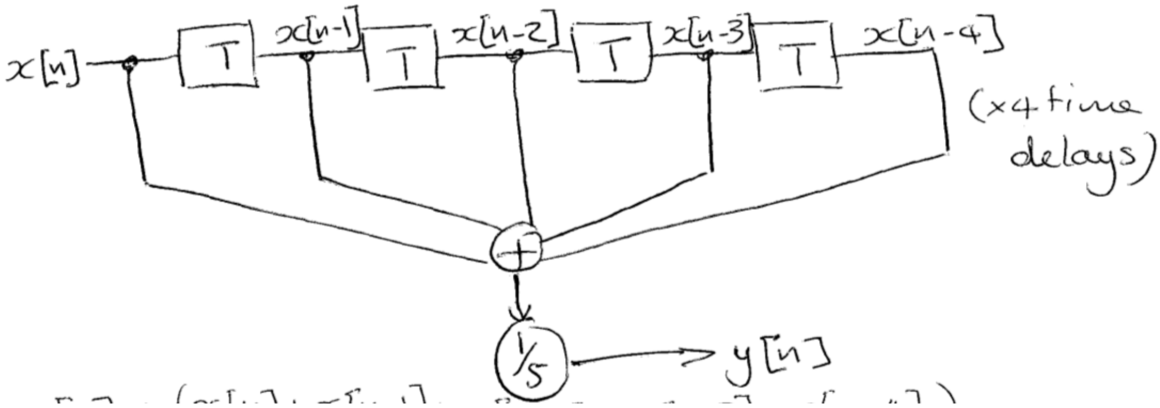
   "If the input consists of a sum of signals then the output is the sum of the responses to those signals."

A digital signal processing system that is LTI can be analysed more easily using the response to simpler individual signal components which can then be combined for the analysis of more complicated system inputs.

(b) Draw the system diagram using time delays, addition and multiplication for a moving average filter,

$$y[n] = \frac{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]}{5}$$

Solution The system diagram for the above filter is given by:



(c) Calculate the impulse response of this moving average filter (5 stage) showing all steps:
Solution The impulse response is given by:

n	$x[n]$	$x[n-1]$	$x[n-2]$	$x[n-3]$	$x[n-4]$	$h[n] = y[n]$
$n < 0$	0	0	0	0	0	0
0	1	0	0	0	0	$1 \times \frac{1}{5} = \frac{1}{5}$
1	0	1	0	0	0	$1 \times \frac{1}{5} = \frac{1}{5}$
2	0	0	1	0	0	$1 \times \frac{1}{5} = \frac{1}{5}$
3	0	0	0	1	0	$1 \times \frac{1}{5} = \frac{1}{5}$
4	0	0	0	0	1	$1 \times \frac{1}{5} = \frac{1}{5}$
5	0	0	0	0	0	0
$n > 5$	0	0	0	0	0	0

Thus, the impulse response is given by:

$$h[n] = y[n] = \begin{cases} \frac{1}{5} & \text{for} \quad 0 \leq n < 5 \\ 0 & \text{elsewhere} \end{cases}$$

(d) Convolve, showing all calculation steps this moving average filter with a unit step function,

$$u[n] = \begin{cases} 1 & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0 \end{cases}.$$

Solution Convolution

$$\begin{aligned} f[n] &= u[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} u[k]h[n-k] \end{aligned}$$

But

$$u[n] = \begin{cases} 1 & k \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore

$$\begin{aligned} f[n] &= \sum_{k=-\infty}^{-1} 0 \times h[n-k] + \sum_{k=0}^{\infty} 1 \times h[n-k] \\ &= \sum_{k=0}^{\infty} h[n-k]. \end{aligned}$$

We can use this result to calculate the convolution.

n			$y[n]$
-1	$h[-1]$	$= 0$	$= 0$
0	$h[0]$	$= \frac{1}{5}$	$= \frac{1}{5}$
1	$h[0] + h[1]$	$= \frac{1}{5} + \frac{1}{5}$	$= \frac{2}{5}$
2	$h[0] + h[1] + h[2]$	$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$	$= \frac{3}{5}$
3	$h[0] + h[1] + h[2] + h[3]$	$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$	$= \frac{4}{5}$
4	$h[0] + h[1] + h[2] + h[3] + h[4]$	$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$	$= 1$
5	$h[0] + h[1] + h[2] + h[3] + h[4] + h[5]$	$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + 0$	$= 1$
6	$h[0] + h[1] + h[2] + h[3] + h[4] + h[5] + h[6]$	$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + 0 + 0$	$= 1$
...	$h[0] + h[1] + h[2] + h[3] + h[4] + h[5] + h[6] + ...$	...	$= 1$

(e) $y[n] = \frac{1}{4}\left(x[n+1] + x[n] + x[n-1] + x[n-2]\right)$ and $y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$ are 4 term non-causal and causal moving average filters respectively. What does *non-causal* mean? (Explain in 1 sentence.) **Solution** Non-causal means that the system does not rely on data from the future, it relies purely on data from the present and past. The first one is non-causal because it includes an $n+1$ data point which effectively means that it is a data value taken from one sample in the future.

(f) The equations for these 4 term moving average filters are known as a difference equations. Most difference equations for Linear Time Invariant (LTI) systems can be put into the form:

$$\sum_{m=0}^{N} a[m]y[n-m] = \sum_{m=0}^{M} b[m]x[n-m].$$

Place the 4 term moving average filters into this more general form. **Solution** Firstly the non-causal equation does not really fit easily into the general equation above, however we can manipulate it slightly

$$y[n] = \frac{1}{4}\left(x[n+1] + x[n] + x[n-1] + x[n-2]\right)$$

can be converted to a form that enables it to be expressed in general form if we shift all the calculations one time step into the past like so:

$$y[n-1] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right).$$

So that now we let $N = 1$, $a[0] = 0$ and $a[1] = 1$ then $M = 3$ and $b[m] = \frac{1}{4}$ for $0 \leq m < 4$ so that we get

$$y[n-1] = \sum_{m=0}^{M} \frac{1}{4} x[n-m].$$

If we substitute the known variables and take the fraction out of the summation we get

$$y[n-1] = \frac{1}{4} \sum_{m=0}^{3} x[n-m].$$

The second filter is causal $y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$ and can therefore be expressed more easily in general form where

$$y[n] = \frac{1}{4} \sum_{m=0}^{3} x[n-m].$$

(g) Convolve a 2 term causal moving average filter $y[n]$ with the following signal:

$$f[n] = \begin{cases} 2 & \text{when} \quad n = 3, \\ 0.5 & \text{when} \quad 0 \leq n \leq 2, \\ 0.7 & \text{when} \quad 4 \leq n \leq 7, \\ 0 & \text{everywhere else.} \end{cases}$$

Show all essential working and steps. **Solution** A 2 term moving average filter is of the form:

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

This is to be convolved with $x[n] = f[n]$ ($f[n]$ is given in the question):

$$y[n] = f[n] * h[n]$$

where $h[n]$ is the impulse response of the filter which we can calculate using the previous steps followed above to obtain:

$$h[n] = \begin{cases} \frac{1}{2} & 0 \leq n < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Convolution can be calculated using the convolution sum formula:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

to obtain:

n	$x[n]$	$x[n-1]$	$y[n]$
-3	0	0	$\frac{1}{2}(0+0) = 0$
-2	0	0	0
-1	0	0	0
0	0.5	0	$\frac{1}{2}(0.5+0) = 0.25$
1	0.5	0.5	0.5
2	0.5	0.5	0.5
3	2	0.5	1.25
4	0.7	2	1.35
5	0.7	0.7	0.7
6	0.7	0.7	0.7
7	0.7	0.7	0.7
8	0	0.7	0.35
9	0	0	0