Introduction to the z-Transform

Contents

1 Introduction

Laplace is useful for the solution of differential equations. In contrast to this, z-transforms are useful for the solution of difference equations. Recall that a difference equation can be considered to be a discrete approximation of differential equations. z-transforms are another representation that can be considered to enable a frequency domain representation.

The unilateral z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{1}$$

where n is the discrete time index, $x[n]$ is the time domain signal or other similar component such as an impulse reponse.

The bilateral z-transform is similar except the sum is from $-\infty$,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{2}$$

which can be considered to allow for the case where $x[n]$ is not zero before $n = 0$.

The z-transform

- Transforms a digital signal to a frequency respresentation
- Used for stability analysis using the *poles* and *zeros*
- Also used for frequency analysis

2 z-Transform Calculation

How to calculate or determine the z-transform of a given signal? Similar to Laplace, there are tables of z-transforms for given signals. They can also be calculated from basic first principles, as can be seen by the following example.

Example Determine the z-transform of the following,

$$x[n] = \begin{cases} 1 & \text{for} \quad n \geq 0; \\ 0 & \quad \text{elsewhere.} \end{cases} \tag{3}$$

which is the unit step function, *i.e.* $x[n] = u[n]$.

Answer Using the definition for the unilateral z-transform we have

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}. \tag{4}$$

Here, we can substitute in the term for $x[n]$ which is 1 for all positive values of n, to give

$$X(z) = \sum_{n=0}^{\infty} 1 \times z^{-n} = \sum_{n=0}^{\infty} z^{-n}. \tag{5}$$

The sum can be expanded in this way $X(z) = z^{-0} + z^{-1} + z^{-2} + x^{-3}...$ $= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...$

Geometric series can have the following forms $y = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ and $y = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$.

The resulting z-transform can be seen to be a geometric series of the form

$$X(z) = \sum_{k=0}^{\infty} ar^k. \tag{6}$$

If we let $a = 1$ and $r = z^{-1}$ so that

$$X(z) = \sum_{k=0}^{\infty} 1 \times (z^{-1})^k = \frac{1}{1 - z^{-1}}. \tag{7}$$

Multiplying top and bottom of the right hand side by z gives

$$X(z) = \frac{z}{z-1}. \tag{8}$$

□

Example 2 Determine the z-transform of a square pulse of the form

$$x[n] = \begin{cases} 0.2 & \text{for} \quad 0 \le n \le 5; \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

Answer Substituting $x[n]$ into the definition for the unilateral z-transform we have

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{4} 0.2 z^{-n}. \tag{10}$$

Expanding the sum on the right hand side we have $X(z) = 0.2 \times z^0 + 0.2 \times z^{-1} + 0.2 \times z^{-2} + 0.2 \times z^{-3} + 0.2 \times z^{-4})$ $= 0.2 \times (z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4}).$

Equation (10) can be seen to be a geometric series of the form

$$X(z) = \sum_{k=0}^{p-1} ar^k. \tag{11}$$

If we let $a = 0.2$ and $r = z^{-1}$ we obtain

$$X(z) = \sum_{k=0}^{p-1} 0.2 (z^{-1})^k = 0.2 \frac{1 - (z^{-1})^p}{1 - z^{-1}} = 0.2 \frac{1 - z^{-p}}{1 - z^{-1}}. \tag{12}$$

Substituting in the value for $p = 5$ results in

$$X(z) = 0.2 \frac{1 - z^{-5}}{1 - z^{-1}}. \tag{13}$$

Multiplying top and bottom by z^5 to eliminate the negative powers gives us

$$X(z) = 0.2 \frac{z^5 - 1}{z^5 - z^4} = 0.2 \frac{z^5 - 1}{z^4 (z - 1)}. \tag{14}$$

3 The Meaning of z in the Time Domain

Each z^{-1} can be seen to be equivalent to a time delay and each z can be seen to be a time advance. For instance, if we have a function which is equal to a time shifted unit impulse

$$x[n] = \delta[n-\tau] \tag{15}$$

where the time shifted unit impulse is given by

$$\delta[n-\tau] = \begin{cases} 1 & \text{when} \quad n = \tau; \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

Then the z-transform of $x[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} \delta[n-\tau] z^{-n}. \tag{17}$$

Property of the unit impulse is:

$$f[n]\delta[n-\tau] = \begin{cases} f[\tau] & \text{where} \quad n = \tau; \\ 0 & \text{elsewhere.} \end{cases}$$

Sifting property of the unit impulse is:

$$\sum_{n=-\infty}^{\infty} f[n]\delta[n-\tau] = f[\tau].$$

Equation (17) can then be seen to simplify to

$$X(z) = z^{-\tau} \tag{18}$$

due to the sifting property of the unit impulse. This indicates that the z-transform of a time shifted unit impulse corresponds to a time delay of τ time samples. This is important because it shows that if there is a signal or similar at a point in time τ, then this would correspond to having a z-transform consisting of a z term raised to the negative power of the time delay.

4 Inverse z-Transform

The inverse z-transform can be found, from first principles, in the reverse set of steps that were used for the forward z-transform from first principles. For instance, let us consider the following example

Example inverse z-transform Consider the following z-domain function:

$$X(z) = \frac{z}{z-0.5}. \tag{19}$$

Determine the inverse z-transform from first principles.

Answer Recall the geometric series formula

$$y = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}. \tag{20}$$

This can be seen to be similar if the z-domain function in (19) divide through by z, top and bottom which gives

$$X(z) = \frac{1}{1-0.5z^{-1}}. \tag{21}$$

If we consider that $a = 1$ and $r = 0.5z^{-1}$ we can substitute these into the geometric series formula so that

$$X(z) = \sum_{k=0}^{\infty} (0.5z^{-1})^k. \tag{22}$$

Bringing the k inside the brackets we have

$$X(z) = \sum_{k=0}^{\infty} 0.5^k z^{-k}. \tag{23}$$

This can be expanded like so

$$\begin{aligned} X(z) &= 0.5^0 z^0 + 0.5^1 z^{-1} + 0.5^2 z^{-2} + 0.5^3 z^{-3} + 0.5^4 z^{-4} ... \\ &= 1 + 0.5z^{-1} + 0.25z^{-2} + 0.125z^{-3} + 0.0625z^{-4} ... \end{aligned} \tag{24}$$

We previously noted that each z^{-1} is a time delay of 1 time instance. This means that the $X(z)$ can be seen to be equivalent to a series of time delayed unit impulses. For the time domain we would have

$$x[0] = 1,\ x[1] = 0.5,\ x[2] = 0.25,\ x[3] = 0.125,\ x[4] = 0.0625, ... \tag{25}$$

Considering this sequence, it should be possible to observe that the corresponding time domain function is given by

$$x[n] = (0.5)^n. \tag{26}$$

□

Example 2 Find the signal corresponding to the z-Transform:

$$X(z) = \frac{z^2 - 0.2}{z(z - 0.2)} \tag{27}$$

Solution Remember the geometric series formula: $s = \sum\limits_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$. We therefore need to find form of $X(z)$ to easily find r, a and n. Dividing through by z^2

$$X(z) = \frac{1 - 0.2z^{-2}}{1 - 0.2z^{-1}}. \tag{28}$$

So that $r = 0.2z^{-1}$, $a = 1$ and $n = 2$ resulting in:

$$X(z) = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{1} (0.2z^{-1})^k = 1 + 0.2z^{-1}. \tag{29}$$

Therefore the original signal, $x[n]$ is given by $x[0] = 1$ and $x[1] = 0.2$. □

As noted in the beginning, similar to Laplace transforms, z-transforms are more often determined with the use of tables of z-transforms of commonly found time domain functions. Sometimes a z-domain function consists of a more complicated function often consisting of a fraction with polynomials in the numerator and denominator. These fractions can then be split using the method of partial fractions. These will not be considered here, just yet.

4.1 Inverse z-Transform

The inverse z-Transform $\mathcal{Z}^{-1}(X(z))$ is given by

$$x[n] = \mathcal{Z}^{-1}(X(z)) = \frac{1}{2\pi j} \int X(z) z^{n-1} \mathrm{d}z$$

- The inverse z-Transform is **not usually computed directly**.
- Instead the z-Transform is split into parts using partial fractions
- And then the inverse z-Transform of the parts are found using a table of z-Transform pairs.

4.2 (Unilateral) z-Transform Pairs

Lynn and *Fuerst* give the following table of z-Transform pairs:

Signal $x[n]$	**z-Transform** $X(z)$
$\delta[n]$	1
$u[n] = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$	$\frac{z}{z-1}$
$r[n] = \begin{cases} n & \text{for} \quad n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$	$\frac{z}{(z-1)^2}$
$a^n u[n]$	$\frac{z}{z-a}$
$(1-a^n)u[n]$	$\frac{z(1-a)}{(z-a)(z-1)}$
$\cos(n\Omega_0)u[n]$	$\frac{z(z-\cos(\Omega_0))}{z^2-2z\cos(\Omega_0)+1}$
$\sin(n\Omega_0)u[n]$	$\frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1}$
$a^n\sin(n\Omega_0)u[n]$	$\frac{az\sin(\Omega_0)}{z^2-2az\cos(\Omega_0)+a^2}$

Need to separate z-domain function using partial fractions into parts, in the form of the expressions on the right hand side of the above.

Partial Fractions *Example* Decompose the following function into partial fractions:

$$\frac{1}{(z+3)(z-2)}.$$

Solution Let

$$\frac{1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}.$$

Then $A(z-2) + B(z+3) = 1$. So that

$$Az - 2A + Bz + 3B = 1$$

$$z(A+B) - 2A + 3B = 1$$

Therefore $z(A+B) = 0 \Rightarrow A = -B$ and $-2A + 3B = 1$ so that $-2A - 3A = 1$ giving $A = -\frac{1}{5}$ and $B = \frac{1}{5}$.

Check:

$$\frac{A}{z+3} + \frac{B}{z-2} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2} = \frac{\frac{1}{5}(5)}{(z+3)(z+2)} = \frac{1}{(z+3)(z+2)}.$$

□

Cover up method *Example* Decompose the following function into partial fractions:

$$\frac{z}{(z+3)(z-2)}.$$

Solution Let $\frac{z}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$. To find $A \Rightarrow z+3 = 0 \Rightarrow z = -3$.

$$A = \left.\frac{z}{(z+3)(z-2)}\right|_{z=-3} = \frac{-3}{-3-2} = \frac{3}{5}.$$

To find $B \Rightarrow z-2 = 0 \Rightarrow z = 2$.

$$B = \left.\frac{z}{(z+3)(z-2)}\right|_{z=2} = \frac{2}{2+3} = \frac{2}{5}.$$

Hence

$$\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}.$$

Check:

$$\frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2} = \frac{\frac{3}{5}(z-2) + \frac{2}{5}(z+3)}{(z+3)(z-2)} = \frac{\frac{3}{5}z - \frac{6}{5} + \frac{2}{5}z + \frac{6}{5}}{(z+3)(z-2)} = \frac{z}{(z+3)(z-2)}.$$

□

Inverse z-Transform *Example* Find the inverse z-Transform of:

$$X(z) = \frac{1}{(z+3)(z-2)} = \frac{1}{5}\left(\frac{1}{z-2} - \frac{1}{z+3}\right). \tag{30}$$

Solution Re-writing (30) to

$$X(z) = \frac{z^{-1}}{5}\left(\frac{z}{z-2} - \frac{z}{z+3}\right). \tag{31}$$

Enables us to find inverse z-Transforms for the two terms inside the brackets:

$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and

$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n].$$

The two terms are multiplied by z^{-1} which is equivalent to a time delay hence the final signal is given by:

$$x[n] = \mathcal{Z}^{-1}(X(z)) = \frac{1}{5}\left(2^{(n-1)}u[n-1] - ((-3)^{(n-1)})u[n-1]\right).$$

□

Inverse z-Transform *Example* Find the inverse z-Transform of:

$$X(z) = \frac{z}{(z+3)(z-2)}.$$

Solution *From earlier the partial fraction expansion is given by:* $\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}$. **(i)** However it is more convenient if we divide both sides by z first. Hence

$$\frac{X(z)}{z} = \frac{1}{(z+3)(z-2)}.$$

The Right Hand Side (RHS) has partial fractions (see earlier slide):

$$\frac{X(z)}{z} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2}.$$

Multiplying both sides by z then gives:

$$X(z) = \frac{1}{5}\left(\frac{-z}{z+3} + \frac{z}{z-2}\right).$$

(ii) We saw earlier:

$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and

$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n]$$

so that

$$\begin{aligned} x[n] =& \mathcal{Z}^{-1}(X(z)) \\ =& \frac{1}{5}\left(2^{(n)}u[n] - ((-3)^{(n)})u[n]\right). \end{aligned}$$

□

4.3 Inverse z-Transform via Long Division

The numerator and the denominator of the z-Transform can be divided using algebraic long division to find coefficients that correspond to the original signal.

Example Given $H(z) = \frac{z}{(z-1)(z+2)} = \frac{z}{z^2+z-2}$, determine the coefficients.

Solution Via algebraic or polynomial long division:

$$
\begin{array}{r l}
 & z^{-1} - z^{-2} + 3z^{-3} - 5z^{-4} \ldots \\
z^2 + z - 2 & \overline{)\, z} \\
 & \underline{z \ \ + 1 \ \ - 2z^{-1}} \\
 & \quad - 1 \ \ + 2z^{-1} \\
 & \quad \underline{-1 \ \ - z^{-1} + 2z^{-2}} \\
 & \qquad 3z^{-1} - 2z^{-2} \\
 & \qquad \underline{3z^{-1} + 3z^{-2} - 5z^{-3}} \\
 & \qquad\quad - 5z^{-2} + 5z^{-3} \\
 & \qquad\qquad \ldots
\end{array}
$$

So the coefficients of the original signal are given by:
$x[0] = 0$, $x[1] = 1$, $x[2] = -1$, $x[3] = 3$, $x[4] = -5$, etc.
This can be checked by performing the inverse z-Transform on $H(z)$.
Expansion with partial fractions gives: $H(z) = \frac{1}{3}\left(\frac{z}{z-1} - \frac{z}{z+2}\right)$
Inverse z-Transform: $x[n] = \mathcal{Z}^{-1}(H(z)) = \frac{1}{3}\left(u[n] - (-2)^n u[n]\right)$
Then $x[0] = \frac{1}{3}(1-1) = 0$, $x[1] = \frac{1}{3}(1+2) = 1$, $x[2] = \frac{1}{3}(1-4) = -1$, $x[3] = \frac{1}{3}(1+8) = 3$, $x[4] = \frac{1}{3}(1-15) = -5$, etc.
This confirms the long division result.

Inverse z-Transform *Example* Find the inverse z-Transform of:

$$X(z) = \frac{0.5z}{z^2 - z + 0.5} \tag{32}$$

Solution The table of z-Transform pairs has the following definition:

$$\mathcal{Z}^{-1}\left(\frac{az\sin(\Omega_0)}{(z^2 - 2az\cos(\Omega_0) + a^2)}\right) = a^n \sin(n\Omega_0)u[n]. \tag{33}$$

Therefore we can try to equate the terms inside (33) and (32).

$$\text{In the numerator: } a\sin(\Omega_0) = 0.5, \text{ and in the denominator } a^2 = 0.5$$

$$\Rightarrow a = \sqrt{0.5}, \text{ then } \sin(\Omega_0) = 0.5/\sqrt{0.5}, \Rightarrow \Omega_0 = \sin^{-1}(0.5/\sqrt{0.5}) = \frac{\pi}{4}.$$

We can therefore *plug* these values into the result of (33) to find the inverse z-Transform of (32):

$$x[n] = \mathcal{Z}^{-1}\left(\frac{0.5z}{z^2 - z + 0.5}\right) = (\sqrt{0.5})^n \sin(n\pi/4)u[n].$$

□

5 Stability

A linear system is said to be stable if it has: **A Bounded Output *for* A Bounded Input**.

Bounded means the signal does not exceed a particular value. A system is not usually very useful if it goes to ±infinity. System stability is expressed using a function of the impulse response $h[n]$ of the system:

$$\sum_{-\infty}^{\infty} |h[n]| < \infty \tag{34}$$

where $|-h[n]| = |h[n]|$. This ensures that the system is bounded and will not be larger than infinity for some input. If equation (34) is true then the system can be described as being BIBO **stable**.

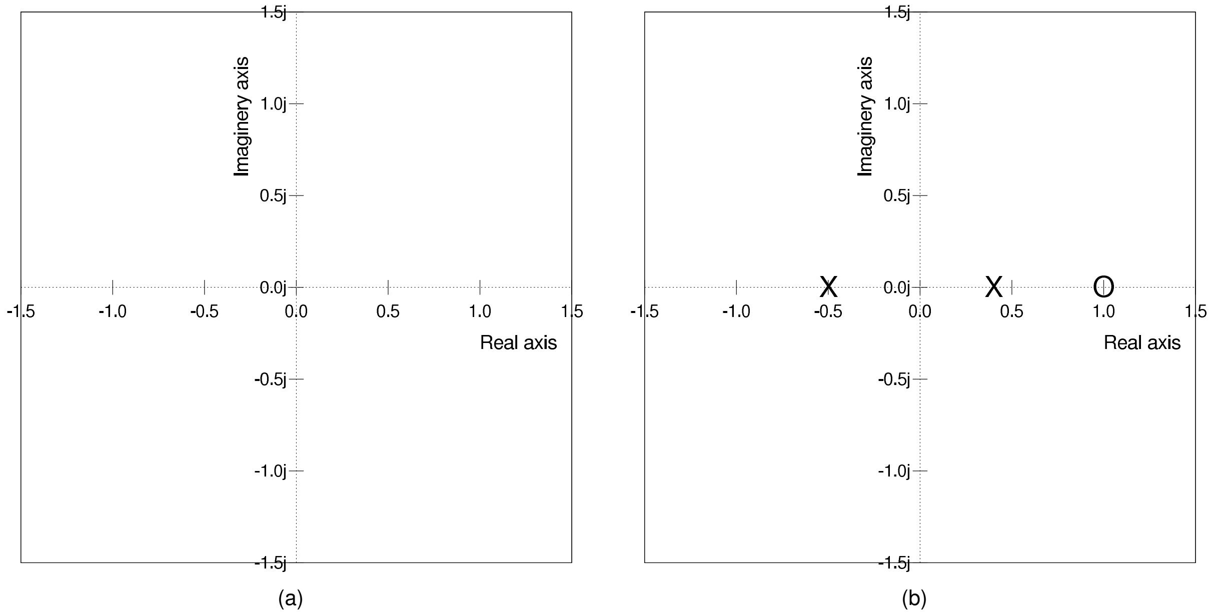


Figure 1: On the left we have an illustration of the z-plane consisting of a real axis (horizontal) and an imaginery axis (vertical). On the right is a z-plane representation of a zero and two poles located at $1 + 0.j$, $-0.5 + 0j$ and $0.4 + 0j$ respectively.

5.1 The z-Transform and Stability

The z-Transform can be used to determine if a system is stable. The z-Transform results in a rational function consisting of a numerator $N(z)$ and a denominator $D(z)$, *i.e.*

$$X(z) = \frac{N(z)}{D(z)} \tag{35}$$

The z-domain function, $X(z)$ can represent a number of different things such as: a system input, a system output or a system transfer function.

The stability of $X(z)$ can be found by the roots of $N(z)$ and $D(z)$:

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z - z_1)(z - z_2)(z - z_3)...}{(z - p_1)(z - p_2)(z - p_3)...} \tag{36}$$

A z-Transform can be represented *graphically*, as shown in Fig. 1. The z-Plane which has the following properties,

- The z-Plane is complex $(a + jb)$ where $j = \sqrt{-1}$
- The vertical axis (↑) is imaginary (b)
- The horizontal axis (→) is real (a)
- The z-Plane is also known as an Argand diagram

The z-plane is useful for depicting the roots of the numerator and the denominator. In particular, the roots of the numerator are z_1, z_2, z_3... known as **zeros**. In addition, the roots of the denominator are p_1, p_2, p_3... known as **poles**.

The **zeros** are values of z that make $X(z) \to 0$ For example, if $z = z_1$ then

$$\begin{aligned} X(z = z_1) =& \frac{N(z = z_1)}{D(z = z_1)} = \frac{K(z_1 - z_1)(z_1 - z_2)(z_1 - z_3)...}{(z_1 - p_1)(z_1 - p_2)(z_1 - p_3)...} \\ =& \frac{K \times 0 \times (z_1 - z_2)(z_1 - z_3)...}{(z_1 - p_1)(z_1 - p_2)(z_1 - p_3)...} = 0 \end{aligned} \tag{37}$$

The **poles** are values of z that make $X(z) \to \infty$ For example, if $z = p_1$ then

$$\begin{aligned} X(z = p_1) =& \frac{N_1}{D_1} = \frac{K(p_1 - z_1)(p_1 - z_2)(p_1 - z_3)...}{(p_1 - p_1)(p_1 - p_2)(p_1 - p_3)...} \\ =& \frac{K(p_1 - z_1)(p_1 - z_2)(p_1 - z_3)...}{0 \times (p_1 - p_2)(p_1 - p_3)...} = \frac{K...}{0} = \infty \end{aligned} \tag{38}$$

Each **zero**: z_1, z_2, z_3, ... is represented by a **circle**: **O** in the z-plane. For each **pole**: p_1, p_2, p_3, ... these are represented by a **cross**: **X** in the z-plane.

For example, given the following z-domain function

$$X(z) = \frac{z-1}{(z+0.5)(z-0.4)} \tag{39}$$

which has a single zero located at

$$z_1 = 1 + 0j \tag{40}$$

which is determined from the numerator of $X(z)$. The denominator has two roots corresponding two two poles from the roots of the denominator:

$$p_1 = -0.5 + 0j; \tag{41}$$

$$p_2 = 0.4 + 0j. \tag{42}$$

These are illustrated in Fig. 1(b). □

Stability is determined by the location of the **poles** in the z-plane. This can be illustrated with the following example.

Consider the following z-domain transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(z-a)} \tag{43}$$

We can easily determine time domain representation of this function. From the table of z-Transform pairs we have

$$\mathcal{Z}^{-1}\left(\frac{z}{z-a}\right) = a^n u[n] \tag{44}$$

Therefore, if we let

$$H(z) = z^{-1}\left(\frac{z}{z-a}\right) \tag{45}$$

And as z^{-1} is a unit delay hence:

$$x[n] = a^{n-1}u[n-1] \tag{46}$$

So that $x[0] = 0$, $x[1] = 1$, $x[2] = a$, $x[3] = a^2$, $x[4] = a^3$ etc. In other words, we have the following relation

$$x[n] = a^{n-1}u[n-1] \tag{47}$$

Now consider what might happen to this function if $a = 0.99$

- The result is the time domain function is decreasing and tending to zero ($x[n] \to 0$) when $a < 1$.

This is illustrated with the plots in Figs. 2(a) and 2(b).

We may also consider what might happen if, *e.g.* $a = 1.01$:

- The results of this is that the time domain impulse response is increasing and tending to infinity ($x[n] \to \infty$) when $a > 1$.

This is also illustrated, in Figs. 2(c) and 2(d).

For a system with $x[n] = a^{n-1}u[n-1]$ we can observe the following

- Decreasing and tending to zero ($x[n] \to 0$) when $a < 1$;
- Increasing and tending to infinity ($x[n] \to \infty$) when $a > 1$.

These observations are true more generally:

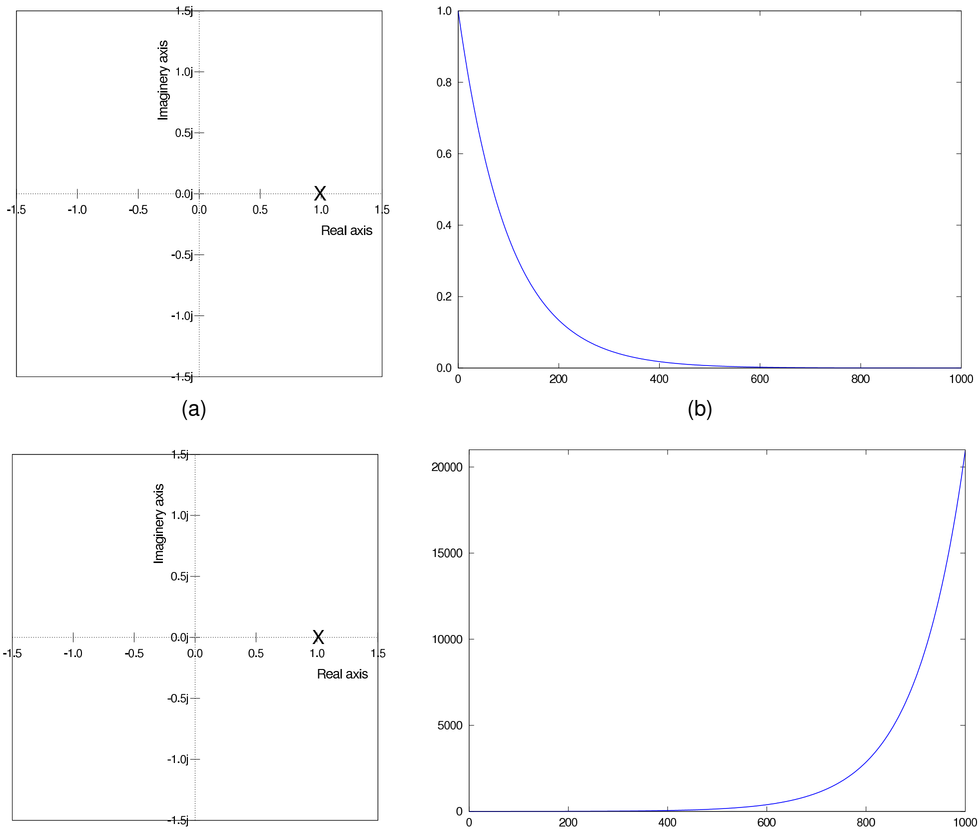
$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)...}{(z-p_1)(z-p_2)(z-p_3)...}. \tag{48}$$

If **magnitude** *of any pole (p_i) is greater than 1 then it will tend to infinity.*

The unit circle is a useful concept to help indicate whether a system is stable or not.

- A unit circle is drawn on the z-plane. This is illustrated in Fig. 3.
- If any pole is outside of the unit circle then the system is **not stable**.

If the **magnitude** *of any pole (p_i) is greater than 1 then it will tend to infinity.*



(a) (b)

(c) (d)

Figure 2: Illustration of what happens to the time domain impulse response in (46) derived from the transfer function in (43). Top illustrates the case for when $a < 1$ and the bottom illustrates what happens when $a > 1$. The left column are the z plane representation of the poles in the transfer function in (43) and on the right are the corresponding time domain impulse responses.

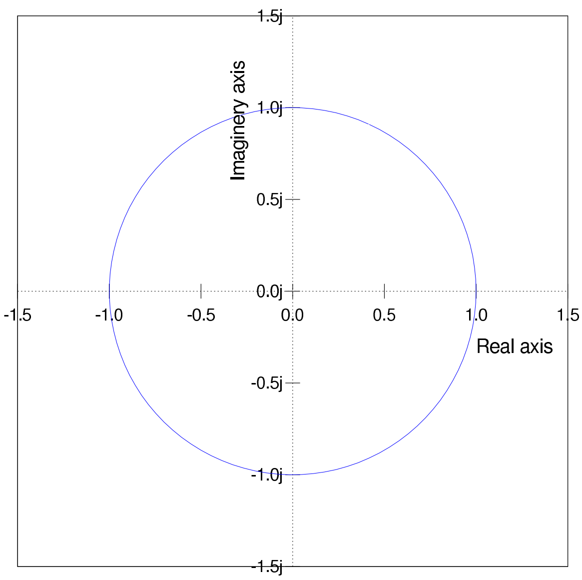


Figure 3: Illustration of the z-plane with the unit circle included. Any pole outside of the unit circle will indicate a source of instability.

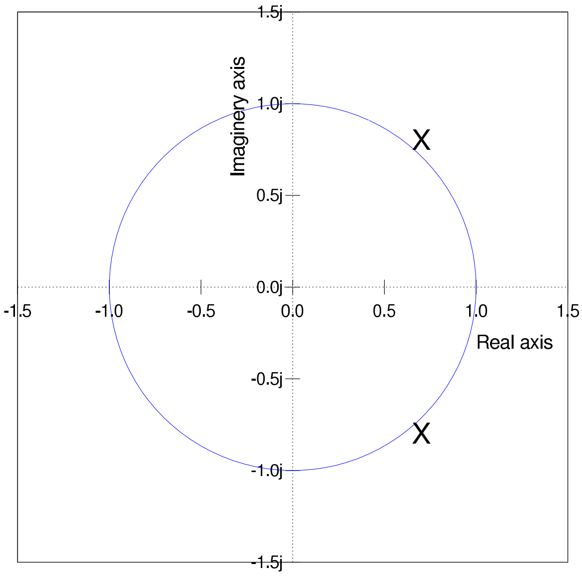


Figure 4: z-Plane poles and zeros for (49).

Stability: *Magnitude Example* Determine whether the following system is stable:

$$H(z) = \frac{1}{(z - 0.7 + 0.8j)(z - 0.7 - 0.8j)} \tag{49}$$

Solution The system has two poles:

$$p_1 = 0.7 - 0.8j \text{ and } p_2 = 0.7 + 0.8j.$$

Distance from the origin given by the magnitude:

$$r = \sqrt{0.7^2 + 0.8^2} = 1.063 > 1.$$

These poles, illustrated in Fig. 4 are beyond the unit circle, therefore this system is **not stable**. □

Stability: *Magnitude Example* **2** Determine whether the following system is stable:

$$H(z) = \frac{1}{(z - 0.5 - 0.5j)(z - 0.5 + 0.5j)} \tag{50}$$

Solution The system has two poles, given by the following

$$p_1 = 0.5 + 0.5j \text{ and } p_2 = 0.5 - 0.5j$$

Distance from origin of poles is given by:

$$r = \sqrt{0.5^2 + 0.5^2} = 0.707 < 1.$$

These poles which are illustrated in Fig. 5 are inside the unit circle, there we can say that the system is **stable**. □

The z-Plane and The Zeros

The z-Plane zeros do **not** determine stability. They can be located anywhere in the z-Plane without directly affecting stability. If a **zero** is located at the origin then there is a time advance of a signal. If there are more zeros than poles then the system starts before $n = 0$ and is therefore **not causal**. It is usually desirable to have the same number of poles and zeros in a system to:

- Ensure minimum delay or time lag;
- Ensure the system is causal.

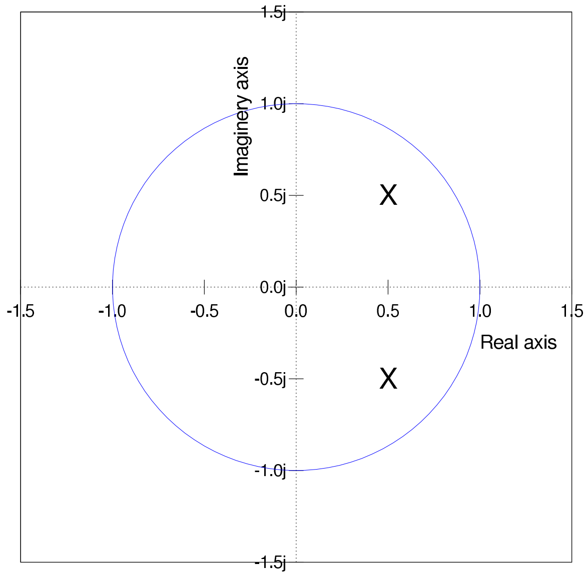


Figure 5: z-Plane poles for (50).

Example: The inverse z-Transform of:

$$H(z) = \frac{1}{z - 0.4} = z^{-1}\left(\frac{z}{z - 0.4}\right) \tag{51}$$

is given by (using the table of z-Transform pairs):

$$x[n] = 0.4^{n-1}u[n-1], \tag{52}$$

which has a delay of 1 time interval. If we provide $H(z)$ with a zero at the origin (*i.e.* $z_1 = 0$) so that:

$$H(z) = \frac{z - z_1}{z - 0.4} = \frac{z}{z - 0.4} \tag{53}$$

then the inverse z-Transform is given by:

$$x[n] = 0.4^n u[n], \tag{54}$$

which has **no time delay**.

6 Solving Difference Equations

This section considers the problem of solving difference equations. Partial fractions are often useful for their solution too. Difference equations are equations similar to, for example

$$y_{n+1} - y_n = 1. \tag{55}$$

Alternative but equivalent notation to $y[n+1] - y[n] = 1.$

Here y is a function of the discrete time index. The solution of a difference equation takes the recursive definition to a non-recursive definition. Recursive means the signal is defined in terms of itself for past or future values. For example, for the above we can have

$$y_n = y_{n+1} - 1. \tag{56}$$

Example 1 First order by the method of partial fractions. Consider the following difference equation

$$y_{n+1} - 2y_n = 1. \tag{57}$$

Solve the above difference equation with the initial condition $y_0 = 1$.

Solution The aim here is to solve the difference equation, first by converting it to the z-domain, then solving it using algebra in the z-domain to determine an expression for $Y(z)$ and then taking inverse z-transform to determine y_n.

Multiplying each of the terms in (57 by z^{-n} and summing gives

$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} - 2\sum_{n=0}^{\infty} y_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}. \tag{58}$$

Each of these terms can be seen to be similar to the equation for the unilateral z-transform, *i.e.*

$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=0}^{\infty} x[n] z^{-n} \tag{59}$$

where instead here $x[n]$ is taking on different forms.

- The left hand term is the z-transform of y_{n+1};
- The middle term is the z-transform of $-2y_n$;
- The right hand term is the z-transform of the unit step, $u[n]$.

So that for the left hand term we

$$\mathcal{Z}\left\{y_{n+1}\right\} = zY(z) - zy_0 \tag{60}$$

where y_0 is the time domain value of y at time $n = 0$.

> Left shift theorem can be found in tables of z-transforms.
> Left shift by 1 time step, *i.e.* $x[n+1]$ is given by:
> $\mathcal{Z}\left\{x[n+1]\right\} = zX(z) - zx[0]$.
> Left shift by 2 time steps, *i.e.* $x[n+2]$ is given by:
> $\mathcal{Z}\left\{x[n+2]\right\} = z^2X(z) - z^2x[0] - zx[1]$.

The z-transform of the middle term is

$$\mathcal{Z}\left\{-2y_n\right\} = -2Y(z). \tag{61}$$

The z-transform of the right hand side is the z-transform of the unit step which is given by

$$\mathcal{Z}\left\{1\right\} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}. \tag{62}$$

The z-transform is an LTI operation so that the results of these transforms can be combined together with the principle of superposition so that

$$zY(z) - zy_0 - 2Y(z) = \frac{z}{z-1}. \tag{63}$$

Recall, the aim is to determine an expression for $Y(z)$. Therefore, factoring out $Y(z)$ yields

$$Y(z)(z-2) - zy_0 = \frac{z}{z-1}. \tag{64}$$

Substituting in the initial condition value for $y_0 = 1$

$$Y(z)(z-2) - z = \frac{z}{z-1}. \tag{65}$$

Solving for $Y(z)$ by first adding z to both sides

$$Y(z)(z-2) = \frac{z}{z-1} + z = \frac{z + z(z-1)}{z-1} = \frac{z + z^2 - z}{z-1} = \frac{z^2}{z-1}. \tag{66}$$

Now dividing both sides by $(z-2)$ gives us

$$Y(z) = \frac{z^2}{(z-1)(z-2)}. \tag{67}$$

This is now a reasonably succinct expression for $Y(z)$. It might be possible to find an inverse transform of this expression from a table of z-transforms directly. However we will split this fraction of polynomials in z into partial fractions, using the method of partial fractions.

Partial fractions are often used to determine some expressions that can be more easily inverse transformed. However, here the numerator has the same power as the denominator. For the method of partial fractions to work, the numerator power needs to be lower than the denominator power. Therefore we need to adjust the expression that we are going to split into partial fractions. Thus, the expression that we will determine the partial fractions can be given by the following

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)} \tag{68}$$

where we have divided through by z on both sides. After we have determined the partial fraction expansion, the z can be brought back by multiplying both sides by z.

Applying the method of partial fractions, we first need to consider splitting the polynomial into each of its roots with

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2}. \tag{69}$$

To determine A and B, we can first obtain a common denominator and then equate the numerator of the resulting expression with the numerator of the original expression in (68), *i.e.* z.

Thus,

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}. \tag{70}$$

Equating the numerators then gives us

$$z = A(z-2) + B(z-1). \tag{71}$$

We can solve for A by letting $z = 1$ which will set the factor of B to zero to give

$$1 = A(1-2) + B(0) = A(-1). \tag{72}$$

Therefore

$$A = -1. \tag{73}$$

Similarly, setting $z = 2$ yields

$$2 = A(0) + B(1) = B. \tag{74}$$

Therefore

$$B = 2. \tag{75}$$

Substituting the values for A and B back into the partial fraction expansion in (69) gives us

$$\frac{Y(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-2}. \tag{76}$$

Multiplying both sides by z

$$Y(z) = \frac{-1z}{z-1} + \frac{2z}{z-2}. \tag{77}$$

Both terms on the right hand side are readily available in tables of inverse z-transforms.

$$\boxed{\begin{aligned} \mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} &= u[n] \\ \mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\} &= a^n u[n] \end{aligned}}$$

For the left hand side we have

$$\mathcal{Z}^{-1}\{Y(z)\} = y_n. \tag{78}$$

For the first term on the right hand side we have

$$\mathcal{Z}^{-1}\left\{-1\frac{z}{z-1}\right\} = -u_n \tag{79}$$

i.e. negative 1 for $n \geq 0$. For the second term on the right hand side

$$\mathcal{Z}^{-1}\left\{2\frac{z}{z-2}\right\} = 2 \times 2^n u_n. \tag{80}$$

Putting these together gives us

$$y_n = -u_n + 2 \times 2^n u_n = -u_n + 2^{n+1} u_n. \tag{81}$$

We can now readily determine the value of y_n for any value of n without having to perform any recursive calculations. For example, for $n = 10$ we have:

$$y_{10} = -1 + 2^{11} = 2047. \tag{82}$$

We can check to see if this solution for $n = 10$ is correct with a short script in Matlab, for instance

```
N=10;
y=zeros(1,N+1);
y(1)=1;
for n=1:N
   y(n+1)=1+2*y(n);
end
```

The values calculated in y can then be inspected and are given by

```
y =
   1    3    7    15    31    63    127    255    511    1024    2047
```

The solution to the difference equation is preferable, particular in cases where much larger numbers are required. In this particular equation, the result would also be huge due to the resulting power in the formula. □

The above is an example of finding the solution to a first order difference equation. However higher orders are possible too of course.

Example 2 Solve the following 2nd order difference equation

$$y_{n+2} = y_{n+1} + 6y_n \tag{83}$$

with initial conditions given as $y_0 = 1$ and $y_1 = 2$.

Answer Taking the z-transform of the above terms yields

$$z^2Y(z) - z^2y_0 - zy_1 = zY(z) - zy_0 + 6Y(z) \tag{84}$$

where the left shift theorem by 2 steps and 1 step have been used, as noted earlier. Substituting in the initial values

$$z^2Y(z) - z^2 - 2z = zY(z) - z + 6Y(z) \tag{85}$$

and then rearranging to put $Y(z)$ on the left hand side gives us

$$z^2Y(z) - zY(z) - 6Y(z) = z^2 + 2z - z. \tag{86}$$

Combining like terms and factoring out $Y(z)$ gives us

$$z^2Y(z) - zY(z) - 6Y(z) = Y(z)(z^2 - z - 6) = z^2 + z. \tag{87}$$

Dividing through by the $Y(z)$ factor and finding the roots gives us

$$Y(z) = \frac{z(z+1)}{(z+2)(z-3)}. \tag{88}$$

Quadratic formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$(z^2 - z - 6) = (z+2)(z-3)$
where $a = 1,\ b = -1,$ and $c = -6.$

Thus, determining the partial fraction expansion is required. Again, the numerator has the same power of z as the denominator, therefore dividing through by z and putting in the partial fractions with unknown constants in the numerators gives us

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z+2)(z-3)} = \frac{A}{z+2} + \frac{B}{z-3}. \tag{89}$$

To determine A and B, determine the common denominator for the right hand side and then equate the numerators giving us

$$z + 1 = A(z - 3) + B(z + 2). \tag{90}$$

Letting $z = -2$ yields

$$-2 + 1 = A(-5) \tag{91}$$

$$\therefore A = \frac{1}{5}. \tag{92}$$

For B, let $z = 3$ yields

$$3 + 1 = B \times 5 \tag{93}$$

$$\therefore B = \frac{4}{5}. \tag{94}$$

Substituting the values of these constants into the partial fractions gives us

$$\frac{Y(z)}{z} = \frac{\frac{1}{5}}{z+2} + \frac{\frac{4}{5}}{z-3}. \tag{95}$$

Mutliplying through by z

$$Y(z) = \frac{\frac{1}{5}z}{z+2} + \frac{\frac{4}{5}z}{z-3}. \tag{96}$$

The inverse z-transform of each term can then be found, as before to give

$$\mathcal{Z}^{-1}\left\{Y(z)\right\} = y_n = \frac{1}{5}(-2)^n u_n + \frac{4}{5}3^n u_n. \tag{97}$$

This has a more interesting form that can be used, as shown previously, to determine values of y_n for different values of n directly rather than having to determine each value of y_n recursively. □

6.1 Summary

The solution of difference equations as shown here can be considered to follow a similar set of steps which are:

1. Determine the z-transform for the difference equation;
2. Solve the z-domain equation using algebra and determine an expression for $Y(z)$ or similar;
3. Take the inverse z-transform to determine the solution of the difference equation.

Exercise Solve the difference equation

$$a_{n+2} - 4a_n = 1, \tag{98}$$

with initial conditions $a_0 = 0$ and $a_1 = 2$.

Answer You should find

$$a_n = -\frac{1}{3}u_{n-1} + \frac{3}{2}2^{n-1} + \frac{5}{6}(-2)^{n-1}. \tag{99}$$