Filter Structures

Digital Signal Processing

Notes

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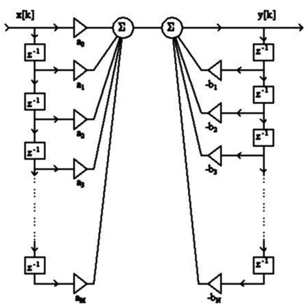
# Filter Structures

Notes




Known as Direct Form I realisation,
with appearance of two filters in series:

- FIR-like stage (sometimes called a Moving-Average (MA) filter)

  … *followed by* …

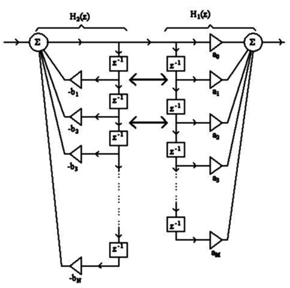
- a feedback stage (sometimes called an Auto-Regressive (AR) filter)

## Direct Form I

- Filter structure requires $(N + M)$ values to be stored by the unit delays
    - *Represented in the diagram by the formula* $z - 1$.
- A more efficient realisation can be made...

Notes

## Alternative Structure




- First rearrange system to put the AR stage first,
- Points identified by the heavy arrows contains exactly the same data.
- Smaller column of $z - 1$ unit delay storage locations now redundant:
    - Removing them, means we only need $\max(M, N)$.

Redrawing without redundant delays, gives Direct Form II or state-variable realisation of the system.
*This is left as an exercise.*

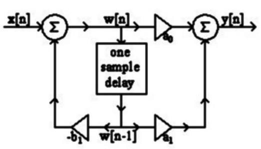
Notes

## Analyzing 1st Order IIR

A general 1st-order digital filter can be viewed as being two devices:

- a feedback part and
- a feedforward part,

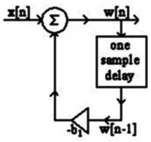
Connected in series (or cascaded).




Notes

## Feedback part

Input sequence $= x[n]$; output sequence $= w[n]$




Notice minus sign on the coefficient $b_1$ and the absence of a coefficient in the upper signal path.

Notes

## Feedback part

The difference equation of this part is:

$$w[n] = x[n] - b_1 w[n-1]$$

... which z-transforms using the delay property into:

$$W(z) = X(z) - z^{-1} b_1 W(z)$$

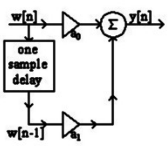
i.e. $W(z) = \frac{X(z)}{1+b_1 z^{-1}}$ ...leading to a transfer function given by:

$$H_{back}(z) = \frac{W(z)}{X(z)} = \frac{1}{1+b_1 z^{-1}}$$

Notes

## Feedforward part

Input sequence $= w[n]$; output sequence $= y[n]$




Notice the presence of coefficients in both upper & lower signal paths.

Notes

## Feedforward part

The difference equation of this part is:

$$y[n] = a_0 w[n] + a_1 w[n-1]$$

...which z-transforms using the delay property into:

$$Y(z) = a_0 W(z) + a_1 z^{-1} W(z)$$

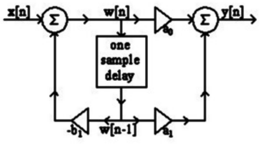
... leading to a transfer function given by:

$$H_{fwd}(z) = \frac{Y(z)}{W(z)} = a_0 + a_1 z^{-1}$$

Notes

## Combined Filter

Input $= x[n]$; output $= y[n]$; ($w[n]$ is an internal state variable).




Notes

## Combined Filter

The combined transfer function of the complete filter is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)}$$

i.e. combined transfer function of **devices in cascade** is:

**the product of their individual transfer functions,**

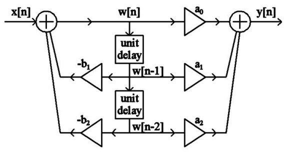
yielding

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

Notes

# Analysis of a Biquadratic Filter

Biquadratic filter:




Feedforward part:

$$y[n] = a_0 w[n] + a_1 w[n-1] + a_2 w[n-2]$$

z-transform:

$$\begin{aligned} Y(z) &= a_0 W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) \\ &= W(z)\left(a_0 + a_1 z^{-1} + a_2 z^{-2}\right) \end{aligned}$$

Notes

# Analysis of a Biquadratic Filter

Feedback part:

$$\begin{aligned} w[n] &= x[n] - b_1 w[n-1] - b_2 w[n-2] \\ w[n] + b_1 w[n-1] + b_2 w[n-2] &= x[n] \end{aligned}$$

z-transform:

$$W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) = X(z)$$

$$W(z)\left(1 + b_1 z^{-1} + b_2 z^{-2}\right) = X(z)$$

$$W(z) = \frac{X(z)}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

Notes

# Analysis of a Biquadratic Filter

Combining them:

$$Y(z) = \left(a_0 + a_1 z^{-1} + a_2 z^{-2}\right) \frac{X(z)}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} = \frac{a_0 z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}$$

Notes

## Difference Equation

We can derive the difference equation in the same way as for the first-order filter:

$$\frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

$$Y(z)\left(1 + b_1 z^{-1} + b_2 z^{-2}\right) = X(z)\left(a_0 + a_1 z^{-1} + a_2 z^{-2}\right)$$

$$y[n] + b_1 y[n-1] + b_2 y[n-2] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2]$$

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] - b_1 y[n-1] - b_2 y[n-2]$$

Notes

## Higher-Order Filters

Transfer function of a general IIR filter can be written

**Ratio of two polynomials in $z$**

*Already seen for first and second order filters*

$$\begin{aligned} H(z) &= \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_M z^{-M}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + a_N z^{-N}} \\ &= z^{-(M-N)} \frac{a_0 z^M + a_1 z^{M-1} + a_2 z^{M-2} + \ldots + a_M}{z^N + b_1 z^{N-1} + b_2 z^{N-2} + \ldots + a_N} \end{aligned}$$

Leading $z^{-(M-N)}$ just keeps algebra correct when $M$ and $N$ are different.

Notes

## Higher-Order Filters

The two big polynomials can be factorised:

$$H(z) = a_0 z^{-(M-N)} \frac{(z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)}$$

- Leading $a_0$ keeps algebra correct;
- $\{z_i\}$ numerator terms are the zeros,
- $\{p_k\}$ denominator terms are the poles.

This formulation allows us to implement filter as a combination of low-order blocks, either in series (cascade) or in parallel.

Notes

## Cascade Realisation

- If coefficients of numerator and denominator polynomials are real then
  - Poles and zeros of $H(z)$ must be either real or complex conjugate pairs.
  - Enables splitting of filter in two:
    - First and second order (biquadratic) filters:

$$H(z) = a_0 z^{-(M-N)} \left(\frac{z-z_1}{z-p_1}\right)\left(\frac{z-z_2}{z-p_2}\right)\ldots\left(\frac{(z-z_m)(z-z_m^*)}{(z-p_n)(z-p_n^*)}\right)\ldots$$

Notes

## Cascade Realisation

Splitting transfer function into cascaded 1st and 2nd order sections can be very beneficial:

- In practice coefficients of a digital filter implemented to certain precision (number of bits).
- Rounding errors in a single coefficient can affect location of every pole
  - Potentially making system unstable
  - *(for stability all poles must lie within unit circle)*.
- Factorising system into 1st and 2nd order sections can:
  - Make sure error in particular coefficient of smaller section affect (at most) a conjugate pair of poles
  - Be much easier to predict and to control.

Notes

## Cascade Realisation

- For first-order section, the impact on the coefficient is can be seen from $|p_n|$.
- For second-order (biquadratic section) with conjugate poles (and conjugate zeros), first need to multiply it out:

$$\left(\frac{(z-z_m)(z-z_m^*)}{(z-p_n)(z-p_n^*)}\right) = \frac{z^2 - 2\mathrm{Re}(z_m)\times z + |z_m|^2}{z^2 - 2\mathrm{Re}(p_n)\times z + |p_n|^2}$$

... where $\mathrm{Re}()$ means the real-part of.

Notes

## Cascade Realisation

Two issues:

1. $|p_n| < 1$ (poles inside the unit circle) which directly affects denominator's constant coefficient.
2. Also $|\mathrm{Re}(p_n)| < |p_n|$ (otherwise $p_n$ cannot exist) which constrains the coefficient of $z$ in the denominator.

**Note:** Before working out cascade realisation, have to factorise the polynomials and it may not be easy to determine the poles' locations accurately, though software e.g. MATLAB makes this less of a problem.

Notes

## Parallel Realisation

**Realisation using a number of first and second order sections in parallel**

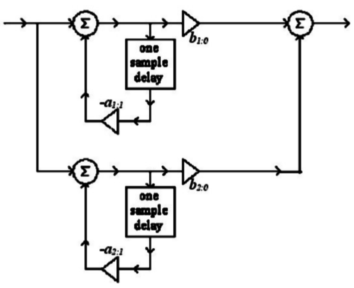
- If transfer function denominator is factorised,
- Transfer function is in a form suitable for inversion by partial fractions:

$$\frac{H(z)}{z} = z^{-(M-N)}\frac{a_0 z^M + a_1 z^{M-1} + a_2 z^{M-2} + ... + a_M}{z(z-p_1)(z-p_2)...(z-p_N)}$$
$$= \frac{C_0}{z} + \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + ... + \frac{C_N}{z-p_N}$$

- Parallel sections have same input signal and with outputs summed together.

Notes

## Parallel Realization




Similar to cascade realisation:

- Parallel realization decouples terms so that numerical errors in one coefficient affects one conjugate pair of poles
  - Much easier to predict and control

Notes

## Exercise

Notes

Determine for the following transfer function

$$H(z) = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}$$

1. A cascade and
2. A parallel realisation

using only first-order structures. Also:

3. Sketch block diagrams which result and compare with block diagram you would get from implementing the above as a single biquadratic structure.

## Summary

Notes

- Filter design examples provided;
- Different filter realization methods considered.

Notes