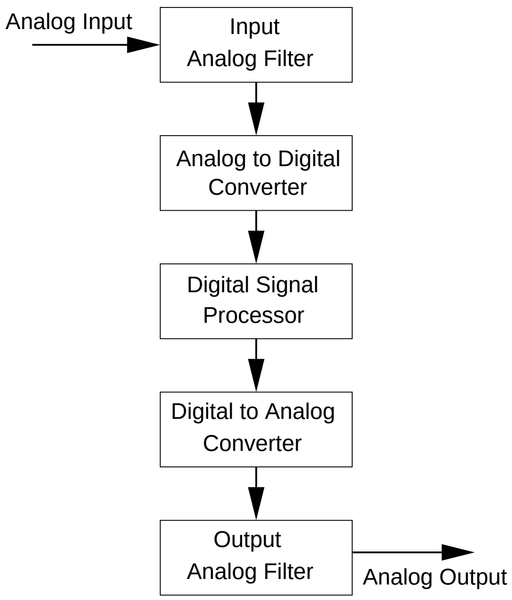
Signal Sampling

Contents

1 Digital Signals

1.1 A Typical Digital Signal Processing

- Signal sampling is performed by an Analogue to Digital Converter or ADC to create a digital signal from an analogue signal



- The analogue signal is reconstructed from the digital signal by a Digital to Analogue Converter (DAC).

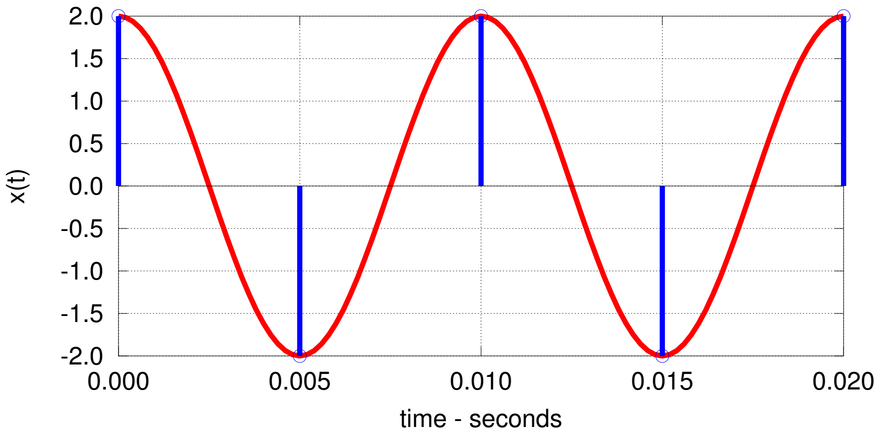
2 Sampling, Aliasing and Nyquist

2.1 Minimum Sampling Frequency

The value of the minimum sampling frequency is given by the sampling theorem which states that a signal has to be sampled at twice the maximum frequency present in the signal. Otherwise the digital signal will not contain sufficient information to enable the original analogue signal to be reconstructed. This also means that:

- An analogue signal at frequency f_i should be sampled at least $f_s = 2 \times f_i$

Here is an example of a signal that has been sampled at a frequency that is sufficiently large enough to enable it to be properly represented in digital form and then back in analogue form upon reconstruction:



This analogue signal (cosine wave) with frequency f_i has been sampled at

$$f_s = 2f_i.$$

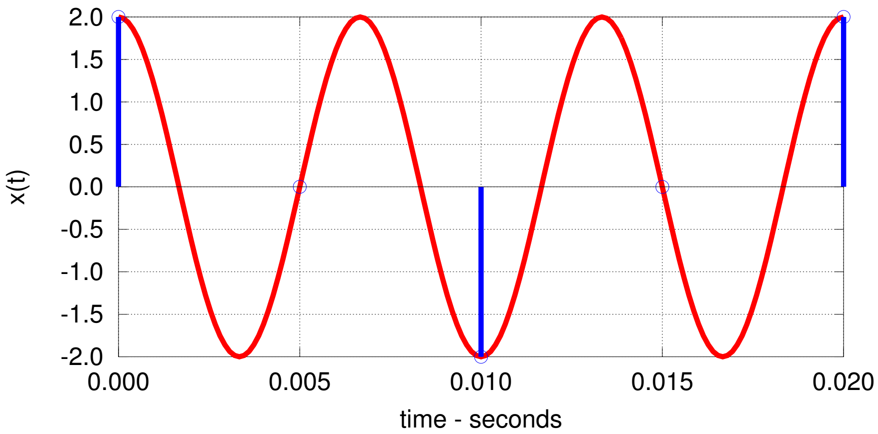
Higher sampling frequencies could also be used to enable the sampled signal to be properly represented in digital form.

2.2 Under Sampling

If a signal is sampled at a frequency less than $2f_i$ it would be insufficiently sampled. This means that it would not be sampled enough to enable the digital signal processor to process the signal correctly. Furthermore it would be impossible to reconstruct the original signal in analogue form.

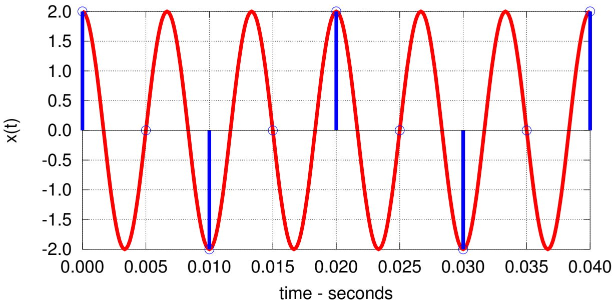
- If an analogue signal is undersampled where $f_i > f_s/2$ then the digital signal will not contain sufficient information to reconstruct the analogue signal

For example, the discrete samples here are insufficient to recreate the original signal.

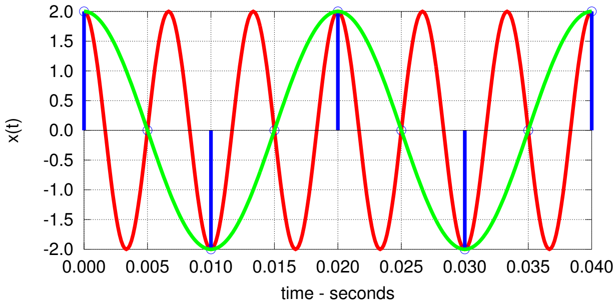


2.3 Aliasing (Errors)

Aliasing occurs when the sampling frequency is not high enough $f_s < 2f_i$ or (equivalently) the sampled analogue signal has too high frequency $f_i > f_s/2$. For example, here is a signal with frequency 150Hz. On the left it has been sampled insufficiently at a frequency of 200Hz. On the right a curve that joins the digital samples has been included illustrating one of the aliasing frequencies that occur. These aliasing frequencies are mistaken by the digital signal processor as true signal components which they are not. They are also present upon reconstruction back to the analogue form.



$f_i = 150$Hz



$f_{\text{alias}} = 50$Hz

$f_s = 200$Hz

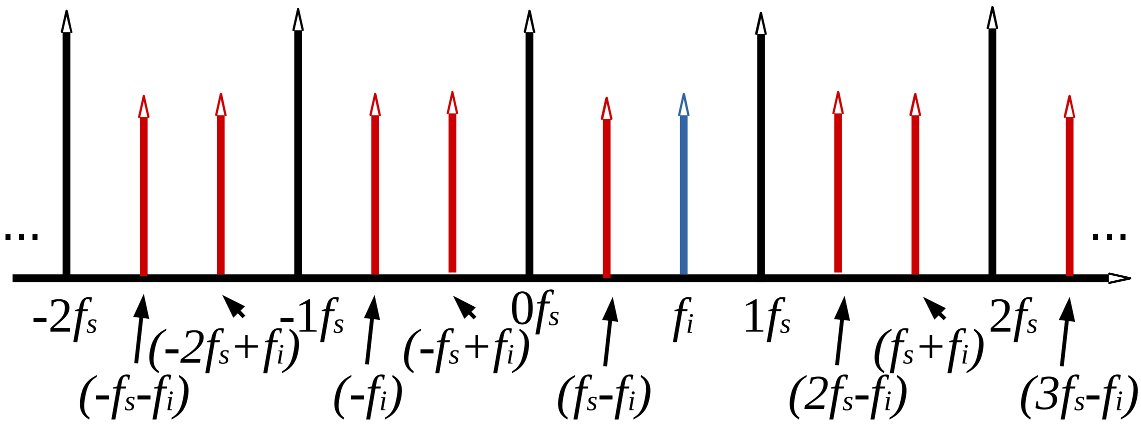
An aliasing frequency occurs at $f_s = 200$Hz because the signal sampling process represents a signal in a way that is ambiguous where other frequencies could also come about because of the way the signal has been sampled.

If a signal has been sampled at f_sHz then the sampled signal now appears at regular intervals from $-\infty$ up to $+\infty$.

$$f_{\text{alias}} \text{ is one of } \{(f_i + n \times f_s) \text{ or } (n \times f_s - f_i) : \text{ for } -\infty \le n \le \infty\}.$$

In the example above we have $f_s = 200$Hz and $f_i = 150$Hz. There are an infinite number of components now present in the digitally sampled signal. The aliasing error frequency of $f_{\text{alias}} = 50$Hz occurs at $n = 1$ and $1 \times f_s - f_i = 200 - 150 = 50$Hz.

This is illustrated below.



2.4 Sampling and Nyquist

A digital signal is sampled f_s times a second.

- The **Nyquist frequency**, also known as the **folding frequency** is

$$\frac{f_s}{2}.$$

- The **Nyquist rate** is different from the *Nyquist frequency*. The Nyquist rate is given by:

$$2 \times B$$

  where B is the bandwidth of the signal.

- The **sampling frequency** should be greater than the Nyquist rate:

$$f_s > 2B$$

to prevent **aliasing** errors.

3 Spectrum

A signal is rarely composed of a single frequency and we will often consider a signal to consist of a range of frequencies. The difference between the highest $f_{\max}$ and lowest frequency f_{low} in the signal is then referred to as the Bandwidth, B *i.e.*

$$B = f_{\max} - f_{\text{low}}.$$

The lowest frequency is often 0Hz, *i.e.* $f_{\text{low}} = 0$ so that the Bandwidth is equal to the maximum frequency *i.e.*

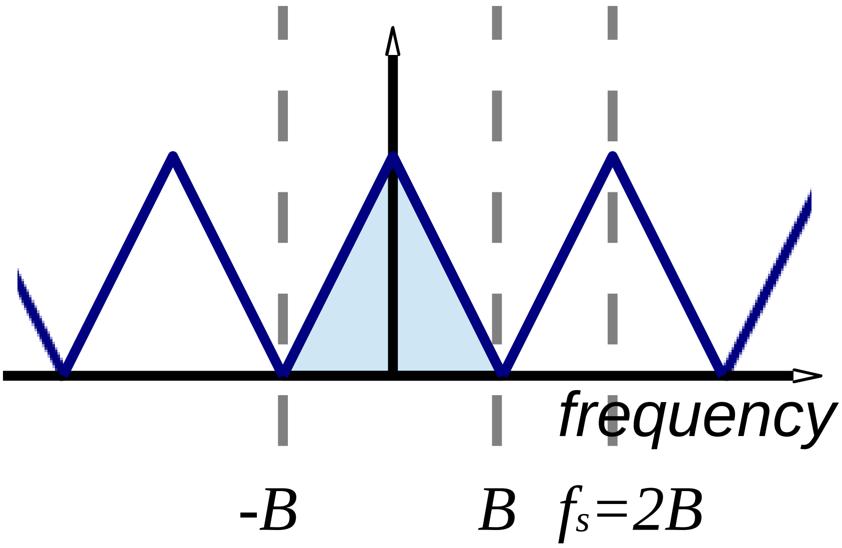
$$B = f_{\max} \qquad \text{when} \qquad f_{\text{low}} = 0.$$

The spectra of sampled signals also contains:

- Negative frequencies from $-f_s/2$ to 0 Hertz
- Original signal frequencies from 0 Hertz to $f_s/2$ Hertz
- Images of the negative and original signal frequencies

3.1 Spectrum: no aliasing

If $f_s = 2B$ frequency spectrum of digital signal will be:

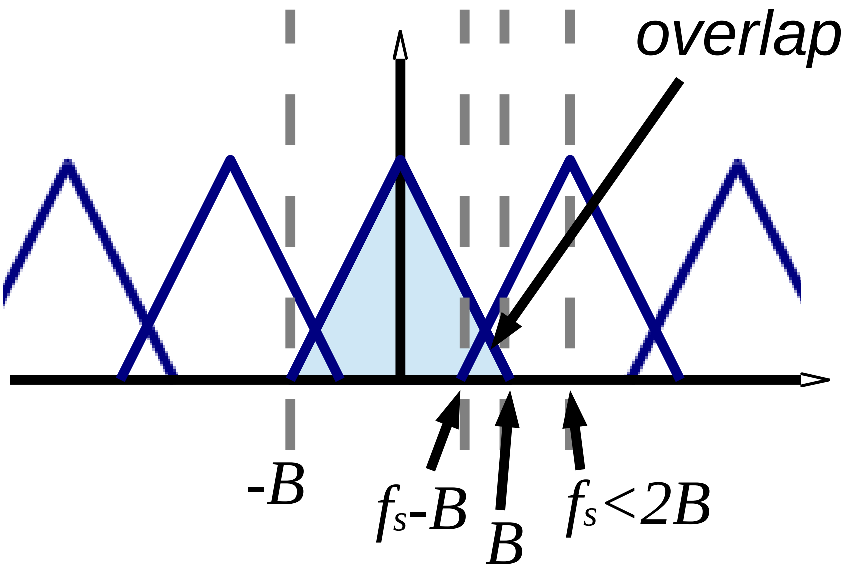


$f_s/2$ is known as folding frequency because all frequencies from 0Hz to BHz fold over to BHz to $2B$Hz:

Creating a mirror image of the frequencies.

3.2 Spectrum: with aliasing

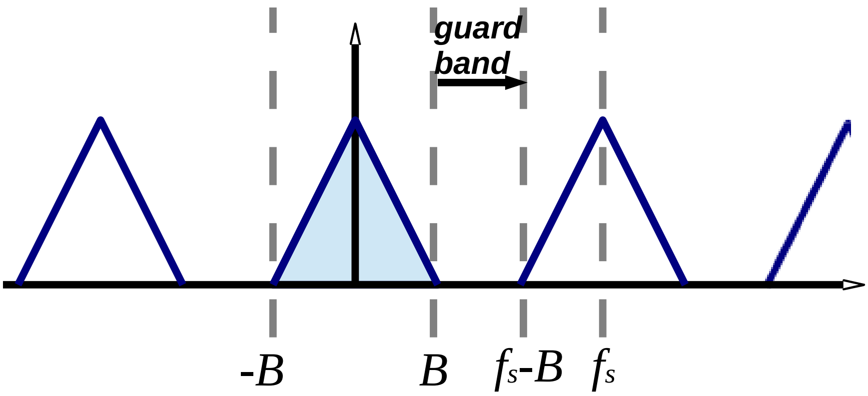
If the sampling frequency is not high enough, *i.e.* $f_s < 2B$ then the frequency spectrum of digital (sampled) signal will be:



Aliasing error frequencies occur in the overlap region between $(f_s - B)$Hz and BHz. These overlapping frequencies cause problems because it is impossible to know which parts of the signal are the true signal content.

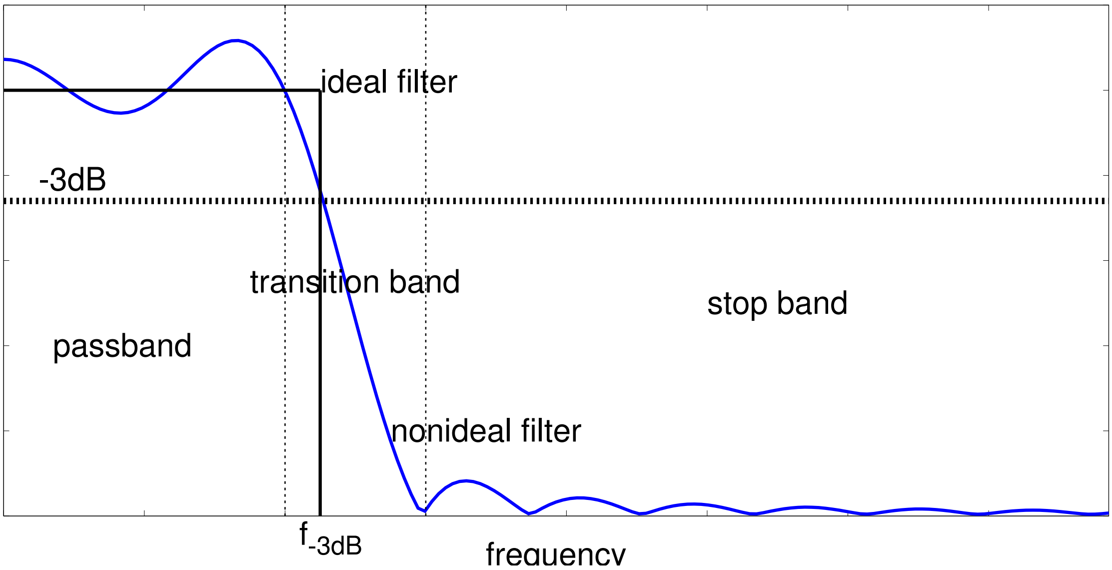
3.3 Spectrum: $f_s > 2B$

The sampling theorem requires sampling to be greater than twice the maximum frequency in the signal. Earlier sampling at exactly twice the maximum frequency was illustrated. However if the signal is sampled at an even higher sampling frequency, *i.e.* if $f_s > 2B$ then there is a gap known as a guard band between B Hz and $(f_s - B)$Hz like so:



This guard band is really useful because a signal usually consists of a range of frequencies that should not be attenuated. Sampling into digital form means that all frequencies above the maximum frequency *of interest* have to be removed using an analogue anti-aliasing filter. However real filters are not ideal and there is a transition band between the cut off frequency and the frequencies at which have been attenuated sufficiently that they are below the minimum quantisable level by the ADC.

This means that the guard band can be useful because the antialiasing filter and (also) reconstruction filters will not be ideal filters and will have finite width transition bands between the pass and stop bands:



Here you can see $f_{-3\mathrm{dB}}$ or below is the frequency at which the filter is typically designed to allow frequencies to *pass*. However $f_{-3\mathrm{dB}}$ is not the frequency at which all frequencies *above* are completely blocked. This is due to the non-ideal response. An *ideal* response would have a sudden drop at this -3dB cut off frequency but real world filters can not be designed with this sudden drop. Therefore there is a transition band that covers the range of frequencies which transition between the pass and stop bands.

4 Fourier Series

Fourier theory can be really useful to help us understand why the spectrum of a signal contains negative frequencies and why a sampled signal contains images of the signal.

4.1 Fourier Series

Discrete Fourier series approximation of periodic digital signal $x[n]$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \exp\left(-\frac{j2\pi kn}{N}\right).$$

with $0 \leq n < N$. $X[k]$ represents the spectral components or harmonics for $0 \leq k < N$. The original signal can be reconstructed with:

$$x[n] = \sum_{k=0}^{N-1} a_k \exp\left(-\frac{j2\pi kn}{N}\right).$$

The complex exponential can appear to be a bit intimidating. However we can use Euler's formula to convert a complex exponential into what might seem a simpler form in terms of a complex number consisting of two trigonometric terms. **Euler's formula** is often used to expand the complex exponentials:

$$\exp\left(-\frac{j2\pi kn}{N}\right) = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right).$$

So the signal $x[n]$ is approximated by a combination of cosine and complex sinusoidal functions.

4.2 Negative Frequencies

- Negative frequencies are the mirror image of the frequencies in the signal.

Negative frequencies are interesting because they appear in the frequency spectrum of a signal. They also appear as part of the repeated images of a sampled signal.

To understand negative frequencies it can be useful to consider the following question:

- How to represent a real signal (i.e. no $j = \sqrt{-1}$) with a combination of cosine and complex sinusoidal functions?

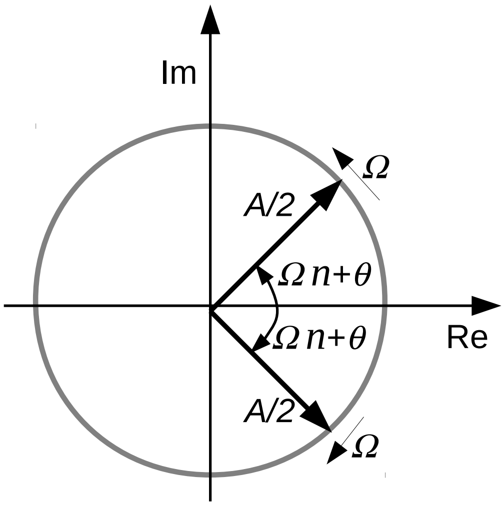
- **Only use the real part of the complex exponential.**

Example
Represent $\cos(\Omega n + \theta)$ with complex exponentials where $\Omega = \frac{2\pi k}{N}$.
Answer
Representing $\cos(\Omega n + \theta)$ with complex exponentials:

$$\begin{aligned} A\cos(\Omega n + \theta) &= \frac{1}{2}\left(A\left(\cos(\Omega n + \theta) + j\sin(\Omega n + \theta)\right) + A\left(\cos(\Omega n + \theta) - j\sin(\Omega n + \theta)\right)\right) \\ &= \frac{A}{2}\underbrace{\exp(j\Omega n + \theta)}_{\text{positive frequencies}} + \frac{A}{2}\underbrace{\exp(-j\Omega n + \theta)}_{\text{negative frequencies}} \end{aligned}$$

- The **positive frequencies** travel anti-clockwise
- The **negative frequencies** travel clockwise
- Positive and negative frequencies project same value onto **real (Re) axis**.

Complex Plane:



where $\Omega = \frac{2\pi k}{N}$. So this helps to illustrate that representing the a simple real cosine signal using complex exponentials requires a complex exponential with a positive frequency in combination with another complex exponential with a negative frequency. This will be true for other signals and furthermore frequency is usually defined in terms of Fourier theory which converts a signal in the time domain to a signal in the Fourier domain using complex exponentials.

5 Sampling Theorem and the Fourier Spectrum of Sampled Signals

We have considered the origin of the negative frequencies in a frequency spectrum of a signal. We will now consider the origin of the multiple images of a sampled signal. First we need to consider the sampling theorem more mathematically.

Sampling Theorem : An analogue signal $x(t)$ with bandwidth B is uniquely specified by its samples $x[n] = x(mT)$, $m = 0, \pm1, \pm2, ...$ with sampling period $T = 1/f_s$ and sampling frequency f_s where

$$f_s > 2B.$$

The sampled signal can be described by a train of unit impulse functions:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

here we are using the unit impulse function:

$$\delta(t - nT) = \begin{cases} 1 & \text{when} \quad t = nT \\ 0 & \text{elsewhere.} \end{cases}$$

The analogue signal $x(t)$ can be multiplied by $x_\delta(t)$ to get

$$\begin{aligned} x[n] = x(t)x_\delta(t) &= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT), \end{aligned}$$

using sifting property of delta function $x(t)\delta(t-nT) = x(nt)\delta(t-nT)$. We will now convert this train of impulse functions to the Fourier domain using the Fourier transform.

Recall the Fourier and the inverse Fourier transforms:

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp\left(-j2\pi ft\right)\mathrm{d}t \quad \text{and} \quad x(t) = \int_{-\infty}^{\infty} X(f)\exp\left(-j2\pi ft\right)\mathrm{d}f.$$

To determine the Fourier transform of the discretely sampled signal $x[n] = x(t)x_\delta(t)$ we will consider each part individually. The Fourier transform of the original unsampled analogue signal is given by:

$$\mathcal{F}(x(t)) = X(f).$$

The Fourier Transform of the impulse train $x_\delta(t)$ is given by:

$$\mathcal{F}(x_\delta(t)) = X_\delta(f) = \frac{1}{T}\sum_n \delta(f-nf_s)$$

Fourier (or inverse) Transform of multiplication is convolution with symbol $*$, therefore the Fourier transform of $x(t) \times x_\delta(t)$ is

$$\mathcal{F}(x(t) \times x_\delta(t)) = X(f) * \delta(f-nf_s).$$

Convolution of any function e.g. $G(f)$ with an impulse function $\delta(f-nf_s)$ has the effect of shifting the function to a different frequency given by $f-nfs$ *i.e.*

$$G(f) * \delta(f-nf_s) = G(f-nf_s).$$

Using above we get:

$$\begin{aligned} X(f) * X_\delta(f) &= X(f) * \left[\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(f-nf_s)\right] \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f-nf_s) \end{aligned}$$

$\therefore$ Fourier Transform of sampling process $x(t)x_\delta(t)$:

$$X(f) * X_\delta(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f-nf_s)$$

shows spectrum of $x(t)$ is repeated at intervals spaced by nf_s.

For analogue signal with frequency f_i, the sampled signal's frequency content for the positive frequency will be

$$..., f_i-2f_s, f_i-f_s, f_i, f_i+f_s, f_i+2f_s, ...$$

Sampling images of negative frequencies of signal:

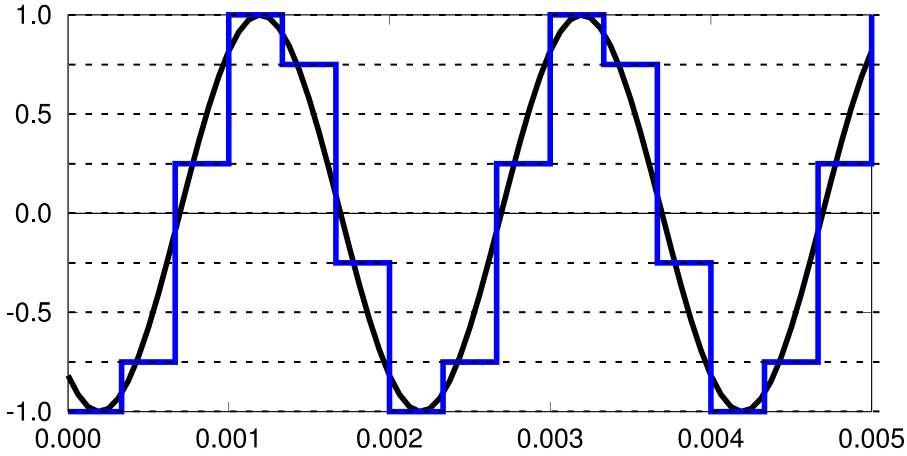
$$..., -2f_s-f_i, -f_s-f_i, -f_i, f_s-f_i, 2f_s-f_i, ...$$

Combining positive and negative frequencies:

$$((f_i+n\times f_s), (n\times f_s-f_i): \text{ for } -\infty \le n \le \infty)$$

6 Reconstruction

Assuming the sampling of the analogue signal has been performed at a high enough frequency then the frequency images of the original signal still have to be removed when the analogue signal is reconstructed.
A reconstruction filter can help remove the noisy components that may otherwise manifest as a series of steps.



These steps are the quantisation steps of the sampling process.

7 Summary

Analogue to Digital Conversion (ADC) and Digital to Analogue Conversion (DAC) are important steps that convert and reconstruct an analogue signal to and from the digital domain.
Aliasing errors may occur if the signal is not sampled frequently enough.
Furthermore the digital representation is ambiguous with images of the original signal that need to be removed upon reconstruction to an analogue form.