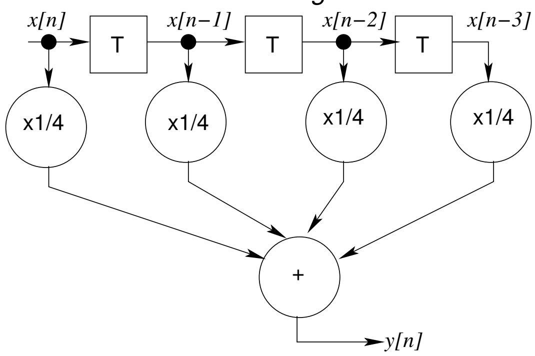
Solutions for: Introduction to IIR Filters Tutorial

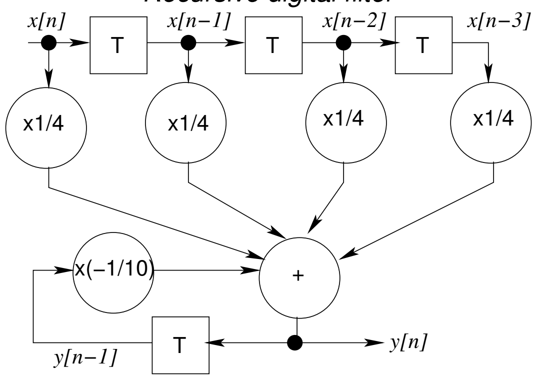
1. How is a recursive digital filter different from a non-recursive digital filter? Explain in a few sentences and draw two example system diagrams, one of a recursive filter and one of non-recursive filter.
   Solution A recursive filter uses past output values ($y[n-i]$) for the calculation of the current output $y[n]$:

   - *Recursive Filter Example*
   $$y[n] = 0.5y[n-1] + 0.5x[n].$$

   A non-recursive filter only uses input values $x[n-i]$ where $i \geq 0$:

   - *Non-recursive Filter Example*
   $$y[n] = 0.5x[n-1] + 0.5x[n].$$





2. Design a bandpass filter using the pole-zero placement method with:
   - centre frequency $\Omega_0 = \pi/2$
   - bandwidth $\Omega_{bw} = \pi/8$
   - complete attenuation at $\Omega_{r1} = 0$ and $\Omega_{r2} = \pi$

   Solution Bandpass filter has x2 poles at $\pm\Omega_0 = \pm\pi/2$ radians. Therefore
   $$H(z) = K\frac{\text{zeros}}{(z - r\exp(j\pi/2))(z - r\exp(-j\pi/2))}$$

   The radii of the poles are given by:
   $$r \cong 1 - \frac{\Omega_{\text{bw}}}{2} = 1 - \frac{\pi/8}{2} = 0.80365;$$

   and the zeros are at $\Omega_{\text{r1}} = 0$ and $\Omega_{\text{r2}} = \pi$, so that
   $$H(z) = K\frac{(z - \exp(j\Omega_{\text{r1}}))(z - \exp(j\Omega_{\text{r2}}))}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

   As

- $\exp(\Omega_{r1}) = \exp(j0) = \cos(0) + j\sin(0) = 1 - j0 = 1$
- $\exp(\Omega_{r2}) = \exp(j\pi) = \cos(\pi) + j\sin(\pi) = -1 + j0 = -1$,

then the transfer function becomes:

$$H(z) = K\frac{(z-1)(z+1)}{(z - 0.80365\exp(j\pi/2))(z - 0.80365\exp(-j\pi/2))}.$$

Using Euler's identity,

- $\exp(j\pi/2) = \cos(\pi/2) + j\sin(\pi/2) = +j$
- and $\exp(j\pi/2) = \cos(\pi/2) - j\sin(\pi/2) = -j$,

so that

$$H(z) = K\frac{(z-1)(z+1)}{(z - 0.80365j)(z + 0.80365j)}.$$

3. Calculate the difference equation for the above system.
Solution Recall that $H(z) = \frac{Y(z)}{X(z)}$,

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{(z-1)(z+1)}{(z - 0.80365j)(z + 0.80365j)} = K\frac{z^2 - 1}{z^2 + 0.64585}.$$

Then

$$Y(z)(z^2 + 0.64585) = X(z)K(z^2 - 1).$$

Remembering that each z^{-1} is a unit **delay**, so that each z is a unit **advance**, then the difference equation is:

$$y[n+2] + 0.64585y[n] = K(x[n+2] - x[n])$$

which can be made causal by making $n = n - 2$ so that

$$y[n] + 0.64585y[n-2] = K(x[n] - x[n-2]).$$

K is not known, but can be used to make the peak pass band gain to be **unity**.

4. Calculate and sketch the frequency response for the above filter from the z-plane representation.
Solution Using the generalized difference equation form:

$$H(\Omega) = \frac{\sum\limits_{k=0}^{M} b[k]\exp(-jk\Omega)}{1 + \sum\limits_{k=1}^{N} a[k]\exp(-jk\Omega)}.$$

So that (using Euler's identity):

$$H(\Omega) = \frac{K(1 - \cos(2\Omega) + j\sin(2\Omega))}{1 + 0.64585(\cos(2\Omega) - j\sin(2\Omega))}$$

which has magnitude frequency response:

$$\mathrm{Mag}(H(\Omega))^2 = \frac{K((1 - \cos(2\Omega))^2 + \sin^2(2\Omega))}{(1 + 0.64585\cos(2\Omega))^2 + \sin^2(2\Omega)}.$$

5. For the above system, with sampling frequency $500Hz$:

   (a) What is the bandpass centre frequency?
   Solution Converting from digital to analogue, we know that 2π digital frequency corresponds to the analogue sampling frequency ($f_s = 500Hz$). Therefore we can use $f = \frac{\text{digital frequency}}{2\pi} \times f_s$.
   Therefore centre frequency:

$$f_0 = \frac{\pi/2}{2\pi}f_s = 125Hz.$$

(b) The bandwidth?
Solution Bandwidth:

$$f_{\mathrm{bw}} = \frac{\pi/8}{2\pi} f_s = 31.25Hz.$$

6. Design a digital bandstop filter using pole-zero placement method with following parameters:

   - Centre frequency $\Omega_0 = \pi/10$ radians (complete attenuation)
   - Bandstop width, $\Omega_w = 2\Omega_{\mathrm{cf}} = \pi/20$ radians

   Solution

   - Complete attenuation at $\Omega_0 = \pi/10$, therefore x2 zeros (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$:

   $$H(z) = K\frac{(z - \exp(j\pi/10))(z - \exp(-j\pi/10))}{\mathrm{poles}}$$

   - Centre frequency at $\Omega_0 = \pi/10$ radians, therefore x2 poles (complex-conjugate pair) at $\pm\Omega_0 = \pm\pi/10$,

   $$H(z) = K\frac{(z - \exp(j\pi/10))(z - \exp(-j\pi/10))}{(z - r\exp(j\pi/10))(z - r\exp(-j\pi/10))}$$

   - The poles are **scaled** with radius r to control **the width of the band stop**,

   $$r \cong 1 - \frac{\Omega_{\mathrm{w}}}{2} = 1 - \frac{\pi/20}{2} = 0.92146$$

   - resulting in:

   $$H(z) = K\frac{(z - \exp(j\pi/10))(z - \exp(-j\pi/10))}{(z - 0.92146\exp(j\pi/10))(z - 0.92146\exp(-j\pi/10))}$$

   - Transfer function is then (using Euler's identity like before):

   $$H(z) = K\frac{z^2 - 1.9021z + 1}{z^2 - 1.7527z + 0.84909}$$

   - As before, each z is a **unit advance**, so

   $$y[n+2] - 1.7527y[n+1] + 0.84909y[n] = K(x[n+2] - 1.9021x[n+1] + x[n])$$

   letting $n = n - 2$, making it causal:

   $$y[n] - 1.7527y[n-1] + 0.84909y[n-2] = K(x[n] - 1.9021x[n-1] + x[n-2]).$$

7. Convert the following single pole low pass analog filter into a digital filter (z-plane form transfer function) with digital cut-off frequency $\Omega_{\mathrm{cf}} = 0.3\pi$ using the bilinear transformation method:

   $$H(s) = \frac{\omega_{\mathrm{cf}}}{s + \omega_{\mathrm{cf}}}.$$

   Solution

   (a) Calculate **analogue cut-off frequency** ω_{cf} from **digital cut-off frequency** $\Omega_{\mathrm{cf}} = 0.2\pi$:

   $$\omega_{\mathrm{cf}} = 2f_{\mathrm{s}}\tan(\Omega_{\mathrm{cf}}/2) = 2f_{\mathrm{s}}\tan(0.1\pi) = 2f_{\mathrm{s}}A$$

   (b) Therefore **analogue transfer function**:

   $$H(s) = \frac{2f_{\mathrm{s}}A}{s + 2f_{\mathrm{s}}A}$$

   (c) Apply **bilinear transformation**: $s = 2f_{\mathrm{s}}\frac{1-z^{-1}}{1+z^{-1}}$:

   $$H(z) = \frac{2f_{\mathrm{s}}A}{2f_{\mathrm{s}}\frac{1-z^{-1}}{1+z^{-1}} + 2f_{\mathrm{s}}A} = \left(\frac{2f_{\mathrm{s}}}{2f_{\mathrm{s}}}\right)\frac{A(1+z^{-1})}{(1-z^{-1}) + A(1+z^{-1})}$$

(d) The z-transform transfer function of the filter is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A + Az^{-1}}{1 + A + (A-1)z^{-1}} \tag{1}$$

where $A = \tan(0.1\pi)$.

8. Is the digital filter for the above stable?
Solution Rearranging to determine **the poles** for **stability analysis** gives:

$$H(z) = \frac{A}{1+A}\frac{z+1}{z+\frac{A-1}{1+A}}.$$

- So there is 1 pole at $z + \frac{A-1}{1+A} = 0$ or $z = -\frac{A-1}{1+A}$.
- Remember $A = \tan(0.1\pi)$, so the pole is: $z = -0.50953$,
- *the magnitude is less than 1, so the filter is stable.*

9. Calculate the time domain difference equation from the z-plane representation of the transfer function.
Solution Multiplying both sides by both denominators of equation (1) results in

$$Y(z)\left\{1 + A + (A-1)z^{-1}\right\} = X(z)\left\{A + Az^{-1}\right\}$$

Remembering that each z^{-1} is a unit **delay**, so that

$$(1+A)y[n] + (A-1)y[n-1] = Ax[n] + Ax[n-1]$$

Dividing through by $(1+A)$ and rearranging gives

$$y[n] = \frac{A}{1+A}\left(x[n] + x[n-1]\right) - \frac{A-1}{1+A}y[n-1],$$

where $A = \tan(0.1\pi)$.
This is now a difference equation we can use to filter a signal.

10. Calculate and sketch the magnitude frequency response for the above filter, using the bilinear transformation method.
Solution The analogue transfer function from step 2 in earlier slide was:

$$H(s) = \frac{2f_s A}{s + 2f_s A}$$

The s-plane variable s can be replaced by the Fourier complex frequency variable $j\omega$,

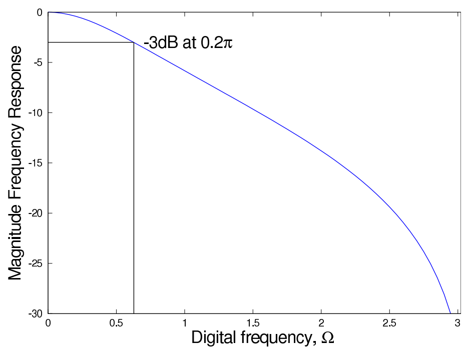
$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{2f_s A}{j\omega + 2f_s A}.$$

The Fourier frequency can then be converted to the digitial frequency Ω using $\omega = 2f_s \tan\left(\frac{\Omega}{2}\right)$ (the bilinear transformation of the frequencies):

$$H(\Omega) = H(\omega)\Big|_{\omega = 2f_s \tan\left(\frac{\Omega}{2}\right)} = \frac{2f_s A}{j2f_s \tan\left(\frac{\Omega}{2}\right) + 2f_s A} = \frac{A}{j\tan\left(\frac{\Omega}{2}\right) + A}$$

again where $A = \tan(0.1\pi)$. So the magnitude frequency response calculated directly from the Bilinear transformation is:

$$|H(\Omega)| = \sqrt{\frac{A^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2 + A^2}} = \sqrt{\frac{(\tan(0.1\pi))^2}{\left(\tan\left(\frac{\Omega}{2}\right)\right)^2 + (\tan(0.1\pi))^2}}$$



The designed cut-off frequency $\Omega_{\mathrm{cf}} = 0.2\pi$ is confirmed by this plot which can be found manually by calculating the decibel value of $20\log_{10}|H(\Omega)|$ at $\Omega = 0.2\pi$.

11. List the advantages and disadvantages of recursive filters in comparison to non-recursive filters.
Solution

Characteristic	**IIR**	**FIR**
Multiplications	least	most
Coefficient quantification sensitivity	can be high	very low
Overflow errors	can be high	very low
Stability	by design	always
Linear phase	no	always
Simulate analog filter	yes	no
Coefficient memory	least	most
Design complexity	moderate	simple

adapted from "Understanding digital signal processing" by R. G. Lyons