Solutions for: z-Transforms Tutorial

1. Find the z-Transform of a step function:

$$x[n] = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \quad \text{elsewhere} \end{cases}.$$

Solution

$$\begin{aligned} X(z) =& \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} \\ =& (z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + ...) \\ =& \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + ...\right) \end{aligned}$$

This is a geometric series of the form:

$$s = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

where $a = 1$ and $r = z^{-1}$ so that

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

2. Find the z-Transform of a square pulse:

$$x[n] = \begin{cases} 0.2 & \text{for} \quad 0 \leq n < 5 \\ 0 & \quad \text{elsewhere} \end{cases}.$$

Solution

$$\begin{aligned} X(z) =& \sum_{n=0}^{\infty} x[n] z^{-n} = 0.2 \sum_{n=0}^{4} z^{-n} \\ =& 0.2(z^0 + z^{-1} + z^{-2} + z^{-3} + z^{-4}) \\ =& 0.2\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}\right) \end{aligned}$$

This is a geometric series of the form:

$$s = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$$

where $a = 0.2$, $n = 5$ and $r = z^{-1}$ so that

$$X(z) = 0.2\frac{1 - z^{-5}}{1 - z^{-1}} = 0.2\frac{z^5 - 1}{z^5 - z^4} = 0.2\frac{z^5 - 1}{z^4(z-1)}.$$

3. Find the signal corresponding to the z-transform:

$$X(z) = \frac{z}{z - 0.5}.$$

Solution Remember the geometric series formula: $s = \sum\limits_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. Need to find the form of $X(z)$ to easily find r and a... Dividing the numerator and denominator by z gives

$$X(z) = \frac{1}{1 - 0.5z^{-1}}.$$

So that $r = 0.5z^{-1}$ and $a = 1$ then

$$X(z) = \sum_{k=0}^{\infty} \left(0.5z^{-1}\right)^k = 1 + 0.5z^{-1} + (0.5z^{-1})^2 + (0.5z^{-1})^3 + (0.5z^{-1})^4 + ...$$
$$= 1 + 0.5z^{-1} + 0.25z^{-2} + 0.125z^{-1} + 0.0625z^{-4} + ...$$

Remembering that each z^{-1} is a delay of 1 time instance, the signal $x[n]$ is then given by the coefficients for each time instance, *i.e.* $x[0] = 1$, $x[1] = 0.5$, $x[2] = 0.25$, $x[3] = 0.125$, $x[4] = 0.0625$ *etc.*

4. Find the signal corresponding to the z-Transform:

$$X(z) = \frac{z^2 - 0.2}{z(z - 0.2)}.$$

Solution Remember the geometric series formula: $s = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$. Need to find the form of $X(z)$ to easily find r, a and n... Dividing through by z^2 gives

$$X(z) = \frac{1 - 0.2z^{-2}}{1 - 0.2z^{-1}}.$$

So that $r = 0.2z^{-1}$, $a = 1$ and $n = 2$ resulting in:

$$X(z) = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{1} (0.2z^{-1})^k = 1 + 0.2z^{-1}.$$

Therefore the original signal, $x[n]$ is given by $x[0] = 1$ and $x[1] = 0.2$.

5. Decompose the following function into partial fractions:

$$\frac{1}{(z+3)(z-2)}$$

Solution Let

$$\frac{1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}.$$

Then

$$A(z-2) + B(z+3) = 1.$$

So that

$$Az - 2A + Bz + 3B = 1$$
$$z(A+B) - 2A + 3B = 1$$

Therefore $z(A + B) = 0 \Rightarrow A = -B$ and $-2A + 3B = 1$ so that $-2A - 3A = 1$ giving $A = -\frac{1}{5}$ and $B = \frac{1}{5}$.
Check:

$$\frac{A}{z+3} + \frac{B}{z-2} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2} = \frac{\frac{1}{5}(5)}{(z+3)(z+2)} = \frac{1}{(z+3)(z+2)}$$

6. Decompose the following function into partial fractions: (cover-up method)

$$\frac{z}{(z+3)(z-2)}$$

Solution Let $\frac{z}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$. To find $A \Rightarrow z + 3 = 0 \Rightarrow z = -3$.

$$A = \left.\frac{z}{(z+3)(z-2)}\right|_{z=-3} = \frac{-3}{-3-2} = \frac{3}{5}.$$

To find $B \Rightarrow z - 2 = 0 \Rightarrow z = 2$.

$$B = \left.\frac{z}{(z+3)(z-2)}\right|_{z=2} = \frac{2}{2+3} = \frac{2}{5}.$$

Hence
$$\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}.$$

Check:
$$\frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2} = \frac{\frac{3}{5}(z-2) + \frac{2}{5}(z+3)}{(z+3)(z-2)} = \frac{\frac{3}{5}z - \frac{6}{5} + \frac{2}{5}z + \frac{6}{5}}{(z+3)(z-2)} = \frac{z}{(z+3)(z-2)}.$$

7. Find the inverse z-Transform of:
$$X(z) = \frac{1}{(z+3)(z-2)}$$

Solution Re-writing
$$X(z) = \frac{z^{-1}}{5}\left(\frac{z}{z-2} - \frac{z}{z+3}\right). \tag{1}$$

Enables us to find inverse z-Transforms for the two terms inside the brackets:
$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and
$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n].$$

The two terms are multiplied by z^{-1} which is equivalent to a time delay hence the final signal is given by:
$$x[n] = \mathcal{Z}^{-1}(X(z)) = \frac{1}{5}\left(2^{(n-1)}u[n-1] - ((-3)^{(n-1)})u[n-1]\right).$$

8. Find the inverse z-Transform of:
$$X(z) = \frac{z}{(z+3)(z-2)}$$

Solution *From earlier the partial fraction expansion is given by:* $\frac{z}{(z+3)(z-2)} = \frac{\frac{3}{5}}{z+3} + \frac{\frac{2}{5}}{z-2}$. **(i)** However it is more convenient if we divide both sides by z first. Hence
$$\frac{X(z)}{z} = \frac{1}{(z+3)(z-2)}.$$

The Right Hand Side (RHS) has partial fractions (see earlier slide):
$$\frac{X(z)}{z} = \frac{-\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2}.$$

Multiplying both sides by z then gives:
$$X(z) = \frac{1}{5}\left(\frac{-z}{z+3} + \frac{z}{z-2}\right).$$

(ii) We saw earlier:
$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u[n]$$

and
$$\mathcal{Z}^{-1}\left(-\frac{z}{z+3}\right) = -((-3)^n)u[n]$$

so that
$$\begin{aligned} x[n] =& \mathcal{Z}^{-1}(X(z)) \\ =& \frac{1}{5}\left(2^{(n)}u[n] - ((-3)^{(n)})u[n]\right). \end{aligned}$$

9. Use algebraic long division to find the coefficients of the following transfer function:

$$H(z) = \frac{z}{z^2 + z - 2}$$

Solution Via algebraic or polynomial long division:

$$\begin{array}{r l}
 & z^{-1} - z^{-2} + 3z^{-3} - 5z^{-4} ... \\
z^2 + z - 2 \lceil & \overline{z \qquad\qquad\qquad\qquad} \\
 & \underline{z \;\; +1 \;\; -2z^{-1}} \\
 & \quad -1 \;\; +2z^{-1} \\
 & \quad \underline{-1 \;\; -z^{-1} + 2z^{-2}} \\
 & \qquad 3z^{-1} - 2z^{-2} \\
 & \qquad \underline{3z^{-1} + 3z^{-2} - 5z^{-3}} \\
 & \qquad\quad -5z^{-2} + 5z^{-3} \\
 & \qquad\qquad ...
\end{array}$$

So the coefficients of the original signal are given by:
$x[0] = 0$, $x[1] = 1$, $x[2] = -1$, $x[3] = 3$, $x[4] = -5$, *etc.*
This can be checked by performing the inverse z-Transform on $H(z)$.
Expansion with partial fractions gives: $H(z) = \frac{1}{3}\left(\frac{z}{z-1} - \frac{z}{z+2}\right)$
Inverse z-Transform: $x[n] = \mathcal{Z}^{-1}(H(z)) = \frac{1}{3}\left(u[n] - (-2)^n u[n]\right)$
Then $x[0] = \frac{1}{3}(1-1) = 0$, $x[1] = \frac{1}{3}(1+2) = 1$, $x[2] = \frac{1}{3}(1-4) = -1$, $x[3] = \frac{1}{3}(1+8) = 3$, $x[4] = \frac{1}{3}(1-15) = -5$, etc.
This confirms the long division result.

10. Find the inverse z-Transform of:

$$X(z) = \frac{0.5z}{z^2 - z + 0.5}$$

Solution The table of z-Transform pairs has the following definition:

$$\mathcal{Z}^{-1}\left(\frac{az\sin(\Omega_0)}{(z^2 - 2az\cos(\Omega_0) + a^2)}\right) = a^n \sin(n\Omega_0)u[n]. \tag{2}$$

Therefore we can try to equate the terms inside (2) and the equation from the earlier question.

$$\text{In the numerator: } a\sin(\Omega_0) = 0.5, \text{ and in the denominator } a^2 = 0.5$$

$$\Rightarrow a = \sqrt{0.5}, \text{ then } \sin(\Omega_0) = 0.5/\sqrt{0.5}, \Rightarrow \Omega_0 = \sin^{-1}(0.5/\sqrt{0.5}) = \frac{\pi}{4}.$$

We can therefore *plug* these values into the result of (2) to find the inverse z-Transform:

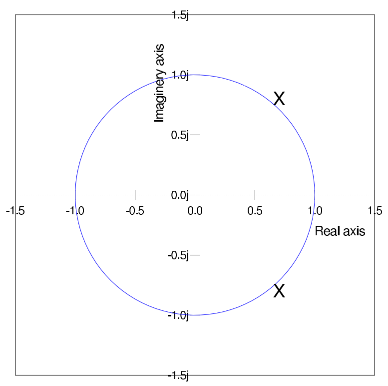
$$x[n] = \mathcal{Z}^{-1}\left(\frac{0.5z}{z^2 - z + 0.5}\right) = (\sqrt{0.5})^n \sin(n\pi/4)u[n].$$

11. What are the poles and zeros for the following z-Transform? Sketch them on a z-plane diagram.

$$X(z) = \frac{1}{(z - 0.7 + 0.8j)(z - 0.7 - 0.8j)}$$

Solution The system has two poles at:

$$p_1 = 0.7 - 0.8j \text{ and } p_2 = 0.7 + 0.8j.$$



12. What does BIBO stable mean and is the function (from previous question) BIBO stable?
Solution BIBO stands for Bounded Input Bounded Output. A linear system is stable if if has:

   - **A Bounded Output *for* A Bounded Input**

13. What are the poles and zeros for the following z-Transform? Sketch them on a z-plane diagram.

$$X(z) = \frac{1}{(z-0.5+0.5j)(z-0.5-0.5j)}$$

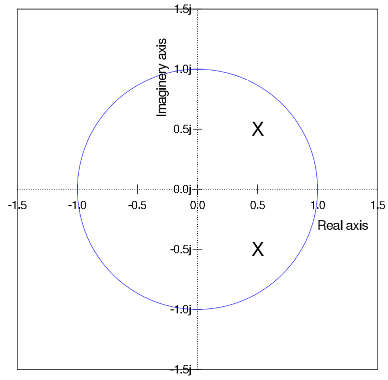
Is it BIBO stable?
Solution The system has two poles at:

$$p_1 = 0.5 + 0.5j \text{ and } p_2 = 0.5 - 0.5j$$

The distance from the origin of these poles is given by the magnitude:

$$r = \sqrt{0.5^2 + 0.5^2} = 0.707 < 1.$$

These poles are inside the unit circle, therefore this system is **stable**.



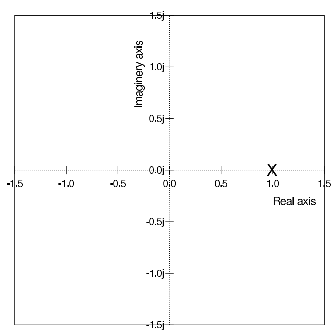
14. When is the system with the following transfer function BIBO stable?

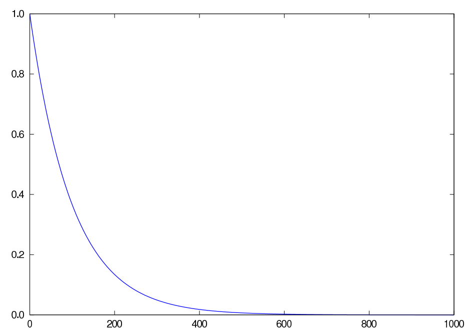
$$H(z) = \frac{1}{z-a}.$$

Solution

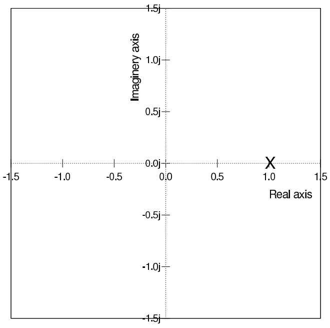
- From the table of z-Transform pairs:
- $\mathcal{Z}^{-1}\left(\frac{z}{z-a}\right) = a^n u[n]$
- Therefore let $H(z) = z^{-1}\left(\frac{z}{z-a}\right)$
- z^{-1} is a unit delay hence:
- $x[n] = a^{n-1}u[n-1]$
- So that $x[0] = 0$, $x[1] = 1$, $x[2] = a$, $x[3] = a^2$, $x[4] = a^3$ *etc.*

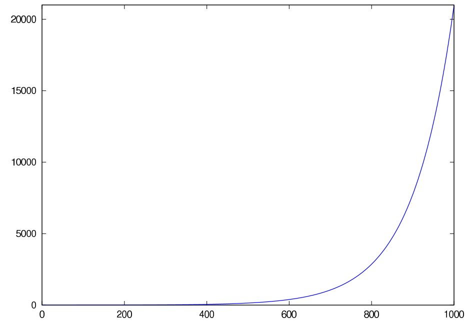
- $x[n] = a^{n-1}u[n-1]$
- $x[0] = 0$, $x[1] = 1$, $x[2] = a$, $x[3] = a^2$, $x[4] = a^3$ *etc.*
- If $a = 0.99$, decreasing and tending to zero ($x[n] \to 0$) when $a < 1$





- If $a = 1.01$, increasing and tending to infinity ($x[n] \to \infty$) when $a > 1$





These observations are true more generally:

*If the **magnitude** of any pole (p_i) is greater than 1 then the system will tend to infinity.*

15. Find the inverse z-Transform of:
$$H(z) = \frac{1}{z - 0.4}$$
Solution The inverse z-Transform is given by (using the table of z-Transform pairs):
$$x[n] = 0.4^{n-1}u[n-1],$$
which has a delay of 1 time interval.

16. What effect (in the time domain) does adding a zero at the origin have on the z-Transform in the previous equation?
Solution If we provide $H(z)$ in the previous question with a zero at the origin (*i.e.* $z_1 = 0$) so that:
$$H(z) = \frac{z - z_1}{z - 0.4} = \frac{z}{z - 0.4}$$
then the inverse z-Transform is given by:
$$x[n] = 0.4^n u[n],$$
which has **no time delay**.