Introduction to (Discrete) Fourier Analysis

Contents

1 Discrete Fourier Transform (Discrete Fourier Series)

Periodic digital signal $x[n]$ can be represented by Discrete Fourier Series. Line spectrum coefficients can be found by using the *analysis equation*:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right)$$

where $X[k]$ is the k spectral component or harmonic and N is the number of sample values in each period of the signal.

Regeneration of the original signal $x[n]$ can be found using the *synthesis equation*:

$$x[n] = \sum_{k=0}^{N-1} X[k]\exp\left(\frac{j2\pi kn}{N}\right).$$

In fact the $\frac{1}{N}$ can be used on the regeneration of the signal instead, which is what we will normally do outside of this chapter.

1.1 Finding Line Spectra

- Remember Euler's identity:

$$\exp\left(\frac{-j2\pi kn}{N}\right) = \underbrace{\overbrace{\cos\left(\frac{2\pi kn}{N}\right)}^{\text{real part}}}_{\mathrm{Re}(\cdot)}\underbrace{\overbrace{-j\sin\left(\frac{2\pi kn}{N}\right)}^{\text{imaginary part}}}_{\mathrm{Im}(\cdot)}$$

$$\mathrm{Re}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = \cos\left(\frac{2\pi kn}{N}\right)$$
$$\mathrm{Im}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) = -j\sin\left(\frac{2\pi kn}{N}\right)$$

- The real $\mathrm{Re}(X[k])$ and imaginary $\mathrm{Im}(X[k])$ components can be calculated individually

1.2 Finding Line Spectra *with a computer*

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right)$$

Two loops:

- A loop over n
- A loop over k

Pseudo-code (*assumes $x[n]$ is real only, no imaginary components*):

1. Given $x[n]$ and N:
2. Create $X[k]$ size N and set all elements to zero $X[k] = 0$
3. For k=0 to N-1 % *outer loop*
   (a) For n=0 to N-1 % *inner loop*
      i. $\mathrm{Re}(X[k]) = X[k] + x[n] \times \cos(2\pi kn/N)$ % *real part*
      ii. $\mathrm{Im}(X[k]) = X[k] - x[n] \times \sin(2\pi kn/N)$ % *imaginary part*
   (b) End For
   (c) Let $\mathrm{Re}(X[k]) = \mathrm{Re}(X[k])/N$ and Let $\mathrm{Im}(X[k]) = \mathrm{Im}(X[k])/N$
4. End For

1.3 Finding Line Spectra *Example 1*

Let $N = 2$ and $x[0] = 7$, $x[1] = 1$. Find $X[k]$. To start find the real part

- *To start* Let $n = 0$ and $k = 0$, then

$$\mathrm{Re}(X[0]) = \mathrm{Re}(X[0]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 0 \times 0/2) = 7$$

- Increment n, so $n = 1$, then

$$\mathrm{Re}(X[0]) = \mathrm{Re}(X[0]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 0 \times 1/2) = 8$$

- Divide by N: $\mathrm{Re}(X[0]) = \mathrm{Re}(X[0])/N = 4$
- *If we increment n any more then it will be larger than $N - 1$ so*
  Let $n = 0$ and increment k so that $k = 1$, then

$$\mathrm{Re}(X[1]) = \mathrm{Re}(X[1]) + x[0] \times \cos(2\pi kn/2) = 0 + 7 \times \cos(2\pi 1 \times 0/2) = 7$$

- Increment n, so $n = 1$, then

$$\mathrm{Re}(X[1]) = \mathrm{Re}(X[1]) + x[1] \times \cos(2\pi kn/2) = 7 + 1 \times \cos(2\pi 1 \times 1/2) = 6$$

- Divide by N: $\mathrm{Re}(X[1]) = \mathrm{Re}(X[1])/N = 3$

So the real part of $X[k]$ is given by $\mathrm{Re}(X[0]) = 4$ and $\mathrm{Re}(X[1]) = 3$.

Finding the imaginary part of $X[k]$...

- *To start* Let $n = 0$ and $k = 0$, then

$$\mathrm{Im}(X[0]) = \mathrm{Im}(X[0]) - x[0] \times \sin(2\pi kn/2) = 0 + 7 \times \sin(2\pi 0 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

$$\mathrm{Im}(X[0]) = \mathrm{Im}(X[0]) - x[1] \times \sin(2\pi kn/2) = 0 + 1 \times \sin(2\pi 0 \times 1/2) = 0$$

- Divide by N: $\mathrm{Im}(X[0]) = \mathrm{Im}(X[0])/N = 0$
- *If we increment* n *any more then it will be larger than* $N-1$ *so* Let $n = 0$ and increment k so that $k = 1$, then

$$\mathrm{Im}(X[1]) = \mathrm{Im}(X[1]) - x[0] \times \sin(2\pi kn/2) = 0 + 7 \times \sin(2\pi 1 \times 0/2) = 0$$

- Increment n, so $n = 1$, then

$$\mathrm{Im}(X[1]) = \mathrm{Im}(X[1]) - x[1] \times \sin(2\pi kn/2) = 0 + 1 \times \sin(2\pi 1 \times 1/2) = 0$$

- Divide by N: $\mathrm{Im}(X[0]) = \mathrm{Im}(X[0])/N = 0$

So the imaginary part of $X[k]$ is given by $\mathrm{Im}(X[0]) = 0$ and $\mathrm{Im}(X[1]) = 0$.
So the line spectra for signal:

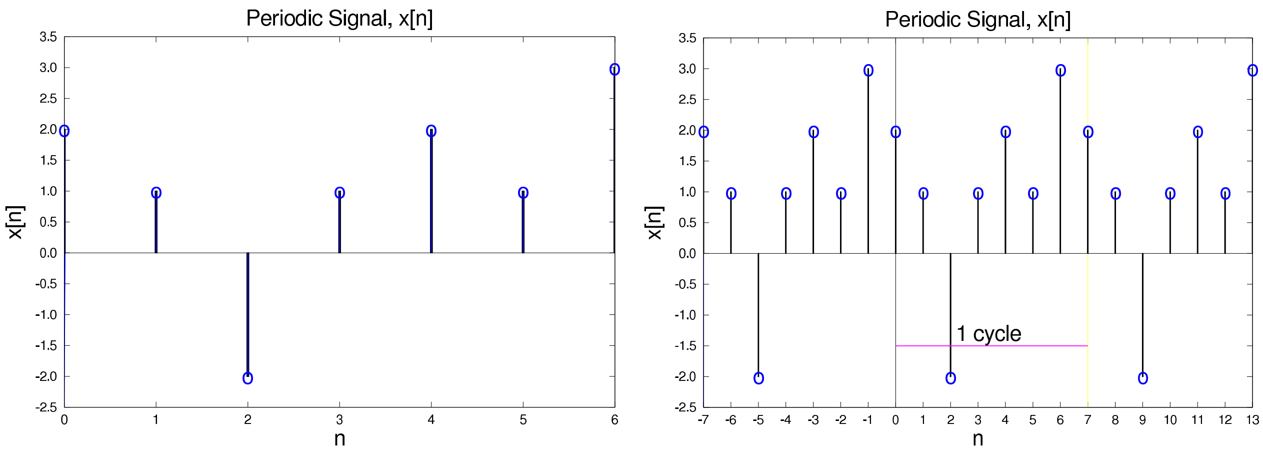
$$x = (7 \;\; 1)$$

are

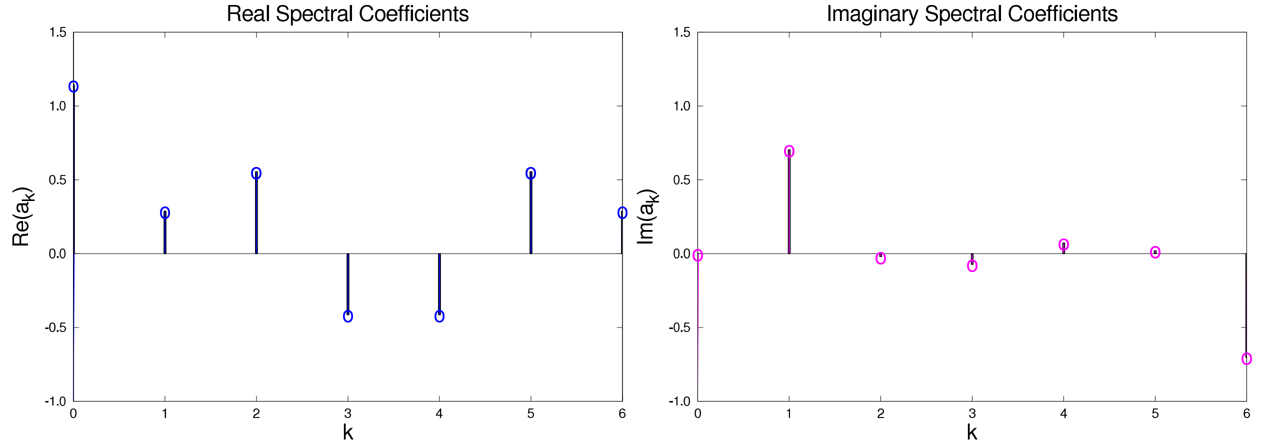
$$a = (4 \;\; 3).$$

1.4 Finding Line Spectra *Example 2*

This time let $N = 7$ so that $x = (+2 \;\, +1 \;\, -2 \;\, +1 \;\, +2 \;\, +1 \;\, +3)^{\mathrm{T}}$ or $x[0] = +2$, $x[1] = +1$, ..., $x[6] = +3$. A plot of this *periodic* signal is given by

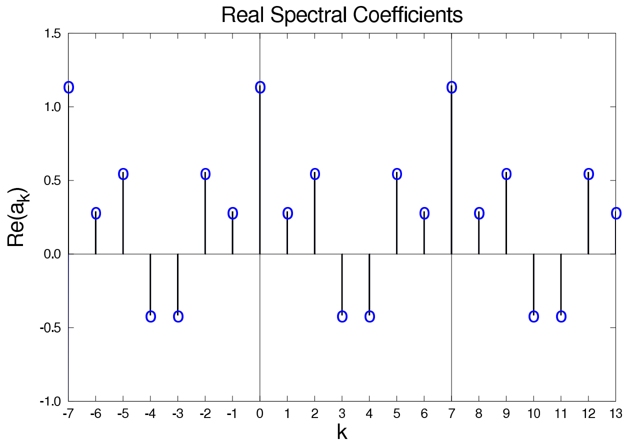


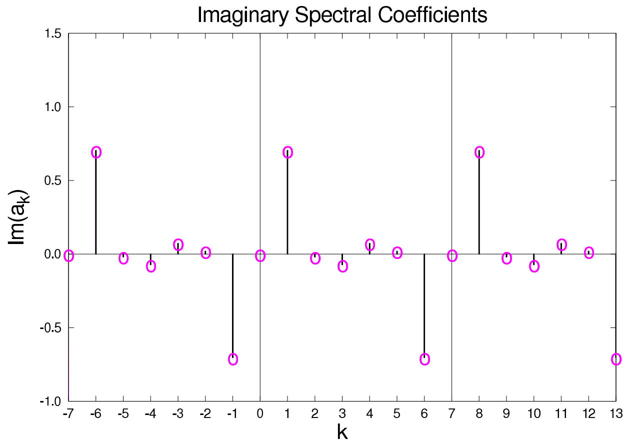
The line spectra in this case are found by a computer program using the algorithm described earlier.



Notice: (for the real coefficients) the mirror image, where $X[1] = X[6]$ and $X[2] = X[5]$ etc. For the imaginary coefficients a similar effect, except for a change of sign, *e.g.* $X[1] = -X[6]$.

The line spectra are also periodic. The line spectra have the same periodicity as the original signal.





1.5 Magnitude and Phase of Line Spectra

The real and imaginary components of the transformed signal contain important information about the signal. However it is not always convenient to have a complex representation for the data. Instead the magnitude of the transformed signal is often used as it enables a single magnitude value to represent each frequency component. Furthermore it carries information about the power of the signal as we will see shortly.

The magnitude can be calculated with:

$$\mathrm{Mag}(X[k]) = \sqrt{\mathrm{Re}(X[k])^2 + \mathrm{Im}(X[k])^2}.$$

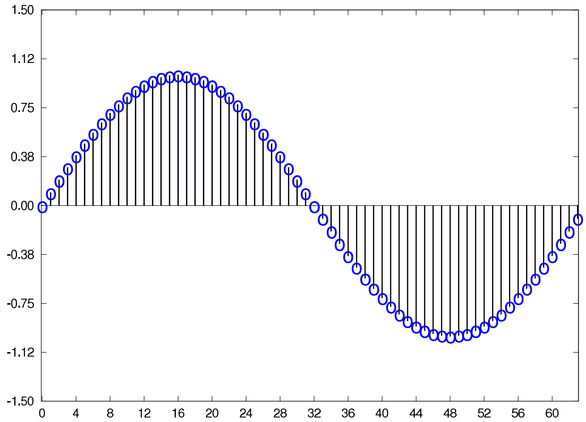
The phase can also be a useful property of the Fourier transformed data. The phase is calculated with:

$$\phi(X[k]) = \tan^{-1}\left(\frac{\mathrm{Im}(X[k])}{\mathrm{Re}(X[k])}\right).$$

- The magnitude indicates the relative strength of the signal at different frequencies;
- The phase indicates the phase angle of the signal at different frequencies.

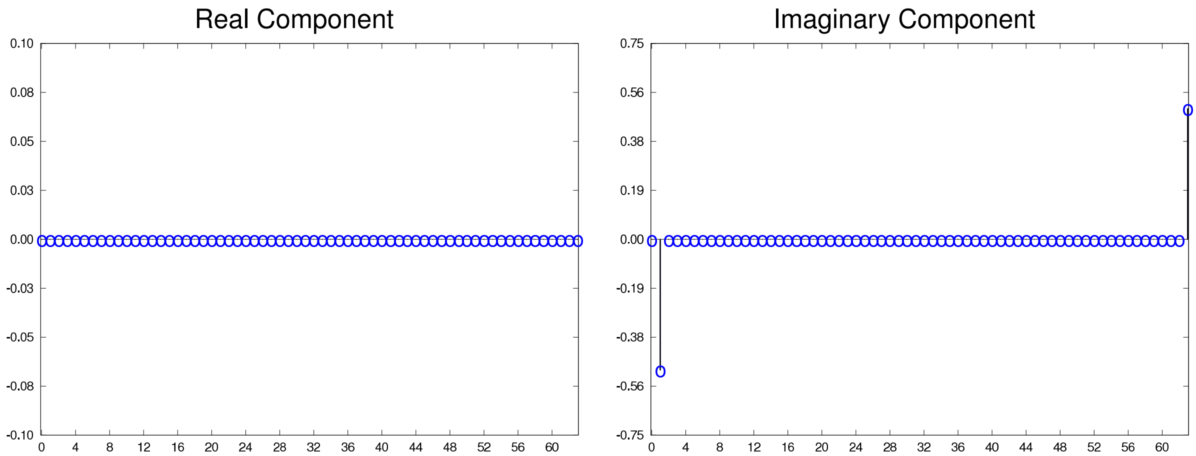
1.6 Magnitude and Phase *Example* 1

Consider a signal consisting of a single cycle of a sine wave



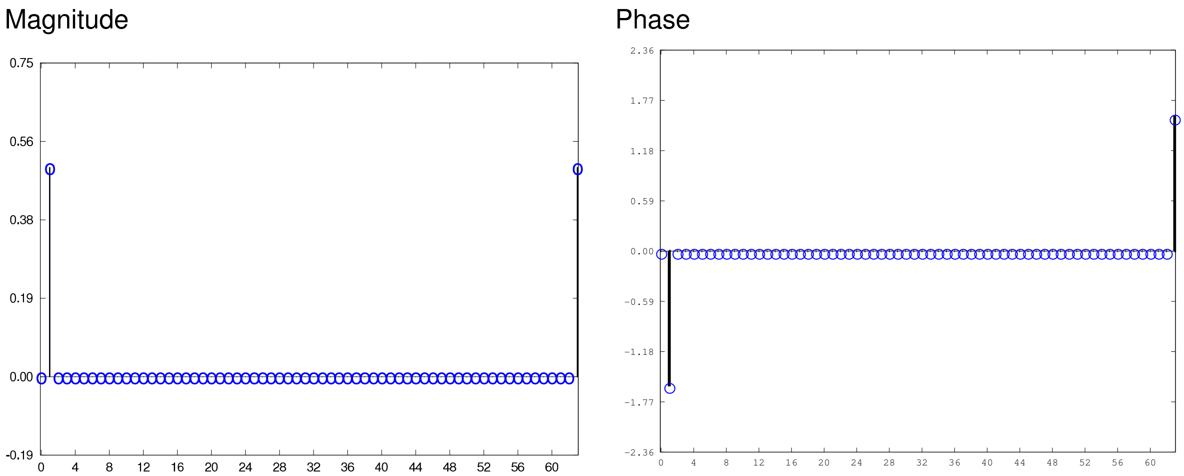
$$x[n] = \sin(2\pi n/64)$$

The real and imaginary components contain separate information about the frequency content of the signal. The sine function is not present in the real part.



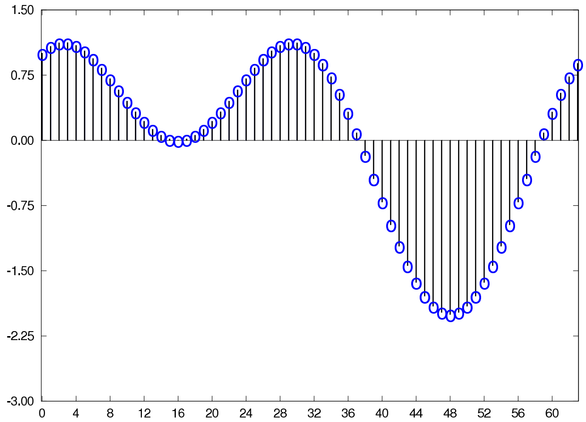
$$x[n] = \sin(2\pi n/64)$$

The magnitude provides a convenient overview of the frequency content of the signal.



1.7 Magnitude and Phase *Example* 2

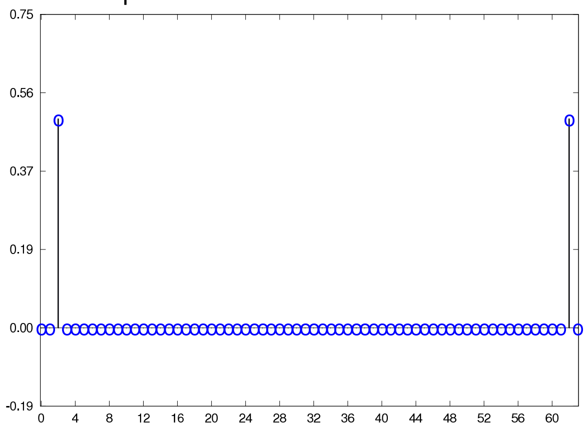
Original Signal:



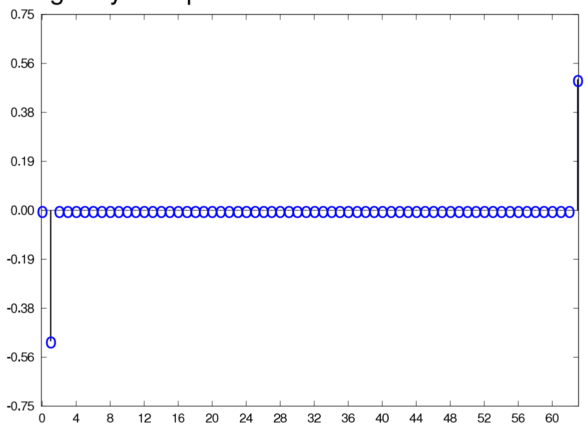
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine part of the signal is present in the real part.

Real Component



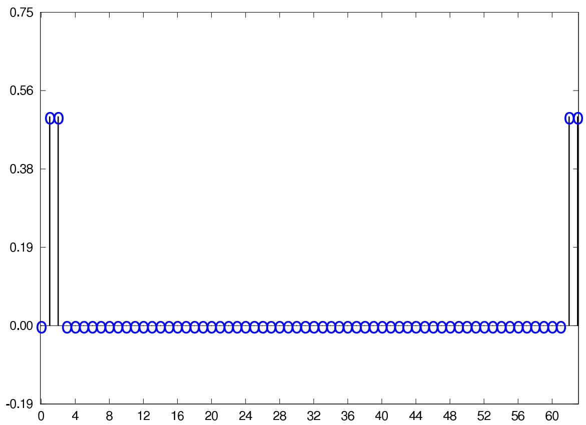
Imaginary Component



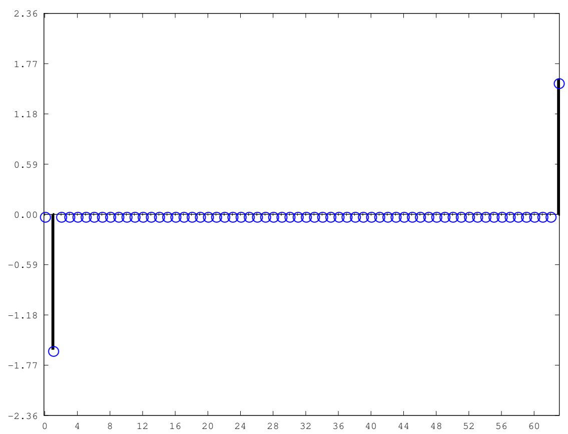
$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32)$$

The cosine frequency is not present in the phase as it has zero phase. The sine part is present as it has $\pm\pi/2$ phase.

Magnitude

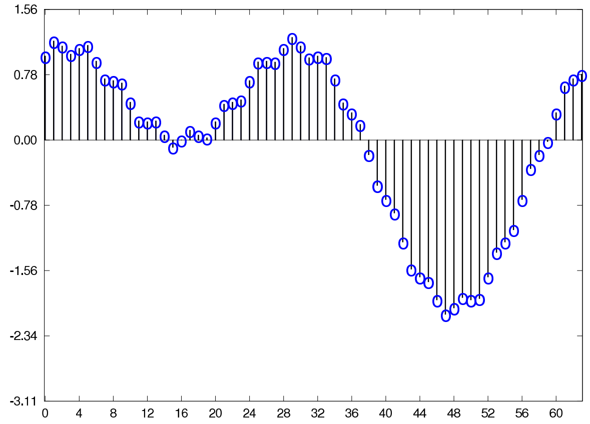


Phase



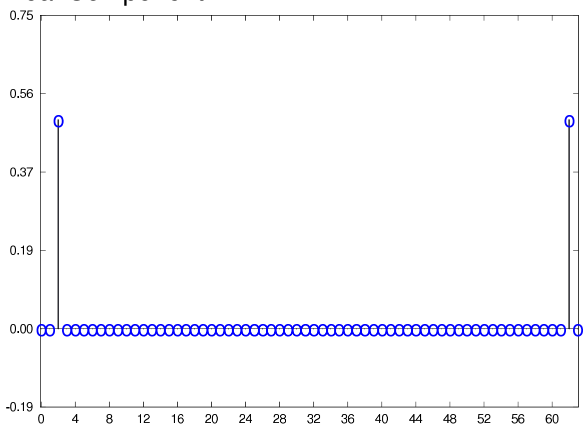
1.8 Magnitude and Phase *Example* 3

Original Signal:

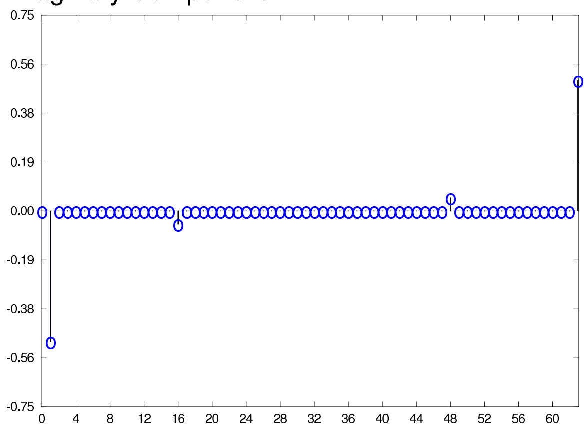


$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32) + 0.1\sin(2\pi n/4)$$

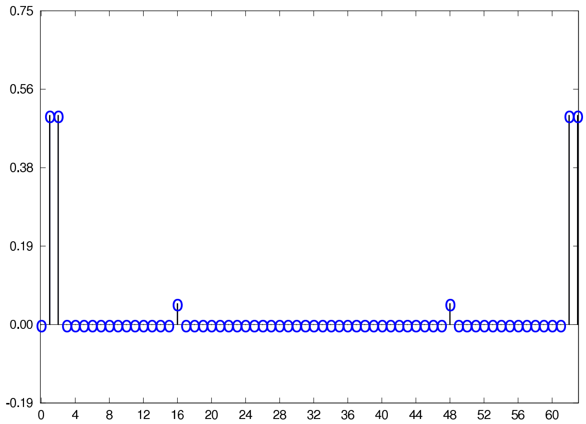
Real Component



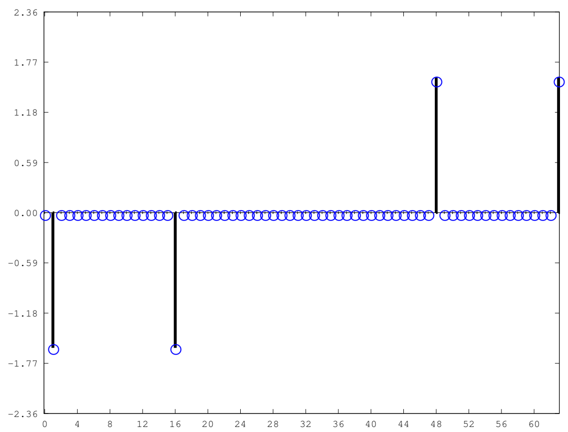
Imaginary Component



Magnitude

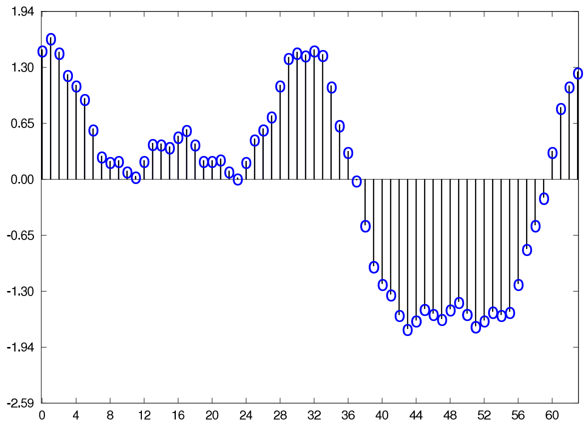


Phase



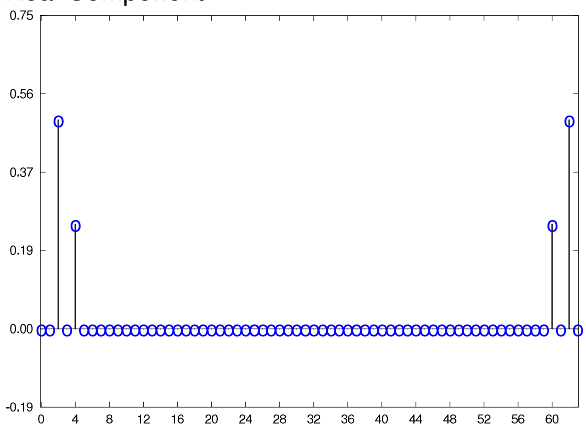
1.9 Magnitude and Phase *Example* 4

Original Signal:

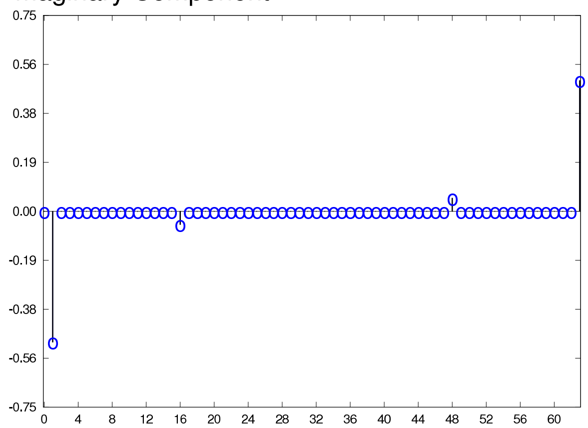


$$x[n] = \sin(2\pi n/64) + \cos(2\pi n/32) + 0.1\sin(2\pi n/4) + 0.5\cos(2\pi n/16)$$

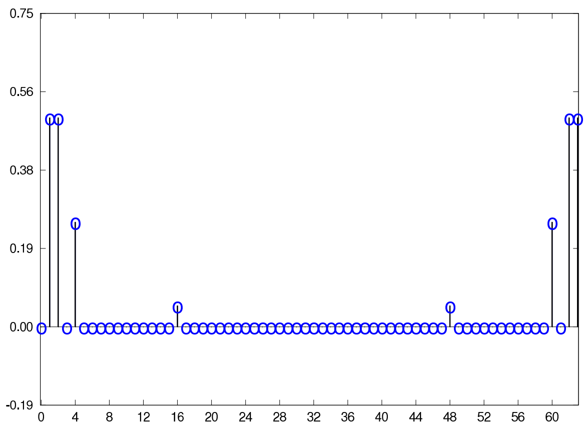
Real Component



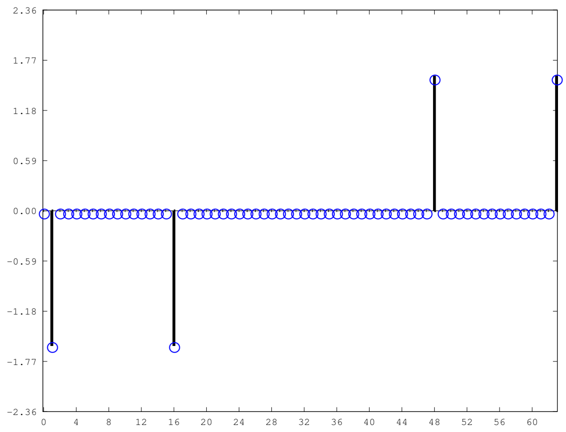
Imaginary Component



Magnitude

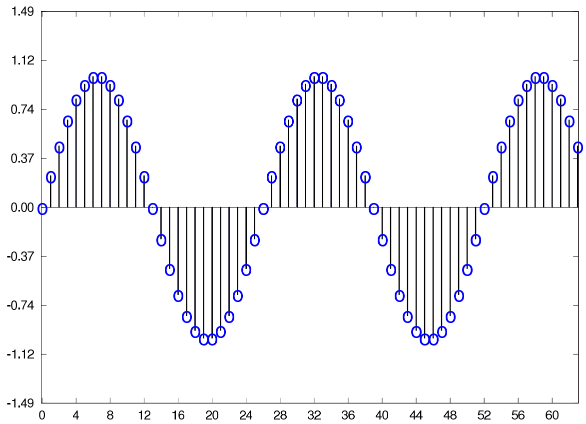


Phase



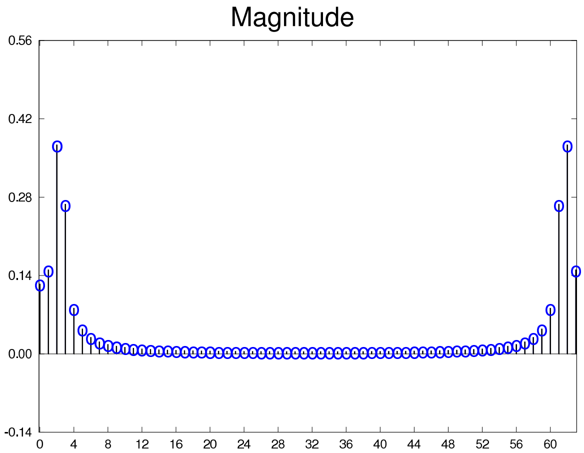
1.10 Discontinuities

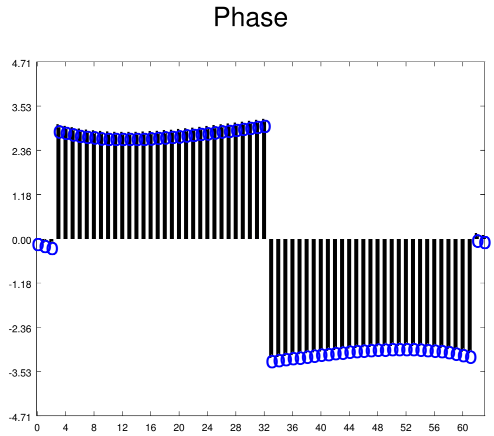
Original Signal:



$$x[n] = \sin(2\pi n/26)$$

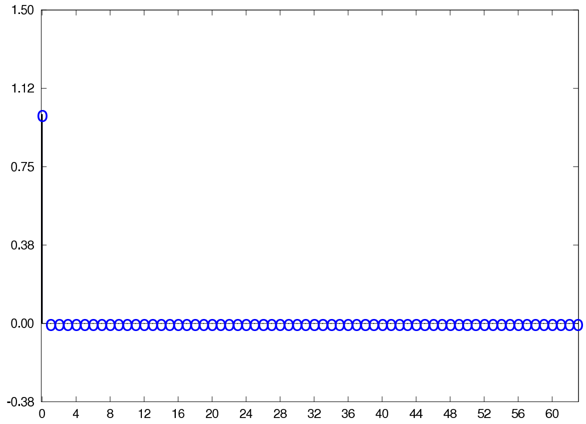
- This signal is not periodic as the sine function is analyzed over $\sim 2\frac{2}{5}$ periods. The end of the signal does not join up with the beginning, resulting in a discontinuity.
- The discontinuity has many frequency components with different phases.





1.11 Impulse Function

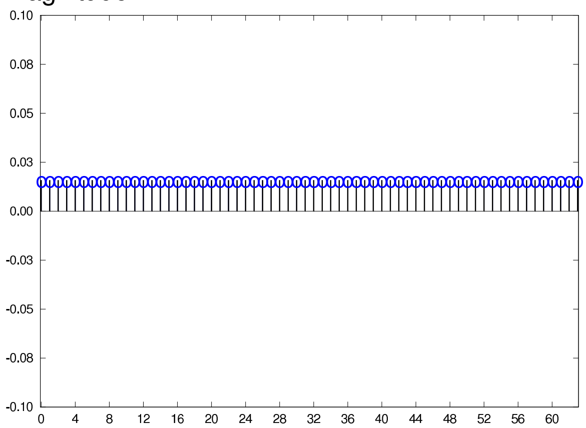
Original Signal:



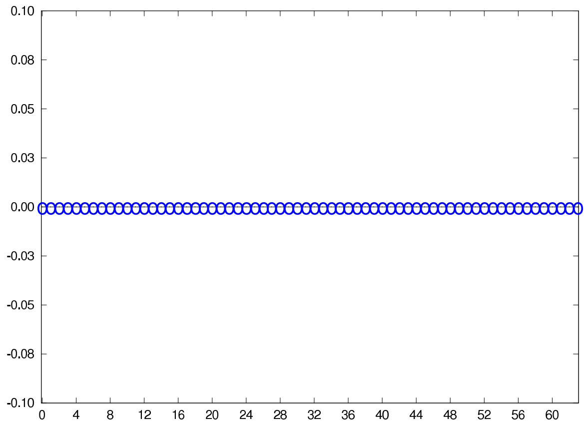
$$x[n] = \delta[0]$$

- The line spectra of a (periodic) impulse function is composed of all frequencies
- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Zero phase because the function is even, *i.e.* $x[n] = x[-n]$, frequency response composed cosine functions only.

Magnitude

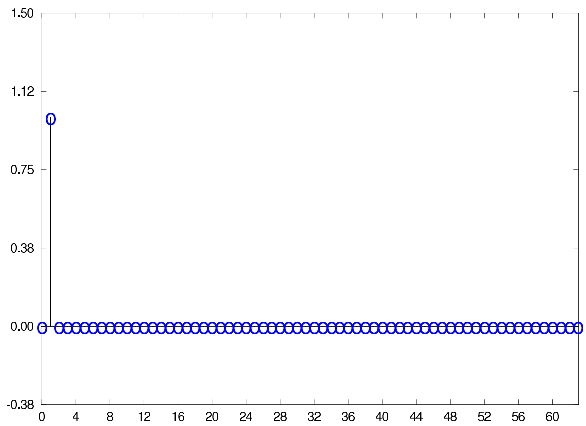


Phase



Shifted Impulse Function

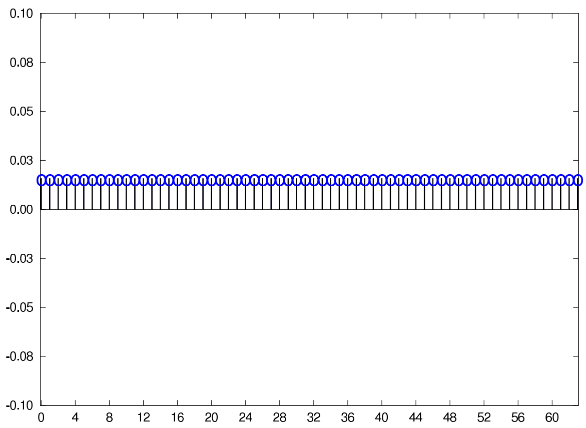
Original Signal

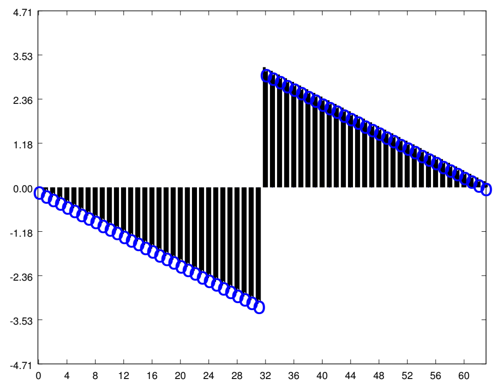


$$x[n] = \delta[n-1]$$

- The line spectra of a (periodic) impulse function is composed of all frequencies

- This illustrates the usefulness of an impulse function in characterizing a system's frequency response
- Phase components present when odd function, *i.e.* $x[n] = -x[-n]$, composed of sine functions only.





1.12 Useful Properties

Parseval's theorem Equates the total power of a signal in the time and frequency domains:

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \sum_{k=0}^{N-1}(\mathrm{Mag}(X[k]))^2$$

Example
Impulse function, $\delta[0] = 1$

$$\frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2 = \frac{1}{N}$$

and

$$\sum_{k=0}^{N-1}(\mathrm{Mag}(X[k]))^2 = N \times \left(\frac{1}{N}\right)^2 = \frac{1}{N}$$

which are equal.

Other Useful Properties

$x[n] \leftrightarrow X[k]$ symbolizes $X[k]$ is the discrete Fourier Series of $x[n]$.

Linearity: If $x_1[n] \leftrightarrow X_1[k]$ and $x_2[n] \leftrightarrow a_2[k]$ then

$$w_1x_1[n] + w_2x_2[n] \leftrightarrow w_1a_1[k] + w_2a_2[k]$$

Time-shifting (invariance): If $x[n] \leftrightarrow X[k]$ then

$$x[n-n_0] \leftrightarrow X[k]\exp(-j2\pi kn_0/N),$$

i.e. The shift is just a phase shift and does not affect the magnitude.

2 The Discrete Time Fourier Transform

2.1 Aperiodic Digital Sequences

- Different analysis and synthesis equations are necessary for aperiodic sequences, known as the **Discrete Time Fourier Transform** (DTFT) for aperiodic digital sequences,

$$X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n)$$

- and the **inverse DTFT** for aperiodic digital sequences

$$x[n] = \mathcal{F}^{-1}(X(\Omega)) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\exp(j\Omega n)\mathrm{d}\Omega.$$

- *Note: $X(\Omega)$ is a continuous function. It is also periodic which is a result of the ambiguities in discretely sampled signals.*

Comparing the DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n), \tag{1}$$

with the Discrete Fourier Series analysis equations:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp\left(\frac{-j2\pi kn}{N}\right). \tag{2}$$

We can see that $\Omega = \frac{2\pi k}{N}$. Furthermore n has also been taken to $\pm\infty$ and because of this the Fourier Transform is no longer divided by N (otherwise $X(\Omega)$ would be zero) so that $X(\Omega)$ can in some way be equated with $NX[k]$.

2.2 DTFT *Impulse Function Example*

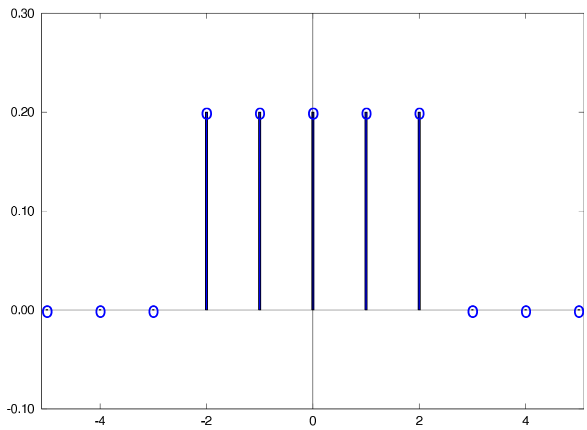
The DTFT of the impulse function $\delta[0]$:

$$\begin{aligned} x[n] &= \delta[0] \\ \therefore\ X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[0]\exp(-j\Omega n) \\ &= \exp(-j\Omega \times 0) \\ &= 1. \end{aligned}$$

In other words, the DTFT of an impulse function consists of all frequencies. Similar to the Discrete Fourier Series representation of a periodic impulse function, calculated earlier.

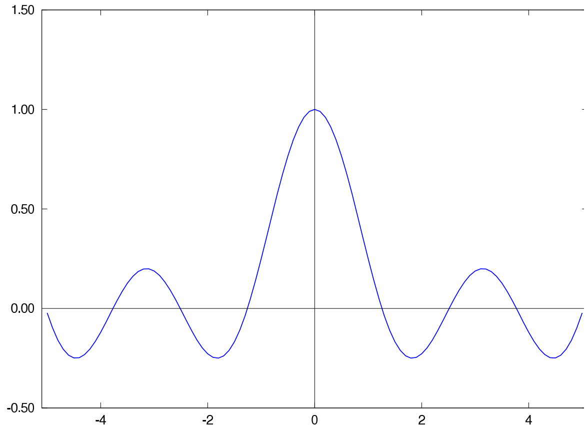
2.3 DTFT *Boxcar Example*

If $x[n] = \begin{cases} 0.2 & \text{if} \quad -2 \le n \le 2, \\ 0 & \text{otherwise.} \end{cases}$



Then

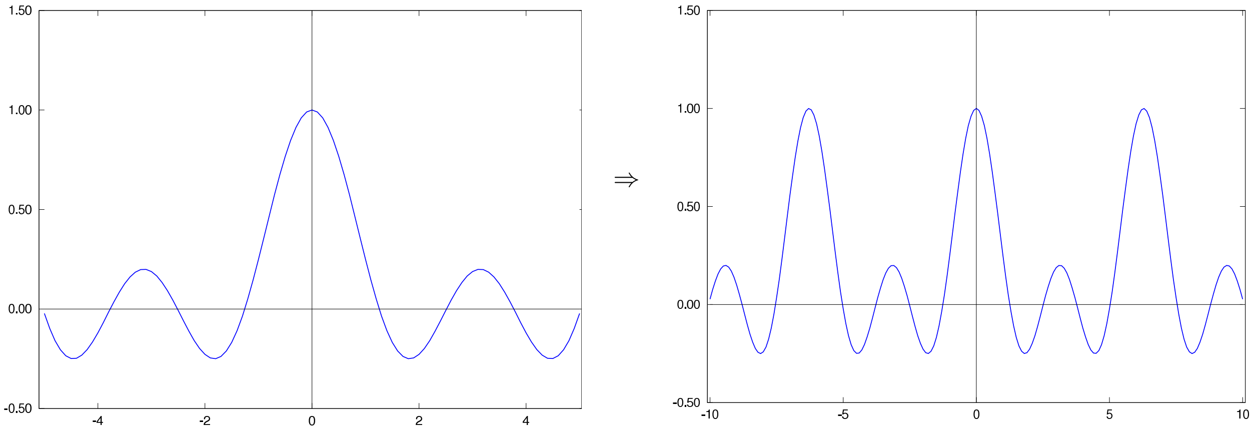
$$X(\Omega) = 0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).$$



$$
\begin{aligned}
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega n) &= \sum_{n=-2}^{2} 0.2\exp(-j\Omega n) \\
&= 0.2 \times (\exp(j\Omega 2) + \exp(j\Omega 1) + \exp(-j\Omega 0) + \exp(-j\Omega 1) + \exp(-j\Omega 2)) \\
&= 0.2 \times (\cos(\Omega 2) + j\sin(\Omega 2) + \cos(\Omega 1) + j\sin(\Omega 1) + 1 \\
&\quad + \cos(\Omega 1) - j\sin(\Omega 1) + \cos(\Omega 2) - j\sin(\Omega 2)) \\
&= 0.2 \times (2\cos(\Omega 2) + 2\cos(\Omega 1) + 1).
\end{aligned}
$$

2.4 Periodicity of the DTFT

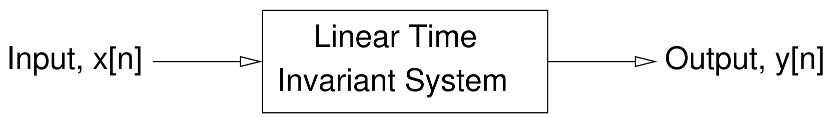
Also note that the DTFT of an aperiodic signal is periodic.



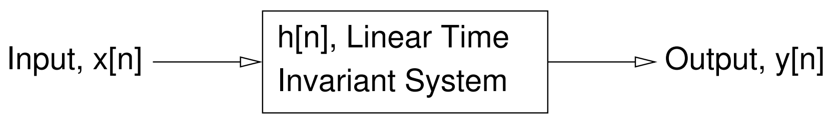
The periodicity is every 2π periods, a result of the sampling in the digitisation process.

2.5 Frequency Response of LTI Systems

An LTI system has an input $x[n]$ and an output $y[n]$:



Recall that an LTI system has an impulse response, $h[n]$:



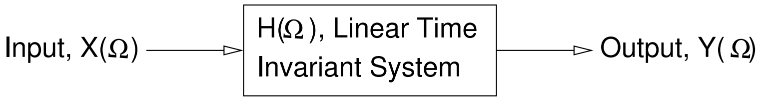
which describes the response of the system when an impulse function is given as the input. The impulse response is useful as it can be used to calculate the output signal for a given input signal:

$$y[n] = x[n] * h[n]$$

where $*$ is convolution *NOT* multiplication.

Frequency Domain

An LTI system can also be described in the frequency domain:



where

- The input frequency domain signal is $X(\Omega) = \mathcal{F}(x[n])$,
- The output frequency domain signal is $Y(\Omega) = \mathcal{F}(y[n])$
- The LTI system is described by $H(\Omega) = \mathcal{F}(h[n])$ which is known as the *frequency response* of the system and is the Fourier Transform of the impulse response.

In the frequency domain, the output can be calculated more easily:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

where multiplication *IS* used here. In other words, *convolution* is performed by multiplication in the frequency domain.

The frequency domain convolution (multiplication) equation:

$$Y(\Omega) = X(\Omega) \times H(\Omega),$$

can be re-arranged so that:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

so if we want to find the frequency response of a system then we can find it via this equation or via the Fourier transform of the time domain representation $h[n]$.

Recall the general form of LTI difference equations:

$$\sum_{m=0}^{N} a[m]y[n-m] = \sum_{m=0}^{M} b[m]x[n-m].$$

Using the linearity and time-shifting properties of Fourier transforms we can convert it to an expression using frequency domain terms:

$$\sum_{m=0}^{N} a[m]\exp(-jk\Omega)Y(\Omega) = \sum_{m=0}^{M} b[m]\exp(-jk\Omega)X(\Omega).$$

Therefore the frequency response of a system can also be described by

$$H(\Omega) = \frac{\sum\limits_{m=0}^{M} b[m]\exp(-jm\Omega)}{\sum\limits_{m=0}^{N} a[m]\exp(-jm\Omega)}.$$

This equation can be used to directly find the frequency response of a system even if only the coefficients $a[m]$ and $b[m]$ are known.

2.6 Frequency Response *Example*

Q. A moving average filter has $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$. Find the frequency response of this filter.
A. We can find the frequency response by using the coefficients:

- There is only 1 output coefficient, $a[0] = 1$.
- There are 3 input coefficients, $b[0] = b[1] = b[2] = \frac{1}{3}$

- Therefore

$$H(\Omega) = \frac{\frac{1}{3}\sum_{m=0}^{M}\exp(-jm\Omega)}{\exp(-j0\Omega)} = \frac{1}{3}(1+\exp(-j\Omega)+\exp(-j2\Omega))$$
$$= \frac{1}{3}(1+\cos(\Omega)-j\sin(\Omega)+\cos(2\Omega)-j\sin(2\Omega))$$
$$= \frac{1}{3}(1+\cos(\Omega)+\cos(2\Omega)-j(\sin(\Omega)+\sin(2\Omega)))$$

- Magnitude:

$$\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}\left((1+\cos(\Omega)+\cos(2\Omega))^2+(\sin(\Omega)+\sin(2\Omega))^2\right)}$$

- Phase:

$$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega)+\sin(2\Omega))}{(1+\cos(\Omega)+\cos(2\Omega))}\right)$$

The magnitude can be simplified using:

- $2\sin(\Omega)\sin(2\Omega) = \cos(\Omega)-\cos(3\Omega)$
- $2\cos(\Omega)\cos(2\Omega) = \cos(\Omega)+\cos(3\Omega)$
- $\sin^2(\Omega) = \frac{1-\cos(2\Omega)}{2}$
- $\sin^2(2\Omega) = \frac{1-\cos(4\Omega)}{2}$
- $\cos^2(\Omega) = \frac{1+\cos(2\Omega)}{2}$
- $\cos^2(2\Omega) = \frac{1+\cos(4\Omega)}{2}$

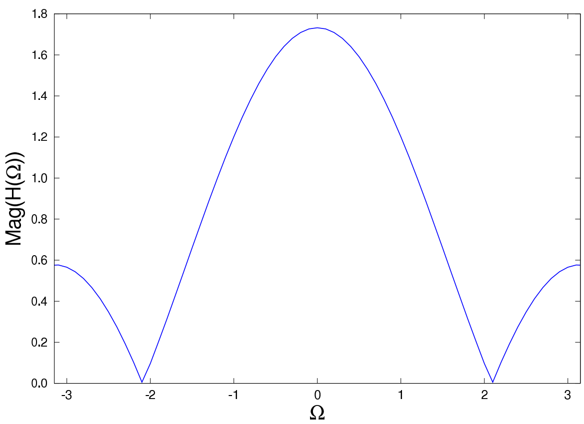
Resulting in:

- $\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3+2(2\cos(\Omega)+\cos(2\Omega)))}$

Plotting the Moving Average Filter (k=3) Frequency Response Example

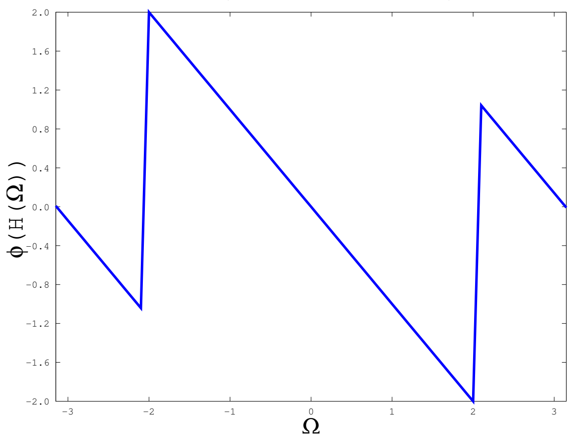
Magnitude

$\text{Mag}(H(\Omega)) = \sqrt{\frac{1}{3}(3+2(2\cos(\Omega)+\cos(2\Omega)))}$



Phase

$\phi(H(\Omega)) = \tan^{-1}\left(-\frac{(\sin(\Omega)+\sin(2\Omega))}{(1+\cos(\Omega)+\cos(2\Omega))}\right)$



3 Derivation of the DFT

A discretely sampled signal $x[n] : n = 0, ..., N-1$ is:

$$x_s(t) = \sum_{n=0}^{N-1} x[n]\delta(t-nT)$$

where T is the sampling interval. Fourier transform is:

$$X_s(f) = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f n T).$$

This is periodic, i.e. $X_s(f) = X_s(f + 1/T)$.

Discretizing $X_s(f)$ requires looking at regularly spaced values of f, i.e. $f_k = k \times f_0$ for some f_0:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f_k n T) = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi k f_0 n T).$$

Discrete sampling in frequency domain implies periodicity in time domain, i.e. $x[n + N] = x[n]$, $\therefore$ assume $f_0 = \frac{1}{NT}$, then:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi k n}{N}\right)$$

This is the DFT. The inverse DFT is given by:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \exp\left(+j\frac{2\pi k n}{N}\right).$$

4 Overview of Fourier Techniques

There are $\times 4$ **Fourier techniques**:

- Fourier Transform: continuous and aperiodic (CA)
- Fourier Series: continuous and periodic (CP)
- Discrete-time Fourier Transform: discrete and aperiodic (DA)
- Discrete-time Fourier Series: discrete and periodic (DP)

In terms of equations, we can also compare the four different Fourier techniques:

- Fourier Transform: continuous and aperiodic (CA)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) \mathrm{d}t \qquad x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t) \mathrm{d}\omega$$

- Fourier Series: continuous and periodic (CP)

$$x_k = \tfrac{1}{T} \int_{a}^{a+T} x(t) \exp(-j2\pi k t/T) \mathrm{d}t \qquad x(t) = \sum_{k=-\infty}^{\infty} x_k \exp(j2\pi k t/T)$$

- **DTFT**: Discrete-time Fourier Transform: discrete and aperiodic (DA)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n) \qquad x[n] = \tfrac{1}{2\pi} \int_{2\pi} X(\Omega) \exp(j\Omega n) \mathrm{d}\Omega.$$

- **DFT**: Discrete-Time Fourier Series: discrete and periodic (DP)

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi k n/N) \qquad x[n] = \tfrac{1}{N} \sum_{k=0}^{N-1} X[k] \exp(j2\pi k n/N)$$

We have looked at two of the above Fourier techniques; the two based solely in the sampled domain for the non-Fourier domain part, *i.e.* sampled signals. In summary the things that we have investigated as part of this chapter have included:

- An introduction to frequency domain analysis
- Discrete Fourier Series (actually the Discrete Fourier Transform or DFT).
- Magnitude and Phase
- The Fourier Transform for aperiodic digital sequences (actually the Discrete Time Fourier Transform or DTFT).