Question 5

(a)

$$
\begin{array}{r|l}
 & x + 1 \\
x^3 - 4x^2 + 4x + 6 & x^4 - 3x^3 + 5x^2 - 17x + 6 \\
 & x^4 - 4x^3 + 4x^2 + 6x \\
\hline
 & x^3 + x^2 - 23x + 6 \\
 & x^3 - 4x^2 + 4x + 6 \\
\hline
 & 5x^2 - 27x
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{1}{5}x + \frac{7}{25} \\
5x^2 - 27x & x^3 - 4x^2 + 4x + 6 \\
 & x^3 - \frac{27}{5}x^2 \\
\hline
 & \frac{7}{5}x^2 + 4x \\
 & \frac{7}{5}x^2 - \frac{189}{25}x \\
\hline
 & \frac{289}{25}x + 6
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{125}{289}x - \frac{213825}{83521} \\
\frac{289}{25}x + 6 & 5x^2 - 27x + 0 \\
 & 5x^2 + \frac{750}{289}x \\
\hline
 & -\frac{8553}{289}x + 0 \\
 & -\frac{8553}{289}x - \frac{1282950}{83521} \\
\hline
 & \frac{1282950}{83521}
\end{array}
$$

$$\frac{24137569}{32073750}x + \frac{501126}{1282950}$$

$$\frac{1282950}{83521} \;\Big)\; \frac{289}{25}x + 6$$

$$\frac{289}{25}x + 0$$

$$6$$

$$6$$

$$0$$

The last non-zero remainder is always the GCD. Thus, the GCD of $f(x)$ and $g(x)$ is

$$\frac{1282950}{83521}.$$

(b) def polynomial GCD (f(x), g(x)) :

```
r = f(x) mod g(x)
r.list = [ ]
while r > 0 :
    a = f(x)
    b = g(x)
    r = a mod b
    append . r.list (r)
    b → a
    r → b
return r.list [-2]      # returns second last element
                          of the list which is the GCD
```

(c) $f(x) = x^4 - 3x^3 + 5x^2 - 17x + 6$

$g(x) = x^3 - 4x^2 + 4x + 6$

polynomial GCD (f(x), g(x)) :

r.list = [ ]

$r = 5x^2 - 27x$

while $r = 5x^2 - 27x > 0$ (TRUE)

$a = x^4 - 3x^3 + 5x^2 - 17x + 6$

$b = x^3 - 4x^2 + 4x + 6$

$r = 5x^2 - 27x$

append $(5x^2 - 27x)$

r.list = $[5x^2 - 27x]$

$a = x^3 - 4x^2 + 4x + 6$

$b = 5x^2 - 27x$

while $r = 5x^2 - 27x > 0$ (TRUE)

$a = x^3 - 4x^2 + 4x + 6$

$b = 5x^2 - 27x$

$r = \frac{289}{25}x + 6$

append $(\frac{289}{25}x + 6)$

r.list = $[5x^2 - 27x, \frac{289}{25}x + 6]$

$a = 5x^2 - 27x$

$b = \frac{289}{25}x + 6$

while $r = \frac{289}{25}x + 6 > 0$ (TRUE)

$a = 5x^2 - 27x$

$b = \frac{289}{25}x + 6$

$r = \frac{1282950}{83521}$

append $\left(\frac{1282950}{83521}\right)$

r.list $= \left[5x^2 - 27x, \frac{289}{25}x + 6, \frac{1282950}{83521}\right]$

$a = \frac{289}{25}x + 6$

$b = \frac{1282950}{83521}$

while $r = \frac{1282950}{83521} > 0$ (TRUE)

$a = \frac{289}{25}x + 6$

$b = \frac{1282950}{83521}$

$r = 0$

append (0)

r.list $= \left[5x^2 - 27x, \frac{289}{25}x + 6, \frac{1282950}{83521}, 0\right]$

$$a = \frac{1282950}{83521}$$

$$b = 0$$

while $r = 0 > 0$ (FALSE)

return r.list[-2] $= \frac{1282950}{83521}$

(d) We can write the gcd $(f(x), g(x))$ as

$$\frac{1282950}{83521} = (5x^2 - 27x) - \left(\frac{289}{25}x + 6\right)\left(\frac{125}{289}x - \frac{21382}{8352}\right.$$

from the Euclidean algorithm. (by rearranging terms

Then, we can substitute

$$\frac{289}{25}x + 6 = (x^3 - 4x^2 + 4x + 6) - (5x^2 - 27x)\left(\frac{1}{5}x + \frac{7}{25}\right)$$

also by ~~substituting~~ / rearranging terms.

$$\frac{1282950}{83521} = (5x^2 - 27x) - (x^3 - 4x^2 + 4x + 6)\left(\frac{125}{189}x - \frac{213825}{83521}\right)$$
$$+ (5x^2 - 27x)\left(\frac{1}{5}x + \frac{7}{25}\right)\left(\frac{125}{289}x - \frac{213825}{83521}\right)$$

$$\frac{1282950}{83521} = (5x^2 - 27x)\left(1 + \left(\frac{1}{5}x + \frac{7}{25}\right)\left(\frac{125}{289}x - \frac{21382}{83521}\right.\right.$$
$$- (x^3 - 4x^2 + 4x + 6)\left(\frac{125}{189}x - \frac{213825}{83521}\right)$$

We can now substitute (by rearranging)

$$5x^2 - 27x = x^4 - 3x^3 + 5x^2 - 17x + 6 - (x+1)(x^3 - 4x$$
$$+ 4x + 6)$$

$$\frac{1282950}{83521} = \Big((x^4 - 3x^3 + 5x^2 - 17x + 6) - (x+1)(x^3 - 4x^2 + 4x + 6) \Big)\left(1 + \left(\tfrac{1}{5}x + \tfrac{7}{25}\right)\left(\tfrac{125}{289}x - \tfrac{213825}{83521}\right)\right)$$

$$- (x^3 - 4x^2 + 4x + 6)\left(\frac{125}{189}x - \frac{213825}{83521}\right)$$

$$\frac{1282950}{83521} = (x^4 - 3x^3 + 5x^2 - 17x + 6)\left(1 + \left(\tfrac{1}{5}x + \tfrac{7}{25}\right)\left(\tfrac{125}{289}x - \tfrac{213825}{83521}\right)\right) + (x^3 - 4x^2 + 4x + 6)$$

$$\left(-(x+1)\left(1 + \left(\tfrac{1}{5}x + \tfrac{7}{25}\right)\left(\tfrac{125}{289}x - \tfrac{213825}{83521}\right)\right) - \left(\tfrac{125}{189}x - \tfrac{213825}{83521}\right)\right)$$

Therefore, we can write

$$p(x) = 1 + \left(\tfrac{1}{5}x + \tfrac{7}{25}\right)\left(\tfrac{125}{289}x - \tfrac{213825}{83521}\right)$$

$$q(x) = -(x+1)\left(1 + \left(\tfrac{1}{5}x + \tfrac{7}{25}\right)\left(\tfrac{125}{289}x - \tfrac{213825}{83521}\right)\right) - \left(\tfrac{125}{189}x - \tfrac{213825}{83521}\right)$$

where $\gcd(f(x), g(x)) = f(x)p(x) + g(x)q(x)$

(e) We perform the extended Euclidean algorithm on

$$x^3-4x^2+4x+6 \mod x^4-3x^3-5x^2-17x+6$$

$$x+1$$

$$x^3-4x^2+4x+6 \;\big)\; x^4-3x^3-5x^2-17x+6$$

$$x^4-4x^3+4x^2+6x$$

$$x^3-9x^2-23x+6$$

$$x^3-4x^2+4x+6$$

$$-5x^2-27x+0$$

so

$$x^4-3x^3-5x^2-17x+6=\underbrace{(x^3-4x^2+4x+6)}_{g(x)}\underbrace{(x+1)}_{q}+\underbrace{(-5x^2-27x)}_{r} \quad \text{—①}$$

$$-\frac{1}{5}x+\frac{47}{25}$$

$$-5x^2-27x+0 \;\big)\; x^3-4x^2+4x+6$$

$$x^3+\frac{27}{5}x^2+0$$

$$-\frac{47}{5}x^2+4x+6$$

$$-\frac{47}{5}x^2-\frac{1269}{25}x+0$$

$$\frac{1369}{25}x+6$$

$$x^3-4x^2+4x+6=(-5x^2-27x)\left(-\frac{1}{5}x+\frac{47}{25}\right)+\left(\frac{1369}{25}x+6\right) \quad \text{—②}$$

~~We can no longer proceed because $-5x^2-27x$ cannot~~

$$
\begin{array}{r}
-\frac{125}{1369}x - \frac{905325}{1874161} \\
\frac{1369}{25}x+6 \enclose{longdiv}{-5x^2 - 27x + 0} \\
-5x^2 - \frac{750x}{1369} \\
\hline
-\frac{36213}{1369}x + 0 \\
-\frac{36213}{1369}x - \frac{5431950}{1874161} \\
\hline
\frac{5431950}{1874161}
\end{array}
$$

$$(-5x^2-27x) = \left(\frac{1369}{25}x+6\right)\left(-\frac{125}{1369}x - \frac{905325}{1874161}\right) + \left(\frac{5431950}{1874161}\right) \quad — ③$$

The remainder here $\frac{5431950}{1874161}$ is the GCD. Now we can rewrite and find a linear combination of x^3-4x^2+4x+6 and $x^4-3x^3-5x^2-17x+6$ to find the multiplicative inverse.

Rearranging equation ③

$$\frac{5431950}{1874161} = (-5x^2-27x) - \left(\frac{1369}{25}x+6\right)\left(-\frac{125}{1369}x - \frac{905325}{1874161}\right)$$

Rearranging ② and substituting into ③