# ANALYSIS OF FOURIER METHODS FOR NAVIER-STOKES EQUATION

By

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**Abstract.**

This is the first in a series of three papers in which spectral and pseudospectral methods are analyzed for a class of time dependent nonlinear partial differential equations. In this paper, we prove optimal order of convergence of the Fourier-Galerkin method for the 2-D Navier-Stokes equation in various energy norms and in $L^p$-norms of the velocity field. The optimal order of convergence of the Fourier-collocation method is also proved. We briefly outline the extension of these results to 3-D Navier-Stokes equation in the original form and the rotational form. Semigroup formulations and the variation of constants formula are essential tools in the analysis.

## § 0 Introduction

Spectral and pseudospectral methods for the full incompressible Navier-Stokes equation provides an example where a large number of calculations have been done but the analysis of the methods is still unsatisfactory. It is now proper to say that spectral and pseudospectral methods, using trigonometric series, Chebyshev and Legendre polynomials, are among the most effective and popular numerical methods for the incompressible fluid flow in simple geometries.

For the analysis of these methods, with full time dependent models, there have been two papers by Hald [5], [6] which we should mention. [5] appears to be the first attempt to give the order of convergence of the classical Fourier method for the full Navier-stokes equation (hereafter abbreviated NSE). Probably due to the fact that energy estimates was the essential tool, Hald did not get the optimal order of convergence in the sense that the error of the numerical solution is of the same order as the error in the initial approximation. Also the error constant he had grows exponentially in time which is not necessary because the flows he considered was stable in time. In [6], the same type of analysis was carried out for the Fourier-collocation method with stream function-vorticity formulation. There the nonlinear terms drop out and the analysis boils down to a careful treatment of the aliasing error. The final results have the same deficiencies mentioned above and more importantly the analysis does not apply to the formulation with primitive variables which is used much more in practice.

The analysis is much more complete for the steady NSE. Important work has been down by Canuto, Maday, Quarteroni and their co-workers using variational techniques [1], [8], [9], [10]. Two fundamental ingredients in their analysis are: the generalized Babuska-Brezzi condition for the mixed formulation and the basic approximation theory for the spectral expansions and interpolations in Sobolev spaces [1], [3], the later being used constantly in our analysis. An abstract framework of Brezzi, Rappaz and Raviart plays the role of bridging the gap between linear and nonlinear problems [2].

It is the purpose of this series of papers to analyze convergence properties of the most commonly used spectral type of methods. We use the semigroup formulation and the variation of constant formula. The argument is an extension of the usual one for proving local existence of solutions to an evolutionary equation [7]. We apply this argument to the semi-discretized equation. The observation is a standard one in numerical analysis. By consistency, the true solution to the continuous problem satisfies the numerical scheme approximately. This enables us to work in a neighborhood of the true solution. Once we have a good initial approximation, it will never get much worse. And an a priori estimates of the numerical approximation follows from that of the true solution.

This approach should be useful for a wide class of numerical methods and evolutionary equations which admit a semigroup formulation. In [4], some general results are proved upon assuming a very weak stability condition. These results are then applied to the Burgers' equation.

There has been some discussion in the literature about whether aliasing error would make the Fourier-collocation method unstable if it is used for the original form of the NSE (see equation (1.1)). Indeed the rotational form of the NSE was suggested to avoid the "aliasing instability"[14]. The error estimates proved later in this paper shows rigorously that at least for smooth flow, aliasing should not be a problem even if the original form of the NSE is used.

The analysis starts with the Fourier-Galerkin method. In § 1, we put the NSE in a dynamical form and prove some a priori estimates for the 2-D NSE. In § 2, we analyze the Fourier-Galerkin method in various energy norms and a $L^p$-norm of the velocity field. Here the a priori estimates for the numerical solution follows in the same fashion as for the true solution. In § 3, we deal with the Fourier-collocation method. Here more work has to be done to obtain uniform a priori estimates for the numerical solutions. Following that, optimal order of convergence is proved in the $H^1$-norm of the velocity.

In this work, I am especially benefited from the discussion with Dr. Okamoto when I was visiting IMA last fall. I enjoyed very much the exciting environment created by the visiting members and the IMA staff. It is my pleasure to thank Dr. Okamoto and the IMA staff. I would also like to thank Professor Engquist for his interest and helpful comments on this work, and for arranging my visit to IMA.

## § 1. Functional Setting of the Navier-Stokes Equation and Some A Priori Estimates

### 1.1 The Navier-Stokes Equation (NSE).

We adopt the notations and some standard results on NSE from Temam [16].

The NSE, in velocity-pressure formulation, reads,

$$
(1.1)\qquad \begin{cases}
\dfrac{\partial u}{\partial t} + u \cdot \nabla u - \Delta u + \nabla p = 0 \quad , \qquad \text{on} \qquad \Omega = (-\pi, \pi)^2 \\[2ex]
\nabla \cdot u = 0 \\[2ex]
\text{periodic boundary conditions} \\[2ex]
u(0, x) = a(x)
\end{cases}
$$

Here $u$ stands for velocity, $p$ for pressure. The constant density and viscosity are set to be 1 for simplicity.

The spaces we will work with are: for $m \in \mathbb{N}$

$$
(1.2)\qquad H_p^m(\Omega) = \{u, \qquad u = \sum_{k \in \mathbb{Z}^2} c_k \, e^{ik\cdot x} \quad , \bar{c}_k = c_{-k} \ , \sum_{k \in \mathbb{Z}^2} |k|^{2m} |c_k|^2 < \infty \ \}
$$

$$
(1.3)\qquad \dot{H}_p^m(\Omega) = \{u \in H_p^m(\Omega), \quad c_o = 0\}
$$

The subscript $p$ stands for "periodic". For any $\alpha \in \mathbb{R}$, we define $H_p^\alpha(\Omega)$ by duality and interpolation in the usual way. $\| \cdot \|_s$ will denote the usual $H^s$-norm for $s \in \mathbb{R}$. We may use various equivalent definitions of the Sobolev norms. To represent the velocity field, we need Sobolev spaces for vector-valued functions. But for simplicity we will use the same notations as those for the scalar functions. This should not cause any confusion. We introduce also

$$
\begin{aligned}
V &= \{u \in H^1_p(\Omega), \qquad \text{div } u = 0\} \\
H &= \{u \in H^0_p(\Omega), \qquad \text{div } u = 0 \quad \text{in the weak sense}\} \\
\dot V &= \{u \in \dot H^1_p(\Omega), \qquad \text{div } u = 0\} \qquad (1.4) \\
\dot H &= \{u \in \dot H^0_p(\Omega), \qquad \text{div } u = 0 \text{ in the weak sense}\} \\
G &= \{u \in H^0_p(\Omega), \quad u = \nabla q \ , \ q \in H^1_p(\Omega)\}
\end{aligned}
$$

Note that $G$ is the orthogonal complement of $H$ in $H^0_p(\Omega)$.

For $u(t) \in \ C([0,T], \ H^s(\Omega))$ , we set

$$|||u(t)|||_s = \sup_{0\le t\le T} \|u(t)\|_s \tag{1.5}$$

### The Projection Operator $P$

$P$ denotes the projection operator $H^0_p(\Omega) \to H$ under the usual $L^2$-inner product. If $f \in \ H^0_p$ , and $f = \sum_{k\in\mathbb{Z}^2} f_k\, e^{ik\cdot x}$, it is easy to see that

$$Pf = \sum_{k\in\mathbb{Z}^2, k\neq 0} (1 - \frac{k\cdot k^T}{|k|^2})\, f_k\, e^{ik\cdot x} + f_0$$

We note that $P$ commutes with derivation.

**The Stokes Operator A**

For $u \in \mathcal{D}(A) = \{u \in \dot{H}, \quad \Delta u \in \dot{H}\} = \dot{H}_p^2(\Omega) \cap H$, define

$$Au = -\Delta u \tag{1.6}$$

$A$ can be extended as a positive definite, self-adjoint operator on $\dot{H}$. Thus we can define the powers of $A$, $A^\alpha$, for $\alpha \in \mathbb{R}$. In fact,

$$\mathcal{D}(A^\alpha) = \{u \in \ \dot{H}_p^{2\alpha}(\Omega)\,, \quad \text{div } u = 0\}$$

is a closed subspace of $\dot{H}_p^{2\alpha}(\Omega)$. Moreover, there exists a constant $\tilde{c}$ (depending on $\alpha$), such that,

$$\tilde{c}^{-1}\|u\|_{2\alpha} \leq \|A^\alpha u\|_0 \leq \tilde{c}\|u\|_{2\alpha}, \qquad \text{for } u \in \mathcal{D}(A^\alpha) \tag{1.7}$$

These results can be proved easily by using the explicit formulas for $A$ and $A^{-1}$ established in [16].

For our analysis, it is essential that the equation admits a semigroup formulation. It is easy to see that $-A$ generates an analytic semigroup on $\dot{H}$, denoted by $\{e^{-At}\}_{t\geq 0}$, with the following estimates:

$$\|A^\alpha e^{-At}\| \leq Mt^{-\alpha}e^{-\delta t} \quad , \qquad t > 0\,, \alpha > 0 \tag{1.8}$$

for some constants $M$ and $\delta$.

Without loss of generality, we can assume $u(t) \in \dot{H}$ for $0 \le t < +\infty$ (See [16]). By using the Stokes operator, we can write the NSE as a dynamical system in the space of divergence free vector fields.

$$
(1.9) \qquad \begin{cases} u(t): \quad [0, +\infty) \to \dot{V} \\[2ex] \dfrac{du}{dt} + Au + Fu = 0 \\[2ex] u(0) = a \in \dot{V} \end{cases}
$$

where $Fu = P(u \cdot \nabla u)$ is the nonlinear term.

In the following analysis, we will use $C$ to denote a generic constant independent of the data. It may have different values in different places. We use $K(\cdot), L(\cdot), L'(\cdot)$ etc. to denote constants depending on various norms of the data and we will try to indicate the dependence. Some of the specific constants will be denoted by $\tilde{c}$. (see (1.7)).

### 1.2 Some A Priori Estimates for the NSE.

The following lemma is used frequently in the analysis. Many inequalities of this type can be found in Chapter 7 of Henry [7]. The special version we use, is proved in [13].

LEMMA 1.1. Let $T, \alpha, \beta, \nu$ be positive constants. $0 < \nu < 1$. Then for any continuous function $f: \ [0, T] \to [0, \infty)$

satisfying

$$
(1.10) \qquad f(t) \le \alpha + \beta \int_0^t (t-s)^{-\nu} f(s) ds \quad , \quad 0 \le t \le T
$$

We have

$$f(t) \leq c\alpha \ \exp\{c\beta^{\frac{1}{1-\nu}} t\} \quad , \quad 0 \leq t \leq T \tag{1.11}$$

with a positive constant $c$ which depends only on $\nu$.

We also recall the standard estimates for the trilinear form [16],

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i D_i v_j w_j dx$$

$$|b(u,v,w)| \leq c\|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3} \tag{1.12}$$

with $m_1, m_2, m_3$ satisfying:

$$m_1, m_2, m_3 \geq 0$$

$$m_1 + m_2 + m_3 > 1$$

$$\text{or} \qquad m_1 + m_2 + m_3 = 1 \quad \text{but} \quad m_1, m_2, m_3 \neq 1$$

The next lemma summarizes some standard estimates.

LEMMA 1.2. There exist positive constants $C$ and $\delta$ , and a function $k(\cdot)$, such that for $a \in \dot{V}, \quad t > 0$

$$\|u(t)\|_0 \leq c\|a\|_0 e^{-\delta t} \tag{1.13a}$$

$$\|A^{\frac{1}{4}} u(t)\|_0 \leq \|A^{\frac{1}{4}} a\|_0 \ \exp\{c\|a\|_0^2 - \delta t\} \tag{1.13b}$$

$$\|A^{\frac{1}{2}} u(t)\|_0 \leq \|A^{\frac{1}{2}} a\|_0 \ \exp\{c\|a\|_0^4 - \delta t\} \tag{1.13c}$$

and for $a \in \mathcal{D}(A)$

$$\|Au(t)\|_0 \le k(\|a\|_2)\, e^{-\delta t} \tag{1.13d}$$

This lemma is proved in [13] for no-slip boundary condition. The proof works equally well for periodic boundary condition.

Lemma 1.3 is an extension of Lemma 1.2.

LEMMA 1.3. For $r > \frac{1}{2}$, there exist constants $C$, $\delta$ and a function $L'_r(a)$ depending on various norms of $a$ , such that for $a \in \mathcal{D}(A^r)$,

$$\|A^r u(t)\|_0 \le C\|A^r a\|_0 \exp\{L'_r(a) - \delta t\} \tag{1.14}$$

where for $\frac{1}{2} < r < \frac{3}{4}$, $L'_r(a)$ is a function of $\|A^{\frac{1}{2}}a\|_0 \exp(c\|a\|_0^4)$ , for $\frac{3}{4} \le r < 1$ , $L'_r(a)$ is a function of $\|A^{\frac{1}{4}}a\|_0 \; \exp(c\|a\|_0^2)$ , for $1 \le r \le \frac{5}{4}$, $L'_r(a)$ is a function of $\|A^{\frac{3}{4}}a\|_0$ and $\|A^{\frac{1}{4}}a\|_0 \; \exp(c\|a\|_0^2)$, and for $r > \frac{5}{4}$ , $L'_r(a)$ is a function of $\|A^{\frac{5}{4}}a\|_0$ , $\|A^{\frac{3}{4}}a\|_0$ and $\|A^{\frac{1}{4}}a\|_0 \; \exp(c\|a\|_0^2)$ .

*Proof.* By using variation of constants formula, we can write (1.10) formally in the following form

$$u(t) = e^{-At}a - \int_0^t e^{-A(t-s)}\, Fu(s)ds \tag{1.15}$$

Hence

$$A^r u(t) = e^{-At}A^r a - \int_0^t A^r e^{-A(t-s)}\, Fu(s)ds$$

$$= e^{-At}A^r a - \int_0^t A^\beta e^{-A(t-s)}\, A^{r-\beta}\, Fu(s)ds$$

Here $\beta > 0$.

*Case 1.* $\frac{3}{4} \le r < 1$. Choose $\beta$, such that $r < \beta < 1$. We claim that

$$\|A^{r-\beta}Fu(s)\|_0 \le \|A^{r-\beta}P(u\cdot\nabla u)\|_0 \le c\|u\|_{\frac{1}{2}}\|u\|_{2r}\,. \tag{1.16}$$

We prove (1.16) by duality argument. Let $\omega \in \dot{H}^0_p(\Omega)$, Set $m_1 = \frac{1}{2}\,,\ m_2 = 2r-1\,,\ m_3 = 2(\beta - r)$ in (1.12), we get

$$(A^{r-\beta}Fu(s),\,\omega) = (u\nabla u\,,\,PA^{r-\beta}\omega) \le c\|u\|_{\frac{1}{2}}\|u\|_{2r}\|\omega\|_0$$

This proves (1.16). Now from (1.8) and (1.16) we obtain

$$\|A^r u(t)\|_0 \le \|e^{-At}A^r a\|_0 + \int_0^t \|A^\beta e^{-A(t-s)}\|\ \ \|A^{r-\beta}Fu(s)\|_0 ds$$

$$\le Me^{-\delta t}\|A^r a\|_0 + CM\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|u\|_{\frac{1}{2}}\|u\|_{2r}ds$$

$$\le Me^{-\delta t}\|A^r a\|_0 + L(a)\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|A^r u\|_0 ds$$

where $L(a) = c\|A^{\frac{1}{4}}a\|_0\ \exp(c\|a\|_0^2)$, see (1.13b).

Let $\varphi(t) = e^{\delta t}\|A^r u(t)\|_0$, use Lemma 1.1, we get

$$\varphi(t) \le c\|A^r a\|_0 \quad \exp\{L(a)^{\frac{1}{1-\beta}}t\}$$

Hence,

$$\|A^r u(t)\|_0 \le c\|A^r a\|_0 \ \exp\{L(a)^{\frac{1}{1-\beta}} - \delta\}t \quad .$$

In order to get a decaying factor, we use the following trick which I learned from Okamoto [13].

For any $t_0 > 0$, carry out the above procedure for $t > t_0$, we get

$$\|A^r u(t)\|_0 \le c\|A^r u(t_0)\|_0 \ \exp\{L(u(t_0))^{\frac{1}{1-\beta}} - \delta\}(t-t_0)$$

$$\le c\,\|A^r u(t_0)\|_0 \ \exp\{\delta t_0\} \ \exp\{L(u(t_0))^{\frac{1}{1-\beta}} - \delta\}t$$

$$\le c\,\|A^r a\|_0 \ \exp\{L(a)^{\frac{1}{1-\beta}} t_0 \ \exp\{L(u(t_0))^{\frac{1}{1-\beta}} - \delta\}t$$

where $L(u(t_0)) = \|A^{\frac14} u(t_0)\|_0 \ \exp c\|u(t_0)\|_0^2 \le \|A^{\frac14} a\|_0 \ \exp\{\tilde{c}\|a\|_0^2 - \delta t_0\}$

Put

$$t_0 = \max\left\{0,\ \frac{1}{\delta}\log\frac{\|A^{\frac14}a\|_0 \exp(\tilde{c}\|a\|_0^2)}{(\frac{\delta}{2})^{1-\beta}}\right\}$$

then

$$L(u(t_0))^{\frac{1}{1-\beta}} \le \frac{\delta}{2}$$

$$t_0 \le c\|A^{\frac14}a_0\|e^{c\|a\|_0^2}$$

Therefore for $t \ge t_0$ ,

$$\|A^r u(t)\|_0 \le c\|A^r a\|_0 \exp\{L(a)^{\frac{1}{1-\beta}} t_0\} \exp\{-\frac{\delta}{2}t\}$$

$$\le c\|A^r a\|_0 \exp\{L'(a) - \frac{\delta}{2}t\|$$

where $L'(a) = L(a)^{\frac{1}{1-\beta}} t_0$

For $t \leq t_0$,

$$\|A^r u(t)\|_0 \leq c\|A^r a\|_0 \exp\{L(a)^{\frac{1}{1-\beta}} t_0\} \exp(-\delta t)$$

$$= c\|A^r a\|_0 \exp\{L'(a) - \delta t\}$$

Lemma 1.3 is proved in this case by setting

$$L'_r(a) = L(a)^{\frac{1}{1-\beta}} \cdot c \cdot \|A^{\frac{1}{4}} a\|_0 \exp(c\|a\|_0^2)$$

*Case 2.* $1 \leq r \leq \frac{5}{4}$ Set $\beta = r - \frac{1}{2}$

Notice that for $u \in \dot{V}$

$$\int_\Omega u \cdot \nabla u \; dx = 0 \tag{1.17}$$

Therefore

$$\|A^{\frac{1}{2}} F(u)\|_0 \leq \|A^{\frac{1}{2}} P(u \cdot \nabla u)\|_0 \leq c\|u \cdot \nabla u\|_1$$

$$\leq c\|D(u \cdot \nabla u)\|_0$$

In the last step, Poincare inequality was used because of (1.17).

By Sobolev Imbedding Theorem, we have

$$H_p^{\frac{3}{2}}(\Omega) \hookrightarrow W^{1,4}(\Omega), \qquad H_p^{3/2}(\Omega) \hookrightarrow L^\infty(\Omega)$$

Therefore from

$$\|A^{\frac{1}{2}}Fu(s)\|_0^2 \le \|D(u\cdot\nabla u)\|_0^2 \le c\big(\int_\Omega |\nabla u|^4\,dx + \int_\Omega |u\cdot D^2 u|^2\,dx\big)$$

$$\le c(\|u\|_{W^{1,4}}^4 + \|u\|_{L^\infty}^2\|u\|_2^2)$$

we obtain,

$$\|A^{\frac{1}{2}}Fu(s)\|_0 \le c\|u(s)\|_{3/2}\,\|u(s)\|_2 \le c\|u(s)\|_{3/2}\,\|A^r u(s)\|_0$$

$$\le L(a)\|A^r u(s)\|_0$$

where we have $L(a) = c\|A^{\frac{3}{4}}a\|_0 \exp\{L'_{\frac{3}{4}}(a)\}$ from Case 1. From these we get,

$$\|A^r u(t)\|_0 \le \|A^r e^{-At}a\|_0 + M\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|A^{\frac{1}{2}}Fu(s)\|_0\,ds$$

$$\le Me^{-\delta t}\|A^r a\|_0 + cL(a)\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|A^r u(s)\|_0\,ds$$

Now the lemma can be proved in this case by the same arguments as in Case 1.

*Case 3.* $\quad r > \frac{5}{4}$

$$\|A^r u(t)\|_0 \le \|A^r e^{-At}a\|_0 + \int_0^t \|A^{\frac{3}{4}}e^{-A(t-s)}\|\;\|A^{r-\frac{3}{4}}Fu(s)\|_0\,ds$$

In this case, we use the Moser type of calculus inequalities [11]. For $s > 1$, there is a constant $c_s$, such that for any $f, g \in H^s(\Omega)$,

$$\|f\cdot g\|_s \le c_s(\|f\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f\|_s)$$

Note that $H^{2r-\frac{3}{2}} \hookrightarrow L^\infty$ , $H^{\frac{5}{2}} \hookrightarrow W^{1,\infty}$ and the imbeddings being continuous. Therefore we have,

$$\|A^{r-\frac{3}{4}}Fu(s)\|_0 \leq c\|u(s)\cdot\nabla u(s)\|_{2r-\frac{3}{2}} \leq c\|u\|_{L^\infty}\|\nabla u\|_{2r-\frac{3}{2}}+$$

$$+\|\nabla u\|_{L^\infty}\,\|u\|_{2r-\frac{3}{2}}$$

$$\leq c\|u(s)\|_{\frac{5}{2}}\,\|u(s)\|_{2r} \leq L(a)\|A^r u(s)\|_0$$

where $L(a) = c\|A^{\frac{5}{4}}a\|_0\exp\{L'_{\frac{5}{4}}(a)\}$ from Case 2.

Again Lemma 1.3 can be proved by the same arguments used in Case 1.

*Case 4.* $\frac{1}{2} < r < \frac{3}{4}$

We first use duality argument to estimate $\|Fu(s)\|_0$ . For $\varphi \in H_p^0(\Omega)$, set $m_1 = 1$, $m_2 = 2r-1$, $m_3 = 0$ in (1.12). We get

$$(Fu(s),\varphi) = (u(s)\nabla u(s),\, P\varphi) \leq c\|u(s)\|_1\|u(s)\|_{2r}\|\varphi\|_0$$

Therefore

$$\|Fu(s)\|_0 \leq c\|u(s)\|_1\,\|u(s)\|_{2r}$$

From (1.15) we obtain,

$$\|A^r u(t)\|_0 \leq \|e^{-At}A^r a\|_0 + \int_0^t \|A^r e^{-A(t-s)}\|\,\|Fu(s)\|_0\,ds$$

$$\leq M\,\|A^r a\|_0 + CM\int_0^t (t-s)^{-r}e^{-\delta(t-s)}\|u(s)\|_1\,\|A^r u(s)\|_0\,ds$$

$$\leq M\,\|A^r a\|_0 + CML(a)\int_0^t (t-s)^{-r}e^{-\delta(t-s)}\|A^r u(s)\|_0\,ds$$

Here $L(a) = \|A^{\frac{1}{2}}a\|_0\exp(c\|a\|_0^4)$ from (1.13c). The argument used in Case 1 can also be used to prove the lemma in the present case. Now the proof is complete. □

## § 2. Analysis of Fourier-Galerkin Method for NSE

**2.1 Fourier-Galerkin Method.**

Let

$$S_N = \{\varphi(x), \quad \varphi = \sum_{-N\leq k_1,k_2\leq N} c_k \, e^{ik\cdot x} \}$$

$$P_N = S_N \times S_N \tag{2.1}$$

$$\dot{P}_N = P_N \cap \dot{H}^0_p(\Omega) \quad , \quad \dot{S}_N = S_N \cap \dot{H}^0_p(\Omega)$$

$$V_N = V \cap P_N, \qquad \dot{V}_N = \dot{V} \cap P_N$$

Denote also by $P_N$ the projection operator from $L^2$ to $P_N$ in the $L^2$-inner product. It is easy to see $P_N P = P P_N$. Therefore we have $P_N(H) \subset V_N$, i.e. $P_N\varphi \in V_N$, for any $\varphi \in H$.

The semi-discrete Fourier-Galerkin approximation to (1.1) can be formulated as

$$\left[\begin{array}{l} \text{Find } u_N(t) \in C^1\left([0,+\infty),\ P_N\right), \quad p_N(t) \in C\left([0,+\infty), \dot{S}_N\right) \\[2ex] \text{such that for } v_N \in P_N\,, \quad \varphi_N \in S_N, \quad t > 0 \\[2ex] (\dfrac{\partial u_N}{\partial t} - \Delta u_N + u_N \cdot \nabla u_N + \nabla p_N, \quad v_N) = 0 \\[2ex] (\nabla \cdot u_N, \quad \varphi_N) = 0 \\[2ex] u_N(0) = P_N a \end{array}\right. \tag{2.2}$$

(2.2) can also be written as a dynamical system in $V_N$

$$\text{(2.3)} \qquad \begin{cases} u_N(t) \in V_N \quad , \quad \text{for } t \geq 0 \\ \\ \dfrac{du_N}{dt} + Au_N + F_N u_N(t) = 0 \\ \\ u_N(0) = P_N a \end{cases}$$

where $F_N u_N(s) = P_N F u_N(s) = P_N P(u_N \cdot \nabla u_N(s))$

Notice that for (2.3) we have

$$\text{(2.4)} \qquad \frac{d}{dt}\int_\Omega u_N \, dx = 0$$

Thus if $a \in \dot{V}$, then $u_N(t) \in \dot{V}$ for all $t \geq 0$ and (2.3) actually defines a dynamical system in $\dot{V}_N$ .

From the variation of constants formula, we get an integral form of (2.3):

$$\text{(2.5)} \qquad u_N(t) = e^{-At} P_N a - \int_0^t e^{-A(t-s)} F_N \, u_N(s) \, ds$$

The Fourier projection operator $P_N$ has the nice property that it commutes with derivation. Because of this, all the a priori estimates we have mentioned for NSE carry over to its Fourier approximation. We will not list these results here as they are exactly the same as those for the true solution. But we will use them in the error analysis.

### 2.2 Error Analysis in the Energy Norms.

In this subsection, we will prove optimal order of convergence for the Fourier-Galerkin method, in the sense that the error in the numerical solution is of the same order as that of the projection operator $P_N$.

First of all, let's recall the basic approximation properties of $P_N$. We have for $0 \leq \mu \leq m, \quad u \in H_p^m(\Omega)$,

$$\|u - P_N u\|_\mu \leq \frac{c}{N^{m-\mu}} \|u\|_m$$

where $C$ is some constant independent of $N$ and $u$ .

THEOREM 2.1. For $m \geq \frac{1}{2}$, $0 \leq r < \frac{1}{2}$, there exists a constant $C$, such that for $a \in \mathcal{D}(A^m)$, $t \geq 0$, we have

$$\|u(t) - u_N(t)\|_{2r} \leq CN^{2(r-m)} \|a\|_{2m} \exp \{L'_m(a) - \delta t\} \tag{2.6}$$

where $L'_m(a)$ was defined in Lemma 1.3.

*Proof.* Let $e_N(t) = u(t) - u_N(t)$. By (1.15) and (2.5) one has

$$e_N(t) = e^{-At}(a - P_N a) - \int_0^t e^{-A(t-s)}(Fu(s) - F_N u_N(s))\, ds$$

$$= e^{-At}(a - P_N a) - \int_0^t e^{-A(t-s)}(Fu(s) - P_N Fu(s))\, ds$$

$$- \int_0^t e^{-A(t-s)}(P_N Fu(s) - F_N u_N(s))\, ds$$

$$= I_1 - I_2 - I_3$$

Roughly speaking, the estimates of $I_1$ and $I_2$ involve consistency argument, whereas the estimates of $I_3$ involve stability argument.

$$I_1 = e^{-At}(a - P_N a)$$

$$\|I_1\|_{2r} \le \tilde{c}\ \|A^r e^{-At}(a - P_N a)\|_0$$

$$\le cN^{2(r-m)}\ \|A^m e^{-At} a\|_0 \le CN^{2(r-m)} e^{-\delta t}\ \|a\|_{2m}$$

$$I_2 = (I - P_N) \int_0^t e^{-A(t-s)} F u(s)\, ds$$

$$= (I - P_N)(e^{-At} a - u(t))$$

Hence

$$\|I_2\|_{2r} \le \tilde{c} \|A^r I_2\|_0 \le C\ N^{2(r-m)}(e^{-\delta t}\|a\|_{2m} + \|u(t)\|_{2m})$$

$$\le CN^{2(r-m)}\ \|a\|_{2m}\ \exp\ \{L'_m(a) - \delta t\}\ .$$

by Lemma 1.2 and Lemma 1.3 ,

$$I_3 = \int_0^t e^{-A(t-s)}(P_N\, F u(s) - P_N F\, u_N(s))\, ds$$

$$= P_N \int_0^t e^{-A(t-s)}(P(u \cdot \nabla u) - P(u_N \cdot \nabla u_N))\, ds$$

$$= P_N \int_0^t e^{-A(t-s)} P(e_N \cdot \nabla u + u_N \cdot \nabla e_N)\, ds$$

For $0 \le r < \frac{1}{2}$, we can choose $\beta$, such that $r + \frac{1}{2} < \beta < 1$. Then

$$A^r I_3 = P_N \int_0^t A^\beta e^{-A(t-s)} A^{r-\beta} P(e_N \cdot \nabla u + u_N \cdot \nabla e_N)\, ds$$

We estimate $A^{r-\beta}P(e_N \cdot \nabla u)$ by duality argument. Let $w \in H_p^0$, set $m_1 = 2r, \quad m_2 = 0, \quad m_3 = 2(\beta - r)$ in (1.12)

$$(A^{r-\beta}P(e_N \cdot \nabla u),\, w) = (e_N \cdot \nabla u,\ A^{r-\beta}Pw) \le c\ \|e_N\|_{2r}\ \|u\|_1\ \|w\|_0$$

Therefore,

$$\|A^{r-\beta}P(e_N \cdot \nabla u)\|_0 \le c\|e_N(s)\|_{2r}\,\|u(s)\|_1$$

Next, we estimate $A^{r-\beta}P(u_N \cdot \nabla e_N)$. Let $w \in H_p^0(\Omega)$. Notice that $u_N \in V$. Using integration by parts and set $m_1 = 1,\ m_2 = 2(\beta - r) - 1,\ m_3 = 2r$, we get

$$(A^{r-\beta}P(u_N \cdot \nabla e_N),\, w) = (u_N \cdot \nabla e_N,\ A^{r-\beta}Pw)$$

$$= -(u_N \cdot \nabla(A^{r-\beta}Pw),\ e_N) \le c\|u_N\|_1\ \|w\|_0\ \|e_N\|_{2r}$$

Hence

$$\|A^{r-\beta}P(u_N \cdot \nabla e_N)\|_0 \le c\|u_N\|_1\ \|e_N\|_{2r}$$

Therefore

$$\|A^{r-\beta}P(e_N \cdot \nabla u + u_N \cdot \nabla e_N)\|_0 \le c\ (\|u_N\|_1 + \|u\|_1)\ \|e_N\|_{2r}$$

$$\le L(a)\ \|e_N(s)\|_{2r}$$

where $L(a) = c\ \|a\|_1\ \exp\{c\ \|a\|_0^4\}$ by Lemma 1.2.

These give us

$$\|e_N(t)\|_{2r} \le CN^{2(r-m)} \ \exp\{L'_m(a) - \delta t\} \quad \|a\|_{2m} +$$

$$+ L(a) \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|e_N(s)\|_0 \, ds$$

Let $\varphi(t) = \|e_N(t)\|_{2r} e^{\delta t}$, then

$$\varphi(t) \le CN^{2(r-m)} \|a\|_{2m} \ \exp L'_m(a) + L(a) \int_0^t (t-s)^{-\beta} \varphi(s) \, ds$$

Use Lemma 1.1, we get

$$\varphi(t) \le CN^{2(r-m)} \|a\|_{2m} \exp L'_m(a) \ \exp\{L(a)^{\frac{1}{1-\beta}} t\}$$

Observe that from Lemma 1.3, $\|u(t)\|_{2m}$ and $L(u(t))$ decrease as $t$ increases. Therefore we can use the same trick as we did in Lemma 1.3, to get

$$\|e_N(t)\|_{2r} \le CN^{2(r-m)} \ \|a\|_{2m} \ \exp\{L'_m(a) - \delta t\}. \qquad \square$$

THEOREM 2.2. For $m \ge \frac{1}{2}$, there exists a constant $C$, such that for any $a \in \mathcal{D}(A^m)$, $t \ge 0$, we have

$$\|u(t) - u_N(t)\|_1 \le CN^{1-2m} \ \|a\|_{2m} \ \exp\{L'_m(a) - \delta t\} \tag{2.7}$$

where $L'_m(a)$ was defined in Lemma 1.3

*Proof.* As in Theorem 2.1, we estimate $I_1, I_2$ and $I_3$ separately.

$$\|I_1\|_1 \leq CN^{1-2m} \ \|A^m e^{-At} a\|_0 \leq CN^{1-2m} \ \|a\|_{2m} e^{-\delta t}$$

$$\|I_2\|_1 \leq CN^{1-2m} \ \|A^m (e^{-At} a - u(t))\|_0 \leq CN^{1-2m} \ \|a\|_{2m} \ \exp\{L'_m(a) - \delta t\}$$

$$\|I_3\|_1 \leq C \int_0^t \|A^{\frac{3}{4}} e^{-A(t-s)}\| \ \|A^{-\frac{1}{4}}(Fu_N(s) - Fu(s))\|_0 \, ds$$

$$\|A^{-\frac{1}{4}}(Fu_N(s) - Fu(s))\|_0 \leq A^{-\frac{1}{4}} P_N(u_N \cdot \nabla e_N)\|_0 + \|A^{-\frac{1}{4}} P_N(e_N \cdot \nabla u)\|_0$$

Use duality argument, let $\omega \in H_p^0(\Omega)$ , and $m_2 = 0 \ \ m_3 = \frac{1}{2}$ in (1.12), we get

$$(A^{-\frac{1}{4}} P_N(e_N \cdot \nabla u), \omega) \ = \ (e_N \cdot \nabla u \ , \ A^{-\frac{1}{4}} P_N \omega) \leq C \ \|e_N\|_1 \ \|u\|_1 \ \|\omega\|_0$$

Thus,

$$\|A^{-\frac{1}{4}} P_N(e_N \cdot \nabla u)\|_0 \leq c\|e_N\|_1 \ \|u\|_1 \ \ \leq c \, L(a) \ \|e_N(s)\|_1$$

where $L(a) = \|a\|_1 \ \exp(c \ \|a\|_0^4)$ .
Similarly, $\|A^{-\frac{1}{4}} P_N(u_N \cdot \nabla e_N)\|_0 \leq c \ \|u_N\|_1 \ \|e_N\|_1 \leq c \ L(a) \| \, e_N(s)\|_1$
Therefore,

$$\|I_3\|_1 \leq c \, L(a) \int_0^t (t-s)^{-\frac{3}{4}} e^{-\delta(t-s)} \ \|e_N(s)\|_1 \, ds$$

In summary, we have,

$$\|e_N(t)\|_1 \leq c \, N^{1-2m} \ \|a\|_{2m} \ \exp\{L'_m(a) - \delta t\} \ +$$

$$+ \, cL(a) \int_0^t (t-s)^{-\frac{3}{4}} e^{-\delta(t-s)} \ \|e_N(s)\|_1 \, ds$$

As in Theorem 2.1, we get

$$\|e_N(t)\|_1 \leq CN^{1-2m} \|a\|_{2m} \exp\{L'_m(a) - \delta t\}. \qquad \square$$

THEOREM 2.3. For $r > \frac{1}{2}$, $m \geq r$, there exists a constant $C$, such that for $a \in \mathcal{D}(A^m)$, $t \geq 0$.

$$\|u(t) - u_N(t)\|_{2r} \leq CN^{2(r-m)} \|a\|_{2m} \exp\{L'_m(a) - \delta t\} \ . \tag{2.8}$$

where $L'_m(a)$ was defined in Lemma 1.3

*Proof.* First we assume $\frac{1}{2} < r < \frac{3}{4}$. Again we have

$$\|I_1\|_{2r} + \|I_2\|_{2r} \leq CN^{2(r-m)} \|a\|_{2m} \exp\{L'_m(a) - \delta t\}.$$

The estimates of $I_3$ in Theorem 2.2 give us

$$\|A^{-\frac{1}{4}} P_N(e_N \cdot \nabla u + u_N \cdot \nabla e_N)\|_0 \leq c(\|u\|_1 + \|u_N\|_1) \|A^{\frac{1}{2}} e_N\|_0$$

$$\leq cL(a) \|e_N(s)\|_{2r} \ .$$

So we have

$$\|e_N(t)\|_{2r} \leq C\, N^{2(r-m)} \|a\|_{2m} \exp\{L'_m(a) - \delta t\} +$$

$$cL(a) \int_0^t (t-s)^{-(r+\frac{1}{4})} e^{-\delta(t-s)} \|e_N(s)\|_{2r}\, ds$$

and (2.8) follows.

The case when $r \geq \frac{3}{4}$ is readily proved by combining the techniques in Lemma 1.3 and Theorem 2.1. We omit the details. □

REMARK. It is easy to check that the pressure term $p(t,x)$ and its Fourier-Galerkin approximation $P_N(t,x)$ satisfy respectively the following Poisson equation with periodic boundary condition:

$$\begin{aligned} -\Delta p = & \quad \nabla \cdot (u \cdot \nabla u) \\ -\Delta p_N = & \quad P_N(\nabla \cdot (u_N \cdot \nabla u_N)) \end{aligned}$$

and

$$\int_\Omega p(t,x)\, dx = \int_\Omega p_N(t,x)\, dx = 0$$

Once we have the estimates on the velocity, convergence of pressure follows from standard arguments for elliptic equations. We refer to [10] for some results.

**2.3 Estimates for Inhomogeneous Equation and the Smoothing Effect of the NSE.**

We first indicate briefly how results in the previous subsection can be generalized when a forcing term is added to the NSE. Again, by applying the operators $A$ and $P$, we can write the NSE as a dynamical system in the space of divergence free vector fields.

$$\text{(2.10)} \qquad \begin{cases} u(t) \in \dot{V} \qquad \text{for } t \geq 0 \\ \dfrac{du}{dt} + Au + Fu(t) = P\, f(t) \\ u(0) = a \in \dot{V} \end{cases}$$

where $f(t):\ [0,\infty) \to \dot{H}^0_p(\Omega)$ is the forcing term. Without loss of generality, we have assumed

$$\int_\Omega f(t,x)\,dx = 0$$

(See chapter 1 of Temam [16]). Therefore $u(t) \in \dot{V}$ for $t > 0$ if $a \in \dot{V}$ .

We recall the following standard results on the regularity of the solution.

LEMMA 2.1. Assume for $m \geq 1,\quad a \in \dot{H}^m_p(\Omega), f(t) \in C\,([0,+\infty),\ \dot{H}^{m-1}_p(\Omega))$. Then there exists a function $L(t)$, such that for $t \geq 0$,

$$\|u(t)\|_m \leq L(t) \tag{2.11}$$

where $L(t)$ depends on $\|a\|_m$ and $|||f(t)|||_{m-1} = \sup_{0\leq s\leq t} \|f(s)\|_{m-1}$

The Fourier-Galerkin approximation $u_N(t)$ of (2.10) satisfies the equation:

$$\begin{cases} u_N(t) \in \dot{V}_N \\ \dfrac{d\,u_N}{dt} + A\,u_N + F_N(t) = P_N\,Pf(t) \\ u_N(0) = P_N\,a \end{cases} \tag{2.12}$$

Now we have

THEOREM 2.4. Assume $a \in \dot{H}^m_p(\Omega)$, $f \in C([0,+\infty),\dot{H}^{m-1}_p(\Omega))$ for $m \geq 1,\quad 0 \leq r \leq m$. Then there exists a function $L(t)$, depending on the data, but independent of $N$, such that

$$\|u(t) - u_N(t)\|_r \leq L(t)N^{r-m} \tag{2.13}$$

The proof of this theorem uses exactly the same ideas as the ones in Theorem 2.1, 2.2, and 2.3. Therefore we omit the details.

Next we give some results which illustrate the smoothing properties of the NSE. For this purpose, we need the following lemma from Henry [7].

LEMMA 2.2. If $0 \leq \alpha, \beta < 1,\ 0 < T < +\infty$ and for $0 \leq t \leq T$

$$\varphi(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta}\varphi(s)\,ds \tag{2.14}$$

where $a, b$ are positive constants. Then there exists a constant $c(\beta, b, T)$, depending on $\beta, b, T$ only, such that

$$\varphi(t) \leq \frac{at^{-\alpha}}{1-\alpha}\,c(\beta, b, T) \qquad \text{for } 0 \leq t \leq T \tag{2.15}$$

We will assume that $a \in \dot{V}$. In this case, by modifying Okamoto's proof of Proposition 6.3 in [13], we have

LEMMA 2.3. For $0 \leq t \leq T$, there exists a constant $C$, such that for $a \in \dot{V}$, $f \in C([0,T], \dot{H}^1_p(\Omega))$ $0 < t \leq T$

$$\|Au(t)\|_0 \leq L(a)e^{-\delta t}t^{-\frac{1}{2}} + C \sup_{0\leq s\leq t} \|f(s)\|_1 \tag{2.16}$$

where $L(a)$ is a function of $\|a\|_1$ .

For the Fourier-Galerkin approximation, we have,

THEOREM 2.5. Assume $a \in \dot{V}$, $f(t) \in C([0,T], \dot{H}^1_p(\Omega))$. Then there exists a constant $C$ such that for $0 < t \leq T$,

$$\|u(t) - u_N(t)\|_1 \leq \frac{L'}{N}t^{-\frac{1}{2}} \tag{2.17}$$

where $L'$ depends on the data.

*Proof.* Carry out the same procedure as in Theorem 2.1 and use Lemma 2.3, we get

$$\|e_N(t)\|_1 \le \frac{L(a)}{N}(t^{-\frac{1}{2}} + L) + L(a)\int_0^t (t-s)^{-\frac{3}{4}} \|e_N(s)\|_1 \, ds$$

Let $\varphi(t) = \|e_N(t)\|_1$, use Lemma 2.2 for $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$, we obtain

$$\|e_N(t)\|_1 \le \frac{L'}{N} t^{-\frac{1}{2}} \qquad \text{for } 0 < t \le T$$

This procedure gives a constant $L'$ depending on $\|a\|_1$ and $|||f(t)|||_1$ . □

REMARK. We have analyzed the NSE only in 2-D. In 3-D, the main difficulty is to establish regularity for the true solution. This can be circumvented by either restricting ourselves to a short time interval, or assuming some a priori estimates for the true solution and the numerical approximation. If we admit these, the foregoing analysis routinely carries over to 3-D. Of course a more relevant problem would be to assume suitable smoothness for the true solution only, and then prove that the numerical solution, which solves a nonlinear ordinary differential equation, exists and satisfies some a priori estimates independent of $N$. These require the ideas presented in § 3, see subsection 3.3.

### 2.4 Error Estimates in $L^p$-norms $(1 < p < +\infty)$.

To illustrate the advantage of this approach, we will analyze in this section the Fourier-Galerkin method in $L^p$-norms $(1 < p < +\infty)$. Other approaches for spectral methods have been using energy conservation or energy estimates which seems to be a rather restricted tool.

In this subsection only, we will change our notation. All the functions and vector fields dealt with are periodic in space, so we will drop the subscript "$p$" which was used to indicate functions being periodic. Instead $p$ will indicate that $L^p$-norm is used. For example, for $k \in \mathbb{N}$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega), \qquad D^\alpha u \in L^p(\Omega), \quad \text{for } |\alpha| \le k\}$$

$$X_p = \text{ the closure of } C^\infty \text{ divergence-free periodic vector fields in } L^p(\Omega).$$

Similarly we define $\dot{W}^{k,p}(\Omega)$ and $\dot{X}_p$. Define the Stokes operator $A_p$ by: $\mathcal{D}(A_p) = \dot{W}^{2,p}(\Omega) \cap \dot{X}_p$

$$A_p u = -\Delta u \qquad \text{for } u \in \mathcal{D}(A_p)$$

LEMMA 2.4. The Stokes operator $-A_p$ generates an analytic semigroup in $\dot{X}_p$.

This follows directly from the corresponding statement for Laplace operator.

LEMMA 2.5. The Projection operator $P_N$ is uniformly bounded from $L^p$ to $L^p$, i.e. there is a constant $C$, such that for all $N \geq 1$ .

$$\|P_N\| \leq C$$

For a proof, see Chapter 7 of [17].

Denote the semigroup generated from $-A_p$ by $\{e^{-A_p t}\}_{t \geq 0}$. The standard estimates for analytic semigroups are satisfied by $\{e^{-A_p t}\}_{t \geq 0}$.

$$\|A_p^\alpha e^{-A_p t}\| \leq M_\alpha t^{-\alpha} \qquad , \qquad \text{for } \alpha \geq 0$$

Here and in this subsection, the operator norms are evaluated in the space $L(\dot{X}_p \to \dot{X}_p)$. In the following, $A_p$ will be abbreviated as $A$.

THEOREM 2.6. Assume $a \in W^{2,p}$ , $f \in C([0,T], W^{1,p})$ , $1 < p < +\infty$.

$$R = |||u(s) - P_N u(s)|||_{L^p} + |||f(s) - P_N f(s)|||_{L^p}$$

Then there exist constants $L, L'$ and $N_o$, depending on T and the data $a$ and $f$ only, such that for $N > N_o$, $\quad 0 \leq t \leq T$,

$$\|u(t)\|_{L^\infty} + \|u_N(t)\|_{L^\infty} \leq L \tag{2.18a}$$

$$\|u(t) - u_N(t)\|_{L^p} \leq L'R \tag{2.18b}$$

*Proof.* The proof of the first part of the theorem, namely the $L^\infty$ boundedness, is rather lengthy. The ideas used are very similar to the ones used in Theorem 3.1 below. Therefore we will not produce it here and we proceed to prove the second part of the theorem. Again let $e_N = u - u_N$

$$e_N(t) = e^{-At}(a - P_N a) + \int_0^t e^{-A(t-s)}(Pf(s) - P_N Pf(s))\, ds$$

$$- \int_0^t e^{-A(t-s)}(P(u \cdot \nabla u) - P_N P(u \cdot \nabla u))\, ds$$

$$- \int_0^t e^{-A(t-s)}(P_N P(u \cdot \nabla u) - P_N P(u_N \cdot \nabla u_N))\, ds$$

$$= I_1 + I_2 + I_3 + I_4$$

$$\|I_1\|_{L^p} \leq \|e^{-At}\| \ \|a - P_N a\|_{L^p} \leq C\ R$$

$$\|I_2\|_{L^p} \leq \int_0^t \|e^{-A(t-s)}\| \ \|P\| \ \|f - P_N f\|\, ds \leq CR$$

Here we have used the fact that $P$ is bounded from $L^p$ to $L^p$. Notice that

$$\int_0^t e^{-A(t-s)} P(u \cdot \nabla u)\, ds = e^{-At} a + \int_0^t e^{-A(t-s)} P\, f(s)\, ds - u(t)$$

$$\int_0^t e^{-A(t-s)} P_N(u \cdot \nabla u)\, ds = e^{-At} P_N a + \int_0^t e^{-A(t-s)} P_N\, P\, f(s)\, ds - P_N u(t)$$

Therefore,

$$\|I_3\|_{L^p} \le CR$$

$$\|I_4\|_{L^p} \le \|P_N\| \int_0^t \|A^{\frac{1}{2}}\, e^{-A(t-s)}\|\ \|A^{-\frac{1}{2}}\, P(u\cdot\nabla u - u_N\cdot\nabla u_N)\|_{L^p}\, ds$$

$$\le C \int_0^t (t-s)^{-\frac{1}{2}}\ \|A^{-\frac{1}{2}}\, P(e_N\cdot\nabla u + u_N\cdot\nabla e_N)\|_{L^p}\ ds$$

For any $\varphi \in \dot{L}^q$, $\quad q = \frac{p}{p-1}$ ,

$$\|\nabla(A^{-\frac{1}{2}}\, P\varphi)\|_{L^q} \le c\ \|A^{\frac{1}{2}}\, A^{-\frac{1}{2}}\, P\varphi\|_{L^q} \le c\ \|P\varphi\|_{L^q} \le c\ \|\varphi\|_{L^q}$$

Integration by parts, we get,

$$(A^{-\frac{1}{2}}\, P(e_N\cdot\nabla u), \varphi) = -(e_N\cdot\nabla(A^{-\frac{1}{2}} P\varphi), u)$$

$$\le \|u\|_{L^\infty}\ \|e_N\|_{L^p}\ \|\nabla(A^{-\frac{1}{2}}\, P\varphi)\|_{L^q} \le c\ \|u\|_{L^\infty}\ \|e_N\|_{L^p}\ \|\varphi\|_{L^q}$$

Therefore,

$$\|A^{-\frac{1}{2}}\, P(e_N\cdot\nabla u)\|_{L^p} \le C\ \|u\|_{L^\infty}\ \|e_N\|_{L^p}$$

Similarly,

$$\|A^{-\frac{1}{2}}\, P(u_N\cdot\nabla e_N)\|_{L^p} \le C\ \|u_N\|_{L^\infty}\ \|e_N\|_{L^p}$$

Hence,

$$\|I_4\|_{L^p} \le CL \int_0^t (t-s)^{-\frac{1}{2}}\ \|e_N(s)\|_{L^p}\, ds$$

Altogether we have

$$\|e_N(t)\|_{L^p} \leq CR + CL \int_0^t (t-s)^{-\frac{1}{2}} \|e_N(s)\|_{L^p}\, ds$$

Apply Lemma 1.1, we get

$$\|e_N(t)\|_{L^p} \leq L'R \qquad \text{for } 0 \leq t \leq T, \qquad \square$$

REMARK. For any $\varphi \in S_N$ we have

$$\|u - P_N u\|_{L^p} \leq \|u - \varphi\|_{L^p} + \|P_N u - P_N \varphi\|_{L^p} \leq c\, \|u - \varphi\|_{L^p}$$

Therefore,

$$\|u - P_N u\|_{L^p} \leq c \inf_{\varphi \in S_N} \|u - \varphi\|_{L^p} \tag{2.19}$$

Estimates on the right hand side of (2.19) can be found in Nikol'skii [12]. More notations are needed to state the results. This should be a good excuse for not quoting the results here.

## § 3. Fourier-Collocation Method for NSE

### 3.1 Fourier-Collocation Method.

For a continuous $2\pi$-periodic function $\varphi(x)$ on $\Omega = [-\pi, \pi]^2$, we define its Fourier interpolatant $P_c\varphi(x)$ to be

$$P_c\varphi(x) = \sum_{-N \le k_1, k_2 \le N} \varphi_k \, e^{ik\cdot x} \quad \in \; S_N \tag{3.1a}$$

$$\varphi_k = \frac{1}{(2N+1)^2} \sum_{-N \le j_1, j_2 \le N} \varphi(x_j) \, e^{-ix_j \cdot k} \tag{3.1b}$$

where $k = (k_1, k_2)$ , $j = (j_1, j_2)$, $x_j = (j_1 h \, , j_2 h)$, $h = \frac{2\pi}{2N+1}$ .

We recall the following results in approximation theory [3].

Lemma 3.1. For $u \in H_p^m(\Omega)$, $m > 1$, $0 \le r \le m$, we have

$$\|u - P_c u\|_r \le C N^{r-m} \, \|u\|_m \tag{3.2}$$

for some constant $C$ independent of $N$ and $u$.

In contrast to the Fourier-Galerkin method, now we have to work in the undotted spaces. This is because, for a function which averages to zero on $\Omega$, its Fourier interpolatant usually does not preserve that property. For this reason, we will introduce a stronger Stokes operator.

The Stokes operator $A$ now is defined to be

$$\mathcal{D}(A) = H_p^2 \cap H \tag{3.3a}$$

$$Au = -\Delta u + u \qquad \text{for } u \in \mathcal{D}(A) \tag{3.3b}$$

It is easy to check the following properties of $A$. $A$ is a densely defined closed operator on $H$. It can be extended to a positive definite, self-adjoint operator. Thus its fractional powers $A^\alpha$, $\alpha \in \mathbb{R}$, are well-defined. Indeed,

$$\mathcal{D}(A^\alpha) = \{u \in H_p^{2\alpha}(\Omega), \qquad \operatorname{div} u = 0\} \tag{3.4}$$

is a closed subspace of $H_p^{2\alpha}(\Omega)$, and there exists a constant $\tilde{c}$, depending on $\alpha$, such that

$$\tilde{c}^{-1}\,\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le \tilde{c}\,\|u\|_{2\alpha}, \qquad \text{for } u \in \mathcal{D}(A^\alpha) \tag{3.5}$$

$-A$ generates an analytic semigroup on $H$, denoted by $\{e^{-At}\}_{t\geq 0}$, with the usual estimates:

$$\|A^\alpha e^{-At}\| \le M_\alpha t^{-\alpha} e^{-\delta t} \quad , \quad \alpha \geq 0 \tag{3.6}$$

for some constants $M_\alpha$ and $\delta$. Here the operator is viewed as from $H$ to $H$. Somewhere in the context, we will use operator norms from $H^s$ to $H^s$, $s \neq 0$. We will point out this in the appropriate places.

By variation of constants formula, (1.1) with a forcing term $f(t)$, is equivalent to the integral form

$$u(t) = e^{-At}a - \int_0^t e^{-A(t-s)}\{Fu(s) - u(s) - Pf(s)\}\,ds \tag{3.7}$$

where $Fu(s) = P(u \cdot \nabla u)$

The Fourier-Collocation approximation of the NSE can be formulated as the following:

$$
(3.8) \qquad \begin{cases} \text{Find } u_N(t): \ [0,T] \to P_N, \quad p_N(t): \ [0,T] \to \dot{S}_N \quad , \quad \text{such that} \\ \\ at\ x_j = (j_1 h,\ j_2 h),\ -N \le j_1, j_2 \le N \\ \\ \dfrac{\partial u_N}{\partial t}(x_j) - (\Delta u_N)(x_j) + u_N(x_j) \cdot (\nabla u_N)(x_j) + (\nabla p_N)(x_j) = f(x_j) \\ \\ (\nabla \cdot u_N)(x_j) = 0 \\ \\ u_N(0) = P_c\, a \end{cases}
$$

It can also be written as a dynamical system in $V_N$

$$
(3.9) \qquad \begin{cases} U_N(t) \in V_N, \qquad \text{for } t \ge 0 \\ \\ \dfrac{d u_N}{dt} + A u_N + F_N u_N - u_N = P\ P_c\, f \\ \\ u_N(0) = P_c\ a \end{cases}
$$

where $F_N u_N = P\, P_c(u_N \cdot \nabla u_N)$

Using variation of constants formula, we get an integral form of (3.9)

$$
U_N(t) = e^{-At}\, P_c\, a - \int_0^t e^{-A(t-s)} \{F_N\, u_N(s) - u_N(s) - P\, P_c f(s)\}\, ds
$$

In the next subsection, we will establish some a priori estimates for $u_N(t), \qquad t \ge 0.$

### 3.2 A Priori Estimates for the Fourier-Collocation Approximation.

LEMMA 3.2. Assume $u(t) \in C([0,T], H_p^3 \cap H), f(t) \in C([0,T]. H_p^{3/2} \cap H)$. Then there exist $t_0, N_0$, depending only on $|||u(s)|||_3$, $|||f(s)|||_{3/2}$, such that for $N > N_0$, (4.9) has a solution $u_N(t) \in V_N$, for $0 \le t \le t_0$. Moreover, we have

$$\|u(t) - u_N(t)\|_{\frac{5}{2}} \le \frac{c}{\sqrt{N}} (|||u(s)|||_3^2 + |||f(s)|||_{\frac{3}{2}} + \|a\|_3), \quad 0 \le t \le t_0 \tag{3.11}$$

*Proof.* Define $S^0 = \{v(t) \in C([0,t_0], V_N), \quad v(0) = P_c a,$

$$\sup_{0 \le t \le t_0} \|v(t) - u(t)\|_{5/2} \le 1\} \tag{3.12}$$

For $\omega \in \ C([0,t_0],\ H_p^{\frac{5}{2}}(\Omega))$ we define

$$|w| = \sup_{0 \le t \le t_0} \|\omega\|_{5/2} \tag{3.13}$$

For $v \in S^0$, define for $0 \le t \le t_0$

$$Gv(t) = e^{-At} P_c a - \int_0^t e^{-A(t-s)} \{PP_c(v \cdot \nabla v) - v - PP_c f\}\, ds. \tag{3.14}$$

Now (3.9) or (3.10) is equivalent to the fixed point equation for $G$:

$$v(t) = Gv(t) \qquad\qquad v \in S^0\ . \tag{3.15}$$

We will show that by choosing $t_0$ appropriately, $G$ defines a contraction on $S^0$. For this purpose, we first check how much (3.15) is satisfied by $u(t)$, the true solution of NSE. From (3.7), and (3.14) we obtain

$$u(t) = Gu(t) + e^{-At}(a - P_c a) + \int_0^t e^{-A(t-s)} P(f - P_c f)\, ds$$

$$- \int_0^t e^{-A(t-s)}\, P\{u \cdot \nabla u - P_c(u \cdot \nabla u)\}\, ds$$

$$= Gu(t) + R_1 + R_2 - R_3$$

$$\|R_1\|_{5/2} \leq \tilde{c}\, \|A^{5/4} e^{-At}(a - P_c a)\|_0 \leq c\, \|a - P_c a\|_{5/2} \leq \frac{c}{\sqrt{N}}\, \|a\|_3$$

$$\|R_2\|_{5/2} \leq \tilde{c} \int_0^t \|A^{3/4} e^{-A(t-s)} A^{\frac{1}{2}}\, P\,(f - P_c f)\|\|_0\, ds$$

$$\leq c \int_0^t (t-s)^{-\frac{3}{4}}\, \|f(s) - P_c\, f(s)\|_1\, ds \leq \frac{c}{\sqrt{N}}\, |||f|||_{3/2}$$

$$\|R_3\|_{5/2} \leq \tilde{c} \int_0^t \|A^{3/4}\, e^{-A(t-s)} A^{\frac{1}{2}}\, P\{u \cdot \nabla u - P_c(u \cdot \nabla u)\}\|_0\, ds$$

$$\leq c \int_0^t (t-s)^{-\frac{3}{4}}\, \|u \cdot \nabla u - P_c(u \cdot \nabla u)\|_1\, ds$$

$$\leq \frac{c}{\sqrt{N}} \int_0^t (t-s)^{-\frac{3}{4}}\, \|u \cdot \nabla u\|_{3/2}\, ds$$

Note that $H^{3/2}(\Omega)$ is a Banach algebra, hence

$$\|u \cdot \nabla u\|_{3/2} \leq c\, \|u\|_{3/2}\, \|\nabla u\|_{3/2} \leq c\|u\|_3^2$$

Thus,

$$\|R_3\|_{5/2} \le \frac{c}{\sqrt{N}} \;\; |||u|||_3^2$$

Therefore,

$$|Gu - u| \le \frac{R}{\sqrt{N}} \tag{3.16}$$

where $R = c(|||u|||_3^2 + |||f|||_{3/2} + \|a\|_3)$

Now let $v \in S^0$, then $\|v\|_{5/2} \le \|u\|_{5/2} + 1 \le |||u|||_3 + 1$, for $0 \le t \le t_0$.

$$\|Gv-u\|_{5/2} \le \|R_1 + R_2 - R_3\|_{5/2} + \|Gv - Gu\|_{5/2}$$

$$\le \frac{R}{\sqrt{N}} + \tilde{c} \int_0^t \|A^{3/4} e^{-A(t-s)}\| \; \|A^{\frac{1}{2}}\{PP_c(u \cdot \nabla u - v \cdot \nabla v) + v - u\}\|_0 \, ds$$

$$\|A^{\frac{1}{2}} PP_c(u \cdot \nabla u - v \cdot \nabla u)\|_0 \le c \|P_c(u \cdot \nabla u - v \cdot \nabla v)\|_1 \le c \|u \cdot \nabla u - v \cdot \nabla v\|_{3/2}$$

$$\le c_1 (\|u\|_{5/2} + \|v\|_{5/2}) \; \|u - v\|_{5/2} \le L \; \|u - v\|_{5/2} \le L$$

where $L = c_1(2\,|||u|||_3 + 2)$. Again we have used the property that $H^{3/2}(\Omega)$ is a Banach algebra. Also

$$\|A^{\frac{1}{2}} (u - v)\|_0 \le \tilde{c} \; \|u - v\|_{5/2} \le \tilde{c}$$

Thus we get,

$$\|Gv - u\|_{5/2} \le \frac{R}{\sqrt{N}} \; + \tilde{c}\, M(L + \tilde{c}) \int_0^t (t-s)^{-\frac{3}{4}} \, ds$$

$$\le \frac{R}{\sqrt{N}} + 4\tilde{c}\, M(L + \tilde{c})\, t_0^{1/4}$$

Therefore,

$$|Gv - u|_{5/2} \leq \frac{R}{\sqrt{N}} + 4\tilde{c}M(L+\tilde{c})\, t_0^{1/4}$$

Now we choose $t_0, N_0$, such that

$$4\tilde{c}\, M(L+\tilde{c}) t_0^{1/4} < \frac{1}{2} \tag{3.17a}$$

$$\frac{R}{\sqrt{N}_0} \leq \frac{1}{2} \tag{3.17b}$$

Note that $t_0, N_0$ only depend on $|||u(t)|||_3$ , $|||f(t)|||_{3/2}$ and some universal constants. Thus we have for $N > N_0$ ,

$$|Gv - u| < 1$$

Notice that $Gv$ satisfies a linear differential equation

$$\begin{cases} \dfrac{d}{dt} Gv + A\, Gv + P\, P_c (v \cdot \nabla v) - v = P\, P_c\, f \\ \\ Gv|_{t=0} \quad = P_c\, a \in V_N \end{cases}$$

So it is clear that for $0 \leq t \leq t_0\,, Gv(t) \in V_N$. Therefore $Gv \in S^0$.

Next we show that under condition (3.17a), $G$ actually defines a contraction on $S^0$. Let $v, w \in S^0$, then for $0 \leq t \leq t_0$

$$\|Gv - Gw\|_{5/2} \leq \tilde{c} \int_0^t \|A^{3/4}\, e^{-A(t-s)} A^{\frac{1}{2}} \{P P_c (v \cdot \nabla v - w \cdot \nabla w) +$$

$$+ w - v\} \, \|_0 \, ds$$

$$\leq \tilde{c}\, M(L+\tilde{c}) \int_0^{t_0} (t-s)^{-3/4} \;\; \|w - v\|_{5/2} \, ds$$

Thus,

$$|Gv - Gw| \leq 4\tilde{c}\, M(L + \tilde{c})\; t_0^{1/4}\; |w - v| < \frac{1}{2}\, |v - w|$$

We conclude that there exists a unique fixed point of $G$ in $S^0$. Therefore, (4.10) can be solved for $0 \leq t \leq t_0$. We denote this solution by $u_N(t)$, and set $e_{N(t)} = u(t) - u_N(t)$. It is easy to see that

$$\|e_N(t)\|_{5/2} \leq \frac{R}{\sqrt{N}} + \tilde{c} \int_0^t \; \|A^{3/4}\, e^{-A(t-s)}\, \| \; \|A^{\frac{1}{2}}\{P\, P_c(u \cdot \nabla u - u_N \cdot \nabla u_N) +$$

$$+ u_N - u\} \; \|_0 \; ds$$

$$\leq \frac{R}{\sqrt{N}} + \tilde{c}\, M(L + \tilde{c}) \int_0^t (t - s)^{-\frac{3}{4}} \; \|e_N(s)\|_{5/2} \; ds$$

In Lemma 1.1, set $\beta = \tilde{c}\, M(L + \tilde{c}), \nu = \frac{3}{4}$, then we have for $0 \leq t \leq t_0$.

$$\|e_N(t)\|_{5/2} \leq c_0 \frac{R}{\sqrt{N}} \exp\{c_0 \beta^4 t_0\} \leq \frac{R}{\sqrt{N}}\, r \tag{3.18}$$

where $c_0$ is a constant, $r = c_0 \exp\{\frac{c_0}{4096}\}$. We used (3.17a) in the last step. This completes the proof of the lemma. □

Next theorem is a global version of Lemma 3.2.

THEOREM 3.1. Assume $u(t) \in C([0, T], H_p^3 \cap H)$. $f(t) \in C([0, T],\; H_p^{3/2}(\Omega))$. Then there exist $N_0(T)$, $L(T)$, depending only on $T$, $|||u(t)|||_3$ and $|||f(t)|||_{3/2}$, such that for $N > N_0(T)$, (3.9) has a unique solution $u_N(t)$. Moreover, we have for $t \leq T$.

$$\|u(t) - u_N(t)\|_{5/2} \leq \frac{L(T)}{\sqrt{N}} \tag{3.19}$$

*Proof.* We use an induction argument to construct $u_N(t)$. Let $t^n = n\,t_0$, $t_0$ is defined in Lemma 3.2. The lemma shows that our construction works for one time step, with step size $t_0$. Assume this can be done for $n$-steps, with the same step size $t_0$, and the following estimate

$$\|u(t) - u_N(t)\|_{5/2} \leq r(Mr+1)^{n-1}\frac{R}{\sqrt{N}} < 1 \qquad \text{for } t^{n-1} \leq t \leq t^n\,. \tag{3.20}$$

where $r$ and $R$ are defined in the lemma. We plan to show that the same thing can be done for $t^n \leq t \leq t^{n+1}$, with an estimate similar to (3.20)

$$\text{Define } \; S^n = \{v(t) \in C([t^n, t^{n+1}], V_N), \quad v(t^n) = u_N(t^n)\,,$$
$$\sup_{t^n \leq t \leq t^{n+1}} \|v(t) - u(t)\|_{5/2} \leq 1\}$$

For $v \in C([t^n,\ t^{n+1}],\ H_p^{5/2}(\Omega))$ define

$$|v| = \sup_{t^n \leq t \leq t^{n+1}} \|v\|_{5/2}$$

and for $v \in S^n$, $t^n \leq t \leq t^{n+1}$ , define

$$Gv(t) = e^{-A(t-t^n)}u_N(t^n) - \int_{t^n}^{t} e^{-A(t-s)}\{PP_c(v\cdot\nabla v) - v - P\,P_c f\}\,ds \tag{3.21}$$

We want to show that $G$ defines a contraction on $S^n$. First we check that $u(t)$ satisfies the fixed point equation $v = Gv$ approximately. In fact, as in the lemma, we have,

$$u(t) = Gu(t) + e^{-A(t-t^n)}(u(t^n) - u_N(t^n)) + \int_{t^n}^{t} e^{-A(t-s)}P(f - P_c\,f)\,ds$$

$$- \int_{t^n}^{t} e^{-A(t-s)}\,P\{u\cdot\nabla u - P_c\,u\cdot\nabla u\}\,ds$$

$$= Gu(t) + R_1' + R_2' - R_3'$$

$$\|R_1'\|_{5/2} \leq \|e^{-A(t-t^n)}\|\ \ \|u(t^n) - u_N(t^n)\|_{5/2} \leq Mr(Mr+1)^{n-1}\frac{R}{\sqrt{N}}$$

We have used the operator norm from $H^{5/2}$ to $H^{5/2}$ and the induction assumption (3.20).

The estimates of $R_2'$ and $R_3'$ are the same as those for $R_2$ and $R_3$ in the lemma,

$$\|R_2'\|_{5/2} + \|R_3'\|_{5/2} \leq \frac{R}{\sqrt{N}}$$

Therefore, for $t^n \leq t \leq t^{n+1}$

$$\|u(t) - Gu(t)\|_{5/2} \leq (Mr+1)^n \frac{R}{\sqrt{N}}$$

Next, assume $v, w \in S^n$, then with the same argument as in the lemma, we get for $t^n \leq t \leq t^{n+1}$,

$$\|Gv - u\|_{5/2} \leq \|Gv - Gu\|_{5/2} + (Mr+1)^n \frac{R}{\sqrt{N}}$$

$$\leq (Mr+1)^n \frac{R}{\sqrt{N}} + \tilde{c}M(L+\tilde{c}) \int_{t^n}^{t} (t-s)^{-\frac{3}{4}} \ \|u - v\|_{5/2} \, ds$$

$$\leq (Mr+1)^n \frac{R}{\sqrt{N}} + 4\,\tilde{c}M(L+\tilde{c})\ t_0^{1/4}$$

$$\|Gv - Gw\|_{5/2} \leq 4\,\tilde{c}M(L+\tilde{c})\ t_0^{1/4}\ |v - w|$$

Now if we choose $N_0$, such that

$$(Mr+1)^n \frac{R}{\sqrt{N_0}} < \frac{1}{2} \tag{3.22}$$

then for $N > N_0$, we have

$$|Gv - u| \leq 1$$

$$|Gv - Gw| < \frac{1}{2}\ |v - w|$$

Hence $G$ is a contraction on $S^n$.

Note that since we are considering a finite time interval $[0, T]$, $n \leq \frac{T}{t_0}$ and $(Mr + 1)^n R$ is bounded by a constant $L'(T)$. Condition (3.22) is satisfied if we choose $N_0(T) > 2L'(T)$.

(3.20) follows by exactly the same argument as in the lemma. Now by induction principle we have proved (3.19).

The global solution of (3.10) or (3.9), again denoted by $u_N(t)$, can be built by patching together the local solutions we have constructed on each interval $[t^n, t^{n+1}], n = 0, 1, 2, \ldots$. Its uniqueness and smoothness (in time) follows from standard arguments for ODE. $\square$

### 3.3 Error Estimates for the Fourier-Collocation Method.

We first prove a simple lemma.

LEMMA 3.3. For any $\varphi \in P_{2N}$, we have

$$\|P_c\varphi\|_0 \leq 9\ \|\varphi\|_0 \tag{3.23}$$

where $P_c$ denotes the interpolation operator in $P_N$.

*Proof.* Let $\varphi = \sum_{-2N \leq k_1, k_2 \leq 2N} \hat{\varphi}_k\ e^{ik \cdot x}$

then,

$$P_v\, \varphi = \sum_{-N \leq k_1, k_2 \leq N} \tilde{\varphi}_k\ e^{ik \cdot x}$$

Set $\hat{\varphi}_k = 0$ , for $|k_1|$ or $|k_2| > 2N$. Then we have

$$\tilde{\varphi}_k = \sum_{-1 \leq \ell_1, \ell_2 \leq 1} \hat{\varphi}_{k+(2N+1)\,\ell}$$

$$\|P_c \varphi\|_0^2 = 2\pi \sum_{|k| \leq N} \tilde{\varphi}_k^2 = 2\pi \sum_{|k| \leq N} \Big( \sum_{-1 \leq \ell_1, \ell_2 \leq 1} \hat{\varphi}_{k+(2N+1)\ell} \Big)^2$$

$$\leq 18\pi \sum_k \sum_{-1 \leq \ell_1, \ell_2 \leq 1} (\hat{\varphi}_{k+(2N+1)\,\ell})^2 \leq 81 \cdot 2\pi \sum_k \hat{\varphi}_k^2$$

$$= 81 \; \|\varphi\|_0^2 \, .$$

$\square$

THEOREM 3.2. (Error Estimates in $H^1$-norm)

Assume for $m \geq 3$, $u(t) \in C\left([0,T]\,,\, H_p^m(\Omega)\right), f(t) \in C\left([0,T],\, H_p^{m-1}(\Omega)\right)$. Then there exist $L'$, $N_0(T)$, such that for $N \geq N_0(T)$, $0 \leq t \leq T$,

$$\|u(t) - u_N(t)\|_1 \leq \frac{L'}{N^{m-1}} \left( |||u(t)|||_m^2 + \|a\|_m + |||f(t)|||_{m-1} \right) \tag{3.24}$$

where $L'$ and $N_0(T)$ depend on $|||u(t)|||_3$, $|||f(t)|||_{3/2}$ and $T$ only.

*Proof.* By Theorem 3.1, there is a $N_0(T)$, such that for $N > N_0(T)$, (3.9) has a unique solution on $[0, T]$, and

$$|||u_N(t)|||_{5/2} \leq |||u(t)|||_{5/2} + 1$$

Let $e_N(t) = u(t) - u_N(t)$, then

$$\begin{aligned}
e_N(t) =& e^{-At}(a - P_c a) + \int_0^t e^{-A(t-s)}\, P(f - P_c f)\, ds - \\
& - \int_0^t e^{-A(t-s)}\, P(u \cdot \nabla u - P_c(u \cdot \nabla u))\, ds + \int_0^t e^{-A(t-s)}(u - u_N)\, ds \\
& - \int_0^t e^{-A(t-s)} P\, P_c(u \cdot \nabla u - u_N \cdot \nabla u_N)\, ds \\
& = I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

$$\|I_1\|_1 = \|e^{-At}(a - P_c\, a)\|_1 \leq M\|a - P_c\, a\|_1 \leq \frac{C}{N^{m-1}}\, \|a\|_m.$$

Here the operator norm is taken from $H^1$ to $H^1$.

$$\|I_2\|_1 = \| \int_0^t e^{-A(t-s)} P(f - P_c f)\, ds \|_1$$

$$\leq \tilde{c} \int_0^t \|A^{\frac{1}{2}} e^{-A(t-s)}\| \ \|P(f - P_c f)\|_0 \, ds$$

$$\leq \frac{c}{N^{m-1}} \int_0^t (t-s)^{-\frac{1}{2}} \ \|f(s)\|_{m-1} \, ds \leq \frac{c}{N^{m-1}} \||f|\|_{m-1}$$

$$\|I_3\|_1 = \| \int_0^t e^{-A(t-s)} P(u \cdot \nabla u - P_c(u \cdot \nabla u))\, ds \|_1$$

$$\leq \tilde{c} \int_0^t \|A^{\frac{1}{2}} e^{-A(t-s)}\| \ \|u \cdot \nabla u - P_c(u \cdot \nabla u)\|_0 \, ds$$

$$\leq \frac{c}{N^{m-1}} \int_0^t (t-s)^{-\frac{1}{2}} \ \|u \cdot \nabla u\|_{m-1} \, ds \leq \frac{c}{N^{m-1}} \ \||u(t)|\|_m^2$$

$$\|I_4\|_1 = \| \int_0^t e^{-A(t-s)} e_N(s)\, ds \|_1 \leq \tilde{c} \int_0^t \|A^{\frac{1}{2}} e^{-A(t-s)}\| \ \|e_N(s)\|_0 \, ds$$

$$\leq \tilde{c} M \int_0^t (t-s)^{-\frac{1}{2}} \ \|e_N(s)\|_1 \, ds$$

$$\|I_5\|_1 = \| \int_0^t e^{-A(t-s)} P\, P_c (u \cdot \nabla u - u_N \cdot \nabla u_N)\, ds \|_1$$

$$\leq \tilde{c} \int_0^t \|A^{\frac{1}{2}} e^{-A(t-s)}\| \ \|P_c (u \cdot \nabla u - u_N \cdot \nabla u_N)\|_0 \, ds$$

By Lemma 3.3, note that $P_c^2 = P_c$ , we have

$$\|P_c(u\cdot\nabla u - u_N\cdot\nabla u_N)\|_0 \le C\,\|P_c(u\cdot\nabla u) - u_N\cdot\nabla u_N\|_0$$

$$\le C\,(\|u\cdot\nabla u - P_c u\cdot\nabla u\|_0 + \|u\cdot\nabla u - u_N\cdot\nabla u_N\|_0)$$

$$\le \frac{C}{N^{m-1}}\,\|u\|_m^2 + C\,\|\,u\cdot\nabla e_N + e_N\cdot\nabla u_N\|_0$$

$$\le \frac{C}{N^{m-1}}\,\|u\|_m^2 + C\,(\|u\|_{5/2} + \|u_N\|_{5/2})\,\|e_N\|_1$$

$$\le \frac{C}{N^{m-1}}\,\|u\|_m^2 + CL\,\|e_N\|_1$$

where $L = 2\,|||u(t)|||_{5/2} + 1$. Therefore

$$\|I_5\|_1 \le \frac{C}{N^{m-1}}\,|||u|||_m^2 + CL\int_0^t (t-s)^{-\frac{1}{2}}\,\|e_N(s)\|_1\,ds$$

Altogether, we get for $0 \le t \le T$,

$$\|e_N(t)\|_1 \le \frac{C}{N^{m-1}}\,R_m + (\tilde{c}\,M + CL)\int_0^t (t-s)^{-\frac{1}{2}}\,\|e_N(s)\|_1\,ds$$

where $R = |||u(t)|||_m^2 + \|a\|_m + |||f(t)|||_{m-1}$

By Lemma 1.1, we obtain,

$$\|e_N(t)\|_1 \le \frac{L'}{N^{m-1}}\,R_m \qquad \text{for } 0 \le t \le T$$

where $L'$ is another constant depending on $L$ and $T$. □

Remark 1. The smoothing property of NSE can be explored in this framework, allowing us to obtain the same order of convergence under weaker regularity assumption on the initial data, e.g. $a \in H_p^{m-1}(\Omega)$. Some results in this direction were given for Fourier-Galerkin method in § 2. Also a version of Theorem 3.1 and Theorem 3.2 can be proved for $u(t) \in H^{2+\epsilon}(\Omega), \epsilon > 0,\ 0 \le t \le T$. This is nearly the weakest regularity assumption to make $P_c(u \cdot \nabla u)$ well-defined.

REMARK 2. Whether an optimal $L^2$ estimate is valid remains to be an open problem. This is important because the $L^2$ norm measures the natural kinetic energy.

### 3.3 Remarks on 3-D NSE and NSE in Rotational Form.

First we briefly outline an answer to the problem posed at the end of subsection 2.3. As the arguments used in the previous subsection are essentially independent of the dimension, we will state some results without proof.

Note that the formulation of the NSE and the Fourier-Collocation method in 3-D are analogous to those in 2-D. Moreover, Theorem 3.1 and 3.2 extend directly to 3-D.

THEOREM 3.3. For any $0 < \epsilon' < \epsilon$, assume $u(t) \in C\left([0,T],\ H_p^{5/2+\epsilon}\right), f(t) \in C\left([0,T],\ H_p^{3/2+\epsilon}\right)$. Then there exists a constant $N_0(T)$, such that for $N > N_0(T)$, the 3-D analogue of (3.9) has a unique solution $u_N(t)$ on $[0,T]$. Furthermore, we have

$$|||u_N(t)|||_{5/2+\epsilon'} \le |||u(t)|||_{5/2+\epsilon'} + 1$$

THEOREM 3.4. Assume that for $m > 5/2$, $u(t) \in C\left([0,T],\ H_p^m(\Omega)\right), f(t) \in C\left([0,T],\ H_p^{m-1}(\Omega)\right)$. Then there exist constants $N_0(T),\ L'(T)$, such that for $N > N_0(T)$, $\qquad 0 \le t \le T$.

$$\|u(t) - u_N(t)\|_1 \le \frac{L'(T)}{N^{m-1}} \left(|||u(s)|||_m^2 + \|a\|_m + |||f(s)|||_{m-1}\right). \tag{3.25}$$

Next we turn into the rotational form of the NSE which reads:

$$
(3.26) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \times \omega - \Delta u = \nabla(p + \dfrac{1}{2}|u|^2) \\ \\ \omega = \nabla \times u, \quad \nabla \cdot u = 0 \\ \\ \text{periodic boundary condition} \\ \\ u(0) = a \end{cases}
$$

where $\omega$ is the vorticity. It was suggested by Orszag [14] that when Fourier- Collocation method is used for this equation, aliasing instability does not occur because a quadratic integral is conserved by the numerical solution. Our analysis has shown that aliasing instability does not occur even if the Fourier-Collocation method is used for the NSE in its original form, at least when viscosity is not small. But because of this historical reason, it is worthwhile to comment on the convergence properties for (3.26).

It is straightforward to see that the arguments used for (1,1) and its 3-D analogue are directly applicable for (3.26). Similar results as Theorem 3.3 and 3.4 can be obtained for (3.26) with little change of the proofs. We will not do this here. Instead, we remark that the semi-conservation property of the numerical solution (in the absence of viscous effects and time-discretization errors) makes it possible to use Hald's idea. With Hald's argument, it might be difficult to give optimal estimates. This can be avoided by using the variation of constants formula after obtaining a $L^\infty$-norm bound from Hald's argument. Again we omit the details.

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