Disclaimer: the notebook is based on Ben Moll's superb lecture notes on distributional macroeconomics and some of my thoughts (like one in subsubsection 1.3.2)

1 Deterministic optimal control problems

Consider the following deterministic optimal control problem throughout the chapter:

$$v(x_0) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} r(x(t), \alpha(t)) dt$$

under the law of motion for the state

$$\dot{x}(t) = f(x(t), \alpha(t))$$

with state vector $x \in \mathcal{X} \subset \mathbb{R}^N$ and $\alpha \in A \subset \mathbb{R}^M$. Note that the system is autonomous since r and $\dot{x}$ does not depend on t explicitly, while it depends on $x(t)$ and $\alpha(t)$.

1.1 Hamiltonian optimality principle

Definition 1.1 (current-value Hamiltonian)**.**

$$H(x, \alpha, \lambda) = r(x, \alpha) + \lambda^T f(x, \alpha) \tag{1}$$

where $\lambda \in \mathbb{R}^N$ is called the co-state vector that depends on time.

Remarks 1.1. Note that this can be considered as the continuous version of Lagrangians.

Proposition 1.1 (Pontryagin's maximum principle)**.** α is a solution iff it satisfies:

$$
\begin{aligned}
H_\alpha(x(t), \alpha(t), \lambda(t)) = 0 \\
\dot{\lambda}(t) = \rho\lambda(t) - H_x(x(t), \alpha(t), \lambda(t))
\end{aligned}
$$

with the boundary condition for co-state variable $\lambda(t)$, i.e., transversality condition:

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T)x(T) = 0$$

Examples 1.1 (Neoclassical growth model)**.** Consider the example

$$\max_{\{c(t)\}_{t\geq 0}} \int e^{-\rho t}u(c(t))dt$$

such that

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t) \tag{2}$$

The control is c, where the state is k. The corresponding Hamiltonian is $H(c, k, \lambda) = u(c) + \lambda[F(k) - \delta k - c]$. By appling Pontryagin's maximum principle, we have

$$
\begin{aligned}
\lambda &= u'(c) \\
\dot{\lambda} &= \lambda(\rho + \delta - F(k))
\end{aligned}
$$

Substitution yields

$$u''(c)c' = u'(c)(\rho + \delta - F(k))$$

Using the **relative risk aversion coefficient** $\sigma(c) = -u''(c)c/u'(c)$, we have

$$\frac{c'}{c} = \frac{1}{\sigma(c)}(F'(k) - \rho - \delta)$$

where $1/\sigma(c)$ is called the **intertemporal elasticity of substitution** (IES).

1.2 Hamilton-Jacobi-Bellman equation

Proposition 1.2 (Generic HJB equation)**.** The value function v at optimal satisfies the following HJB equation:

$$\rho v(x) = \max_{\alpha\in A} r(x, \alpha) + \nabla v(x)^T f(x, \alpha) \tag{3}$$

where $\nabla x(t) = f(x(t), \alpha(t))$.

Remarks 1.2 (Hamiltonian to HJB equation)**.** From (1), define the maximized Hamiltonian $\overline{H}(x, \lambda)$:

$$\overline{H}(x, \lambda) = \max_{\alpha \in A} \left[r(x, \alpha) + \lambda^T f(x, \alpha) \right] \tag{4}$$

Then the HJB equation (3) can be defined as

$$\rho v(x) = \overline{H}(x, \nabla v(x)) \tag{5}$$

i.e., at optimal policy, the co-state vector is identical with the vector of shadow values.

1.3 Numerical solution of HJB equations

Consider Example 1.1 with $f(x, \alpha) = F(k) - \delta k - c$ and $r(x, \alpha) = c^{1-\sigma}/(1-\sigma)$ where $x = k$ and $\alpha = c$. The corresponding HJB equation is

$$\rho v(k) = \max_c u(c) + v'(k)(F(k) - \delta k - c) \tag{6}$$

Taking FOC, note that we have

$$u'(c) = v'(k) \tag{7}$$

$c = (u')^{-1}(v'(k))$ for each k.

Discretizing k with index $i = 1, ..., I$ gives

$$\rho v_i = u(c_i) + v_i'(F(k_i) - \delta k_i - c_i) \tag{8}$$

1.3.1 Upwind scheme and discretization of ∇v

Note that v is concave in k and the drift of state variables can be either positive or negative. Basic idea of upwind scheme: use forward difference when the drift of the state variable $f(x, \alpha)$ is positive, backward difference when it is negative. Let $v'_{i,B}$ and $v'_{i,F}$ be v' computed from backward difference and forward difference respectively.

Define the follwings:

$$s_{i,F} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,F}) \tag{9}$$

$$s_{i,B} = F(k_i) - \delta k_i - (u')^{-1}(v'_{i,B}) \tag{10}$$

Using the upwind scheme, the HJB eqaution can be rewritten as

$$\rho v_i = u(c_i) + \frac{v_{i+1} - v_i}{\Delta k} s^+_{i,F} + \frac{v_i - v_{i-1}}{\Delta k} s^-_{i,B} \tag{11}$$

for each i. This can be written in a compact matrix form

$$\rho \mathbf{v} = \mathbf{u} + \mathbf{A}(\mathbf{v})\mathbf{v} \tag{12}$$

such that the ith row of $\mathbf{A}(\mathbf{v})$ is defined as

$$\begin{bmatrix} 0 & \cdots & 0 & -\frac{s^-_{i,B}}{\Delta k} & \frac{s^-_{i,B}}{\Delta k} - \frac{s^-_{i,F}}{\Delta k} & \frac{s^-_{i,F}}{\Delta k} & 0 & \cdots & 0 \end{bmatrix} \tag{13}$$

where the non-zero elements above are located in $i-1, i, i+1$th columns respectively.

1.3.2 Discretization of $u(c)$

One thing left to find is $\mathbf{u}$. This can be computed from the derivative of v by the FOC in (7), which yields $u(c) = u((u')^{-1}(v'(k)))$. To compute the derivative, Ben Moll and Jesus Fernandez-Villaverde use the following discretization scheme:

$$v'_i = v'_{i,F} 1(s_{i,F} > 0) + v'_{i,B} 1(s_{i,B} < 0) + \overline{v}'_i 1(s_{i,F} < 0 < s_{i,B}) \tag{14}$$

where $\overline{v}'_i := u'(F(k_i) - \delta k_i)$, i.e., the steady state such that $s_i = 0$.

Here's my take on this. I argue that there is no need to arbitrarily define $\overline{v}'_i$ to deal with the case when s_i is taken to be zero. Note that we need $\overline{v}'_i$

only for $u(c)$ when the current state is close enough to the steady state, i.e., $s_i \approx 0$ by taking $s_i = 0$. On the other hand, note that when $s_i = 0$, we can directly compute the consumption value by the law of motion $f(k, c) = 0$ from (2) without relying on v':

$$c_i = F(k_i) - \delta(k_i) \tag{15}$$

Hence, instead of defining v'_i and $u(c(v'_i))$ accordingly, we can use the following consumption approximation scheme:

$$c_i = c(v'_{i,F})1(s_{i,F} > 0) + c(v'_{i,B})1(s_{i,B} < 0) + [F(k_i) - \delta(k_i)]1(s_{i,F} < 0 < s_{i,B}) \tag{16}$$

so that $u(c(v'_i))$ can be now replaced with $u(c_i)$, which ameliorates the need of defining an arbitary discretization scheme for v'_i when s_i is close to zero.

1.3.3 (Optional) Discretization of $\text{tr}(\Delta v)$

For stochastic HJBE with Brownian motions for state evolution, discretization for $v''(k)$ is needed. We use the following second order central method to approximate v'':

$$v''_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta k)^2} \tag{17}$$

This can be written in a compact matrix form

$$\mathbf{L}_2\mathbf{v}$$

such that

$$\mathbf{L}_2 = \frac{1}{(\Delta k)^2} \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{P \times P} \tag{18}$$

Note that $\mathbf{L}_2$ matrix is independent of $\mathbf{v}$.

1.3.4 (Optional) Discretization of $\mathrm{tr}(\Delta v)$, under reflecting barrier

Suppose that reflecting barriers are imposed, i.e., for $v'(k_1) = v'(k_P) = 0$. Then, this implies that we have

$$v_0 = v_1 \tag{19}$$

$$v_P = v_{P+1} \tag{20}$$

v'' using (17) at k_1 and k_P are then $v'(k_1)$ and $v'(k_P)$. Thus, $\mathbf{L}_2$ in (18) under the reflecting barrier conditions becomes

$$\mathbf{L}_2 = \frac{1}{(\Delta k)^2} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}_{P \times P} \tag{21}$$

1.3.5 Solving HJB equations with iterative methods

Note that u and A all depend on v nonlinearly. To solve this, we use iterative Euler methods as follows.

Definition 1.2 (Explicit Euler method)**.** Update v^{n+1} given v^n according to

$$\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^n = u(c_i^n) + (v_i^n)'(k_i)(F(k_i) - \delta k_i - c_i^n) \tag{22}$$

where $c_i^n := (u')^{-1}[(v^n)'(k_i)]$

Definition 1.3 (Implicit Euler method)**.** Update v^{n+1} given v^n according to

$$\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^{n+1} = u(c_i^n) + (v_i^{n+1})'(k_i)(F(k_i) - \delta k_i - c_i^n) \tag{23}$$

where $c_i^n := (u')^{-1}[(v^n)'(k_i)]$

For each step n, v^{n+1} can be computed by solving

$$\begin{aligned} \frac{1}{\Delta}(\mathbf{v}^{n+1} - \mathbf{v}^n) + \rho \mathbf{v}^{n+1} &= \mathbf{u} + \mathbf{A}_n \mathbf{v}^{n+1} \\ \left[\left(\rho + \frac{1}{\Delta} \right) \mathbf{I}_I - \mathbf{A}_n \right] \mathbf{v}^{n+1} &= \mathbf{u} + \frac{1}{\Delta} \mathbf{v}^n \end{aligned} \tag{24}$$

2 Optimal control under diffusion processes

Consider the following problem throughout the chapter

$$v(x_0) = \max_{\{\alpha(t)\}_{t\geq 0}} \mathbf{E}_0 \int_0^\infty e^{-\rho t} r(x(t), \alpha(t)) dt \tag{25}$$

with the law of motion for the state

$$dx(t) = f(x(t), \alpha(t), t) \tag{26}$$

2.1 Stochastic HJB eqautions

Proposition 2.1 (Stochastic HJB with Brownian law of motion)**.** Suppose that the law of motion in (26) follows Brownian motion, i.e.,

$$dx(t) = f(x(t), \alpha(t))dt + \sigma(x(t))dW(t) \tag{27}$$

then the corresponding HJB eqaution is

$$\rho v(x) = \max_\alpha r(x, \alpha) + \nabla v(x) \cdot f(x, \alpha) + \frac{1}{2}\mathrm{tr}(\Delta_x v(x)\sigma^2(x) \tag{28}$$

For scalar cases, we have

$$\rho v(x) = \max_\alpha r(x, \alpha) + v'(x) f(x, \alpha) + \frac{1}{2} V''(x)\sigma^2(x) \tag{29}$$

Lemma 2.1. Suppose that the state space x can be expressed in x_1, x_2 that represent endogenous and exogenous states respectively such that the law of motions are

$$dx_1 = \bar{f}(x_1, x_2, \alpha)dt \tag{30}$$

$$dx_2 = \bar{\mu}(x_2) + \bar{\sigma}(x_2)dW \tag{31}$$

then the corresponding HJB equation is

$$\begin{aligned}\rho v(x_1, x_2) = \max_\alpha r(x_1, x_2, \alpha) &+ v_1(x_1, x_2)\bar{f}(x_1, x_2, \alpha) + v_2(x_1, x_2)\bar{\mu}(x_2) \\ &+ \frac{1}{2} v_{22}(x_1, x_2)\bar{\sigma}^2(x_2)\end{aligned} \tag{32}$$

Examples 2.1 (Linear RBC Model with log utility)**.** Consider the real business cycle model with the following expected lifetime payoff

$$v(k_0, z_0) = \max_{\{c(t)\}_{t\geq 0}} \mathbf{E}_0 \int_0^\infty e^{-\rho t} u(c(t)) dt \tag{33}$$

subject to the following laws of motions:

$$dk = (zF(k) - \delta k - c)dt \tag{34}$$

$$dz = \bar{\mu}(z)dt + \bar{\sigma}(z)dW \tag{35}$$

Note that the evolution of k is endogeneous. This yields the HJB equation of

$$\begin{aligned} \rho v(k,z) = \max_c u(c) + v_k(k,z)[zF(k) - \delta k - c] + v_z(k,z)\mu(z) \\ + \frac{1}{2} v_{zz}(k,z)\sigma^2(z) \end{aligned} \tag{36}$$

Suppose that the utility function is a log-utility function $u(c) = 1/c$ and production is linear in k, i.e., $zF(k) = zk$.

We claim that the optimal consumption is $c = \rho k$; the 'proof' is done by guess and verify for v, $v(k,z) = \nu(z) + \kappa \log k$. The FOC $u'(c) = v_k(k,z)$ yields

$$c = \frac{k}{\kappa}$$

Substituting this to the HJB equation yields

$$\rho[v(z) + \kappa \log k] = \log k - \log \kappa + \frac{\kappa}{k}[zk - \delta k - k/\kappa] + \nu'(z)\mu(z) + \frac{1}{2}\nu''(z)\sigma^2(z) \tag{37}$$

Collect all the terms involving $\log k$: we have $\rho\kappa \log k = \log k$, i.e., $\rho\kappa = 1$ which implies that $c = k/\kappa = \rho k$.

We also impose reflecting barrier boundary conditions with respect to the stochastic $z \in [\underline{z}, \overline{z}]$, i.e.,

$$v'(k, \underline{z}) = 0 \tag{38}$$

$$v'(k, \overline{z}) = 0 \tag{39}$$

for all k. In FD scheme with P grids on z, this corresponds to

$$v_{i,0} = v_{i,1} \tag{40}$$

$$v_{i,P} = v_{i,P+1} \tag{41}$$

for all ith grid on k.

Examples 2.2 (Neoclassical growth model with stochastic k)**.** Consider the neoclassical growth model from Example 1.1, with a Brownian flavour added to the evolution of k. Now the model solves for the following expected lifetime payoff

$$v(k_0) = \max_{\{c(t)\}_{t\geq 0}} \mathbf{E}_0 \int_0^\infty e^{-\rho t} u(c(t)) dt \tag{42}$$

subject to the stochastic law of motions:

$$dk = f(k, c)dt + \sigma dW \tag{43}$$

where $f(k, c) := F(k) - \delta k - c$.

Then, the corresponding HJB equation is

$$\rho v(k, z) = \max_c u(c) + v'(k)[F(k) - \delta k - c] + \frac{\sigma^2}{2} v''(k) \tag{44}$$

Taking FOC, we have an identical relation as (7), i.e., $u'(c) = v'(k) \Leftrightarrow c = (u')^{-1}(v'(k))$.

We also impose reflecting barrier boundary conditions with respect to the stochastic $k \in [\underline{k}, \overline{k}]$, i.e.,

$$v'(\underline{k}) = 0 \tag{45}$$

$$v'(\overline{k}) = 0 \tag{46}$$

In FD scheme with P grids, this corresponds to

$$v_0 = v_1 \tag{47}$$

$$v_P = v_{P+1} \tag{48}$$

The discretized stochastic HJBE yields the system

$$\rho \mathbf{v} = \mathbf{u} + \left(\mathbf{A}(\mathbf{v}) + \frac{\sigma^2}{2} \mathbf{L}_2 \right) \mathbf{v} \tag{49}$$

Thus, the corresponding implicit Euler method is to solve for $\mathbf{v}^{n+1}$ in the following system at each nth step:

$$\begin{aligned} \frac{1}{\Delta}(\mathbf{v}^{n+1} - \mathbf{v}^n) + \rho \mathbf{v}^{n+1} &= \mathbf{u} + \left(\mathbf{A}_n + \frac{\sigma^2}{2} \mathbf{L}_2 \right) \mathbf{v}^{n+1} \\ \left[\left(\rho + \frac{1}{\Delta} \right) \mathbf{I}_I - \mathbf{A}_n - \frac{\sigma^2}{2} \mathbf{L}_2 \right] \mathbf{v}^{n+1} &= \mathbf{u} + \frac{1}{\Delta} \mathbf{v}^n \end{aligned} \tag{50}$$