Holger Sieg

Carnegie Mellon University

Econometrics III: Problem Set # 2
Single Agent Dynamic Discrete Choice

INSTRUCTIONS:

This problem set was originally created by Ron Goettler. The objective of this problem set is to estimate a machine replacement problem (similar to Rust 1987) using Rust's nested fixed-point algorithm, and using Hotz-Miller's conditional choice probability method. Please answer the following question and hand in the solution by Monday, Feb 25, 2008.

THE MODEL:

Firms each use one machine to produce output in each period. The machines age, becoming more likely to breakdown, and in each time period the firms have the option of replacing the machines. Let a_t be the age of the machine at time t and let the expected current period payoffs from using a machine of age a_t be given by:

$$\Pi(a_t, i_t, \epsilon_{0t}, \epsilon_{1t}) = \begin{cases} \theta_1 a_t + \epsilon_{0t} & \text{if } i_t = 0 \\ R + \epsilon_{1t} & \text{if } i_t = 1 \end{cases},$$

where $i_t = 1$ if the firm decides to replace the machine at t, R is the net cost of a new machine, and ϵ_t are time specific shocks to the payoffs from replacing and not replacing. Assume ϵ_t are i.i.d. logit errors (i.e., type 1 extreme value). This payoff function may be interpreted as a normalized profit function, or a cost function.

Assume a simple state evolution equation for age:

$$a_{t+1} = \begin{cases} \min\{5, a_t + 1\} & \text{if } i_t = 0 \\ 1 & \text{if } i_t = 1 \end{cases},$$

That is, if not replaced, the machine ages by one year, up to a maximum of 5 years; after 5 years the machine's age is fixed at 5 until replacement. If replaced in the current year, the machine's age next year is 1. Hence, a_t is one of five possible values: 1, 2, 3, 4, or 5.

Part A: Rust's Nested Fixed Point Algorithm

1. Write down the sequence problem for a firm maximizing the EDV of future payoffs (assume an infinite horizon).

2. Write down Bellman's equation (the functional equation) for the value function of the firm. Use the "alternative-specific" value functions method from lecture, i.e., define $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$, where $\bar{V}_0(a_t)$ is the firm's value at time t if it chooses not to replace the machine and $\bar{V}_1(a_t)$ is the firm's value at time t if it replaces the machine. The functional equation should be a mapping from this pair of equations to itself. (For the parts that follow, remember from class that there are (at least) two ways to write the functional equation, and that one of them will be easier to use than the other.)

3. On the computer, write a procedure that solves this dynamic programming problem given values of the parameters (θ_1, R). Assume $\beta = .9$. Your procedure should iterate the contraction mapping on the two alternative-specific value functions. Each of these functions is represented by a five element vector since there are five values of a. Iterate until the V functions do not change very much (say 10^{-6}). Remember that given the logit error assumption there is an analytic solution to the expectation of the max in these equations and that Euler's constant is approximately 0.5775. (You could also iterate on the "smooth value function" $\bar{V}(a_t)$ that is not alternative-specific, and then derive $\bar{V}_1(a_t)$, where $\bar{V}_0(a_t)$ from $\bar{V}(a_t)$ and Π.)

4. Solve the model for the parameters $(\theta_1 = -1, R = -3)$. Suppose $a_t = 2$. Will the firm replace its machine in period t? For what value of $\epsilon_{0t} - \epsilon_{1t}$ is the firm indifferent between replacing its machine or not? What is the probability (to an econometrician who doesn't observe the ϵ's) that this firm will replace its machine? What is the EDV of future payoffs for a firm at state $\{a_t = 4, \epsilon_{0t} = 1, \epsilon_{1t} = 1.5\}$? (Remember the constant term has been normalized out so this EDV could be negative.)

5. Download the data for this problem set from the course web page. The dataset has two columns: a_t and i_t. Consider this to be cross-sectional data (i.e., there is only one data point per firm). Estimate (θ_1, R) using maximum likelihood. The best way to do this is to write a function that returns $\ln(L)$, where L is the likelihood function

and then use a package (e.g. gauss or matlab) to maximize the this function (or minimize the negative log-likelihood). Also compute (asymptotic) standard errors. The function itself should follow these steps:

(a) Start with arbitrary (θ_1, R). That is, these should be the arguments of the function.

(b) Solve the dynamic programming problem given these parameters. That is, compute the functions $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$ using your procedure from part 3.

(c) Using $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$, compute the probability of replacement for each possible a_t.

(d) Compute the likelihood of each observation using the above probabilities. Form the log-likelihood function by summing the logs of the likelihoods across observations (i.e., assume that each data point is an independent observation).

(e) Return $\ln(L)$ for a maximization routine or $-\ln(L)$ for a minimization routine.

6. Describe (you do NOT have to do this on the computer) how you would need to change your model (either the dynamic programming problem, the estimation procedure, or both) to accommodate the following perturbations:

(a) Consider an alternative empirical model. Suppose firms differ in their value of θ_1. Proportion α of firms have $\theta_1 = \theta_{1A}$, proportion $(1-\alpha)$ have $\theta_1 = \theta_{1B}$. How would you change both the dynamic programming problem and the likelihood function?

(b) What if you had the model in (a) but you have panel data (i.e., multiple observations on each firm)? Write down the likelihood function.

(c) What if instead of firms differing in θ_1, machines differ in θ_1 (i.e., when a firm replaces an old machine, the new machine may have $\theta_1 = \theta_{1A}$ (with probability α) or it may have $\theta_1 = \theta_{1B}$ (with probability $1-\alpha$)? Now how would you need to change your program (go back to the original data with one observation per firm)?

Part B: Hotz-Miller Algorithm

1. Using the data, calculate the machine replacement probabilities at each state a_t using the average replacement rate in the sample ("cell averages"). These estimates are nonparametric estimates of the replacement probabilities. Also calculate the standard errors of your estimates (using the Bernoulli formula from first year statistics).

2. Define the normalized conditional value function as $\Psi_1(a_t) = \bar{V}_1(a_t) - \bar{V}_0(a_t)$. Write down the Hotz and Miller inversion formula for this model. That is, write down the formula for $\Psi_1(a_t)$ as a function of the conditional choice probabilities. Calculate the normalized conditional value functions using the choice probabilities you estimated above.

3. Write down current period utility (i.e., payoff) for each choice in terms of the parameters and the current state. Call this u_0 and u_1. Define $w_{kt}(a_t) = E\left[\epsilon_{kt} | a_t, d_{kt} = 1\right]$ for $k \in \{0, 1\}$. In the special case of the logit, $w_{kt} = \ln(p_{kt})$, where $\gamma = 0.5775$ is Euler's constant. You can work this out for yourself mathematically, but it is tedious.

   What is the formula for expected utility at time t conditional on reaching state a_t, as a function of current period utilities (u_0, u_1), the conditional choice probabilities, $Pr(i_{it} = 1|a_t)$, and w_{0t} and w_{1t}? Call the vector of expected utilities (a vector across states a_t) $\underline{\mathrm{u}}$.

4. Construct the two 5x5 conditional transition matrices F_0 and F_1 which give the transition probabilities conditional on the firm's $\{0, 1\}$ replacement choice. Write them down.

5. Using the estimated conditional choice probabilities and the F_0 and F_1 matrices above, calculate the 5x5 unconditional transition matrix of states (which takes into account the probability of replacement as well as the transition probabilities conditional on replacement). Call this matrix T and write it down.

6. Write down the formula for the conditional value functions, $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$, in terms of current period utilities (u_0, u_1), expected utility $(\underline{\mathrm{u}})$, the unconditional transition

probabilities T, the discount factor ($\beta = 0.9$), and the conditional transition probabilities F_0 and F_1. On the computer, write a procedure which takes a parameter vector (θ_1, R) as inputs and returns the conditional value functions using this formula.

7. Estimate (θ_1, R) using maximum likelihood. Again, write a function that returns $-\ln(L)$ if using a minimization routine, or $\ln(L)$ if using a maximization routine. Also compute standard errors. Do not bother adding correction for the fact that you are using a two-step procedure (the conditional choice probabilities were estimated nonparametrically in the first stage). The function itself should follow these steps:

   (a) Start with arbitrary (θ_1, R). That is, these should be the arguments of the function.

   (b) Use the procedure that you wrote above to calculate $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$.

   (c) Using $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$ compute the probability of replacement for each possible a_t.

   (d) Compute the likelihood of each observation using the above probabilities. Form the log likelihood function by summing the logs of the likelihoods across observations (i.e., assume that each data point is an independent observation).

   (e) Return $\ln(L)$ for a maximization routine or $-\ln(L)$ for a minimization routine.

8. Compare your results here in part B with those of Part A.