## More algebraic spaces

Algebraic spaces have dense open subschemes

Set of alg space : set of $K$-pts $\forall$ fields $K$

$A$ alg space.

$|A| :=$ underlying set

$V \subset |A|$ is closed if it is the set of pts of a closed alg. subspace

An algebraic space is a quotient of a scheme by an étale equivalence

$S =$ base scheme , $U =$ étale atlas

$$\begin{tikzcd} U\times_A U \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} U\times_A U \arrow[d] \\ U\times_S U \end{tikzcd} \qquad \begin{tikzcd} U\times_A U \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\text{étale}"] & U \arrow[r] & A \\ U \end{tikzcd}$$

**Stein's Factorization Thm** Any étale map factors as an open immersion followed by a finite étale map.

$$\begin{tikzcd} X \arrow[r, "\text{ét}"] \arrow[d, hook, "\text{open immersion}"'] & Y \\ Z \arrow[ur, "\text{finite étale}"'] \end{tikzcd}$$

Want Quotients of schemes by finite étale equivalence relations exist (as schemes)

Prove by reducing to affine case.

Example Equivalence relation comes from a finite étale group $G \curvearrowright \operatorname{spec} A$

$$\begin{tikzcd} \operatorname{spec} A \arrow[d] \\ \operatorname{Spec} A^G \end{tikzcd}$$

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$A \rightrightarrows A_1$ $A_0 =$ equalizer (where the maps agree)

Take spec $A_0$

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$X$ alg. space

$Et(X)$ = small étale site

Objects are algebraic spaces étale over $X$

Morphisms are $X$ morphisms

$Et'(X)$ objects are schemes étale over $X$

$Et'(X) \subset Et(X)$ induces an isomorphism of topoi

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What is $\mathcal{O}_X$?

From above, we only need to define sheaves over the schemes in $Et'(X)$.

if $Y \to X$ is a scheme then the $Y$-valued points of $\mathcal{O}_X$ are $\Gamma(\mathcal{O}_Y, Y_{zar})$ gives a ring sheaf on $X$.

- Sheaves of $\mathcal{O}_X$ modules make sense.

$$U \times_X U \rightrightarrows U \to X$$

Sheaves on $X$ are the same as "equivariant sheaves on $U$"

Equivariant sheaves. $U \times U \overset{p_1}{\underset{p_2}{\rightrightarrows}} U$

$(\mathcal{F}, \varphi)$ $\mathcal{F}$ a sheaf on $U$

$\varphi : p_1^* \mathcal{F} \xrightarrow{\sim} p_2^* \mathcal{F}$

$$\begin{array}{c} U \times_X U \times_X U \\ \downarrow\downarrow\downarrow \\ U \times_X U \end{array}$$

Pullback Satisfies Cocycle Condition

(Quasi) coherent sheaves can be defined. They agree with the usual definitions on schemes.

Sheaves basically behave the same on alg. spaces or schemes.

<u>Representability</u> "effectivity of deformations" <u>Contraction</u>

Deformation theory of your sheaf is nice $\Rightarrow$ representable by an alg. space.
[moduli spaces of canonically polarized varieties]

If you can locally contract a subvariety then you can globally contract it to get an algebraic space.
<u>Contraction</u> surjective, proper birational - morphism.

<u>Representability</u>.

$F$ étale sheaf.

$F(K)$ "$K$ pts of algebraic space"

Fix some $x \in F(K)$. We want an étale neighborhood of $x$.

<u>$1^{st}$</u> Get a formal neighborhood

$T_x F = F(k[\varepsilon]/\varepsilon^2)$

We can define all higher order neighborhoods using Artin rings.
Using hypothesis on $F$, I get a formal neighborhood of $x$ in $F$.

<u>$2^{nd}$</u> Formal neighborhoods descend to an étale neighborhood.

$F(k[[x]])$ "=" compatible elements of $F(k[x]/x^n)$ $\forall n$

<u>Contraction</u>

$$\begin{array}{ccc} Y & \subset & X \\ \downarrow & & \downarrow \\ Y' & \subset & X' \end{array}$$