Lecture # 4 2/26.

S (Spec $\mathbb{C}$) finite type over a field

$$\begin{tikzcd} Sch_S \arrow[r] \arrow[dr, bend right] & Fun(Sch_S^{op}, sets) \\ & Fun(Aff_S^{op}, sets) \end{tikzcd}$$

Q what is the image?

$F: Sch_S^{op} \to Sets$ is represented by a separated scheme iff

(1) • F is a (big Zariski) Sheaf

(2) • $\Delta: F \to F \times F$ is a closed immersion.

(3) • $U_i \to F$ open immersion
Schemes $\coprod U_i \to F$ is surjective

Ex $Gr(K, d)$ is a scheme

Ex $Hilb_{X/S}$ (1), (2) is easy but (3) is hard.

Rewrite (1), (2) and (3) in the big Étale topology

(1) F is a big étale sheaf

(2) $\Delta: F \to F \times F$ is ~~a closed immersion~~ represented by schemes

(3) $\underset{\substack{\uparrow \\ \text{scheme}}}{U} \to F$ is surjective étale.

Recall
$$\begin{tikzcd} U_i \times_F U_j \arrow[r] \arrow[d] & U_i \arrow[d] \\ U_j \arrow[r] & F \end{tikzcd}$$
if U_i, U_j affine we want the intersection of U_i and U_j to be affine
$$\begin{tikzcd} \overset{\text{affine}}{U_i \times_F U_j} \arrow[r, hook] \arrow[d] & \overset{\text{affine}}{U_i \times U_j} \arrow[d] \\ F \arrow[r, hook, "\Delta"] & F \times F \end{tikzcd}$$

$$\begin{tikzcd} T\times_G F \arrow[r] \arrow[d] & F \arrow[d] \\ T \arrow[r] & G \end{tikzcd}$$

$\begin{matrix} F \\ \downarrow \\ G \end{matrix}$ is represented by schemes if for any scheme T, the fiber product $T\times_G F$ is a scheme.

Q: Is this a scheme?

A: No. It is an algebraic space.

Def F, G presheaves, $F \to G$ is étale/surjective if

(1) $F \to G$ represented by schemes

(2) For

$$\begin{tikzcd} F\times_G T \arrow[r] \arrow[d] & F \arrow[d] \\ T \arrow[r] & G \end{tikzcd}$$

[if T is a scheme $\Rightarrow$ (by (1)) $F\times_G T$ is a scheme

then $\begin{matrix} F\times_G T \\ \downarrow \\ T \end{matrix}$ is étale/surjective.

If P is a property of morphisms that is preserved under base change and étale local on target then P is a property of schemes étale local.

What is not preserved:

- projective
- affine
- quasicompact
- separated
- proper (?) Olson says yes, Artin says no

Ex. scheme

$$G \circlearrowright X \text{ scheme} \quad \text{then} \quad X/G \text{ is an algebraic space}$$

free action

discrete group

Take the presheaf $T \mapsto X(T)/G$ and sheafify.

example take $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda \subseteq \mathbb{C}$

$\mathbb{C}/\Lambda$ is an algebraic space

E (elliptic curves) is a scheme but $\mathbb{C}/\Lambda \not\cong E$ as algebraic spaces.

not separated

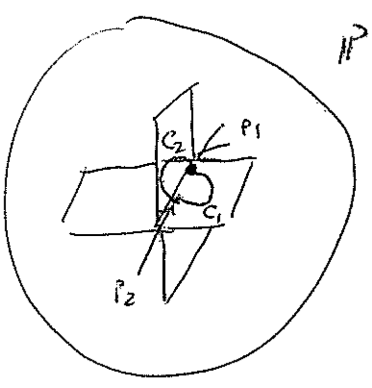
$$(\mathbb{C}/\Lambda)^{an} \cong E^{an}$$

Q: Why is $\mathbb{C}/\Lambda$ not separated?

$$\begin{tikzcd} \mathbb{C}\times\Lambda \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C}/\Lambda \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\times\Lambda \arrow[r] \arrow[d] & \mathbb{C}\times\mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda \arrow[r, "\Delta"] & \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda \end{tikzcd}$$

$\mathbb{C}\times\Lambda$ should be a closed subscheme

Hironaka: Z - 3 fold that is proper, smooth but not projective.



$\mathbb{P}^3$

$C_1, C_2 \subseteq \mathbb{P}^3$

$C_1 \cap C_2 = \{P_1, P_2\}$

$Bl_{C_1}\mathbb{P}^3 \supset \tilde{C}_2$

$$\begin{tikzcd} Bl_{\tilde{C}_2} Bl_{C_1}\mathbb{P}^3 \arrow[d] & \pi^{-1}(P_1) \arrow[l, hook'] \arrow[dl] \\ \mathbb{P}^3 \ni P_1 & \end{tikzcd}$$

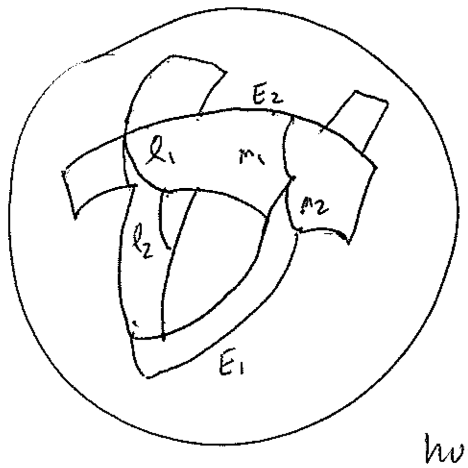
$U_1 = Bl_{\tilde{C}_2} Bl_{C_1}\mathbb{P}^3 \setminus \pi^{-1}(P_1)$

$U_2 = Bl_{\tilde{C}_1} Bl_{C_2}\mathbb{P}^3 \setminus \pi^{-1}(P_2)$

$\pi^{-1}(\mathbb{P}^3 \setminus (P_1 \cup P_2)) \subseteq U_1, \subseteq U_2$

$Z = U_1 \cup_{\pi^{-1}(-)} U_2$

Z



$l_1 + l_2 \sim m_2$

$m_1 + m_2 \sim l_1$

?

$l_1 + l_2 + m_1$

$\Rightarrow l_2 + m_1 \sim 0$ X

not projective, if it were we could find some non-zero hyperplane section.

$$\begin{array}{c}\sigma \\ \circlearrowright\end{array} \quad C_1, C_2 \subset \mathbb{P}^3 \qquad \sigma^2 = 1$$

$C_1 \xleftrightarrow{\sigma} C_2$

$P_1 \xleftrightarrow{\sigma} P_2$ $\rightsquigarrow \sigma \curvearrowright Z \quad l_2 \xleftrightarrow{\sigma} m_1$

Z/σ is not a scheme!

$\bar{l} = \mathrm{Im}(l_2) = \mathrm{im}(m_1)$

irreducible

Let $D \subset Z/\sigma$ be a divisor s.t. $D \cap \bar{l}$ properly

$\bar{D} \subset Z/\sigma$

$\bar{D} \cap (l_2 + m_1)$ properly

$\bar{D}\,(\underset{\substack{\| \\ 0}}{l_2 + m_1}) > 0$

Claim Hilb_Z is not a scheme

Suppose it was:

$$\left\{ \begin{array}{c} [\Gamma \subset Z] \\ \dim \Gamma = 0 \\ \ell(\Gamma) = 2 \\ \text{length} \quad \sigma(\Gamma) = \Gamma \end{array} \right\} \subseteq \mathrm{Hilb}_Z$$

would also be represented by schemes

But this space is Z/σ.

But :

Thm (Artin) X finite type $/S$ then $\mathrm{Hilb}_{X/S}$ is an algebraic space