Grothendieck Topologies + Sites

Def Let C be a category. A Grothendieck (pre) topology on C is a collection of sets of morphisms $\{U_i \to X\}_{i \in I}$ for each $X \in Ob(C)$ s.t. "coverings of X"

- $\forall X$ $\{X \xrightarrow{id} X\}$ is a covering ($\forall$ $\{Y \xrightarrow{\sim} X\}$)
- (locality) if $\{U_i \to X\}_{i \in I}$ and $\{V_{ij} \to U_i\}_{j \in J_i}$ are coverings then so is $\{V_{ij} \to U_i \to X\}_{ij}$
- (base change) for any $Y \to X$ in C and any covering $\{U_i \to X\}_{i \in I}$ all the pullbacks exist:

$$\begin{array}{ccc} V_i & \to & U_i \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

Moreover $\{V_i \to Y\}_{i \in I}$ is a covering of Y

Def Presheaf Functor $C^{op} \to set$. It is separated if for every covering $\{U_i \to X\}_{i \in I}$, $F(X) \to \prod_{i \in I} F(U_i)$ is injective

A presheaf is a sheaf if $F(X) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_X U_j)$ is an equalizer diagram. (going through each arrow gives you the same thing)

Def Site category with Grothendieck Topology

Fix a site C, then we have:

There are adjoints

Presheaves on C

$\downarrow \to \uparrow$ forget

separated presheaves on C

$\downarrow \to \uparrow$ forget

sheaves on C $\leftarrow$ sheafification

Ex On any category C

indiscrete topology: only $\{X \xrightarrow{\sim} Y\}$ are coverings. All presheaves are sheaves.

(2) Canonical topology

A presheaf is a sheaf $\Leftrightarrow$ representable ie $Hom(-, X)$ for some $X \in C$.

Ex A topology is subcanonical if it's coarser than the canonical topology. i.e. all representable functors are sheaves.

Any topological space T gives a category of open subsets of T, with topology $\{U_i \to X\}_{i\in I}$ is covering iff $\cup U_i = X$ (subcanonical)

Fix a scheme S. Small Zariski site

$$\mathcal{C} = \left\{ \begin{array}{c} X \\ \downarrow \\ S \end{array} \text{ morphisms } \cancel{\text{open embedding}} \text{ of schemes} \right\} \quad \mathrm{Hom}_{\mathcal{C}}\left(\begin{smallmatrix} X \\ \downarrow \\ S\end{smallmatrix}, \begin{smallmatrix} Y \\ \downarrow \\ S\end{smallmatrix}\right) = \text{open embeddings over } S. \quad X \hookrightarrow Y$$

$$\begin{tikzcd} \{U_i \arrow[r] \arrow[dr] & X\}_{i\in I} \arrow[d] \\ & S \end{tikzcd}$$ is cover iff $\cup U_i = X$

Big Zariski site $\mathcal{C} = \mathrm{Sch}/S$ $\mathrm{Hom}_{\mathcal{C}}\left(\begin{smallmatrix} X \\ \downarrow \\ S\end{smallmatrix}, \begin{smallmatrix} Y \\ \downarrow \\ S\end{smallmatrix}\right) = \{$arbitrary morphisms of schemes over $S\}$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

$$\begin{tikzcd} \{U_i \arrow[r] \arrow[dr] & X\} \arrow[d] \\ & S \end{tikzcd}$$ is covering iff each $U_i \to X$ is an open embedding and $\coprod_{i\in I} U_i \to X$ is surjective

Big/small Étale sites - replace open embedding by étale morphism

Def An algebraic space is a sheaf $\mathcal{F}$ on the big étale site s.t. $\mathcal{F}$ is the sheafification of the presheaf $T \to X(T)/R(T)$ for some étale equivalence relation $R \hookrightarrow X \times X$

(i.e. R a scheme, $R \hookrightarrow X \times X$ closed immersion such that each projection $R \to X$ is étale and for every scheme T, $R(T) \subset X(T) \times X(T)$ is an equivalence relation on the set $X(T)$.)

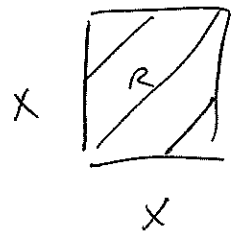
Example Let $X = \mathrm{Spec}\, \mathbb{C}[t, t^{-1}] = Y$. Fix $n > 1$ $n \in \mathbb{Z}$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \arrow[d, "\ell", "\text{raise to } n\text{th power}"'] \\ X \arrow[r, "\ell"] & Y \end{tikzcd}$$

$R = X \times_Y X \Rightarrow R \to X \times X$ is an étale equivalence relation

Example continued.

The sheafification of $T \mapsto X(T)/R(T)$ in large étale topology is just $Hom_{\mathcal{C}}(-, Y)$ i.e. is "Y" by abuse of notation.



Fibred Categories.

Def A *category over $\mathcal{C}$* is just a category $\mathcal{F}$ $\downarrow$ functor $P_{\mathcal{F}}$. $\mathcal{C}$

in $\mathcal{F} \xrightarrow{P_{\mathcal{F}}} \mathcal{C}$ we say a morphism in $\mathcal{F}$, $Y \to Z$ is *cartesian* if whenever

$$\begin{tikzcd} X \arrow[r, "f", "\exists!"'] \arrow[rr, bend left] & Y \arrow[r] & Z \end{tikzcd} \quad (1)$$

$$\begin{tikzcd} P_{\mathcal{F}}(X) \arrow[r] \arrow[rr, bend left] & P_{\mathcal{F}}(Y) \arrow[r] & P_{\mathcal{F}}(Z) \end{tikzcd}$$

$\exists!$ f s.t.
(1) Triangle (1) commutes.
(2) $P_{\mathcal{F}}(f) = g$.

Def A ~~fibred~~ *fibered category over $\mathcal{C}$* is a category over $\mathcal{C}$ s.t. every morphism downstairs admits a pullback, i.e. a cartesian morphism in $\mathcal{F}$ that maps to it under $P_{\mathcal{F}}$.

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\exists", "\text{cartesian}"'] & Z \\ \bullet \arrow[r] & P_{\mathcal{F}}(Z) \end{tikzcd}$$

Fibered categories over $\mathcal{C}$ form a 2-category.

Objects = Fibered categories over $\mathcal{C}$.

1-Morphisms
$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\Pi"] \arrow[dr, "P_{\mathcal{F}}"'] \arrow[r, phantom, "2", shift right=3] & & \mathcal{G} \arrow[dl, "P_{\mathcal{G}}"] \\ & \mathcal{C} & \end{tikzcd}$$
Π preserves cartesian morphisms

2-morphisms:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, bend left, "\pi_1"] \arrow[rr, bend right, "\pi_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] \arrow[dr, "p_{\mathcal{F}}"'] & & \mathcal{G} \arrow[dl, "p_{\mathcal{G}}"] \\ & \mathcal{C} & \end{tikzcd}$$

α is nat'l transformation $\pi_1 \Rightarrow \pi_2$ s.t. for any $X \in Ob(\mathcal{F})$

$\alpha_X : \pi_1(X) \to \pi_2(X)$ is a morphism in $\mathcal{G}$ that $p_{\mathcal{G}}$ sends to $id_{p_{\mathcal{F}}(X)}$.

Example Comma category Fix $Y \in Ob\,\mathcal{C}$

$\mathcal{C}/Y$ = Category of all $\begin{matrix} X \\ \downarrow \\ Y \end{matrix}$ $\qquad \mathcal{C}/Y \xrightarrow{p} \mathcal{C}$ is a fibered category

$p \downarrow \quad p\left(\begin{matrix} X \\ \downarrow \\ Y \end{matrix}\right) = X$

$\mathcal{C}$

2-Yoneda Lemma. For any other fibered category $\mathcal{F} \to \mathcal{C}$ there's an equivalence of categories $Hom_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) \to \mathcal{F}(X)$

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Category whose objects are 1-morphisms