Constructing $\overline{\mathcal{M}}_g$

4/30

S scheme, k alg closed, $g \geq 2$

Def A prestable curve over S is a proper flat morphism $\pi: C \to S$ s.t. $\forall$ points $\bar{s} \to S$, $C_{\bar{s}}$ is a connected nodal curve

π is stable if for all components C_i of $C_{\bar{s}}$, $2(g(C_i)) - 2 + \#\text{ nodes} > 0$

Def Let $\overline{\mathcal{M}}_g$ be the fibered category:

Objects $(S, f: C \to S)$ S scheme, $f: C \to S$ stable curve of genus g

Morphisms $(S', f': C' \to S') \to (S, f)$ are cartesian diagrams

$$\begin{tikzcd} C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Thm $\overline{\mathcal{M}}_g$ is a Deligne-Mumford Stack

Step 1 $\overline{\mathcal{M}}_g$ is a stack.

prop Let $\pi: C \to S$ a family of stable curves. Then $\omega_{C/S}^{\otimes 3}$ is relatively very ample.

Prop 4.4.12 + $\omega_{C/S}^{\otimes 3}$ very ample $\Rightarrow$ satisfies descent.
(embed objects in a Hilbert scheme, which satisfies descent)

Step 2 $\overline{\mathcal{M}}_g$ is an algebraic stack.

Representable diagonal $\quad ? = \mathrm{Isom}_S(C, C')$

we need alg space

$$\begin{tikzcd} S(?) \arrow[r, shift left] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[l, shift left] \arrow[d, "\left(\begin{smallmatrix} C & C' \\ \downarrow & \downarrow \\ S, & S \end{smallmatrix}\right)"] \\ \overline{\mathcal{M}}_g \arrow[r, "\Delta"] & \overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_g \end{tikzcd}$$

So we want to show: $\mathrm{Isom}_S(C, C')$ is representable.

$$\begin{array}{l} H \\ \downarrow \\ S \end{array} \quad \text{Hilbert scheme of } C \times_S C' \text{ containing isomorphisms}$$

Isomorphisms $\sigma : C_{S'} \to C'_{S'} \longleftrightarrow \Gamma \subseteq (C \times_S C')_{S'}$

s.t. $\Gamma \xrightarrow{\sim} C_{S'}$, $\Gamma \xrightarrow{\sim} C'_{S'}$ $\Big\}$ open condition

Therefore $\exists\ \underset{\text{scheme}}{H^\circ} \underset{\text{open}}{\subseteq} H$ s.t. $H^\circ \cong \mathrm{Isom}_S(C, C')$

$\Rightarrow$ Diagonal is representable

Step 3. $\overline{\mathcal{M}}_g$ admits a smooth cover

If C is smooth, then $\omega_C^{\otimes 3}$ has degree $3(2g-2)$. $\Rightarrow$ $h^0(\omega_C^{\otimes 3}) = 6g-6-g+1 = 5g-5$.

Look at $\mathcal{H}'$ = Hilbert scheme of genus g curves in $\mathbb{P}^{5g-6}$

$$\begin{tikzcd} \mathcal{S} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathcal{W} \arrow[r, phantom, "\underset{\text{open}}{\subseteq}"] & \mathcal{H}' \end{tikzcd}$$

stable

$\mathcal{W} \to \overline{\mathcal{M}}_g$ smooth surjection.

Fiber over C is a choice of

$\mathbb{P}^{5g-5} \to \mathbb{P}\,\omega^{\otimes 3}$