UCLA Number Theory Seminar 11/23

Calabi-Yau varieties and Shimura varieties.

Defn. X sm. proj. variety is Calabi-Yau (CY)

if $\underbrace{\Omega^d_X}_{\text{canonical bdle}} \cong \mathcal{O}_X$, where $d = \dim X$.

examples. Elliptic curves / AV's

K3's ...

more "generic": quintic 3-fold $\subseteq \mathbb{P}^4$

Recent (last few decades) interest comes from physics, e.g. mirror symmetry: curve counting, derived categories, ...

What about <u>arithmetic</u>?

Attractor Mechanism.

(Moore, Brunner - Rogenkamp)

Defn. X is an attractor if

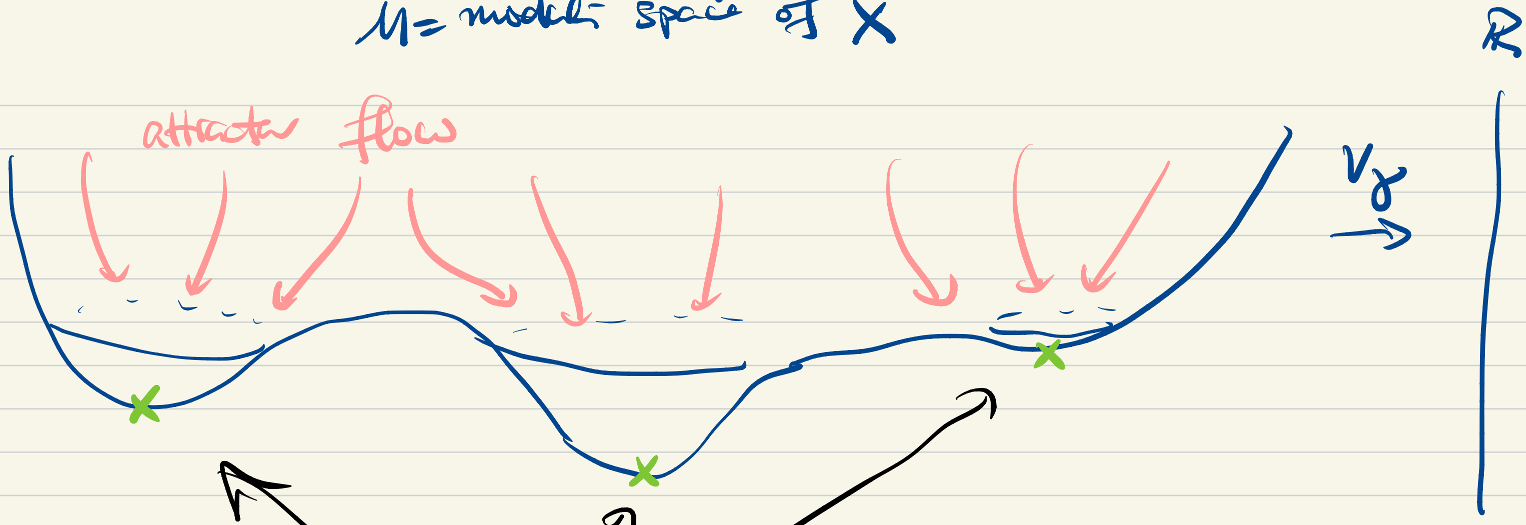
$\exists\ \gamma \in H^d(X, \mathbb{Q})$ s.t. $\gamma \perp \underbrace{H^{d-1,1}}_{\text{(n-1, 1) - piece of Hodge decomp.}}$

Recall. $H^d(X, \mathbb{C}) = \overset{\text{1-dimensional}}{H^{d,0}} \oplus H^{d-1,1} \oplus \cdots$

dim of the "moduli space" of X.

$\therefore$ picks out discrete points.

$\widetilde{M}$ = univ. cover of moduli space of X



attractor points for the class γ.

$$\left(V_\gamma = \frac{|\langle \gamma, \Omega \rangle|^2}{\int_X \Omega \wedge \bar{\Omega}} \right)$$

Conj (Moore, '98) If X is an attractor then it's defined $/ \bar{\mathbb{Q}}$

(i.e. has model $/\bar{\mathbb{Q}}$).

Moore verified this in several cases, e.g. Abelian 3-folds, K3 × Elliptic curve, finite quotients of such ...

__Observe.__ Can make same conjecture for Calabi-Yau variation of Hodge structures (CYVHS), i.e. VHS with $h^{d,0}=1$. (wt d).

__Where do these come from?__

__Gross:__ canonical CYVHS on many __Shimura varieties__ "symmetric spaces"

__Thm__ (L) for all weight 3 CYVHS in Gross' list, attractor pts are CM.

e.g. obtain simple parametrization of CM points on E_6, E_7 Shimura varieties, analogues of "Shioda-Inose K3's".

Thm (L-Tripathy) $\exists$ infinite family of CY's in odd dim $\neq 1,3,5,9$ for which Attractor Conj fails. In $d = 1,3,5,9$ case, the Conj holds and the moduli spaces are Shimura varieties.

Pf sketch. "Dolgachev CY's"

$\mathbb{P}^1 \ni x \longmapsto \{x \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1\} \subseteq \mathrm{Sym}^{2n-3}\mathbb{P}^1 = \mathbb{P}^{2n-3}$

Consider

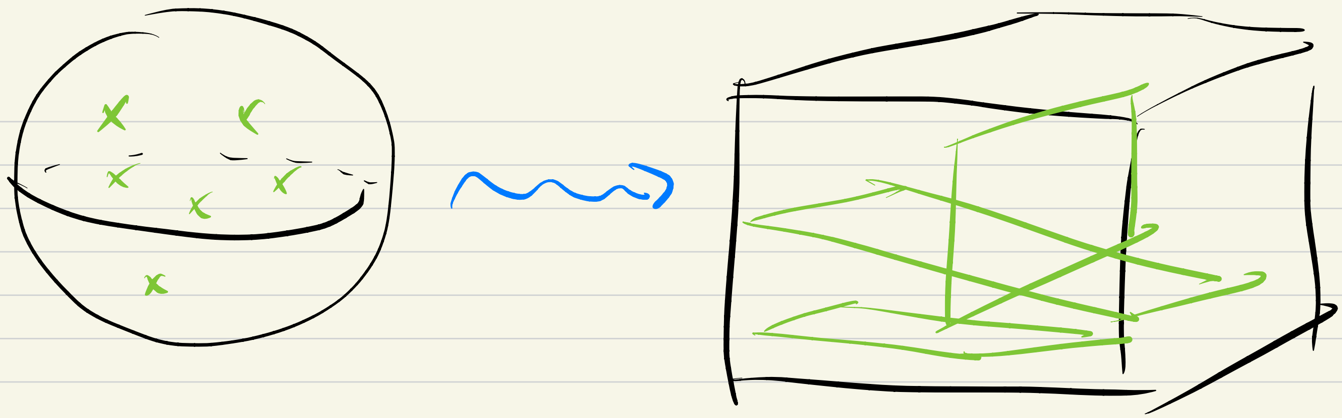
$$M_{0,2n} := \left\{ \begin{array}{c} 2n \text{ points} \\ \text{on } \mathbb{P}^1 \end{array} \right\}_{(x_i)_{i=1,\dots,2n}} \longrightarrow \left\{ \begin{array}{c} 2n \text{ hyperplanes} \\ \text{in } \mathbb{P}^{2n-3} \end{array} \right\}_{(H_i)_{i=1,\dots,2n}} := \mathcal{H}$$

$C =$ cyclic n-fold cover of $\mathbb{P}^1$ branched at x_i's

$X' =$ cyclic n-fold cover of $\mathbb{P}^{2n-3}$ branched at H_i's

Thm (Sheng-Xu-Zuo) (i) X' admits crepant resolution X, obtaining max'l family of CY $(2n-3)$-folds.

(ii) Hodge structure of $X^\bullet$ closely related to that of C. $\square$



From construction, have

$$\begin{tikzcd} x \in \mathcal{H} \arrow[rr] \arrow[dr] & & \mathcal{A}_g \\ & P(x) \in Sh \arrow[ur] & \end{tikzcd} \qquad (g = n^2)$$

Sh: a particular unitary Shimura variety.

<u>Key lemma</u>. Attractor condition + algebraicity $\Rightarrow$

$$P(x) \sim A_1 \times A_2$$

isogenous.

A_1: CM by $\mathbb{Q}(\zeta)$, ζ primitive n^{th} root of 1.

$\therefore$ assuming Attractor conj, such $x \in P(\mathcal{H}) \cap$ countable union of special Shimura subvarieties.

Conj. (Zilber - Pink) if $V \subset Sh$ s.t.

$$\{ V \cap \bigcup_{\Sigma} Sh' \} \text{ dense in } V,$$

then $V \subset$ proper Shimura subvar. of Sh.

set of Sh. subvar. of codim > codim V

(unlikely intersection)

Quick aside on Shimura varieties

e.g. moduli of abelian varieties with extra structures.

Has lots of symmetries ...

For us: has beautiful collection of special subvarieties, e.g. CM points, AV's w/ extra endom's, etc.

Zilber - Pink: "unlikely intersections" are all explained by such special subvarieties.

Back to proof. (Assume Attractor Conj)

Have $\mathcal{H} \xrightarrow{P} Sh$, and lemma $\Rightarrow$

all attractor points lie on

$$(\star) \qquad \mathcal{H} \cap \bigcup_{\Sigma} Sh' \subset \mathcal{H}.$$

locus where $P(x)$ splits up to isog.

Prop. (i) att. pts dense on $\mathcal{H}$

(ii) the intersections in $(\star)$ are unlikely. (except for $d=1,3,5,9$)

(iii) $\mathcal{H}$ Hodge generic, i.e. not contained in proper Shimura subvariety.

This proposition finishes our proof. $\square$

Other differences between CY's w/ and w/o Shimura moduli

(1) Shimura moduli $\Longleftrightarrow$ dense set of CM points

$\Longleftrightarrow$ $\exists$ notion of "isogenies" between CY's

$\Longleftrightarrow$ mirror has no non-trivial GW invariants.

Concretely, can one prove for explicit families of generic CY3's (e.g. mirror quintic) that no isogenies should exist (i.e. Weil #'s different)?

Conclusion & Further Questions

* These results suggest that the Attractor Conjecture holds iff the moduli space is a Shimura variety.

<u>Note.</u> physically significant!
"mathematically" $\Longleftrightarrow$ mirror of X has no non-trivial Gromov-Witten invariants.

* <u>Expect</u>:

de Jong?
(Klingler?)

Shimura moduli $\Longleftrightarrow$ CM points dense in moduli space
(analogue of Coleman's Conj.)

$\Longleftrightarrow$ $\exists$ good notion of <u>isogenies</u>

<u>Concrete Question</u>. Can one prove for explicit families that no isogenies exist? e.g. by proving Weil #'s occur fin'ly many times on moduli space (e.g. mirror Quintic)?