UCLA, May.3.2021. CM points on compact special orthogonal groups and theta lifts mod p

§1. Introduction

Ergodic theory may have interesting application in algebraic number theory.
An example: let ℓ be an odd prime, for $\beta\in\mathbb{Z}_\ell$, with ℓ-adic expansion

$$\beta= \beta_0+\beta_1\ell+\beta_2\ell^2+\cdots \quad (\beta_i\in[0,\ell-1]),\ \text{put}\ x_n(\beta):=\frac{\beta_0+\cdots+\beta_{n-1}\ell^{n-1}}{\ell^n}\in[0,1].$$

Then one has

Theorem 0. Suppose $\gamma_1,\dots,\gamma_t\in\mathbb{Z}_\ell$ $(t\geq 1)$ linearly independent over $\mathbb{Z}$. Then for almost all $\beta\in\mathbb{Z}_\ell$, the sequence of vectors $X_n(\beta):=(x_n(\beta\gamma_1),\dots,x_n(\beta\gamma_t))\in[0,1]^t$ is uniformally distributed.

$$T^\gamma=\sum_{n=0}^{\infty}(T-1)^n\binom{\gamma}{n}.$$

($\overset{\text{Sinnott}}{\iff}$ the powers series $T^{\gamma_1},\dots,T^{\gamma_t}$ algebraically independent in $\mathbb{F}_\ell[[T-1]]$)

$\Rightarrow$ the completed torus $\widehat{\mathbb{G}}_{m,\mathbb{F}_\ell}=\mathrm{Spf}(\mathbb{F}_\ell[[T-1]])$ is (Zar) dense in $\mathrm{Spec}(\mathbb{F}_\ell[T^{\gamma_1},\dots,T^{\gamma_t}])$.

From this we deduce

Theorem (Ferrero-Washington) $\mathrm{ord}_\ell\, h(\mathbb{Q}(\zeta_{\ell^n}))=\lambda n+\nu+\underset{(\mu=0)}{\underline{0}}\cdot\ell^n$ (for $n>>1$)
(with integers λ,ν)

We want to find similar results for other groups/objects.

§2. Main Result.

Fix a quadratic space (W,Q) with $W=\mathbb{Z}^{2n}$ $(n\geq 2)$ and $Q=\mathrm{diag}(\delta_1,\dots,\delta_{2n})$, $\delta_1,\dots,\delta_{2n}$ positive integers. Write $G=SO(Q)/\mathbb{Q}$. Fix a maximal torus $H\subset G$ consisting of $\mathrm{diag}(g_1,\dots,g_n)$ with g_i of size 2×2. We assume H, G are split at ℓ. (condition on H may be relaxed).

We will study the following objects and maps among them:

$$\begin{tikzcd}
CM := H(\mathbb{Q})\backslash H(\mathbb{A}_f) \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_0"'] & X := G(\mathbb{Q})\backslash G(\mathbb{A}_f) \arrow[d, "\varphi_2"] \\
 & Z := G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\tilde{G}^{der}(\mathbb{A}_f)
\end{tikzcd}$$

with $\tilde{G}^{der}(\mathbb{A}_f):=G(\mathbb{A}_f^{\ell})\times G(\mathbb{Q}_\ell)^{der}$. $H(\mathbb{A}_f)$ acts on these objects on the left and $G(\mathbb{A}_f)$ acts on the right. Moreover X is cpt, Z is cpt and abelian group.

Next consider $\underset{\text{cpt, open}}{K\subset G(\mathbb{A}_f)}$, $\underset{\text{cpt, open}}{U\subset H(\mathbb{A}_f)/H(\mathbb{Q})}$ and finite subset $R\subset H(\mathbb{Q})\backslash H(\mathbb{A}_f)$ $\left(\begin{array}{c}\text{elements distinct mod}\\ H(\mathbb{Q})Z_G(\mathbb{Q}_\ell)\end{array}\right)$

Then we put

$$x\longmapsto (\varphi_1(\gamma x))_{\gamma\in R}$$

$$\begin{tikzcd}
CM_K = CM/K \arrow[r, "\bar{\varphi}_1"] \arrow[dr, "\bar{\varphi}_0"'] & X_K^R=(X/K)^R \arrow[d, "\bar{\varphi}_2"] \\
 & Z_K^R=(Z/K)^R
\end{tikzcd}$$

Note that X_K^R, Z_K^R finite sets with discrete top.

Like in Sinnott's theorem on the density of torus orbits, we have

Theorem 1 (Z.)

Let $\mathcal{L} \subset CM$ be a $G(\mathbb{Q}_\ell)$-orbit and $\bar{\mathcal{L}} \subset CM_K$ its image. Then for all but finitely many $\bar{x} \in \bar{\mathcal{L}}$, we have $\bar{\varphi}_1(U\bar{x}) = \bar{\varphi}_2^{-1}(U\bar{\varphi}_0(\bar{x}))$ $(\subset X_K^R)$

Intuitively ($\#R=1$) the fibre of $\bar{\varphi}_2$ is a $\tilde{G}^{der}(\mathbb{A}_f)$-orbit and the above result says that the U-orbits $\bar{\varphi}_1(U\bar{x})$ is as large as possible (for a.h.f.m $\bar{x} \in \bar{\mathcal{L}}$)

§3. Application

The above result generalizes a theorem of C. Cornut and V. Vatsal (2005) where they treat the case $(G = B^\times,\ H = K^\times)$ for B quaternion algebra with centre $F/\mathbb{Q}$ t.r.
K/F quadratic t.i. with $K \hookrightarrow B$

They don't need condition that B is definite at ∞. However their proof still relies on the case that B is definite.

<u>Consequences of Cornut-Vatsal</u>

① Mazur's conjecture on non-torsionness of ~~(Heegner)~~ Heegner points (Cornut, Vatsal)

② non-vanishing of Rankin-Selberg L-value using Gross-Zagier formula (Cornut-Vatsal)

③ non-vanishing mod p of Yoshida lifts (Hsieh-Namikawa)

Ferrero-Washington $\rightsquigarrow$ Cornut-Vatsal
$\Downarrow$
H. Hida on μ-invariant of Hecke L-functions

Theorem 1 motivated from our generalisation of ③

<u>Some notations</u>: a symplectic space (W', Q') with $\begin{cases} W' = \mathbb{Z}^{2n} = W'_+ \oplus W'_- & \text{maximal isotropic submodules.} \\ Q' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \end{cases}$

Write $G' = Sp(Q')/\mathbb{Q}$. So $G(\mathbb{A}) \times G'(\mathbb{A})$ acts via Weil representation on the space $S(W \otimes W'_+(\mathbb{A}))$ of Bruhat-Schwartz functions on $W \otimes W'_+(\mathbb{A})$.

We fix then an odd prime $p \nmid (\delta_1 \cdots \delta_{2n} \ell)$ and isomorphism $\mathbb{C} = \bar{\mathbb{Q}}_p$. Write $\mathcal{O}/\mathbb{Z}_p$ finite flat ext with maximal ideal $\wp$. Write V_λ for algebraic rep of G of weight $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ (with resp. to H)

Theorem 2. (Z.)

Let $F : G(\mathbb{A}_f) \to V_\lambda(\mathcal{O})$ be a p-integral automorphic form of $G(\mathbb{A}_f)$ of weight λ of level $G(\hat{\mathbb{Z}})$

Suppose that

(1) $p > \max(n, \lambda_1-\lambda_2, \lambda_2-\lambda_3, \cdots, \lambda_{n-1}-\lambda_n, \lambda_n)$

(2) $F \not\equiv 0 \pmod{p}$ and the (automorphic) representation of $\tilde{G}^{der}(\mathbb{A}_f)$ generated by $F \pmod{p}$ is irreducible

(3) technical conditions on ~~ℓ~~, G and $H(\mathbb{A}_f) \to \mathbb{Z}_\ell$ is surjective.

Then we can construct an explicit element $\phi_\lambda \in S(W \otimes W'_+(\mathbb{A}))$ such that the theta lift $\Theta_{\phi_\lambda}(F)$ of F by ϕ_λ from G to G' is a (Siegel) automorphic form of weight $\tilde{\lambda} = (\lambda_1+n, \cdots, \lambda_n+n)$ of level $\Gamma_0(2\, \mathrm{l.c.m}(\delta_1, \cdots, \delta_{2n}))$, of character χ_Q (quadratic char asso to Q) whose Fourier coefficients $\in V_{\tilde{\lambda}}(\mathcal{O})$ and moreover $\Theta_{\phi_\lambda}(F) \not\equiv 0 \pmod{p}$.

Remark: this gives a theta lift mod p:

$$\mathcal{A}(G, \bar{\mathbb{F}}_p) \dashrightarrow \mathcal{A}(G', \bar{\mathbb{F}}_p).$$

Idea of proof: thm 2 $\Leftarrow$ non-vanishing mod p of certain Bessel periods

$\Leftarrow$ non-vanishing mod p of ~~certain~~ toric integrals.

More precisely, suppose the ℓ-Sylow subgp of $(\mathcal{O}/\mathfrak{p})^\times$ is of order ℓ^s. For any $k > s$, write

$$H(\hat{\mathbb{Z}})_{\ell^k} := \ker(H(\hat{\mathbb{Z}}) \to H(\mathbb{Z}_\ell) \to H(\mathbb{Z}/\ell^k)), \quad \Gamma_k = H(\mathbb{Q}) \backslash H(\mathbb{A}_f) / H(\hat{\mathbb{Z}})_{\ell^k}.$$

we need to show

Prop: For a map $\tilde{F}: G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K \to \mathcal{O}/\mathfrak{p}$ not invariant under ~~translation~~ right translation by $\tilde{G}^{der}(\mathbb{A}_f)$ (non-Eisenstein). Then for a char $\psi: \Gamma_k \to \mu_{\ell^\infty}$ and for a.b.f.m. $\bar{x} \in \bar{\mathcal{L}}$, there is an element $a = a(\bar{x}, \psi) \in H(\mathbb{A}_f)$ such that

$$\sum_{\tau \in \Gamma_k} \psi(\tau) \tilde{F}(a\tau\bar{x}) \neq 0.$$

Proof: we apply Theorem 1, set $R = \Gamma_k$. We know that f.a.b.m. $\bar{x} \in \bar{\mathcal{L}}$, $\bar{\varphi}_1(U\bar{x}) = \bar{\varphi}_2^{-1}(U\bar{\varphi}_0(\bar{x}))$.

$$\exists\ y_1 \neq y_2 \in X_K \text{ with } \begin{cases} \varphi_2(y_1) = \varphi_2(y_2) \\ \tilde{F}(y_1) \neq \tilde{F}(y_2) \end{cases}$$

($\in \varphi_0(x)K$, can assume this by assumption (3)) up to replace R by some hR for $h \in H(\mathbb{A}_f)$)

Write $R = \{\tau_1, \cdots, \tau_r\}$ and fix $w_1, \cdots, w_r \in \varphi_2^{-1}(\varphi_0(x))$.

$$\text{Thm 1} \Rightarrow \bar{\varphi}_1(Ux) = \bar{\varphi}_2^{-1}(U\bar{\varphi}_0(x))$$

$$\Rightarrow \exists\ a_1 \neq a_2 \in U, \text{ s.t. } \bar{\varphi}_1(a_i x) = (y_i, w_2, w_3, \cdots, w_r) \quad (i=1,2)$$

Thus

$$\sum_{i=1}^r \psi(\tau_i) \tilde{F}(a_1\tau_i x) - \sum_{i=1}^r \psi(\tau_i) \tilde{F}(a_2 \tau_i x)$$

$$= \psi(\tau_1)(\tilde{F}(a_1\tau_1 x) - \tilde{F}(a_2\tau_1 x)) = \psi(\tau_1)(\tilde{F}(y_1) - \tilde{F}(y_2)) \neq 0. \quad \boxtimes$$

§4. Sketch of proof for Thm 1.

For simplicity, we assume $G(\mathbb{Q}_\ell) = SO\begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}(\mathbb{Q}_\ell)$ and $H(\mathbb{Q}_\ell)$ diagonal matrices

We can find a finite set I such that for each $i \in I$, there is an element $x_i \in \mathcal{L}$, a one-parameter unipotent subgp $\mathcal{U}_i(\mathbb{Q}_\ell) \subset G(\mathbb{Q}_\ell)$ (conjugate to $\left(\begin{array}{c|c} \begin{smallmatrix}1 & & s\\ & \ddots & \vdots\\ & & s\end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix}1 & s \cdots s\\ & \ddots & \vdots \\ & & 1\end{smallmatrix}\end{array}\right)$) and a compact open subset $\tilde{K} \subset \mathbb{Q}_\ell$, such that there is a decomposition

$$U\mathcal{L}K = \bigcup_{i\in I}\bigcup_{n\in\mathbb{N}}\bigcup x_i\,\mathcal{U}_i\!\left(\frac{\tilde{K}}{\ell^n}\right)K = \bigcup_{i\in I}\bigcup_{n\in\mathbb{N}}\bigcup x_i\,\mathcal{U}_i\!\left(\frac{1}{\ell^n}\right)K. \qquad (*)$$

~~Fix a~~ one-parameter unipotent subgp $V=\{u(t)\}_{t\in\mathbb{Q}_\ell} \subset G(\mathbb{Q}_\ell)$ and $\Delta(V) \subset G(\mathbb{A}_f)^R$ diagonal image of V. Then we have Ratner's theorems (on orbit closures and ergodic measures)

Theorem: (1) For any $x \in CM$, for almost all $\gamma \in H(\mathbb{Q})\backslash H(\mathbb{A}_f)$,

$\bar{\varphi}_1(\gamma x V)$ is dense in $\bar{\varphi}_2^{-1}(\gamma\,\bar{\varphi}_0(x))$.

(2) In this case, $\forall\, f: X_K^R \to \mathbb{C}$, we have ($s \in \mathbb{Q}_\ell$)

$$\lim_{|s|\to+\infty}\frac{1}{\lambda(s\tilde{K})}\int_{s\tilde{K}} f\circ\bar{\varphi}_1(\gamma x\, u(t))\,dt = \int_{\bar{\varphi}_2^{-1}(\gamma\bar{\varphi}_0(x))} f(z)\,d\mu_{\gamma\bar{\varphi}_0(x)}(z) \qquad (**)$$

Now we take $f = \mathbb{1}_{g_0 K^R}$ a characteristic function, apply $\int_U d\gamma$ to both sides of (**)

$$\lim_{|s|\to+\infty}\frac{1}{\lambda(s\tilde{K})}\int_{s\tilde{K}} dt\ \underbrace{\int_U f\circ\bar{\varphi}_1(\gamma x\, v(t))\,d\gamma}_{=:\ A(f,\,x v(t))} = RHS = B(f,x)$$

$$= \mu(\{\gamma\in U \mid \bar{\varphi}_1(\gamma x\, v(t)) = g_0\})$$

$$= \begin{cases} I(g_0)\neq 0 & ;\ \bar{\varphi}_0(x) \in U\bar{\varphi}_2(g_0) \\ 0 & ;\ o/w. \end{cases}$$

$A(f,x)$ factors through $U\backslash CM/K$ and we can apply $(*)$ to get that:

$\forall\, \varepsilon>0$, there is a compact subset $C(\varepsilon)\subset\mathcal{L}$ s.t. $x\in\mathcal{L}\setminus C(\varepsilon)$

$$\left|\mu(\{\gamma\in U \mid \bar{\varphi}_1(\gamma x) = g_0\}) - I(g_0)\right| < \varepsilon \ \text{ if }\ \bar{\varphi}_0(x)\in U\bar{\varphi}_2(g_0)$$

The image $\overline{C(\varepsilon)} \subset CM_K$ of $C(\varepsilon)$ is compact and discrete, thus is finite. So we get that for all but finitely many $\bar{x}\in\bar{\mathcal{L}}$, we have

$$\bar{\varphi}_1(U\bar{x}) = \bar{\varphi}_2^{-1}(U\bar{\varphi}_0(\bar{x}))$$