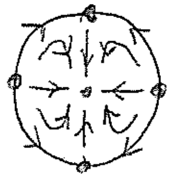
~~Idea~~ **Idea** ← something like Morse theory

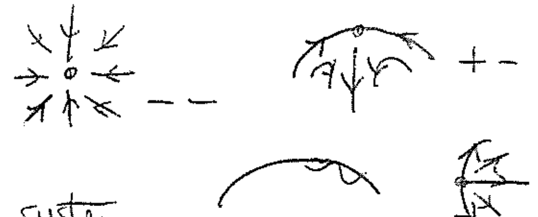
Let's consider a manifold, disc  $\subset \mathbb{R}^2$
on it, a dynamical system  ← colors.

> A. Bialynicki-Birula: Some theorems on actions of alg. groups
> Annals, 1973

~~Intuition~~ **Idea** We can ~~classify~~ stratify the space looking at ending point. We get

$1\times$  $\simeq \mathbb{R}^2$, $2\times \to \cdot \leftarrow \simeq \mathbb{R}^1$ $2\times \cdot \simeq \mathbb{R}^0$ → stratification!

Remarks • Strata are ⟺ fixed points

~~Remarks~~ • Points have some sort of signature that dermines the dimension  $--$ $+-$ $++$

• We can look at this dynamical system as a group action on the manifold.

Aim make everything algebraic. From $\mathbb{G}_m$ (over k alg. closed) action → stratification
(we will draw consequences for base field $= \mathbb{C}$ but this is true anyways.

Notation X smooth scheme, $G = \mathbb{G}_m$ acting on it.

X^G subscheme of fixed points (closure of set of closed (not really) fixed points)

Fogarty: X smooth $\rightsquigarrow$ X^G smooth

$a \in X^G$ closed point $\longrightarrow$ $\mathbb{G}_m$ acts on T_aX

$$T_aX = T_aX^+ \oplus \underbrace{T_aX^0}_{= T_aX^G} \oplus T_aX^-$$

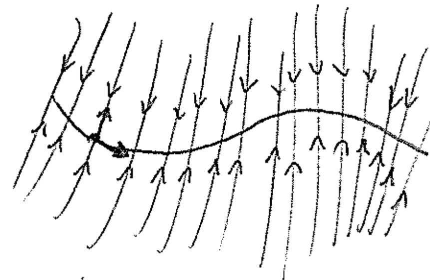
> **Remind** $\mathbb{G}_m \curvearrowright$ Vector space
>
> $\rightsquigarrow V = \bigoplus_{a_i \in \mathbb{Z}} V_{a_i}$ $\quad \mathbb{G}_m \curvearrowright V_{a_i}$
>
> $t(x) = t^{a_i} \cdot x$
>
> $V = \underset{a_i > 0}{V^+} \oplus \underset{a_i = 0}{V^0} \oplus \underset{a_i < 0}{V^-}$

Thm X smooth complete scheme, $\mathbb{G}_m \curvearrowright X$, $X^G = \bigcup_{i=1}^{n} X_i^G$ conn. comp. $\boxed{\text{II}}$

There is a stratification $\bigcup_{i=1}^{r} X_i^+ = X$ in nonsing. locally closed, with maps $\varphi_i : X_i^+ \to X_i^G$ s.t.

- $X_i^G \hookrightarrow X_i^+$
- $\varphi_i|_{X_i^G} = \mathrm{Id}_{X_i^G}$
- $\varphi_i : X_i^+ \to X_i^G$ is a G-fibration (~ vector bundle)
- if $a \in X_i^G$, $T_a X_i^+ = T_a X^+ \oplus T_a X^0$

(the same with $+ \rightsquigarrow -$)



Pf $X_i^+ = \{x \in X \text{ s.t. } \lim_{g \to 0} g \cdot x \in X_i^G\}$ + many checks

Remarks • $X_i^+ \pitchfork X_i^- = X_i^G$

- $\forall a \notin X^G$, $\exists!\, i \neq j$, $a \in X_i$, $a \in X_i^-$
- Notation: $N_i^+ = \operatorname{codim} \begin{matrix} X_i^+ \\ \downarrow \\ X_i^G \end{matrix} = \dim T_a X^+$ for $a \in X_i^G$, $N_i^- = \operatorname{codim} \begin{matrix} X_i^- \\ \downarrow \\ X_i^G \end{matrix} = \dim T_a X^-$ for $a \in X_i^G$ $\rightsquigarrow N_i^+ + N_i^- + \dim X^G = \dim X$.

Corollary 1 $\exists!\, i$ for which $N_i^- = 0$ ($\Leftrightarrow X_i^- = X_i^G$)

Pf. $\exists!\, i$ for which X_i^+ is dense in $X \Rightarrow N_i^+ + \dim X^G = \dim X \Rightarrow N_i^- = 0$.

Corollary 2 X complete irreducible, $X \neq X^G \Rightarrow X^G$ not connected

Pf $\exists!\, i$ with $N_i^{+,-} = 0$, $\exists j$ with $N_j^- = \dim X - \dim X^G \to$ different

Corollary 3 X complete, X^G = all isolated points $\Rightarrow$ affine stratification of X ($X = \bigcup \mathbb{A}^{n_i}$) $\rightarrow$
- All Chow groups
- Topological properties (s.t. Euler charact.)
- (Co)Homology

We can say something more:

Prop. 1 X complete, X^G isolated points $\Rightarrow$ affine stratification is symmetrical $\left(\#\{\mathbb{A}^k\} = \#\{\mathbb{A}^{\dim X - k}\}\right)$

Prop. 2 X complete, X^G isolated points $\Rightarrow$ $X = \bigcup \mathbb{A}^{n_i}$ $\quad \forall\, 0 \le k \le \dim X,\ \exists\, n_i = k$.

Cor. X complete $\Rightarrow$ $\#\{X^G\} > \dim X$.

Pf. isolated pts $\to$ Prop. 2
not $\to$ ≥ 1 dimensional.

Examples Projective space $\leftarrow$ plenty of different actions!

Example 1 $\mathbb{G}_{m,t} \curvearrowright \mathbb{P}^2$ as $t \mapsto \begin{bmatrix} t & & \\ & 1 & \\ & & 1 \end{bmatrix}$ fixed pts $[1\,0\,0]$
whole line $[0\,a\,b]$!

Better, $\mathbb{G}_{m,t} \curvearrowright \mathbb{P}^2$ $\begin{bmatrix} t^2 & & \\ & t & \\ & & 1 \end{bmatrix}$

In general $\mathbb{G}_{m,t} \curvearrowright \mathbb{P}^n$ $t \mapsto \begin{bmatrix} t^n & & & \\ & t^{n-1} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ fixed pts

		$\dim T_a X^+$	$T_a X^0$	$T_a X^-$
X_0^G	$[1\,0 \ldots 0]$	0	0	n
X_1^G	$[0\,1\,0 \ldots 0]$	1	0	$n-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$n-1$		
X_n^G	$[0 \ldots 0\,1]$	n	0	0

X_i^+	X_i^-
$\mathbb{A}^0$	$\mathbb{A}^n$
$\mathbb{A}^1$	$\mathbb{A}^{n-1}$
$\vdots$	$\vdots$
$\mathbb{A}^n$	$\mathbb{A}^0$

✓

Example 2 Conics in $\mathbb{P}^2$, a $\mathbb{P}^5$

$\begin{bmatrix} t^2 & & \\ & t & \\ & & 1 \end{bmatrix}$ acts on this too $\quad x \; y \; z$

$$\begin{aligned} x^2 &\mapsto t^4 x^2 \\ xy &\mapsto t^3 xy \\ y^2 &\mapsto t^2 y^2 \\ xz &\mapsto t^2 xz \\ yz &\mapsto t\, yz \\ z^2 &\mapsto z^2 \end{aligned}$$

Fixed.	$\dim T_a X^+$	$T_a X^0$	$T_a X^-$	X_i^+	X_i^-	
x^2	0	0	5	$\mathbb{A}^0$	$\mathbb{A}^5$	
y	1	0	4	$\mathbb{A}^1$	$\mathbb{A}^4$	
$y^2+\mu xz$	2	1	2	$E \xrightarrow{\mathbb{C}^2} \mathbb{P}^1$	$F \xrightarrow{\mathbb{C}^2} \mathbb{P}^1$	$\leftarrow$ vect bundle
yz	4	0	1	$\cancel{E' \xrightarrow{\mathbb{C}^2} \mathbb{P}^1}$ $\mathbb{A}^4$	$\cancel{F' \xrightarrow{\mathbb{C}^2} \mathbb{P}^1}$ $\mathbb{A}^1$	
z^2	5	0	0	$\mathbb{A}^5$	$\mathbb{A}^0$	

there is a better choice! $\begin{bmatrix} t^3 & & \\ & t & \\ & & 1 \end{bmatrix}$

$$\left.\begin{aligned} x^2 &\longmapsto t^6x^2 \\ xy &\longmapsto t^4xy \\ y^2 &\longmapsto t^2xy \\ xz &\longmapsto t^3xz \\ yz &\longmapsto tyz \\ z^2 &\longmapsto z^2 \end{aligned}\right\} \to 6 \text{ fixed curves}$$

$\longrightarrow$ affine stratificati

<u>Future</u> I'm working on a new compactification of ~~comple~~ twisted cubics and I ~~have~~ would like to apply this.... still don't have smoothness (maybe I'll never have it)