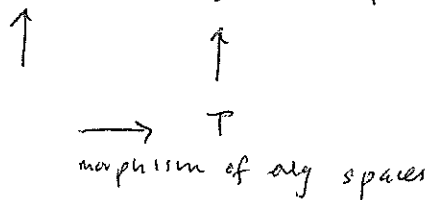


Coarse Moduli spaces

\mathcal{X}/S algebraic stack (étale topology of schemes over S)

(1) \mathcal{X} is a stack

(2) $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ representable by algebraic spaces.



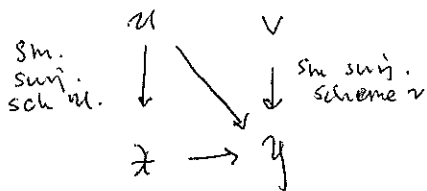
(3) $X \rightarrow \mathcal{X}$ surjective smooth morphism from a scheme.

$f: T \rightarrow \mathcal{X}$ we can talk about properties of f .

alg space

$f: \mathcal{X} \rightarrow \mathcal{Y}$ morphism of algebraic stacks
representable by algebraic spaces

Suppose $f: \mathcal{X} \rightarrow \mathcal{Y}$ morphism of algebraic stacks



$u \times_{\mathcal{Y}} v \rightarrow v$
morphism of alg spaces

So if (P) is a property of morphisms of alg. spaces which is (Smooth) local on the ~~source~~ ^{source} + target then we say f has (P) if $u \times_{\mathcal{Y}} v \rightarrow v$ does.

Properties of morphisms

- separated
- proper.

Separated + Properness

$X \rightarrow S$ map of schemes

$X \rightarrow X \times_S X$ closed immersion

} separated.

This is not good for algebraic stacks as it is almost never true

Ex Let G be an algebraic group. G/k .

$\text{Spec } k \xrightarrow{\text{given by trivial bundle over a pt}} BG \rightarrow BG \times BG$
 $\downarrow \text{Spec } k$
 $\text{Spec } k$

$\begin{matrix} G \\ \text{Isom}(X, X) \end{matrix} \rightarrow \text{Spec}(k) \xrightarrow{(X, X)} BG \times BG$
 $\downarrow \quad \quad \downarrow$
 $BG \xrightarrow{f} BG \times BG$

* If f was a closed immersion, so would the pullback.

Lemma:

$f: \mathcal{X} \rightarrow \mathcal{Y}$ alg ~~stacks~~ ~~spaces~~

$\Delta_f: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces.

Proof: \mathcal{X} want to be an alg space.

$\begin{matrix} \mathcal{X} & \xrightarrow{\Delta_f} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad} & T \times_{\mathcal{Y}} T \end{matrix}$
 (where T is an alg space)
 Δ_f is representable by algebraic spaces.

$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(T) = \mathcal{X}(T) \times_{\mathcal{Y}(T)} \mathcal{X}(T)$
 $\downarrow \alpha$
 $\alpha = (x, x', \alpha)$
 $x, x' \in \mathcal{X}(T)$
 $\alpha: f(x) \xrightarrow{\sim} f(x')$
 groupoids

T' point of $\mathcal{X}(T')$ is $\mathcal{X}(T') \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}(T'), \alpha}$

$\tilde{x} \in \mathcal{X}(T'), T' \xrightarrow{\varphi} T$

$(\tilde{x}, \tilde{x}, \text{id}: \tilde{x} \rightarrow \tilde{x})$

$(x|_{T'}, x'|_{T'}, \alpha|_{T'}: f(x|_{T'}) \xrightarrow{\sim} f(x'|_{T'}))$

This all amounts to: $x|_{T'} \xrightarrow{\sim} x'|_{T'}$ s.t. those under f is $\alpha|_{T'}$.

$$\begin{array}{ccc}
 \mathcal{X} \times_{\mathcal{Y}}^T \mathcal{X} \xrightarrow{\alpha} \text{Isom}_{\mathcal{X}}(x, x') & \Rightarrow & \mathcal{X} \times_{\mathcal{Y}}^T \mathcal{X} \text{ is an algebraic space} \\
 \downarrow & \square & \downarrow \\
 T \xrightarrow{\alpha} \text{Isom}_{\mathcal{Y}}(f(x), f(x')) & & (\mathcal{X} \text{ alg stack, } \text{Isom}(x, x') \text{ alg space, } \\
 & & x, x' \in \mathcal{X}(T))
 \end{array}$$

$$\begin{array}{ccc}
 \text{Isom}(x, x') & \rightarrow & T \\
 \downarrow & & \downarrow \\
 \mathcal{X} & \rightarrow & \mathcal{X} \times \mathcal{X}
 \end{array}
 \quad (x, x')$$

$X \rightarrow X \times_S X$ closed immersion \Leftrightarrow proper (for schemes)

Def $f: \mathcal{X} \rightarrow \mathcal{Y}$ morphism of alg. stacks

① f is separated if Δ_f is proper.

(if Δ_f is repr by alg spaces and when you base change it is an alg space with proper map ~~to itself~~.)

② f is proper if f is separated, finite type and universally closed.

Example A/k abelian variety

$BA \rightarrow \text{spec}(k)$ is proper.

Fact

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$$

$g \circ f$ proper, f surjective, g separated

$$\begin{array}{ccc}
 A & \rightarrow & \text{spec}(k) \\
 \downarrow & & \downarrow \\
 BA & \rightarrow & BA \times BA
 \end{array}
 \Rightarrow g \text{ is proper.}$$

$$\begin{aligned}
 \text{Let } \mathcal{X} &= \text{spec}(k) \\
 \mathcal{Y} &= BA \\
 \mathcal{Z} &= \text{spec}(k)
 \end{aligned}$$

Amounts to show $g: BA \rightarrow \text{spec}(k)$ is separated.

Coarse moduli spaces

Def \mathcal{X}/S algebraic stack.

A coarse moduli space for \mathcal{X} is a morphism:

$$\pi: \mathcal{X} \rightarrow X, \quad X \text{ alg. space such that}$$

(1) π is initial among maps from \mathcal{X} to alg. spaces.

(2) For every $k = \bar{k}$, $\pi_0(\mathcal{X}(k)) \xrightarrow[\text{bijection}]{\sim} X(k)$

Examples

① Moduli of smooth curves of genus 0. \mathcal{M}_0 .

$$\mathcal{M}_0 \cong [* / \mathrm{PGL}_2] = B\mathrm{PGL}_2.$$

Coarse moduli space = $*$.

more generally, BG where G is any alg. group (G/k).

$BG \rightarrow \mathrm{Spec}(k)$ is a coarse moduli space.

In general given a stack \mathcal{X} , can form: $\pi_0(\mathcal{X}) : (\mathrm{Sch}/S)^{\mathrm{op}} \rightarrow \mathrm{sets}.$

$$\pi_0(BG) = *$$