The Shafarevich conjecture for hypersurfaces in abelian varieties
(joint work with Brian Lawrence)

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Faltings' Theorem (Shafarevich's Conjecture)

Let n be a natural number. Let S be a finite set of primes.

We say A an abelian variety over $\mathbb{Q}$ has good reduction at p if $A_{\mathbb{Q}_p}$ extends to a smooth proper scheme over $\mathbb{Z}_p$.

Theorem (Faltings)

There exist finitely many abelian varieties A over $\mathbb{Q}$, with good reduction at all primes not in S, up to isomorphism.

(Number fields too - $\mathbb{Q}$ just for simplicity of notation.)

Same expected for many other types of varieties. Known for a few: K3 surfaces (André, She), del Pezzo surfaces (Scholl), flag varieties (Javanpeykar and Loughran), complete intersections of Hodge level at most 1 (Javanpeykar and Loughran), surfaces fibered smoothly over a curve (Javanpeykar), and Fano threefolds (Javenpeykar and Loughran).

Main Theorem

Fix a single abelian variety A over $\mathbb{Q}$ of dimension n with good reduction away from S.

We say a smooth hypersurface $H \subset A$ has good reduction at p if the closure of H in $A_{\mathbb{Z}_p}$ remains smooth.

Every smooth hypersurface H represents a class $[H]$ in the Neron-Severi group of A. Fix ϕ an ample class.

Theorem 1 (Lawrence-S)

Assume $n \geq 4$. There are only finitely many smooth hypersurfaces $H \subseteq A$ representing ϕ, with good reduction at all primes not in S, up to translation.

Main Theorem ($n = 3$ case)

Consider the sequence of numbers

$$1, 5, 20, 76, 285, 1065, \ldots$$

and associated sequence of binomial coefficients

$$\binom{1+5}{1}, \binom{5+20}{5}, \binom{20+76}{20}, \binom{76+285}{76}, \binom{285+1065}{285}, \ldots$$

Theorem 2 (Lawrence-S)

Assume $n = 3$. Assume that the intersection number $\phi \cdot \phi \cdot \phi$ is not a multiple of any of the binomial coefficients on this list but the first. Then there are only finitely many smooth hypersurfaces $H \subseteq A$ representing ϕ, with good reduction at all primes not in S, up to translation.

Prior work of Lawrence and Venkatesh

Theorem (Lawrence-Venkatesh)

Fix n and d. Assume they are both sufficiently large according to some complicated formula. Fix a finite set S of primes.

Then the set of hypersurfaces $H \subset \mathbb{P}^n_{\mathbb{Q}}$, of degree d, with good reduction at all primes not in S, is not Zariski dense in the moduli space of all degree d hypersurfaces.

Same strategy + new ideas = stronger result (in a slightly different setting).

The Lawrence-Venkatesh method

Consider a variety X such that we want to show finiteness of $X(\mathbb{Z}[1/S])$. (e.g. $X =$ moduli space of smooth hypersurfaces).

Suppose we have a smooth proper map $f : Y \to X$. (e.g. universal family)

Key tool is Galois action on the cohomology of the fiber $H^i(Y_x, \mathbb{Q}_p)$. We want to show

- Only finitely many Galois representations can occur as $H^i(Y_x, \mathbb{Q}_p)$ for $x \in X(\mathbb{Z}[1/S])$. (Falting-Serre).
- The Galois representations varies a lot as x changes (p-adic Hodge theory.)

The cohomology $H^i(Y_x, \mathbb{Q}_p)$ carries an action of the fundamental group $\pi_1(X)$ of X. (Even $H^i(Y_x, \mathbb{Q})$ does.) This action controls how the Galois representation varies. Need this to be "big".

The Lawrence-Venkatesh method

For $f : Y \to X$ smooth proper such that $H^i(Y_x, \mathbb{Q})$ has dimension N, obtain a representation $\rho : \pi_1(X) \to GL_N(\mathbb{Q})$.

We say the *monodromy* of ρ is the Zariski closure of the image of ρ (algebraic group w/ representation).

Lawrence and Venkatesh method proves $X(\mathbb{Z}[1/S])$ is not Zariski dense given a smooth proper $f : Y \to X$, under numerical conditions on the Hodge numbers of $H^i(Y_x, \mathbb{Q})$, plus the assumption that ρ has *big monodromy*. (e.g. SL_N, Sp_N, SO_N sufficiently big.)

Example: $X =$ moduli space of smooth hypersurfaces in $\mathbb{P}^n_{\mathbb{Z}}$, Y=universal family, $i = n - 1$.

Want to do Noetherian induction, but lose big monodromy.

What's better about hypersurfaces in abelian varieties

Let $Y \to X$ be a family of hypersurfaces in a fixed abelian variety A, parameterized by X. (e.g. X moduli space of hypersurfaces in A.)

We can construct *many* representations of $\pi_1(X)$ from Y. Two ways:

- Fix χ a 1-dimensional character of $\pi_1(A)$. Then we have twisted cohomology $H^{n-1}(Y_x, \chi)$. The fundamental group $\pi_1(X)$ also acts on this. Take ρ_χ this representation.
- Fix m a natural number. Consider $[m] : A \to A$ the multiplication-by-m map. Then $\pi_1(X)$ acts on $H^{n-1}([m]^{-1} Y_x, \mathbb{Q})$. So does $A[m]$. These actions commute. Take ρ_χ the π_1-rep on the eigenspace of $A[m]$ with eigenvalue χ.

Suffices to show, for each nonconstant family $Y \to X$, for *some* χ, the rep ρ_χ has big monodromy. Then induct.

We do this!

Geometric theorem

Fix A an abelian variety of dimension n, $Y \to X$ a smooth family of hypersurfaces in A with (ample) class ϕ. Assume:

- Y is not preserved under translation by a nontrivial element of A.
- Y is not the constant family, translated by a section of A over X.
- $n \geq 4$, or $n = 3$ and ϕ^3 not one of our binomial coefficients from earlier, or $n = 2$ and ϕ^2 neither a power of 2 greater than 4, $\binom{2k}{k}$ for $k > 2$, nor 56.

Then for some character χ, ρ_χ has monodromy containing SO_N, SL_N, or Sp_N. (In fact, for almost all χ).

Let X = Zariski closure inside moduli space of smooth hypersurfaces in A of the set of hypersurfaces with good reduction outside S (or an irreducible component). Let Y be the quotient of the universal family of hypersurfaces on X, by any points which preserve it up to translation.

- By construction, Y is not preserved up to translation by any nontrivial point of A.
- If our desired statement is false and there are infinitely many hypersurfaces with good reduction up to translation, then Y is not the constant family up to translation.
- If the original ϕ^n was not a multiple of any of the forbidden values, then ϕ^n of the quotient is not equal to any of the forbidden values.

Because it satisfies all three assumptions, some ρ_χ has big monodromy. Then by Lawrence-Venkatesh, integral points are not Zariski dense in X. This contradicts the fact that X was the Zariski closure of the integral points, so our desired statement must be true.

Tannakian monodromy groups

Krämer and Weissauer defined a *Tannakian monodromy group* associated to a smooth variety $Z \subseteq A$.

- If you know what a perverse sheaf is: They also associate a group to a perverse sheaf on A. We view varieties as perverse sheaves by taking the constant sheaf on that variety.

Concretely, they

- proved for almost all χ, $H^i(Z, \chi) = 0$ unless $i = \dim Z$
- proved for almost all χ, $\dim H^{\dim Z}(Z, \chi) = N =$ Euler characteristic of Z times $(-1)^{\dim Z}$.
- defined an algebraic group $G_Z \subseteq GL_N$ that acts naturally on $H^{\dim Z}(Z, \chi)$.

How should we think about the Tannakian monodromy group?

Should think of G_Z as like a monodromy group, but over the "space" of possible characters χ.

The usual monodromy group of ρ_χ controls variation in cohomology $H^{n-1}(Y_x, \chi)$ as we vary a point x (and fix χ).

- Controls Galois representations.
- Controls Hodge structures.
- Sato-Tate.

The Tannakian monodromy group of Y_x controls variation of $H^{n-1}(Y_x, \chi)$ as we vary a character χ (and fix x).

- Same theorems hold.

How do Krämer & Weissauer define the Tannakian group?

They define a suitable category of perverse sheaves.

They define an operation, *sheaf convolution*, on this category. This uses three maps $pr_1, pr_2, m : A \times A \to A$. The formula is:

$$K * L = Rm_*(pr_1^* K \otimes pr_2^* L).$$

This operation behaves like tensor product - makes the category into a symmetric monoidal abelian category.

Tannakian category: any symmetric monoidal abelian category satisfying some axioms. Theorem: These are always the category of representations of a unique pro-algebraic group.

G_Z = Tannakian group associated to subcategory generated by the constant sheaf on Z, acting on representation associated to the constant sheaf on Z.

Geometric strategy

This strategy has two steps:

1. For a smooth hypersurface $H \subset A$, we prove the Tannakian monodromy group G_H contains SL_N, SO_N, or Sp_N.
   - Assuming H is not invariant under translation by any nontrivial element of A.
   - Assuming $(n, [H]^n)$ avoids list of forbidden values.
2. For any family $Y \to X$ of varieties in A, where the Tannakian monodromy group G_{Y_η} of the *generic* fiber Y_η contains SL_N, SO_N, or Sp_N, we prove that for almost all χ, the usual monodromy group of ρ_χ contains SL_N, SO_N, or Sp_N.
   - Assuming Y is not a constant family, up to translation by a section of A over X.

Proof of (2)

Key idea (topology): The fundamental group of fiber of a fibration is a normal subgroup of the fundamental group of the total space.

AG version: The fundamental group of the generic fiber of any morphism is a normal subgroup of the fundamental group of the total space.

AG version 2: The monodromy group of a representation, restricted to the generic fiber of a morphism, is a normal subgroup of the monodromy group of that representation.

Strategy: Make a big Tannakian group in which G_{Y_η} and ρ_χ for generic χ are both normal subgroups.

Group theory: Normal subgroups of GL_N, GO_N, GSp_N contain SL_N, SO_N, Sp_N or are contained in scalars.

Proof of (1)

Krämer proved a series of wonderful theorems controlling the Tannakian group using the "characteristic cycle". Especially effective for smooth varieties, hypersurfaces.

For any smooth hypersurface H, not translation-invariant, conclude: G_H contains as a normal subgroup a *simple* algebraic group acting by an irreducible *minuscule* representation.

Minuscule representations: Characters of the maximal torus appearing form a single orbit under the Weyl group.

- Classified!
- One for each simple algebraic group and character of the center of that simple group
- Example: Rep $\wedge^k$ of SL_m for $1 \leq k \leq m$.

Eliminate all pairs of a simple group w/ a minuscule representation, except classical group w/ standard representation.

How to use big monodromy

Let $N = \dim H^{n-1}(Y_x, \chi) = \phi^n$. Let $\mathbb{Q}_\chi$ =coefficient field of χ. Let $\mathbb{Q}_{\chi,p} = p$-adic completion.

For each $x \in X(\mathbb{Z}[1/S])$, $H^{n-1}(Y_x, \chi)$ is a Galois representation.

- This is a representation $\rho_{\chi,x}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $GL_N(\mathbb{Q}_{\chi,p})$.
- $\rho_{\chi,x}$ is unramified away from $S \cup \{p\}$.
- For every prime ℓ not in $S \cup \{p\}$, Frobenius at ℓ acts on $\rho_{\chi,x}$.
- The characteristic polynomial of Frobenius has coefficients integers in $\mathbb{Q}_\chi$ and roots of size $\ell^{(n-1)/2}$.

Faltings-Serre: There are finitely many such (semisimple) representations.

Period Maps

Suppose instead we knew there were finitely many Hodge structures among the integral points.

We would want to show a Torelli-type theorem. (There are only finitely many $x \in X$ such that $H^{n-1}(Y_x, \chi)$ has a given Hodge structure.)

Tool: Period map from the universal cover $\tilde{X}$ of X to period domain D.

Ax-Schanuel (Bakker-Tsimerman) implies a very general Torelli-type statement. For any subvariety of D, of codimension $> \dim X$, projection to X of its inverse image in $\tilde{X}$ is not Zariski dense.

Needs monodromy to act transitively on D, can pass to smaller period domain if not.

p-adic Hodge theory

Crystalline cohomology: $H^{n-1}(Y_x, \chi) \otimes \mathbb{Q}_p$ is a $\mathbb{Q}_{\chi,p}$ vector space with a (semilinear) Frobenius action. Depends only on $x \mod p$.

p-adic de Rham cohomology: For each $x \in X$ with given reduction mod p, we get a Hodge filtration on this vector space. This comes from a p-adic analytic period map $X \to D$.

p-adic Hodge theory: This vector space + filtration + Frobenius is determined up to isomorphism by the Galois representation $\rho_{\chi,x}$.

Bakker-Tsimerman theorem implies p-adic version by clever LV argument with formal power series.

Problem 1: Frobenius centralizer (easy). Problem 2: semisimplification (hard).