Midterm 1
Math 31B-2, Winter 2020

Name:	**UID:**

Directions—Please read carefully!

- You are allowed **50 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- No notes, books, your own scratch papers, calculators, cell phones, computers, or other electronic aids are allowed.
- In order to receive full credit, you must **show your work or explain your reasoning**; your final answer is less important than the reasoning you used to reach it. Correct answers without work will receive little or no credit.
- Unless otherwise indicated, please simplify your answers.
- You can use the backs of pages as scratch papers, but **only those written in the front of pages** will be graded.
- Please write neatly. Illegible answers will be assumed to be incorrect. **Circle or box your final answer** when relevant.

Good luck!

Question	Points	Score
1	12	
2	18	
3	22	
4	20	
5	28	
Total:	100	

1. You do not need to provide explanation for the following questions.

(3) (a) Suppose the population in an area grows exponentially. Order the time it takes for the following three scenarios using $<, >, =$. (eg. A<B=C means B and C take the same time while A takes less time.)
(A) The population grows from 100 to 150. 50 %
(B) The population grows from 150 to 200. <50 %
(C) The population grows from 150 to 225. 50 %

exp. function : percentage change over a fixed time is constant

$\Rightarrow A = C > B$

(3) (b) True or False: Let $f(x)$ be an invertible function. If $f(x)$ is increasing, then $f^{-1}(x)$ is also increasing.

True.

$x_1 > x_2 \Rightarrow y_1 = f(x_1) > f(x_2) = y_2$

$\rightsquigarrow y_1 > y_2 \Rightarrow x_1 > x_2$

(3) (c) True of False: We can use L'Hôpital's rule to determine the limit

$$\lim_{x\to 0} \frac{e^x}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1.$$

False. "$\frac{1}{0}$" not indeterminate form

(3) (d) Which integral represents the length of the curve $y = \sin x$ between 0 and $\frac{\pi}{6}$?

$$\int_0^{\frac{\pi}{6}} \sqrt{1+\sin^2 x}\, dx \qquad \int_0^{\frac{\pi}{6}} \sqrt{1+\cos^2 x}\, dx$$

arc length $= \int_a^b \sqrt{1+f'(x)^2}\, dx$

2. Find the following limits.

(5) (a) $\lim_{x\to 0} \frac{\tan 4x}{\tan 5x}$ $\quad$ "$\frac{0}{0}$"

$$\underset{L'H}{=} \lim_{x\to 0} \frac{4\sec^2 4x}{5\sec^2 5x}$$

$$= \boxed{\frac{4}{5}} \quad \left(\sec(0) = \frac{1}{\cos(0)} = 1 \right)$$

(13) (b) $\lim_{x\to\infty} (1 + \frac{1}{x})^x$

$$= \lim_{x\to\infty} e^{x \cdot \ln(1+\frac{1}{x})}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$$

$$= e^1$$

$$= \boxed{e}$$

$$\left(\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \right)$$

3. Let $f(x) = \frac{1}{2}(x + \frac{1}{x})$ be a function with domain $(0, \infty)$ and range $[1, \infty)$.

(10) (a) Is $f(x)$ an invertible function? If so, explain your reasoning. If not, explain your reasoning, find a restricted domain on which $f(x)$ is invertible, and explain why $f(x)$ is invertible on your restricted domain.

No. $f(2) = f(\frac{1}{2}) \Rightarrow f$ is not one-to-one

When restricted to $[1,\infty)$, $f(x)$ is one-to-one

$f'(x) = \frac{1}{2}(1 - \frac{1}{x^2}) \geq 0$ for $x \leq 1$

$\Rightarrow f$ is increasing on $[1,\infty)$

$\Rightarrow f$ is one-to-one on $[1,\infty)$

(12) (b) Write down the formula for the inverse function $f^{-1}(x)$ with domain $[1, \infty)$ and range being the domain/your choice of restricted domain of $f(x)$. (Hint: You will need to use the quadratic formula: the two roots of ax^2+bx+c are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2}$, and choose the correct root accordingly.)

$y = \frac{1}{2}(x + \frac{1}{x})$

$\rightsquigarrow 2yx = x^2 + 1$

$\rightsquigarrow x^2 - 2yx + 1 = 0$

$\rightsquigarrow x = \dfrac{2y \pm \sqrt{4y^2 - 4}}{2}$

$= y \pm \sqrt{y^2 - 1}$

My restricted domain is $x \in [1,\infty)$,

So $x = y + \sqrt{y^2-1}$

(20) 4. Evaluate the indefinite integral

$$\int \frac{1}{x^3 + x^2 - x - 1}\, dx.$$

$$\frac{1}{x^3+x^2-x-1} = \frac{1}{(x-1)(x^2+2x+1)} = \frac{1}{(x-1)(x+1)^2}$$

Write $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\rightsquigarrow 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

$x = -1 ; \quad 1 = -2C$

$x = 1 : \quad 1 = 4A$

$x = 0 ; \quad 1 = A - B - C$

$\Rightarrow A = \frac{1}{4}, \; B = -\frac{1}{4}, \; C = -\frac{1}{2}$

$$\int \frac{1}{x^3+x^2-x-1}\, dx = \int \left(\frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} - \frac{1}{2} \frac{1}{(x+1)^2} \right) dx$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| + \frac{1}{2} \cdot \frac{1}{x+1} + C$$

5. Recall that the inverse sine function $\sin^{-1}(x)$ has domain $[-1,1]$ and range $[-\frac{\pi}{2},\frac{\pi}{2}]$.

(5) (a) $\cos(\sin^{-1}(x)) = ?$

$$\sin^{-1}x \in [-\tfrac{\pi}{2},\tfrac{\pi}{2}] \Rightarrow \cos(\sin^{-1}(x)) \geq 0$$

$$\cos(\sin^{-1}x) = \sqrt{1-\sin^2(\sin^{-1}x)} = \sqrt{1-x^2}$$

(8) (b) Use the fact that $\sin^{-1}(x)$ is the inverse function of $\sin(x)$, derive

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$$

(15) (c) Evaluate the indefinite integral

$$\int \sin^{-1}(x)\,dx.$$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$