Motivation

When the objects of a moduli problem have auts there is no fine moduli space.

Example $\mathcal{M}_{1,1}$

$$\left\{ \begin{array}{c}\text{elliptic curves} \\ \text{over } \mathbb{C}\end{array} \right\} / \cong \quad \overset{j}{\underset{\simeq}{\longleftrightarrow}} \quad \mathbb{A}^1_j$$

$\mathcal{E}_t \subset \mathbb{P}^2 \times \mathbb{A}^1_t \setminus \{0\}$ $\qquad$ $\mathcal{E}_t : Y^2 Z = X^3 - tZ^3$

$\downarrow$ $\qquad$ $j(\mathcal{E}_t) = 0.$

$\mathbb{A}^1_t \setminus \{0\}$ $\qquad$ $\mathbb{A}^1_t \setminus \{0\} \to \mathbb{A}^1_j$

constant map to 0.

$\mathcal{E}_0 \times (\mathbb{A}^1_t \setminus \{0\})$, $E_0 = y^2 = x^3 - 1$

$\downarrow$ $\qquad$ $(t = 1)$

$\mathbb{A}^1_t \setminus \{0\} \longrightarrow \mathbb{A}^1_j$ constant map to 0.

* if $\mathbb{A}^1_j$ was a fine moduli space, these families should be the same.

We will show these families are distinct by showing the generic fibers are distinct.

Claim $\mathcal{E}_{t,\mathbb{C}(t)} \cong E_{0,\mathbb{C}(t)}$

Pf $\mathbb{C}(t) \hookrightarrow K$ alg closure $\qquad$ $f: \mathcal{E}_{t,\mathbb{C}(t^{1/6})} \cong E_{0,\mathbb{C}(t^{1/6})}$

$t^{1/6} \in K$ 6^{th} root $\qquad$ $(x,y) \mapsto (t^{-1/3}x, t^{-1/2}y)$

$I = \{$isomorphisms $E_{t, \mathbb{C}(t^{1/6})} \cong E_{0, \mathbb{C}(t^{1/6})}\}$

The automorphisms of E_0 act on this [free and transitive] (torsor?)

$G = Gal(K/\mathbb{C}(t))$ acts on here by acting the defining equations.

These actions commute.

There would need to be a Galois invariant isomorphism on I.

Suppose $g \in G$, $g(t^{1/6}) = \chi(g) t^{1/6}$ where $\chi(g)$ is a 6^{th} root of unity

Action of g on f; $g \cdot (f) : (x,y) \mapsto (\chi(g)^{-2} x, \chi(g)^{-3} y) \mapsto (\chi(g)^{-2} t^{-1/3} x, \chi(g)^{-3} t^{-1/2} y)$

[inverse of g acting on here]

G acts on f (acts on coefficient) Use g^{-1} to act on equations, then use f, then we have the identity because g acts as the identity on $E_{0, \mathbb{C}(t^{1/6})}$

None of this is defined over $\mathbb{C}(t)$.

("there is a 6-1 cover that is birational but it doesn't descend")

$\mathbb{Z}/6$ automorphism group on E_t.

$M_{1,1} : Sch^{op} \to sets$

$S \mapsto \left\{ \{\begin{smallmatrix} E \\ \downarrow \\ S \end{smallmatrix}\}_S \text{ smooth, proper w/ section, fibers elliptic curves} \right\} / \cong$

If $\exists$ a fine moduli space $Hom(-, M)$ should be isomorphic to such a functor.

$\mathcal{M}_{1,1} : Sch^{op}/_{\mathbb{C}} \longrightarrow Gpds$

cat whose objects are morphisms of elliptic curves

morphisms are isomorphisms

"Smooth Deligne - Mumford stack"

In a groupoid, morphisms are invertible

Conditions :

(1) Gluing properties

(2) "Algebraic" (Algebraic spaces ?)

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schemes as functors - motivation for how to think of stacks.

$X/_S$ scheme (or generally $X \in \mathrm{ob}\,\mathcal{C}$)

$h_X = \mathrm{Hom}(-, X) : Sch_S^{op} \to$ sets

*

Yoneda's Lemma. $\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \text{sets})$

$X \longmapsto h_X$

This is a fully faithful embedding

(The homs are in bijection. $\mathrm{Hom}(X, Y) \longleftrightarrow \mathrm{Hom}(h_X, h_Y)$)

$\mathrm{Hom}(h_X, F) \simeq F(X)$

$X \xrightarrow{id} X$

$$h_X(Y) \downarrow F(Y)$$

Natural transformation.

Setup S affine scheme

$\mathcal{C} = \mathrm{Aff}_S$ category of affine S-schemes

X/S a scheme $\rightsquigarrow h_X : \mathrm{Aff}_S^{op} \to \mathrm{Sets}$

What about h_X is a result of X being a scheme.

Def $f : F \to G$ $F, G : \mathcal{C}^{op} \to \mathrm{sets}$ is relatively representable if

$\forall g : h_T \to G$, $T \in \mathcal{C}$

$$\begin{tikzcd} F \arrow[r, "f"] & G \\ F \times_G h_T \arrow[u] \arrow[r, "\varphi"] & h_T \arrow[u, "g"] \end{tikzcd}$$

$\left(F \times_G h_T \right)(B) = \underbrace{F(B) \times h_T(B)}_{G(B)}$

Fiber product of sets.

is representable. ($\iff \simeq h_R$ for some R)

Def F, G $\mathcal{C}^{op} \to \mathrm{sets}$ $f : F \to G$ morphism is an open (resp. closed) embedding if (1) f is relatively representable

(2) φ from above is open (resp. closed).

$\forall T \in \mathcal{C}$ and every such map. $h_T \to G$.

Def $F : \mathrm{Aff}_S^{op} \to \mathrm{Sets}$ is a (big) <u>Zariski sheaf</u> if $\forall U \in \mathrm{Aff}_S$

and $\forall U = \bigcup U_i$ open affine cover
$$\begin{array}{c} \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \\ \uparrow \\ F(U) \end{array}$$
is exact.

injects and surjects onto things that agree on both maps.

"Gluing property" agreeing on overlaps.

Proposition $F : \mathrm{Aff}_S^{op} \to$ sets is represented by an separated S-scheme

iff (1) F is a Zariski sheaf

(2) $\Delta : F \to F \times F$ is an affine closed immersion (embedding)

(3) There exists $X_i \in \mathrm{Aff}_S$. $\pi_i : h_{X_i} \to F$ s.t. (open embeddings)

$\coprod h_{X_i} \to F$ surjective map of sheaves.

(basically a covering principle.)

$\forall$ $s \in F(U)$ $\exists$ open cover of U, restriction ---

Moreover $h_{(-)} :$ (Sep S-schemes) $\to$ (functors w/ (1), (2), (3))

is an equivalence