Log geometry (Amand P.)

Essentially, generalization of scheme theory.

Basic objects $\underline{X}$ scheme underlying + $\alpha: M_X \to \mathcal{O}_X$

sheaf of monoids; mult. monoid; monoid map. comm. monoids

PRELOG STRUCTURE

D. Abr.: "special dust over X"

def. log-structures are pre-log structures s.t. $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$

(idea: things that have more then 1 preimage have to vanish; sort of the branches of the logarithm)

Fact pre-log str. gives an associated log-str.

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^* \arrow[r, hook] \arrow[d] & M_X \arrow[d] \arrow[dr, "\alpha"] & \\ \mathcal{O}_X^* \arrow[r] & M_X^a \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

sort of product. assoc. log. str.

Example $\underline{X}$ smooth, $D \subset X$ divisor

$$M_X(U) = \{ g \in \mathcal{O}_X(U) \mid g|_{U \setminus D} \in \mathcal{O}^* \} \subset \mathcal{O}_X(U)$$

(the case D normal crossing is going to be "log smooth")

This is very related to logarithmic differentials, that in fact is the motivation for the theory.

Example (building blocks) P monoid, R a ring $\Rightarrow R[P]$ ring, then

$\underline{X} = \operatorname{spec}(R[P])$, $\alpha: P \to R[P]$

e.g. $P = \mathbb{N}$, $R = \mathbb{Z}$, $\underline{X} = \operatorname{spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$ $\quad \alpha: P \to R[P]$, $e_1 \mapsto t$ (is as before $D \subset X$ where $D = \{0\}$, $X = \mathbb{A}^1_{\mathbb{Z}}$)

Morphisms $f: \underline{X} \to \underline{Y}$ morph. of schemes, $\alpha_Y: M_Y \to \mathcal{O}_Y$ log str. on Y.

we can pull it back to $f^{-1}(M_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \rightsquigarrow$ call it $f^* M_Y \to \mathcal{O}_X$ "inverse image"

A map between $(\underline{X}, M_X)$ and $(\underline{Y}, M_Y)$ is $f: \underline{X} \to \underline{Y}$ and $f^\flat: f^* M_Y \to M_X$ (flat)

$$\begin{tikzcd} f^* M_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M_X \arrow[d] \\ f^{-1}(\mathcal{O}_Y) \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

<u>Building blocks of morphisms</u> two monoids $\theta: Q \to P$, R ring,

we get $R[Q] \to R[P]$ $\rightsquigarrow$ we get morph. of log schemes $(\operatorname{spec} R[P], P) \to (\operatorname{spec} R[Q]$

There is a lot of toric stuff going on.

"The log structures-morphisms that are made (etale locally) by these building blocks, are locally toric varieties/morphisms."

<u>Ex.</u> $\underline{X} = \operatorname{Spec} \dfrac{k[x_1, \dots, x_n]}{(x_1 \cdot x_2 \cdots x_n)}$ $\qquad \mathbb{N}^r \to \mathcal{O}_X$, $e_i \mapsto x_i$

$p = \operatorname{spec}(k) \xhookrightarrow[f]{\text{origin}} \underline{X}$ $\qquad f^*(M_X) = k^* \oplus \mathbb{N}^r \longrightarrow \mathcal{O}_p = k$

$(a, v) \longmapsto a$ if $v = 0$, 0 if $v \neq 0$

this k^* comes from "logification" of the pre log $(\mathbb{N}^r \to \mathcal{O}_{\underline{X}})$

this is called a <u>log pt</u> of rk r. <u>standard</u> when $r=1$

Let's restrict $\alpha: M_X \to \mathcal{O}_X$, because otherwise there are too many patologies.

Restrictions coming from geometry. $\qquad P^{gp} = \{(a,b) \in P^2\} / (a,b) \sim (c,d) \Leftrightarrow \exists s \mid s+a+d = s+b+c$

- P monoid is <u>integral</u> if $P \to P^{gp}$ is injective
- P is <u>fine</u> if it is fin. gen.
- P <u>saturated</u> if $n \cdot p \in P$ $(p \in P^{gp}) \Rightarrow p \in P$.

<u>Ex.</u> $P = \mathbb{N} \setminus \{1\}$, $\operatorname{spec}(P \to k[P]) = \operatorname{Spec} k[x^2, x^3]$ cusp.
here $1 \in P^{gp} = \mathbb{Z}$ so this is not saturated.

<u>Chart</u> $(\underline{X}, M_X)$ a log scheme, P monoid. A <u>chart</u> for M_X is
(const. sheaf) $P_X \to M_X \xrightarrow{\alpha} \mathcal{O}_X$ s.t. $P_X^a \to \mathcal{O}_X$ is isom. to M_X

<u>Similarly</u> $f: \underline{X} \to \underline{Y}$ a map. A chart is $Q, P, Q \xrightarrow{\theta} P$ where

$$\begin{tikzcd} P_X \arrow[r] & M_X \\ Q_X \arrow[u, "\theta"] \arrow[r] & M_Y \arrow[u, "f^b"'] \end{tikzcd}$$

<u>Differentials</u> (Log smooth curves)

(Grothendieck point of view on derivations)

$$\begin{tikzcd} T_0 \arrow[r, "\varphi"] \arrow[d] & X \arrow[d, "f"] \\ T_1 \arrow[r] & Y \end{tikzcd}$$

T_1 nilpot. thickening of T_0 with $I^2=0$

we want to define $\Omega^1_{X/Y}$ (enhancement of $\Omega^1_{\underline{X}/\underline{Y}}$, still an $\mathcal{O}_{\underline{X}}$-module)

(X, Y log shemes, T_0, T_1 schemes)

In the classical setting

we consider two liftings g_1, g_2

$$\begin{tikzcd} T_0 \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\ T_1 \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r, "\psi"] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{O}_{T_0} & \psi^{-1}(\mathcal{O}_X) \arrow[l] \arrow[dl, shift right, "g_1^\#"'] \arrow[dl, shift left, "g_2^\#"] \arrow[d, leftrightarrow] \\ \mathcal{O}_{T_1} \arrow[u] & \psi^{-1}(\mathcal{O}_Y) \arrow[l] \end{tikzcd}$$

$\rightsquigarrow g_1^\# - g_2^\# : \psi^{-1}(\mathcal{O}_X) \to J$ derivation in $\mathrm{Der}_Y(X, \mathcal{O}_X)$ "$\partial_{g_1 - g_2}$" we call it

here J in general but often $J \simeq \mathcal{O}_X$ over X.

Now consider the log situation, we want $j^* M_{T_1} \cong M_{T_0}$ so that

$$1 \to 1 + J \to M_{T_1} \to M_{T_0} \to 1$$

$$\begin{tikzcd} M_{T_0} & \psi^{-1}(M_X) \arrow[l] \arrow[dl, shift right, "g_1^\flat"'] \arrow[dl, shift left, "g_2^\flat"] \\ M_{T_1} \arrow[u] & \psi^{-1}(M_Y) \arrow[l] \arrow[u, "f^\flat"'] \end{tikzcd}$$

$g_1^\flat - g_2^\flat : \psi^{-1} M_X \to J$ ADDITIVE, such that

(1) $D(ab) = D(a) + D(b)$

(2) $\alpha(a) \cdot D(a) = \partial(\alpha(a))$

(3) $D(a) = 0 \;\; \forall a \in \mathrm{Im}(f^{-1}(M_Y) \to M_X)$

(2) gives $D(a) = \dfrac{\partial(\alpha(a))}{\alpha(a)}$

So D acts as "dlog"

So g_1, g_2 gives (∂, D)

classical derivation

log derivatives-like with inputs in M_X

<u>Point</u> (∂, D) can be constructed locally, they define a sheaf $\mathrm{Der}_Y(X, J)$ and when $J \simeq \mathcal{O}_X$ this is dual to $\Omega^1_{X/Y}$ (cotangent bundle)

<u>Log smooth</u> $f : X \to Y$ is smooth if $\forall$ $\begin{tikzcd} T_0 \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\ T_1 \arrow[r, "\psi"] & Y \end{tikzcd}$ $\exists g : T_1 \to X$ making it commute

sort of evaluative criterion (if $\exists !$, then is étale)

we follow these definitions in the log case.

<u>Kato's thm</u> $f : X \to Y$ is log smooth (étale) iff étale locally on X there are charts (P, Q, θ) s.t.

(a) kernel + torsion of coker of $Q^{gp} \to P^{gp}$ are finite with invertible order (char k ∤ order) (to define étale we need the whole cokernel)

(b) $X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}(\mathbb{Z}[P])$ needs to be étale

(b) is like asking that locally the map is a projection from a toric variety to another.

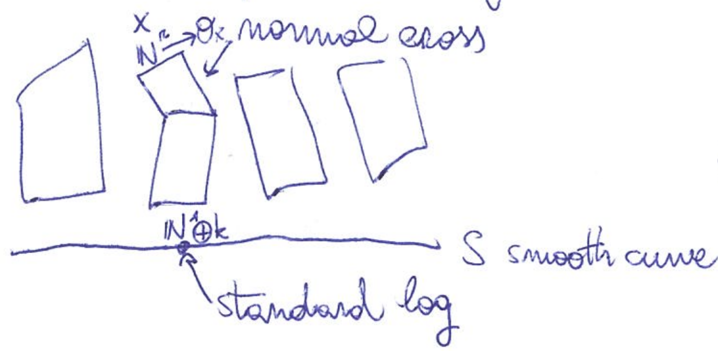
e.g. $\mathbb{N} \setminus \{1\} \Rightarrow P$, $\begin{matrix} \operatorname{spec} k[P] \\ \downarrow \\ pt \end{matrix}$ is log smooth!

def. log curves $f : X \to S$ gen fibers are reduced 1-dim'l connected X, S are fine, saturated, and f is log smooth.

Kato log curves are semistable (log smooth curve is stable if it has no inf. automorphism)

$\rightsquigarrow \overline{M}_g$ is "collection" of stable log smooth curves.

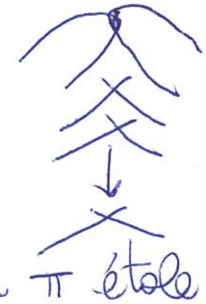
other example



there $X \to pt$ is still log smooth, with $\theta : \mathbb{N}^1 \to \mathbb{N}^2$, $e_1 \mapsto (e_1, \dots, e_1)$ diagonal

If $X \to p, k^* \oplus \mathbb{N}$ does not admit a log structure of Δ diagonal type, then X cannot be smoothed in a smooth total space. (Classically known as d-semistability)

Other example admissible covers problem:



this map is not flat!

but there are natural log structures which make π étale

Mochizuki's thesis is about this; using log geometry admissible covers are just log-étale maps.

What about surfaces? For K3, ~~Olsson~~. Classically one tries to do semistable reduc on families $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ and try to make fibers as less singular as possible.

$\rightsquigarrow$ there is mod. space of log smooth K3 surfaces