Construction of Euler Systems for $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$

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Why Euler system?

V: p-adic representation of $\mathrm{Gal}_{\mathbb{Q}}$

- "Existence" of Euler system for V
  $\Rightarrow$ upper bound on the size of Selmer groups of V
  $\Rightarrow$ Iwasawa Main Conjecture for V
  (comparison of Selmer groups and p-adic L-function)

- Explicit Reciprocity Law (relation with p-adic L-function)
  $\Rightarrow$ construct p-adic L-function

Goal

Construct Euler system for V, for V is attached to a automorphic representation of $G = \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ (non-endoscopic, ordinary at p).

What is an Euler system?

V: p-adic representation of $\mathrm{Gal}_{\mathbb{Q}}$, unramified outside $\Sigma \ni p$
$T \subset V$: $\mathrm{Gal}_{\mathbb{Q}}$-stable lattice
$P_\ell(X; V) = \det(I - X \cdot \mathrm{Frob}_\ell^{-1} \mid V)$
When $V \leftrightarrow$ automorphic representation Π,

$$P_\ell(\ell^{-s}; V) = L(s, \Pi_\ell)^{-1}.$$

Definition

An Euler system for $(T^*(1), \Sigma)$ is a collection of

$$c_{Mp^n} \in H^1(\mathbb{Q}(\mu_{Mp^n}), T^*(1))$$

where M: square-free products of primes $\ell \notin \Sigma$, and $n \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{aligned} &\text{(tame)} \quad \mathrm{Nm}_{\mathbb{Q}(\mu_{Mp^n})}^{\mathbb{Q}(\mu_{\ell Mp^n})} c_{\ell Mp^n} = P_\ell(\mathrm{Frob}_\ell^{-1}; V) \cdot c_{Mp^n} \\ &\text{(wild)} \quad \mathrm{Nm}_{\mathbb{Q}(\mu_{Mp^n})}^{\mathbb{Q}(\mu_{Mp^{n+1}})} c_{Mp^{n+1}} = c_{Mp^n} \end{aligned}$$

Example of an Euler system

$V = \mathbb{Q}_p,\ T = \mathbb{Z}_p,\ \Sigma = \{p\} \rightsquigarrow T^*(1) = \mathbb{Z}_p(1)$

Construction

$$
\begin{array}{lccc}
 & H^1(\mathbb{Q}(\mu_{Mp^n}), \mathbb{Z}_p(1)) & \xleftarrow{\text{Kummer}} & \mathbb{Q}(\mu_{Mp^n})^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \\
(n \geq 1) & c_{Mp^n} & \longleftarrow\!\shortmid & 1 - \zeta_{Mp^n} \\
(n = 0) & c_M & \longleftarrow\!\shortmid & \mathrm{Nm}^{\mathbb{Q}(\mu_{Mp})}_{\mathbb{Q}(\mu_M)}(1 - \zeta_{Mp})
\end{array}
$$

Norm relations

Direct computation:

$$\mathrm{Nm}^{\mathbb{Q}(\mu_{\ell Mp^n})}_{\mathbb{Q}(\mu_{Mp^n})}(1 - \zeta_{\ell Mp^n}) = (1 - \mathrm{Frob}_\ell^{-1}) \cdot (1 - \zeta_{Mp^n}) = \frac{1 - \zeta_{Mp^n}}{1 - \zeta_{Mp^n}^{1/\ell}}$$

$$\mathrm{Nm}^{\mathbb{Q}(\mu_{Mp^{n+1}})}_{\mathbb{Q}(\mu_{Mp^n})}(1 - \zeta_{Mp^{n+1}}) = 1 - \zeta_{Mp^n}, \text{ if } n \geq 1$$

Integral formula for L-function

Starting point: $\displaystyle\int_0^\infty \frac{\theta(it)-1}{2} t^{s/2} dt = \pi^{-s/2}\Gamma(s)\zeta(s)$

- $\mathbb{G}_m$: $\displaystyle\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \chi(x)|x|^s\theta_\phi(x)dx$ represents $L(s,\chi)$
- GL_2: $\displaystyle\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \varphi_\Pi(\begin{pmatrix} x & \\ & 1\end{pmatrix})|x|^{s-1/2}dx$ represents $L(s,\Pi)$
- $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$: $\displaystyle\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi_\Pi(h)E_\phi(h_1,s)d(h_1,h_2)$ represents $L(s,\Pi)$, where $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow G = \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$

Construction in étale cohomology

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi_\Pi(h) E_\phi(h_1, s) d(h_1, h_2)$$

$$\begin{array}{ccc}
c_{Mp^n} & \in & H^1(\mathbb{Q}(\mu_{Mp^n}), T^*(1)) \\
 & & \cap \\
 & & H^d_{\text{ét}}(Y_{G,\overline{\mathbb{Q}}}(K_\Sigma), \mathcal{L}^*(1)) \\
c_{Mp^n} & \in & H^1(\mathbb{Q}(\mu_{Mp^n}), H^d_{\text{ét}}(Y_{G,\overline{\mathbb{Q}}}(K_\Sigma), \mathcal{L}^*(1))) \\
 & & \text{spectral} \updownarrow \text{sequence} \\
 & & H^{d+1}_{\text{ét}}(Y_{G,\mathbb{Q}(\mu_{Mp^n})}(K_\Sigma), \mathcal{L}^*(1)) \\
 & & \updownarrow \\
 & & H^{d+1}_{\text{ét}}(Y_{G,\mathbb{Q}}(K_\Sigma(Mp^n)), \mathcal{L}^*(1))
\end{array}$$

Construction in etale cohomology (cont'd)

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi_\Pi(h) E_\phi(h_1, s) d(h_1, h_2)$$

$$\begin{tikzcd}
\mathcal{S}(\mathbb{A}_f^2) \arrow[d, "Eis"] \arrow[ddd, bend left, "\otimes_{\mathcal{H}(H)}\mathcal{H}(G)"] \\
H^1_{\text{ét}}(Y_{\mathrm{GL}_2}, \mathbb{Q}_p(1)) \arrow[d, "\mathrm{pr}_1^*"] \\
H^1_{\text{ét}}(Y_H, \mathbb{Q}_p(1)) \arrow[d, "pushforward"] \\
H^{d+1}_{\text{ét}}(Y_G, \mathbb{Q}_p(1 + \frac{d}{2}))
\end{tikzcd}$$

Call this map

$$ES \colon \mathcal{S}(\mathbb{A}_f^2) \otimes_{\mathcal{H}(H)} \mathcal{H}(G) \to H^{d+1}_{\text{ét}}.$$

We want to define

$$c_{Mp^n} = ES(\phi_{Mp^n} \otimes \xi_{Mp^n}).$$

What ϕ_{Mp^n}, ξ_{Mp^n} to choose to satisfy (tame) norm relation?

Tame norm relation

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi_\Pi(h) E_\phi(h_1, s) d(h_1, h_2)$$

- When Π has a Whittaker model, can define local integral $Z(\varphi_\ell, s)$ so that
$$Z(\varphi_\ell^0, s) = L(s, \Pi_\ell)$$
where φ_ℓ^0: normalized spherical vector.
- Compute to find Hecke operator ξ_s so that
$$Z(\xi_s \cdot \varphi_\ell^0, s) = 1.$$

Can construct from Z a non-zero Hecke-equivariant

$$\mathfrak{Z}\colon \mathcal{S}(\mathbb{Q}_\ell^2) \otimes \mathcal{H}(G) \to \Pi_\ell^\vee$$

$$\mathfrak{Z}(\phi \otimes \xi)(\varphi_\ell) \\ =< F_\phi, Z(\xi \cdot \varphi_\ell, s) >$$

so that $\mathfrak{Z}(\phi \otimes \xi_s) = (*) \cdot L(s, \Pi_\ell)^{-1} \cdot \mathfrak{Z}(\phi \otimes \mathrm{ch}\, G(\mathbb{Z}_\ell))$ for all suitable ϕ.

Tame norm relation (cont'd)

$$\mathfrak{Z}(\phi \otimes \xi_s) = (*) \cdot L(s, \Pi_\ell)^{-1} \cdot \mathfrak{Z}(\phi \otimes \mathrm{ch}\, G(\mathbb{Z}_\ell))$$

- Gan–Gross–Prasad conjecture $\Rightarrow$ ES_ℓ and $\mathfrak{Z}$ are equal up to scalar
- ϕ_{Mp^n} is chosen suitably to normalize $(*)$
- ξ_{Mp^n} is chosen so that its ℓ-component is ξ_0