Def $f: \mathcal{X} \to \mathcal{Y}$ mor of stacks is <u>representable</u>

if $\forall$ scheme U & mor $y: U \to \mathcal{Y}$

$\mathcal{X} \times_{\mathcal{Y}, y} U$ is an alg space

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y},y} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ U \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

Lem If $f: \mathcal{X} \to \mathcal{Y}$ representable

then $\forall$ alg space V & mor $y: V \to \mathcal{Y}$

$\mathcal{X} \times_{\mathcal{Y}, y} V$ is an alg. space.

Def A stack $\mathcal{X}/S$ is an <u>alg stack</u> if

(i) $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable

(ii) $\exists$ sm surj mor $\pi: X \to \mathcal{X}$ w/ X a sch.

Lem $\mathcal{X}/S$ stack $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable iff

$\forall$ S-sch U & $u_1, u_2 \in \mathcal{X}(U)$

the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on $Sch(U)$ is an alg space.

Pf

$$\begin{tikzcd} \underline{\mathrm{Isom}}(u_1,u_2) \arrow[r] \arrow[d] & U \arrow[d, "u_1 \times u_2"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

$\square$

It makes sense to talk about (ii) w/ (i)

(i) $\Rightarrow$ Every $t: T \to \mathcal{X}$ is rep.
(T sch)

If $u: U \to \mathcal{X}$ is another mor, $U \times_{u,\mathcal{X},t} T$ is alg sp. by (i)
(U sch)

$$\begin{tikzcd} U \times_{u,\mathcal{X},t} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S T \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

Exmp X alg sp G/S smooth sch. $\curvearrowright X$.

Define $[\cancel{X}/G]$

Obj : (T, P, π) (i) T S-sch

(ii) P a $G_T := G \times_S T$ - torsor on the big ét site of T

(iii) $\pi : P \to X \times_S T$ G_T - equivariant mor of sheaves on (Sch/T)

Mor : $(f, f^b) : (T', P', \pi') \to (T, P, \pi)$

$$\begin{tikzcd} T' \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & T \arrow[dl] \\ & S & \end{tikzcd}$$

$f^b : P' \to f^* P$ isom. of G_T - torsor on (Sch/T')

s.t.

$$\begin{tikzcd} P' \arrow[rr, "f^b"] \arrow[dr, "\pi'"'] & \arrow[d, phantom, "\circlearrowleft"] & f^* P \arrow[dl, "f^*\pi"] \\ & X \times_S T' & \end{tikzcd}$$

Claim $[X/G]$ is a stack

WTS $[X/G]$ is an alg stack.

Let T be an S-sch. (T, P_i, π_i) $(i = 1, 2)$ $\in [X/G]$

Simplified as (P_i, π_i)

$I := \mathrm{Isom}((P_1, \pi_1), (P_2, \pi_2))$:

$$T'/T \longmapsto \begin{tikzcd} P_1|_{T'} \arrow[rr, "\simeq"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & P_2|_{T'} \arrow[dl] \\ & \cancel{X}_{T'} & \end{tikzcd} \text{ as } G_{T'} \text{ - torsors}$$

Fact S base sch. Y/S alg space (Sch/S) w/ ét top

$g : F \to Y$ a mor of sheaves ~~U sch~~

$$\begin{tikzcd} F \times_Y U \arrow[r] \arrow[d] & U \arrow[d, "\text{ét surj}"] \\ F \arrow[r] & Y \end{tikzcd}$$

U - sch

$F \times_Y U$ alg sp $\Rightarrow$ F alg sp.

By Fact, to verify that I is an alg sp. we can replace T by an ét. covering. Hence we may assume P_1 and P_2 are trivial. Fix

$$\begin{tikzcd} \sigma_i : P_i \arrow[rr, "\simeq"] \arrow[dr, "\pi_i"'] & & G_T \arrow[dl, "\rho_i"] \\ & X_T & \end{tikzcd}$$

When P_1 & P_2 are trivial we have

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_T \arrow[d] \\ X_T \arrow[r, "\Delta"] & X_T \times_T X_T \end{tikzcd}$$

(It's essentially about how to identify P_1 & P_2)

Thus I is an alg sp.

- A sm covering of $[X/G]$ is given by $g : X \longrightarrow [X/G]$ defined by (G_X, ρ)
  - $G_X := G \times X$ trivial torsor
  - $\rho : G_X = G \times X \to X$ action map.

Def P property of mor of alg sp. which is stable w.r.t ~~sm.~~ top. A rep mor of alg stacks $f: \mathcal{X} \to \mathcal{Y}$ <u>has property P</u> if $\forall$ $Y \to \mathcal{Y}$ Y alg sp.

$$\mathcal{X} \times_{\mathcal{Y}} Y \to Y \text{ of alg sp. has } P$$

Exmp étale, sm of rel dim d, separated. ...

Def An alg stack $\mathcal{X}/S$ is <u>Deligne–Mumford</u> if $\exists$ rep. ét surj $X \to \mathcal{X}$ w/ X a scheme.

We have another characterization of being DM

Thm $\mathcal{X}/S$ an alg stack.

$\mathcal{X}$ is DM $\Leftrightarrow$ $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.

Prop $[X/G]$ is DM iff $\forall\ \bar{s}: \operatorname{Spec}(k) \to S$, $x \in [X/G](k)$
the stabilizer group scheme $G_x \subset G_{\bar{s}}$ is ét $/\bar{s}$

k : alg closed

$$\begin{array}{ccc} G_{\bar{s}} & \longrightarrow & G \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & S \end{array}$$