Anand: Unirationality of some moduli spaces of curves

Question: when is M_g unirational?

(unirational: $\mathbb{P}^N \xrightarrow{\text{dominates}} M_g$)

$(f_1, \cdots, f_N) \leftarrow$ free parameters

general genus g curve has coeffs in these free parameters.

First instance: (F Severi 1915) "M_g is probably unirational"

Severi: M_g is unirational for $g \leq 10$.

Idea: Cram the curve into the plane $\mathbb{P}^2$

$g=1 \rightsquigarrow j$ invariant

$g=2 \rightsquigarrow$ all hyperelliptic - vary the branch pts.

$g=3$: General $g=3$ curve is a plane quartic (canonical model)

$\binom{4+2}{2} - 1 \rightsquigarrow \mathbb{P}^{14}$ linear system

$g=4$: Introduce nodes. A degree 5 curve has $p_a = \binom{5-1}{2} = 6$

Hence we want a degree 5 curve with 2 nodes.

Fix the nodes $\longrightarrow$ 6 linear conditions on coeffs

$$\rightsquigarrow \mathbb{P}^{20-6} = \mathbb{P}^{14}$$

Have we covered the general genus 4 curve?

Question: When does the general genus g curve have a map to $\mathbb{P}^r$ ($\mathbb{P}^2$) of degree d.

Brill-Noether theory answers this.

$$\rho(g,r,d) = g - (r+1)(g-d+r)$$

<u>Theorem</u>: If $\rho \geq 0$, then the general genus g curve has a map to $\mathbb{P}^r$ of degree d.

$$W^r_{d,g} = \{(C,L) \mid \deg L = d,\ h^0(L) \geq r+1\}.$$

$$\begin{array}{c} W^r_{d,g} \\ \downarrow \\ M_g \end{array} \quad \text{when is this map dominant?}$$

<u>$g=5$</u>: $\rho(5,2,5) < 0$

$\Rightarrow$ can't use quintics again

$\rho(5,2,6) \geq 0$

Sextics with 5 nodes.

$\dim M_5 = 12$

Can we fix the position of the <u>5 nodes</u>?

$\mathbb{P}^{27} = |\mathcal{O}_{\mathbb{P}^2}(6)|$, 5 nodes fixed impose $3 \times 5 = 15$

$\mathbb{P}^{12}$ of quintics singular at the 5 fixed nodes

But the 5 fixed nodes in $\mathbb{P}^2$ have moduli (when there were 2 we could move them)

Hence we may not hit the general curve.

(In fact $\mathbb{P}^{12} \overset{\text{dominates}}{\dashrightarrow} M_5$).

<u>$g=6$</u>: $\rho(6,2,6) \geq 0$. Sextics with 4 nodes.

You can fix the location of the 4 nodes

$$\mathbb{P}^{15} \overset{\text{dominant}}{\dashrightarrow} M_6$$

$g=7$: $\rho(7,2,7)$. septics with 8 nodes......

Main Diagram: Severi Variety $= \{$plane curves of degree d with δ-nodes$\}$

$$\begin{tikzcd} V_{d,\delta} \arrow[r, "m"] \arrow[d, "h"] & M_g \\ (\mathbb{P}^2)^{(\delta)} & \end{tikzcd}$$

this map is dominant when $\rho(g,r,d) \geq 0$.

$d \geq \left[\frac{2}{3}g\right]+2$.

dominant when general δ-"singular conditions" are independent

space of δ points in $\mathbb{P}^2$

(1) $\dim|\mathcal{O}(d)| \geq 3\delta$

(2) $d \geq \left[\frac{2}{3}g\right]+2$

Severi thought this worked and stopped at $g=10$. But these two inequalities are competing. We know
$\delta = p_a - g = \binom{d-1}{2} - g$
Hence these conditions are both met only for $g \leq 10$

$g=7$ (ctd): Here the nodes have to change location

Fix 8 nodes $\Rightarrow \mathbb{P}^{11}$ of septics

But M_7 has dim 18.

So we vary the location of the nodes to get more dimensions.

$g=10$: $d=9$, 18 nodes will determine (gen)

a unique genus 10 curve.

Focus on the location of nodes

$\boxed{\text{Segre 1938}}$: Question: Maybe $\mathbb{P}^2$ sucks. Try other surfaces?

Perhaps: $\exists$ S & a linear system $|L|$ on S, s.t $|L| \xrightarrow{m} M_g$ is dominant.

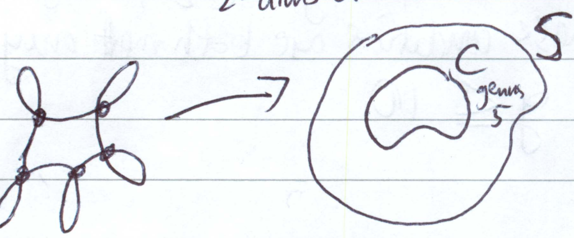
$g=6$: Sextics with 4 nodes

$S = Bl_4 \mathbb{P}^2 \quad |L| = |6H - 2\sum E_i|$

$g=5$: Sextics with 5 nodes

$S = Bl_5 \mathbb{P}^2$

$\uparrow$ 2-dims of moduli



$|3H - \sum^5 E_i|$

$S \hookrightarrow \mathbb{P}^4$

degree 4 surface in $\mathbb{P}^4$ contains C

degree del Pezzo $\rightarrow$ complete intersections of 2 quadrics in $\mathbb{P}^4$ (pencil of quadrics P)

$C \hookrightarrow \mathbb{P}^4$ canonical model....

$\boxed{\text{Segre:}}$ Answer: If $|L|$ on a surface S dominates M_g then $g \leq 6$ & S is rational.

This is the first instance of a negative result.

Fast forward to the 80's....

~~$h^0(mK_{M_g})\sim$~~ <u>Theorem</u> (Eisenbud, Harris, Mumford)

M_g is of general type for $g \geq 24$.

i.e $h^0(mK_{M_g}) \sim$ poly of degree $\dim M_g$.

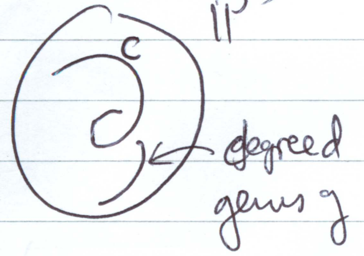
i.e A general curve of $g \geq 24$ is not an element of a pencil of curves on a surface

There are some cases left:

<u>$11 \leq g \leq 14$</u>: Underlying all proofs is same surface

a complete intersection $S \xhookrightarrow{\text{canonical}} \mathbb{P}^n$, $\mathcal{O}_S(1) = K_S$.

<u>Chang-Ran</u>: $M_{11,12,13}$ are unirational: <u>Idea</u> work in $\mathbb{P}^3$



$\rho(g,3,d) \geq 0 \longrightarrow$ There exists a vector bundle F on $\mathbb{P}^3$ of rank 2 s.t C is the zero locus of a section

Construct F as cohomology of $\mathcal{O}(-1)^{\oplus a} \to \mathcal{O}^{\oplus b} \to \mathcal{O}(1)^{\oplus c}$

Then....

Grandmaster: <u>A. Verra</u>: M_{14} is unirational & my technique works for $M_{11,12,13}$.

<u>Idea</u>: $W_{8,14}$ = curves with a g_8: $C \xrightarrow[14]{8:1} \mathbb{P}^1$

$$\begin{array}{c} W_{8,14} \\ \downarrow \\ M_{14} \end{array}$$

consider: $|K_C - L|$, degree 18 in $\mathbb{P}^6$

$C \xhookrightarrow{|K_C - 4|} \mathbb{P}^6$ degree 18

$h^0(I_C(2)) = 5, \quad Q_1, \dots, Q_5$

$Q_1 \cap \dots \cap Q_5 = C \cup D$ ← there is some D

D - genus 8, degree 14

$h^0(I_D(2)) = 7$

degree, genus, unirational

<u>Mukai+Verra</u>:

$$\begin{tikzcd}
 & \mathrm{Pic}_{14,8} \arrow[dl, bend right] \arrow[d] \\
\mathrm{Gr}(5,7) \text{ bundle} \arrow[d] \arrow[dr, "\text{dominantly}"] & M_8 \\
\mathrm{Pic}_{14,8} & M_{14}
\end{tikzcd}$$

unirational

<u>Mukai</u>: General genus 8 curve is a linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$

M_{15} - rationally connected

M_{16} - uniruled

$M_{17, \dots, 21}$ <u>open</u> (Kodaira dim unknown)

<u>See</u>: Verra's paper in "the handbook of moduli"