Alg. stacks — a few examples.

Last time we developed the notion of an alg. stack:

$$\begin{array}{c} \mathcal{M} \\ \downarrow \\ Sch/_S \end{array}$$

- is a stack if 1) any $\begin{array}{c} T \\ \downarrow \\ S \end{array}$, $\xi, \eta \in \mathcal{M}(T)$ then $\underline{Isom}_S(\xi, \eta)$ is a sheaf in the étale/sm. top.
  2) Every descent datum is effective.
- is an alg. stack if 1) $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable. (by an alg. space).
  2) $\exists$ sm/étale morphism $\pi : X \to \mathcal{M}$, $X \downarrow$ scheme.
- is a DM stack if 1) Δ is formally unramified. (Roughly means no object has infinitesimal automorphisms.)

__Example__: $[X/G]$ quotient stack

X an alg space, G smooth group scheme S $\circlearrowright X$.

As a category $[X/G]$ has objects G-torsors (also called principal G-bundles).

These are triples $(T, \mathcal{P}, \pi)$ s.t. 1) $\begin{array}{c} T \\ \downarrow \\ S \end{array}$ is an S-scheme

2) $\mathcal{P}$ is a G_T-bundle

3) $\pi : \mathcal{P} \to X_T$ G_T-~~[illegible]~~ equivariant.

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dashed, "\pi"'] \arrow[d] \arrow[loop right, "G \text{ action}"] \\ X_T \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

Morphisms are (f, f^b) s.t. $\begin{tikzcd} T' \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\supseteq"] & T \arrow[dl] \\ & S & \end{tikzcd}$, $\begin{tikzcd} \mathcal{P}' \arrow[rr, "f^b"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\supseteq"] & f^*\mathcal{P} \arrow[dl] \\ & X \times_S T' & \end{tikzcd}$ and $\underbrace{f^b \text{ is isom.}}$ This is crucial to make sure $[X/G]$ is fibered in groupoids.

The projection $[X/G] \to Sch/_S$ is $(T, \mathcal{P}, \pi) \mapsto T$.

Assume G is an affine group scheme, $S = \operatorname{Spec} \mathbb{C}$.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\pi"] \arrow[d] & X \\ T & \end{tikzcd} \quad G$$

__Rmk__: $G \curvearrowright X$ acts freely ~~[illegible]~~ Then $\exists$ X/G scheme..

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$$\begin{array}{ccc} [X/G] & \longrightarrow & Sch/(X/G) \\ \begin{matrix} P \xrightarrow{\pi} X \\ \downarrow \\ T \end{matrix} & \longmapsto & \begin{matrix} P/G = T \\ \downarrow \\ X/G \end{matrix} \end{array}$$

can be inv.

This shows X/G represents $[X/G]$.

Main case of interest

$[*/G] = BG$ the classifying stack.

To see $[X/G]$ is an alg. stack: Recall

Lemma: $\mathcal{M}$ stack$/S$. Then $\Delta: \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is repr. iff $\forall U \to S$, $u_1, u_2 \in \mathcal{M}(U)$ then $\underline{Isom}(u_1, u_2)$ on U is an alg. space.

$I = \underline{Isom}((P_1, \pi_1), (P_2, \pi_2)) : Sch/_T \to Sets$

$$\begin{matrix} T' \\ \downarrow \\ T \end{matrix} \longmapsto \left\{ \begin{matrix} \text{isom of } G_T\text{-spaces} \\ \begin{tikzcd} P_1|_{T'} \arrow[rr, "\cong"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\supseteq"] & P_2|_{T'} \arrow[dl, "\pi_2"] \\ & X_T & \end{tikzcd} \end{matrix} \right\}.$$

Fact: if Y/S is alg space, F is a sheaf on $(Sch/S)_{ét}$, $g: F \to Y$

Then if $\exists$ ét surj $\underset{\text{scheme}}{U} \longrightarrow Y$ s.t. $F \times_Y U$ is alg space $\Rightarrow F$ is alg. space. (Exercise in book.)

We can apply this to show ~~[illegible]~~ I as above is an alg. space.
It is also possible to furnish an étale atlas for $[X/G]$ so it is indeed an alg stack.

Second example of interest: $\mathcal{M}_g$.

assume $g \geq 2$.

$\mathcal{M}_g$ objects: Families $\begin{matrix} \mathcal{C} \\ \downarrow f \\ S \end{matrix}$ of smooth genus g curves.

Morphisms: {Cartesian squares

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

The forgetful functor $\begin{matrix} \mathcal{C} \\ \downarrow \\ S \end{matrix} \mapsto S$ gives $\mathcal{M}_g \to Sch/S$ as a structure morphism.

Actually $\mathcal{M}_g$ is a stack. (DM).

Want to show $\mathcal{M}_g = [X/G]$ for some X & G.

Take X to be a closed subscheme of some Hilbert scheme, $G = GL(n)$ acting on it.

Step 1: Observation:

Lemma: $\begin{matrix} \mathcal{C} \\ \downarrow f \\ S \end{matrix} \in \mathcal{M}_g$ $(\omega_{C/S})^{\otimes 3} = L_{C/S}$.

(i) $f_* L_{C/S}$ is locally free of rk $5g - 5$.

(ii) $f^* f_* L_{C/S} \to L_{C/S}$ is ~~[illegible]~~ surj.

$C \hookrightarrow \mathbb{P}(f_* L_{C/S})$ is a closed emb.

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Construct :

$\tilde{\mathcal{M}}_g : Sch \longrightarrow Sets$

$S \longmapsto \left\{ \begin{matrix} \mathcal{C} \\ \downarrow f \\ S \end{matrix} , \ \sigma : f_* L_{C/S} \to \mathcal{O}_S^{5g-5} \right\} / \cong .$

representable : Take $\left(\begin{matrix} \mathcal{C} \\ \downarrow f \\ S \end{matrix} , \sigma \right) \in \tilde{\mathcal{M}}_g$.

$$\iota : \mathcal{C} \xrightarrow[\text{lemma}]{} \mathbb{P}(f_* L_{C/S}) \xrightarrow{\mathbb{P}\sigma} \mathbb{P}^{5g-6}$$

Observe : The Hilb poly of such an embedding is $(6g-6)m - g + 1$

$\mathcal{H}' = \text{Hilb}$ w/ Hilb poly in $\mathbb{P}^{5g-6}$.

Get a locally closed subscheme $\tilde{H} \subset \mathcal{H}'$ by imposing pts yield sm. gen g curves.

Get a map $F : \tilde{M}_g \to \tilde{H}$ isom.

Get an action $PGL(5g-5) \curvearrowright \tilde{\mathcal{M}}_g$ and construct $\mathcal{M}_g$ as quotient.

action is ~~free~~. not nec. free.

i.e. $\mathcal{M}_g = [\tilde{\mathcal{M}}_g / PGL(5g-5)]$. is an alg. stack.

We can also show $\mathcal{M}_g$ is DM.

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