Grothendieck Sites, Fibered Categories

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This talk has two parts, both aimed at generalizing the notion of sheaves on a topological space.

From "topological spaces" to "sites"

Define a **Grothendieck (pre-)topology** on a category: isomorphisms, locality, base change. Define a **site**. [Mention Grothendieck topologies with sieves, in contradistinction to the Grothendieck pretopologies we're discussing.]

Presheaves (which do *not* depend on topology!), separated presheaves, and sheaves. Left-adjoints to forgetful functors. Sheafification.

Indiscrete topology (only isomorphisms): all presheaves are sheaves. **Canonical topology** (sheaf iff representable). A sheaf over one topology is automatically a sheaf over a coarser topology. We will study only subcanonical topologies (sheaf if representable).

Any topological space has a site of open subsets. Sheaves with usual meaning.

More examples, over a scheme S: Small/big Zariski, étale, fppf (each $U_i \to U$ is flat and locally of finite presentation), fpqc ($\coprod U_i \to U$ is faithfully flat, and every quasi-compact open subset of U is the image of some quasi-compact open subset of $\coprod U_i$) sites. [Recall a morphism of schemes is **faithfully flat** iff it's flat and surjective. It's **étale** iff it's flat of relative dimension one and with regular geometric fibers. It is **quasi-compact** iff preimages of quasi-compact sets are quasi-compact. It is **quasi-separated** iff the diagonal morphism is quasi-compact. It is **locally of finite presentation** iff its restriction to any open affine subcheme upstairs mapping to an open affine subscheme downstairs corresponds to a finitely presented ring homomorphism. It is of **finite presentation** iff it's locally of finite presentation, quasi-compact, and quasi-separated.]

Algebraic space: a certain kind of sheaf F on big étale site. Two equivalent definitions. Definition 1: (i) there exists an "atlas," i.e., an étale surjective morphism $U \to F$, and (ii) the diagonal Δ_F morphism is representable, quasi-compact, and separated. (Notation for Yoneda embedding is surpressed.) Definition 2: sheafification on big étale site of $T \mapsto X(T)/R(T)$ for some *étale equivalence relation* $R \subset X \times X$ (closed immersion so that each projection $R \to X$ is étale, and for each T, $R(T) \subset X(T) \times X(T)$ is an equivalence relation in the set-theoretic sense). Example: X/G where G is a discrete group acting freely on X.

From "presheaves" to "fibered categories"

Categories over other categories. **Cartesian morphisms** (a.k.a. "pullbacks") upstairs.

The 2-category of **fibered categories** over a base category $\mathcal{C}$. Objects: fibered categories (pullbacks aways exist). Morphisms: Functors over $\mathcal{C}$ preserving Cartesian morphisms. 2-morphisms: base-preserving natural transformations (i.e., mapping to identity morphisms in $\mathcal{C}$). Notation: If $\mathcal{F} \xrightarrow{p_\mathcal{F}} \mathcal{C}$ and $\mathcal{G} \xrightarrow{p_\mathcal{G}} \mathcal{C}$ are fibered categories, then the **fiber** $\mathcal{F}(X)$ is the 1-subcategory of $\mathcal{F}$ consisting of objects over $X \in \mathcal{C}$ and *morphisms over* id_X, and $\mathrm{HOM}_\mathcal{C}(\mathcal{F}, \mathcal{G})$ is the 1-category of morphisms of fibered categories $\mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$.

Morally, properties of a morphism of fibered categories (being fully faithful, or equivalence of categories) can be checked locally, i.e., at the level of $\mathcal{F}(X) \to \mathcal{G}(X)$ for each object $X \in \mathcal{C}$.

Category fibered in P's = fibered category $\mathcal{F} \xrightarrow{p_\mathcal{F}} \mathcal{C}$ such that, for each object $X \in \mathcal{C}$, the 1-category $\mathcal{F}(X)$ is P. (Morally, we identify sets with *discrete* categories, i.e., only identity morphisms.)

Representable category fibered in sets: comma category $\mathcal{C}/Y \to \mathcal{C}$, for any object $Y \in \mathcal{C}$. For any object $X \in \mathcal{C}$, the 1-category $(\mathcal{C}/Y)(X)$ is the *discrete* category on the set $\mathrm{Hom}_\mathcal{C}(X, Y)$. **Two-Yoneda lemma**: there's an *equivalence of categories* (nobody said isomorphism!)

$$\mathrm{HOM}_\mathcal{C}(\mathcal{C}/X,\, \mathcal{F}) \xrightarrow{\mathsf{eval}_{\mathsf{id}_X}} \mathcal{F}(X)$$

given by evaluation at the object $\mathsf{id}_X : X \to X$ in $\mathcal{C}/X$. For objects X and Y in $\mathcal{C}$, there's in fact an *isomorphism* of discrete categories

$$\mathrm{HOM}_\mathcal{C}(\mathcal{C}/X, \mathcal{C}/Y) \to \mathrm{Hom}_\mathcal{C}(X, Y).$$

Presheaves as categories fibered in sets. In fact, an *equivalence of categories*:

$$\text{(presheaves on } \mathcal{C}) \xrightarrow{\Gamma} \text{(categories fibered in sets over } \mathcal{C}).$$

So we can think of fibered categories as generalizations of *pre*sheaves. Over a site, what special kinds of fibered categories correspond to *separated sheaves* and *sheaves*? Answer: *prestacks* and *stacks*, with *descent* replacing the equalizer condition for sheaves (to be discussed next time).

A fibered category is **fibered in groupoids** iff all its morphisms are Cartesian.

Split fibered category: Fibered category F with subcategory K ("splitting" or "cleavage"), with same objects and identity morphisms as F, but only Cartesian morphisms, such that pullbacks are unique. Every fibered category is equivalent (*qua* fibered category) to one that admits such a splitting.

In a category $\mathcal{C}$ with finite fiber products, a **groupoid object**

$$(X_{\mathsf{ob}}, X_{\mathsf{iso}}, \mathsf{src}, \mathsf{tar}, \mathsf{iden}, \mathsf{inv}, \mathsf{mult})$$

gives rise to a category fibered in groupoids, $\{X_{\mathsf{ob}}/X_{\mathsf{iso}}\} \to \mathcal{C}$.

Fiber product of categories fibered in groupoids: diagrams of "2-commutative" triangles and squares, which commute only up to 2-isomorphisms.