Algebraic Stacks.

Def $f: \mathcal{X} \to \mathcal{Y}$ morphism of stacks is representable if $\forall \underset{\text{Sch.}}{\underset{\uparrow}{U}} \to \mathcal{Y}$

$$\begin{tikzcd} U \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ U \arrow[r] & \mathcal{Y} \end{tikzcd}$$

$U \times_{\mathcal{Y}} \mathcal{X}$ is an alg. space.

Lemma U alg. space. Then $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

Def A stack $\mathcal{X}/S$ is an algebraic stack if

(1) $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable

(2) $\exists$ sm surj. $\pi: \underset{\text{sch}}{X} \to \mathcal{X}$

Rmk 1) (1) $\Rightarrow$ Every $t: \underset{\text{Sch}}{\underset{\uparrow}{T}} \to \mathcal{X}$ is representable

2) If $u: \underset{\text{sch}}{U} \to \mathcal{X}$ then $U \times_{\mathcal{X} \times_S \mathcal{X}} T$ is an alg space.

$$\begin{tikzcd} U \times_{\mathcal{X} \times_S \mathcal{X}} T \arrow[r] \arrow[d] & U \times_S T \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

Lemma $\mathcal{X}/S$ stack. $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable iff $\forall$ S-sch U & $u_1, u_2 \in \mathcal{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on Sch/U is an alg. space.

Pf

$$\begin{tikzcd} \underline{\mathrm{Isom}}(u_1, u_2) \arrow[r] \arrow[d] & U \arrow[d, "u_1 \times u_2"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

<u>Example</u> X alg space. G/S smooth group scheme $\curvearrowright X$.

$[X/G]$ <u>Category</u>: Objects (T, P, π)

(1) T S-scheme (e.g. principal G-bundle)

(2) P $G_T := G \times_S T$ - torsor on the ét site of T

(3) $\pi : P \to X \times_S T$ G_T- equivariant
$$X \times_S T = X_T$$

mor $(f, f^\flat)$ $(T', P', \pi') \longrightarrow (T, P, \pi)$

$$\begin{tikzcd} T' \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & T \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[rr, "f^\flat", "\simeq"'] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & f^* P \arrow[dl] \\ & X_{T'} & \end{tikzcd}$$

<u>Claim</u> $[X/G]$ is a stack

<u>WTS</u> $[X/G]$ is an algebraic stack

$X \to [X/G]$

Let T be a scheme. $(T, P_i, \pi_i) \in [X/G](T)$

Let $I = \underline{Isom}((P_1, \pi_1), (P_2, \pi_2))$

$$T'/T \longmapsto \left\{ \begin{tikzcd} P_1|_{T'} \arrow[r, "\simeq"] \arrow[dr] & P_2|_{T'} \arrow[d] \\ & X_{T'} \end{tikzcd} \text{ as } G_{T'}\text{-torsors} \right\}$$

<u>Fact</u> Y/S alg. space Sch/S with étale topology

$g: F \to Y$ mor of schemes

$$\begin{tikzcd} F \times_Y U \arrow[r] \arrow[d] & U \arrow[d, "\text{ét surj.}"] \\ F \arrow[r] & Y \end{tikzcd}$$
$U \leftarrow$ sch

$F \times_Y U$ alg space $\Rightarrow$ F alg. space.

Replace T by an ét covering s.t. P_1 & P_2 are trivial. Then

fix $\sigma_i : P_i \xrightarrow{\sim} G_T$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & G_T \arrow[d] \\ {} & {} \end{tikzcd}$$

Def P property of mor. of alg. space which is stable w.r.t. smooth topology.

A representable mor. of alg stacks:

$f: \mathcal{X} \to \mathcal{Y}$ has property P if $\forall\ Y \to \mathcal{Y}$ (Y alg space)

$\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ of alg space has P.

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} Y & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ Y & \to & \mathcal{Y} \end{array}$$

__Examples__ étale, smooth, separated, unramified.

__Def__ An alg. stack is a __Deligne-Mumford__ stack if $\exists$ a representable ét surjective $X \to \mathcal{X}$ where X is a scheme.

__Thm__ $\mathcal{X}$ DM stack $\iff$ $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.

__Prop__ $[X/G]$ is DM $\iff$ $\forall\ s: \mathrm{Spec}(k) \to S$, $t \in [X/G](k)$

the Stabilizer group scheme $G_t \subset G_s$ is ét $/_{\bar{s}}$. (k alg. closed).