Ramification of Hilbert Eigenvariety at Classical Points

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AMS Sectional Meeting at UW Madison
Special Session on Geometry and Topology in Arithmetic
September 15, 2019

What is an Eigenvariety?

Goal

Study the geometry of eigenvariety (for Hilbert modular forms)

An **eigenvariety** is a rigid analytic space parametrizing p-adic modular forms (up to scalar multiples) which are Hecke eigen, finite-slope, and overconvergent.
A p-**adic modular form** is a p-adic limit of modular forms.

Example of p-adic modular forms

Let $k \geq 4$ be an integer, $p \geq 3$ a prime.
Let

$$G_k(q) := \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

be the Eisenstein series of weight k, level 1, where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.
Let $G_k^{(p)}(q) := G_k(q) - p^{k-1} G_k(q^p)$, which is of weight k and level p.
Then for each $r \geq 1$,

$$\begin{aligned} k \equiv k' \mod (p-1)p^{r-1} &\implies d^{k-1} \equiv d^{k'-1} \mod p^r, \forall p \nmid d \\ &\implies G_k^{(p)} \equiv G_{k'}^{(p)} \mod p^r \end{aligned}$$

Given $\alpha = (\alpha_r) \in \mathbb{Z}/(p-1) \times \mathbb{Z}_p = \lim_{r \to \infty} \mathbb{Z}/(p-1)p^{r-1}$, the topological limit

$$G_\alpha^{(p)} = \lim_{r \to \infty} G_{\alpha_r}^{(p)} \in \mathbb{C}_p[\![q]\!]$$

is a p-adic modular form of weight $\alpha \in \mathbb{Z}/(p-1) \times \mathbb{Z}_p$ and tame (prime-to-p) level 1.

Hecke eigen, finite-slope and overconvergent

- $\exists$ Hecke operators T_ℓ, S_ℓ, U_p on the space of (p-adic) modular forms. If $f = \sum_{n\geq 0} a_n q^n$ is of weight α, tame level N, then for each prime $\ell \nmid Np$
  - $T_\ell(f) = \sum_{n\geq 0}(a_{n\ell} + \ell^{\alpha-1} a_{n/\ell})q^n$.
  - S_ℓ has no explicit formula in terms of q-expansion....
  - $U_p(f) = \sum_{n\geq 0} a_{np} q^n$ plays an important role.

  Hecke eigen = simultaneous eigenvector for all Hecke operators
- The **slope** is the p-adic valuation of the U_p-eigenvalue. Finite-slope = U_p-eigenvalue is non-zero
- **Overconvergence** is defined geometrically, so that U_p-operator acting on the space of p-adic overconvergent forms is a compact operator.

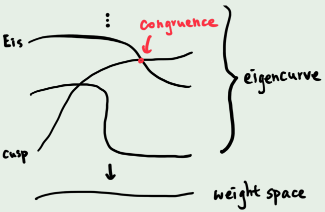
Eigenvariety

Given the notion "p-adic family of modular forms parametrized by weight" and a compact Hecke operator, one can formally construct an eigenvariety $\mathcal{E}$ through Buzzard's eigenvariety machine (2007).

There is the **weight map** wt from the eigenvariety to the **weight space**.

Example 1

The first example of an eigenvariety was the eigencurve, constructed by Coleman–Mazur in 1998 for elliptic modular forms.



Main Question (vague)

Can one describe the ramification locus of the weight map wt?

Galois representation

Let f be a p-adic modular form which is Hecke eigen, finite-slope, and overconvergent.
Write λ_ℓ for the T_ℓ-eigenvalue, and μ_ℓ for the S_ℓ-eigenvalue of f.
Then there exists a Galois representation

$$\rho\colon\ \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}_p)$$

such that

$$\det(X - \rho(\mathrm{Frob}_\ell)) = X^2 - \lambda_\ell X + \ell\mu_\ell.$$

for all $\ell \nmid Np$, where N is the tame level of f.

Main Question

Can one describe the ramification locus of the weight map wt, in terms of the associated Galois representation?

Hilbert modular forms

We will directly treat the more general case of Hilbert modular forms. Hilbert modular forms are

- Automorphic forms for the group GL_2/F, where F is a totally real field of degree d.
- Can also be constructed using sections of automorphic vector bundles on Hilbert modular variety, which parametrizes abelian varieties of dim. d with $\mathcal{O}_F$-action.
- Weight $= ((k_\tau)_{\tau:\ F\hookrightarrow\mathbb{R}}, w) \in \mathbb{Z}^{d+1}$.

Main Result

- F: totally real field of degree d.
- p: a rational prime which splits completly in F.
- f: a classical cuspidal Hilbert Hecke eigenform of finite slope and of weight $(\underline{k}, w)$.
- $x \in \mathcal{E}$: point on the cuspidal Hilbert eigenvariety $\mathcal{E}$ corresponding to f.
- ρ_f: $\mathrm{Gal}_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$: the Galois representation associated to f.

Theorem 2 (H.)

The point x is ramified $\iff$ $\exists$ *prime $v \mid p$ of F such that*

- $\rho_f|_{\mathrm{Gal}_{F_v}}$ *splits, and*
- *f has v-slope* $= \frac{w+k_v-2}{2}$.

Remark: For classical f, v-slope is between $\frac{w-k_v}{2}$ and $\frac{w+k_v-2}{2}$.

Idea for proving Theorem 2

The point x is ramified.

$\Updownarrow$ Almost by definition

$\exists$ generalized Hecke eigenform f' with
the same eigenvalues and weight as f,
but which is not a scalar multiple of f

$\Updownarrow$ Classicality Thm. (Tian–Xiao)

f is in the image of Θ-operator.

$\Downarrow$ Companion forms

$\exists$ prime $v \mid p$ of F such that

- $\rho_f|_{\mathrm{Gal}_{F_v}}$ splits
- f has v-slope $= \frac{w+k_v-2}{2}$.

$\Uparrow$ Theorem 3

Θ-operator

For each p-adic prime v of F,
we have Θ_{v,k_v-1}-operator on overconvergent forms,

$$\Theta_{v,k_v-1}(\sum_{\xi\in \text{a lattice of } F} a_\xi q^\xi) = \sum_\xi v(\xi)^{k_v-1} a_\xi q^\xi,$$

(For $F = \mathbb{Q}$, $\Theta_{p,k-1}(\sum a_n q^n) = \sum n^{k-1} a_n q^n$, i.e., $\Theta_{p,k-1} = \left(q\frac{d}{dq}\right)^{k-1}$.)
which

- increases v-weight by $2k_v - 2$, preserves v'-weight for $v' \neq v$, and
- increases v-slope by $k_v - 1$, preserves v'-slope for $v' \neq v$.

Define $\Theta := \bigoplus_v \Theta_{v,k_v-1}$.

Main Result (Cont'd)

Theorem 3 (H.)

Let $T_x\mathcal{E}_{\mathrm{wt}(x)}$ be the tangent space of the fiber of wt *at x. Then*

$$\dim T_x\mathcal{E}_{\mathrm{wt}(x)} \geq \#\{v\colon\ \rho_f|_{\mathrm{Gal}_{F_v}} \textit{ splits and } f \textit{ has } v\textit{-slope} = \frac{w+k_v-2}{2}\}.$$

Idea for proving Theorem 3

Denote by Σ the set $\{v\colon\ \rho_f|_{\mathrm{Gal}_{F_v}}$ splits and f has v-slope $= \frac{w+k_v-2}{2}\}$.
Define suitable Galois deformation functors D, D^0 with

$$\begin{array}{ccc} T_x\mathcal{E} & \subset & D(\bar{k}(x)[\varepsilon]/\varepsilon) \\ \cup & & \cup \\ T_x\mathcal{E}_{\mathrm{wt}(x)} & \subset & D^0(\bar{k}(x)[\varepsilon]/\varepsilon). \end{array}$$

$\mathcal{E}$ is equidimensional of dimension $d+1 \implies$ Suffices to prove

$$\dim D(\bar{k}(x)[\varepsilon]/\varepsilon^2) - \dim D^0(\bar{k}(x)[\varepsilon]/\varepsilon^2) \le d+1-\#\Sigma.$$

Can show: Let $\widetilde{\rho} \in D(\bar{k}(x)[\varepsilon]/\varepsilon^2)$, $v \in \Sigma$.
If $\widetilde{\rho}|_{\mathrm{Gal}_{F_v}}$ has a constant Hodge–Tate weight $\frac{w-k_v}{2}$,
then $\widetilde{\rho}|_{\mathrm{Gal}_{F_{v'}}}$ has constant Hodge–Tate weights for all $v' \in \Sigma$.