Ken [illegible]

Lecture #3. 2/19

Descent $M_{1,1} : \mathrm{Sch}^{op} \to$ sets not representable so consider

$\mathcal{M}_{1,1}$: **Objects.** $(S, (E, e))$ S scheme, E elliptic curve over S

Morphisms (f, g) $f : S' \to S$, $g : E' \to E$

$$\begin{array}{ccc} E' & \to & E \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

pullback: $E' = E \times_S S'$

$p : \mathcal{M}_{1,1} \to \mathrm{Sch}$

$(S, (E, e)) \mapsto S$ (forgetting E) $\Rightarrow \mathcal{M}_{1,1}$ fibred Cat / Schm.

Fibers are families

So $\mathcal{M}_{1,1}$ is a category fibered in groupoids

$$\begin{array}{c} F \\ \downarrow p \\ C \end{array}$$

$X \in ob(C)$

C/X

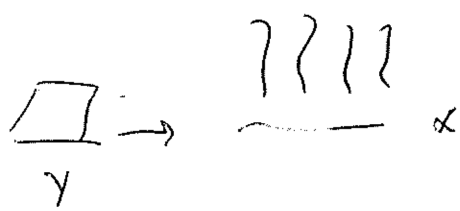
2-Yoneda Lemma. $\mathrm{Hom}_C(C/X, F) \to F(X)$

$(C/X \xrightarrow{g} F) \mapsto g(id_X)$

is an equivalence of categories.

In our example: $C = \mathrm{Sch}$, $F = \mathcal{M}_{1,1}$

Let $X \in ob(C)$



$Y \to X$ choose any pull-back

Definition (fppf site) X, Sch_X

Define a grothendieck Topology on X.

$\{U_i \to U\}$ is a covering if each $U_i \to U$ flat and locally of finite presentation, and $\coprod U_i \to U$ is surjective.

$X \in Sch_S$ functor of points $h_X : Sch_S^{op} \to set$

Main Technical Theorem

h_X is a sheaf in Sch_S with the fppf topology.

Corollary This defines a fully faithful functor from Sch into (Sheaves on Sch_{ET}) Big Étale site

Main steps. 1) can assume $S = spec\, \mathbb{Z}$

2) Enough to prove: for every faithfully flat $V \to U$ of locally finite presentation, $h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$ is exact.

Pick a cover $\{U_i \to U\}$

$$\begin{array}{ccccc} h_X(U) & \to & \prod_i h_X(U_i) & \rightrightarrows & \prod_{i,j} h_X(U_i \times_U U_j) \\ \| & & \uparrow\wr & & \uparrow\wr \\ h_X(U) & \to & h_X(V) & \rightrightarrows & h_X(V \times_U V) \end{array}$$

Pick $V = \coprod_{i \in I} U_i$

3) Enough to prove 2 for U, V affine.

4) Can also assume that X is affine.

Now, $U = spec\, A$ $V = spec\, B$ $X = spec\, R$.

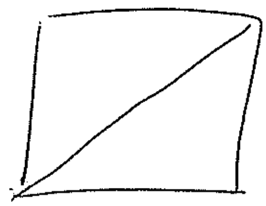
$\hom(R, A) \to \hom(R, B) \rightrightarrows \hom(R, B \otimes_A B)$

<u>Lemma</u> $A \xrightarrow{f} B$ faithfully flat A-algebra, then for any A-module M, the following is exact. $M \to M_B \overset{b\otimes 1}{\underset{1\otimes b}{\rightrightarrows}} M_{B\otimes_A B}$

$(M_B = B \otimes_A M)$

Note, just going to use for $M = A$.

$$A \to B \overset{b\otimes 1}{\underset{1\otimes b}{\rightrightarrows}} B\otimes_A B$$



<u>Actual</u> <u>Proof</u>. Tensor again w/ B.

$$b_1 \otimes b_2 \otimes b_3 \longmapsto b_1 \otimes b_2 b_3$$

$$B \otimes_A B \otimes_A B \longrightarrow B \otimes_A B$$

<u>General</u> <u>Stuff</u> <u>about</u> <u>Descent</u>.

<u>Setup</u>
- C is a category w/ finite fiber products (sch, sch_S).
- $p : F \to C$ fibered category.
- For each $f : X \to Y$, <u>choose</u> pullback functor $f^* : F(Y) \to F(X)$

<u>Def</u> For $\{X_i \to Y\}_{i\in I}$ morphisms in C, define

$F(\{X_i \to Y\})$ to be the following category

<u>Objects</u>. $(\{E_i\}_{i\in I}, \{\sigma_{ij}\}_{i,j\in I})$ where $E_i \in F(X_i)$ and

$\sigma_{ij} : pr_1^* E_i \to pr_2^* E_j$ is an isomorphism in $F(X_i \times_Y X_j)$

For all $i, j, k \in I$ ... [cocycle condition commutative diagram]

Morphisms $\{E_i\}_{i\in I}$ $\{\sigma_{ij}\}_{i,j\in I}$ and collection $\{g_i : E_i' \to E_i\}$ in $F(X_i)$

s.t.

$$\begin{tikzcd} pr_1^* E_i' \arrow[r, "pr_1^* g_i"] \arrow[d, "\sigma_{ij}'"] \arrow[dr, phantom, "\circlearrowright"] & pr_1^* E_i \arrow[d, "\sigma_{ij}"] \\ pr_2^* E_j' \arrow[r, "pr_2^* g_j"] & pr_2^* E_j \end{tikzcd}$$

$$\mathcal{E} : F(Y) \longrightarrow F(\{X_i \longrightarrow Y\})$$

Def $\{X_i \to Y\}$ is of effective descent for Y if $\mathcal{E}$ is an equivalence of categories.

Ex: (1) sheaves on a topological space (gluing data)

(2) $\mathcal{M}_{1,1}$

Def Let C be a site. A category fibered in groupoids is a stack if $\forall x \in ob(C)$ and any covering $\{X_i \to X\}_{i\in I}$ the functor $F(X) \longrightarrow F(\{X_i \to X\})$ is an equivalence of Categories.