Quasicoherent sheaves on alg stacks (1/5)

Stack is a fibered category in groupoids where descent works.

A stack is **algebraic** if: • the diagonal map

$$\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X} \text{ representable}$$

$$\bullet \; X \xrightarrow[\text{surj.}]{\text{sm}} \mathcal{X}$$

At the beginning we were dealing with functors

Example $\mathcal{M} : Sch/S \to Sets$

$$X \longmapsto \left\{ \begin{matrix} \text{families} \\ \text{over} \\ X \end{matrix} \right\}$$

$$Mor(-, M) : Sch/S \to \cancel{Sets} \longleftarrow \text{replaced with } \underline{\text{groupoids}}$$

$$X \longmapsto mor(X, M)$$

We ended up with fibered categories:

Example. Fibered Category

$$\begin{matrix} \mathcal{F} = \left\{ \begin{matrix} \text{all families over} \\ \text{any base} \end{matrix} \right\} & \left\{ \begin{matrix} E \\ \downarrow \\ X \end{matrix} \right\} \\ \downarrow & \downarrow \\ Sch/S & X \end{matrix}$$

$$\begin{matrix} Sch/M = \left\{ \begin{matrix} X \to M \\ \searrow_S \swarrow \end{matrix} \right\} \\ \downarrow \qquad\qquad \downarrow \\ Sch_S \qquad\qquad X \searrow S \end{matrix}$$

Remark The kind of information in $\mathcal{F}$ and in M is slightly different.

$\mathcal{X}$ alg stack.

We want sheaves on $\mathcal{X}$.

Idea: To define sheaves on $\mathcal{X}$, we construct sheaves on all $X \to \mathcal{X}$ (scheme X)

$$\mathcal{F} \text{ (sheaf on } Z)$$
$$\downarrow$$
$$Z \text{ (scheme)}$$

schemes mapping to Z : We can pull back $\mathcal{F}$ to all of these and so $\mathcal{F}$ is equivalent to understanding all of these compatibility conditions.

Def $AS(\mathcal{X})$ category of alg spaces over $\mathcal{X}$

Objects: (T, t) where : $T \xrightarrow{t} \mathcal{X}$ morphism (T alg space)

morphisms: $(T', t') \xrightarrow{(f, f^b)} (T, t)$

$$\begin{tikzcd} & T \arrow[dr, "t"] & \\ T' \arrow[ur, "f"] \arrow[rr, "t'"'] & & \mathcal{X} \end{tikzcd}$$

f^b is a natural transformation between $t \circ f$ and t'.

$\text{Lis-Ét}(\mathcal{X}) \subset Sch/\mathcal{X} \subset AS(\mathcal{X})$

Def $\text{Lis-Ét}(\mathcal{X})$ is a category where

Obj: (T, t) with T scheme, t smooth

morphisms: $(T, t) \xrightarrow{(f, f^b)} (T', t')$, f étale.

Def A covering of (T, t) is $\{(T_i, t_i), f_i\}$ that is a covering.

$$\begin{tikzcd} T_i \arrow[r, "f_i"] \arrow[dr, "t_i"'] & T \arrow[d, "t"] \\ & \mathcal{X} \end{tikzcd}$$

Def $\mathcal{X}_{Lis\text{-}\acute{E}t}$ is the category of sheaves over $Lis\text{-}\acute{E}t(\mathcal{X})$ with this topology.

Rmk This is a <u>topos</u>.

Def $\mathcal{C}$ category is a topos if it is equivalent to the category of sheaves over a certain category.

Remark $(T, t) \in ob(Lis\text{-}\acute{E}t(\mathcal{X}))$

$$\acute{E}t(t) \hookrightarrow Lis\text{-}\acute{E}t(\mathcal{X})$$

$$(T', u) \rightsquigarrow (T', t \circ u)$$

$T' \xrightarrow{u} T$ étale

$$\begin{tikzcd} T' \arrow[r, "u"] \arrow[dr] & T \arrow[d, "t"] \\ & \mathcal{X} \end{tikzcd}$$

Example $\mathcal{O}_{\mathcal{X}}$ is the sheaf sending $(T, t) \longmapsto \Gamma(T, \mathcal{O}_T)$

Remark We can define sheaves of groups, rings, Λ-modules (Λ a sheaf).

So we know how to construct $\mathcal{O}_{\mathcal{X}}$ modules and we want to pass to quasi-coherent sheaves.

Def $\mathcal{O}_{\mathcal{X}}$ module $\mathcal{F}$ is called <u>Cartesian</u> if $(f, f^b) : (T', t') \to (T, t)$

$$\begin{matrix} (T', t') & \to & (T, t) \\ \uparrow & & \uparrow \\ \mathcal{F}(T', t') & & \mathcal{F}(T, t) \end{matrix}$$

$$f^* \mathcal{F}_{(T,t)} = f^{-1} F_{(T,t)} \otimes_{f^{-1}\mathcal{O}_T} \mathcal{O}_{T'} \xrightarrow{\sim} F_{(T', t')}$$

Def A qcoh sheaf on $\mathcal{X}$ is a cartesian sheaf F for which $F_{(T,t)}$ is qcoh/T for every $(T, t) \in Lis\text{-}\acute{E}t(\mathcal{X})$.

Remarks A cartesian $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ is qcoh $\iff$ $F_{(X,x)}$ is qcoh for a single $X \to \mathcal{X}$ smooth, surjective

Remark If $\mathcal{X}$ is <u>Deligne</u> - <u>Mumford</u> (DM) we can define $\acute{E}t(\mathcal{X})$, $\mathcal{X}_{\acute{e}t}$

Fact

$$\begin{array}{ccc} \mathcal{X}_{\acute{e}t} & \xrightarrow{\sim} & \mathcal{X}_{lis\text{-}\acute{e}t} \\ \cup & & \cup \\ \mathrm{qcoh}(\mathcal{X}_{\acute{e}t}) & \xrightarrow{\sim} & \mathrm{qcoh}(\mathcal{X}_{lis\text{-}\acute{e}t}) \end{array}$$

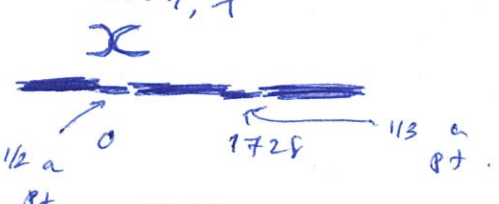
Example. $\mathcal{M}_{1,1} \longrightarrow \underline{\mathbb{A}^1_j}$

$\mathcal{M}_{1,1} \quad \not\simeq \quad \mathrm{Sch}/_{\mathbb{A}^1_j}$

What is the difference?

$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z} \neq \{1\} = \mathrm{Pic}(\mathbb{A}^1_j)$

Idea $\mathcal{M}_{1,1}$



Subtracting gives us "1/6" of a pt.

$\mathbb{Z}/6\mathbb{Z}$

Mumford proves:

Étale maps to $\mathcal{M}_{1,1}$ are those ramified with order 2 over 0. order 3 over 1728.

But every ell. curve with a marked pt has an aut of order 2.

$\rightsquigarrow \mathbb{Z}/12\mathbb{Z}$.