Coarse Moduli spaces Part II.

Points of an algebraic stack.

$\mathcal{X}/s$ alg stack.

$\mathrm{Spec}(K) \to \mathcal{X}$

Def A point of $\mathcal{X}$ is an equivalence class of such maps from fields.
$|\mathcal{X}|$ denotes the set of points.

Two points are equivalent if $\exists$ $\mathrm{Spec}(M)$ and maps:

$$\begin{tikzcd} \mathrm{Spec}(M) \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\ \mathrm{Spec}(L) \arrow[r] & \mathcal{X} \end{tikzcd}$$

making this diagram commute.

Rmk $\mathcal{X} = X$ a scheme. then $|\mathcal{X}|$ = set of points of X.
$x \in X$ $\quad \mathrm{Spec}(k(x)) \to X$

Fact $\exists$ unique topology on $|\mathcal{X}|$ such that
(1) $\forall$ morphism $\mathcal{X} \to \mathcal{Y}$ of stacks, $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous.
(2) $\forall$ $\mathcal{U} \to \mathcal{X}$ flat, locally of finite presentation
$\mathcal{U}$ alg. space then $|\mathcal{U}| \to |\mathcal{X}|$ continuous open map.

Dimension

X/s an algebraic space. $\quad \mathcal{U} \to X$ étale surjection from scheme $\mathcal{U}$
$x \in X/s$ $\quad x: \mathrm{Spec}(k) \to X$ $\quad u \mapsto x$
$\dim_x(X) = \dim_u(\mathcal{U})$

- $f : \underset{x}{\overset{}{X}} \to \underset{y}{Y}$ alg spaces. ($x \longmapsto y$)

$x \in |X|$

$\dim_x(f) := \dim_x(X_y)$

Now let $\mathcal{X}/S$ be an algebraic stack.

$\pi : \mathcal{U} \to \mathcal{X}$, $\mathcal{U}$ alg space, $u \in \mathcal{U}$

$\dim_u(\pi) = \dim_{(u,u)}(\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U})$

$$\begin{tikzcd} (u,u) \in \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "\pi"] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Let $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ be a smooth surjection where $\mathcal{U}$ is an alg. space. ($u \longmapsto x$)

Def $\dim_x(\mathcal{X}) = \dim_u(\mathcal{U}) - \dim_u(\pi)$

$\dim(\mathcal{X}) = \sup_{x \in |\mathcal{X}|} \dim_x(\mathcal{X})$

Example. $[X/G]$ has smooth cover $X \longrightarrow [X/G]$

$$\begin{tikzcd} X \times G = X \times_{[X/G]} X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X \arrow[r] & {[X/G]} \end{tikzcd}$$

$$\dim_{\bar{x}}[X/G] = \dim X - \dim_x(\pi)$$
$$= \dim X - \dim G.$$

E.G. $\dim BG = -\dim(G)$.

Def $\mathcal{X}/S$ alg. stack.

A coarse moduli space is an alg space X, $\pi : \mathcal{X} \to X$ s.t.

(1) π is initial among maps to alg spaces.

(2) $\forall k = \bar{k}$, $\pi_0(\mathcal{X})(k) \xrightarrow{\sim} X(k)$

Examples. G reductive group $\overset{\text{linear}}{\curvearrowright} X \subset \mathbb{P}^n_k$

① $[X^{ss}/G] \to X^{ss}/\!/G$ "good moduli space"

② $[X^s/G] \xrightarrow{\sim} X^s/\!/G$

$$\begin{tikzcd} X^s/\!/G \\ X^s \arrow[u, "G\text{-bundle}"'] \end{tikzcd}$$

$\mathbb{G}_m \curvearrowright \mathbb{A}^1$

$[\mathbb{A}^1/\mathbb{G}_m](T)$ = line bundle
$$\begin{tikzcd} \mathcal{L} \arrow[d] \\ T \arrow[u, bend right, "s \in \Gamma(T,\mathcal{L})"'] \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathbb{G}_m \text{ bundle}"'] & \mathbb{A}^1 \\ T \end{tikzcd}$$

corresponds to $s \in \Gamma(T,\mathcal{L})$

Corresponds to $\mathcal{L}$

- $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{spec}(k)$ Good moduli space.
- $[\mathbb{A}^1 \setminus \{0\}/\mathbb{G}_m] \xrightarrow{\sim} \operatorname{Spec}(k)$

$\mathcal{M}_{1,1}(S) = $
$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ S \arrow[u, bend right, "o"'] \end{tikzcd}$$
family of elliptic curves

$\mathcal{M}_{1,1} \to \mathbb{A}^1_j$ is a coarse moduli space.

$$\begin{tikzcd} \mathcal{M}_{1,1} \arrow[r] \arrow[d] & Y \\ \mathbb{A}^1_j \arrow[ur, dashed] \end{tikzcd}$$

alg space.

"Legendre Family"

$\mathcal{M}_{1,1}$

$$\begin{tikzcd} \mathcal{E}_\lambda : y^2 = x(x-1)(x-\lambda) \arrow[d] \\ B = \mathbb{A}^1 \setminus \{0,1\} \end{tikzcd}$$

$f: B \to Y$, $S_3 \curvearrowright B$

$\lambda \to \frac{1}{\lambda}$

$\lambda \mapsto \frac{1}{1-\lambda}$

$$f: \begin{tikzcd} B \arrow[r] \arrow[d] & Y \\ \mathbb{A}^1_j \arrow[ur] \end{tikzcd}$$

Root stack.

$$\begin{tikzcd} \mathcal{L} \arrow[d] \\ X \end{tikzcd}, \quad s \in \Gamma(\mathcal{L})$$

Def $\sqrt[r]{(\mathcal{L}, s)/X}\,(T)$ rth-root stack.

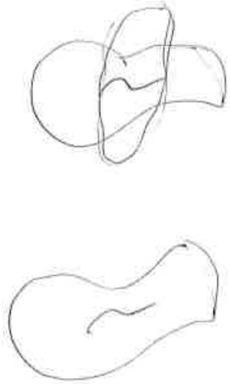
Objects: (f, m, φ, t) :
$f: T \to X$
m line bundle on T
$\varphi: m^{\otimes r} \xrightarrow{\sim} f^*\mathcal{L}$
$t \in \Gamma(m)$ s.t. $t^{\otimes r} \longmapsto f^*(s)$

Alg. stack.

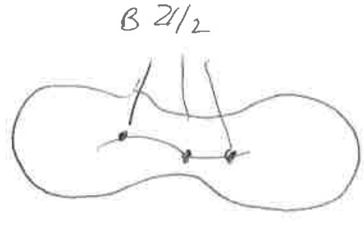
Fact $\sqrt[r]{(\mathcal{L},s)/X} \longrightarrow X$ is a coarse moduli space.

$$\begin{array}{c} \mathcal{L}^{\otimes m} \\ \downarrow \\ X \end{array} \qquad s \in \Gamma(\mathcal{L}^{\otimes m})$$

$$\begin{tikzcd}
\mathbb{Z}/m \circlearrowright \tilde{X} = \operatorname{Spec}\left(\bigoplus_{i=0}^{m-1} (\mathcal{L}^*)^i\right) \arrow[d, "\mathbb{Z}/m \text{ branched cover}"'] \arrow[r] & {[\tilde{X}/\mathbb{Z}/m]} \arrow[dl] \\
X & \wr\!\wr \\
 & \sqrt[m]{(\mathcal{L}^m, s)/X}
\end{tikzcd}$$



$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array}$$



$[\tilde{X}/\mathbb{Z}/2]$

Thm (Keel-Mori)

Alg stack $\mathcal{X}/S$ locally finite presentable with finite inertia stack.

implied by separatedness, e.g. Deligne-Mumford.

then $\exists$ coarse moduli space $\pi : \mathcal{X} \to X$ and moreover:

alg. space.

(1) X/S locally finite type

(2) $\mathcal{X}/S$ separated $\Rightarrow$ X/S separated

(3) π is proper, $\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}}$

(4)
$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
X' \arrow[r, "\text{flat}"'] & X
\end{tikzcd}$$

coarse moduli space.