Moduli of Fontaine-Laffaille modules and mod p local-global compatibility

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$X = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_\infty(\prod_\ell U_\ell)$ modular curve.

Cohomology

$G_{\mathbb{Q}} \times \mathbb{T}$ ↺

$$H^1(X_{U,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)[\lambda] = \rho \otimes \bigotimes_\ell \pi_\ell(\rho|_{G_{\mathbb{Q}_\ell}})^{U_\ell}$$

$G_{\mathbb{Q}}$ ↺

- $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ (cuspidal) system of (good) Hecke eigenvalues.
- $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ Galois representation associated to λ.
- $\pi_\ell(\rho|_{G_{\mathbb{Q}_\ell}})$ smooth $\mathrm{GL}_2(\mathbb{Q}_\ell)$ representation, associated by (classical) local Langlands correspondence.

$G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$ $G_{\mathbb{Q}}$ $GL_2(\mathbb{Q}_p)$

$$\lim_{U_p} H^1(X_{U_pU^p}, \overline{\mathbb{Q}}_p)[\lambda] = \rho \otimes \pi_p(\rho|_{G_{\mathbb{Q}_p}}) \otimes \cdots$$

Mod p cohomology (Emerton): $\bar{\rho}$ ordinat.

$H^1(X_{U,\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)[\bar{\lambda}] = \bar{\rho} \otimes \bigotimes_\ell \pi_\ell(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})^{U_\ell}$

$G_{\mathbb{Q}}$ $\mathbb{T}$

- $\bar{\lambda} : \mathbb{T} \to \overline{\mathbb{F}}_p$ system of mod p Hecke eigenvalues.
- $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ Galois representation associated to $\bar{\lambda}$.
- $\pi_p(\bar{\rho}|_{G_{\mathbb{Q}_p}})$ much more subtle than the rest: mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (Breuil).

Mod p completed cohomology

$$\widetilde{H}^1(X_{U^p})[\bar{\lambda}] \stackrel{\text{def}}{=} \lim_{\substack{\longrightarrow \\ U_p}} H^1(X_{U_pU^p}, \overline{\mathbb{Q}}_p)[\bar{\lambda}] = \bar{\rho} \otimes \pi_p(\bar{\rho}|_{G_{\mathbb{Q}_p}}) \otimes \cdots$$

$G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$

G/F^+ rank n definite unitary group for CM extension F/F^+:

- $G(F_w^+) \cong \mathrm{GL}_n(F_w^+)$ and F_w^+ unramified for all $w \mid p$. $(\simeq \mathrm{GL}_n(\mathbb{Q}_{p^f}))$
- Fix $v \mid p$ and $U^v \subset G(\mathbb{A}_{F^+}^{\infty,v})$ open compact.

Have 0 dim. Shimura variety:

$$X_{U_v U^v} = G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty})/U_v U^v.$$

(finite set pts)

Spaces of algebraic automorphic forms

- $H^0(X_{U_v U^v}, \overline{\mathbb{F}}_p)$ has $\mathbb{T}$-action.
- $\overline{\lambda} : \mathbb{T} \to \overline{\mathbb{F}}_p$ has associated $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$.

Mod p completed cohomology

$$\widetilde{H}^0 \stackrel{\text{def}}{=} \lim_{\substack{\longrightarrow \\ U_v}} H^0(X_{U_v U^v}, \overline{\mathbb{F}}_p)[\overline{\lambda}]$$

$\circlearrowleft$ $G(F_w^+) = \mathrm{GL}_n(\mathbb{Q}_{p^f})$

Dream: $\widetilde{H}^0 = \pi_p(\overline{\rho}|_{G_{F_v^+}}) \otimes \cdots$, mod p local Langlands for $\mathrm{GL}_n(F_v^+)$.

Questions

1. $\widetilde{H}^0$ depend only on $\overline{\rho}_v \stackrel{\text{def}}{=} \overline{\rho}|_{G_{F_v^+}}$?
2. $\widetilde{H}^0$ determines $\overline{\rho}_v$?

$\mathrm{GL}_n(\mathbb{Z}_{p^f})$

- Have some control of $\mathrm{GL}_n(\mathcal{O}_{F_v^+})$-socle of $\widetilde{H}^0$ ("weights of mod p automorphic forms", discrete information) (joint work with D. Le, B. Levin, S. Morra).
- $\overline{\rho}_v$ has non-trivial moduli. Need to extract "continuous information".

E.g. $\overline{\rho}_v = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ $\rightsquigarrow$ classified $\mathbb{P}(H^1(G_{\mathbb{Q}_{p^f}}, \chi_2\chi_1^{-1}))$

$\chi_i : G_{\mathbb{Q}_{p^f}} \to \mathbb{F}_p^\times$

usually dim f.

$f > 1$: $\exists \infty$ ext classes

Known cases of question 2

$\overline{\rho}_v$ is niveau 1, Fontaine-Laffaille of weight λ ($\in$ lowest alcove), sufficiently generic. Then $\widetilde{H}^0$ determines $\overline{\rho}_v$ for:

- $\mathrm{GL}_2(\mathbb{Q}_{p^f})$: Breuil-Diamond.
- $\mathrm{GL}_3(\mathbb{Q}_{p^f})$: Herzig-Le-Morra ($f = 1$), Enns ($f > 1$), assuming $F(\lambda) \in \mathrm{soc}\, \widetilde{H}^0$. ($\Rightarrow$ $\overline{\rho}_v$ is FL)
- $\mathrm{GSp}_4(\mathbb{Q}_p)$ (Enns).
- $\mathrm{GL}_n(\mathbb{Q}_p)$ (Park-Qian), conditional. On behavior of soc $\widetilde{H}^0$.

Niveau 1 means extensions of characters, and FL restricts ordering on inertial weights. $\overline{\rho} = \begin{pmatrix} x_1 & & * \\ & \ddots & \\ & & x_n \end{pmatrix}$, $\rho = \begin{pmatrix} \omega^{a_1} & & * \\ & \ddots & \\ & & \omega^{a_n} \end{pmatrix}$ $0 \le a_1 \le a_2 \le \dots \le a_n < p-1$.

- $\mathrm{GL}_2(K)$, any K and $\overline{\rho}_v$ (Scholze).
- $\mathrm{GL}_n(K)$, restrictive $\overline{\rho}_v$ (Liu).

Main Theorem (Le-L.-Morra-Park-Qian)

Recall $G(F_v^+) = \mathrm{GL}_n(\mathbb{Q}_{p^f})$. Assume

- $F(\lambda)$ is in the socle of $\widetilde{H}^0$, λ is $6n$-deep in the lowest alcove.
- $\overline{\rho}$ satisfies Taylor-Wiles hypothesis.

Then

1. $\mathrm{GL}_n(\mathbb{Z}_{p^f})$-socle of $\widetilde{H}^0$ determines the stratum $\mathcal{S}$ of $\overline{\rho}_v$ in moduli of Fontaine-Laffaille representations.
2. If $\mathcal{S}$ is "non-degenerate", $\widetilde{H}^0$ determines $\overline{\rho}_v$.

- $\lambda = (\lambda_{\sigma,i})_\sigma \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(\mathbb{F}_{p^f},\overline{\mathbb{F}}_p)}$ give rise to irrep $F(\lambda)$ of $\mathrm{GL}_n(\mathbb{F}_{p^f})$ (or $\mathrm{GL}_n(\mathbb{Z}_{p^f})$) (Serre weight). if $\lambda_{\sigma,i} - \lambda_{\sigma,i+1} < p$.
- Second part applies to a Zariski open in space of FL-extensions with given semisimplification $\overline{\rho}_v^{ss}$.
- Conjecture: "non-degenerate" is superfluous (linear algebra problem)!

Main ingredients

1. Stratification on moduli of FL modules.
2. Interpretation of stratum in terms of $\mathrm{GL}_n(\mathbb{Z}_{p^f})$-socle ($\Leftarrow$ Le-L.-Levin-Morra).
3. Extract normalized Hecke operators mod p: "Hecke functions" on FL moduli evaluated at $\overline{\rho}_v$.

- Second and third ingredient ultimately come from probing (char 0 version of) $\widetilde{H}^0$ with various *tame inertial type* τ.
- Values of Hecke functions supply the needed "continuous data" to pin down $\overline{\rho}|_{G_{F_v}}$.

Trend: often reducing char 0 operators U_p mod p gives 0!

tame $\tau : I_{\mathbb{Q}_{p^f}/P_{\mathbb{Q}_{p^f}}} \to GL_n(\overline{\mathbb{Z}}_p)$ finite image, can be extended to $W_{\mathbb{Q}_{p^f}}$.
$\exists$ finitely many tame τ.

Mod p Fontaine-Laffaille modules for $\mathbb{Q}_{p^f}$

Fontaine-Laffaille module M: over $\mathbb{F}$ $\quad R \quad$ $\mathbb{F}$-alg R

- Projective rank n over $\mathbb{F}_{p^f} \otimes_{\mathbb{F}} \mathbb{F}$. $\quad \mathbb{F}_{p^f} \otimes_{\mathbb{F}} R =$
- Decreasing filtration $\mathrm{Fil}_\bullet$ of amplitude $< p-1$.
- Frobenius semi-linear $\varphi : \mathrm{gr}_\bullet M \xrightarrow{\sim} M$. $\quad$ $x \mapsto x^p$ on $\mathbb{F}_{p^f}$ $\quad$ R-lin, similar for $\mathbb{F}_{p^f}$.

Filtration jumps recorded by tuple of integers λ (automatically a lowest alcove weight). $\quad \lambda = (\lambda_{\sigma,1}, \dots, \lambda_{\sigma,n}) \quad 0 < \lambda_{\sigma,1} - \lambda_{\sigma,n} < p$.

$\lambda_{\sigma,i} > \lambda_{\sigma,i+1}$

FL functor: $M \rightsquigarrow$ representations $G_{\mathbb{Q}_{p^f}} \to \mathrm{GL}_n(\mathbb{F})$.

Fix λ regular. Moduli of FL modules of weight λ, $\mathcal{FL}_\lambda$, have simple description.

Let

$$B = TU, B_- = TU_-$$

be opposite Borels and W Weyl group for GL_n.

$\{M, \mathrm{Fil}$ + basis adapted to $\mathrm{Fil}\}$ $\quad (GL_n)^f$

$$\mathcal{F}\ell_\lambda = \{M, \varphi : \mathrm{gr}_{\mathrm{Fil}} M \to M\} = \left[B^f \backslash (GL_n)^f \right]_{\varphi}$$

$$(b_1, \dots, b_f) \quad (g_1, \dots, g_f)$$

$$= (\dots, b_i g_i \, \pi(b_{i+1})^{-1}, \dots)$$

$$\pi : B \to T.$$

$$(\lambda GL_n)^f \xrightarrow{T\text{-torsor}} \left[(\lambda GL_n)^f / T \right] \quad \text{shifted conjugation}$$

$$\left[B^f \backslash (GL_n)^f \right]_{\varphi} = \left[(\lambda GL_n)^f / T \right]$$

$$(\lambda GL_n)^f \xrightarrow{T\text{-torsor}} (B \backslash GL_n)^f$$

- Get stratification by pulling back translates (by W) of Schubert stratification on $B\backslash\mathrm{GL}_n$. Stratum = common refinement of all translates Schubert stratification
- Locus of $\overline{\rho}_v$ with given semisimplification is union of stratum.
- Examples:

$f = n$

$$\begin{array}{c} U\backslash \mathrm{GL}_n \\ \downarrow \\ \mathrm{Fl}_X = [(U\backslash \mathrm{GL}_n)/T] \\ \vee \\ [B_-/T] = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix} \xrightarrow{FL} \begin{pmatrix} w^{x_1} & & \\ & \ddots & \\ & & w^{x_n} \end{pmatrix} \end{array}$$

Potentially crystalline Emerton-Gee stack

- τ tame inertial type.
- $\mathcal{X}^\tau$ moduli stack of potentially crystalline representations with (parallel) Hodge-Tate weights $(0, \cdots, n-1)$ and inertial type τ.
- Inertial local Langlands: $\tau \rightsquigarrow \sigma(\tau)$, (usually) irreducible Deligne-Lusztig representation of $\mathrm{GL}_n(\mathbb{F}_{p^f})$.
- There are $|W|^f$ types τ such that $F(\lambda)$ interacts nicely with $\sigma(\tau)$: "obvious" type τ.
- Le-L-Levin-Morra: analyze $\mathcal{X}^\tau$ in terms of subvarieties of affine flag variety and hence control corresponding pot.crystalline deformation rings.

τ

$\exists$ natural stratification, related to $\mathrm{loc}\,\mathrm{Fl}^{\circ}$.

$\mathrm{Fl}_\lambda \hookrightarrow \mathcal{X}^{\tau}_{\mathbb{F}_p} \Rightarrow \mathcal{X}^{\tau} = \{$ p-crystalline reps of $G_{\mathbb{Q}_p}$ with type τ. $\}$

$$\begin{array}{ccc} & & \mathcal{X}^{\tau} \\ & & \downarrow \\ \mathrm{Spec}\,\mathbb{F}_p & \longrightarrow & \mathrm{Spf}\,\mathbb{Z}_p \end{array}$$

$\exists$ for many choice of τ. $(\geq |W|^f)$.

$\mathrm{Fl}_\lambda \hookrightarrow \mathcal{X}^{\tau} \hookleftarrow (\mathcal{X}^{\tau})^{\mathrm{rig}}$

$\mathcal{H}(\sigma(\tau))$

$\mathcal{X}^{\tau,\mathrm{rig}}(\overline{\mathbb{Q}}_p) = \{$ Radic reps p-crystalline $\}$

$\downarrow$

WD = moduli of Weil-Deligne reps

Proposition

Suppose τ is an obvious type. There is a natural stratification of $\mathcal{X}^{\tau}_{\overline{\mathbb{F}}_p}$ such that

- Strata controls the number of weights in the mod p reduction of $\overline{\sigma(\tau)}$ which are modular in $\widetilde{H}^0$.
- Induces translated Schubert stratification on $\mathcal{FL}_{\lambda}$.

- Taylor-Wiles patching thickens $\widetilde{H}^0$ to sheaves in $\mathcal{X}^{\tau}$, whose support determine modularity.
- By varying τ, this gives the first part of Main Theorem!

$f=1$: $\exists$ $n!$ Grander of Schubert stratification on $B\backslash GL_n$.

Hecke algebra

- $\mathcal{H}(\sigma(\tau)) = \mathrm{End}(\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_n(\mathbb{Z}_{p^f})}^{\mathrm{GL}_n(\mathbb{Q}_{p^f})}(\sigma(\tau))$: polynomial ring in $\leq n$-variables U_i (generalization of classical U_p operator for $\Gamma_1(p)$).
- $\mathcal{H}(\sigma(\tau))$ gives functions on $\mathcal{X}^{\tau,\mathrm{rig}}$, have reciprocity law for action on (char 0) completed cohomology (Caraiani-Emerton-Geraghty-Gee-Paskunas-Shin).

$\mathrm{Hom}(\mathrm{c}\text{-}\mathrm{Ind}_{\mathrm{GL}_n(\mathbb{Z}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \sigma, \tilde{H}^0)$

$\mathcal{H}(\sigma(\tau))$ $\longrightarrow$ $R_v^\tau[\frac{1}{p}]$

$\sigma(\tau) = \mathrm{Ind}_{I_w}^{\mathrm{GL}_n(\mathbb{Z}_p)} \chi_1 \otimes \dots \otimes \chi_n$

$\mathrm{GL}_n(\mathcal{O}_p)$

$\tilde{H}^0 \circlearrowleft R_v^\tau$ = local Galois defm ring for $\bar{\rho}_v$

reciprocity = 2 actions of $\mathcal{H}(\sigma(\tau))$ coincide

Proposition

Suppose $F(\lambda)$ is the unique Jordan-Holder factor of $\overline{\sigma(\tau)}$ which is modular.
$\bar{\rho}$ is most generic relative τ.
Then:

- $|U_i| = |p^{\kappa_i}|$ is constant.
- The value of U_i/p^{κ_i} can be extracted by applying $\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_{p^f})}(\bullet, \widetilde{H}^0)$ to a diagram of smooth $\mathrm{GL}_n(\mathbb{Q}_{p^f})$-representations over $\overline{\mathbb{F}}_p$.

Phenomena: $\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_{p^f})}(\bullet, \widetilde{H}^0)$ annihilates $\mathrm{c}-\mathrm{Ind}F(\mu)$ for non-modular $F(\mu)$, and U_i/p^{κ_i} is an isomorphism in a localized category.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ V & & V \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(V, \tilde{H}^0) \arrow[rr, dashed, "\sim"] & & \mathrm{Hom}(V, \tilde{H}^0) \\ & \mathrm{Hom}(V, \tilde{H}^0) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Hecke functions for $\mathrm{GL}_n(\mathbb{Q}_p)$

- $w \in W$, $S \subset \{1, \cdots, n\}$ w-stable.
- $f_{w,S}(A) = \prod_{i \in S} \frac{\det(Aw)_{[k,n],[k,n]}}{\det(Aw)_{[k+1,n],[k+1,n]}}$