[Feb]ruary 4, 2014 - AGLS on Stacks - Introduction

I. Motivation.

Problem: When a moduli problem's objects have automorphisms, typically won't be finely represented by a scheme.

E.g. $M_{1,1} : Sch_{/\mathbb{C}}^{op} \longrightarrow Sets.$

$$S \longmapsto \left\{ \begin{array}{c} \mathcal{E} \\ \downarrow \uparrow e \\ S \end{array} \text{ smooth, proper, geometric fibers elliptic curves} \right\} / \cong .$$

- Obvious guess for a moduli space of elliptic curves: $\mathbb{A}^1_j$ - the j-line.

$$\{ \text{elliptic curves}/\mathbb{C} \} / \cong \xrightarrow[\text{bijection}]{j} \mathbb{A}^1_j .$$

- But $\mathbb{A}^1_j$ does not classify families:

①
$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & \mathbb{P}^2 \times (\mathbb{A}^1_t \setminus \{0\}) \\ {\scriptstyle [0:1:0]} \uparrow \downarrow & & \\ \mathbb{A}^1_t \setminus \{0\} & & \end{array} , \quad \mathcal{E}_t : Y^2 Z = X^3 - tZ^3.$$

②
$$\begin{array}{c} E_0 \times \mathbb{A}^1_t \setminus \{0\} \\ \downarrow \\ \mathbb{A}^1_t \setminus \{0\} \end{array} , \quad E_0 : Y^2 Z = X^3 - Z^3$$

$j(\mathcal{E}_t) = j(E_0) = 0 \quad \forall t.$

$\rightsquigarrow$ induced morphisms by ① and ②

$$\mathbb{A}^1_t \setminus \{0\} \longrightarrow \mathbb{A}^1_j \quad \text{are constant} = 0.$$

But,

<u>Claim</u>: $\mathcal{E}_t \not\cong E_0 \times \mathbb{A}^1_t \setminus \{0\}$ as families, in fact generic fibers

$$\mathcal{E}_{t,\mathbb{C}(t)} \not\cong E_{0,\mathbb{C}(t)}.$$

<u>Pf.</u> · Fix $\mathbb{C}(t) \hookrightarrow K$ alg. closure, $t^{1/6} \in K$. Then

$$f: \mathcal{E}_{t,\mathbb{C}(t^{1/6})} \cong E_{0,\mathbb{C}(t^{1/6})}$$

$$f: (x,y) \longmapsto (t^{-1/3}x, t^{-1/2}y)$$

$$y^2 = x^3 - t.$$

$$\left(\frac{y}{t^{1/2}}\right)^2 = \left(\frac{x}{t^{1/3}}\right)^3 - 1.$$

Set $I = \{\text{isomorphisms } \mathcal{E}_{t,K} \simeq E_{0,K}\}$ ⟲ $\mathrm{Aut}(E_0)$ $(= \mathbb{Z}/6)$ free transitive.

⟳ $G = \mathrm{Gal}(K/\mathbb{C}(t))$.

These actions commute.

$\Rightarrow$ Suffice to show f not fixed by G so that there are no elements in I defined over $\mathbb{C}(t)$.

Do this by explicit calculation of action of any $g \in G$ on f:

If $g \cdot t^{1/6} = \chi(g) t^{1/6}$, $\chi(g)$ 6-th root unity, $g \cdot f : (x,y) \longmapsto (\chi(g)^{-2} t^{-1/3} x, \chi(g)^{-3} t^{-1/2} y)$.

Rem. Since $\mathbb{A}^1_j$ is a "coarse moduli space" for elliptic curves, this proves there is no ~~fine~~ representing scheme for $M_{1,1}$.

Rem. Many other examples of interesting moduli problems w/auts, $M_g, g \geq 2$, Vect bundles/variety, etc....

Ultimate solution: Consider

$$\mathcal{M}_{1,1} : Sch^{op}/\mathbb{C} \longrightarrow Gpds.$$

$$S \longmapsto \left\{ e \begin{array}{c} \mathcal{E} \\ \uparrow\downarrow \\ S \end{array} \text{ family ell. crves} \right\}$$

don't mod out by auts!

To do this in a useful way, need to:

(0) Understand in what sense $M_{1,1}$ is a functor (fibred categories)

(1) Understand the glueing or sheaf-like properties (descent, stacks) of $\mathcal{M}_{1,1}$.

(2) Understand in what sense $\mathcal{M}_{1,1}$ is algebraic (algebraic stacks)

II. Schemes as functors

- Scheme X/S determined by $h_X = \mathrm{Hom}(-,X) : Sch_S^{op} \to Sets$.

  (Yoneda lemma says for cat. $\mathcal{C}$,

  $$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, Sets) \\ X &\longmapsto h_X \end{aligned}$$ fully faithful

  $\mathrm{Hom}(h_X, F) \simeq F(X)$)

- As a warm up for alg. spaces + stacks, let's build category of schemes up in terms of affine schemes.
  - Silly but instructive.

- <u>Set-up</u>: S affine scheme. Aff_S = affine schemes$/S$.

<u>Q</u>: Given $F : \mathrm{Aff}_S^{op} \to Sets$, when is it representable by a scheme?

- <u>Preliminary defs.</u>

<u>Def.</u> $f : F \to G$ morphism of functors $\mathcal{C}^{op} \to Sets$, called <u>relatively representable</u> if $\forall\, g : h_T \to G$, $T \in \mathcal{C}$

$$\begin{tikzcd} F \times_G h_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d, "f"] \\ h_T \arrow[r] & G \end{tikzcd}$$

fiber product $F \times_G h_T$ is representable

<u>Def.</u> $F, G : \mathrm{Aff}_S^{op} \to \mathrm{Sets}$, $f : F \to G$ is an <u>affine open (resp. closed) embedding</u> if:

(1) f relatively representable.

(2) $\forall T \in \mathrm{Aff}_S$, $g : h_T \to G$. the morphism is an open (resp. closed) embedding of affine schemes.

$$\begin{tikzcd} F \times_G h_T \arrow[r] \arrow[d] & F \arrow[d, "f"] \\ h_T \arrow[r, "g"'] & G \end{tikzcd}$$

<u>Def.</u> $F : \mathrm{Aff}_S^{op} \to \mathrm{Sets}$ is a <u>big Zariski sheaf</u> if $\forall U \in \mathrm{Aff}_S$, $U = \bigcup U_i$ open cover by affines,

$$F(U) \longrightarrow \prod F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

is exact.

Proposition: $F: \mathrm{Aff}_S \to \mathrm{Set}$ representable by a separated S-scheme iff:

(1) F is a big Zariski sheaf.

(2) $\exists\, X_i \in \mathrm{Aff}_S$, $\pi_i : h_{X_i} \to F$ affine open embeddings, s.t. $\coprod_i h_{X_i} \to F$ surjective morphism of Zariski sheaves.

(3) $\Delta : F \to F \times F$ is an affine closed embedding.

The functor

$$h_{(-)} : \begin{pmatrix} \text{separated} \\ S\text{-schemes} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{functors w/} \\ (1)-(3) \end{pmatrix}$$

is an equivalence.

Example: Fun to check $\mathbb{P}^n_{\mathbb{Z}}$ is a separated scheme by using this criteria.

Rem. Can get a similar criteria for arbitrary schemes.