Coarse Moduli spaces

$\mathcal{X}/S$ algebraic stack (étale topology of schemes over S)

(1) $\mathcal{X}$ is a stack

(2) $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ representable by algebraic spaces.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \\ {} \arrow[u] \arrow[r, "\text{morphism of alg spaces}"'] & T \arrow[u] \end{tikzcd}$$

(3) $X \to \mathcal{X}$ surjective smooth morphism from a scheme.

$f: \underset{\substack{\uparrow \\ \text{alg space}}}{T} \longrightarrow \mathcal{X}$ we can talk about properties of f.

$f: \mathcal{X} \to \mathcal{Y}$ morphism of algebraic stacks
representable by algebraic spaces

Suppose $f: \mathcal{X} \to \mathcal{Y}$ morphism of algebraic stacks

$$\begin{tikzcd} \mathcal{U} \arrow[d, "\substack{\text{sm.} \\ \text{surj.} \\ \text{sch'm.}}"'] \arrow[dr] & V \arrow[d, "\substack{\text{sm surj.} \\ \text{scheme } V}"] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

$\mathcal{U} \times_{\mathcal{Y}} V \to V$
morphism of alg spaces

So if (P) is a property of morphisms of alg. spaces which is (smooth) local on the source + target then we say f has (P) if $\mathcal{U} \times_{\mathcal{Y}} V \to V$ does.

Properties of morphisms
- separated
- proper.

Separated & Properness

$X \to S$ map of schemes

$X \to X \times_S X$ Closed immersion $\Big\}$ Separated.

This is not good for algebraic stacks as it is almost never true

Ex Let G be an algebraic group. G/k.

$BG = [\overset{\operatorname{Spec} k}{\overset{\shortparallel}{*}} / G]$

given by trivial bundle over a pt

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] & BG \arrow[r] \arrow[d] & BG \times BG \\ & \operatorname{Spec} k & \end{tikzcd}$$

$$\begin{tikzcd} \overset{G}{\overset{\shortparallel}{\operatorname{Isom}(x,x)}} \arrow[r] \arrow[d] & \operatorname{Spec}(k) \arrow[d, "{(x,x)}"] \\ BG \arrow[r, "f"] & BG \times BG \end{tikzcd}$$

* If f was a closed immersion, so would the Pullback.

Lemma:

$f : \mathcal{X} \to \mathcal{Y}$ alg stacks

$\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces.

Proof:

$$\begin{tikzcd} T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

want to be an alg space

T alg space

T'

Yoneda Lemma

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(T) = \mathcal{X}(T) \times_{\mathcal{Y}(T)} \mathcal{X}(T)$$

groupoids

$\Downarrow$ a

$a = (x, x', \alpha)$

$x, x' \in \mathcal{X}(T)$

$\alpha : f(x) \xrightarrow{\sim} f(x')$

T' point of $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is $\mathcal{X}(T') \times_{\Delta,\ \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}(T'),\ a} \operatorname{Hom}(T', T)$

$\tilde{x} \in \mathcal{X}(T')$, $T' \xrightarrow{\varphi} T$

$\downarrow$

$(\tilde{x}, \tilde{x}, \mathrm{id} : \tilde{x} \to \tilde{x})$

$(x|_{T'}, x'|_{T'}, \alpha|_{T'} : f(x|_{T'}) \xrightarrow{\sim} f(x'|_{T'}))$.

This all amounts to: $x|_{T'} \xrightarrow{\sim} x'|_{T'}$ s.t. image under f is $\alpha|_{T'}$.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \alpha} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Isom}_{\mathcal{X}}(x, x') \arrow[d] \\ T \arrow[r, "\alpha"] & \mathrm{Isom}_{\mathcal{Y}}(f(x), f(x')) \end{tikzcd}$$

$\Rightarrow$ $\mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} T$ is an algebraic space

($\mathcal{X}$ alg stack, $\mathrm{Isom}(x, x')$ alg space, $x, x' \in \mathcal{X}(T)$)

$$\begin{tikzcd} \mathrm{Isom}(x, x') \arrow[r] \arrow[d] & T \arrow[d, "(x, x')"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

$X \to X \times_S X$ closed immersion $\Leftrightarrow$ proper (for schemes)

<u>Def</u> $f : \mathcal{X} \to \mathcal{Y}$ morphism of alg. stacks

① f is <u>separated</u> if Δ_f is proper.

(if Δ_f is repr by alg spaces and when you base change it is an alg space with proper map ~~to $\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}$~~).

$$\left(\begin{tikzcd} * \arrow[r] \arrow[d] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ \bigcirc \arrow[r] & T \arrow[u] \end{tikzcd}\right) \text{ proper}$$

② f is <u>proper</u> if f is separated, finite type and universally closed.

<u>Example</u> A/k abelian Variety

$BA \to \mathrm{spec}(k)$ is proper.

<u>Fact</u>

$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ $\quad$ $\underbrace{g\circ f \text{ proper, } f \text{ surjective, } g \text{ separated}}$ $\Rightarrow$ g is proper.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{spec}(k) \arrow[d] \\ BA \arrow[r] & BA \times BA \end{tikzcd}$$

Let $\mathcal{X} = \mathrm{spec}(k)$
$\mathcal{Y} = BA$
$\mathcal{Z} = \mathrm{spec}(k)$

Amounts to show $g : BA \to \mathrm{spec}(k)$ is separated.

Coarse moduli spaces

Def $\mathcal{X}/S$ algebraic stack.

A coarse moduli space for $\mathcal{X}$ is a morphism:

$\pi\ \mathcal{X} \to X$, X alg. space such that

(1) π is initial among maps from $\mathcal{X}$ to alg. spaces.

(2) For every $k = \bar{k}$, $\pi_0(\mathcal{X}(k)) \xrightarrow[\text{bijection}]{\sim} X(k)$

Examples

① Moduli of smooth curves of genus 0. $\mathcal{M}_0$.

$\mathcal{M}_0 \simeq [*/PGL_2] = BPGL_2$.

Coarse moduli space $= *$.

more generally, BG where G is any alg. group (G/k).

$BG \to \mathrm{Spec}(k)$ is a coarse moduli space.

In general given a stack $\mathcal{X}$, can form: $\pi_0(\mathcal{X}) : (\mathrm{Sch}/S)^{op} \to \mathrm{sets}$.

$\pi_0(BG) = *$