Equivariant localization, parity sheaves, and cyclic base change

Tony Feng

MIT/IAS

September 17, 2020

Global Langlands correspondence

Notation:

- $F =$ global function field, e.g. $\mathbb{F}_\ell(t)$
- $G =$ reductive group over F, e.g. SL_n
- $k = \overline{\mathbb{F}}_p$ (coefficients), $p \neq \mathrm{char}(F)$

Global Langlands correspondence

Notation:

- F = global function field, e.g. $\mathbb{F}_\ell(t)$
- G = reductive group over F, e.g. SL_n
- $k = \overline{\mathbb{F}}_p$ (coefficients), $p \neq \mathrm{char}(F)$

Vincent Lafforgue constructed

$$\left\{ \begin{array}{c} \text{irreducible cuspidal} \\ \text{automorphic representations} \\ \text{of } G \text{ over } k \end{array} \right\} \to \left\{ \begin{array}{c} \text{Langlands parameters} \\ \mathrm{Gal}(F^s/F) \to {}^L G(k)/\sim \end{array} \right\}.$$

Does it have expected properties?

Global Langlands correspondence

$$\left\{\begin{array}{c}\text{irreducible}\\ \text{automorphic representations}\\ \text{of } G \text{ over } k\end{array}\right\} \to \left\{\begin{array}{c}\text{Langlands parameters}\\ \mathrm{Gal}(F^s/F) \to {}^LG(k)/\sim\end{array}\right\}.$$

Langlands functoriality:

Global base change

$$\left\{\begin{array}{c}\text{irreducible}\\ \text{automorphic representations}\\ \text{of } G \text{ over } k\end{array}\right\} \xrightarrow{\mathrm{GLC}(G)} \left\{\begin{array}{c}\text{Langlands parameters}\\ \mathrm{Gal}(F^s/F) \to {}^L G(k)/\sim\end{array}\right\}.$$

Suppose

- H reductive over F,
- E/F field extension, $G := \mathrm{Res}_{E/F}(H_E)$.

There's a diagonal embedding $\phi_{\mathrm{BC}} \colon {}^L H \to {}^L G$.

Global base change

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{automorphic representations} \\ \text{of } G \text{ over } k \end{array} \right\} \xrightarrow{\mathrm{GLC}(G)} \left\{ \begin{array}{c} \text{Langlands parameters} \\ \mathrm{Gal}(F^s/F) \to {}^L G(k)/\sim \end{array} \right\}.$$

Suppose

- H reductive over F,
- E/F field extension, $G := \mathrm{Res}_{E/F}(H_E)$.

There's a diagonal embedding $\phi_{\mathrm{BC}} \colon {}^L H \to {}^L G$.

Theorem 1 (Existence of global base change)

Assume p is good for $\widehat{G}$ and $\mathrm{Gal}(E/F) \approx \mathbb{Z}/p\mathbb{Z}$.
Then $\rho \in \mathrm{Im}\ \mathrm{GLC}(H) \implies \phi_{\mathrm{BC}} \circ \rho \in \mathrm{Im}\ \mathrm{GLC}(G)$.

Base change

Not optimal: should be true even with characteristic 0 coefficients.

- Follows from bijectivity of GLC for $G = \mathsf{GL}_n$ by L. Lafforgue.
- Known for GL_n over number fields by Arthur-Clozel.

All proofs use the trace formula.

Base change

Not optimal: should be true even with characteristic 0 coefficients.

- Follows from bijectivity of GLC for $G = \mathsf{GL}_n$ by L. Lafforgue.
- Known for GL_n over number fields by Arthur-Clozel.

All proofs use the trace formula.

Novelty for general G: can have

$$\begin{matrix} f, f' \text{ generating } \textit{isomorphic} \\ \text{automorphic representations} \end{matrix} \to \text{different } L\text{-parameters.}$$

Indistinguishable by the trace formula!

Local Langlands correspondence

Notation:

- F_v = local function field, e.g. $\mathbb{F}_\ell((t))$.
- H = reductive group over F_v.

Local Langlands correspondence

Notation:

- F_v = local function field, e.g. $\mathbb{F}_\ell((t))$.
- H = reductive group over F_v.

Genestier-Lafforgue constructed:

$$\left\{\begin{matrix}\text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k\end{matrix}\right\} / \sim \longrightarrow \left\{\begin{matrix}\text{semi-simple} \\ \text{Langlands parameters} \\ \mathrm{Weil}(F_v) \to {}^L H(k)/\sim\end{matrix}\right\}.$$

Local base change

Notation:

- E_v/F_v extension, $\mathrm{Gal}(E_v/F_v) \approx \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \mathrm{Res}_{E_v/F_v}(H_{E_v})$

$$\left\{\begin{matrix}\text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k\end{matrix}\right\} \overset{\mathrm{BC}}{\dashrightarrow} \left\{\begin{matrix}\text{irreducible representations} \\ \text{of } G(F_v) \text{ over } k\end{matrix}\right\}$$

$\mathrm{Im}\,(\mathrm{BC})$ should be $\{\Pi \colon \Pi \circ \sigma \approx \Pi\} \implies G \rtimes \sigma \curvearrowright \Pi$.

$\mathrm{BC}^{-1}(\Pi) = ?$

Local base change

Tate cohomology. $\sigma \curvearrowright \Pi$

$$0 = \sigma^p - 1 = (\sigma - 1)(\underbrace{1 + \sigma + \ldots + \sigma^{p-1}}_{N})$$

Local base change

Tate cohomology. $\sigma \curvearrowright \Pi$

$$0 = \sigma^p - 1 = (\sigma - 1)(\underbrace{1 + \sigma + \ldots + \sigma^{p-1}}_{N})$$

$$T^0(\Pi) := \frac{\ker(\sigma - 1 \mid \Pi)}{\operatorname{Im}(N \mid \Pi)}$$

$$T^1(\Pi) := \frac{\ker(N \mid \Pi)}{\operatorname{Im}(\sigma - 1 \mid \Pi)}$$

Treumann-Venkatesh Conjecture

Conjecture (Treumann-Venkatesh)

Let Π *be an irreducible* σ*-fixed representation of* $G(F_v)$*. Then any irreducible subquotient of* $T^i(\Pi)$ *transfers under LLC to*

$$\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}_p} k.$$

Treumann-Venkatesh Conjecture

Conjecture (Treumann-Venkatesh)

Let Π be an irreducible σ-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}_p} k.$$

Theorem 2 (F.)

Assume p is good for $\widehat{G}$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under the Genestier-Lafforgue correspondence to $\Pi^{(p)}$.

Treumann-Venkatesh Conjecture

Conjecture (Treumann-Venkatesh)

Let Π be an irreducible σ-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}_p} k.$$

Theorem 2 (F.)

Assume p is good for $\widehat{G}$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under the Genestier-Lafforgue correspondence to $\Pi^{(p)}$.

Previously proved by Ronchetti for depth zero supercuspidals of GL_n induced from cuspidal Deligne-Lusztig representations.

Summary of Lafforgue's correspondence

Γ = a group.

Excursion algebra $\mathrm{Exc}(\Gamma, \widehat{G}) \quad \sim \quad$ Functions on $\mathrm{Hom}(\Gamma, \widehat{G})/\widehat{G}$.

$$\{\mathrm{Exc}(\Gamma, \widehat{G}) \to k\} \leftrightarrow \{\text{semi-simple L-parameters } \Gamma \to \widehat{G}(k)\}.$$

$[G] := G(F)\backslash G(\mathbf{A}_F)$.

Idea: construct $\mathrm{Exc}(\mathrm{Gal}(F^s/F), \widehat{G}) \curvearrowright C_c^\infty([G]; k)$.

The excursion algebra

Can present $\mathrm{Exc}(\Gamma, \widehat{G})$ explicitly by generators and relations.

Generators: $S_{n,f,(\gamma_i)_{i=1,\ldots,n}}$

The excursion algebra

Can present $\mathrm{Exc}(\Gamma, \widehat{G})$ explicitly by generators and relations.

4.2.2. *Relations.* Next we describe the relations. (Compare [Laf18a, §10].)

(i) $S_{\emptyset, f, *} = f(1_G)$.

(ii) The map $f \mapsto S_{I,f,(\gamma_i)_{i\in I}}$ is a k-algebra homomorphism in f, i.e.

$$S_{I,f+f',(\gamma_i)_{i\in I}} = S_{I,f,(\gamma_i)_{i\in I}} + S_{I,f',(\gamma_i)_{i\in I}},$$
$$S_{I,ff',(\gamma_i)_{i\in I}} = S_{I,f,(\gamma_i)_{i\in I}} \cdot S_{I,f',(\gamma_i)_{i\in I}},$$

and

$$S_{I,\lambda f,(\gamma_i)_{i\in I}} = \lambda S_{I,f,(\gamma_i)_{i\in I}} \text{ for all } \lambda \in k.$$

(iii) For all maps of finite sets $\zeta \colon I \to J$, all $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^I / \widehat{G}_k)$, and all $(\gamma_j)_{j \in J} \in \Gamma^J$, we have

$$S_{J,f^\zeta,(\gamma_j)_{j\in J}} = S_{I,f,(\gamma_{\zeta(i)})_{i\in I}}$$

where $f^\zeta \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^J / \widehat{G}_k)$ is defined by $f^\zeta((g_j)_{j\in J}) := f((g_{\zeta(i)})_{i \in I})$.

(iv) For all $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^I / \widehat{G}_k)$ and $(\gamma_i)_{i\in I}, (\gamma_i')_{i\in I}, (\gamma_i'')_{i\in I} \in \Gamma^I$, we have

$$S_{I \sqcup I \sqcup I, \widetilde{f}, (\gamma_i)_{i\in I} \times (\gamma_i')_{i\in I} \times (\gamma_i'')_{i\in I}} = S_{I,f,(\gamma_i(\gamma_i')^{-1}\gamma_i'')_{i\in I}},$$

where $\widetilde{f} \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^{I\sqcup I \sqcup I} / \widehat{G}_k)$ is defined by

$$\widetilde{f}((g_i)_{i\in I} \times (g_i')_{i\in I} \times (g_i'')_{i\in I}) = f((g_i(g_i')^{-1} g_i'')_{i\in I}).$$

Actions of the excursion algebra

How to construct $\mathrm{Exc}(\Gamma, \widehat{G}) \curvearrowright$?

Actions of the excursion algebra

How to construct $\mathrm{Exc}(\Gamma, \widehat{G}) \curvearrowright$?

Tannakian construction: given a family of functors

$$\mathrm{Rep}(\widehat{G}^I) \xrightarrow{H_I} \mathrm{Mod}(\Gamma^I)$$

Actions of the excursion algebra

How to construct $\mathrm{Exc}(\Gamma, \widehat{G}) \curvearrowright$?

Summary of Lafforgue's correspondence

Where does this structure come from?

$F \leftrightarrow X$ smooth projective curve.

Summary of Lafforgue's correspondence

Where does this structure come from?

Geometric Satake equivalence:

$$\mathrm{Rep}_k(\widehat{G}) \cong P_{G(\mathcal{O}_v)}(\underbrace{G(F_v)/G(\mathcal{O}_v)}_{\mathrm{Gr}_G}).$$

$$\begin{array}{ccc} \mathrm{Sht}_I & & \left\{ \begin{array}{c} (x_i)_{i\in I} \in X^I(S) \\ \mathcal{E} = G\text{-bundle over } X \times S \\ \varphi\colon \mathcal{E}|_{X\times S - \bigcup_{i\in I}\Gamma_{x_i}} \xrightarrow{\sim} {}^{\sigma}\mathcal{E}|_{X\times S-\bigcup_{i\in I}\Gamma_{x_i}} \end{array} \right\} \\ \big\downarrow & & \big\downarrow \\ X^I & & \{x_i\}_{i\in I} \end{array}$$

Lafforgue's correspondence

$$\begin{aligned} \mathrm{Sht}_{\emptyset} = \mathrm{Bun}_G(\mathbb{F}_\ell) &\overset{Weil}{\sim} G(F)\backslash G(\mathbf{A}_F)/\prod G(\mathcal{O}_v) \\ \implies H_{\emptyset}(\mathbb{1}) = H^0_c(\mathrm{Bun}_G(\mathbb{F}_\ell); k) &\sim C_c([G]/\prod G(\mathcal{O}_v)) \end{aligned}$$

$\implies$ get $\mathrm{Exc}(\Gamma, {}^L G) \curvearrowright C_c(\mathrm{Bun}_G(\mathbb{F}_\ell))$.

Equivariant localization

S = set with $\mathbb{Z}/p\mathbb{Z} \approx \langle \sigma \rangle$-action.

$$T^0 C_c(S) \cong T^0 C_c(S^\sigma).$$

Equivariant localization and automorphic forms

$$T^0 C_c(S) \cong T^0 C_c(S^\sigma).$$

Take $S = [\mathrm{Bun}_G(\mathbb{F}_\ell)] \implies S^\sigma = [\mathrm{Bun}_H(\mathbb{F}_\ell)]$.

$$T^0 C_c(\mathrm{Bun}_G(\mathbb{F}_\ell)) \cong T^0 C_c(\mathrm{Bun}_H(\mathbb{F}_\ell)).$$

Equivariant localization and excursion operators

Excursion operators:

$$\begin{tikzcd}
T^0 H^0_c(\mathrm{Sht}_G; \mathrm{Sat}(\mathbb{1})) \arrow[r] \arrow[d, equal] & T^0(\mathrm{Sht}_G; \mathrm{Sat}(W)) \arrow[r] \arrow[d, dashed, "?"] & \ldots \arrow[d, dashed, "?"] \\
T^0 H^0_c(\mathrm{Sht}_H; \mathrm{Sat}(\mathbb{1})) \arrow[r] & T^0(\mathrm{Sht}_H, \mathrm{Sat}(\mathrm{Res}(W))) \arrow[r] & \ldots
\end{tikzcd}$$

Equivariant localization and excursion operators

Excursion operators:

$$\begin{tikzcd}
T^0 H^0_c(\mathrm{Sht}_G; \mathrm{Sat}(\mathbb{1})) \arrow[r] \arrow[d, equal] & T^0(\mathrm{Sht}_G; \mathrm{Sat}(W)) \arrow[r] \arrow[d, dashed, "?"] & \dots \arrow[d, dashed, "?"] \\
T^0 H^0_c(\mathrm{Sht}_H; \mathrm{Sat}(\mathbb{1})) \arrow[r] & T^0(\mathrm{Sht}_H, \mathrm{Sat}(\mathrm{Res}(W))) \arrow[r] & \dots
\end{tikzcd}$$

Want to identify all steps of the excursion.

- **Topological** aspect: relating (Tate) cohomology of a space with (Tate) cohomology of its fixed points.
- **Representation-theoretic** aspect: geometric interpretation of restriction functor on dual groups.

Tate cohomology

Suppose $\langle\sigma\rangle \approx \mathbb{Z}/p \curvearrowright X$, $\mathcal{F} \in D^b_\sigma(X;k)$.

$$T^i(X;\mathcal{F}) :=$$
$$H^i(\mathrm{Tot}(\ldots \xrightarrow{N} C^*(X;\mathcal{F}) \xrightarrow{1-\sigma} C^*(X;\mathcal{F}) \xrightarrow{N} C^*(X;\mathcal{F}) \xrightarrow{1-\sigma} \ldots))$$

Smith theory

(Smith, Quillen, Treumann, etc.)

$$T^i(X; \mathcal{F}) \cong T^i(X^\sigma; \mathcal{F}|_{X^\sigma}).$$

Representation-theoretic aspects

Treumann-Venkatesh construct *Brauer homomorphism*

$$\begin{tikzcd}
\text{Hecke algebra for } G \arrow[r] \arrow[d, equal] & \text{Hecke algebra for } H \arrow[d, equal] \\
\text{Representation ring of } {}^L G \arrow[r] & \text{Representation ring of } {}^L H
\end{tikzcd}$$

Representation-theoretic aspects

Treumann-Venkatesh construct *Brauer homomorphism*

$$\begin{tikzcd} \text{Hecke algebra for } G \arrow[r] \arrow[d, equal] & \text{Hecke algebra for } H \arrow[d, equal] \\ \text{Representation ring of } {}^L G \arrow[r] & \text{Representation ring of } {}^L H \end{tikzcd}$$

With Lonergan we construct a *categorification*

$$\begin{tikzcd} \text{Hecke category for } G \arrow[r] \arrow[d, equal] & \text{Hecke category for } H \arrow[d, equal] \\ \text{Representation category of } {}^L G \arrow[r] & \text{Representation category of } {}^L H \end{tikzcd}$$

using recent tools in geometric representation theory

- parity sheaves (Juteau-Mautner-Williamson)
- Smith-Treumann theory (Treumann, Leslie-Lonergan).