Problem Set 3 Math

AA274: Principles of Robotic Autonomy — **Chi Zhang**

Stanford University — SUNet ID: czhang94

Winter 2018 — Date: February 28, 2018

Problem 1: Extended Kalman Filter

(i) Keep in mind that $\theta(t)$ is a function of t although we assume a zero-order hold on the control input, *i.e.* V and ω are considered constants *w.r.t.* time over a time interval of length dt.

$$\begin{aligned} x_t &= x_{t-1} + \int_0^{dt} \dot{x}_{t-1} d\tau \\ &= x_{t-1} + \int_0^{dt} V_t \cos(\theta_{t-1} + \omega_t \tau) d\tau \\ &= x_{t-1} + \frac{V_t}{\omega_t} \sin(\theta_{t-1} + \omega_t \tau) \Big|_0^{dt} \\ &= x_{t-1} + \frac{V_t}{\omega_t} \left[\sin(\theta_{t-1} + \omega_t dt) - \sin(\theta_{t-1})\right] \end{aligned} \tag{1}$$

$$\begin{aligned} y_t &= y_{t-1} + \int_0^{dt} \dot{y}_{t-1} d\tau \\ &= y_{t-1} + \int_0^{dt} V_t \sin(\theta_{t-1} + \omega_t \tau) d\tau \\ &= y_{t-1} - \frac{V_t}{\omega_t} \cos(\theta_{t-1} + \omega_t \tau) \Big|_0^{dt} \\ &= y_{t-1} - \frac{V_t}{\omega_t} \left[\cos(\theta_{t-1} + \omega_t dt) - \cos(\theta_{t-1})\right] \end{aligned} \tag{2}$$

$$\begin{aligned} \theta_t &= \theta_{t-1} + \int_0^{dt} \dot{\theta}_{t-1} d\tau \\ &= \theta_{t-1} + \omega_t \int_0^{dt} d\tau \\ &= \theta_{t-1} + \omega_t dt \end{aligned} \tag{3}$$

Given the expressions above, it is trivial to compute Jacobians G_x, G_u. Note that $\partial x_t / \partial \omega_t$ and $\partial y_t / \partial \omega_t$ in G_u are a little complicated:

$$\frac{\partial x_t}{\partial \omega_t} = \frac{V_t}{\omega_t^2} \left[\sin(\theta_{t-1}) - \sin(\theta_{t-1} + \omega_t dt) + \omega_t dt \cos(\theta_{t-1} + \omega_t dt)\right] \tag{4}$$

$$\frac{\partial y_t}{\partial \omega_t} = \frac{V_t}{\omega_t^2} \left[\cos(\theta_{t-1} + \omega_t dt) - \cos(\theta_{t-1}) + \omega_t dt \sin(\theta_{t-1} + \omega_t dt)\right] \tag{5}$$

In general, Eq.[1]-[5] work perfectly. However, all of them except Eq.[3] become indeterminate forms when $\omega_t \to 0$ (Turtlebot moves straight without turning). To resolve this issue, it is necessary to evaluate their limits using L'Hospital's rule:

$$\lim_{\omega_t \to 0} x_t = x_{t-1} + \left.\frac{V_t dt \cos(\theta_{t-1} + \omega_t dt)}{1}\right|_{\omega_t=0} = x_{t-1} + V_t dt \cos(\theta_{t-1})$$

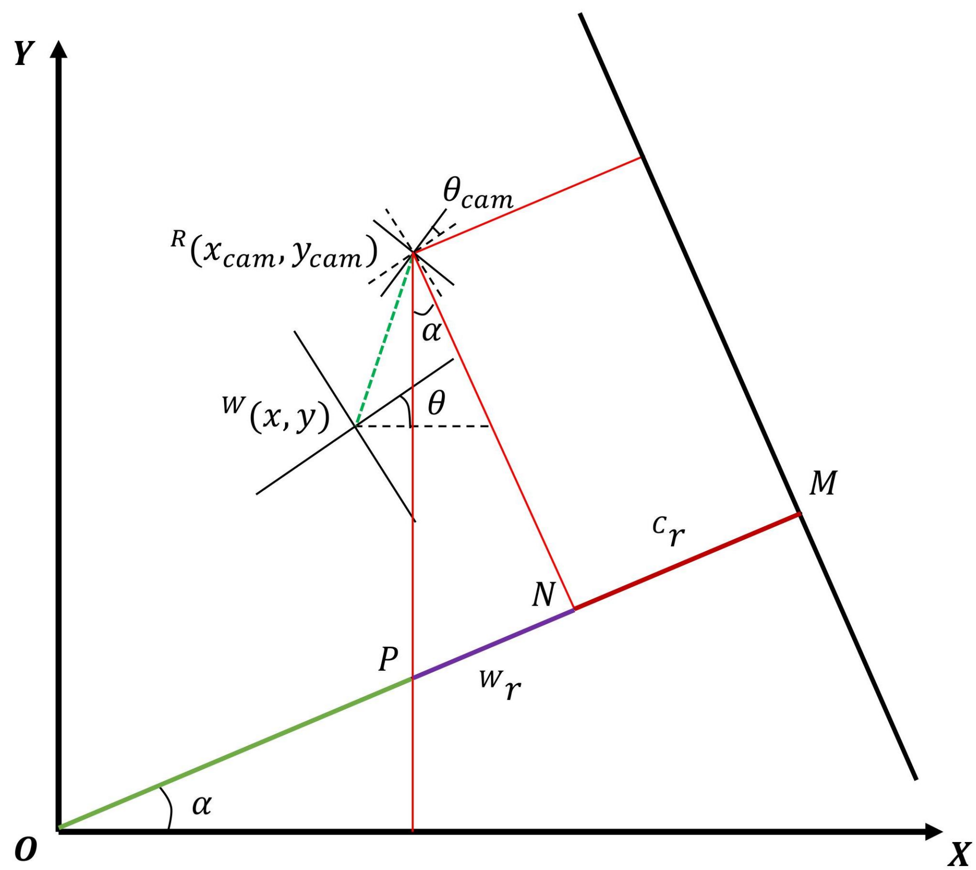
$$\lim_{\omega_t \to 0} y_t = y_{t-1} + \left.\frac{V_t dt \sin(\theta_{t-1} + \omega_t dt)}{1}\right|_{\omega_t=0} = y_{t-1} + V_t dt \sin(\theta_{t-1})$$

$$\lim_{\omega_t \to 0} \theta_t = \theta_{t-1}$$

$$\lim_{\omega_t \to 0} \frac{\partial x_t}{\partial \omega_t} = -\frac{V_t dt^2 \sin(\theta_{t-1})}{2}$$

$$\lim_{\omega_t \to 0} \frac{\partial y_t}{\partial \omega_t} = \frac{V_t dt^2 \cos(\theta_{t-1})}{2}$$

(iii) As shown below, the task is to convert (α, r) from world frame $\{W\}$ to camera frame $\{C\}$. For convenience of computation, we first convert (x, y) coordinates of camera from robot frame $\{R\}$ to world frame $\{W\}$.



Since we know that the transformation matrix between $\{R\}$ and $\{W\}$ is

$$
{}^{W}_{R}T = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}
$$

thus

$$
{}^{W}\begin{bmatrix} x_{\text{cam}} \\ y_{\text{cam}} \\ 1 \end{bmatrix} = {}^{W}_{R}T \; {}^{R}\begin{bmatrix} x_{\text{cam}} \\ y_{\text{cam}} \\ 1 \end{bmatrix} = \begin{bmatrix} x_{\text{cam}}\cos\theta - y_{\text{cam}}\sin\theta + x \\ x_{\text{cam}}\sin\theta + y_{\text{cam}}\cos\theta + y \\ 1 \end{bmatrix}
$$

It is simple to see that

$$
{}^{C}\alpha = {}^{W}\alpha - \theta - \theta_{\text{cam}}
$$

and here are steps to derive ${}^{C}r$:

$$
\begin{aligned}
OP &= \frac{{}^{W}x_{\text{cam}}}{\cos\alpha} \\
PN &= ({}^{W}y_{\text{cam}} - {}^{W}x_{\text{cam}}\tan\alpha)\sin\alpha \\
OP + PN &= \frac{{}^{W}x_{\text{cam}}}{\cos\alpha} - \frac{{}^{W}x_{\text{cam}}\sin^2\alpha}{\cos\alpha} + {}^{W}y_{\text{cam}}\sin\alpha \\
&= {}^{W}x_{\text{cam}}\cos\alpha + {}^{W}y_{\text{cam}}\sin\alpha
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}^{C}r &= |OM - ON| \\
&= \left|{}^{W}r - (OP + PN)\right| \\
&= \left|{}^{W}r - {}^{W}x_{\text{cam}}\cos\alpha - {}^{W}y_{\text{cam}}\sin\alpha\right|
\end{aligned}
$$