MS&E 349: Homework 1

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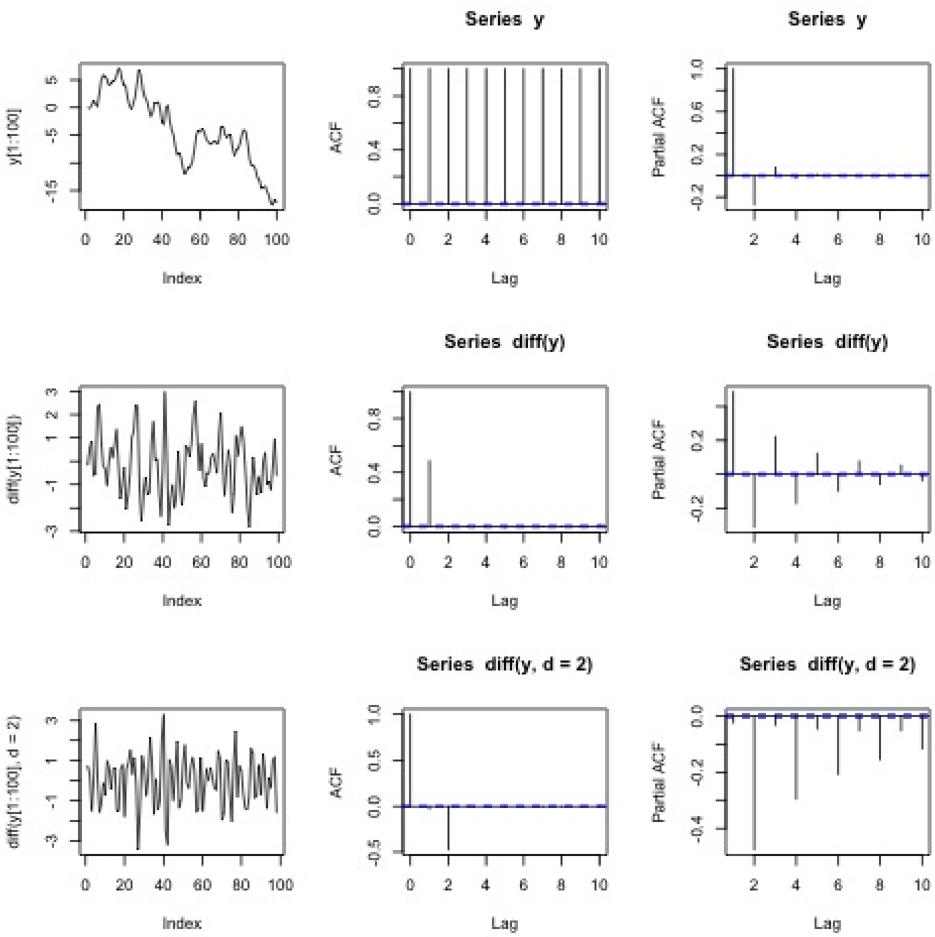
Due 24pm, February 3rd 2021. Submit on Canvas

Please submit one homework assignment per group. The homework solution should be submitted online to Canvas. Please indicate clearly the names of all group members. I prefer that solutions are typed in Latex, but it is also fine to submit scanned copies of handwritten solutions. Include the commented code in an appendix section. Please also submit the executable and commented code.

Theoretical Questions

Question 1 Time Series

The top left panel shows a simulation of an $ARIMA(p, d, q)$ process y_t. The panel below shows the implied series for Δy_t and the bottom left panel shows $\Delta^2 y_t$. The middle columns are the corresponding autocorrelation functions and the right columns shows the partial autocorrelations. (The partial autocorrelations are the regression coefficients of y_t on its past values.)



Based on the presented information, what is your best guess for the values of p,d and q? What are the likely signs of any AR and MA parameters? Explain your answer.

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Question 2 Time Series and GMM

Suppose $\{Y_t : -1 \leq t \leq T\}$ is an observed strictly stationary time series generated by the $AR(1)$ model

$$Y_t = \sum_{i=0}^{\infty} \phi_0^i \epsilon_{t-i}$$

where $|\phi_0| \in (0,1)$ and $\epsilon_t \sim i.i.d.\ N(0,1)$. It can be shown that

$$\frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} Y_{t-1}^2 \\ Y_{t-2}Y_{t-1}\end{pmatrix} \xrightarrow{p} \begin{pmatrix} E[Y_{t-1}^2] \\ E[Y_{t-1}Y_t]\end{pmatrix} = \begin{pmatrix}\gamma(0) \\ \gamma(1)\end{pmatrix}$$

and

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\begin{pmatrix} Y_{t-1} \\ Y_{t-2}\end{pmatrix}\epsilon_t \xrightarrow{d} N\left(0, \lim_{T\to\infty} Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\begin{pmatrix} Y_{t-1} \\ Y_{t-2}\end{pmatrix}\epsilon_t\right)\right)$$

Let $X_t = (Y_t, Y_{t-1}, Y_{t-2})'$ and define the functions:

$$h_1(X_t,\phi) = Y_{t-1}(Y_t - \phi Y_{t-1}) \qquad h_2(X_t,\phi) = Y_{t-2}(Y_t - \phi Y_{t-1}) \qquad h(X_t,\phi) = [h_1(X_t,\phi), h_2(X_t,\phi)]'.$$

1. Show that $\Phi_1 = \Phi_2 = \Phi = \{\phi_0\}$, where

$$\Phi_1 = \{\phi : E[h_1(X_t,\phi)] = 0\} \qquad \Phi_2 = \{\phi : E[h_2(X_t,\phi)] = 0\} \qquad \Phi = \{\phi : E[h(X_t,\phi)] = 0\}$$

2. For $k \in \{1,2\}$, let

$$\hat{\phi}_k = argmin_\phi g_{T,k}(\phi)^2, \qquad g_{T,k}(\phi) = \frac{1}{T}\sum_{t=1}^{T} h_k(X_t,\phi).$$

   It can be shown that

$$\sqrt{T}\left(\hat{\phi}_1 - \phi_0\right) \xrightarrow{d} N(0,\omega_1^2)$$

   where ω_1^2 is some function of ϕ_0. Verify this claim and express ω_1^2 in terms of ϕ_0.

3. It can be shown that

$$\sqrt{T}\left(\hat{\phi}_2 - \phi_0\right) \xrightarrow{d} N(0,\omega_2^2)$$

   where ω_2^2 is some function of ϕ_0. Verify this claim and express ω_2^2 in terms of ϕ_0. Is $\omega_1^2 = \omega_2^2$? If "yes", are $\hat{\phi}_1$ and $\hat{\phi}_2$ asymptotically equivalent? If "no", can $\hat{\phi}_1$ and $\hat{\phi}_2$ be ranked in terms of asymptotic efficiency (A consistent estimator is more efficient than another consistent estimator if it has a smaller asymptotic variance)?

4. Let

$$\hat{\phi}_W = argmin_\phi g_T(\phi)^\top W g_T(\phi), \qquad g_T(\phi) = [g_{T,1}(\phi), g_{T,2}(\phi)]'$$

where W is a (non-zero) symmetric, positive semi-definite 2×2 matrix.

Give conditions (on W and/or $\{Y_t\}$) under which $g_T(\hat{\phi}_W) = 0$.

5. It can be shown that if

$$\begin{pmatrix} 1 \\ \phi_0 \end{pmatrix}^\top W \begin{pmatrix} 1 \\ \phi_0 \end{pmatrix} > 0,$$

then

$$\sqrt{T}\left(\hat{\phi}_W - \phi_0\right) \xrightarrow{d} N(0, \omega_W^2)$$

where ω_W^2 is some function of ϕ_0. Verify this claim and express ω_W^2 in terms of ϕ_0.

6. Find W^*, a value of W for which ω_W^2 is minimal, and propose a feasible estimator $\hat{\phi}$ (i.e. an estimator $\hat{\phi}$ that can be computed without knowledge of ϕ_0) satisfying

$$\sqrt{T}\left(\hat{\phi} - \phi_0\right) \xrightarrow{d} N(0, \omega_{W^*}^2)$$

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Empirical Questions

The data that you need for this exercise is in the homework folder on Canvas.

Question 3 ARIMA

Consider the monthly crude oil prices from January 1986 to February 2016. The original data are from FRED and also available in m-COILWTICO.txt. If you are using Python, you can use the `statsmodels` package package (use `import statsmodels` to import the package).

1. Obtain the time plot of the oil prices and its first differenced series.

2. Based on the plots, is the first differenced series weakly stationary? Why?

3. Use the ADF test to test for non-stationarity in the oil prices and its first differenced series. Report your results.
(Hint: You can use `statsmodels.tsa.stattools.adfuller(x)`)

4. Plot the ACF and PACF functions for the oil price and the differenced series. Draw your conclusion.
   (Hint: You can use `smt.graphics.plot_acf(x, lags=40, alpha=0.05)` and `smt.graphics.plot_pacf(x, lags=40, alpha=0.05)`)

5. Let r_t be the first differenced price series. Test $H_0 : \rho_1 = \cdots = \rho_{12} = 0$ versus H_1: $\rho_i \neq 0$ for some $1 \leq i \leq 12$. Draw your conclusion.
   (Hint: You can use `statsmodels.stats.diagnostic.acorr_ljungbox(x, lags=12, boxpierce=False)`)

6. Build an AR model for r_t, including model checking. (Hint: Use the PACF to determine the order of the AR(p) process.) Refine the model by excluding all estimates with t-ratio less than 1.645. (Hint: the t-ratio is the ratio of the estimate divided by the standard error.) Write down the fitted model.
   (Hint: You can use

   ```
   model = statsmodels.tsa.arima_model.ARIMA(x, order=(6,0,0))
   model_fit = model.fit(disp=0)
   print(model_fit.summary())
   ```

7. Build an ARIMA model for r_t, including model checking. Write down the fitted model.
   (Hint: an ARMA(1,6) might be a good choice and which can be fitted with the command

   ```
   mod = sm.tsa.statespace.SARIMAX(dx, trend='c', order=(1,0,6))
   res = mod.fit(disp=False)
   print(res.summary())
   ```

8. Use the fitted ARMA(1,6) model to compute 1-step to 4-step ahead forecasts of r_t at the forecast origin February, 2016. Also, compute the corresponding 95% interval forecasts. Report the numbers and also include a plot for your forecast.
   (Hint: You can use

   ```
   predict = res.get_prediction(start=361, end=364)
   predict_val = predict.predicted_mean
   predict_ci = predict.conf_int()
   ```

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Question 4 GARCH

Consider the daily returns of Amazon (amzn) stock from January 2, 2009 to December 31, 2014. The simple returns are available from CRSP and in the file `d-amzn3dx0914.txt` (the column with heading `amzn`). Transform the simple returns to log returns. Multiple the log returns by 100 to obtain the percentage returns. Let r_t be the percentage log returns.
You can use the `statsmodels` and `arch` packages (Import these packages with `import statsmodels` and `import arch`.)

1. Is the expected value of r_t zero? Why?

2. Are there any serial correlations in r_t? Why? (Hint: use ACF and PACF plots.)

3. Fit a Gaussian ARMA-GARCH model to the r_t series. Obtain the normal QQ-plot of the standardized residuals, and write down the fitted model. Plot the ACF of the standardized and squared standardized residuals. Is the model adequate?
   (Hint: You can use `statsmodels.api.qqplot(x)` for QQ-plot and and

   ```
   am = arch.arch_model(y, vol='Garch', p=p, q=q)
   am_fit = am.fit()
   print(am_fit.summary())
   ```

   for GARCH(p,q) model)[1]

4. Build an ARMA-GARCH model with Student-t innovations for the $r - t$ series. Perform model checking, including the QQ-plot. Plot the ACF of the standardized and squared standardized residuals. Is the model adequate?
   (Hint: You can use `arch_model(res, p=p, q=q, dist='StudentsT')` for the Student-t innovations)

5. Write down the fitted model.

6. Obtain 1-step to 5-step ahead mean and volatility forecasts using the fitted ARMA-GARCH model with Student-t innovations. Report the numerical values and the plots for your forecast. Comment on the statement: "Returns are not predictable (for short horizons)".
   (Hint: You can use `am_fit.forecast(horizon=5)` for 1-step to 5-step forecasts)

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[1] Please refer `https://buildmedia.readthedocs.org/media/pdf/arch/latest/arch.pdf` for more details.