Prep Math 140 Selection of Problems

Calculus 2 (McGill University)

SOLUTIONS - PREP 101
McGill MATH 140 Final Exam (Selection of problems)

Thank you for attending a PREP 101 session! Best of luck !!

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APRIL 2006

* $$\lim_{x\to\infty} x-\sqrt{x^4-4x} = \lim_{x\to\infty} \frac{(x-\sqrt{x^4-4x})(x+\sqrt{x^4-4x})}{x+\sqrt{x^4-4x}}$$

$$= \lim_{x\to\infty} \frac{x^2-x^4+4x}{x+\sqrt{x^4}\sqrt{1-\frac{4}{x^3}}} = \lim_{x\to\infty} \frac{x^4\left(-1+\frac{1}{x^2}+\frac{4}{x^3}\right)}{x+x^2\sqrt{1-\frac{4}{x^3}}}$$

$$= \lim_{x\to\infty} \frac{x^4\left(-1+\frac{1}{x^2}+\frac{4}{x^3}\right)}{x^2\left(\frac{1}{x}+\sqrt{1-\frac{4}{x^3}}\right)} = \lim_{x\to\infty} \frac{-x^2}{1} = -\infty$$

* $$\lim_{x\to 0} \frac{x^2+1}{1-\cos 2x} = \frac{1}{0^+} = +\infty$$

(Note as $x\to 0^+$ or $x\to 0^-$ $\cos x\to 1^-$ so $1-\cos x\to 0^+$)

* $$\lim_{x\to\ln 2} \frac{e^{2u}+e^{u}-6}{e^{2u}+6e^{u}-16} = \lim_{x\to\ln 2} \frac{(e^u+3)\cancel{(e^u-2)}}{\cancel{(e^u-2)}(e^u+8)} = \frac{2+3}{2+8} = \frac{5}{10} = \frac{1}{2}$$

(Note: $e^{\ln 2} = 2$)

* $$\lim_{x\to-\infty} \frac{\cosh 2x+\sinh 2x}{(\cosh x+\sinh x)^2} = \lim_{x\to-\infty} \frac{\frac{e^{2x}+e^{-2x}+e^{2x}-e^{-2x}}{2}}{\left(\frac{e^{x}+e^{-x}+e^{x}-e^{-x}}{2}\right)^2}$$

$$= \lim_{x\to-\infty} \frac{e^{2x}}{e^{2x}} = 1$$

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* The surface area of a sphere is increasing at 5 cm^2/sec. How fast is the volume changing when $r = 10cm$?

$S = 4\pi r^2 \quad , \quad V = \frac{4}{3}\pi r^3$

$S' = 8\pi r . r' \Rightarrow r' = \frac{S'}{8\pi r} = \frac{5}{8\pi(10)} = \frac{1}{16\pi}$ cm/s $\quad \left(S' = \frac{dS}{dt}, \ r' = \frac{dr}{dt} \right)$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi \ (100) \cdot \frac{1}{16\pi} = 25$ cm^3/sec.

* A cylinder is such that its top and bottom are disks of radius r, its side has area $2\pi rh$ (h = height of cylinder), and its volume is $54\pi \ cm^3$.

Find the dimensions of the cylinder that uses the least amount of material.

Amount of material = $\underbrace{2\pi r^2}_{\text{TOP + bottom}} + \underbrace{2\pi rh}_{\text{side}}$

But $V = \pi r^2 . h = 54\pi \Rightarrow h = \frac{54\pi}{\pi r^2} = \frac{54}{r^2}$

So amount of material = $S(r) = 2\pi r^2 + 2\pi r . \frac{54}{r^2} = 2\pi r^2 + \frac{108\pi}{r}$

where $r > 0$.

$S'(r) = 4\pi r - \frac{108\pi}{r^2} = 0 \Rightarrow 4\pi r = \frac{108\pi}{r^2} \Rightarrow r^2 = 27 \Rightarrow r = \sqrt{27} = 3\sqrt{3}$

$S''(r) = 4\pi + \frac{216\pi}{r^3} \quad S''(\sqrt{27}) > 0 \Rightarrow r = \sqrt{27}$ gives Minimum surface.

$h = \frac{54}{r^2} = \frac{54}{27} = 2$ cm.

APRIL 2007

* find a polynomial $f(x) = Ax^3 + 6x^2 - Bx$ such that f has a local maximum at $x=-1$ & an inflection pt at $x=1$ or show that no such polynomial exists.

$f'(x) = 3Ax^2 + 12x - B$

$f''(x) = 6Ax + 12 = 0 \Rightarrow \quad x = -\frac{2}{A}$ (the only possible inflection pt)

$$\Rightarrow -\frac{2}{A} = 1 \Rightarrow \boxed{A = -2}$$

now: $f'(x) = -6x^2 + 12x - B$

$x=-1$ gives a local max $\Rightarrow$ $x=-1$ critical value $\Rightarrow f'(-1) = 0$

$$-6 - 12 - B = 0 \Rightarrow B = -18$$

Now verify that in this case $x=-1$ gives local max and <u>NOT</u> local min.

$$-6x^2 + 12x + 18 = -6(x^2 - 2x - 3) = -6(x-3)(x+1)$$

$$\Rightarrow f(x) = -2x^3 + 6x^2 + 18x$$

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* Let $f(x) = |3-x|$ show that f is not differentiable at $x=3$

$$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{|3-x|-0}{x-3} = \lim_{x\to 3} \frac{|3-x|}{x-3}$$

$\rightarrow \lim_{x\to 3^-} \frac{|3-x|}{x-3} = \lim_{x\to 3^-} \frac{3-x}{x-3} = -1$

$\rightarrow \lim_{x\to 3^+} \frac{|3-x|}{x-3} = \lim_{x\to 3^+} \frac{-(3-x)}{x-3} = 1$

$\Rightarrow \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = f'(3)$ Does not exist

* Show that $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$

$$\frac{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}}{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}} = \frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}} = \frac{2e^x}{2e^{-x}} = e^x \cdot e^x = e^{2x}.$$

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December 2006

* $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}+\frac{7}{x^2}\right)^x$ Let $y=\left(1-\frac{2}{x}+\frac{7}{x^2}\right)^x$

$$\Rightarrow \ln y = x\ln\left(1-\frac{2}{x}+\frac{7}{x^2}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(1-\frac{2}{x}+\frac{7}{x^2}\right) \quad (\infty . 0)$$

$$= \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}+\frac{7}{x^2}\right)}{1/x} \overset{HR}{=} \lim_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x}+\frac{7}{x^2}}\cdot\left(\frac{2}{x^2}-\frac{14}{x^3}\right)}{-1/x^2}$$

$$= \lim_{x\to\infty}\frac{1}{1-\frac{2}{x}+\frac{7}{x^2}}\left(\frac{2}{x^2}-\frac{14}{x^3}\right)(-x^2)$$

$$= \lim_{x\to\infty}\frac{1}{1-\cancel{\frac{2}{x}}+\frac{7}{x^2}}\left(-2+\cancel{\frac{14}{x}}\right) = -2.$$

$$\Rightarrow \lim_{x\to\infty} y = e^{-2}.$$

* Find the horizontal asym. of $g(x)=2\tanh x+1$

$$\lim_{x\to\infty} 2\tanh x + 1 = \lim_{x\to\infty} 2\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)+1$$

$$= \lim_{x\to\infty} 2\left(\frac{\cancel{e^x}(1-\cancel{e^{-2x}})}{\cancel{e^x}(1+\cancel{e^{-2x}})}\right)+1 = 2+1=3$$

(same procedure for $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} 2\left(\frac{e^{-x}(e^{2x}-1)}{e^{-x}(e^{2x}+1)}\right)+1 = -1$)

$$\Rightarrow y=-1 \text{ H.A at } -\infty,\ y=3 \text{ HA at } \infty$$