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Math 1400 Quiz #1 09/16/22 **Quiz Session 0040-102: 10:15am**

1. Find the Taylor series for $f(x) = \sin x$ around the point $x = \frac{\pi}{3}$. Write out the first 5 non-zero terms of the series and express it in summation (Σ) notation.

Begin by taking derivatives.

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x$$

From here the derivatives repeat so we can stop. Evaluate at the expansion point $x = \pi/6$.

$$f(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}, \quad f'(\tfrac{\pi}{3}) = \tfrac{1}{2}, \quad f''(\tfrac{\pi}{3}) = -\tfrac{\sqrt{3}}{2}, \quad f'''(\tfrac{\pi}{3}) = -\tfrac{1}{2}$$

Now we may use that the Taylor coefficients are given by $a_n = \dfrac{f^{(n)}(c)}{n!}$ *to write the series.*

$$\sin x = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{3}}{2 \cdot 2!}(x - \frac{\pi}{3})^2 - \frac{1}{2 \cdot 3!}(x - \frac{\pi}{3})^3 + \frac{\sqrt{3}}{2 \cdot 4!}(x - \frac{\pi}{3})^4 + \dots$$

To write in summation notation note that the series does not strictly alternate, but we can make it do so by breaking it into even and odd terms.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2 \cdot (2n)!}(x - \frac{\pi}{3})^{2n} + \frac{(-1)^{n+1}}{2 \cdot (2n+1)!}(x - \frac{\pi}{3})^{2n+1}$$

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2. (a) Find the Taylor series of

$$f(x) = \frac{5x}{2-3x^2}$$

around the point $x = 0$.

We can rewrite the function as

$$f(x) = \frac{5x}{2} \cdot \frac{1}{1-\frac{3}{2}x^2}$$

Now use the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ *with the substitution* $x \to \frac{3}{2}x^2$ *to obtain*

$$\frac{1}{1-\frac{3}{2}x^2} = \sum_{n=0}^{\infty} (\frac{3}{2}x)^n = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{2^n}$$

Multiply by $\frac{5x}{2}$ *to complete.*

$$f(x) = \sum_{n=0}^{\infty} \frac{5 \cdot 3^n x^{2n+1}}{2^{n+1}}$$

(b) Determine the values of x for which the series converges.

The series for $\frac{1}{1-x}$ *converges for* $|x| < 1$*. Multiplication by a polynomial does not affect the interval of convergence. Therefore we just must account for the substitution* $x \to -\frac{3}{2}x^2$*. This gives us a new range*

$$|-\frac{3}{2}x^2| < 1 \implies |x| < \sqrt{\frac{2}{3}}$$

(c) Using the series determine the value of $f^{(5)}(0)$. Explain how this value can be read from the Taylor series.

The derivatives at the expansion point are encoded in the Taylor series coefficients. The series is expanded at $x = 0$ *so* $a_n = \frac{f^{(n)}(0)}{n!} \implies f^{(n)}(0) = n! \cdot a_n$*. So in this instance*

$$f^{(5)}(0) = 5! \cdot \frac{5 \cdot 3^2}{2^3}$$

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3. Evaluate the following limit.

$$\lim_{x\to 0} \frac{1-e^{2x^3}}{8x^3-x^4+3x^5}$$

Begin with the expansion of e^x *at* $x=0$ *and substitute* $x \to 2x^3$.

$$e^x = 1 + x + \frac{x^2}{2} + H.O.T.$$
$$e^{2x^3} = 1 + 2x^3 + \frac{4x^6}{2!} + H.O.T.$$

Substituting into the limit gives us

$$\lim_{x\to 0} \frac{1-(1+2x^3+\frac{4x^6}{2!}+H.O.T.)}{8x^3-x^4+3x^5}$$
$$=\lim_{x\to 0} \frac{-2x^3-\frac{4x^6}{2!}-H.O.T.}{8x^3-x^4+3x^5}$$

Now we can factor an x^3 *out of the numerator (remember the H.O.T. is degree greater than 6) and the denominator and cancel.*

$$=\lim_{x\to 0} \frac{-2-\frac{4x^3}{2!}-H.O.T.}{8-x+3x^2} = \frac{-2-0}{8+0} = -\frac{1}{4}$$

Where we note that the H.O.T. approaches 0 at $x \to 0$ *since all terms are polynomials in* x *of degree* > 0.