MATH 140 Practice Set #1

Calculus 1 (McGill University)

MATH 140

These are a few problems to illustrate some types of limits and derivatives.
The level of difficulty of these problems is similar to the exercises in your book.
More problems will be posted throughout the semester.

1. Evaluate each of the following limits:

(a) $\lim\limits_{x \to -2} \dfrac{\frac{x-2}{x+3} + 4}{x+2}$

(b) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 - 3x} - x \right)$

(c) $\lim\limits_{x \to 0} \dfrac{\sin^2(2x)\tan(3x)}{x^3}$

2. Derive each of the following functions:

(a) $f(x) = \sin(3\tan(2x))$

(b) $f(x) = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

(c) $g(x) = \sec^2(5x)$

(d) $g(x) = \dfrac{1 - \tan(2x)}{1 + x^e + e^x}$

(e) $h(x) = (1 + 3^x)\cos(x^3)\, e^{2x - x^3}$

3. (Harder) Evaluate $\lim\limits_{x \to \pi/4} \dfrac{\tan x - 1}{x - \pi/4}$

MATH 140 - Solutions

antoine.prep101@gmail.com

Evaluate each of the following Limits

0/0 Limit

(a) $$\lim_{x\to -2} \frac{\frac{x-2+4(x+3)}{x+3}}{\frac{x+2}{1}} = \lim_{x\to -2} \frac{x-2+4x+12}{x+3}\cdot\frac{1}{x+2}$$

$$= \lim_{x\to -2} \frac{5x+10}{(x+3)(x+2)} = \lim_{x\to -2}\frac{5(x+2)}{(x+3)(x+2)} = \lim_{x\to -2}\frac{5}{x+3} = 5$$

Note: If $f(x) = \frac{x-2}{x+3}$; the above limit is $f'(-2) = \lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}$

$\infty - \infty$ Limit

(b) $$\lim_{x\to\infty} \sqrt{x^2-3x} - x = \lim_{x\to\infty} \sqrt{x^2-3x} - x \cdot \frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x}$$

∞/∞ Limit

$$= \lim_{x\to\infty} \frac{x^2-3x-x^2}{\sqrt{x^2-3x}+x} = \lim_{x\to\infty}\frac{-3x}{\sqrt{x^2-3x}+x}$$

$$= \lim_{x\to\infty}\frac{x(-3)}{\sqrt{x^2(1-3/x)}+x} = \lim_{x\to\infty}\frac{x(-3)}{|x|\sqrt{1-3/x}+x}$$

note: $\sqrt{x^2} = |x|$ & $|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0\end{cases}$

$$= \lim_{x\to\infty}\frac{x(-3)}{x\sqrt{1-3/x}+x} = \lim_{x\to\infty}\frac{\cancel{x}(-3)}{\cancel{x}(\sqrt{1-3/x}+1)} = \lim_{x\to\infty}\frac{-3}{\sqrt{1-3/x}+1} = \frac{-3}{2}.$$

$\frac{3}{\infty}\to 0$

Question: what happens if $x\to -\infty$?

ANS : $\lim_{x\to -\infty} \sqrt{x^2-3x} - x = \infty - (-\infty) = \infty$

0/0 Limit

(c) $$\lim_{x\to 0} \frac{\sin(2x)\cdot\sin(2x)\cdot\sin(3x)}{x\cdot x\cdot x\cdot\cos(3x)}$$

$$= \lim_{x\to 0} \frac{\sin(2x)}{2x}\cdot\frac{\sin(2x)}{2x}\cdot\frac{\sin(3x)}{3x}\cdot\frac{(2)(2)(3)}{\cos(3x)}$$

$$= 1\cdot 1\cdot 1\cdot\frac{12}{\cos 0} = 12$$

Note: $$\lim_{A\to 0}\frac{\sin(A)}{A} = 1.$$

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Derive each of the following functions

(a) $$f'(x) = \cos(3\tan(2x))\cdot(3\tan(2x))'$$
$$= \cos(3\tan(2x))\cdot 3\sec^2(2x)\cdot(2x)'$$
$$= \cos(3\tan(2x))\cdot 3\sec^2(2x)\cdot(2)$$
$$= 6\cos(3\tan(2x))\cdot\sec^2(2x)$$

(b) $$f(x) = \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\left(1+\sqrt{1+\sqrt{x}}\right)'$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{2\sqrt{1+\sqrt{x}}}\cdot(1+\sqrt{x})'$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}$$

(c) $g'(x) = 2\sec(5x)\,(\sec(5x))'$

$= 2\sec(5x)\cdot\sec(5x)\cdot\tan(5x)\cdot(5x)'$

$= 2\sec^2(5x)\cdot\tan(5x)\cdot 5$

$= 10\sec^2(5x)\cdot\tan(5x)$

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(d) $g'(x) = \dfrac{(1-\tan(2x))'(1+x^e+e^x) - (1-\tan(2x))\cdot(1+x^e+e^x)'}{(1+x^e+e^x)^2}$

$$g'(x) = \frac{-2\sec^2(2x)\,(1+x^e+e^x) - (1-\tan(2x))\,(1+ex^{e-1}+e^x)}{(1+x^e+e^x)^2}$$

(e) $h'(x) = \big(\underbrace{(1+3^x)\cos(x^3)}_{\text{product rule}}\big)'\,e^{2x-x^3} + (1+3^x)\cos(x^3)\cdot\left(e^{2x-x^3}\right)'$

$= \overbrace{\left((1+3^x)'\cos(x^3) + (1+3^x)\left(\cos(x^3)\right)'\right)} e^{2x-x^3} + (1+3^x)\cos(x^3)\cdot e^{2x-x^3}\cdot(2-3x^2)$

$= \left(3^x\ln 3\cos(x^3) - (1+3^x)\sin(x^3)\cdot(3x^2)\right)e^{2x-x^3} + (1+3^x)\cos(x^3)\cdot e^{2x-x^3}(2-3x^2)$

$= 3^x\ln 3\cos(x^3)\cdot e^{2x-x^3} - (1+3^x)\sin(x^3)\cdot(3x^2)\cdot e^{2x-x^3} + (1+3^x)\cos(x^3)e^{2x-x^3}(2-3x^2)$

NOTE: $(FG)' = F'G + FG'$

$(FGH)' = F'GH + FG'H + FGH'$

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3. Remember the definition of the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Here $f(x) = \tan x$ & $a = \pi/4$

$$f'(\pi/4) = \lim_{x \to \pi/4} \frac{\tan x - \tan(\pi/4)}{x - \pi/4}$$

$$= \lim_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4} = \sec^2(\pi/4) = (\sqrt{2})^2 = 2.$$