Prac Final 2021 solutions

Calculus (Massachusetts Institute of Technology)

Practice final

18.01, Fall 2021 with solutions

Not due

(The final exam itself will be 3 hours long. It is closed book, no calculators.)

1 Finger exercises

a. Find $\displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{\cos\theta}}\,d\theta$.

Solution. We will use u-substitution. Using $u = \cos\theta$ and $du = -\sin\theta d\theta$:

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{\cos\theta}}\,d\theta &= \int_1^0 \frac{-du}{\sqrt{u}} \\ &= -2u^{1/2}\big|_1^0 \\ &= -2(0^{1/2} - 1^{1/2}) = 2\end{aligned}$$

Where we got the new limits by plugging in $1 = \cos(0)$ for the lower limit and $0 = \cos(\pi/2)$ for the upper limit.

b. Find the derivative of $\cos\big(\cos\theta^2\big)$.

Solution. We need to apply chain rule. We have three layers: $f(g(h(x)))$ where $f(x) = g(x) = \cos\theta$ and $h(x) = \theta^2$. We can write:

$$\begin{aligned}\frac{d}{d\theta}\cos\big(\cos\theta^2\big) &= -\sin\big(\cos\theta^2\big)\cdot\frac{d}{d\theta}\cos(\theta^2) \\ &= -\sin\big(\cos\theta^2\big)\cdot(-\sin(\theta^2))\cdot\frac{d}{d\theta}\theta^2 \\ &= -\sin\big(\cos\theta^2\big)\cdot(-\sin(\theta^2))\cdot 2\theta \\ &= 2\theta\sin(\cos(\theta^2))\sin(\theta^2)\end{aligned}$$

c. Find $\displaystyle\int x\ln x\,dx$.

Solution. We will integrate by parts.

$$u = \ln x, \qquad v = x^2/2$$
$$du = \frac{1}{x}dx, \qquad dv = xdx$$

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\end{aligned}$$

2 Taylor

Let $f(x) = e^{-x^2} + \frac{1}{10}x$.

a. Find the 2nd-order Taylor series of f around $x = 0$.

Solution. Compute

$$f(x) = e^{-x^2} + \frac{1}{10}x, \quad f(0) = 1$$
$$f'(x) = -2xe^{-x^2} + \frac{1}{10}, \quad f'(0) = \frac{1}{10}$$
$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}, \quad f''(0) = -2.$$

Use this in the formula for Taylor series to get

$$f(x) \approx 1 + \frac{1}{10}x - x^2.$$

Alternative method: Recall the Taylor series formula for e^z :

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

and plug in $z = -x^2$ to get the 2nd order Taylor series

$$e^{-x^2} \approx 1 - x^2$$

(we don't need any terms that are higher order in x, so we drop them). The function $\frac{1}{10}x$ is linear, so it is already in the form of a 2nd order Taylor series. Add the two Taylor approximations together to get

$$f(x) \approx 1 - x^2 + \frac{1}{10}x = 1 + \frac{1}{10}x - x^2.$$

b. Suppose that we want to choose x in the range $-1 \leq x \leq 1$ to maximize $f(x)$. It's not possible to find an exact formula for this x. Instead, approximate the best choice of x. Is it closest to .5 or .05 or .005? Explain your reasoning.

Hint: Use the Taylor series from the previous part.

Solution. We can't find an exact value of x because when we try to find the critical points of f in the range $-1 \leq x \leq 1$, we get

$$f'(x) = -2xe^{-x^2} + \frac{1}{10} = 0 \to \frac{1}{10} = 2xe^{-x^2} \to x = ???$$

Instead of maximizing f directly, we maximize its Taylor approximation from part a:

$$T(x) = 1 + \frac{1}{10}x - x^2.$$

The derivative is $T'(x) = \frac{1}{10} - 2x$, which means $x = 0.05$ is a critical point, T is increasing before $x = 0.05$ and decreasing after $x = 0.05$. Thus the maximum of $f(x)$ in the range $-1 \leq x \leq 1$ occurs around $x = 0.05$.

3 Approximating a square root

Let $f(x) = x^3$.

a. Write down the straight line approximation of $f(x)$ around $x = 10$.

Solution. We recall that the straight line approximation of $f(x)$ takes the form:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

(Notice - this is also the first-order Taylor series around $x = 10$). We can plug in $x = 10$:

$$\begin{aligned} f(10) &= (10)^3 = 1000 \\ f'(x) &= 3x^2 \\ f'(10) &= 3(10)^2 = 300 \end{aligned}$$

Then we have:

$$f(10 + \Delta x) \approx 1000 + 300\Delta x$$

b. Use it to approximate $(10.2)^3$.

Solution. 10.2 is close to 10, so we will use the straight line approximation from part a. We can write:

$$\begin{aligned} (10.2)^3 &= f(10.2) \\ &= f(10 + 0.2) \\ &\approx 1000 + 300(0.2) = 1060 \end{aligned}$$

c. Use it to approximate the cube root of 1003.

Solution. Now we will backsolve, using the straight line approximation from part a. Suppose $10 + y$ is the cube root of 1003. We expect y to be positive, since $10^3 = 1000$. We can plug in the approximation and set it equal to 1003:

$$\begin{aligned}
1003 &= (10+y)^3 = f(10+y) \\
1003 &\approx 1000 + 300(y) \\
3 &\approx 300y \\
\frac{1}{100} &\approx y
\end{aligned}$$

Then $10 + y = 10.01$ is our approximation for the cube root of 1003.

4 Center of mass

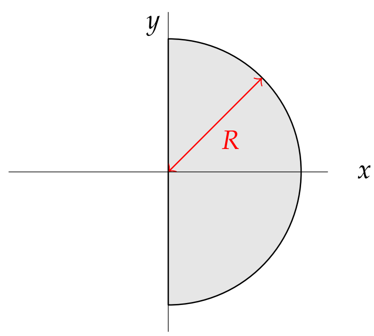
First let's review center of mass. Suppose that we have n particles with masses m_1, m_2, ..., m_n. Suppose that the positions of the particles are (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Then the x coordinate of the center of mass is given by

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}. \tag{$*$}$$

(There is a similar formula for y_{cm}, the y coordinate of the center of mass.)

For instance, if we have two particles with equal mass, then x_{cm} is the average of x_1 and x_2. If there are two particles and $m_2 = 2m_1$, then $x_{\text{cm}} = \frac{1}{3}x_1 + \frac{2}{3}x_2$.

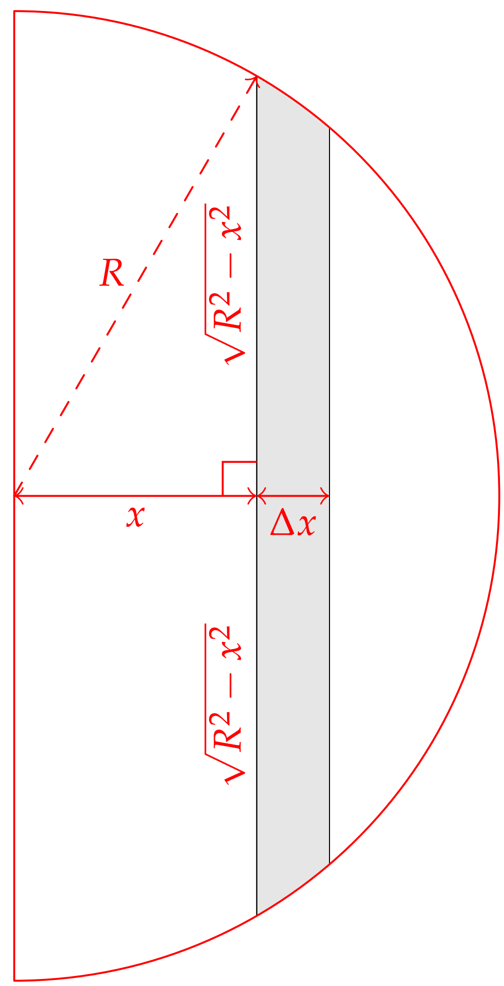
In this problem, we consider a half-disk with radius R:



The half-disk has uniform density σ. In this problem we will find x_{cm}.

a. Write down an integral that corresponds to the numerator of $(*)$.

Solution. The numerator corresponds to the sum of little pieces of mass weighted by their x positions. If we take the slice of the half-disk between x and $x + \Delta x$, then the x position is basically constant for all little pieces of mass in that (approximate) rectangle. See the picture below for a diagram:



The area of the rectangle is $2\sqrt{R^2 - x^2}\Delta x$, where we used that the formula for the top quadrant of the half-disk is $y = \sqrt{R^2 - x^2}$. The area times the density σ gives the mass, so the mass corresponding to the rectangle is $2\sigma\Delta x\sqrt{R^2 - x^2}$. Weight this by the position x and add all of the pieces between $x = 0$ and $x = R$ to get:

$$2\sigma \int_0^R x\sqrt{R^2 - x^2}\, dx$$

b. Evaluate the integral from the last part.

Solution. Perform the substitution $u = R^2 - x^2$, $du = -2x\,dx$. Then the integral becomes

$$\begin{aligned} 2\sigma \int_0^R x\sqrt{R^2 - x^2}\,dx &= 2\sigma \int_{R^2}^0 \sqrt{u}\left(-\frac{1}{2}\right) du \\ &= -\sigma \int_{R^2}^0 \sqrt{u}\,du \\ &= -\sigma \frac{2}{3} u^{3/2}\Big|_{R^2}^0 \\ &= \sigma\frac{2}{3}((R^2)^{3/2} - (0)^{3/2}) = \sigma\frac{2}{3}R^3 \end{aligned}$$

Again - be careful with changing limits and u-substitution.

c. Find the total mass of the half-disk, which corresponds to the denominator of $(*)$.

Solution. The total mass of the half-disk is the area times the density. The area of the half-disk is $\frac{1}{2}\pi R^2$, so the mass is $\sigma\frac{1}{2}\pi R^2$.

d. Find x_{cm}.

Solution.

$$x_{cm} = \frac{\sigma\frac{2}{3}R^3}{\sigma\frac{1}{2}\pi R^2} = \frac{4R}{3\pi}.$$

5 Sketching t^2e^{-t}

The function

$$f(t) = t^2e^{-t} \qquad (t \geq 0) \tag{1}$$

is a frequent visitor in the analysis of linear systems. The goal is to sketch it based on its characteristics.

a. When t is tiny (and positive), is $f(t)$ large and negative, large and positive, small and negative, or small and positive?

Solution. The function t^2 is small and positive and the function e^{-t} is close to 1. Thus, their product is small and positive.

b. What is the slope of $f(t)$ at $t = 0$?

Solution. Calculate $f'(t) = 2te^{-t} - t^2e^{-t}$ and $f'(0) = 0$.

c. When t is large (and positive), is $f(t)$ large and negative, large and positive, small and negative, or small and positive?

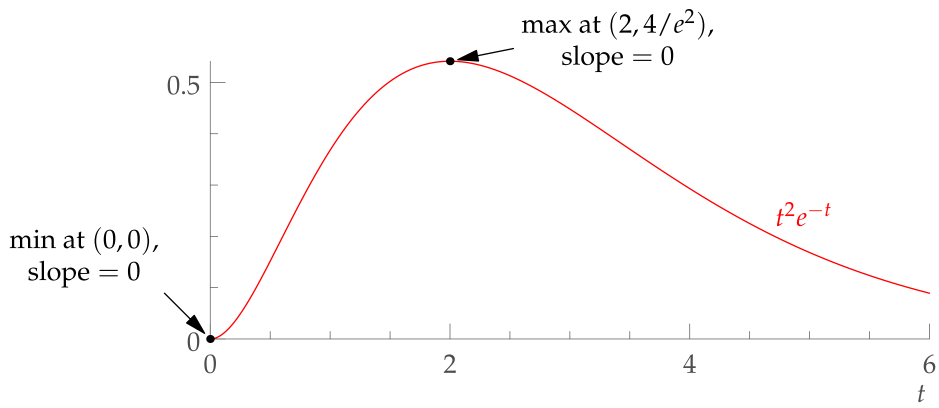
Solution. When t is large and positive, t^2 is large and positive, but e^{-t} is very small and positive. Because exponentials change faster than any polynomial, the exponential wins (once t is large enough). Thus the product is small and positive.

d. Does $f(t)$ have any local maxima or minima? If so, give their location(s), i.e. the t value and $f(t)$ there.

Solution. Find the critical points: $f'(t) = 2te^{-t} - t^2e^{-t} = te^{-t}(2 - t)$ leads to $t = 0$ or $t = 2$. The function has a local minimum at $t = 0$ (since the derivative is positive just after $t = 0$) and a local maximum at $t = 2$ (since the derivative is positive just before $t = 2$ and negative just after $t = 2$).

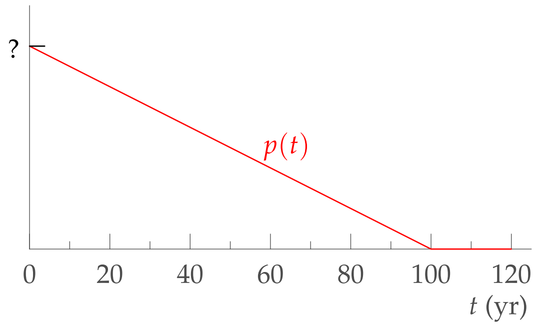
e. Sketch $f(t)$ by sketching a function that satisfies the preceding criteria, labeling any slopes or maxima/minima that you've found.

Solution.


max at $(2, 4/e^2)$,
slope $= 0$
0.5
min at $(0,0)$,
slope $= 0$
0
0
2
4
6
t
t^2e^{-t}

6 Age distribution

Here is a *very* crude but computationally tractable model of the probability density for age (T) in the US population: The density decreases steadily with age from 0 to 100 years.



a. What is the area under the $p(t)$ graph?

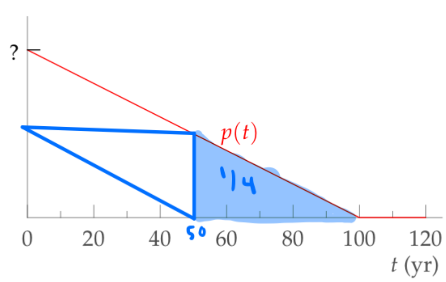
Solution. Since $p(t)$ is a probability density function, the integral $\int_{0\text{yrs}}^{100\text{yrs}} p(t)\,dt =$ the area under the $p(t)$ graph, should be 1.

b. What fraction of the population is at least 50 years old?

Solution. We need to find the integral of the density function:

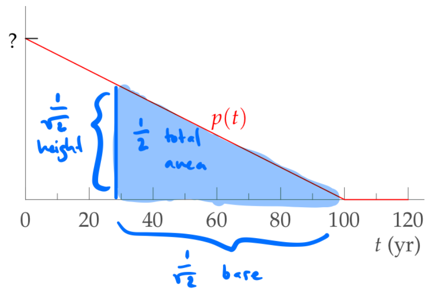
$$\text{proportion of population at least 50 y/o} = \int_{50\text{yrs}}^{100\text{yrs}} p(t)\,dt$$

One solution is to notice that the line $t = 50$ bisects the bottom of the triangle, and the area to the right corresponds to a triangle that has 1/4 of the total area:



c. Find and label the median age on the graph.

Solution. We want to find the line such that 50% of the population is on either side. Consider drawing a line at $t = m$, where m is the median age. Then the triangle to the right of the line is similar to the triangle under all of $p(t)$, but it must have $\frac{1}{2}$ the area. To achieve this, it must have $\frac{1}{\sqrt{2}}$ the base and height as well, as in the diagram:



We must pick m such that $100 - m$ years is $\frac{1}{\sqrt{2}} \approx 0.707$ of 100 years. Then $m \approx 29.3$ years.

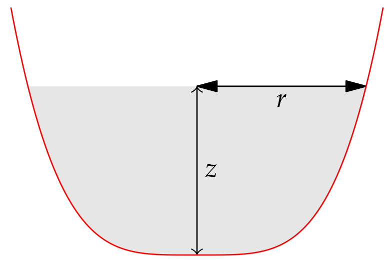
d. What are correct units on the y axis? Using those units, what is $p(0)$ (labeled with a "?" on the axis)?

Solution. Since the area under the curve needs to be unitless, the y axis should be in terms of yrs^{-1}. Then since the integral $\int_{0\text{yrs}}^{100\text{yrs}} p(t)\, dt = 1$ and we know this area is just a triangle, we must have

$$\frac{1}{2}(100 \text{ yrs})\, p(0) = 1$$
$$\Rightarrow p(0) = \frac{1}{50} \text{ yrs}^{-1}$$

7 Water clocks

Here is a side view of a water clock filled partly with water.



i Its cross-sectional radius (r), which varies with height (z), is $r = Az^{1/4}$ where A is a constant.

ii Water drains out of a small hole of radius r_0 at the bottom, at a speed $v = \sqrt{2gz}$.

The goal is to find the ODE for height versus time.

a. When the water is at a height z, how much water drains out in a short time Δt?

$$\Delta V_{\text{out}} \approx$$

Solution. Consider that volume V satisfies $V = Az$ for base area A and height z. Then we can write $\Delta V = A\Delta z$. A corresponds to the area of the hole through which the water is draining, which is πr_0^2. The water drains out at a speed of $v = \sqrt{2gz}$ m / s (check the units!), so $\Delta z = \sqrt{2gz} \cdot \Delta t$. We have our final answer:

$$\Delta V_{\text{out}} \approx \pi r_0^2 \cdot \sqrt{2gz} \cdot \Delta t$$

b. What is the resulting small change in height?

$$\Delta z \approx$$

(careful with the sign).

Solution. We want the relationship between the volume of water V and the height z. We can approximate with cylinders at z of height Δz and radius $r(z)$. Notice that when the water is at height z, we can write:

$$\begin{aligned}\Delta V &= -\Delta z \cdot \pi \cdot r(z)^2 \\ &= -\Delta z \cdot \pi A^2 z^{1/2}\end{aligned}$$

From which we can continue:

$$\begin{aligned}\Delta z &= -\frac{\Delta V}{\pi A^2 z^{1/2}} \\ &= -\frac{\pi r_0^2 \sqrt{2gz}\Delta t}{\pi A^2 z^{1/2}} \\ &= -\frac{r_0^2 \sqrt{2g}\Delta t}{A^2}\end{aligned}$$

c. What is the ODE for z?

$$\frac{dz}{dt} =$$

Hint: You should find that dz/dt is constant.

Solution. We can rearrange terms from the previous part:

$$\begin{aligned}\Delta z &= -\frac{r_0^2\sqrt{2g}}{A^2}\Delta t \\ \frac{dz}{dt} &= -\frac{r_0^2\sqrt{2g}}{A^2}\end{aligned}$$

which is defined in constants r_0, g, A, so $\frac{dz}{dt}$ is constant.

d. Why is it useful for a water clock to have constant dz/dt?

Solution. The height decreases at some constant speed, so we could make markings along the z axis of equal spacing and use the height of the water to accurately indicate how much time has passed.