140final-v1 - past final

Calculus 1 (McGill University)

December 2023
Final Examination

VERSION #: 1

CALCULUS 1
MATH 140 SECTIONS: 001-002-003
DATE: DECEMBER 11, 2023 TIME: 9:00 AM – 12:00 PM

EXAMINER: Dr. Jérôme Fortier ASSOC. EXAMINER: Dr. Dylan Cant

SOLUTIONS

EXAM:	CLOSED BOOK ☒ OPEN BOOK ☐		
	SINGLE SIDED ☐ PRINTED ON BOTH SIDES ☒		
	MULTIPLE CHOICE ANSWER SHEETS	YES ☐	NO ☒
	ANSWER BOOKLET REQUIRED:	YES ☐	NO ☒
	EXTRA BOOKLETS PERMITTED:	YES ☐	NO ☒
	ANSWER ON EXAM:	YES ☒	NO ☐
	SHOULD THE EXAM BE: RETURNED ☒ KEPT BY STUDENT ☐		
CRIB SHEETS:	PERMITTED ☐ Specifications:	NOT PERMITTED ☒	
DICTIONARIES:	TRANSLATION ONLY ☐ REGULAR ☐	NONE ☒	
CALCULATORS:	PERMITTED ☐	NOT PERMITTED ☒	
ANY SPECIAL INSTRUCTIONS:	See page 2.		

<u>Instructions</u>

- This exam has a total of 100 points and consists of two parts:
  - **Part A** is worth 50 points and consists of eleven **multiple choice** questions. Those questions must be answered on the bubble sheet on page 20 of the exam. You may detach that page for answering the questions.
  - **Part B** is worth 50 points and consists of five **written questions**, worth 10 marks each. Those questions must be answered directly on the exam. You may use the back pages for extra space.
- This is a closed book examination. Calculators and smarter devices are forbidden.
- Pages 17–19 are extra space for rough work. If you require them to be graded, you must indicate it very clearly in the question to which it relates, otherwise the graders will not look at them. No extra booklet will be allowed.
- The exam will be scanned and graded on a computer (with Crowdmark). So please do not write anything over the QR code in the corner of any page, and write dark enough and not too close to the edge of the paper.
- Good luck!

Part A – Multiple Choice Questions

Each question in this part counts for 5 points (unless stated otherwise), for a total of 50 points. Fill-in your answers on page 20 of this exam (A1 on line 1, A2 on line 2, and so on). Blacken only one answer per question. Any erasing must be done cleanly. You may use the blank spaces on this part or pages 17–19 for rough work, but no partial credit will be given for that work.

Question A1. $\lim_{x\to 0^+} \dfrac{\sin(x)(e^x - 1)}{x^3} =$

(A) $-1/2$

(B) 1

(C) $\boxed{\infty}$

(D) 0

(E) None of the above.

Question A2. $\lim_{x\to\infty} \sqrt{4+x^2} - x =$

(A) $1/4$

(B) 4

(C) DNE

(D) $\boxed{0}$

(E) None of the above

Question A3. Find the most general antiderivative of $\dfrac{1}{1+x^2} + \sin(x)$.

(A) $\boxed{\arctan(x) - \cos(x) + C}$

(B) $\ln(1+x^2) - \cos(x) + C$

(C) $x - \frac{1}{x} + \cos(x) + C$

(D) $\dfrac{-2x}{(1+x^2)^2} - \cos(x) + C$

(E) None of the above.

Question A4. Price p and demand x for a given product follows the rule

$$(1 + x + x^2)e^p = 1.$$

What is the rate of change dp/dx when $x = 1$?

(A) $\boxed{-1}$

(B) $\ln(1/3)$

(C) $-1/3$

(D) $-1/e$

(E) None of the above

Question A5.
What are the x coordinates of the points on the parabola $x = 4 - y^2$ which are closest to the origin.

(A) $\boxed{1/2}$

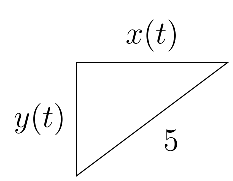
(B) 0

(C) $-1/2$

(D) 4

(E) None of the above

Question A6.
A right angled triangle is transforming in such a way that the length of the hypotenuse is always 5cm long.



What is the ratio y'/x' when $x = 4$? Assume that x' is non-zero.

(A) $\boxed{-4/3}$

(B) $-3/4$

(C) $4/3$

(D) $-3/5$

(E) None of the above

Question A7. Let $f(x) = (x+2)^{\cosh(x)}$. Find $f'(0)$.

(A) $3/2$

(B) $\boxed{1}$

(C) 0

(D) 2

(E) None of the above

The next two questions are part of the same question, and each is worth a fraction of the total 5 points.

Find the equation for the tangent line $y = mx + b$ to the graph of

$$f(x) = \frac{x^2}{1+x^2}$$

at the point $x = 1$.

Question A8. What is m? (2.5 points)

(A) 0

(B) $\boxed{1/2}$

(C) $-1/2$

(D) $3/2$

(E) None of the above

Question A9. What is b? (2.5 points)

(A) $\boxed{0}$

(B) $1/2$

(C) $-1/2$

(D) $3/2$

(E) None of the above

Question A10. $\lim_{x\to 0}\left(\frac{x+1}{x}-\frac{1}{\ln(1+x)}\right)=$

(A) $\sqrt{e}$

(B) 0

(C) 1

(D) $\boxed{1/2}$

(E) Does not exist

Question A11. Which value of k makes the following function continuous?

$$f(x)=\begin{cases}\cos(x-\pi k) & x\le 0\\ \dfrac{\sin(x)-x}{1-\cos(x)} & x>0\end{cases}$$

(A) $k=1/4$

(B) $\boxed{k=1/2}$

(C) $k=1$

(D) $k=2$

(E) None of the above.

Part B – Long answer questions

This part counts five questions, worth 10 marks each. You must justify all your work, with knowledge and notation from this course, unless specified otherwise. You may use the back pages or pages 17–19 for extra space, but do not forgot to tell us clearly when we should have a look at pages 17–19.

Question B1.
A parabola $y = (x - r_1)(x - r_2)$ with roots $r_1 < r_2$ is changing in such a way that the distance between the roots is growing at a constant speed of 1 unit per second. When the distance between the roots is 2, what is the rate of change of the y-coordinate of the vertex?
Note: *The vertex of a parabola $y = (x - h)^2 + k$ is the point (h, k).*
x coordinate of vertex is midpoint of roots, so $x = (r_1 + r_2)/2$. Plug in to formula to get y-coordinate of vertex:

$$y_{\text{vertex}} = \frac{1}{4}(r_2 - r_1)(r_1 - r_2).$$

Let $d = (r_2 - r_1)$ be distance between roots, so:

$$y_{\text{vertex}} = -\frac{d^2}{4}.$$

Differentiate to get:

$$y'_{\text{vertex}} = -\frac{2dd'}{4} = -\frac{2d}{4},$$

When

$$d = 2$$

we get the final answer:

$$y'_{\text{vertex}} = -1.$$

Extra space for ***Question B1***.

Question B2.

A manufacturer wants to design an **open-top** cylindrical container that can hold exactly 1000 cubic centimeters of liquid. What dimensions should be used to minimize the amount of material used in its construction?

Don't forget to show that your answer is a minimum and not a maximum!

Solution:

$$S = 2\pi r h + \pi r^2$$

$$V = \pi r^2 h = 1000$$

Replace h by r using $\pi r^2 h = 1000$,

$$S = \frac{2\pi r \cdot 1000}{\pi r^2} + \pi r^2 = \frac{2000}{r} + \pi r^2$$

Taking the derivative of r

$$\frac{dS}{dr} = \frac{-2000}{r^2} + 2\pi r$$

$$\frac{dS}{dr} = 0$$

get

$$r_0 = \frac{10}{\pi^{1/3}}$$

check $\frac{dS}{dr} < 0$ for $r < r_0$ and $\frac{dS}{dr} > 0$ for $r > r_0$ OR that $\dfrac{d^2S}{dr^2} > 0$ at r_0.

Extra space for ***Question B2***.

Question B3.
Consider the function $f(x) = 2xe^{-x^2} + 1$.

(a) Find the discontinuities, critical points, and inflection points (if any).

discontinuities: none,

critical points:

$$f'(x) = 2(1 - 2x^2)e^{-x^2}, x = \pm\sqrt{\frac{1}{2}}$$

inflection points:

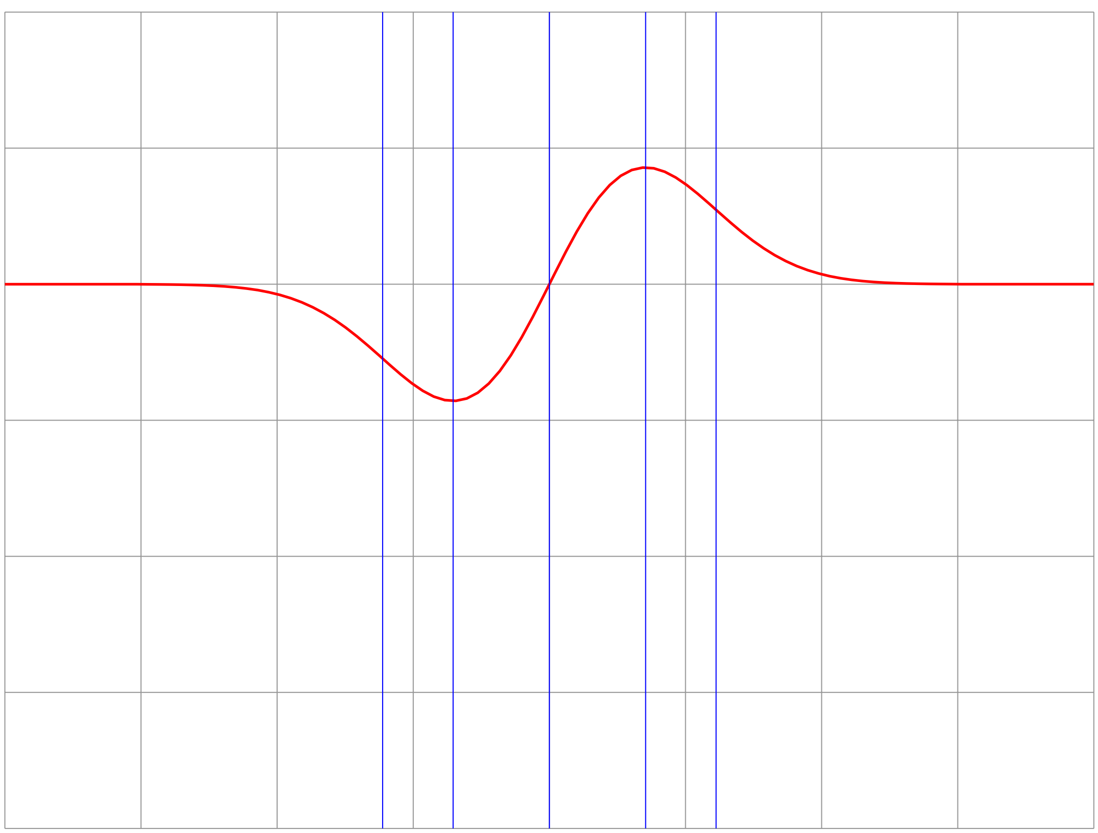
$$f''(x) = 4(2x^2 - 3)e^{-x^2}, x = \pm\sqrt{\frac{3}{2}}$$

(b) Find all horizontal and vertical asymptotes (if any).

$$\lim_{x \to \pm\infty} f(x) = 1$$

Vertical doesn't exist, horizontal asymptotes $y = 1$

(c) Use all the information above to create an accurate sketch of the curve of $f(x)$. Label all critcal points, inflection points and asymptotes, and provide justification for the overall shape of the curve.



Extra space for Question B3.

Question B4.
A particle is travelling along the x-axis in such a way that $x''(t) = 2$. Moreover, suppose that $x(0) = 1$ and $x(1) = 4$.

(a) Find the explicit formula for $x(t)$.

$$x''(t) = 2$$

$$x'(t) = 2t + A$$

$$x(t) = t^2 + At + B$$

plug in $x(0) = 1$, *get*

$$B = 1$$

plug in $x(1) = 1$, *get*

$$1 + A + B = 4$$

so $A = 2$, *so*

$$x(t) = t^2 + 2t + 1$$

(b) Which time t satisfies $x'(t) = 0$?

$$x'(t) = 2t + 2 = 0 \implies t = -1$$

Extra space for Question B4.

Question B5.

Alice and Bob are two runners, who race against each other. They start the race from the same point at time $t = 0$.

Alice wins the race! Use the Mean Value Theorem to rigorously prove that there is a time $t_0 > 0$ when the velocity of Alice was strictly greater than the velocity of Bob.

Denote by $A(t)$ and $B(t)$ the position of Alice and Bob at time t, respectively, and assume that both of those functions are differentiable. Consider the function $h(t) = A(t) - B(t)$.

Let k be the duration of the course. Then saying that Alice wins simply means $A(k) > B(k)$, or in other words, $h(k) > 0$. Since Alice and Bob start together, then we also have $h(0) = 0$. Bu the Mean Value Theorem, there is a time $t_0 \in (0, k)$ when:

$$h'(t_0) = \frac{h(k) - h(0)}{k} > 0.$$

We conclude that $A'(t_0) - B'(t_0) > 0$, or equivalently, $A'(t_0) > B'(t_0)$.

Extra space for Question B5.

*Extra space for **any question**. Do not forget to refer the graders to this page from the relevant question if you want this page to be graded.*

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