CS 181 Spring 2019 Section 5

Margin-Based Classification, SVMs

1 Motivation

The idea for Support Vector Machines is that, for all the linear hyperplanes that exist, we want one that will create the largest distance, or "margin", with the training data. Larger margins tend to improve generalization error. To define the margin, we consider a hyperplane of the form

$$\mathbf{w}^\top \mathbf{x} + w_0 = 0$$

What is the perpendicular distance from an example $\mathbf{x}_i$ to the decision boundary $\mathbf{w}^\top \mathbf{x} + w_0 = 0$?.

For two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on the hyperplane, consider the projection with $\mathbf{w}$:

$$\mathbf{w}^\top(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{w}^\top \mathbf{x}_1 - \mathbf{w}^\top \mathbf{x}_2 = -w_0 - (-w_0) = 0$$

Therefore, $\mathbf{w}$ is orthogonal to the hyperplane. So to get the distance from a hyperplane and an arbitrary example $\mathbf{x}$, we just need the length in the direction of $\mathbf{w}$ between the point and the hyperplane. We let r signify the distance between a point and the hyperplane. Then $\mathbf{x}_\perp$ is the projection of the point onto the hyperplane, so that we can decompose a point $\mathbf{x}$ as

$$\mathbf{x}_\perp + r\frac{\mathbf{w}}{||\mathbf{w}||} = \mathbf{x}$$

Left multiply by $\mathbf{w}^\top$:

$$\mathbf{w}^\top \mathbf{x}_\perp + r\frac{\mathbf{w}^\top \mathbf{w}}{||\mathbf{w}||} = \mathbf{w}^\top \mathbf{x} \Rightarrow r = \frac{\mathbf{w}^\top \mathbf{x} + w_0}{||\mathbf{w}||}$$

Scalar r then gives the signed, normalized distance between a point and the hyperplane. For correctly classified data, we have $y_i = +1$ when this distance is positive and $y_i = -1$ when it is negative. Based on this, we can obtain a positive distance for both kinds of examples by multiplying by y_i. We define the margin of the dataset as the minimum such distance over all examples:

$$\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{||\mathbf{w}||}$$

What is the training problem for support vector machines?

We want the $\mathbf{w}$ and w_0 that maximize the margin:

$$\arg\max_{\mathbf{w}, w_0} \frac{1}{||\mathbf{w}||} \min_i y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)$$

In the hard-margin training problem, we know that the data is linearly separable and therefore any margin (including the optimal) must be greater than 0:

$$\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{||\mathbf{w}||} > 0$$

We can observe that $\mathbf{w}$ and w_0 are invariant to changes of scale. Because of this, it is without loss of generality to impose $\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{||\mathbf{w}||} > 1$. This lets us write the optimization problem as:

$$\arg\max_{\mathbf{w}, w_0} \frac{1}{||\mathbf{w}||} \quad \text{s.t. } \forall i \; y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1$$

What is the corresponding minimization problem for hard-margin training for SVMs?

Explain at a high level why the minimization problem for SVMs is equivalent to the max-margin problem.

We can invert $\mathbf{w}$ to change the max to a min:

$$\arg\min_{\mathbf{w}, w_0} \frac{1}{2}||\mathbf{w}||^2 \quad \text{s.t. } \forall i \; y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1$$

Informally, this is the same as max-margin training because the constraint is binding for the examples closest to the decision boundary. For these examples we have $y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) = 1$. The distance on these examples is $1/||\mathbf{w}||$, and is maximized by minimizing $||\mathbf{w}||^2$.

2 Soft Margin Formulation

For the hard margin formulation, we have been assuming that the data is linearly separable. However, this is not always true, and even if the data is linearly separable, it may not be best to find a separating hyperplane. In optimizing generalization error, there is a tradeoff between the size of the margin and the number of mistakes on the training data.

For the soft margin formulation, we introduce a slack variable $\xi_i \geq 0$ for each i to relax the constraints on each example.

$$\xi_i \begin{cases} = 0 & \text{if correctly classified} \\ \in (0, 1] & \text{correctly classified but inside margin region} \\ > 1 & \text{if incorrectly classified} \end{cases}$$

We can now rewrite the training problem for a soft margin formulation to be

$$\arg\min_{\mathbf{w}, w_0} \frac{1}{2}||\mathbf{w}||^2 + C \sum_{i=1}^{n} \xi_i$$

$$\text{s.t. } \forall i \; y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1 - \xi_i$$

$$\xi_i \geq 0$$

We add a regularization parameter C, that controls how much we penalize violating the hard margin constraints. A large C penalizes these violations and thus “respects” the data closely and has small regularization. A small C does not penalize the sum of slack variables as heavily, relaxing the constraint. This is increasing the regularization.

3 Practice Problems (cover some but not all at section)

1. **Removing Support Vectors and Retraining (Berkeley, Fall '11)**

   Suppose that we train two SVMs, the first containing all of the training data and the second trained on a data set constructed by removing some of the support vectors from the first training set. How does the size of the optimal margin change between the first and second training data? What is a downside to doing this?

2. **Proof that margin is invariant to scalar multiplication**

   In reformulating our max margin
   $$\frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{||\mathbf{w}||}$$
   training problem, we use the fact that the margin is invariant to multiplying $(\mathbf{w}, w_0)$ by any $\beta > 0$. Show that this property is true.

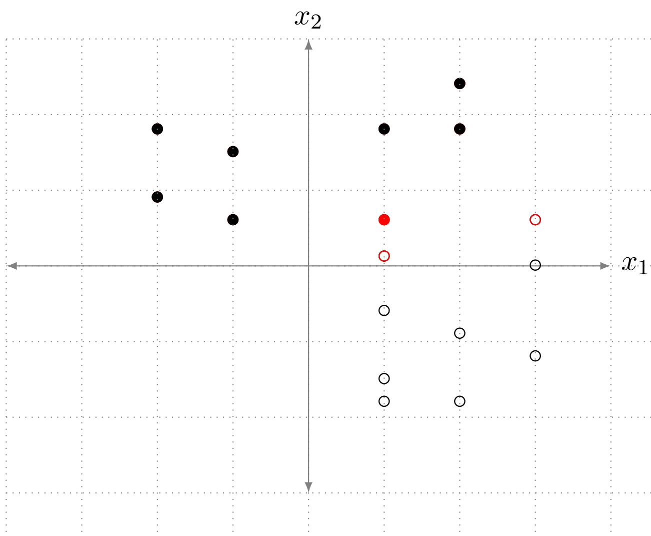
3. **(Berkeley, Fall '11)**

   Consider $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$. Show that the polynomial kernel of degree 2, $K(\mathbf{x}, \mathbf{x}') = (1 + \mathbf{x}^\top \mathbf{x}')^2$ is equivalent to a dot product $\boldsymbol{\phi}(\mathbf{x})^\top \boldsymbol{\phi}(\mathbf{x}')$ where $\boldsymbol{\phi}(\mathbf{x}) = (x_1^2, x_2^2, x_1, x_2, x_1 x_2, 1)$.

4. **Draw Margin Boundary**

In the figures below, the red examples represent the support vectors. All other examples can be assumed to have $\alpha_i = 0$. Draw the margins for the boundary given this information.
i) For the first example, you can assume the hard margin formulation. Draw the decision boundary as well as the two margin boundaries given the support vector.
ii) For the second example, you can assume the soft margin formulation and that all points are correctly classified with the optimal decision boundary. The decision boundary is already given. Draw the two margin boundaries given the support vector.

i)



ii)