CS181 2019 Midterm 1 Review Questions

1. **Linear Regression**

Consider a one-dimensional regression problem with training data $\{x_i, y_i\}$. We seek to fit a linear model with no bias term:

$$\hat{y} = wx$$

a. Assume a squared loss $\frac{1}{2}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2$ and solve for the optimal value of w^*.

b. What is the prediction for some new observation x, without mention of w?

c. Suppose that we have a generative model of the form $\hat{y} = wx + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ and w is known. Given a new x, what is the expression for the probability of $\hat{y}$?
Note: The univariate Gaussian PDF is:

$$\mathcal{N}(a|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(a-\mu)^2}{2\sigma^2}\right)$$

d. Now assume that w is random and that we have a prior on w with known variance s_0^2:

$$w \sim \mathcal{N}(0, s_0^2)$$

Write down the form of the *posterior* distribution over w. Take logs and drop terms that don't depend on the data and prior parameters, but you do not need to simplify further (i.e. you do not need to complete the square to make it look like a normal).

2. **Multiclass Classification**

Suppose that we have a K-class classification scenario with training are 1-hot vectors.

We model this problem using a neural network with d units in a single hidden layer, expressed as a column vector $\boldsymbol{\phi}(\mathbf{x}; \mathbf{W}, \mathbf{w}_0) \in \mathbb{R}^d$, which we write as $\boldsymbol{\phi}$. We take a linear combination of these values and pass them to a softmax function to get a final set of K outputs. Let $\mathcal{C}_k$ represent a 1-hot vector with a 1 in the k^{th} index and let $\mathbf{v}_\ell \in \mathbb{R}^d$ be a column vector of weights:

$$p(\mathbf{y} = \mathcal{C}_k | \mathbf{x}; \{\mathbf{v}_\ell\}_{\ell=1}^K, \mathbf{W}, \mathbf{w}_0) = \frac{\exp(\mathbf{v}_k^\top \boldsymbol{\phi})}{\sum_{\ell'=1}^K \exp(\mathbf{v}_{\ell'}^\top \boldsymbol{\phi})}$$

a. Suppose we add the same global bias to each vector of weights in the final layer, i.e. replace $\mathbf{v}_k^\top \boldsymbol{\phi}$ with $\mathbf{v}_k^\top \boldsymbol{\phi} + v_0$ for some scalar v_0 with the same scalar for all k. Does this increase the expressivity of our model? Why or why not?

b. Write down and simplify the log likelihood of a particular observation $(\mathbf{x}_i, \mathbf{y}_i)$, including constants. Assume that we use a sigmoid activation function (don't need to simplify within the sigmoid, just the sums/logs/exps around it)

$$\boldsymbol{\phi}(\mathbf{x}; \mathbf{W}, \mathbf{w}_0) = \boldsymbol{\sigma}(\mathbf{W}\mathbf{x} + \mathbf{w}_0)$$

c. Consider the scenario of drawing items from a distribution and encoding them in binary for communication. An efficient scheme encodes common items with a short code and rare items with longer codes. The *cross-entropy* $\mathbb{E}_{p(x)}[-\ln q(x)]$ can be interpreted as the expected number of required bits to send a randomly chosen item $x \sim p(x)$ using a code optimized for $q(x)$. For classification, we can use the following as a loss:

$$\mathbb{E}_{p(y|x)}[-\ln q(y|x)]$$

where $p(\mathbf{y}|\mathbf{x})$ is 1 for the true class and is 0 otherwise and $q(\mathbf{y} = \mathcal{C}_k|\mathbf{x})$ is your model's prediction (output of the softmax layer for k^{th} class). Write down the expression for the cross-entropy by unpacking the expectation and writing it as a sum of terms and describe its relationship to the log loss in part (b).

d. When running SGD, we need to compute the gradient of the loss for a single example $(\mathbf{x}, \mathbf{y})$ with respect to each of the parameters. Use the above definition of $\boldsymbol{\phi}$ to compute this gradient for the bias vector $\mathbf{w}_0$ using the original negative log likelihood loss.

$$\frac{\partial}{\partial \mathbf{w}_0} - \ln p(\mathbf{y} = \mathbf{y}_i | \mathbf{x}; \mathbf{W}, \mathbf{w}_0, \{\mathbf{v}_\ell\})$$

You may use $\sigma'()$ for the derivative of the sigmoid function and this intermediate term:

$$\frac{\partial}{\partial \mathbf{v}_j^\top \boldsymbol{\phi}} - \ln p(\mathbf{y} = \mathcal{C}_k | \mathbf{x}; \mathbf{W}, \mathbf{w}_0, \{\mathbf{v}_\ell\}) = p(\mathbf{y} = \mathcal{C}_k | \mathbf{x}; \mathbf{W}, \mathbf{w}_0, \{\mathbf{v}_\ell\}) - y_j$$

3. **Overfitting and Underfitting**

Harvard Insta-Ice Unit (HI2U) has built a robot that can deliver 24-hour shaved ice to student houses. To prevent collisions, they train three different approaches to classify camera images as containing nearby tourists or open space; if the robot identifies a tourist in its path, it is programmed to halt. The performances of the classifiers are:

	Training Accuracy	Testing Accuracy
Classifier A	75.3%	74.8%
Classifier B	80.3%	77.8%
Classifier C	90.2%	60.0%

where Classifier B has a more expressive model class than A, and classifier C has both a more expressive model class and more features than A. All the classifiers have closed-form solutions, so HI2U is pretty sure that the inference is not hindering performance.

a. If you had to choose one: might Classifier A be overfitting or underfitting? Explain your reasoning.

b. If you had to choose one: might Classifier C be overfitting or underfitting? Explain your reasoning.

c. If you had to guess yes or no: might more training examples significantly boost the test-time performance of Classifier A? Classifier C? Explain your reasoning.

Hint: try to relate your reasoning to model bias and model variance.

4. **Neural Networks**

Consider the following non-linearity for use in a neural network:

$$f_{0/1}(z) = \left\{ \begin{array}{ll} 1 & \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

Let $\mathbf{x}$ be a binary feature vector of length 4: $\mathbf{x} \in \{0,1\}^4$. Define neural network A as follows:

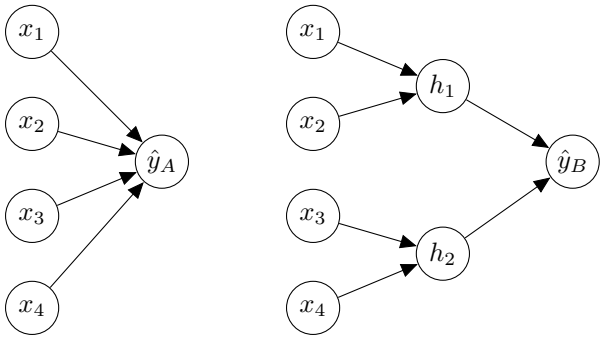
$$\hat{y_A} \leftarrow f_{0/1}(\mathbf{w}^\top \mathbf{x} + w_0)$$

with weight vector $\mathbf{w} \in \mathbb{R}^4$ and bias scalar $w_0 \in \mathbb{R}$.

Let $\mathbf{x}^L = [x_1, x_2]$ and $\mathbf{x}^R = [x_3, x_4]$. Define neural network B as follows:

$$\begin{aligned} h_1 &\leftarrow f_{0/1}(\mathbf{t}^\top \mathbf{x}^L + a) \\ h_2 &\leftarrow f_{0/1}(\mathbf{u}^\top \mathbf{x}^R + b) \\ \mathbf{h} &\leftarrow [h_1, h_2] \\ \hat{y_B} &\leftarrow f_{0/1}(\mathbf{v}^\top \mathbf{h} + c) \end{aligned}$$

with weight vectors $\mathbf{t}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and bias scalars $a, b, c \in \mathbb{R}$. Basically, B can only look at the two halves of the input separately and has an extra layer to merge the transformations on the two halves of the input with another transformation:



a. i. Describe a logical formula on inputs that can be expressed by A but not by B and provide weights for $\mathbf{w}$ and w_0 that implement the formula in A (hint: think about things you may want to do with binary vectors, e.g. `AND`s, `OR`s)

ii. Provide an argument for why B cannot express this formula (we don't expect a rigorous proof, but try to give a complete and convincing argument).

iii. How might you change the architecture of B to fix this issue? What downside might this have?

b. What is the concern about training the networks as currently defined? What change would you make to the network to alleviate this concern?

c. State **two** ways in which a *validation set* can be used when training neural networks (one sentence for each is fine).