Chapter 9

Four Variants of the Vehicle Routing Problem

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9.1 ▪ Introduction

In this chapter, we review four important variants of the *Vehicle Routing Problem* (VRP):

1. *VRP with Backhauls* (VRPB);
2. *Heterogeneous or mixed Fleet VRP* (HFVRP);
3. *Periodic VRP* (PVRP);
4. *Split Delivery VRP* (SDVRP).

Each family plays an important role in the literature and considers a characteristic feature often encountered, both alone and combined, in real-world applications. The resulting VRP variants are non-trivial extensions of the basic VRP, and they all deserve specifically tailored algorithmic approaches for their effective solution.

In the first problem class, the *VRP with Backhauls* (VRPB), the customers are partitioned into two sets, the linehaul and backhaul customers. The linehaul customers require the delivery of goods from the depot, and the backhaul customers have goods to be picked up and transported to the depot. Since a rearrangement of the load inside a vehicle is often impossible or undesirable, e.g., because the vehicle's loading space is only accessible from the rear, a precedence relation on the service of linehaul and backhaul customers is imposed: In a feasible VRPB solution the combined service of linehauls and backhauls in the same route is possible, but all linehaul customers receive their demand before load at backhaul customers is picked up. The VRPB belongs to the more general class of delivery and collection VRP, which clearly introduces considerable improvements over separate distribution and collection services in different routes.

In the second variant, the family of *Heterogeneous or mixed Fleet VRP* (HFVRP), the fleet consists of different vehicles or vehicle types, each with a specific capacity and costs.

The choice of the appropriate vehicle to be assigned to each route can have a considerable impact on the overall cost of the solution both at the operational level, at which the heterogeneous fleet is given, and at the strategic level, at which the optimal fleet composition has to be determined.

The third family we examine is the *Periodic VRP* (PVRP). Here deliveries occur in a multi-day service horizon: each customer has to be assigned to one or several specific days for deliveries, and routing must be performed for each day. Hence, the PVRP comprises three types of decisions: the choice of visiting days for each customer, the clustering of customers into tours on each day, and the routing of the vehicle for each cluster.

The last variant treated in this chapter is the *Split Delivery VRP* (SDVRP), in which customers may be served more than once by dividing the delivery quantity among the different visits. Hence, the solution of SDVRP requires the determination of the number of visits to each customer, the portion of the demand to deliver to the customer, and the routing of the vehicles. Of course, the three above types of decisions are intertwined, and the practical hardness of the SDVRP can be attributed to this interdependency.

Both heterogeneous and backhaul variants were introduced early in the VRP literature. However, despite their practical relevance, they received relatively little attention from the research community in the last decade. Thus, besides from surveying the relatively few recent papers on these two families, our goal is to revive the attention and possibly stimulate additional research on them. On the contrary, the PVRP and SDVRP variants were the subject of intense research efforts in recent years.

Therefore, the material presented in this chapter is structured as follows: Section 9.2 provides a survey on the VRPB referring to a longer time period starting approximately in 2002. Due to good and very recent surveys available for HFVRP, PVRP, and SDVRP, Sections 9.3, 9.4, and 9.5 only provide an update on these variants by discussing the new contributions that were presented afterwards.

In order to make the four sections self-contained, we decided to re-introduce names and acronyms in each section.

9.2 ▪ VRP with Backhauls

The *VRP with Backhauls* (VRPB), also known as the linehaul-backhaul problem, is an extension of the *Capacitated VRP* (CVRP) in which the customer set is partitioned into two subsets. The first one contains the *linehaul customers*, each requiring a given quantity of product to be delivered. The second subset contains the *backhaul customers*, where a given quantity must be picked up. This customer partition is very frequent in practical situations, such as in the grocery industry, where supermarkets and shops are the linehaul customers, and grocery suppliers are the backhaul customers. In this mixed distribution and collection context a significant saving in terms of transportation costs can be achieved by visiting backhaul customers in distribution routes.

More precisely, the VRPB can be stated as the problem of determining a set of vehicle routes visiting all customers such that (i) each vehicle performs one route; (ii) each route starts and ends at the depot; (iii) for each route the total load associated with linehaul and backhaul customers does not exceed, separately, the vehicle capacity; (iv) on each route the backhaul customers, if any, are visited after all linehaul customers; (v) the total distance traveled by the vehicles is minimized. The *precedence constraint* (iv) is practically motivated by the fact that often vehicles are rear-loaded. Hence, the on-board load rearrangement required by a mix of collection and distribution is difficult, or impossible, to carry out at customer locations. Another practical reason is that, in many applications, linehaul customers have a higher service priority than backhaul customers. Note that the mixed vehicle routes, which visit both linehaul and backhaul customers, are implicitly oriented due to precedence constraint (iv).

In general, VRPB denotes the symmetric version of the problem, in which traveling costs between each pair of customers are equal in both directions, while the case in which such costs may be different is indicated as the *Asymmetric VRPB* (AVRPB). Both VRPB and AVRPB are NP-hard in the strong sense, as they generalize the symmetric and asymmetric CVRP, respectively, arising when no backhauls are present. Toth and Vigo [141] present an extensive survey of the literature on VRPB until the end of the last century. Thus, we mainly discuss the more recent contributions. Computational testing for these problems is generally performed on a set of randomly generated, symmetric instances proposed by Goetschalckx and Jacobs-Blecha [69] (called the GJ set) and on symmetric and asymmetric instances derived from CVRP and ACVRP benchmark sets by Toth and Vigo [138, 139, 140], called the TVS and TVA sets.

Exact Methods Few exact approaches have been proposed for VRPB and, to the best of our knowledge, none during the last decade. These are all surveyed in [141], but since they still represent the state of the art for the optimal solution of this problem variant we briefly recall their main characteristics. The first exact method for both VRPB and AVRPB is a Branch-and-Bound introduced in [139] based on an effective Lagrangian bound for the problem. A valid model for VRPB is obtained by transforming it into a directed CVRP in which all the arcs connecting backhaul to linehaul customers are removed as a consequence of precedence constraint (iv) above: to this end any formulation for directed CVRP, such as model VRP1 presented in Chapter 1, can be used. The Lagrangian bound in [139] extends methods previously proposed for the CVRP (see Section 2.2.3 and [141]) and is derived from combinatorial relaxations based on projection and leading to the solution of directed trees over the linehaul and the backhaul customers set separately. The resulting Branch-and-Bound algorithm is able to solve VRPB test instances from the GJ and TVS sets and AVRPB instances from set TVA with up to 70 customers in total. It is worth noting that the Lagrangian bound is also used in [140] as a base for an effective heuristic algorithm for both variants of the problem.

The other existing exact method for VRPB is due to Mingozzi, Giorgi, and Baldacci [96], who developed a set-partitioning-based approach starting in which routes are obtained by connecting paths visiting only either linehaul or backhaul customers. The resulting integer program is possibly solved through a complex procedure making use of dynamic programming and dual approximation of relaxations of the problem. The results of the computational testing show that the approach is capable of solving undirected problems from the GJ and TVS sets with up to 70 vertices.

Heuristics and Metaheuristics The first heuristic for VRPB is an extension of the savings algorithm proposed by Deif and Bodin [49], followed by a few other early approaches reviewed in Toth and Vigo [141]. More recent approximate algorithms for VRPB adopt metaheuristic paradigms. One of the first is by Osman and Wassan [106], who introduce a *Reactive Tabu Search* (RTS) initialized by two savings-based heuristics. The algorithm uses several data structures to speed up the search and is able to obtain new best solutions for the larger instances of the GJ and TVS sets. The RTS algorithm is later combined with an *Adaptive Memory Programming* (AMP) scheme by Wassan [149], further improving its performance both in terms of solution quality and required time. Another *Tabu Search* (TS) approach is proposed by Brandão [31], who initializes it with two interesting construction algorithms: one is based on the solution of open VRP on the two set of customers, and the other one is based on a K-tree relaxation. The overall TS proves competitive with previous methods finding some new best solutions on the GJ and TVS test instances. Ropke and Pisinger [121] extend their *Adaptive Large Neighborhood Search* (ALNS) framework, which they previously used for several other VRP variants, to the VRPB. The unified heuristic is able to solve six different variants of the VRPB obtaining state-of-the-art results for all of them. A *multi-ant colony system* was proposed by Gajpal and Abad [66] in which a combination of route-ants and vehicle-ants are used to construct solutions respecting the maximum number of vehicles: the resulting algorithm achieves good average performance on the GJ and TVS instances. Another effective method for VRPB is the *route promise algorithm* by Zachariadis and Kiranoudis [154], which is based on an efficient implementation of the *variable length bone exchange* local search.

Vidal et al. [145] present a sophisticated *Hybrid Genetic Search with Adaptive Diversity Control* (HGSADC), which is the current best in class not only for the VRPB but also for a wide class of other VRPs. It yields the best solutions on the GJ test set, but is not tested on TVS and TVA instances. HGSADC uses two subpopulations of feasible and infeasible individuals which are encoded by means of a giant tour representation, to allow for the application of simple crossover operators. A key feature of their method lies in the fitness evaluation, which incorporates the contribution of an individual to the diversity of the population (based on the distance to other individuals in the subpopulation). Tests show that this approach helps to avoid premature convergence and to achieve higher solution quality in shorter time compared to traditional approaches for diversity management. In the so-called education phase, route improvement methods are applied and resulting infeasible solutions are repaired with a given probability. Further details on this heuristic approach can be found in Chapter 4.

A few variants of the VRPB are studied in the literature: The first type of modification is related to the relaxation of the precedence constraint (iv) resulting in problems in which backhaul customers may be served either when sufficient space is available in the vehicle, or even in any position of the tour. After some initial interest on these variants, culminated in the paper by Wade and Salhi [147], who present a specialized insertion algorithm, they have not attracted further research efforts. The only notable exception is the paper by Ropke and Pisinger [121], who call these variants the *Mixed VRPB* (MVRPB). Similarly, it seems that the variant with multiple depots has only been considered in [121] and by Salhi and Nagy [125]. Computational experiments for these variants are generally performed by using the GJ instances disregarding precedences and the new ones proposed by Nagy and Salhi [100].

The *VRPB with time windows* has been extensively studied in the late nineties. Thangiah, Potvin, and Sun [137] and Duhamel, Potvin, and Rousseau [58] consider problems both with and without the strict precedence constraint between linehauls and backhauls, for which test instances are also proposed. More recently, this problem was considered by Zhong and Cole [155], whose *guided local search* approach slightly improves some results obtained with previous metaheuristics. Finally, Ropke and Pisinger [121] use their ALNS algorithm to solve the time-window variant of both VRPB and MVRPB.

Another interesting variant of VRPB is that with heterogeneous fleet, for which Tavakkoli-Moghaddam, Saremi, and Ziaee [136] derive a *memetic algorithm* which is also applied to the problem with homogeneous fleet. Wang and Wang [148] consider the variant of VRPB in which the travel speed of the vehicle is time-dependent. To solve the problem they develop a two-phase approach based on the RTS of Wassan [149]. Finally, Anbuudayasankar et al. [6] describe an interesting real-world problem arising in the replenishment of automated teller machines which turn out to be a bi-objective VRPB, in which both cost and duration of the routes are minimized. For this problem the authors develop a simple saving-based heuristic and a *genetic algorithm* which are tested on real-world and randomly generated instances.

9.3 ▪ Heterogeneous or Mixed Fleet VRP

This section considers the variant of the VRP in which a *heterogeneous fleet* of vehicles, each having a possibly different capacity and cost, is available for the distribution activities. The problem has been known since the early VRP literature, and it was first studied in detail by Golden et al. [70]. Several specific variants have been developed over time to consider the two main "features" of this problem. On the one hand, there is the strategic issue of finding the best assortment of vehicles to be used for the long-term sizing of a fleet. This results in *Fleet Size and Mix* (FSM) problems, in which the vehicle fleet is assumed to be unlimited. On the other hand, one can consider the tactical issue of using the most appropriate vehicles from a limited fleet, which results in the *Heterogeneous VRP* (HVRP). We also use the term *Heterogeneous or mixed Fleet VRP* (HFVRP) to identify the whole problem family.

An extensive survey on HFVRP, covering the main results until 2007, is given by Baldacci, Battarra, and Vigo [19]. Therefore, this section is aimed at providing an update of the literature on the subject. In the following, we mainly concentrate on the problems including only capacity constraints but no additional features, which received more attention in the literature. We also briefly review the approaches proposed to include additional constraints such as time windows and multiple depots. Several case studies and applications which included the presence of a heterogeneous fleet, often in combination with other features, are listed in [19], whereas recent ones are discussed in Tarantilis and Kiranoudis [135], Belfiore and Yoshizaki [25], Oppen, Løkketangen, and Desrosiers [104], and Hertz, Uldry, and Widmer [78]. A decision support system that is capable of solving variants of HVRP with and without backhauls is presented by Tütüncü [143]. We finally mention an interesting application of continuous approximation to fleet sizing described in Jabali, Gendreau, and Laporte [82].

In the HFVRP, we are given a set of n customers, each with a demand q_i, and a fleet of vehicles made up of $|P|$ different vehicle types; i.e., the fleet K is partitioned into subsets of homogeneous vehicles $K = K^1 \cup K^2 \cup \cdots \cup K^{|P|}$. Each vehicle type $p = 1,\ldots,|P|$ has capacity Q^p, and may also have a fixed cost FC^p and a specific traveling cost c_{ij}^p along each arc of the graph modeling the road network.

As already mentioned, many specific variants of the problem were studied in the literature, often known with specific acronyms, depending on the available fleet and the type of considered costs. In particular, the following characteristics vary across different problems:

(i) the vehicle fleet may be either *limited*, i.e., at most $|K^p|$ vehicles of type p may be used, or *unlimited*, i.e., $|K^p| \geq n$ for all $p = 1,\ldots,|P|$;

(ii) the *fixed costs* FC^p of the vehicles may be either considered or ignored, i.e., $FC^p = 0$ for all $p = 1,\ldots,|P|$;

(iii) the routing costs on arcs may be *vehicle-dependent*, i.e., possibly different for each vehicle type, or *vehicle-independent*, i.e., $c_{ij}^{p_1} = c_{ij}^{p_2} = c_{ij}$ for all $p_1 \neq p_2$ and for all $(i,j) \in A$, or *site-dependent* when for each customer i, only a limited subset of vehicle types may be used, i.e., c_{ij}^p are equal to $+\infty$ for some $p = 1,\ldots,|P|$.

Table 9.1. *Heterogeneous VRP variants presented in the literature.*

Acronym	Problem Name	Fleet Size	Fixed Costs	Routing Costs
HVRPFD	Heterogeneous VRP with Fixed Costs and Vehicle-Dependent Routing Costs	*Limited*	*Considered*	*Dependent*
HVRPD	Heterogeneous VRP with Vehicle-Dependent Routing Costs	*Limited*	*Ignored*	*Dependent*
FSMFD	Fleet Size and Mix VRP with Fixed Costs and Vehicle-Dependent Routing Costs	*Unlimited*	*Considered*	*Dependent*
FSMD	Fleet Size and Mix VRP with Vehicle-Dependent Routing Costs	*Unlimited*	*Ignored*	*Dependent*
FSMF	Fleet Size and Mix VRP with Fixed Costs	*Unlimited*	*Considered*	*Independent*

Following the naming convention proposed in Baldacci, Battarra, and Vigo [19], we summarize in Table 9.1 the different problem variants that were actually considered in the literature. The variants with time windows are denoted by adding TW to the end of the acronym of the specific problem, those with multiple depot by adding MD at the beginning of the acronym, and those with backhauls by adding B at the end.

All the problems described above are NP-hard, as they are generalizations of the *Capacitated VRP* (CVRP), which arises when only one vehicle type is present.

Computational testing for HFVRP problems is generally conducted by using a benchmark proposed by Golden et al. [70] that includes a set of 20 test instances for FSM, called the GF set. Instances GF1, GF2, and GF7-12 are adapted from VRP ones with explicit distance matrix and 12 to 30 customers, whereas instances GF3-6 and GF13-20 have a Euclidean distance matrix with 20 to 100 customers. The adaptation of these Euclidean test instances to HVRPD, resulting in set GH, is described in Taillard [132], whereas Li, Golden, and Wasil [88] extend them to HVRPFD and propose five new large instances with 200 to 360 customers, called the LH set. Other authors propose new instances for these problems, and we report them in the following text.

Exact Methods Research on exact approaches for HFVRP has been rather limited in the literature and mostly devoted to FSM problems. The first lower bounds for FSMF were proposed by Golden et al. [70], who describe a relaxation whose optimal solution is found by solving a shortest path on a suitably defined auxiliary graph. Other early approaches are described by Baldacci, Battarra, and Vigo [19]. More recently, the same authors in [20] propose bounds for FSM based on a new two-commodity formulation strengthened with specifically designed cut families. The resulting lower bounds are equivalent in quality to previous ones from the literature, but require considerably less computing time when applied to the twelve Euclidean GF instances.

As in other VRP variants, the most successful exact approaches for the HFVRP are currently those based on set partitioning formulations. Pessoa, Poggi, and Uchoa [110] present a Branch-and-Cut-and-Price algorithm for the FSMF in which the columns are associated with the so-called q-routes (see Christofides, Mingozzi, and Toth [41]), as previously done by Choi and Tcha [39]. The q-routes are (not necessarily simple) circuits covering the depot and a subset of customers, whose total demand is equal to q. The formulation is strengthened by using new families of cuts producing an overall improvement of the bound that allows for the optimal solution of instances from the GF set with up to 75 customers.

Currently the most effective exact approach for the solution of the HFVRP is the Branch-and-Cut-and-Price algorithm by Baldacci and Mingozzi [22]. The algorithm extends approaches proposed by the same authors for the CVRP (see, e.g., Baldacci, Christofides, and Mingozzi [21]) and uses both new relaxations of the set partitioning formulation and reduction rules to resize the vehicle fleet that are particularly effective when, as happens in most HFVRP variants, the vehicle fixed cost contribution to the total cost is relevant. The resulting exact method permits the solution of most Euclidean test instances from the literature for FSMF, FSMD, HVRPFD, and HVRPD.

Few other variants of heterogeneous VRP are studied with the purpose of developing exact approaches. A notable exception is the MDHVRPFDTW, i.e., the variant of HVRPFD with multiple depots and time windows, for which Bettinelli, Ceselli, and Righini [28] present a Branch-and-Cut-and-Price which is used to solve test instances with up to 100 customers derived from an existing benchmark proposed for the *VRP with Time Windows* (VRPTW).

Finally, Dondo and Cerdá [52] examine the MDHVRPFDTW and develop a three-phase matheuristic. In the first phase, customers are heuristically grouped into a limited number of clusters. In the next phase, these clusters are assigned to vehicles and depots through the exact solution of a small-sized ILP model. Finally, the detailed routing and scheduling of each tour is determined heuristically in the last phase. The method is applied to some small-sized test instances derived from VRPTW instances.

Heuristics and Metaheuristics Most solution approaches presented so far in the literature are heuristic algorithms. These are often adaptations or extensions of the methods proposed for the main VRP variants, like the CVRP and the VRPTW.

As previously mentioned, the first study of HFVRP is due to Golden et al. [70], who present a construction heuristic based on an adaptation of the *savings algorithm* by Clarke and Wright [43] and the *giant-tour*-based approach of Beasley [23] to FSMF. In this section, we review the most recent contributions to the approximate solution of HFVRP; for other early approaches see Baldacci, Battarra, and Vigo [19].

Li, Tian, and Aneja [90] address the HVRPFD by means of a *Multi-start Adaptive Memory Programming* (MAMP) metaheuristic, which may be seen as an evolution of the heuristic column generation technique of Taillard [132] and Choi and Tcha [39]. A wide set of routes is generated with construction algorithms, followed by *Tabu Search* (TS) and *path relinking*. Then complete solutions are selected by heuristically solving a set covering problem. The MAMP improved the results on seven out of eight GH instances and is also applied to 19 new random instances. The most recent example of such an approach is represented by the method of Subramanian et al. [131]. The algorithm is based on an *iterated local search* with random neighborhood ordering used to generate the large set of routes which are later selected through the set covering. The algorithm is successfully applied to all variants listed in Table 9.1.

Another successful approach are *Memetic Algorithms* (MAs), which combine population-based evolutionary methods and local search and are developed by Prins [117] for FSMF, FSMD, FSMFD, and HVRPD. Each solution in the population is represented as a giant tour visiting all customers that is later optimally split into a feasible solution through a specialized shortest path procedure. Two different mechanisms to favor population diversification are implemented: one based on solution cost difference and the second based on their distance in the solution space. This latter diversification mechanism yields the best results which, for FSMFD, are competitive with the previous one from the literature. As to the remaining variants, the MA results are close to the best ones and generally require less computing effort. Another algorithm of this type is described in Liu, Huang, and Ma [94], who apply it to FSMF and FSMD. The initial population is built by using simple construction algorithms, and specialized crossover and local search operators are implemented. The resulting algorithm generates solutions whose quality is comparable with the best previous methods.

TS approaches, which were initially quite popular for this problem family, are revived by Brandão in [32, 33]. The first paper reports the implementation of a TS for FSMF and FSMD based on standard insertion and swap neighborhoods. To speed up the search, the neighborhoods are appropriately restricted, as they consider only moves in which some of the involved customers are close to each other. The computational results show that the algorithm compares favorably with previous ones from the literature. The TS algorithm is applied in [33] to HVRPD and obtains good results on the GH set, the LH set, and five new random instances.

A *Variable Neighborhood Search* (VNS) algorithm for the solution of all variants with fixed and variable fleet is proposed by Imran, Salhi, and Wassan [80]. They use the giant-tour representation, whose optimal split is again determined through the solution of a shortest path problem on a suitably defined network. The VNS employs six different interchange and shift neighborhoods in the shaking step and six classical customer insertion and arc exchange neighborhoods in the local search step. The computational testing is performed on all test instances from the literature and shows that the algorithm is competitive with previous methods.

The most recent research on heuristics for heterogeneous routing follows the current trend towards hybridization of various frameworks which achieve both a better overall performance and the possibility of handling several variants with the same algorithm. A very good example of such approaches is the method by Duhamel, Lacomme, and Prodhon [57], who propose a combination of a *greedy randomized adaptive search procedure* and an *evolutionary local search*. The resulting algorithm, which is successfully applied to FSMFD, FSMF, FSMD, and HVRPFD, works on a giant-tour representation of a solution. Two implementations of the split procedure are discussed. The testing is conducted on all available instance classes for the considered variants and demonstrates that the method obtains a solution quality comparable to that of existing approaches. The authors also propose a new set of 96 instances based on actual town locations within French counties and real street distances.

The currently most powerful approach for FSM variants is the *Hybrid Genetic Search with Adaptive Diversity Control* (HGSADC) algorithm by Vidal et al. [145], already described in Section 9.2.

Finally, Naji-Azimi and Salari [101] describe an *Integer Linear Programming* (ILP) based procedure which can be used to improve the heuristic solutions found by other methods within relatively short additional computing time.

We conclude this section by examining the heuristics that were developed for other variants of heterogeneous VRP. Most of the early work in this area refers to the inclusion of time windows to FSMF, denoted as FSMFTW, which is first considered in Liu and Shen [92]. In addition to a first algorithm for this problem, they also propose a new large set of 168 instances with TWs, called LS168. Other early approaches for this problem variant are discussed in Baldacci, Battarra, and Vigo [19].

A heuristic column generation algorithm, derived from the exact approach for the multiple depot variant, is applied by Bettinelli, Ceselli, and Righini [28] to FSMFTW. The method is able to improve previous approaches on the LS168 test set. Bräysy et al. [35] introduce a metaheuristic designed to solve large-scale FSMFTW instances. The algorithm constructs an initial solution through a savings approach followed by a local search step. Both steps take into account proximity measures of the involved customers, which avoid considering merging or moving far away customers. The metaheuristic framework is a simplified version of a combined *threshold accepting* and *guided local search* approach previously proposed by Bräysy et al. [34]. Computational testing on the LS168 set, and on a new set of 600 large-scale instances with up to 1000 customers, shows that this fast metaheuristic is capable of obtaining state-of-the-art results in few seconds of CPU time even for the larger instances. Paraskevopoulos et al. [108] discuss a *Reactive Variable Neighborhood TS* (RVNTS) which is tested on both FSMFTW and HVRPFDTW. They use a two-phase approach in which an initial set of high-quality solutions is determined through a parallel construction method, followed by a route elimination procedure to possibly improve the fleet's utilization. In the second phase, the RVNTS is used to further improve routing costs of the selected solutions based on eight classical neighborhoods. Testing for FSMFTW is performed on the LS168 set and shows that the proposed method considerably improves existing solutions. As to HVRPFDTW, the authors again use a benchmark based on LS168, where the best fleet found by Liu and Shen [92] for FSMFTW is imposed as fixed. In this case, the RVNTS also obtains very competitive solutions. The most recent approach for FSMFTW is an *adaptive memory programming* algorithm described by Repoussis and Tarantilis [119]. The algorithm generates a large set of solutions through a semi-parallel construction algorithm followed by a short-term memory TS improvement. Then, the routes are heuristically selected from the adaptive memory and the obtained solutions are improved through an *iterated TS*. Testing on LS168 show the excellent performance of the proposed method.

The MDFSMF, to the best of our knowledge, is only studied by Salhi and Sari [126]. They propose a multi-level heuristic in which successive refinement steps involving increasingly complex neighborhoods are applied sequentially. The testing is performed on instances derived from a benchmark of 23 MDVRP instances and shows a moderate benefit through the introduction of the heterogeneous fleet compared to the homogeneous vehicles case.

Recently, the heterogeneous version of other important variants of VRP have been studied. For example, Li, Leung, and Tian [89] examine the *open* variant of HVRPFD for which they develop a MAMP algorithm and test it on randomly generated instances. In addition, the FSMFD incorporating two-dimensional loading constraints (2L-HVRP) is introduced by Leung et al. [87]. They use a *simulated annealing* algorithm for the routing component, in conjunction with six heuristics for item loading and a local search refining step, to solve test instances derived from 2L-CVRP benchmarks.

9.4 ▪ Periodic Routing Problems

In periodic routing problems customers require repeated visits during the planning horizon. The days of visit for each customer can be chosen from a given set of feasible *visiting patterns*. For example, assuming a weekly planning horizon, a customer that requires two visits and whose feasible patterns are given as (2,4) and (3,5) must either be visited Tuesday and Thursday, or Wednesday and Friday. Thus, periodic routing problems require three types of decisions: (i) the selection of visiting patterns for each customer, (ii) the assignment of the chosen day-customer combinations to tours, and (iii) the routing of vehicles for each day of the planning horizon. The latter includes that the classical VRP constraints are respected, i.e., each route has to start and end at the depot and has to observe capacity and route duration restrictions. The objective is to minimize the total traveled distance, while the number of employed vehicles on each day must not exceed the fleet size.

These problems are denoted as *Period or periodic VRP* (PVRP). For a detailed discussion of modeling techniques of the PVRP, we refer the reader to Francis, Smilowitz, and Tzur [64]. Applications of practical importance arise in waste and recyclables collection (see Angelelli and Speranza [7], Nuortio et al. [103], and Coene, Arnout, and Spieksma [44]), product distribution and collection (see Alegre, Laguna, and Pacheco [1], Claassen and Hendriks [42], and Ronen and Goodhart [120]), maintenance operations (see Hadjiconstantinou and Baldacci [75] and Blakeley et al. [29]), and health care (see Hemmelmayr et al. [77], Pacheco et al. [107], Shao, Bard, and Jarrah [128], and Maya, Sörensen, and Goos [95]).

The PVRP was first mentioned in Beltrami and Bodin [26] and Foster and Ryan [60] and later formalized in Russell and Igo [123] and Christofides and Beasley [40]. Early solution methods for PVRP were mainly simply construction and improvement heuristics (see Russell and Gribbin [122], Tan and Beasley [134], and Gaudioso and Paletta [67]), which were later replaced by metaheuristic approaches (see Chao, Golden, and Wasil [37], Cordeau, Gendreau, and Laporte [45], and Drummond, Ochi, and Vianna Dalessandro [56]). More recently, mathematical programming–based solution methods have been proposed (see Francis and Smilowitz [61], Francis, Smilowitz, and Tzur [62], and Mourgaya and Vanderbeck [99]). For the assessment of the quality of the proposed solution methods, the benchmark instance set presented in Cordeau, Gendreau, and Laporte [45] is well established. It consists of a set of so-called "old" instances proposed in Chao, Golden, and Wasil [37], Christofides and Beasley [40], Russell and Igo [123], and Russell and Gribbin [122], containing between 20 and 417 customers, and a set of ten "new" instances introduced by Cordeau, Gendreau, and Laporte [45] and containing between 48 and 288 customers.

The literature until roughly 2008 was surveyed in great detail by Francis, Smilowitz, and Tzur [64]. The goal of the section at hand is to review the most important advances concerning the PVRP and its variants published afterwards. In Section 9.4.1, we review work on the classical PVRP. Section 9.4.2 is concerned with the *PVRP with Time Windows* (PVRPTW), which constitutes the most studied variant of the PVRP. Further important PVRP variants are addressed in Section 9.4.3, and, finally, an overview of recent PVRP-related case studies is provided in Section 9.4.4.

9.4.1 ▪ The Standard PVRP

Solution methods for the basic PVRP are reviewed in this section.

Exact Methods and Matheuristics Baldacci et al. [18] are, to the best of our knowledge, the first and only authors to propose bounding techniques and an exact algorithm for the PVRP. They use different relaxations of an extended set-partitioning formulation to derive five bounding procedures, which are used to generate a reduced problem that can be solved by an integer programming solver (see Baldacci, Christofides, and Mingozzi [21]). Tests on the old PVRP instances with up to 153 customers and a set of 20 practice-inspired instances with up to 199 customers for a special case of the problem, called the *tactical planning VRP*, show the effectiveness of their method. The obtained lower bounds are tight (on average within 1% of optimality), several test instances were solved, and some new best known solutions were found.

Gulczynski, Golden, and Wasil [74] present a hybrid method for the PVRP and two practice-inspired problem variants. The first variant addresses the reassignment of customers to new routes while restricting the changes to potentially long established delivery patterns. The second aims at maintaining balanced workloads of the different drivers. Their approach combines (i) a *large neighborhood search* that is based on an integer programming model for reassigning multiple customers to new visiting patterns and moving customers to new routes in order to decrease the total distance traveled and (ii) a *record-to-record travel algorithm* for improving the daily routes as introduced in Groër, Golden, and Wasil [71]. The presented algorithm shows a convincing performance on the old PVRP instances.

Cacchiani, Hemmelmayr, and Tricoire [36] present another matheuristic based on fixing and releasing variables in the LP-relaxation of a set-covering formulation of the PVRP. Columns are generated in a heuristic fashion using an *Iterated Local Search* (ILS) algorithm. On the old PVRP test set, the authors achieve a performance that comes close to that of the hybrid *Genetic Algorithm* (GA) of Vidal et al. [144]; however, their algorithm requires clearly higher run times. On the realistic instances of Pacheco et al. [107] (see below), their matheuristic is able to obtain better solution quality than the approaches presented in Alegre, Laguna, and Pacheco [1], Cordeau, Gendreau, and Laporte [45], Hemmelmayr, Doerner, and Hartl [76], and Pacheco et al. [107], again requiring significantly higher computational effort.

Metaheuristics Hemmelmayr, Doerner, and Hartl [76] use a *Variable Neighborhood Search* (VNS) approach with an acceptance criterion based on *Simulated Annealing* (SA) to tackle the PVRP. Their shaking phase is based on random changes of customers' visit patterns and string relocate and cross-exchange operators for perturbation of the routes (see Savelsbergh [127] and Taillard et al. [133]). The subsequent local search applies the 3-opt operator (see Lin [91]). The authors conduct tests on the old and new PVRP instances, obtaining competitive results on all instances. Moreover, they provide new best solutions for almost all instances in extensive tests. Their method shows a good scaling behavior and thus achieves very good results in short run time for the larger instances. Pirkwieser and Raidl [115] present another VNS approach that uses a multi-level refinement strategy to improve scalability and thus be capable of tackling large-scale instances. The proposed method achieves a solution quality comparable to that of Hemmelmayr, Doerner, and Hartl [76] on standard PVRP instances if both methods are run for the same number of iterations. Moreover, the authors report results on a set of larger PVRP instances with up to 576 customers, which are generated following the procedure of Cordeau, Gendreau, and Laporte [45].

Vidal et al. [144] use their *Hybrid Genetic Search with Adaptive Diversity Control* (HGSADC) to address the PVRP, the *Multi-Depot PVRP* (MDPVRP), and many other VRP variants as mentioned in Sections 9.3 and 9.2. For all available PVRP benchmark instances, HGSADC outperforms previous methods producing best average results for 41 of the 42 instances in reasonable run times and providing 20 new best solutions.

A parallel, iterated version of the unified *Tabu Search* (TS) of Cordeau, Laporte, and Mercier [46] for solving VRP, PVRP, MDVRP, the *site-dependent VRP* and the time-window versions of the problems is presented by Cordeau and Maischberger [48]. The TS is embedded in an ILS and exploits the concurrent computation capabilities of modern computers by means of a simple parallel computation framework. It achieves reasonable results on the standard VRP, PVRP, MDVRP, and *VRP with Time Windows* (VRPTW) instances and is quite competitive for the other problems addressed. The computational experiments show that the integration into the ILS framework and the parallelization clearly improve the solution quality compared to the original TS.

Several variants of the PVRP exist, which differ either in the considered objective or the side constraints and the types of solutions which are searched (see Mourgaya and Vanderbeck [98]). The following section reviews the most studied variant, namely the PVRPTW, while other variants are discussed in Section 9.4.3.

9.4.2 ▪ The PVRP with Time Windows

In the PVRPTW, service at each customer must begin within an associated time interval and every vehicle has to leave and return to the depot within a given scheduling horizon.

The PVRPTW was first studied in Cordeau, Laporte, and Mercier [46], who propose a unified TS heuristic for several routing problems with time windows. In the same work the standard PVRPTW benchmark based on the PVRP instances of Cordeau, Gendreau, and Laporte [45] was introduced. The benchmark consists of 20 instances, 10 of them featuring wide time windows and the remainder having tight ones.

Exact Methods and Matheuristics We are not aware of exact methods for PVRPTW; however, a bounding procedure and several matheuristics have been proposed. Pirkwieser and Raidl [114] present a set covering formulation of PVRPTW and use column generation to solve the LP-relaxation of the master problem. The pricing subproblem resembles an *Elementary Shortest Path Problem with Resource Constraints* (ESPPRC) (see Feillet et al. [59] and Irnich and Desaulniers [81]). Columns are generated by solving the ESPPRC both in an exact way using *dynamic programming* and heuristically with a *Greedy Randomized Adaptive Search Procedure* (GRASP) approach. In numerical tests on a partially reduced version of the standard PVRPTW benchmark of Cordeau, Laporte, and Mercier [46] (considering only a subset of customers and a reduced vehicle number), the authors are able to provide strong lower bounds for some instances.

Pirkwieser and Raidl [113] develop a hybrid of their VNS heuristic [111] and a generic integer programming solver using the set covering formulation presented in [114]. The VNS transfers feasible solutions and the currently best one to the solver, which tries to improve the route combinations. If the solver is successful, the improved solution is transferred back to the VNS for further optimization. To test the approach, the authors propose a new set of 30 PVRPTW benchmark instances with a four or six day planning horizon based on the well-known Solomon instances for VRPTW (see Solomon [130]). In Pirkwieser and Raidl [112], this approach is enhanced by using a *Multiple VNS* (MVNS) approach, in which several cooperative VNS threads are running in the framework of a sequential cooperative multi-start search. Both the standalone MVNS and an MVNS solver hybrid are able to significantly improve the results of the pure VNS on the benchmark instances introduced in [113], while the hybrid is able to yield better results for most instances without dominating the pure MVNS. In [116], the authors additionally propose a hybrid between an evolutionary algorithm and the column generation approach of Pirkwieser and Raidl [114], as well as 15 additional test instances with a planning horizon of eight days.

Metaheuristics The unified TS method of Cordeau, Laporte, and Mercier [46] is enhanced in Cordeau, Laporte, and Mercier [47] and proves able to clearly improve upon the results obtained with the original heuristic. As described above, Cordeau and Maischberger [48] present a parallel iterated version of the TS, which yields very competitive results. Pirkwieser and Raidl [111] propose a VNS for the PVRPTW that uses an SA-based acceptance criterion and a random ordering of the shaking neighborhoods. The method can clearly improve the solution quality of state-of-the-art approaches and shows that the random variant in most cases outperforms a fixed ordering of the shaking neighborhoods. Yu and Yang [153] propose an *ant colony optimization* algorithm that improves upon the old solutions presented in Cordeau, Laporte, and Mercier [46], but is inferior compared to the methods in Cordeau, Laporte, and Mercier [47] and Pirkwieser and Raidl [111].

Nguyen, Crainic, and Toulouse [102] propose a hybrid GA with two new crossover operators, one aiming at exploration and the other at exploiting the high-quality features of the parents. To educate the offspring towards feasibility and higher fitness, the TS of Cordeau, Laporte, and Mercier [46] and the VNS of Pirkwieser and Raidl [111] are integrated into the algorithm. Tests on the standard instances described in [46] and [116] show a superior solution quality of the hybrid GA compared to previously published methods, with the GA requiring very high run times. However, the authors show that neither the TS in [46] nor the VNS in [111] are able to achieve the same quality when given the same time limit.

Vidal et al. [146] extend the HGSADC proposed in Vidal et al. [144] to be capable of solving VRPTW, PVRPTW, and the time-window versions of MDVRP and the site-dependent VRP. The education phase of the GA, which makes up for most of the computational effort, is specifically tailored to efficiently handle problems with time windows. The authors apply methods for restricting the neighborhood to be searched, use memories to prevent unnecessary re-evaluations of insertion costs, and use a new procedure for determining the change in time-window and route duration violations in amortized constant time. The latter works for all neighborhood operators that exchange a bounded number of arcs or relocate a bounded number of nodes. For solving larger instances, structural and geographical decomposition phases are applied. In numerical tests on the standard benchmarks of PVRPTW, MDVRPTW, and the site-dependent VRPTW, the method outperforms all existing approaches concerning solution quality while using similar computation time. Finally, the authors propose new larger instances for PVRPTW, MDVRPTW, and the site-dependent VRPTW with 360 to 960 customers using the generation procedure described in Cordeau, Gendreau, and Laporte [45] and Cordeau, Laporte, and Mercier [46].

9.4.3 ▪ Other PVRP Variants

The MDPVRP has gained increasing attention in recent years. It extends the PVRP by a given set of depots with homogeneous vehicles which are fixedly allocated to the depots over the planning horizon. The task is to assign to each customer a visiting pattern and a depot, which serves the customer in all periods. The objective is the minimization of total traveled distance. Hadjiconstantinou and Baldacci [75] introduced the MDPVRP in the context of preventive maintenance of a utility company and presented a four-phase algorithm involving a TS for the routing decision. Parthanadee and Logendran [109] consider an MDPVRP with multiple trips, time windows, limited product supplies, and interdependent depot operations; i.e., customers are not fixedly assigned to depots. The problem is addressed by multiple TS variants, which are tested on sets of practice-inspired instances. The authors find that operating depots interdependently can be beneficial, in particular if product shortages occur.

Vidal et al. [144] show that any MDPVRP can be converted into an equivalent PVRP that involves one period for each period-depot pair of the MDPVRP. They use this insight to solve the MDPVRP with their HGSADC using a new set of MDPVRP instances generated by combining the PVRP and MDVRP instances of Cordeau, Gendreau, and Laporte [45]. Lahrichi et al. [85] present a parallel cooperative search framework for solving rich combinatorial optimization problems, which involves several heuristics and exact methods that address subproblems generated by attribute-based decomposition of the original problem and recombine and improve the partial solutions. The authors test their framework on the MDPVRP instances proposed in [144] using a generalized version of the TS of [46] and HGSADC as solvers and clearly improve upon the solution quality of the stand-alone methods. Rahimi-Vahed et al. [118] propose a Path Relinking (PR) method that can either be used as a stand-alone algorithm or in the cooperative search framework of Lahrichi et al. [85], achieving competitive solution quality on the MDPVRP instances of [144].

Alonso, Alvarez, and Beasley [5] extend the TS for PVRP proposed in Cordeau, Gendreau, and Laporte [45] to be able to solve the *site-dependent multi-trip PVRP*, or more generally, all combinations of VRP with accessibility restrictions, multiple use of vehicles, and multiple periods. To assess the quality of their solution method, they conduct computational tests on available benchmark instances of PVRP, the site-dependent VRP, and the *multi-trip VRP* and CVRP, showing very competitive performance for all problems except for the CVRP (for which they still achieve a reasonable average deviation of 1% from the best known solution).

Smilowitz, Nowak, and Jiang [129] present a method for quantifying the effect of workforce management in route construction, i.e., to place a value on the benefit of visiting a customer repeatedly with the same driver. Thus, they are able to balance the resulting consistency benefits against additional routing costs like, e.g., increased distance. They propose three different PVRP variants that include consistency metrics as a part of the objective function to be compared against a base PVRP minimizing traveled distance. Similar metrics were considered in Francis, Smilowitz, and Tzur [63]; however, they were calculated a posteriori to evaluate routing solutions and the presented models did not attempt to optimize with respect to these measures.

9.4.4 ▪ Case Studies

In recent years, several practical problem settings closely related to the PVRP have been studied in the context of waste management, product distribution, health care, and similar operations. Nuortio et al. [103] study a waste-collection problem in Eastern Finland, which is modeled as a stochastic PVRPTW with a fixed number of vehicles. For solving the problem, they use a guided variable neighborhood thresholding method specifically developed for large-scale VRP. Their method is tested on real-life data of a Finnish company and achieves drastic savings in traveled distance compared to the manual optimization previously employed at the company.

Alegre, Laguna, and Pacheco [1] consider a PVRP in the context of a car manufacturer's raw material collection operations. Contrary to the problems and associated benchmark instances addressed in the literature, the focus here lies on long planning horizons of up to 90 days, resulting in a complex visiting-pattern assignment problem, which is followed by a relatively simple routing decision. As a solution approach, the authors propose a scatter search for determining the collection schedules and use a simple local search method using the cross exchanges of Taillard et al. [133] and Or exchanges of Or [105] to improve the vehicle routes. The proposed approach is competitive with the state-of-the-art methods on standard PVRP instances at the time of publication and outperforms the best performing method of Cordeau, Gendreau, and Laporte [45] on real-world problem instances with long planning horizons.

Pacheco et al. [107] address a variant of the PVRP faced by a Spanish bakery company planning its deliveries to distribution centers, in which delivery to a customer has to take place within a certain time before a deadline associated with the customer. To decide the delivery day for each customer and construct the daily routes, the authors propose a two-phase algorithm, consisting of a GRASP phase with local improvement to generate a set of high-quality and diverse solutions and a PR phase to intensify the search. Their approach is able to outperform the methods in [1, 45] and in Hemmelmayr, Doerner, and Hartl [76] on a set of reality-based instances. Moreover, they show that a reduction of travel distance of about 20% is possible by relaxing the customer delivery deadlines previously used by the company and additionally allowing service one day prior to the deadline.

Finally, Maya, Sörensen, and Goos [95] address a variant of the MDPVRP in the field of providing mobile teaching assistance. The authors develop an auction heuristic enhanced by VNS and GRASP concepts that yields a reduction in traveled distance of more than 20% in comparison to the currently implemented solution. A similar problem is studied in Shao, Bard, and Jarrah [128], where the authors investigate the assignment and routing of therapists providing home health care.

9.5 ▪ VRP with Split Deliveries

In the classical VRP, customers are visited exactly once and each vehicle delivers the entire demand to the respective customer. The *Split Delivery Vehicle Routing Problem* (SDVRP) is a relaxation of the *Capacitated VRP* (CVRP) in the sense that a customer may be visited more than once. In this case the demand is covered by two or more *split deliveries*.

The SDVRP has been introduced by Dror and Trudeau [54, 55]. The interest in the SDVRP, and more generally in VRP variants which allow split deliveries, can be explained by the fact that significant savings in costs and the number of routes are possible compared to the problems without splitting possibility. Therefore, several papers have been devoted to worst-case analyses in order to quantify how much worse a solution without split deliveries can be compared to an SDVRP solution. Moreover, some exact approaches and several heuristics and metaheuristics addressing SDVRP and its variants can be found in the literature.

The SDVRP has been surveyed by Archetti and Speranza [14, 15]. We briefly summarize their results but put the main focus on recent results not included in the surveys. A first step is the classification of SDVRP and its basic variants provided in Section 9.5.1. Hereafter, we present theoretical results in Section 9.5.2 including properties of optimal solutions, computational complexity, and worst-case results. A description of the most recent exact and heuristic approaches is given in Section 9.5.3.

9.5.1 ▪ The SDVRP and Its Basic Variants

For a more formal definition of SDVRP, we follow the notation of Chapter 1: Let 0 be the depot and $N = \{1, 2, \dots, n\}$ be the set of customers together forming the node set $V = \{0, 1, \dots, n\}$. The customer *demands* q_i for all $i \in N$ are assumed to be positive integers. Moreover, the demand q_i for $i \in N$ must not exceed the *vehicle capacity* Q (also an integer number), where an unlimited homogeneous fleet is assumed. Then, for a given cost matrix $(c_{ij})_{i,j \in V}$, the SDVRP is the following problem: determine a set of minimum-cost routes and associated *delivery quantities* so that (i) each route starts at the depot 0, visits a subset of the customers, and ends at the depot 0 again, (ii) the delivery quantities of each route do not exceed the vehicle capacity Q, and (iii) the delivery amounts of all routes serving a customer $i \in N$ sum up to the demand q_i.

Since the vehicle fleet is assumed unlimited (*Unlimited Fleet*, UF) in the SDVRP, it always has a feasible and herewith an optimal solution. The version with *Limited Fleet* (LF) is denoted by SDVRP-LF. A feasible solution to the SDVRP-LF with $|K|$ vehicles exists if and only if $\sum_{i \in N} q_i \le |K|Q$ holds.

For a given solution, let n_i denote the *number of visits* to the customer $i \in N$. In the classical VRP, we have $n_i = 1$ for all $i \in N$, while n_i may be any positive number in the SDVRP.

Split deliveries allow one to handle customers with large demands q_i exceeding the capacity Q. Following Archetti, Savelsbergh, and Speranza [12], SDVRP$^+$ is the problem variant in which at least one customer $j \in N$ has a demand $q_j > Q$. In the SDVRP$^+$, the number n_i of visits to customers $i \in N$ can be any positive integer. In contrast, VRP$^+$ denotes the variant in which the number of visits to every customer $i \in N$ is limited to the minimum number of necessary visits, which is equal to $\lceil q_i/Q \rceil$.

A possible heuristic solution to both the SDVRP$^+$ and VRP$^+$ results from first splitting demands $q_i > Q$ into $\lfloor q_i/Q \rfloor \cdot Q + (q_i \mod Q)$. $\lfloor q_i/Q \rfloor$ routes serve customer i with full loads Q, and only routes for the remaining demands $(q_i \mod Q) \neq 0$, $i \in N$ (if any), have to be planned because the full loads are served by *direct trips* $(0, i, 0)$, a.k.a. *out-and-back tours*. The remaining demands (all not greater than Q) are then served with an optimal SDVRP or VRP solution. We denote by H^{SDVRP^+} the respective heuristics and problem variants resulting from the initial splitting. Similarly, H^{VRP^+} is a restricted H^{SDVRP^+} in which the number of visits to customer $i \in N$ is limited to $\lceil q_i/Q \rceil$.

Finally, it may be desirable to restrict the granularity of demand splits. Gulczynski, Golden, and Wasil [73] introduce a variant in which, for any real number $0 \leq p \leq 1$, a *Minimum Delivery Amount* (MDA) is defined by pq_i. A feasible solution to the SDVRP-MDA$_p$ requires that every route visiting customer i must deliver at least pq_i units. For example, if demand is $q_i = 10$ and $p = 0.2$, not more than five routes must deliver at least two units each. Obviously, $p = 0$ provides full delivery flexibility as modeled by the SDVRP, while for $p > 1/2$ the SDVRP-MDA$_p$ and the VRP coincide. Interesting new variants result for $0 < p \leq 1/2$.

Table 9.2 summarizes the VRP variants defined above.

Table 9.2. *SDVRP and VRP variants used in worst-case analyses.*

VRP Variant	Demand	Num. Visits	MDA	Fleet
SDVRP	$\forall\, i \in N : 1 \leq q_i \leq Q$	any number	0	unlimited
VRP	$\forall\, i \in N : 1 \leq q_i \leq Q$	1	d_i	unlimited
SDVRP-LF	$\forall\, i \in N : 1 \leq q_i \leq Q$	any number	0	limited
SDVRP-MDA$_p$	$\forall\, i \in N : 1 \leq q_i \leq Q$	$1, \dots, \lfloor 1/p \rfloor$	pq_i	unlimited
SDVRP$^+$	$\exists\, j \in N : q_j > Q$	any number	0	unlimited
VRP$^+$	$\exists\, j \in N : q_j > Q$	$\lceil q_i/Q \rceil$	$(q_i \mod Q)$	unlimited
H^{SDVRP^+}	initial splitting $q_i = Q + \cdots + Q + (q_i \mod Q)$	any number	0	unlimited
H^{VRP^+}	initial splitting $q_i = Q + \cdots + Q + (q_i \mod Q)$	$\lceil q_i/Q \rceil$	$(q_i \mod Q)$	unlimited

9.5.2 ▪ Theoretical Results

Several heuristic and exact algorithms benefit from the following properties allowing one to restrict the search for optimal solutions.

Property 1 There exists an optimal solution in which the delivery quantities are integer numbers (see Archetti, Savelsbergh, and Speranza [12] and Archetti, Bouchard, and Desaulniers [9]) (demands and capacity are assumed to be integer).

For a given solution to the SDVRP, let $i_1, i_2, \dots, i_k \in N$ be $k \geq 2$ different *split customers*, i.e., customers served by at least two different routes. If there exist k routes $r_1, \dots, r_k$ such that route r_j contains the customers i_j and i_{j+1} for all $j = 1, 2, \dots, k$ (defining $i_{k+1} := i_1$), then $(i_1, i_2, \dots, i_k, i_1)$ is a *k-split cycle*.

All the following properties are based on the assumption that the routing costs c_{ij} satisfy the *triangle inequality*, i.e., $c_{ij} + c_{jk} \geq c_{ik}$ for all $i, j, k \in V$. Then there exists an optimal solution to the SDVRP ...

Property 2 ...which does not contain k-split cycles, $k \geq 2$ (see Dror and Trudeau [54]).

Property 3 ...where no two routes share more than one split customer (see Dror and Trudeau [55]).

Property 4 ...where each arc (i, j) between two customers $i, j \in N$ is traversed at most once (see Gendreau et al. [68]).

Property 5 ...where for each pair of reverse arcs (i, j) and (j, i) at most one of them is traversed (see Desaulniers [51]).

Property 6 ...where the number of vehicles used is not greater than $2\lceil \sum_{i \in N} q_i / Q \rceil$ (see Archetti, Bianchessi, and Speranza [8]).

Property 7 ...where the number of splits is less than the number of routes (see Archetti, Savelsbergh, and Speranza [12]).

Maximum Savings by Split Deliveries Table 9.3 summarizes the maximum savings results presented in the literature. All results are based on the assumption that the triangle inequality holds for the routing costs. Most of them, except for some special cases, say that savings of up to 50% for the overall routing distance are possible by splitting deliveries compared to optimal solutions of the respective VRP variant with identical input data.

Table 9.3. *Summary of worst-case analyses.*

Objective and Reference	Precondition	Result
Routing Costs $z(\cdot)$		
[12]		$z(\text{VRP}) \leq 2z(\text{SDVRP})$
[12]		$z(\text{VRP}^+) \leq 2z(\text{SDVRP}^+)$
[12]		$z(\text{H}^{VRP^+}) \leq 2z(\text{VRP}^+)$
[12]		$z(\text{H}^{SDVRP^+}) \leq 2z(\text{SDVRP}^+)$
[12]	$Q = 3$	$z(VRP) \leq \frac{3}{2}z(\text{SDVRP})$
[12]	$Q = 3$	$z(VRP^+) \leq \frac{3}{2}z(\text{SDVRP}^+)$
[152]	$0 < p < 1/2$	$z(\text{VRP}) \leq 2z(\text{SDVRP-MDA}_p)$
[152]	$p = 1/2$	$z(\text{VRP}) \leq \frac{3}{2}z(\text{SDVRP-MDA}_p)$
[72]		$z(\text{VRP})/z(\text{SDVRP-LF})$ can be arbitrarily large
Number of Routes $r(\cdot)$		
[13]		$r(\text{SDVRP}) = \lceil \sum_{i \in N} q_i / Q \rceil$
[13]		$r(\text{VRP}) \leq 2r(\text{SDVRP-LF})$

The SDVRP allows, in principle (disregarding Property 1), that each demand is split into any number of smaller deliveries performed by different vehicles. Xiong et al. [152] show that if each delivery amount d can only be split into two equal deliveries of size $\frac{1}{2}d$, maximum savings of $33\frac{1}{3}$% are possible. Interestingly, for every minimum fraction $p \in (0, \frac{1}{2})$ into which the demand is split (i.e., for the SDVRP-MDA$_p$ as defined above) the same maximal saving of 50% is possible as in the SDVRP case. Note finally that all presented bounds in Table 9.3 have been proven to be tight.

Computational Complexity The VRP and SDVRP are NP-hard because they generalize the *Traveling Salesman Problem* (TSP). However, both are polynomially solvable for the case with vehicle capacity $Q = 2$ (see Archetti, Mansini, and Speranza [11]). Note that for the SDVRP with $Q = 2$, H^{SDVRP} provides an optimal solution (Archetti, Mansini, and Speranza [11] name this property *reducibility*). Both VRP and SDVRP remain NP-hard for $Q \geq 3$ even in case of unit demands [11]. The computational complexity of the SDVRP on classes of instances, where the underlying graph is a circle ($\mathcal{O}(n^2)$), a half-line, and a line ($\mathcal{O}(n)$), a star (NP-hard for LF and $\mathcal{O}(n)$ for UF), and a tree (NP-hard), are shown by Archetti et al. [10], where corresponding results for CVRP (with LF and UF) are also presented.

9.5.3 ▪ Solution Methods for the SDVRP

Exact Methods and Matheuristics Early exact algorithms for the SDVRP were already surveyed by Archetti and Speranza [14, 15]: Based on an arc-flow formulation for the SDVRP, Dror, Laporte, and Trudeau [53] provide a first exact solution approach. Valid inequalities and a cutting-plane algorithm to provide lower bounds on the SDVRP are presented by Belenguer, Martinez, and Mota [24]. Lee et al. [86] develop a dynamic programming–based solution approach, Jin, Liu, and Bowden [83] and Liu [93] a two-stage approach, and Jin, Liu, and Eksioglu [84] a *Column Generation* (CG) approach. The most powerful approach is, at the moment, the Branch-and-Cut-and-Price by Archetti, Bianchessi, and Speranza [8]. The authors are able to consistently solve instances with approximately 20– 30 customers, but also some larger instances, where the largest has 144 customers. Hybrid algorithms using *mixed integer programming* to partially optimize solutions are presented by Archetti, Speranza, and Savelsbergh [17], Chen, Golden, and Wasil [38] and Jin, Liu, and Eksioglu [84].

The *SDVRP with Time Windows* (SDVRPTW) is the extension where time-window constraints are added. The extended set covering formulation of Gendreau et al. [68] exploits Property 4 and uses two types of variables, one for all feasible vehicle routes and another one for the delivery quantities provided by each chosen route. The alternative formulation by Desaulniers [51] is an extended set covering problem with variables that represent both routes and associated delivery quantities. Herein, only *extreme delivery patterns* need to be considered; i.e., only one customer (if any) on the route receives a positive quantity below its entire demand, while all other customers are either served with their entire demand or visited only so that the delivery quantity is zero. Combinations of extreme delivery patterns of a route are then convex-combined into regular patterns in the CG master program, which also exploits Property 5. Later Archetti, Bouchard, and Desaulniers [9] improve the approach [51] by introducing a powerful *Tabu Search* (TS) algorithm to solve the CG subproblem and devising several classes of valid inequalities for the SDVRPTW and corresponding separation procedures.

Salani and Vacca [124] introduce the *VRPTW with Discrete Splits* (DSDVRPTW), in which the demand of a customer consists of several items that may be delivered by several vehicles, but items cannot be split further. If all items have weight one (unit demand case), then the resulting problem is an SDVRPTW, while in general the SDVRPTW is a relaxation of the DSDVRPTW. Furthermore, there may be the restriction to deliver only specific *combinations of items*, called *orders*, to a customer at the same time. The set partitioning formulation of the DSDVRPTW is solved with Branch-and-Price, where the master program has partitioning constraints for each item. The subproblem is an elementary shortest path problem with resource constraints (see Irnich and Desaulniers [81]) with one node per order and elementarity requirements with respect to all orders of the same customer. Order-specific service times can easily be integrated. The approach seems to work well for not too small orders but is not suited for solving the SDVRPTW.

Heuristics and Metaheuristics Heuristics developed prior to 2011 are thoroughly surveyed in Archetti and Speranza [15], including local search (see Dror and Trudeau [54] and Dror and Trudeau [55]), TS (see Archetti, Speranza, and Hertz [16]), *memetic algorithm* (see Boudia, Prins, and Reghioui [30]), and *scatter search* (see Mota, Campos, and Corberán [97]). Derigs, Li, and Vogel [50] compare implementations of *simulated annealing*, *threshold accepting*, *record-to-record travel*, attribute-based local *beam search*, and attribute-based *hill climber*, where the last performs best. In a series of works [2, 3, 4], Aleman and coauthors propose several metaheuristics including *adaptive memory programming*, *variable neighborhood descent*, and a population-based algorithm. In the following, we focus on three approaches not surveyed before.

Berbotto, García, and Nogales [27] present a *randomized granular TS*. The approach follows the granular-TS idea (see Toth and Vigo [142]) of restricting the set of edges available for insertion in a local search move to promising edges depending on whether intensification and diversification is desirable. A novelty is that the authors control the set of short edges using a threshold parameter that also depends on the respective route and its residual capacity, the cost, and the number of edges of the current solution. Instead of only one neighborhood traditionally used in TS, the authors consider seven different neighborhood operators together: Three are classical VRP operators (node relocation, node exchange or swap, and 2-opt*; see, e.g., Funke, Grünert, and Irnich [65]), two others are known SDVRP-specific operators (exchange split and delete worst split; see Ho and Haugland [79] and Archetti, Speranza, and Hertz [16]), and the operators *delete split new* and *delete split* are newly introduced. The moves involve three different routes and simultaneously consider the relocation of a node and the modification of delivery quantities. All seven operators are used at the same time, and a randomization mechanism selects the next neighbor solution. Infeasible solutions regarding vehicle capacity are allowed, a capacity correction phase tries to eliminate such infeasibilities, a further improvement phase eliminates 2-split cycles (see Property 2), and an exact TSP solver is applied for optimizing the individual routes. Overall, the approach achieves very remarkable results because it improves many best known solutions on several benchmark sets from the literature.

Wilck and Cavalier [150] develop a two-phase construction heuristic, where in the first phase customer clusters and delivery quantities are determined, and in the second phase routes are formed for each cluster by solving a TSP. The first phase for the demand partitioning is solved by a rule-based procedure that iteratively adds customers to the current partial solution (clusters and delivery quantities). The three main decisions taken are (i) selection of customers with unsatisfied demand for initializing a route, (ii) sharing residual capacities between the routes, and (iii) insertion of one, two, or three customers with unsatisfied demand into existing routes. The three decisions are made using different rules leading to 36 possible construction algorithms. The authors report nine of them being beneficial. The main advantage of the proposed algorithms is that they are fast and compute relatively good solution.

In Wilck and Cavalier [151], the same authors present *genetic algorithms* for the SDVRP. Two variants differing in the fitness computation perform well compared to the older approaches Chen, Golden, and Wasil [38] and Jin, Liu, and Eksioglu [84] but are not fully convincing compared to the newer approach by Berbotto, García, and Nogales [27] because only few and small improvements are achieved often using longer computation times.

9.6 ▪ Conclusions and Future Research Directions

Four important variants of the VRP are reviewed in this chapter. The VRPB and HFVRP received relatively little attention in the recent literature. For both of them further research on exact methods may highlight structural properties and problem-specific classes of valid inequalities that can improve the quality of currently available methods. In fact, in the heterogeneous case the existing approaches are based on general frameworks that may benefit from further insight on the HFVRP properties, whereas the exact methods for VRPB may be updated to current state of the art. Several aspects of these problems merit further attention also from a heuristic point of view. The specific impact of vehicle-dependent travel times in HVRP, or the multi-depot and mixed service variants of VRPB, are relevant examples in this direction.

The PVRP has continuously received attention by the scientific community, and many interesting solution approaches covering exact methods, matheuristics, and metaheuristics have been published in recent years. The practical importance of the problem is further underlined by a high number of case studies that deal with real-life PVRP. An interesting topic for future research could be the development of exact solution approaches for the most popular PVRP variants, namely the *PVRP with time windows* and the *multi-depot PVRP*. Also, new practically inspired variants that, e.g., incorporate consistency constraints (see Smilowitz, Nowak, and Jiang [129]) or the deviation from existing delivery patterns (see Gulczynski, Golden, and Wasil [74]) should be further investigated.

The SDVRP and its variants are well studied in the literature: Many worst-case analyses and also several empirical studies indicate that substantial savings can result when customers are allowed to be visited more than once. On the downside, multiple visits may be undesirable from a customer point of view. Moreover, the algorithmic handling of possible split deliveries requires specifically tailored programming components in both exact and (meta)heuristic approaches. In particular for the latter, it is not yet clear which algorithmic concepts are best suited for representing (partial) tours and solutions so that fundamental operations in local search and population-based heuristics can be performed in an efficient and effective way. We expect further research in this direction.

Bibliography

[1] J. Alegre, M. Laguna, and J. Pacheco, *Optimizing the periodic pick-up of raw materials for a manufacturer of auto parts*, European Journal of Operational Research, 179 (2007), pp. 736–746.

[2] R. E. Aleman, *A guided neighborhood search applied to the split delivery vehicle routing problem*, PhD thesis, Wright State University, Dayton, OH, 2009.

[3] R. E. Aleman and R. R. Hill, *A tabu search with vocabulary building approach for the vehicle routing problem with split demands*, International Journal of Metaheuristics, 1 (2010), pp. 55–80.

[4] R. E. Aleman, X. Zhang, and R. R. Hill, *An adaptive memory algorithm for the split delivery routing problem*, Journal of Heuristics, 16 (2010), pp. 441–473.

[5] F. Alonso, M. J. Alvarez, and J. E. Beasley, *A tabu search algorithm for the periodic vehicle routing problem with multiple vehicle trips and accessibility restrictions*, Journal of the Operational Research Society, 59 (2008), pp. 963–976.

[6] S. P. ANBUUDAYASANKAR, K. GANESH, S. C. LENNY KOH, AND Y. DUCQ, *Modified savings heuristics and genetic algorithm for bi-objective vehicle routing problem with forced backhauls*, Expert Systems with Applications, 39 (2012), pp. 2296–2305.

[7] E. ANGELELLI AND M. G. SPERANZA, *The application of a vehicle routing model to a waste-collection problem: Two case studies*, Journal of the Operational Research Society, 53 (2002), pp. 944–952.

[8] C. ARCHETTI, N. BIANCHESSI, AND M. G. SPERANZA, *A column generation approach for the split delivery vehicle routing problem*, Networks, 58 (2011), pp. 241–254.

[9] C. ARCHETTI, M. BOUCHARD, AND G. DESAULNIERS, *Enhanced branch and price and cut for vehicle routing with split deliveries and time windows*, Transportation Science, 45 (2011), pp. 285–298.

[10] C. ARCHETTI, D. FEILLET, M. GENDREAU, AND M. G. SPERANZA, *Complexity of the VRP and SDVRP*, Transportation Research Part C: Emerging Technologies, 19 (2011), pp. 741–750.

[11] C. ARCHETTI, R. MANSINI, AND M. G. SPERANZA, *Complexity and reducibility of the skip delivery problem*, Transportation Science, 39 (2005), pp. 182–187.

[12] C. ARCHETTI, M. W. P. SAVELSBERGH, AND M. G. SPERANZA, *Worst-case analysis for split delivery vehicle routing problems*, Transportation Science, 40 (2006), pp. 226–234.

[13] ———, *To split or not to split: That is the question*, Transportation Research Part E: Logistics and Transportation Review, 44 (2008), pp. 114–123.

[14] C. ARCHETTI AND M. G. SPERANZA, *The split delivery vehicle routing problem: A survey*, in The Vehicle Routing Problem: Latest Advances and New Challenges, B. L. Golden, S. Raghavan, and E. A. Wasil, eds., vol. 43 of Operations Research/Computer Science Interfaces Series, Springer, New York, 2008, pp. 103–122.

[15] ———, *Vehicle routing problems with split deliveries*, International Transactions in Operational Research, 19 (2012), pp. 3–22.

[16] C. ARCHETTI, M. G. SPERANZA, AND A. HERTZ, *A tabu search algorithm for the split delivery vehicle routing problem*, Transportation Science, 40 (2006), pp. 64–73.

[17] C. ARCHETTI, M. G. SPERANZA, AND M. W. P. SAVELSBERGH, *An optimization-based heuristic for the split delivery vehicle routing problem*, Transportation Science, 42 (2008), pp. 22–31.

[18] R. BALDACCI, E. BARTOLINI, A. MINGOZZI, AND A. VALLETTA, *An exact algorithm for the period routing problem*, Operations Research, 59 (2011), pp. 228–241.

[19] R. BALDACCI, M. BATTARRA, AND D. VIGO, *Routing a heterogeneous fleet of vehicles*, in The Vehicle Routing Problem: Latest Advances and New Challenges, B. L. Golden, S. Raghavan and E. A. Wasil, eds., vol. 43 of Operations Research/Computer Science Interfaces Series, Springer, New York, 2008, pp. 3–27.

[20] ——, *Valid inequalities for the fleet size and mix vehicle routing problem with fixed costs*, Networks, 54 (2009), pp. 178–189.

[21] R. BALDACCI, N. CHRISTOFIDES, AND A. MINGOZZI, *An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts*, Mathematical Programming, 115 (2008), pp. 351–385.

[22] R. BALDACCI AND A. MINGOZZI, *A unified exact method for solving different classes of vehicle routing problems*, Mathematical Programming A, 120 (2009), pp. 347–380.

[23] J. E. BEASLEY, *Route-first cluster-second methods for vehicle routing*, Omega, 11 (1983), pp. 403–408.

[24] J.-M. BELENGUER, M. C. MARTINEZ, AND E. MOTA, *A lower bound for the split delivery vehicle routing problem*, Operations Research, 48 (2000), pp. 801–810.

[25] P. BELFIORE AND H. T. Y. YOSHIZAKI, *Scatter search for a real-life heterogeneous fleet vehicle routing problem with time windows and split deliveries in Brazil*, European Journal of Operational Research, 199 (2009), pp. 750–758.

[26] E. J. BELTRAMI AND L. D. BODIN, *Networks and vehicle routing for municipal waste collection*, Networks, 4 (1974), pp. 65–94.

[27] L. BERBOTTO, S. GARCÍA, AND F. J. NOGALES, *A randomized granular tabu search heuristic for the split delivery vehicle routing problem*, Annals of Operations Research, (2013), pp. 1–21.

[28] A. BETTINELLI, A. CESELLI, AND G. RIGHINI, *A branch-and-cut-and-price algorithm for the multi-depot heterogeneous vehicle routing problem with time windows*, Transportation Research Part C: Emerging Technologies, 19 (2011), pp. 723–740.

[29] F. BLAKELEY, B. ARGÜELLO, B. CAO, W. HALL, AND J. KNOLMAJER, *Optimizing periodic maintenance operations for Schindler elevator corporation*, Interfaces, 33 (2003), pp. 67–79.

[30] M. BOUDIA, C. PRINS, AND M. REGHIOUI, *An effective memetic algorithm with population management for the split delivery vehicle routing problem*, in Hybrid Metaheuristics, T. Bartz-Beielstein, M. Blesa Aguilera, C. Blum, B. Naujoks, A. Roli, G. Rudolph, and M. Sampels, eds., vol. 4771 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2007, pp. 16–30.

[31] J. BRANDÃO, *A new tabu search algorithm for the vehicle routing problem with backhauls*, European Journal of Operational Research, 173 (2006), pp. 540–555.

[32] ——, *A deterministic tabu search algorithm for the fleet size and mix vehicle routing problem*, European Journal of Operational Research, 195 (2009), pp. 716–728.

[33] ——, *A tabu search algorithm for the heterogeneous fixed fleet vehicle routing problem*, Computers & Operations Research, 38 (2011), pp. 140–151.

[34] O. BRÄYSY, W. DULLAERT, G. HASLE, D. MESTER, AND M. GENDREAU, *An effective multi-restart deterministic annealing metaheuristic for the fleet size and mix vehicle routing problem with time windows*, Transportation Science, 42 (2008), pp. 371–386.

[35] O. BRÄYSY, P. P. PORKKA, W. DULLAERT, P. P. REPOUSSIS, AND C. D. TARANTILIS, *A well-scalable metaheuristic for the fleet size and mix vehicle routing problem with time windows*, Expert Systems with Applications, 36 (2009), pp. 8460–8475.

[36] V. CACCHIANI, V. C. HEMMELMAYR, AND F. TRICOIRE, *A set-covering based heuristic algorithm for the periodic vehicle routing problem*, Discrete Applied Mathematics, 163 (2014), pp. 53–64.

[37] I.-M. CHAO, B. L. GOLDEN, AND E. A. WASIL, *An improved heuristic for the period vehicle routing problem*, Networks, 26 (1995), pp. 25–44.

[38] S. CHEN, B. L. GOLDEN, AND E. A. WASIL, *The split delivery vehicle routing problem: Applications, algorithms, test problems, and computational results*, Networks, 49 (2007), pp. 318–329.

[39] E. CHOI AND D. TCHA, *A column generation approach to the heterogeneous fleet vehicle routing problem*, Computers & Operations Research, 34 (2007), pp. 2080–2095.

[40] N. CHRISTOFIDES AND J. E. BEASLEY, *The period routing problem*, Networks, 14 (1984), pp. 237–256.

[41] N. CHRISTOFIDES, A. MINGOZZI, AND P. TOTH, *Exact algorithms for the vehicle routing problem based on spanning tree and shortest path relaxation*, Mathematical Programming, 10 (1981), pp. 255–280.

[42] G. D. H. CLAASSEN AND T. H. B. HENDRIKS, *An application of special ordered sets to a periodic milk collection problem*, European Journal of Operational Research, 180 (2007), pp. 754–769.

[43] G. CLARKE AND J. WRIGHT, *Scheduling of vehicles from a central depot to a number of delivery points*, Operations Research, 12 (1964), pp. 568–581.

[44] S. COENE, A. ARNOUT, AND F. C. R. SPIEKSMA, *On a periodic vehicle routing problem*, Journal of the Operational Research Society, 61 (2010), pp. 1719–1728.

[45] J.-F. CORDEAU, M. GENDREAU, AND G. LAPORTE, *A tabu search heuristic for periodic and multi-depot vehicle routing problems*, Networks, 30 (1997), pp. 105–119.

[46] J.-F. CORDEAU, G. LAPORTE, AND A. MERCIER, *A unified tabu search heuristic for vehicle routing problems with time windows*, Journal of the Operational Research Society, 52 (2001), pp. 928–936.

[47] ——, *An improved tabu search algorithm for the handling of route duration constraints in vehicle routing problems with time windows*, Journal of the Operational Research Society, 55 (2004), pp. 542–546.

[48] J.-F. CORDEAU AND M. MAISCHBERGER, *A parallel iterated tabu search heuristic for vehicle routing problems*, Computers & Operations Research, 39 (2012), pp. 2033–2050.

[49] I. DEIF AND L. D. BODIN, *Extension of the Clarke and Wright algorithm for solving the vehicle routing problem with backhauling*, in Proceedings of the Babson Conference on Software Uses in Transportation and Logistic Management, A. Kidder, ed., Babson Park, FL, 1984, pp. 75–96.

[50] U. DERIGS, B. LI, AND U. VOGEL, *Local search-based metaheuristics for the split delivery vehicle routing problem*, Journal of the Operational Research Society, 61 (2010), pp. 1356–1364.

[51] G. DESAULNIERS, *Branch-and-price-and-cut for the split-delivery vehicle routing problem with time windows*, Operations Research, 58 (2010), pp. 179–192.

[52] R. DONDO AND J. CERDÁ, *A cluster-based optimization approach for the multi-depot heterogeneous fleet vehicle routing problem with time windows*, European Journal of Operational Research, 176 (2007), pp. 1478–1507.

[53] M. DROR, G. LAPORTE, AND P. TRUDEAU, *Vehicle routing with split deliveries*, Discrete Applied Mathematics, 50 (1994), pp. 239–254.

[54] M. DROR AND P. TRUDEAU, *Savings by split delivery routing*, Transportation Science, 23 (1989), pp. 141–145.

[55] ——, *Split delivery routing*, Naval Research Logistics, 37 (1990), pp. 383–402.

[56] L. M. A. DRUMMOND, L. S. OCHI, AND S. VIANNA DALESSANDRO, *An asynchronous parallel metaheuristic for the period vehicle routing problem*, Future Generation Computer Systems, 17 (2001), pp. 379–386.

[57] C. DUHAMEL, P. LACOMME, AND C. PRODHON, *A hybrid evolutionary local search with depth first search split procedure for the heterogeneous vehicle routing problems*, Engineering Applications of Artificial Intelligence, 25 (2012), pp. 345–358.

[58] C. DUHAMEL, J.-Y. POTVIN, AND J.-M. ROUSSEAU, *A tabu search heuristic for the vehicle routing problem with backhauls and time windows*, Transportation Science, 31 (1997), pp. 49–59.

[59] D. FEILLET, P. DEJAX, M. GENDREAU, AND C. GUEGUEN, *An exact algorithm for the elementary shortest path problem with resource constraints: Application to some vehicle routing problems*, Networks, 44 (2004), pp. 216–229.

[60] B. A. FOSTER AND D. M. RYAN, *An integer programming approach to the vehicle scheduling problem*, Operational Research Quarterly, 27 (1976), pp. 367–384.

[61] P. M. FRANCIS AND K. R. SMILOWITZ, *Modeling techniques for periodic vehicle routing problems*, Transportation Research Part B: Methodological, 40 (2006), pp. 872–884.

[62] P. M. FRANCIS, K. R. SMILOWITZ, AND M. TZUR, *The period vehicle routing problem with service choice*, Transportation Science, 40 (2006), pp. 439–454.

[63] ——, *Flexibility and complexity in periodic distribution problems*, Naval Research Logistics, 54 (2007), pp. 136–150.

[64] ——, *The period vehicle routing problem and its extensions*, in The Vehicle Routing Problem: Latest Advances and New Challenges, B. L. Golden, S. Raghavan, and E. A. Wasil, eds., vol. 43 of Operations Research/Computer Science Interfaces Series, Springer, New York, 2008, pp. 73–102.

[65] B. FUNKE, T. GRÜNERT, AND S. IRNICH, *Local search for vehicle routing and scheduling problems: Review and conceptual integration*, Journal of Heuristics, 11 (2005), pp. 267–306.

[66] Y. GAJPAL AND P. L. ABAD, *Multi-ant colony system (macs) for a vehicle routing problem with backhauls*, European Journal of Operational Research, 196 (2009), pp. 102–117.

[67] M. GAUDIOSO AND G. PALETTA, *A heuristic for the periodic vehicle routing problem*, Transportation Science, 26 (1992), pp. 86–92.

[68] M. GENDREAU, P. DEJAX, D. FEILLET, AND G. GUEGUEN, *Vehicle routing with time windows and split deliveries*, Technical Report 2006-851, Laboratoire d'Informatique d'Avignon, Avignon, France, 2006.

[69] M. GOETSCHALCKX AND C. JACOBS-BLECHA, *The vehicle routing problem with backhauls*, European Journal of Operational Research, 42 (1989), pp. 39–51.

[70] B. L. GOLDEN, A. A. ASSAD, L. LEVY, AND F. G. GHEYSENS, *The fleet size and mix vehicle routing problem*, Computers & Operations Research, 11 (1984), pp. 49–66.

[71] C. GROËR, B. L. GOLDEN, AND E. A. WASIL, *A library of local search heuristics for the vehicle routing problem*, Mathematical Programming Computation, 2 (2010), pp. 79–101.

[72] C. GUEGUEN, *Méthodes de résolution exacte pour les problèmes de tournées de véhicules*, PhD thesis, École Centrale Paris, France, 1999.

[73] D. GULCZYNSKI, B. L. GOLDEN, AND E. A. WASIL, *The split delivery vehicle routing problem with minimum delivery amounts*, Transportation Research Part E: Logistics and Transportation Review, 46 (2010), pp. 612–626.

[74] ——, *The period vehicle routing problem: New heuristics and real-world variants*, Transportation Research Part E: Logistics and Transportation Review, 47 (2011), pp. 648–668.

[75] E. HADJICONSTANTINOU AND R. BALDACCI, *A multi-depot period vehicle routing problem arising in the utilities sector*, Journal of the Operational Research Society, 49 (1998), pp. 1239–1248.

[76] V. C. HEMMELMAYR, K. DOERNER, AND R. F. HARTL, *A variable neighborhood search heuristic for periodic routing problems*, European Journal of Operational Research, 195 (2009), pp. 791–802.

[77] V. C. HEMMELMAYR, K. DOERNER, R. F. HARTL, AND M. W. P. SAVELSBERGH, *Delivery strategies for blood products supplies*, OR Spectrum, 31 (2009), pp. 707–725.

[78] A. HERTZ, M. ULDRY, AND M. WIDMER, *Integer linear programming models for a cement delivery problem*, European Journal of Operational Research, 222 (2012), pp. 623–631.

[79] S. C. HO AND D. HAUGLAND, *A tabu search heuristic for the vehicle routing problem with time windows and split deliveries*, Computers & Operations Research, 31 (2004), pp. 1947–1964.

[80] A. IMRAN, S. SALHI, AND N. A. WASSAN, *A variable neighborhood-based heuristic for the heterogeneous fleet vehicle routing problem*, European Journal of Operational Research, 197 (2009), pp. 509–518.

[81] S. IRNICH AND G. DESAULNIERS, *Shortest path problems with resource constraints*, in Column Generation, G. Desaulniers, J. Desrosiers, and M. M. Solomon, eds., Springer, New York, 2005, ch. 2, pp. 33–65.

[82] O. JABALI, M. GENDREAU, AND G. LAPORTE, *A continuous approximation model for the fleet composition problem*, Transportation Research Part B: Methodological, 46 (2012), pp. 1591–1606.

[83] M. JIN, K. LIU, AND R. O. BOWDEN, *A two-stage algorithm with valid inequalities for the split delivery vehicle routing problem*, International Journal of Production Economics, 105 (2007), pp. 228–242.

[84] M. JIN, K. LIU, AND B. EKSIOGLU, *A column generation approach for the split delivery vehicle routing problem*, Operations Research Letters, 36 (2008), pp. 265–270.

[85] N. LAHRICHI, T. G. CRAINIC, M. GENDREAU, W. REI, G. CERASELA CRIŞAN, AND T. VIDAL, *An integrative cooperative search framework for multi-decision- attribute combinatorial optimization*, Technical Report 2012-42, CIRRELT, Montréal, Canada, 2012.

[86] C. G. LEE, M. A. EPELMAN, C. C. WHITE III, AND Y. A. BOZER, *A shortest path approach to the multiple-vehicle routing problem with split pick-ups*, Transportation Research Part B: Methodological, 40 (2006), pp. 265–284.

[87] S. C. H. LEUNG, Z. ZHANG, D. ZHANG, X. HUA, AND M. K. LIM, *A meta-heuristic algorithm for heterogeneous fleet vehicle routing problems with two-dimensional loading constraints*, European Journal of Operational Research, 225 (2013), pp. 199–210.

[88] F. LI, B. L. GOLDEN, AND E. A. WASIL, *A record-to-record travel algorithm for solving the heterogeneous fleet vehicle routing problem*, Computers & Operations Research, 34 (2007), pp. 2734–2742.

[89] X. LI, S. C. H. LEUNG, AND P. TIAN, *A multistart adaptive memory-based tabu search algorithm for the heterogeneous fixed fleet open vehicle routing problem*, Expert Systems with Applications, 39 (2012), pp. 365–374.

[90] X. LI, P. TIAN, AND Y. P. ANEJA, *An adaptive memory programming metaheuristic for the heterogeneous fixed fleet vehicle routing problem*, Transportation Research Part E: Logistics and Transportation Review, 46 (2010), pp. 1111–1127.

[91] S. LIN, *Computer solutions of the traveling salesman problem*, Bell System Technical Journal, 44 (1965), pp. 2245–2269.

[92] F. H. LIU AND S. Y. SHEN, *The fleet size and mix vehicle routing problem with time windows*, Journal of the Operational Research Society, 50 (1999), pp. 721–732.

[93] K. LIU, *A study on the split delivery vehicle routing problem*, PhD thesis, Mississippi State University, Starkville, MS, 2005.

[94] S. LIU, W. HUANG, AND H. MA, *An effective genetic algorithm for the fleet size and mix vehicle routing problems*, Transportation Research Part E: Logistics and Transportation Review, 45 (2009), pp. 434–445.

[95] P. MAYA, K. SÖRENSEN, AND P. GOOS, *A metaheuristic for a teaching assistant assignment-routing problem*, Computers & Operations Research, 39 (2012), pp. 249–258.

[96] A. MINGOZZI, S. GIORGI, AND R. BALDACCI, *An exact method for the vehicle routing problem with backhauls*, Transportation Science, 33 (1999), pp. 315–329.

[97] E. MOTA, V. CAMPOS, AND A. CORBERÁN, *A new metaheuristic for the vehicle routing problem with split demands*, in Evolutionary Computation in Combinatorial Optimization, C. Cotta and J. van Hemert, eds., vol. 4446 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2007, pp. 121–129.

[98] M. MOURGAYA AND F. VANDERBECK, *The periodic vehicle routing problem: Classification and heuristic*, RAIRO - Operations Research, 40 (2006), pp. 169–194.

[99] ——, *Column generation based heuristic for tactical planning in multi-period vehicle routing*, European Journal of Operational Research, 183 (2007), pp. 1028–1041.

[100] G. NAGY AND S. SALHI, *Heuristic algorithms for single and multiple depot vehicle routing problems with pickups and deliveries*, European Journal of Operational Research, 162 (2005), pp. 126–141.

[101] Z. NAJI-AZIMI AND M. SALARI, *A complementary tool to enhance the effectiveness of existing methods for heterogeneous fixed fleet vehicle routing problem*, Applied Mathematical Modelling, 37 (2013), pp. 4316–4324.

[102] P. K. NGUYEN, T. G. CRAINIC, AND M. TOULOUSE, *A hybrid genetic algorithm for the periodic vehicle routing problem with time windows*, Technical Report 2011-25, CIRRELT, Montréal, Canada, 2011.

[103] T. NUORTIO, J. KYTÖJOKI, H. NISKA, AND O. BRÄYSY, *Improved route planning and scheduling of waste collection and transport*, Expert Systems with Applications, 30 (2006), pp. 223–232.

[104] J. OPPEN, A. LØKKETANGEN, AND J. DESROSIERS, *Solving a rich vehicle routing and inventory problem using column generation*, Computers & Operations Research, 37 (2010), pp. 1308–1317.

[105] I. OR AND W. P. PIERSKALLA, *A transportation location-allocation model for regional blood banking*, IIE Transactions, 11 (1979), pp. 86–95.

[106] I. H. OSMAN AND N. A. WASSAN, *A reactive tabu search meta-heuristic for the vehicle routing problem with back-hauls*, Journal of Scheduling, 5 (2002), pp. 263–285.

[107] J. PACHECO, A. ALVAREZ, I. GARCIA, AND F. ANGEL-BELLO, *Optimizing vehicle routes in a bakery company allowing flexibility in delivery dates*, Journal of the Operational Research Society, 63 (2012), pp. 569–581.

[108] D. C. PARASKEVOPOULOS, P. P. REPOUSSIS, C. D. TARANTILIS, G. IOANNOU, AND G. P. PRASTACOS, *A reactive variable neighborhood tabu search for the heterogeneous fleet vehicle routing problem with time windows*, Journal of Heuristics, 14 (2007), pp. 247–254.

[109] P. PARTHANADEE AND R. LOGENDRAN, *Periodic product distribution from multi-depots under limited supplies*, IIE Transactions, 38 (2006), pp. 1009–1026.

[110] A. PESSOA, M. POGGI, AND E. UCHOA, *A Robust Branch-Cut-and-Price Algorithm for the Heterogeneous Fleet Vehicle Routing Problem*, vol. 4525 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2007, pp. 150–160.

[111] S. PIRKWIESER AND G. R. RAIDL, *A variable neighborhood search for the periodic vehicle routing problem with time windows*, in Proceedings of the 9th EU/Meeting on Metaheuristics for Logistics and Vehicle Routing, Troyes, France, 2008.

[112] ——, *Multiple variable neighborhood search enriched with ILP techniques for the periodic vehicle routing problem with time windows*, in Hybrid Metaheuristics, M. J. Blesa, C. Blum, L. Gaspero, A. Roli, M. Sampels, and A. Schaerf, eds., vol. 5818 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2009, pp. 45–59.

[113] ——, *Boosting a variable neighborhood search for the periodic vehicle routing problem with time windows by ILP techniques*, in Proceedings of the 8th Metaheuristic International Conference (MIC 2009), S. Voss and M. Caserta, eds., Hamburg, Germany, 2009.

[114] ——, *A column generation approach for the periodic vehicle routing problem with time windows.*, in Proceedings of the International Network Optimization Conference 2009, Pisa, Italy, 2009.

[115] ——, *Multilevel variable neighborhood search for periodic routing problems*, in Evolutionary Computation in Combinatorial Optimization, P. Cowling and P. Merz, eds., vol. 6022 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2010, pp. 226–238.

[116] ——, *Matheuristics for the periodic vehicle routing problem with time windows*, in Proceedings of Matheuristics, Vienna, Austria, 2010.

[117] C. PRINS, *Two memetic algorithms for heterogeneous fleet vehicle routing problems*, Engineering Applications of Artificial Intelligence, 22 (2009), pp. 916–928.

[118] A. RAHIMI-VAHED, T. G. CRAINIC, M. GENDREAU, AND W. REI, *A path relinking algorithm for a multi-depot periodic vehicle routing problem*, Journal of Heuristics, 19 (2013), pp. 497–524.

[119] P. P. REPOUSSIS AND C. D. TARANTILIS, *Solving the fleet size and mix vehicle routing problem with time windows via adaptive memory programming*, Transportation Research Part C: Emerging Technologies, 18 (2010), pp. 695–712.

[120] D. RONEN AND C. A. GOODHART, *Tactical store delivery planning*, Journal of the Operational Research Society, 59 (2007), pp. 1047–1054.

[121] S. ROPKE AND D. PISINGER, *A unified heuristic for a large class of vehicle routing problems with backhauls*, European Journal of Operational Research, 171 (2006), pp. 750–775.

[122] R. A. RUSSELL AND D. GRIBBIN, *A multiphase approach to the period routing problem*, Networks, 21 (1991), pp. 747–765.

[123] R. A. RUSSELL AND W. IGO, *An assignment routing problem*, Networks, 9 (1979), pp. 1–17.

[124] M. SALANI AND I. VACCA, *Branch and price for the vehicle routing problem with discrete split deliveries and time windows*, European Journal of Operational Research, 213 (2011), pp. 470–477.

[125] S. SALHI AND G. NAGY, *A cluster insertion heuristic for single and multiple depot vehicle routing problems with backhauling*, Journal of the Operational Research Society, 50 (1999), pp. 1034–1042.

[126] S. SALHI AND M. SARI, *A multi-level composite heuristic for the multi-depot vehicle fleet mix problem*, European Journal of Operational Research, 103 (1997), pp. 95–112.

[127] M. W. P. SAVELSBERGH, *The vehicle routing problem with time windows: Minimizing route duration*, ORSA Journal on Computing, 4 (1992), pp. 146–154.

[128] Y. SHAO, J. F. BARD, AND A. I. JARRAH, *The therapist routing and scheduling problem*, IIE Transactions, 44 (2012), pp. 868–893.

[129] K. SMILOWITZ, M. NOWAK, AND T. JIANG, *Workforce management in periodic delivery operations*, Transportation Science, 47 (2013), pp. 214–230.

[130] M. M. SOLOMON, *Algorithms for the vehicle routing and scheduling problems with time window constraints*, Operations Research, 35 (1987), pp. 254–265.

[131] A. SUBRAMANIAN, P. H. V. PENNA, E. UCHOA, AND L. S. OCHI, *A hybrid algorithm for the heterogeneous fleet vehicle routing problem*, European Journal of Operational Research, 221 (2012), pp. 285–295.

[132] É. D. TAILLARD, *A heuristic column generation method for the heterogeneous fleet VRP*, RAIRO - Recherche Opérationnelle, 33 (1999), pp. 1–14.

[133] É. D. TAILLARD, P. BADEAU, M. GENDREAU, F. GUERTIN, AND J.-Y. POTVIN, *A tabu search heuristic for the vehicle routing problem with soft time windows*, Transportation Science, 31 (1997), pp. 170–186.

[134] C. C. R. TAN AND J. E. BEASLEY, *A heuristic algorithm for the period vehicle routing problem*, Omega, 12 (1984), pp. 497–504.

[135] C. D. TARANTILIS AND C. T. KIRANOUDIS, *A flexible adaptive memory-based algorithm for real-life transportation operations: Two case studies from dairy and construction sector*, European Journal of Operational Research, 179 (2007), pp. 806–822.

[136] R. TAVAKKOLI-MOGHADDAM, A. R. SAREMI, AND M. S. ZIAEE, *A memetic algorithm for a vehicle routing problem with backhauls*, Applied Mathematics and Computation, 181 (2006), pp. 1049–1060.

[137] S. R. THANGIAH, J.-Y. POTVIN, AND T. SUN, *Heuristic approaches to vehicle routing with backhauls and time windows*, Computers & Operations Research, 23 (1996), pp. 1043–1057.

[138] P. TOTH AND D. VIGO, *Heuristic algorithms for the vehicle routing problem with backhauls*, in Advanced Methods in Transportation Analysis, L. Bianco and P. Toth, eds., Springer-Verlag, Berlin, 1996, pp. 585–608.

[139] ——, *An exact algorithm for the vehicle routing problem with backhauls*, Transportation Science, 31 (1997), pp. 372–385.

[140] ——, *A heuristic algorithm for the symmetric and asymmetric vehicle routing problems with backhauls*, European Journal of Operational Research, 113 (1999), pp. 528–543.

[141] ——, *VRP with backhauls*, in The Vehicle Routing Problem, P. Toth and D. Vigo, eds., SIAM, Philadelphia, 2002, ch. 8, pp. 195–224.

[142] ——, *The granular tabu search and its application to the vehicle-routing problem*, INFORMS Journal on Computing, 15 (2003), pp. 333–346.

[143] Y. TÜTÜNCÜ, *An interactive GRAMPS algorithm for the heterogeneous fixed fleet vehicle routing problem with and without backhauls*, European Journal of Operational Research, 201 (2010), pp. 593–600.

[144] T. VIDAL, T. G. CRAINIC, M. GENDREAU, N. LAHRICHI, AND W. REI, *A hybrid genetic algorithm for multidepot and periodic vehicle routing problems*, Operations Research, 60 (2012), pp. 611–624.

[145] T. VIDAL, T. G. CRAINIC, M. GENDREAU, AND C. PRINS, *A unified solution framework for multi-attribute vehicle routing problems*, Technical Report 2012-23, CIRRELT, Montréal, Canada, 2012.

[146] ——, *A hybrid genetic algorithm with adaptive diversity management for a large class of vehicle routing problems with time-windows*, Computers & Operations Research, 40 (2013), pp. 475–489.

[147] A. C. WADE AND S. SALHI, *An investigation into a new class of vehicle routing problem with backhauls*, Omega, 30 (2002), pp. 479–487.

[148] Z. WANG AND Z. WANG, *A novel two-phase heuristic method for vehicle routing problem with backhauls*, Computers & Mathematics with Applications, 57 (2009), pp. 1923–1928.

[149] N. A. WASSAN, *Reactive tabu adaptive memory programming search for the vehicle routing problem with backhauls*, Journal of the Operational Research Society, 58 (2007), pp. 1630–1641.

[150] J. H. WILCK IV AND T. M. CAVALIER, *A construction heuristic for the split delivery vehicle routing problem*, American Journal of Operations Research, 2 (2012), pp. 153–162.

[151] ——, *A genetic algorithm for the split delivery vehicle routing problem*, American Journal of Operations Research, 2 (2012), pp. 207–216.

[152] Y. XIONG, D. GULCZYNSKI, D. KLEITMAN, B. L. GOLDEN, AND E. A. WASIL, *A worst-case analysis for the split delivery vehicle routing problem with minimum delivery amounts*, Optimization Letters, 7 (2013), pp. 1597–1609.

[153] B. YU AND Z. Z. YANG, *An ant colony optimization model: The period vehicle routing problem with time windows*, Transportation Research Part E: Logistics and Transportation Review, 47 (2011), pp. 166–181.

[154] E. E. ZACHARIADIS AND C. T. KIRANOUDIS, *An effective local search approach for the vehicle routing problem with backhauls*, Expert Systems with Applications, 39 (2012), pp. 3174–3184.

[155] Y. ZHONG AND M. H. COLE, *A vehicle routing problem with backhauls and time windows: A guided local search solution*, Transportation Research Part E: Logistics and Transportation Review, 41 (2005), pp. 131–144.