Chapter 8

Stochastic Vehicle Routing Problems

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8.1 ▪ Introduction

Vehicle routing problems (VRPs) have been the subject of numerous research studies since Dantzig and Ramser [16] first presented this general class of optimization problems in a practical setting. Since then, the operations research community has devoted collectively a large effort towards efficiently solving these problems, developing both exact and heuristic methods; see Laporte [40]. The majority of these studies have been conducted under the assumption that all the information necessary to formulate the problems is known and readily available (i.e., one is in a deterministic setting). In practical applications, this assumption is usually not verified given the presence of uncertainty affecting the parameters of the problem. Uncertainty may come from different sources, both from expected variations and unexpected events. Such variations can affect various aspects of the problem under study (e.g., stochastic parameters, which entail additional feasibility requirements and extra costs). This is particularly true in the case of VRP models, which are used both at the tactical level and at the operational level to plan and control logistical operations. In this case, uncertainty is present given the time lag separating the moments where routes are planned and executed, considering the informational flow that defines the problem. For example, if customer demands are uncertain and only revealed when customers are visited, a planned route performed by a vehicle on a given day may turn out to be infeasible if the total observed demand for the customers scheduled on the route exceeds the capacity of the vehicle. When such a situation occurs, additional costs often involving additional decisions must be taken to produce a feasible solution. The need to account for such extra costs when solving VRPs entails developing models that explicitly factor in all relevant uncertainty features of the problem.

Therefore, in recent years, there has been a steadily growing interest in both formulating and solving Stochastic Vehicle Routing Problems (SVRPs); see the surveys of Gendreau, Laporte, and Séguin [25] and Cordeau et al. [15]. This interest has been further spurred by the fact that a SVRP cannot simply be solved using a deterministic approximation model. Louveaux [51] showed that deterministic VRP models may produce arbitrarily bad solutions when used to approximate SVRP. Thus, there exists a need for models that specifically produce solutions that are cost-effective in an uncertain environment. To do so, different modeling paradigms are available.

This chapter focuses on models defined using the stochastic programming paradigm; see Birge and Louveaux [10] for a thorough presentation of this field. Using this paradigm, uncertainty is formulated in optimization problems by introducing stochastic parameters in the models. In the case of the VRP, three sets of parameters are commonly considered to be stochastic:

1. stochastic demands: the product volumes to either be collected or delivered at customers are random;

2. stochastic customers: customers are either present or absent with a given probability;

3. stochastic times: both the service times at customers and the traveling times for the vehicles can be considered stochastic.

As is usually done, such parameters are formulated using suitably defined random variables or through the use of scenarios. The stochastic optimization models are then obtained by first determining the informational process that defines how and when the values of the stochastic parameters are observed. Based on this process, the decision variables are defined in stages, according to when parameters become known. Therefore, the a priori decisions group all decisions taken in the first stage, before any stochastic parameters are observed. The *recourse* decisions refer to the decisions taken in the second stage and onwards that detail how solutions are modified or adjusted as more information becomes available. It should be noted that the recourse decisions, and their associated costs, are directly related to the outcomes of the stochastic parameters.

Once the decision variables are determined, there exist two general modeling approaches to formulate SVRPs using the considered paradigm. The first is based on the use of a *recourse function*, which is defined as the average recourse cost for a given a priori solution. An SVRP model can then be formulated by considering the recourse function in the objective of the problem. The optimal solution to such a model would minimize the expected total cost. The second approach requires the inclusion of *probabilistic constraints* in the model. These constraints impose specified limits on the probabilities associated with particular random events. In the context of SVRPs, such constraints may, for example, take the form of a limit on the probability of observing either that the a priori solution is infeasible or that the value of the recourse function becomes higher than a specified threshold. Such models are used to obtain solutions that guarantee that some risks (defined here as the probabilities of observing specific random events) are limited.

It is important to note the relationship between SVRPs and *dynamic vehicle routing problems*, which cover the broader class of routing problems in which information about problem data becomes available over time. Models and solution approaches for dynamic VRPs are discussed in detail in Chapter 11 of this book.

Given that SVRP models may be quite different from one another, depending on the considered stochastic parameters and on how recourse decisions are defined, providing a unified presentation of all solution techniques proposed for all models appears as a difficult task. Instead, we have opted for a more specialized approach. Therefore, we focus this presentation on the VRP with Stochastic Demands (VRPSD) and simple recourse.

Simple recourse implies that once the vehicle capacity is exceeded a return trip to the depot is performed and the capacity of the vehicle is restored. In the VRPSD, it is usually assumed that the demand of each customer is only revealed when the vehicle arrives at the customer's location, but other schemes for the revelation of demands may be envisioned. Such problems occur in a number of applications, e.g., in the delivery and collection of money to and from banks (Bertsimas [7] and Lambert, Laporte, and Louveaux [39]), in beer distribution and garbage collection (Yang, Mathur, and Ballou [71]), and in the home delivery of oil (Chepuri and Homem de Mello [11]).

In Sections 8.2 to 8.4, we present the main modeling paradigms that have been proposed to formulate the VRPSD and we detail some of the traditional exact solutions methods that have been developed for the problem. Section 8.2 is devoted to the a priori paradigm. In Section 8.3, we describe the *reoptimization* paradigm. Finally, in Section 8.4, a modeling approach based on the use of *chance*, or probabilistic, *constraints*, which control the level of risk accepted by the decision maker, is presented. In doing so, our aim is to provide a general illustration of the differences between the solution strategies for SVRP and their deterministic VRP counterparts.

We later devote a section to each of the previously mentioned stochastic parameters: stochastic customers demands (Section 8.5), stochastic customers (Section 8.6), and stochastic times (Section 8.7). Some problems include two types of stochasticity, e.g., stochastic travel times and demands. In such cases, the paper is included in the section which better matched the paper's main scientific contributions. Finally, we conclude in Section 8.8 by discussing future research directions in this field.

8.2 ▪ A Priori Optimization

One of the most common solution frameworks for stochastic routing problems is a priori optimization, a paradigm initially put forward by Bertsimas [6], Jaillet [35], and Bertsimas, Jaillet, and Odoni [9]. It consists of modeling the problem in two stages. In the first stage, a planned, or a priori, solution is designed. In the second stage, the first stage solution is executed while uncertainties are gradually revealed and recourse actions are taken based on a predetermined policy. As previously mentioned, in the VRPSD, customer demands are revealed upon arrival at customer locations. As a result, a vehicle may reach a customer and not have sufficient capacity to serve its realized demand. Such a situation is referred to as a *failure*. Several recourse policies have been proposed for the VRPSD. A *classical* policy is to return to the depot upon failure, offload, and resume collections by following the planned route starting at the point of failure. Several authors have studied this policy, e.g., Christiansen and Lysgaard [12], Gendreau, Laporte, and Séguin [23, 24], Goodson, Ohlmann, and Thomas [27], Hjorring and Holt [29], Laporte, Louveaux, and Van hamme [42], and Lei, Laporte, and Guo [47]. An important managerial advantage of the classical policy is that it yields stable routes which require little alteration in the event of failure. Furthermore, solving the VRPSD under the classical recourse policy provides a benchmark against which alternative policies can be assessed.

Under the a priori paradigm, the aim of the first-stage problem is to design a set of vehicle routes of least total expected cost. The routes must satisfy the following three conditions: (1) each route must start and end at the depot, (2) each customer is visited exactly once by exactly one vehicle, and (3) the expected demand of each route does not exceed the vehicle capacity. The latter constraint was originally imposed by Laporte, Louveaux, and Van hamme [42] in order to avoid the creation of routes that would systematically fail. The objective function is the sum of the planned routing cost and of the expected recourse cost.

We now present the notation pertaining to this section. The VRPSD is defined on a complete undirected graph $G=(V,E)$, where $V=\{0,1,\dots,n\}$ is the vertex set and $E=\{(i,j): i,j\in V,\ i<j\}$ is the edge set. Vertex 0 is the depot at which a set K of identical vehicles of capacity Q are based, whereas the remaining vertices represent customers. Each customer $i\in N=V\setminus\{0\}$ has a non-negative stochastic demand ξ_i to be collected. We further assume that these demands are independent random variables with known distributions with expected values μ_i. A travel cost c_{ij} is associated with each edge $(i,j)\in E$. In what follows we present two a priori optimization models for the VRPSD. In Section 8.2.1, we present the network flow formulation, which was put forward by Laporte, Louveaux, and Van hamme [42]. In Section 8.2.2, we present the set partitioning formulation as described by Christiansen and Lysgaard [12]. For each of these formulations, we present the model, an outline of the proposed solution procedure in the corresponding paper, and a summary of the computational results.

8.2.1 ▪ Network Flow Formulation

Let x_{ij} $(i<j)$ be an integer decision variable equal to the number of times edge (i,j) appears in the first-stage solution. The variable x_{ij} must be interpreted as x_{ji} whenever $i>j$. If $i,j>0$, then x_{ij} can only take the values 0 or 1; if $i=0$, then x_{ij} can also be equal to 2, representing a situation when a vehicle makes a return trip between the depot and vertex j. Furthermore, let $\mathcal{Q}(x)$ denote the expected recourse cost of solution x.

The model is then

$$\text{(8.1)}\quad \text{(VRPSD1)}\quad \text{minimize} \sum_{i<j} c_{ij}x_{ij} + \mathcal{Q}(x)$$

$$\text{(8.2)}\quad \text{s.t.}\quad \sum_{j=1}^{n} x_{0j} = 2|K|,$$

$$\text{(8.3)}\quad \sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad \forall k\in N,$$

$$\text{(8.4)}\quad \sum_{i,j\in S} x_{ij} \le |S| - \left\lceil \sum_{i\in S}\mu_i/Q \right\rceil \qquad \forall S\subset N, 3\le |S|\le n-1,$$

$$\text{(8.5)}\quad 0\le x_{ij}\le 1 \qquad \forall i,j\in N, i<j,$$

$$\text{(8.6)}\quad 0\le x_{0j}\le 2 \qquad \forall j\in N,$$

$$\text{(8.7)}\quad x=(x_{ij}) \text{ integer} \qquad \forall i,j\in V, i<j.$$

The objective function (8.1) consists of minimizing the first-stage travel costs and the expected recourse cost. In this model, constraints (8.2) and (8.3) specify the degree of each vertex, whereas constraints (8.4) eliminate subtours and ensure that the expected demand of any route does not exceed the vehicle capacity.

Given a first-stage solution x, the computation of $\mathcal{Q}(x)$ is separable with the routes. The expected cost of route k depends on its orientation, and thus $\mathcal{Q}(x)$ is expressed as

$$\text{(8.8)}\quad \mathcal{Q}(x) = \sum_{k=1}^{|K|} \min\{\mathcal{Q}^{k,1}, \mathcal{Q}^{k,2}\},$$

where $\mathcal{Q}^{k,\delta}$ denotes the expected recourse cost of route k for orientation δ. Equation (8.8) implies that, given an undirected first-stage solution, the expected recourse cost of each route is computed for each direction, and the cheapest orientation is selected. The computation of $\mathscr{Q}^{k,1}$ for route k defined by $(i_1 = 0, i_2, \ldots, i_{t+1} = 0)$, assuming that $\xi_i \leq Q$ with probability 1 for all i, is given by

$$\mathscr{Q}^{k,1} = 2 \sum_{j=2}^{t} \sum_{l=1}^{j-1} P\Big(\sum_{s=2}^{j-1} \xi_{i_s} \leq lQ < \sum_{s=2}^{j} \xi_{i_s} \Big) c_{0 i_j}. \tag{8.9}$$

In the event of a failure, the vehicle performs a return trip to the depot. Therefore, the first factor in the double summation of (8.9) is the probability of incurring the lth failure at customer i_j. This is multiplied by the cost of the return trip to the depot. The value of $\mathscr{Q}^{k,2}$ is computed likewise by reversing the orientation of route k.

If arbitrary distributions are used for demands, the computation of the probabilities appearing in (8.9) becomes rapidly intractable. Therefore, in most applications, the demand probability distributions adhere to the cumulative property; i.e., the sum of two or more independent random variables with a distribution Ψ yields a random variable with distribution Ψ (albeit with a different mean). A number of well-known probability distributions possess the cumulative property, e.g., normal and Poisson distributions. Therefore, this assumption is not overly restrictive.

The described formulation is solved by the integer L-shaped method proposed by Laporte and Louveaux [41]. This is an extension of the L-shaped method of Van Slyke and Wets [68] for continuous stochastic programs, which is an application of Benders' decomposition [4] to stochastic programming. In what follows, we describe the integer L-shaped algorithm, as applied to the VRPSD.

The integer *L*-Shaped Algorithm. The integer L-shaped algorithm applies Branch-and-Cut to a relaxation for the VRPSD in which the recourse term $\mathscr{Q}(x)$ is bounded below by a variable Θ. In addition, the subtour elimination constraints and the integrality requirements are relaxed. Initially, Θ is set equal to a lower bound L on the expected cost of recourse, and its value Θ^ν is computed for the solution of each subproblem solved at iteration ν. As is standard in Branch-and-Cut, violated subtour elimination constraints are generated dynamically as they are found to be violated, and integrality is gradually recovered by branching. Optimality cuts are generated at feasible integer solutions. In most applications these cuts are local, and it often pays to also impose lower bounds on the recourse function $\mathscr{Q}(x)$, in the form of linear functionals computed on the basis of infeasible intermediate solutions. In the following summary of the algorithm, CP denotes the current problem.

Step 0 Compute L, and set the iteration counter ν equal to 0. Define the CP as a relaxation of VRPSD in which constraints (8.4) and (8.7) are removed, the $\mathscr{Q}(x)$ term of the objective function is replaced with Θ, and the constraint $\Theta \geq L$ is imposed. Set the value of the best known solution to $\bar{z} := \infty$. At this stage, the only pendent node is the initial CP.

Step 1 Select a pendent node from the list. If none exists, stop.

Step 2 Set $\nu := \nu + 1$, and solve CP. Let (x^ν, Θ) be its optimal solution.

Step 3 Check for any violated subtour elimination constraints, and generate them accordingly. At this stage, valid inequalities or Lower Bounding Functionals (LBFs) may also be generated. If a violated constraint is found, add it to the CP and return to Step 2. Otherwise, if $cx^\nu + \Theta \geq \bar{z}$, fathom the current node and return to Step 1.

Step 4 If the solution is not integer, then branch on a fractional variable. Append the corresponding subproblems to the list of pendent nodes, and return to Step 1.

Step 5 Compute $Q(x^\nu)$, and set $z^\nu := cx^\nu + Q(x^\nu)$. If $z^\nu < \bar{z}$, then set $\bar{z} := z^\nu$.

Step 6 If $\Theta \geq Q(x^\nu)$, then fathom the current node and return to Step 1. Otherwise, add an optimality cut defined as

$$\sum_{\substack{0<i<j \\ x_{ij}^\nu=1}} x_{ij} \leq \sum_{0<i<j} x_{ij}^\nu - 1 \tag{8.10}$$

and go to Step 2.

A general lower bound L on $\mathcal{Q}(x)$ is described in Proposition 1 of Laporte, Louveaux, and Van hamme [42]. This bound is based on the computation of the probability of failure on each route taken separately. The recourse cost is bounded below by considering the $|K|$ customers closest to the depot and by partitioning the total demand among the $|K|$ vehicles so as to minimize the total cost.

LBFs based on partial routes were first proposed by Hjorring and Holt [29] for the single-vehicle case. A partial route implies a solution which starts and ends at the depot while containing a set of *connected yet not necessarily sequenced customers*. The recourse associated with a partial route is bounded by treating the set of connected yet not necessarily sequenced customers as a single customer, whose demand is equivalent to the sum of the demands of its customers and is distributed accordingly. Furthermore, one associates with this single customer a distance from the depot which is that of the closest customer to the depot, out of the set of connected yet not necessarily sequenced customers. Partial route-based LBFs were then proposed by Laporte, Louveaux, and Van hamme [42] for the multi-vehicle case. Jabali et al. [32] expanded the definition of the LBFs in a way that better exploits the structural information provided by partial routes.

In the context of the VRPSD, Laporte, Louveaux, and Van hamme [42] found that using the normal distribution to model customer demands is more challenging than using the Poisson distribution. We now present a summary of the best results obtained by Jabali et al. [32], where customer demands ξ_i follow a normal distribution $\mathcal{N}(\mu_i, \sigma_i)$ truncated at zero, and all demands are independently distributed. The instances were generated based on the same principles as in Laporte, Louveaux, and Van hamme [42]. Table 8.1 summarizes the results of the 30 instances that were generated for each combination of the number of vertices and $|K|$, for a total of 270 instances. For each combination, several

Table 8.1. *Results for the integer L-shaped algorithm.*

$\lvert V\rvert$	$\lvert K\rvert$	Solved	CPU (min)	%Gap
60	2	24	23	0.3
70	2	17	43	0.5
80	2	13	30	0.5
50	3	16	115	0.7
60	3	6	46	0.7
70	3	9	29	1.5
40	4	9	21	1.5
50	4	5	23	1.9
60	4	3	82	2.0

fill rates are included, but these details are omitted for summary purposes. In Table 8.1, the columns “Solved”, “CPU (min)”, and “%Gap”, respectively, refer to the number of instances solved to optimality, the average CPU times of the algorithm on these instances, and the average gap obtained by the algorithm over all instances of each category. The computation time limit for any given instance was set to 5 hours. The coefficient of variation of the demand distribution was set equal to 30%. Each instance was solved on one out of two Intel(R) Xeon(R) CPU X5675 3.07 GHz processors of a machine with 96 GB of RAM.

The results indicate that the algorithm is able to efficiently solve instances of relatively large sizes, e.g., up to 80 vertices with two vehicles. However, the efficiency of the algorithm deteriorates with the number of vehicles. For instance, the algorithm was able to solve 24 out of 30 instances with 60 vertices and two vehicles, while it was only able to solve three out of 30 instances with 60 vertices and four vehicles.

8.2.2 ▪ The Set Partitioning Formulation

The set partitioning formulation of the VRPSD relies on path-based description of vehicle flows in the graph previously defined. We therefore define a *route* as a path $(0, z_1, \dots, z_j, 0)$, where $z_1, \dots, z_j \in N$ and $z_h \neq z_{h+1}$ for $h = 1, \dots, j-1$. We say that a route is elementary if $z_1, \dots, z_j$ are different; otherwise we say that the route is *non-elementary*. A route is said to be feasible if $\sum_{h=1}^{j} \mu_{z_h} \leq Q$. For each feasible and elementary route r, let c_r denote its expected cost. Let Ω_e denote the set of all feasible and elementary routes. Furthermore, let a_{ir} be a parameter equal to 1 if route r visits customer i and to 0 otherwise. Finally, let λ_r be a variable which takes the value 1 if route r is chosen and 0 otherwise. The set partitioning formulation for the VRPSD can then be written as

$$\text{(8.11)} \qquad \text{(VRPSD2)} \quad \text{minimize} \sum_{r \in \Omega_e} c_r \lambda_r$$

$$\text{(8.12)} \qquad \text{s.t.} \quad \sum_{r \in \Omega_e} a_{ir} \lambda_r = 1 \qquad \forall i \in N,$$

$$\text{(8.13)} \qquad \lambda_r \in \{0, 1\} \qquad \forall r \in \Omega_e.$$

The objective function (8.11) minimizes the total expected distribution cost. Constraints (8.12) guarantee that each customer is visited exactly once by one vehicle. Constraints (8.13) ensure that the decision variables are binary.

Assuming each customer’s demand is only revealed upon reaching it, the total expected cost c_r of a route can be decomposed into two elements. The first element is the deterministic travel cost of the route, while the second is the recourse cost. Since the former is straightforward to compute, we focus on the computation of the latter. Given that each ξ_i follows an independent probability distribution, Christiansen and Lysgaard [12] showed that the probability that the total demand along a path $\{0, z_1, \dots, z_j\}$ is less than lQ is only dependent on the total demand along this path. This implies that this probability is independent of the demand allocation along the path. This observation is used to compute the recourse cost.

Given an elementary path $\{0, z_1, \dots, z_j\}$, let μ and σ denote for this path the total demand expected value and standard deviation, respectively. Let $F(\mu, \sigma^2, lQ) = P(\sum_{h=1}^{j} \xi_{z_h} \leq lQ)$ define the probability that the cumulative demand at customer z_j is less than or equal to lQ. Furthermore, let σ_{z_j} denote the standard deviation of the demand of customer z_j.

The expected recourse cost for customer z_j is then computed as

$$\text{ERC}(\mu,\sigma^2,z_j) = 2c_{0z_j}\sum_{l=1}^{\infty} F(\mu-\mu_{z_j},\sigma^2-\sigma^2_{z_j},lQ) - F(\mu,\sigma^2,lQ). \tag{8.14}$$

Equation (8.14) is approximated by replacing ∞ with a sufficiently large number. The summation provides the expected number of failures at customer z_j. Therefore, the above computation expresses the total expected failure cost caused by customers along the path. This is used to establish a dominance criterion among paths characterized by (μ,σ^2,z_j) and the set of customers visited by each path.

The Branch-and-Price Algorithm. The solution procedure is based on the Dantzig–Wolfe decomposition and column generation. The master problem considered at any given iteration is established by (i) considering only a subset, denoted by Ω, of the set of all feasible routes, which is enlarged by allowing non-elementary routes; (ii) relaxing the integrality constraints (8.13); (iii) changing the set partitioning constraints to set covering constraints (8.12); and (iv) introducing the coefficient a_{ir}, which denotes the number of times that customer i is visited by route r. The master problem is then

$$\text{(VRPSD2 } M_P) \quad \text{minimize} \sum_{r\in\Omega} c_r \lambda_r \tag{8.15}$$

$$\text{s.t.} \quad \sum_{r\in\Omega} a_{ir}\lambda_r \geq 1 \qquad \forall i \in N, \tag{8.16}$$

$$\lambda_r \geq 0 \qquad \forall r \in \Omega. \tag{8.17}$$

The set Ω contains feasible non-elementary routes without 2-cycles (i,j,i). Note that the computation of the expected failure cost is not affected by allowing non-elementary routes. Following the usual principles of column generation, the dual prices $(\pi_1,\ldots,\pi_n)$ obtained from solving M_P are used in a subproblem in the search for one or more routes (columns) with negative reduced costs. If such columns are found, they are added to the master problem, which is then reoptimized. The steps of column generation are repeated until no columns with negative reduced costs can be identified. At this point the current solution is the optimal solution for the master problem.

If the master problem solution is integer and constraints (8.16) are satisfied with equality, then the current solution is optimal for the original problem. If the current solution is fractional, branching is then performed. In the following subsection, we describe the column generation procedure proposed by Christiansen and Lysgaard [12].

The column generation subproblem is solved by dynamic programming. The authors assume that the expected demand and variance of each customer have integer values. Furthermore, an upper bound $V_{\max}$ on the total variance on any feasible elementary route is established by solving a 0-1 knapsack problem.

In order to solve the column generation subproblem, the authors use graph $G_S = (V_S, A_S)$, where V_S has $(n+1)QV_{\max}+1$ vertices. Except for source vertex $v(0,0,0)$, each vertex is denoted by $v(\mu,\sigma^2,i)$ for $\mu = 1,\ldots,Q$, $\sigma^2 = 1,\ldots,V_{\max}$, and $i = 0,\ldots,n$. Each vertex represents a particular set of paths from 0 to i, with a total expected demand of μ and a total variance of σ^2. The arc set A_S is constructed as follows:

Step 1 $A_S := \emptyset$.

Step 2 For $i = 1, \ldots, n$, add an arc from $v(0,0,0)$ to $v(\mu_i, \sigma_i^2, i)$ and set its cost to $c_{0i} + \mathrm{ERC}(\mu_i, \sigma_i^2, i) - \pi_i$.

Step 3 For each ordered pair $i, j \in N$, $i \neq j$, each $\mu = 1, \ldots, Q-1$, and $\sigma^2 = 1, \ldots, V_{\max}$, add an arc from $v(\mu, \sigma^2, j)$ to $v(\mu + \mu_i, \sigma^2 + \sigma_i^2, i)$ (provided that $\mu + \mu_i \leq Q$ and $\sigma^2 + \sigma_i^2 \leq V_{\max}$) and set its cost to $c_{ij} + \mathrm{ERC}(\mu + \mu_i, \sigma^2 + \sigma_i^2, i) - \pi_i$.

Step 4 For each $\mu = 1, \ldots, Q$, each $\sigma^2 = 1, \ldots, V_{\max}$, and each $j = 1, \ldots, n$, add an arc from $v(\mu, \sigma^2, j)$ to $v(\mu, \sigma^2, 0)$ and set its cost to c_{j0}.

The shortest path in G_S from $v(0,0,0)$ to $v(\mu, \sigma^2, 0)$, for all $\mu = 1, \ldots, Q$ and $\sigma^2 = 1, \ldots, V_{\max}$, is the route of least reduced cost among routes with total demand μ and total variance equaling σ^2. Therefore, this shortest path solves the column generation subproblem. This can be done in $O(n^2 Q V_{max})$ time and also in the case where 2-cycles are prohibited.

The algorithm was tested under the assumption that demands follow Poisson distributions. Since this assumption implies that $\mu = \sigma^2$, the number of vertices in G_S reduces to $(n+1)Q+1$. The considered test instances were derived from the instances of Augerat et al. [2] and of Christofides and Eilon [13] having at most 60 customers. The expected demand of each customer was set to the deterministic value of the demand in the original instance. Computational results showed that the proposed approach could solve 18 of the 40 benchmark instances in running times up to 20 minutes on a Pentium Centrino 1500MHz computer with 480MB of RAM. In general, the algorithm yielded better results on instances with tight capacity constraints.

Recently, Gauvin, Desaulniers, and Gendreau [22] reimplemented the Christiansen and Lysgaard method using the most recent techniques for solving the column generation subproblem, including bidirectional labeling, a combination of 2-cycle elimination with ng-routes, and the application of a tabu search heuristic. They also introduced a new aggregate dominance rule and added capacity and subset-row inequalities dynamically in order to strengthen the linear relaxation of the master problem. Their computational results show that their algorithm is much more effective than the Christiansen and Lysgaard original code: it solves 38 out of the 40 instances of the benchmark set, compared to 18, and CPU times for previously closed instances are in general much smaller. For example, an instance with 60 customers and 15 vehicles was solved in less than 10 seconds. These computational results are summarized in Table 8.2, where the results are aggregated according to the number of customers in the instances: instances ranging from 16 to 50 customers (Category 16-50) and from 51 to 60 customers (Category 51-60). In Table 8.2, the columns “Max Veh”, “Number”, and “Solved”, respectively, refer to the maximum number of vehicles used in the obtained solution, the number of instances in the category, and the number of these instances that were solved, while the columns “B&B nodes”, “cuts”, and “CPU(s)” are, respectively, the average number of nodes in the search tress, cuts added

Table 8.2. *Results for the Branch-and-Price algorithm.*

Category	Max Veh	Number	Solved	B&B nodes	Cuts	CPU(s)
16-50	10	29	29	2.0	77.0	70.4
51-60	15	11	9	10.2	165.5	160.1
16-60	15	40	38	4.3	101.6	91.7

by the algorithm, and average computational times over all instances. Experiments were performed on a Intel i7-2600 processor with 3.4 GHz and 16 GB of RAM.

It is interesting to recall that the integer L-shaped algorithm could solve within reasonable CPU time instances with 80 vertices and two vehicles, as well as some instances with 60 vertices and four vehicles. These results together with those just presented indicate that the Branch-and-Price algorithm works well on instances with a large number of vehicles, while the integer L-shaped algorithm performs effectively on instances with a relatively small number of vehicles.

8.3 ▪ The Reoptimization Model

Considering the informational process traditionally assumed when formulating the VRPSD (i.e., each demand becoming known when a vehicle arrives at the associated customer location), a company may decide to sequence the customers in each route in a dynamic fashion (i.e., as the demands are revealed during operations). As such, when considering the overall decisions made when solving the VRPSD, i.e., the assignment of customers to vehicles and both the routing and the replenishment decisions that determine the sequence of vertices visited by the vehicles to produce feasible routes, a company looking to minimize the average cost of each route (i.e., the average distance traveled) would benefit from making the latter decisions dynamically; see Secomandi [58]. As more and more efficient information and communication technologies become accessible to companies, the prospect of applying such dynamic models to perform routing activities becomes increasingly attractive; see Psaraftis [56] and Chapter 11 of this book.

In this section, we first present the general stochastic shortest path problem (SSPP), originally developed by Secomandi in [58, 59], to formulate the case where routes are constructed for the VRPSD using a reoptimization approach. This model is obtained by making no particular assumptions concerning either the routing or replenishment decisions to be made. For example, a vehicle may service the observed demand of a customer in more than one visit, scheduled consecutively or not. Therefore, as opposed to the a priori paradigm, where fixed routes are used during operations by applying reactive recourse actions when replenishments at the depot are necessary (i.e., whenever a vehicle cannot service the demand observed at a particular customer), the reoptimization approach sequences routes dynamically by deciding at each step (occurring when the vehicle leaves a particular location) which customer to visit next and whether a replenishment action should be performed (i.e., a visit to the depot). Therefore, this approach defines a more flexible and proactive recourse strategy when compared to the classical reactive recourse strategy detailed in Section 8.2. We conclude by presenting the general solution approaches developed to solve such dynamic optimization problems.

Before describing the stochastic shortest path formulation, there is an important point to be made. Since the reoptimization approach was first introduced in Dror, Laporte, and Trudeau [17] and formulated as a Markov Decision Process (MDP), the majority of solution methods that have been proposed to solve the VRPSD under this recourse strategy consider the particular case where a single vehicle is located at the depot; see the studies of Secomandi [58, 59], Secomandi and Margot [60], and Novoa and Storer [54]. This fact can be explained by the complexity associated with the problem of solving the multiple vehicle case (i.e., the extremely large number of possible states defined in the dynamic formulation). In order to present all modeling paradigms in a unified way, a general assumption is therefore made here with regard to the assignment decisions. As such, it is assumed that for each vehicle considered in the problem, the subset of customers to be serviced by the vehicle has been determined beforehand (i.e., all assignment decisions have been made prior to conducting the routing operations). These decisions may reflect specific choices made by the company or may be the result of a clustering model. With regard to the a priori paradigm, the assignment decisions can be viewed in the present context as defining the a priori plan to be used. In all cases, by fixing the assignment decisions, the problem decomposes by vehicle.

We now present the model as it is defined for a single vehicle in Secomandi [58]. A vehicle of capacity Q is located at the depot and is available to service the customers in the set N. Let ξ_i for all $i \in N$, define again the random variables that are used to represent the demands of the customers. It is assumed here that these variables are all discrete and are defined such that for all $i \in N$, $p_i(k) = Pr\{\xi_i = k\}$, where the support is $[0, \mathcal{K}]$ with $\mathcal{K} \leq Q$. Furthermore, all random variables are considered to be independent from each other and their realizations are only observed on the first visit of the vehicle at each customer location. The vehicle must start and end its route at the depot. While serving a particular customer, if the available capacity is either reached or exceeded, a return trip to the depot is performed to replenish the capacity up to level Q. It is important to stress that in the present problem, if a failure in service occurs, once the return trip to the depot is performed, the vehicle does not necessarily resume service at the location of the failure. Therefore, all observed demands can be served through multiple visits to the associated customers that are not necessarily scheduled consecutively. The problem consists of finding a dynamic routing policy $\mathbf{p}$ for the vehicle such that all customer demands are serviced and the expected distance traveled is minimized.

The SSPP model proposed by Secomandi [58, 59] represents a special case of the MDP. It is defined as a discrete-time dynamic system where the transitions between two states depend on a given control. Let the state of the system be defined as vector $\mathbf{x} = (l, d_l, j_1, \ldots, j_n)$, where $l \in \{0, 1, \ldots, n\}$ represents the current location of the vehicle, $d_l \in \{0, 1, \ldots, Q\}$ defines the residual capacity of the vehicle before performing the service at l, and values $j_i \in \{?, 0, 1, \ldots, \mathcal{K}\}$, for all $i \in N$, represent the amount of unserviced demand for each customer considered (where ? represents the demand of a customer yet to be visited for which the only available information is the associated distribution of its demand). As such, the starting and finishing states for the system are respectively defined as $(0, Q, ?, ?, \ldots, ?)$ and $(0, Q, 0, 0, \ldots, 0)$. Given a state $\mathbf{x}$, an associated control set is defined as follows: $U(\mathbf{x}) = \{\{m \in \{1, 2, \ldots, n\} \mid j_m \neq 0\} \cup \{0\}\} \times \{a : a \in \{0, 1\}\}$. Therefore, a control $u \in U(\mathbf{x})$ takes the form of a pair of values (m, a). The value m represents the next vertex to be visited along the route. This vertex can either be a customer that has yet to be visited or one that has been visited but that has part of its demand still unserviced. Vertex m can be visited either directly (i.e., $a = 0$) or by first performing a replenishment visit to the depot (i.e., $a = 1$). It should be noted that m can also be set to the depot to account for the case where all demands have been serviced and the system enters the terminal state.

When transiting from one state to another, if the system moves from $\mathbf{x} = (l, d_l, j_1, \ldots, j_l, \ldots, j_m, \ldots, j_n)$ to $\mathbf{x}' = (m, d_m, j_1, \ldots, j'_l, \ldots, j'_m, \ldots, j_n)$, assuming with no loss of generality that $l \leq m$, then the transition cost under control $u \in U(\mathbf{x})$ is defined as

$$g(\mathbf{x}, u, \mathbf{x}') = \begin{cases} c_{lm} & \text{if } u = (m, 0), \\ c_{l0} + c_{0m} & \text{if } u = (m, 1). \end{cases} \tag{8.18}$$

In such a transition, given control u, the capacity in state $\mathbf{x}'$ becomes

$$d_m = \begin{cases} \max(0, d_l - j_l) & \text{if } u = (m, 0), \\ \max(d_l + Q - j_l, Q) & \text{if } u = (m, 1). \end{cases} \tag{8.19}$$

If customer m has not yet been visited in state $\mathbf{x}$ (i.e., $j_m = ?$), then the value of its unserviced demand in the following state (i.e., j'_m) is set to be the realization of the random variable ξ_m; otherwise this value is already known exactly (i.e., $j'_m = j_m$). In the case of j'_l, the following update is applied: $j'_l = \max(0, j_l - d_l)$. The transition probabilities are then defined as

$$p_{\mathbf{xx}'} = \begin{cases} 1 & \text{if } j_m \text{ is known,} \\ Pr\{\xi_m = j'_m\} & \text{otherwise.} \end{cases} \tag{8.20}$$

The SSPP consists of finding an optimal sequence of controls that will bring the system to the termination state. The number of transitions that the system will take to reach the termination state is a random variable, denoted by R, which is dependent on both the random demands of the VRPSD and the controls that are used. Let $\mathbf{x}_k$ define the state of the system at stage k and $\mathbf{x}_R$ define the termination state; then a policy is defined as follows: $\mathbf{p} = \{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{R-1}\}$. For each stage k, $\mathbf{u}_k$ defines a function that associates a control $\mathbf{u}_k(\mathbf{x}_k) \in U_k(\mathbf{x}_k)$ with all possible states $\mathbf{x}_k$. Using the transition cost defined in (8.18), the objective function can now be defined as

$$J_R^{\mathbf{p}}(\mathbf{x}) = E\left[\sum_{k=0}^{R-1} g(\mathbf{x}_k, \mathbf{u}_k(\mathbf{x}_k), \mathbf{x}_{k+1}) \mid \mathbf{x}_0 = \mathbf{x}\right] \tag{8.21}$$

for all possible states $\mathbf{x} \in S$, where set S defines the state space. The expectation is formulated according to the probability distribution of the underlying Markov chain $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_R\}$, which again depends on both the initial state and on the considered policy $\mathbf{p}$. The optimal R-stage cost-to-go function from $\mathbf{x}$ is defined as

$$J_R^{\star}(\mathbf{x}) = \min_{\mathbf{p}} J_R^{\mathbf{p}}(\mathbf{x}). \tag{8.22}$$

It can be shown that (8.22) satisfies Bellman's equation, and we can therefore write

$$J_R^{\star}(\mathbf{x}) = \min_{u_k \in U_k(\mathbf{x}_k)} \sum_{\mathbf{x}_{k+1} \in S} p_{\mathbf{x}_k \mathbf{x}_{k+1}}(u_k)\{g(\mathbf{x}_k, u_k, \mathbf{x}_{k+1}) + J_R^{\star}(\mathbf{x}_{k+1}) \mid \mathbf{x}_k = \mathbf{x}\} \tag{8.23}$$

for all $\mathbf{x} \in S$. Provided (8.23) is known for all possible states $\mathbf{x} \in S$, then an optimal control associated with each state and at each stage k (i.e., $\mathbf{u}_k^{\star}(\mathbf{x})$) can be obtained by solving the following problem:

$$\mathbf{u}_k^{\star}(\mathbf{x}) = \arg \min_{u_k \in U_k(\mathbf{x}_k)} \sum_{\mathbf{x}_{k+1} \in S} p_{\mathbf{x}_k \mathbf{x}_{k+1}}(u_k)\{g(\mathbf{x}_k, u_k, \mathbf{x}_{k+1}) + J_R^{\star}(\mathbf{x}_{k+1}) \mid \mathbf{x}_k = \mathbf{x}\}.$$

Thus, one obtains an optimal policy for the problem $\mathbf{p}^{\star} = \{\mathbf{u}_0^{\star}, \mathbf{u}_1^{\star}, \ldots, \mathbf{u}_{R-1}^{\star}\}$.

As presented in Secomandi [58], neuro-dynamic programming has successfully been applied to solve the SSPP. Given the size of the state space S, exact computation of the cost-to-go function (8.22) is prohibitive from a computational perspective. Neuro-dynamic programming methods rely on the use of approximation functions, obtained using simulation techniques and parametric approximations, to compute (8.22). Such approximation functions are then used in specific solution strategies, such as the Policy Iteration Algorithm (PIA), to heuristically produce policies for the considered problem.

Assuming that an initial policy is available, the PIA searches to improve it through an iterative process. Therefore, at iteration ν of the PIA, let the considered policy be

$\mathbf{p}^\nu = \{\mathbf{u}_0^\nu, \mathbf{u}_1^\nu, \ldots, \mathbf{u}_{R-1}^\nu\}$. Policy $\mathbf{p}^\nu$ is first evaluated by solving the following equation for all states $\mathbf{x} \in S$:

$$J^{\mathbf{p}^\nu}(\mathbf{x}) = \sum_{\mathbf{x}_{k+1} \in S} p_{\mathbf{x}_k \mathbf{x}_{k+1}}(\mathbf{u}_k^\nu(\mathbf{x}_k))\{g(\mathbf{x}_k, \mathbf{u}_k^\nu(\mathbf{x}_k), \mathbf{x}_{k+1}) + J^{\mathbf{p}^\nu}(\mathbf{x}_{k+1}) \mid \mathbf{x}_k = \mathbf{x}\}.$$

An improvement step is then performed to produce the next policy to consider $\mathbf{p}^{\nu+1} = \{\mathbf{u}_0^{\nu+1}, \mathbf{u}_1^{\nu+1}, \ldots, \mathbf{u}_{R-1}^{\nu+1}\}$. This step is defined by solving the following problem again for all states $\mathbf{x} \in S$:

$$\mathbf{u}_k^{\nu+1}(\mathbf{x}) = \arg \min_{u_k \in U_k(\mathbf{x}_k)} \sum_{\mathbf{x}_{k+1} \in S} p_{\mathbf{x}_k \mathbf{x}_{k+1}}(u_k)\{g(\mathbf{x}_k, u_k, \mathbf{x}_{k+1}) + J^{\mathbf{p}^\nu}(\mathbf{x}_{k+1}) \mid \mathbf{x}_k = \mathbf{x}\}.$$

When both the evaluation and improvement steps are performed exactly, the PIA produces a sequence of policies that converges to the optimal one $(\mathbf{p}^0, \mathbf{p}^1, \ldots, \mathbf{p}^\star)$. However, as previously mentioned, in an effort to speed up the solution process, neuro-dynamic programming methods perform these steps using approximations.

Several solution approaches, based on the policy improvement strategy, have been successfully applied to solve the SSPP. Rollout algorithms have been proposed by Secomandi [58, 59] in this context. Starting from an initial policy obtained through an a priori route for the VRPSD, a rollout algorithm performs a single policy improvement step using an approximation of the cost-to-go function for each considered state. The particularity of these algorithms is that the improvement step is performed using the states that are generated while the procedures are executed. By proceeding in this way, the solution times are considerably reduced. However, this is done at the expense of the precision of the improvement step. Enhancements to this method were proposed by Novoa and Storer [54] by performing a two-improvement step approach which applies pruning rules on the set of controls considered, but also uses multiple initial policies derived from different a priori routes to produce a better upper bound on the routing cost.

Finally, Secomandi and Margot [60] proposed applying a partial reoptimization strategy based on the idea of obtaining an optimal policy for a restricted set of states for the MDP. By doing so, the VRPSD is heuristically solved. In the model proposed in [60], it was assumed that when a failure happens at a particular customer the vehicle, after performing a partial delivery, returns to the depot to replenish its capacity and resumes service at the customer where the failure occurred. Considering this special case of the problem, the authors proposed two strategies aimed at reducing the state space. Given an a priori route, these strategies are obtained by using either disjoint or overlapping blocks of customers along the considered route. When disjoint blocks are used, states are generated such that the sequence of customer blocks in the a priori route is followed but the order in which the customers in each block are visited may be dynamic. As for the strategy using overlapping blocks, states are produced such that each customer along the a priori route may be visited in a dynamic fashion within a sliding window defined according to the considered blocks. Based on these two strategies, heuristic algorithms are developed for the problem.

Given that the dynamic formulations for the VRPSD have been applied to the single-vehicle case, it is hard to compare the results obtained using these models with those reported for the a priori models presented earlier. However, a particular route can always be evaluated using either an a priori or a dynamic recourse formulation. Such experimentations were performed by Secomandi [58] to assess the potential improvements, in terms of the average cost of routes, when using the SSPP model proposed. Using the generator originally developed by Gendreau, Laporte, and Séguin [23], instances of different sizes

Table 8.3. *Relative improvements of RP compared to HP.*

	n		
$\overline{F}$	5	10	15
1.0	8.0401	8.0968	8.8898
1.5	5.9113	6.2811	8.1915
2.0	5.4249	5.2688	7.1950

were produced, $n \in \{5, 10, 15\}$, where the expected number of route failures $\overline{F}$ was set to $\{1.0, 1.5, 2.0\}$. For each considered combination, a total of five instances were randomly generated. All instances were first solved to produce a Rollout Policy (RP), obtained by applying the rollout algorithm proposed in [58]. A Heuristic Policy (HP) was then constructed by running a traveling salesman heuristic that produced a route while ignoring the stochastic demands. The route was then evaluated using the classical a priori recourse strategy. Table 8.3 reports the results presented in [58], which correspond to the percentage improvements in the average distances traveled when applying RP compared to HP. Over all instances tested, the average percentage improvement reported was 7.0332. These results show that as the number of customers to be served increases, so does the average improvement observed. Also, when varying the expected number of route failures, one observes that as the value of $\overline{F}$ increases the average improvement decreases.

Finally, considering the size of the problems solved, given the complexity of the SSPP, the instances initially considered by Secomandi [58, 59] were relatively small (i.e., $5 \leq n \leq 15$). The enhancements proposed by Novoa and Storer [54] were able to improve the algorithms to solve larger instances (i.e., $n \in \{5, 8, 20, 30, 40, 60\}$). However, the current state-of-the-art strategies available are the heuristic algorithms proposed by Secomandi and Margot [60], which were shown to efficiently solve instances with up to 100 customers in a few seconds, thus clearly illustrating the advantages associated with the partial reoptimization strategy.

8.4 ▪ Probabilistic Formulation

Accounting for stochastic demands in routing problems is often driven by the paradigm of guaranteeing that a given route will not fail with a given probability. This paradigm is useful when the violation of the capacity constraint is not well defined (Stewart and Golden [62]). Therefore, the Vehicle Routing Problem with Stochastic Demands and Probabilistic Constraints (VRPSDPC) does not explicitly account for the worst-case scenarios, but rather ensures that the feasibility of routes is achieved within a predetermined threshold. Laporte, Louveaux, and Mercure [43] proposed an exact algorithm for the stochastic location-routing problem, which consists of simultaneously locating a depot among a set of potential sites, as well as routing vehicles to service customers with stochastic demands, thus modeling and solving a broader problem than the VRPSD. For consistency reasons, we will omit the location decisions. In addition to the previously defined notation, we impose that the probability that a planned route fails at least once does not exceed β. Finally, we define $V_\beta(S)$ as the minimum number of vehicles required to serve $S \subseteq N$, so that the probability of failure in S does not exceed β. This definition implies that $V_\beta(S)$ is the smallest integer satisfying

$$P\left(\sum_{i \in S} \xi_i > Q V_\beta(S)\right) \leq \beta.$$

In what follows we present the single location adaptation of the model presented in Laporte, Louveaux, and Mercure [43]:

$$\text{(8.24)} \quad \text{(VRPSDPC)} \quad \text{minimize} \sum_{i<j} c_{ij} x_{ij}$$

$$\text{(8.25)} \quad \text{s.t.} \quad \sum_{j=1}^{n} x_{0j} = 2|K|,$$

$$\text{(8.26)} \quad \sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad \forall k \in N,$$

$$\text{(8.27)} \quad \sum_{\substack{i\in S, j\in \bar{S} \\ \text{or} \\ i\in \bar{S}, j\in S}} x_{ij} \geq 2V_\beta(S) \qquad \forall S \subset N, 3 \leq |S| \leq n-1,$$

$$\text{(8.28)} \quad 0 \leq x_{ij} \leq 1 \qquad \forall i,j \in N, i<j,$$

$$\text{(8.29)} \quad 0 \leq x_{0j} \leq 2 \qquad \forall j \in N,$$

$$\text{(8.30)} \quad x = (x_{ij}) \ \text{integer} \qquad \forall i,j \in V, i<j.$$

In this model, constraints (8.25) and (8.26) specify the degree of each vertex. Constraints (8.27) are probabilistic connectivity constraints; their interpretation is that if it is known that at least $V_\beta(S)$ vehicles will visit the customer set S, then there must be at least $2V_\beta(S)$ crossings between S and $\bar{S}$. Therefore, constraints (8.27) are an extension of the classical deterministic connectivity constraints used in the VRP; see Laporte, Nobert, and Desrochers [46].

The customer demands are assumed to be independently and identically distributed random variables of finite mean μ_i and finite variance σ_i^2. If $|S|$ is sufficiently large or if customer demands follow a normal distribution, then $\sum_{i\in S} d_i$ is normally distributed. In both cases, $V_\beta(S)$ can be expressed as

$$V_\beta(S) = \left\lceil \frac{z_\beta(\sum_{i\in S}\sigma_i^2)^{1/2} + \sum_{i\in S}\mu_i}{Q} \right\rceil,$$

where z_β is the order β fractile of the standard normal distribution. Laporte, Louveaux, and Mercure [43] proposed a Branch-and-Bound algorithm to solve the VRPSDPC. As previously mentioned, the VRPSPC reduces to a deterministic VRP. Therefore, we will not present the results of Laporte, Louveaux, and Mercure [43], because state-of-art algorithms for the VRP are likely to outperform the reported results. Furthermore, given that this model can be solved as a deterministic problem, the VRPSDPC is thus much easier to tackle than the previously presented VRPSD models.

It should be noted that Laporte, Louveaux, and Mercure [43] present another model that minimizes the sum of the depot operating costs, vehicle fixed costs, planned routing costs, and expected failure costs. Furthermore, they consider two operational modes with respect to demand revelation and failures. The first resembles the one described in Section 8.2; i.e., customer demand is only revealed upon arriving at the customer location and failures occur whenever vehicle capacity is exceeded. The second operating mode considers that a priori information with respect to customer demand is available. Since it is very often desired that drivers take the same route every day on a routine basis, the authors assume that the routes determined in the first stage will remain unchanged. However, this operating mode differs from the previous one, in that a vehicle never proceeds directly from one customer to the next if it is known in advance that its capacity will be exceeded.

The VRPSDPC was also studied by Golden and Yee [26], who introduced probabilistic constraints considering several demand distributions. The authors also study the case of correlated demands. Stewart and Golden [62] present a three-index formulation to the VRPSDPC, where the probabilistic constraints are non-linear. The authors then show that the probabilistic constraints may be linearized for several distributions, e.g., Poisson, binomial, negative binomial, and gamma. The authors propose two heuristic algorithms to solve the problem. The first is based on the Clarke and Wright algorithm, while the second makes use of Lagrange multipliers.

Bastian and Rinnooy Kan [3] further studied the VRPSDPC for the single-vehicle case and showed that the problem is equivalent to the time-dependent TSP.

8.5 ▪ Stochastic Demands

The VRPSD is by far the most studied variant of SVRP. The first study on this subject is attributed to Tillman [67], who addressed the problem in the multi-depot context. The author proposed a solution procedure based on an extension of the Clarke and Wright savings heuristic [14]. Dror and Trudeau [18] furthered the development of heuristics based on the same algorithm and studied the impact of route failures on the average routing cost. The authors were the first to show that the average cost of a route is dependent on its orientation. One of the first dissertations dealing with the VRPSD is that of Bertsimas [6], which derives a series of bounds, asymptotic results, and theoretical properties for problems where demands are unit random variables (i.e., for customer i, $\xi_i = 1$ with probability p_i and $\xi_i = 0$ with probability $1 - p_i$). This was followed by Dror, Laporte, and Trudeau [17], who derived a general discussion on possible formulations for the VRPSD and studied the properties of these models. Noteworthy theoretical results included the extension of properties that were originally established in the context of the probabilistic TSP by Jaillet [34, 35] and Jaillet and Odoni [36]. Specifically, in an optimal solution of the VRPSD,

1. an optimal route may intersect itself;
2. considering a Euclidean problem, the visits defined by the optimal routes do not necessarily reflect the order in which the customers appear on the convex hull of vertices;
3. if considered separately, the segments of an optimal route are not necessarily optimal.

The first two results provide insights in the structure of optimal solutions of the VRPSD. As for the third result, its impact is mainly on the design of solution procedures.

Following these initial studies, as presented in Sections 8.2–8.4, several papers, providing various optimization models and solution approaches, have been proposed for the VRPSD. In this section, we complete the presentation of these methods by focusing on the studies that consider either alternative definitions of the recourse decisions or problem characteristics other than the ones previously presented (i.e., added problem dimensions or extra constraints).

Regarding the recourse definition, we have already seen that the manner in which both the routing and replenishment decisions are made leads to different formulations for the VRPSD. Recall that the a priori paradigm assumes that both decisions are determined before any information regarding demands is available (i.e., routes are fixed and the replenishment decisions follow the simple recourse strategy), while the reoptimization paradigm allows both decisions to be made dynamically as demands are observed.

Yang, Mathur, and Ballou [71] defined an in-between approach where routes are fixed beforehand and replenishment decisions are made dynamically. As a result, preventive restocking can be performed in order to reduce the average routing cost. Under these principles, the authors showed that, for a fixed sequence of visits to customers, the optimal replenishment decisions can be formulated as a threshold-based policy. Following such a policy, a vehicle, performing the considered sequence, makes a preventive return to the depot if its residual capacity when leaving a particular customer is below the associated customer's threshold. Two heuristics were presented for the single- and multi-vehicle cases. These procedures were tested on instances that included from 10 to 60 customers with different route length constraints.

An extension to the simple recourse strategy was proposed by Ak and Erera [1], where fixed routes are paired a priori to serve their assigned customers. Following this strategy, which the authors refer to as the *paired locally coordinated operating scheme*, routes are fixed and then paired before observing the informational flow (i.e., demands become known). In each pair, routes are assigned a tour type (i.e., one to Type I and the other to Type II). The vehicle performing the route assigned to Type I visits the customers according to the fixed sequence and simply returns to the depot if a failure occurs (thus finishing the tour). All unvisited customers are appended to the end of the Type II route. The other vehicle, performing the route assigned to Type II, serves all customers assigned to it in the fixed sequence plus the unvisited customers of the Type I route. In this case, failures are handled by applying the simple recourse strategy. The rationale behind this approach is that, at the expense of added coordination between vehicles, the total average cost of a solution can be reduced by pooling the capacity of vehicles in pairs. A tabu search procedure was proposed to solve the VRPSD under this recourse strategy. Instances sized from 20 to 150 customers were solved, and the results showed improvements in terms of the expected cost of routes operated using the proposed recourse strategy compared to the simple one. These improvements ranged from approximately 1% to 17% according to the size of the instances and the probability distributions chosen for the demands.

This type of paired-recourse strategy was applied to the case of the VRPSD and split deliveries by Lei, Laporte, and Guo [48]. Using the a priori paradigm, the authors proposed a two-stage model where routes are constructed and paired in the first stage and then used in the second stage to serve customers while allowing demands to be split among paired routes. A large neighborhood search heuristic was developed to solve the problem.

Finally, multi-stage formulations have been proposed for the VRPSD by Hvattum, Løkketangen, and Laporte [30, 31]. The authors considered a problem where customer demands (requests) appear over a given time period. Routes are therefore constructed dynamically by integrating the requests in the tours at predefined stages. In this case, the recourse decisions are defined in multiple stages and correspond to the adjustments made to the routes as additional demands are observed. The authors proposed two heuristic strategies: a sample-based algorithm that applies scenario decomposition to the model [30] and a Branch-and-Regret procedure [31].

In addition to providing alternative recourse formulations, recent studies have also focused on extending VRPSD formulations to include additional problem characteristics. Tatarakis and Minis [66] proposed two dynamic programming models to compute the expected cost of a fixed route for the multiple product VRPSD. In this case, customers have a distinct stochastic demand associated with each product. Therefore, when operating a fixed route, a failure occurs when at least one of the product demands for a given customer cannot be serviced by the vehicle. Two variants of this problem were solved: the compartmentalized load case (i.e., each product is loaded into a distinct compartment on a vehicle) and the unified load case (i.e., all products are loaded in a single compartment on a vehicle). The authors showed that, for one vehicle and two products, the optimal replenishment decisions for a fixed route take the form of a threshold-based policy similar to the one proposed by Yang, Mathur, and Ballou [71]. This result was then generalized by Pandelis, Kyriakidis, and Dimitrakos [55] to the case of an arbitrary number of products. Considering this problem, Mendoza et al. [53] proposed a formulation based on the a priori paradigm for the compartmentalized load case. The authors developed a memetic algorithm and solved instances with up to 484 customers.

Given the presence of route failures in the context of the VRPSD, when a route is operated, its actual duration tends to be longer than planned. Therefore, time requirements, related either to the service of customers or to the workload performed by the vehicles, can become important issues. Several studies have consequently focused on introducing time constraints in VRPSD formulations.

Erera, Savelsbergh, and Uyar [20] addressed the case where both route duration requirements, imposed by driver work rules, and hard time-window constraints are considered when solving the VRPSD. The authors proposed a two-stage model based on the a priori paradigm. In the first stage, a set of fixed routes are created for a subset of customers deemed regular (i.e., ruling out customers that have a low probability of placing a request). In the second stage, given the set of planned routes and the observed demands, a set of operational routes are created to enforce the constraints. As described in [20], routes are constructed in the second stage so as to retain, as much as possible, the characteristics defined by the planned routes (i.e., the assignment and the sequence of customers in the routes).

Erera, Morales, and Savelsbergh [19] showed that, when constructing a priori routes for the VRPSD, imposing tour duration constraints that ensure feasibility for all demand realizations can be done efficiently by solving an adversarial optimization problem. The authors also investigated the impact that such constraints have on the size of the fleet necessary to obtain feasible solutions for the problem.

Lei, Laporte, and Guo [47] proposed an a priori formulation for the VRPSD in the presence of soft time-window constraints. Whenever a vehicle fails to meet a customer's time window, an additional cost equivalent to a route failure is assumed (thus yielding the case where the customer is served by a separate direct trip from the depot). Under this assumption, an efficient method for computing the total expected cost of a fixed route was developed and embedded within an adaptive large neighborhood heuristic. The proposed algorithm was shown to be efficient on a set of modified Solomon instances.

Finally, Goodson, Ohlmann, and Thomas [28] considered route duration limits in the context of the VRPSD. The problem was formulated using the reoptimization strategy. The authors developed a series of rollout heuristics to solve the model and presented numerical results showing the relative superiority of the proposed methods over both a rolling horizon solution strategy and fixed-route approaches.

8.6 ▪ Stochastic Customers

The Vehicle Routing Problem with Stochastic Customers (VRPSC) considers that customers are present with a given probability. Traditionally, the VRPSC is modeled as a two-stage stochastic programming problem. The first stage consists of determining the routes that adhere to the VRP constraints. Given the realization of present and absent customers, the second stage solution is to follow up the routes set by the first stage, while skipping the absent customers. Unless mentioned otherwise the problem assumes deterministic demand.

We will first present some of the relevant literature on the Traveling Salesman Problem with Stochastic Customers (TSPSC), a special case of the VRPSC. This problem was introduced by Jaillet [34], who derived several models, bounds, and properties. Rossi and Gavioli [57] and Jézéquel [37] adapted the Clarke and Wright heuristic [14] to the TSPSC. Further heuristic algorithms were proposed by Bertsimas [6] and Bertsimas and Howell [8]. An exact Branch-and-Cut algorithm for the problem was developed by Laporte, Louveaux, and Mercure [45].

In the VRPSC, customer demand is often considered to be of unit size. This setting was studied by Jézéquel [37] and by Jaillet and Odoni [36]. The latter found that, even for symmetric distances, the cost of a solution depends on the orientation of travel. Jaillet and Odoni [36] also stated that large vehicle capacities may yield large solution costs. Bertsimas [6] describes several properties, bounds, and heuristics for the problem. Waters [70] studied the problem with general integer demands and considered three operating policies. The first follows the planned routes, regardless of the presence or absence of customers. The second follows the planned route while skipping absent customers, and the third reoptimizes the route whenever the absence of a customer is observed.

The Vehicle Routing Problem with Stochastic Customers and Demands (VRPSCD) combines stochastic customers and stochastic demands. The VRPSCD was mentioned by Jézéquel [37] and Jaillet and Odoni [36], but it was formalized by Bertsimas [7]. As for the VRPSC, the VRPSCD is formulated as a two-stage stochastic programming model. The first stage designs routes that visit all customers, the set of present customers is revealed before routes are executed, and customer demands are revealed upon the arrival of the vehicle. The routes are followed while skipping the absent customers, and the vehicle returns to the depot to replenish when its capacity is exhausted. The computation of the objective function has been shown to be difficult. Séguin [61] and Gendreau, Laporte, and Séguin [23] proposed the first exact algorithm for the problem. They computed efficient bounds on the first-stage travel times and solved the problem with the integer L-shaped algorithm. Gendreau, Laporte, and Séguin [24] later developed a tabu search algorithm for the problem, exploiting an approximation that facilitated the computation of the objective function. Finally, Benton and Rossetti [5] considered a version of the VRPSCD in which route reoptimization is allowed when demands are revealed.

Sungur et al. [63] considered the Courier Delivery Problem with uncertainty on the presence of customers and service times. Customers have soft time windows while a hard constraint is considered on the route duration. Uncertainty is represented by scenarios. The objective is to produce a master plan that maximizes the number of served customers, minimizes the total time spent by couriers and penalties for early or late arrival at customers, and maximizes a measure of consistency of the routes for each scenario. The authors proposed a two-phase heuristic where in the first phase an initial master plan is obtained by an insertion heuristic. Recourse subproblems are then solved by an insertion heuristic for each scenario where absent customers are skipped. In the second phase, an iterative procedure provides feedback from the scenarios insertion algorithm to the master insertion algorithm regarding unserved customers. The master plan heuristic gives priority to unserved customers, and new subproblems for each scenario are solved. Tabu search is used to improve the solutions of the insertion heuristics of the scenario subproblems.

8.7 ▪ Stochastic Travel Times

In the classical VRP, travel times are assumed to be deterministic and are usually assumed to be linearly correlated with the distance. Therefore, travel time cost is often referred to as distance. However, in many real-life applications a sizeable degree of variability in travel times is observed, and thus static, deterministic travel times do not represent an accurate approximation of actual travel time. Malandraki and Daskin [52] classified potential causes of variability in travel times into two components. The first component covers temporal variations resulting from hourly, daily, weekly, or seasonal cycles in the average traffic volumes. Because these variations are of a repetitive nature, they can be modeled deterministically by incorporating time-dependent travel times; see, e.g., Malandraki and Daskin [52]. The second component of variations is due to accidents, weather conditions, or other random events; these are the focus of this section. Furthermore, variability may be observed in service times. This variability is fundamental since it influences vertex departures and consequently the resulting travel times. In the remainder of this section, we present an overview of the literature on VRPs with stochastic travel or service times.

The VRP with stochastic travel and service times was first introduced by Laporte, Louveaux, and Mercure [44]. These authors considered the a priori framework, whereby routes are designed before random travel and service times are revealed. After the realization of the random travel and service times, the vehicles follow their a priori routes. No capacity constraints were considered, but vehicles incur a penalty if the route duration exceeds a given deadline. The penalty is proportional to the elapsed route duration in excess of the deadline. The authors presented a chance constraint model and two recourse models. The chance constraint model follows principles similar to those presented in Section 8.4. The objective is to minimize the routing costs, while ensuring that the probability of exceeding the route duration deadline does not exceed β. This feasibility requirement is incorporated in the subtour elimination constraints. Furthermore, the authors propose a three-index simple recourse model and a two-index recourse model. The first-stage decisions of the recourse models consist of determining the number of vehicles and their routes. In the second stage, random variables corresponding to stochastic travel and service times are realized and penalties are incurred for excess duration. A general Branch-and-Cut algorithm was proposed for all three models. Test instances with up to 20 customers and two to five travel time scenarios were solved.

Kenyon and Morton [38] later studied the VRP with uncertainty in travel and service times. The authors proposed two a priori models and, as in Laporte, Louveaux, and Mercure [44], they considered uncapacitated vehicles. The two models correspond to two different objectives. The first minimizes the maximum expected completion time of all routes, while the second maximizes the probability that the operation is completed no later than a pre-specified target time. A three-index formulation is proposed for both models. The authors showed that the solution of the first model with a single vehicle (i.e., the TSP with stochastic travel time) is equivalent to solving the TSP in which the travel and service times are replaced by their means. Considering the second model with one vehicle, the authors developed a methodology based on solving a deterministic non-linear integer program whose continuous relaxation is a convex program. The solution method embeds a Branch-and-Cut scheme within a Monte Carlo sampling-based procedure for the multi-vehicle case. The performance of the algorithm was validated on a 28-vertex instance.

Lambert, Laporte, and Louveaux [39] studied the problem of designing vehicle routes to collect deposits from bank branches and deliver them to a central office. Considering route duration constraints and uncapacitated vehicles, the objective of the problem is to minimize vehicle fixed costs, vehicle costs, and penalty cost for lost interest. The authors considered that congestion may occur with some probability on some arcs and worked with two travel time scenarios for subsets of the arcs. The problem was cast into an a priori two-index model and solved by adapting the Clarke and Wright heuristic. The algorithm was tested on two data sets of instances with 29 and 44 vertices. These data were derived from a practical application.

Several heuristics were developed for the VRP with stochastic travel time or service time. Lei, Laporte, and Guo [49] considered capacity constraints, stochastic service times, and a maximum route duration limit. The objective is to minimize the sum of travel costs, expected service cost, and expected recourse cost. The authors provide a closed form expression for the expected cost of a given route in case of stochastic service times and propose a generalized variable neighborhood search heuristic to solve the problem. The latter heuristic was shown to outperform both a variable neighborhood descent and a variable neighborhood search heuristic. Jabali et al. [33] studied the capacitated VRP with time-dependent travel times, route duration limits, and stochastic service times. In this case, the stochastic service times were modeled as unexpected delays at customer locations by considering a single disruption per route. A tabu search algorithm was adapted to the problem and tested on instances from Augerat et al. [2].

The VRP with soft time windows and stochastic travel times was studied by a number of authors. The uncapacitated version of the problem was considered by Wang and Regan [69]. Li, Tian, and Leung [50] addressed a stochastic capacitated VRP with soft time windows and stochastic service and travel times. All random variables were assumed to be independent and normally distributed. They considered a chance constraint for each time window and another for the route duration. The authors also proposed a two-stage program with recourse, where a penalty is incurred in the case of time-window or route duration violations. They adapted a tabu search–based heuristic for the solution of both models, while using Monte Carlo simulation to evaluate the random events. The VRP with soft time windows and stochastic travel times was also studied by Taş et al. [64]. These authors considered the objective of minimizing customer inconvenience, expressed as expected earliness and lateness at customers locations, and operational costs, expressed as expected driver overtime and vehicle and travel costs. Travel times were assumed to follow a gamma distribution. The authors proposed a three-phase tabu search–based algorithm. In the first phase, a solution is constructed and then improved with respect to the total operating cost, thus yielding an initial feasible solution. This solution is improved by tabu search in the second phase. In the third phase, a post-optimization procedure, accounting for customer inconvenience, is applied to further improve the solution obtained by the tabu search algorithm. In a more recent paper, Taş et al. [65] applied a column generation procedure to optimally solve the problems proposed in Taş et al. [64]. The master problem is modeled as a classical set partitioning problem. The pricing subproblem, for each vehicle, corresponds to an elementary shortest path problem with resource constraints. The column generation procedure is embedded within a Branch-and-Price algorithm.

Errico et al. [21] were the first to propose a formulation for the VRP with stochastic service times and hard time windows, which is considerably more difficult than the case with soft time windows. The difficulty arises from the fact that, given a route, the probability distributions of arrival times at customers locations have to be truncated because of the hard time windows, thus prohibiting the use of convolution properties when summing the random variables. The problem was cast into a chance constraint framework that considers a minimum success probability of the set of vehicle routes. The model is based on the set partitioning approach for the VRP and relies on a transformation of the minimum success chance constraint into a constraint that allocates a probabilistic resource among routes. A Branch-and-Cut-and-Price algorithm was developed. In the column (route) generation subproblem, one accounts for the consumption of probabilistic resource by extending the label dimension and by providing specialized dominance rules.

8.8 ▪ Conclusions and Future Research Directions

In the VRP, stochasticity is observed in customer demands, customer presence, and travel or service times. Formulating this stochasticity may follow several paradigms. Three main modeling paradigms emerge from the literature: a priori optimization, reoptimization, and probabilistic modeling. The choice of a paradigm depends on the informational flow and on the operational policies that govern the application.

The a priori optimization paradigm follows a two-stage model, in which an a priori solution is determined in the first stage. This solution is executed in the second stage while uncertainties are revealed and recourse actions are taken according to a chosen policy. The choice of the recourse policy is paramount in determining the expected total cost of the solutions. These models constitute fundamental building blocks in the SVRP literature. Their results usually serve as benchmarks to more elaborate approaches.

The reoptimization paradigm assumes that routing decisions are reoptimized as stochastic events unfold. In this context, dynamic models are likely to outperform a priori optimization. This superiority stems from the fact that the solution space considered by a priori models is contained in the solution space considered by dynamic models. The sizeable solution space of the latter is precisely the reason why studying dynamic models in absolute terms is a challenging task.

The probabilistic paradigm is steered by the notion that solutions should guarantee a certain level of service. In the context of SVRP, quality of service is measured through the probability of route failure. Therefore, probabilistic models guarantee a certain level of protection against undesired outcomes.

As detailed in Sections 8.5 to 8.7, most of the research on SVRPs has focused on the existence of a single stochastic parameter. Realistic extensions could focus on simultaneously considering a number of stochastic parameters. The interactions between these parameters will yield challenging models. Furthermore, most models consider independent random variables, yet in applications random events are often correlated; e.g., bad weather or road accidents may impact the travel times on a number of arcs at the same time. Finally, many important variants of the VRP have not yet been addressed in a stochastic context.

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