A FLUX IDENTITY OF SCALAR CONSERVATION LAWS FOR LAGRANGIAN FLUX CALCULATION VIA DONATING VOLUMES IN 3 DIMENSION

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3-D flux calculation problem

Given a simple surface patch γ^o, a time-dependent velocity field u(x,t),an initial time t_0 and time interval k, we are interested in the flux that crossed γ^o during time interval $[t_0, t_0+k]$.
We denote the boundary of γ^o to γ_0.
iIf f is further differentiable, we arrive at the scalar conservation law

$$\frac{\partial f}{\partial t} + \bigtriangledown \cdot (f\mathbf{u}) = 0. \tag{1}$$

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Definition 1.1 (Winding numbers)

Let γ be a closed curved surface and $\mathbf{x} \notin \gamma$ be a fixed point. The winding number of γ around $\mathbf{x}$, written $\omega(\gamma, \mathbf{x})$, is the number of positive orientation of a Lagrangian partical p through γ minus its number of negative, as p moves from its initial position $\mathbf{x}$ to ∞. The orientation of the curved surface can be defined by γ_o, which gives user ability to define the orientation. Moreover, if the curved surface is bilateral, the orientation will be well-defined, thus we can determine whether it is positive by the orientation of the normal vector.

Certainly, a closed curved surface which is homeomorphic to a unit ball can be oriented and then the winding number of it is well defined.

Definition 1.2 (Donating volume)

For a given velocity field $\mathbf{u}(\mathbf{x},t)$, the donating volume associated with a surface patch χ over time interval$(t_0 - k, t_0)$, denoted by $D_\chi(t_0, k)$, is the open curvilinear polyhedron enclosed in χ, its timeline $\phi_{t_0}^{-k}(\chi)$, and the streak surface $\Phi_{t_0}^{-k}(\partial_\chi) = \cup_{N\in\partial_\chi}\Phi_{t_0}^{-k}(N)$. $D_\chi(t_0, k)$ is canonical if it is simply connected; then it is positively oriented if the normal vector of χ points into $\text{ext}(\partial D_\chi)$, and negatively oriented otherwise.

$$
\begin{aligned}
&D^n_{\gamma^o}(t_0, k) := \{\mathbf{x} \in R^2 : \omega(\gamma_D, \mathbf{x}) = n\} \\
&\gamma_D := (\gamma^o) \cup \gamma_0 \cup \Psi_{t_0+k}^{-k}(\gamma_0) \cup \phi_{t_0+k}^{-k}(\gamma^o) \cup \phi_{t_0+k}^{-k}(\gamma_0)
\end{aligned}
\tag{2}
$$

Given time t_0 and time interval k, we can define DV on any injective surface patch δ^o.

Theorems we need

Theorem 2.1 (Hopf Theorem)

Let a point $x_0 \in R3$ be given. Two closed curves γ_1 and γ_2 are freely homotopic in $R3 \setminus \{x_0\}$ if and only if $\omega(\gamma_1, x_0) = \omega(\gamma_2, x_0)$.

Theorem 2.2 (Reynolds' transport theorem)

let $M(\tau) : \tau \in [0, k]$ denote a moving closed curve in the space and $g(\cdot, \tau) : M(t) \to R$ a scalar function. We have

$$\frac{d}{d\tau} \int_{M(\tau)} g(\mathbf{x}, \tau) d\mathbf{x} = \int_{M(\tau)} \frac{\partial g(\mathbf{x}, \tau)}{\partial \tau} d\mathbf{x} + \int_{\partial M(\tau)} g(s, \tau) V_n(s, \tau) ds, \tag{3}$$

where $V_n(s, \tau)$ is the normal speed of the boundary of moving closed curve $\partial M(\tau) : \tau \in (0, k)$.

Conclusions in 3-D

Theorem 3.1 (Index-by-Index equivalence of flux sets and DVs)

Any DV of a fixed simple curved surface γ^o is index-by-index equivalent to the flux set of γ^o,

$$\forall n \in \mathbb{Z}, \forall k > 0, D^n_{\gamma^o}(t_0, k) = F^n_{\gamma^o}(t_0, k) \tag{4}$$

if $\mathbf{n}^{\mathbf{DV}}_{\boldsymbol{\gamma}^{\mathbf{o}}}$, the unit normal vector is positive.
For another cases of $\mathbf{n}^{\mathbf{DV}}_{\boldsymbol{\gamma}^{\mathbf{o}}}$, $D^n_{\gamma^o}(t_0, k) = -F^n_{\gamma^o}(t_0, k)$

Theorem 3.2 (the flux identity for arbitrary DVs)

Let f be a scalar function conserved by a nonautonomous flow $\mathbf{u}(\mathbf{x}, t)$ such that (1) holds. The DV $D_{\gamma^o}(t_0, k)$ of a fixed simple curved surface γ^o as in Definition 1.2 satisfies

$$\forall k > 0, \int_{t_0}^{t_0+k} \int_{\gamma^o} f \mathbf{u} \cdot \mathbf{n}_{\gamma^o} ds dt = \sum_{n \in Z} n \int_{D^n_{\gamma^o}(t_0,k)} f(\mathbf{x}, t_0) d\mathbf{x} \quad (5)$$

Homotopy class

Lemma 3.1

The set of generating curved surface of DVs for a simple open curve

$$\mathring{\tau}_{\gamma^\circ}(t_0, k) := \{\tau_{\gamma^\circ}(t_0 + \xi, k - \xi) : \xi \in [0, k]\} \tag{6}$$

is a homotopy class in R^3.

Index-by-Index equivalence of flux sets and DVs.

We use tools: homotopy class and Hopf Thm. We denote a new flow map to prove it. To make it clear, we make the DV move but the point be static, and then we show that it will pass through γ_d from γ^o. $\square$

Lemma 3.2 (The flux identity for canonical DVs)

Let f be a scalar function conserved by a nonautonomous flow $\mathbf{u}(\mathbf{x}, t)$. If the DV $D_{\gamma^o}(t_0, k)$ is canonical, then

$$\int_{t_0}^{t_o+k} \int_{\gamma^o} f\mathbf{u} \cdot \mathbf{n}_{\gamma^o} dsdt = \pm \int_{D_{\gamma^o}(t_0,k)} f(\mathbf{x}, t_0) d\mathbf{x} \tag{7}$$

where $\mathbf{n}_{\gamma^o}$ is the unit normal vector of DV, and $\pm$ is determined by that conanical volume.

Proof.

tools:

Reynolds' Thm

hyperbolic conservation law

divergence Thm.

□

Some hints

Lemma 3.3

A streakline does not intersect itself if the associated time interval is small enough. More precisely, there exists $k_0 > 0$ such that for all $k < k_0$ the streakline $\Psi_{t_0}^{-k}(x_0)$ is either the singleton or a simple curve.

Lemma 3.4

Let $\mathbf{x_1} \neq \mathbf{x_2}$, $k > 0$, t_0 be given and denote by $|\mathbf{x_1} - \mathbf{x_2}|$ the distance beween $\mathbf{x_1}$ and $\mathbf{x_2}$. Then

$$k < \frac{|x_1 - x_2|}{|u|_{max}} \Rightarrow \Psi_{t_0}^{-k}(x_1) \cup \Psi_{t_0}^{-k}(x_2) = \emptyset \tag{8}$$

where $|\mathbf{u}_{max}|$ is the maximum velocity magnitude within time interval $(t_0 - k, t_0)$.

Lemma 3.5

If $x_0 \neq x_1$ *and* $\phi_{t_0}^{-k}(x_1) = M \in \Psi_{t_0}^{-k}(x_0)$, *then* $\Psi_{t_0}^{-k}(x_1)$ *and* $\Psi_{t_0}^{-k}(x_0)$ *overlaps for the curve segment* $\widetilde{Mx_0}$.

Lemma 3.6

Given $k > 0, t_0$, *and two seedings locations* $x_1 \neq x_2$ *with distinct streaklines. These streaklines do not intersect if* x_1 *and* x_2 *are sufficiently close:*

$$x_1 \to x_2 \to \Psi_{t_0}^{-k}(x_1) \cap \Psi_{t_0} - k(x_2) = \emptyset \tag{9}$$

Theorem 3.3

For all point $\mathbf{x}$ *in* R^3 *and given time* t_0, *there is a time interval* k *and a neighborhood of* $\mathbf{x}$ *such that the sreaklines of any points of the neighborhood of* $\mathbf{x}$ *neither self-intersect nor intersect with other streaklines.*

A map

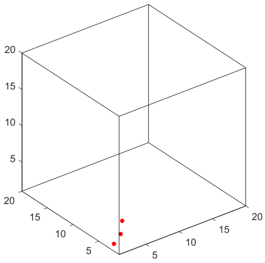

Figure: the preimage of cube: the bottom side is the preimage of γ^o, the boardside is the preimage of the streak surface, and the top margin is the preimage of path surface.

the flux identity for arbitrary DVs

hints:

finite covering Thm

inclusion-exclusion Thm

$$\Psi_{t_0+k}^{-k}(\gamma^o) = \cup_{i=1}^{n} b_i - \cup_{1\le i<j\le n} b_i \cap b_j + \cup_{1\le s<i<j\le n} b_i \cap b_j \cap b_s - \cdots + (-1)^n b_1 \cap b_2 \cap \cdots b_n \quad (10)$$

flux identity

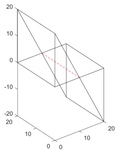

Figure: γ^o intersects with time surface

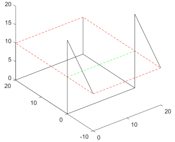

Figure: streak surface intersects with time surface

the flux identity for arbitrary DVs

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$$\Psi_{t_0+k}^{-k}(\gamma^o) = \cup_{i=1}^{n} b_i - \cup_{1\leq i<j\leq n} b_i \cap b_j + \cup_{1\leq s<i<j\leq n} b_i \cap b_j \cap b_s - \cdots + (-1)^n b_1 \cap b_2 \cap \cdots b_n \quad (11)$$

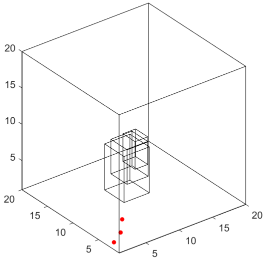

Figure: coverings:the big one is the preimage of DV, and the small ones are the preimage chosen coverings, and finite coverings theorem ensure that they will cover all the cuboid. If we discuss the meeting of those coverings, it is the meeting of those cuboids, and still a cuboid as the picture shows. It satisfies the form of DV.

static situation

We discuss it following two kinds of situations:

- If it is of the measure zero
- If it is not of the measure zero