# A FLUX IDENTITY OF SCALAR CONSERVATION LAWS FOR LAGRANGIAN FLUX CALCULATION VIA DONATING VOLUMES IN 3 DIMENSION

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## 1 Introduction

Given a fixed simple curved surface $\gamma^o$ and a time-dependent vector field $w : R^3 \times R \to R^3$, the total flux of the flux density vector $w(\boldsymbol{x}, t)$ through $\gamma^o$ within a time interval $[t_0, t_0 + k]$ can be expressed as

$$\int_{t_0}^{t_0+k} \int_{\gamma^o} \boldsymbol{w}(\gamma(s), t) ds dt, \tag{1.1}$$

where $\gamma(s)$ and $\boldsymbol{n_{\gamma^o}}$ are a parametrization and the normal vector of $\gamma^o$, respectively. We also refer to the total flux in (1.1) as the Eulerian flux integral of $\boldsymbol{w}$ through $\gamma^o$.

## 2 notations

In this section, we need to find a way to prove such a flux identity conclusion, which will make us generalize the cases in $R^2$.

To make it clear, let us make some definitions like the work in $R^2$.

### 2.1 The flow map

First we note that if f is further differentiable, we arrive at the scalar conservation law

$$\frac{\partial f}{\partial t} + \bigtriangledown \cdot (f \boldsymbol{u}) = 0. \tag{2.1}$$

The ordinary differential equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{2.2}$$

admits a unique solution for any given innitial time $t_0$ and any innitial position $p_0 \in R^D$ if the time-dependent velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is continuous and Lipschitz continue in space. This uniqueness ensures a flow map $\phi : R^D \times R \times R \to R^D$ that takes the innitial position $p_0$ and the time increment $\pm k$, and returns $p(t_0 \pm k)$, the position of $p$ at the final time $t_0 \pm k$:

$$\phi_{t_0}^{+k}(p) := p(t_0 + k) = p(t_0) + \int_{t_0}^{t_0+k} \boldsymbol{u}(p(t), t)dt \tag{2.3}$$

$$\phi_{t_0}^{-k}(p) := p(t_0 - k) = p(t_0) + \int_{t_0}^{t_0-k} \boldsymbol{u}(p(t), t)dt \tag{2.4}$$

We can also use the shorthand notation $\phi_{t_0+k}^{-k}(M)$ when $t_0$ and $k$ are clear from the context, where M is a simple curved surface. We call it the timeline.

## 2.2 Pathlines and Streaklines

One common characteristic curve of the flow map is the pathline, a curve generated by following a single particle in a time interval,

$$\Phi_{t_0}^{\pm k}(p) = \{\phi_{t_0}^{\pm\tau}(p) : \tau \in (0, k)\}. \tag{2.5}$$

We can define it in a location. $\Phi_{t_0}^{\pm k}(M) := \phi_{t_0}^{\pm\tau}(M) : \tau \in (0, k)$, where M is a simple curve.

A backward streakline is the loci of all particles that will pass continuously through a fixed seeding location M,

$$\Psi_{t_0+k}^{-k}(M) := \{\phi_{t_0+\tau}^{-\tau}(M) : \tau \in (0, k)\}, \tag{2.6}$$

and a forward streakline is the loci of all particles that have passed M,

$$\Psi_{t_0-k}^{+k}(M) := \{\phi_{t_0-\tau}^{+\tau}(M) : \tau \in (0, k)\}, \tag{2.7}$$

where the time increment k > 0.

## 2.3 Curves, Curved surface and Winding numbers

**Definition 2.1** (Winding numbers). *Let $\gamma$ be a closed curved surface and $\boldsymbol{x} \notin \gamma$ be a fixed point. The winding number of $\gamma$ around $\boldsymbol{x}$, written $\omega(\gamma, \boldsymbol{x})$, is the number of positive orientation of a Lagrangian partical p through $\gamma$ minus its number of negative, as p moves from its initial position $\boldsymbol{x}$ to $\infty$. The orientation of the curved surface can be defined by $\gamma_o$, which gives user ability to define the orientation. Moreover, if the curved surface is bilateral, the orientation will be well-defined, thus we can determine whether it is positive by the orientation of the normal vector.*

**Definition 2.2** (Flux sets). *The flux set of index n through a simple curved surface $\gamma^o$ over the time interval $(t_0, t_0 + k)$, written $F_{\gamma^o}^n(t_0, k)$, is the loci of all the fluxing particles of index n at time $t_0$. Given time t and time interval k, we can denote the flux number of a point $x_0$ to the times it come through the given curved surface $\gamma^o$.*

**Definition 2.3** (Donating volume). *For a given velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, the donating volume associated with a surface patch $\chi$ over time interval$(t_0 - k, t_0)$, denoted by $D_\chi(t_0, k)$, is the open curvilinear polyhedron enclosed in $\chi$, its timeline $\phi_{t_0}^{-k}(\chi)$, and the streak surface $\Phi_{t_0}^{-k}(\partial_\chi) = \cup_{N\in\partial_\chi} \Phi_{t_0}^{-k}(N)$.*

*$D_\chi(t_0,k)$ is canonical if it is simply connected; then it is positively oriented if the normal vector of $\chi$ points into ext($\partial D_\chi$), and negatively oriented otherwise.*

$$\begin{aligned} D^n_{\gamma^o}(t_0,k) &:= \{\boldsymbol{x} \in R^2 : \omega(\gamma_D, \boldsymbol{x}) = n\} \\ \gamma_D &:= (\gamma^o) \cup \gamma_0 \cup \Psi^{-k}_{t_0+k}(\gamma_0) \cup \phi^{-k}_{t_0+k}(\gamma^o) \cup \phi^{-k}_{t_0+k}(\gamma_0) \end{aligned} \tag{2.8}$$

*Given time $t_0$ and time interval k, we can define DV on any injective surface patch $\delta^o$.*

An injective surface patch is a continuous map $\gamma : (0,1) \times (0,1) \to R^3$, and the boundary is simple when $\gamma$ is injective. Owing to the map, we can find an Jordan curve which is the boundary of simple curved surface. It can easily modify the simple curved surface. Denote the boundary Jordan curve of the curved surface to $\gamma_0$, then $\gamma_0$ is also the boundary of the map $\gamma$. It suggests that the map of $\gamma_0$ is $\gamma : (1,i) \to R^3$ or $(0,i) \to R^3$ or $(i,1) \to R^3$ or $(i,0) \to R^3$ (i $\in$ [0,1]). Because boudary of a square is connected, the Jordan curve is also connected. If the map is simple, we will know the boundary of the map is Jordan curve. We denote the image of the map $\gamma$ to $\gamma^o$ and the boundary of $\gamma^o$ to $\gamma_0$.

Because the surface is obtained from a simple map, it can be oriented.

The map can be showed as Figure 1.

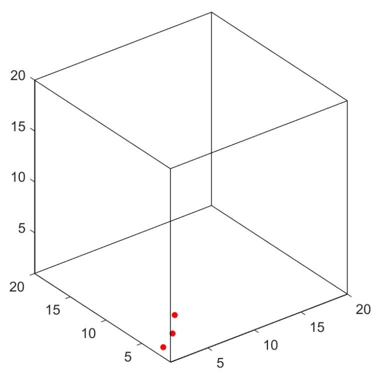


Figure 1: the preimage of cube: the bottom side is the preimage of $\gamma^o$, the boardside is the preimage of the streak surface, and the top margin is the preimage of path surface.

The orientation of a curved surface $\gamma^o$ corresponds to the ordinary definition. If the orientation is pointed to $\infty$, then denote the orientation of that surface to be positive, otherwise, denote the orientation of that surface to be negative.

Certainly, a closed curved surface which is homeomorphic to a unit ball can be oriented and then the winding number of it is well defined.

## 2.4 Homotopy and the Hopf theorem

A closed curved surface in topological space $\chi$ is a continuous map $\zeta$:from the boundary of the unit cube to $\chi$. A homotopy of closed curved surface in $\chi$ is a family of closed curved surface

$\gamma_t : (x, y, z) \to \chi, x, y, z \in [0, 1]$, and at least one variable is 1 or 0. The $xy$ flat is static, such as $\gamma_t(s_1, s_2, 0), \forall s_1, s_2 \in [0, 1]$ are independent of time and the associated map $\mathrm{H}:[0,1]^3 \to \chi$ defined by $\mathrm{H}(\mathrm{s_1}, \mathrm{s_2}, 0, \mathrm{t}) = \gamma_\mathrm{t}(\mathrm{s_1}, \mathrm{s_2})$ is continuous. The relation of homotopy on simple curved surface with fixed generated curve in any space is an equivalence relation. The equivalence class of closed curved surface under this equivalence relation is called the homotopy class of $\zeta$.

**Theorem 2.1** (Hopf Theorem)**.** *Let a point $x_0 \in R3$ be given. Two closed curves $\gamma_1$ and $\gamma_2$ are freely homotopic in $R3 \setminus \{x_0\}$ if and only if $\omega(\gamma_1, x_0) = \omega(\gamma_2, x_0)$.*

### 2.5 Reynolds transport theorem on the homotopy class of moving volumes

A moving closed curves which is not self-intersected in $R^3$ is a homotopy class of volumes.

$$\mathring{\Gamma} = \{\mathring{\gamma}(\tau) : \tau \in [0, k]\}, \tag{2.9}$$

where (1) each volume $\mathring{\gamma} : [0, 1] \times [0, 1] \times [0, 1] \to R^3$ is an oriented simple curved surface. (2) the map associated with the homotopy is smooth in time, and (3) all the simple curved surface have a consistent orientation.

A moving volume in the space is a family of regular subsets of $R^3$,

$$\{M(\tau) : \tau \in [0, k]\},$$

such that $\partial M(\tau) : \tau \in [0, k]$ is a set of moving simple volumes in $R^3$.

**Theorem 2.2** (Reynolds' transport theorem)**.** *let $M(\tau) : \tau \in [0, k]$ denote a moving closed curve in the space and $g(\cdot, \tau) : M(t) \to R$ a scalar function. We have*

$$\frac{d}{d\tau} \int_{M(\tau)} g(\boldsymbol{x}, \tau) d\boldsymbol{x} = \int_{M(\tau)} \frac{\partial g(\boldsymbol{x}, \tau)}{\partial \tau} d\boldsymbol{x} + \int_{\partial M(\tau)} g(s, \tau) V_n(s, \tau) ds, \tag{2.10}$$

*where $V_n(s, \tau)$ is the normal speed of the boundary of moving closed curve $\partial M(\tau) : \tau \in (0, k)$.*

*Proof.* The 3 dimensional situation has been proved, and equation (2.10) is what I mean. The reason why the function makes it is that the formula to devirate a function. □

## 3 Analysis

Many useful identities of two streaklines are given by other papers, so we concentrate on the intersections and self-intersections of the sreaklines.

Because the time may be different, even though two particles come from different positions at the innitial time, their streaklines still can be intersect. Even for one streakline, we cannot deny that it can self-intersect. Although one particle can never simultaneously at 2 positions on $\gamma^o$ which denotes the image set of that injective simple patch, $\Psi^k_{t_0-k}(p_0)$ can intersect several times with $\gamma^o$. Set 2 points $p, q$ on $\gamma^o$, we will know the streaklines of them may be intersect.

Streakline intersections and cusps are closely related to critical points of the advecting flow. Fortunately, Sard's theorem states that these intersection points are of measure zero, and thus they do not affect the validity of integral equations such as the flux identity.

## 3.1 The index-by-index equivalence of flux sets and donating volumes

Owing to the continuity of velocity field and time, the donating volume we defined in the definition (2.3) is connected and smooth, so it is homeomorphism with a unit sphere. And unit sphere is a closed curve which we can determine the orientation, which means it is a bilateral surface. So owing to the last word in the definition (2.1), winding number can be used.

We can set the similar map like the definition we mentioned. we denote the map $\gamma_{timesurface}$ : $[0,1]^2 \times 1$. And the map must be injective because of the only result of the ordinary function. It is also noted that the map's boundary is the same as the map $\gamma_{timesurface}$.

**Definition 3.1.** *For a nonempty DV, let*$[n^-, n^+]$ *denote the largest interval such that* $D^n_{\gamma^o} \neq \emptyset$ *for each* $n \in [n^-, n^+]$. *The unit normal vector of a simple curved surface* $\gamma^o$ *induced by a nonempty DV, written* $\boldsymbol{n}^{DV}_{\gamma^o}$, *is hte normal vector that agrees with the outward unit normal of* $D^{n^+}_{\gamma^o}$, *if* $n^+ > 0$; *otherwise* $\boldsymbol{n}^{DV}_{\gamma^o}$ *is the one that disagrees with the outward unit normal of* $D^{n^-}_{\gamma^o}$.

We set the difinition here only to imply the choosen orientation of the given surface $\gamma^o$.

**Definition 3.2** (proper intersection)**.** *Two surface patch* $\gamma_1$ *and* $\gamma_2$ *intersect properly if for all* $\epsilon_0 > 0$,$s_I \in (s_1, s_2)$, *and* $s'_I \in (s'_1, s'_2)$ *such that*

- $\gamma_1(s_I) = \gamma_2(s'_I) = x_I$;
- *there is* $|\epsilon| \not> \epsilon_0$, $\gamma_1(s_I - \epsilon)$ *and* $\gamma_1(s_I + \epsilon)$ *belongs to the two different half spaces determined by the tangent vector of* $\gamma_2$ *at* $s_I$.
- *there is* $|\epsilon| \not> \epsilon_0$, $\gamma_2(s_I - \epsilon)$ *and* $\gamma_2(s_I + \epsilon)$ *belongs to the two different half spaces determined by the tangent vector of* $\gamma_1$ *at* $s_I$.

*Two surface patch* $\gamma_1$ *and* $\gamma_2$ *intersect improperly if there is* $\epsilon_0 > 0$,$s_I \in (s_1, s_2)$, *and* $s'_I \in (s'_1, s'_2)$ *such that*

- $\gamma_1(s_I) = \gamma_2(s'_I) = x_I$;
- *for all* $|\epsilon| \not> \epsilon_0$, $\gamma_1(s_I - \epsilon)$ *and* $\gamma_1(s_I + \epsilon)$ *belongs to the two different half spaces determined by the tangent vector of* $\gamma_2$ *at* $s_I$.
- *for all* $|\epsilon| \not> \epsilon_0$, $\gamma_2(s_I - \epsilon)$ *and* $\gamma_2(s_I + \epsilon)$ *belongs to the two different half spaces determined by the tangent vector of* $\gamma_1$ *at* $s_I$.

*Me must note that improper intersection does not affect the winding number and it is the situation which is the same as the improper intersection does not exist.*

**Lemma 3.1.** *The set of generating curved surface of DVs for a simple open curve*

$$\mathring{\tau}_{\gamma^o}(t_0, k) := \{\tau_{\gamma^o}(t_0 + \xi, k - \xi) : \xi \in [0, k]\} \tag{3.1}$$

*is a homotopy class in* $R^3$.

*Proof.* Construct a closed curved surface via concatenation as

$$\zeta_r := \gamma^o \cup \gamma_0 \cup \phi^k_{t_0}(\gamma_0) \cup \Psi^k_{t_0}(\gamma^o)$$

It is easy to see that $\Psi_{t_0+k}^{-(k-\tau k)}(\gamma^o)$ are a diminishing loci that will pass $\gamma^o$ during the continuously shortened time intervals. The set of the preimages of $\gamma^o$ are also continuous with respect to $r$ and all DVs are homotopy with a circle in $R^3$. Therefore, the map $H(\boldsymbol{x}, t) = \zeta_t(\boldsymbol{x})$ is continuous and $[\zeta_r]$ is a homotopy class. Hence we derive this conclusion. □

Let us view other article, and there are several useful hints we can use.

**Lemma 3.2.** *A streakline does not intersect itself if the associated time interval is small enough. More precisely, there exists $k_0 > 0$ such that for all $k < k_0$ the streakline $\Psi_{t_0}^{-k}(x_0)$ is either the singleton or a simple curve.*

*Proof.* The proof has been given by [1]. □

**Lemma 3.3.** *Let $\boldsymbol{x_1} \neq \boldsymbol{x_2}$, $k > 0$, $t_0$ be given and denote by $|\boldsymbol{x_1} - \boldsymbol{x_2}|$ the distance beween $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$. Then*

$$k < \frac{|x_1 - x_2|}{|u|_{max}} \Rightarrow \Psi_{t_0}^{-k}(x_1) \cup \Psi_{t_0}^{-k}(x_2) = \emptyset \tag{3.2}$$

*where $|\boldsymbol{u}_{max}|$ is the maximum velocity magnitude within time interval $(t_0 - k, t_0)$.*

*Proof.* The proof has been given by [1]. □

**Theorem 3.1** (Index-by-Index equivalence of flux sets and DVs)**.** *Any DV of a fixed simple curved surface $\gamma^o$ is index-by-index equivalent to the flux set of $\gamma^o$,*

$$\forall n \in \mathrm{Z}, \forall k > 0, D_{\gamma^o}^n(t_0, k) = F_{\gamma^o}^n(t_0, k) \tag{3.3}$$

*if $\boldsymbol{n_{\gamma^o}^{DV}}$, the unit normal vector is positive.*
*For another cases of $\boldsymbol{n_{\gamma^o}^{DV}}$, $D_{\gamma^o}^n(t_0, k) = -F_{\gamma^o}^n(t_0, k)$*

*Proof.* Consider an arbitrary fixed point $\boldsymbol{x_0}$, denote a new flow map $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{x_0}, t)$. It suffices that equation (3.3) holds. As for $\boldsymbol{x_0}$, it is static, and for any points on $\gamma^o$, their velocity is $-\boldsymbol{u}(\boldsymbol{x_0}, t)$. the velocity of other points on the boundary of that DR which are $\notin \gamma^o := \gamma_d$ is $\boldsymbol{v}$. Let $\boldsymbol{n_0}$ be the integer such that $\boldsymbol{x_0} \in D_{\gamma^o}^{n_0}(t_0, k)$. According to the definition (2.3), the winding number of it equals $n_0$, and because of the last sentence in definition (2.3), when it comes to $\tau_{D_{\gamma^o}(t_0+k,0)}$, the winding number around $\boldsymbol{x_0}$ is zero because it is outside of that DV.

If $n_0 = 0$, we will know $\boldsymbol{x_0} \notin$ DV, the Hopf Theorem implies that deforms the homotopically to $\tau_{D_{\gamma^o}(t_0+k)}$ in $R^3 - \{x_0\}$. Hence the flux number of $\boldsymbol{x_0}$ is zero.

If $n_0 \neq 0$, because when it comes to $\tau_{D_{\gamma^o}(t_0+k)}$ in $R^3 - \{x_0\}$, the winding number of $x_0$ is 0, so the points around $\boldsymbol{x_0}$ must be crossed by the moving simple curved surface, therefore, the winding number of the points around $\boldsymbol{x_0}$ must change when the moving curved surface cross it. Furthermore, the new flow map is a diffeomorphism, hence $\gamma_d$ will never cross $\boldsymbol{x_0}$, and the only part can cross $\boldsymbol{x_0}$ is $\gamma^o$. When the generating curve cross $\gamma^o$, the winding number of $\boldsymbol{x_0}$ will minus 1. When it comes to $\gamma(t_0 + k, 0)$, the winding number of $\boldsymbol{x_0}$ must change to 0. Then it must be in $F_{\gamma^o}^{n_0}(t_0, k)$, hence we will gain the identity. □

## 3.2 The flux identity

The degree of a DV $D_{\gamma^o}$ is the maximun integer N such that $|n| > N$ implies $D^n_{\gamma^o} = \emptyset$ for all $n \in Z$.

**Lemma 3.4** (The flux identity for canonical DVs)**.** *Let $f$ be a scalar function conserved by a nonautonomous flow $\mathbf{u}(\boldsymbol{x}, t)$. If the DV $D_{\gamma^o}(t_0, k)$ is canonical, then*

$$\int_{t_0}^{t_o+k} \int_{\gamma^o} f\boldsymbol{u} \cdot \boldsymbol{n}_{\gamma^o} ds dt = \pm \int_{D_{\gamma^o}(t_0,k)} f(\boldsymbol{x}, t_0) d\boldsymbol{x} \tag{3.4}$$

*where $\boldsymbol{n}_{\gamma^o}$ is the unit normal vector of DV, and $\pm$ is determined by that conanical volume.*

*Proof.* We need to prove the situation $D_{\gamma^o} = D^{+1}_{\gamma^o}$, and $D^{-1}_{\gamma^o}$ is similar. For that we have,

$$\begin{aligned}
&\frac{d}{d\tau} \int_{D_{\gamma^o}(t_0+\tau,k-\tau)} f(\boldsymbol{x}, t_0 + \tau) d\boldsymbol{x} \\
&= \int_{D_{\gamma^o}(t_0+\tau,k-\tau)} \frac{\partial f(\boldsymbol{x}, t_0 + \tau)}{\partial \tau} d\boldsymbol{x} + \int_{\partial D_{\gamma^o}(t_0+\tau,k-\tau)} f(s, t_0 + \tau) V_n(s, t_0 + \tau) ds, \\
&= -\int_{D_{\gamma^o}(t_0+\tau,k-\tau)} \nabla \cdot (f\boldsymbol{u}) d\boldsymbol{x} + \int_{\partial D_{\gamma^o}(t_0+\tau,k-\tau)} f(s, t_0 + \tau) V_n(s, t_0 + \tau) ds, \\
&= \int_{\partial D_{\gamma^o}(t_0+\tau,k-\tau)} f(s, t_0 + \tau)(V_n(s, t_0 + \tau) - \boldsymbol{u} \cdot \boldsymbol{n}_{\boldsymbol{\gamma^o}}) ds \\
&= -\int_{\gamma^o} f(s, t_0 + \tau) \boldsymbol{w}(s, t_0 + \tau) \cdot \boldsymbol{n}_{\gamma^o} ds;
\end{aligned}$$

where the first step follows from the Reynolds transport Theorem (2.2), the second step from the hyperbolic conservation law (2.1), and the third step from the divergence theorem. It is important to distinguish the normal speed of the moving closed Jordan curve in homotopy class from those of the underlying ow. Because all Lagrangian particles in the initial DV $D_{\gamma^o}(t_0, k)$ are passively advected by the velocity eld u, we have $V_n = u \cdot n$ for every point in $\partial D_{\gamma^o}(t_0 + \tau, k - \tau) \setminus \gamma^o$. In comparison, $V_n = 0$ holds everywhere on the stationary curved surface $\gamma^o$. Therefore, $V_n - u \cdot n = u \cdot n$ holds on $\gamma^o$ and $V_n - u \cdot n = 0$ holds everywhere else on $\partial D_{\gamma^o}(t_0 + \tau, k - \tau)$. In the last step, the unit normal of $\gamma^o$ is exactly the same as the outward unit normal of $D_{\gamma^o}(t_0 + \tau, k - \tau)$. Then, integrating the equation in $[t_0, t_0 + k]$ yields

$$\begin{aligned}
&\int_0^k [\frac{d}{d\tau} \int_{D_{\gamma^o}(t_0+\tau,k-\tau)} f(\boldsymbol{x}, t_0 + \tau) d\boldsymbol{x}] d\tau = -\int_{t_0}^{t_0+k} \int_{\gamma^o} f(s, t) \boldsymbol{u}(s, t) \cdot \boldsymbol{n}_{\gamma^o} ds dt \\
\iff &\int_{D_{\gamma^o}(t_0+k,0)} f(\boldsymbol{x}, t_0 + k) d\boldsymbol{x} - \int_{D_{\gamma^o}(t_0,k)} f(\boldsymbol{x}, t_0) d\boldsymbol{x} = -\int_{t_0}^{t_0+k} \int_{\gamma^o} f\boldsymbol{u} \cdot \boldsymbol{n}_{\gamma^o} ds dt \\
\iff &\int_{D_{\gamma^o}(t_0,k)} f(\boldsymbol{x}, t_0) d\boldsymbol{x} = \int_{t_0}^{t_0+k} \int_{\gamma^o} f\boldsymbol{u} \cdot \boldsymbol{n}_{\gamma^o} ds dt
\end{aligned}$$

where it follows the fact that $D_{\gamma^o}(t_0 + k, 0) = \emptyset$. ∎

**Theorem 3.2.** *For all point $\boldsymbol{x}$ in $R^3$ and given time $t_0$, there is a time interval $k$ and a neighberhood of $\boldsymbol{x}$ such that the sreaklines of any points of the neighberhood of $\boldsymbol{x}$ neither self-intersect nor intersect with other streaklines.*

*Proof.* On the one hand, we can delete the streakline's self-intersections by reduce the time interval. Owing to the lemma 3.2, for any point $x_0$ in $R^3$ and a fixed innitial time $t_0$, there is a small time interval $k_0$ which can make the streakline of it does not self-intersect. Moreover, from the proof of the lemma 3.2, we can select a small neighberhood of $x_0$, the streaklines of the points in which do not self-intersect because when any point $\boldsymbol{y} \to \boldsymbol{x}$, it will not intersect and they will be packed nicely.

On the other hand, there is always packed nicely when we consider about a small neighberhood of a point's streaklines. Meanwhile, from the lemma 3.3, we can reduce time interval to prevent the streaklines of two points from intersecting by choosing the time interval $k < \frac{|x_1 - x_2|}{|\boldsymbol{u}|_{max}}$ because the tangent vector of streakline is the velocity of the field $\boldsymbol{u}$. Thus we prevent the intersections of two streaklines.

Then we take out the infimun of the time interval and then we get the neiberhood of $\boldsymbol{x}$ and it satisfies the demand of the theorem 3.2. □

**Theorem 3.3.** *If $x_0 \neq x_1$ and $\phi_{t_0}^{-k}(x_1) = M \in \Psi_{t_0}^{-k}(x_0)$, then $\Psi_{t_0}^{-k}(x_1)$ and $\Psi_{t_0}^{-k}(x_0)$ overlaps for the curve segment $\widetilde{Mx_0}$.*

*Proof.* The proof can be viewed in [1]. □

Then we need to generize this Lemma. We need to prove that we can split any kind of DV into some canonical spieces. We will do this in the next part.

**Theorem 3.4** (the flux identity for arbitrary DVs)**.** *Let $f$ be a scalar function conserved by a nonautonomous flow $\boldsymbol{u}(\boldsymbol{x}, t)$ such that (2.1) holds. The DV $D_{\gamma^o}(t_0, k)$ of a fixed simple curved surface $\gamma^o$ as in Definition 2.3 satisfies*

$$\forall k > 0, \int_{t_0}^{t_0+k} \int_{\gamma^o} f \boldsymbol{u} \cdot \boldsymbol{n}_{\gamma^o} ds dt = \sum_{n \in Z} n \int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x}, t_0) d\boldsymbol{x} \tag{3.5}$$

*Proof.* For a sufficiently small time increment, we can make the streak surface not to self-intersect. The streak surface will not be canonical if the streakline of the point on $\gamma_0$ self-intersects or intersect with other streakline of the point on $\gamma_0$. To make it clear, we prove this following three steps.

We take out all of the neighberhood of the points on $\gamma_0$ following the lemma 3.2. Note that $\gamma_0$ is closed, owing to the finite covering theorem, there are finite covering which can cover $\gamma_0$, and each of the covering corresponds to a point which also corresponds to a time interval.

We choose the infimum of those time interval. Because such time interval are finite, we can choose the smallest one. From the way we choose the point and time interval we will know the strealine of all points on $\gamma_0$ are not self-interval. Finally we have this conclusion.

We can also prevent the intersections of different points on $\gamma^o$ by reducing the time interval.

In conclusion, we can prevent the streak surface's self-intersections within a small time interval.

Therefore, we can split time interval into sufficiently small so that for any i, $\psi_{t_0+k_i+k_{i+1}}^{-k_{i+1}}(\gamma^o)$ is a donating volume, and $\Psi_{t_0+k_i+k_{i+1}}^{-k_{i+1}}(\phi_{t_0+k_i}(\gamma^o))$ will not self-intersect.

We can use the theorem 3.2 to any point x in the DV. In short, there is a neighberhood of x which are all neither self-intersect nor intersect with other sreaklines. We use it to all points on $\phi_{t_0}^s(x), x \in \gamma^o, s \in (0,k)$.

For any points we choose there is an open neighber of it which streaklines are not self-intersect or intersect with others, so each of the volumes is normal. We denote the neighberhood as $a_i$, and the time is $t_i$, and the time interval is $k_i$. Then the volume is

$$\Psi_{t_i+k_i}^{-k_i}(a_i) \cup \partial a_i \cup \phi_{t_i+k_i}^{-k_i}(a_i) \cup a_i \cup \phi_{t_i+k_i}^{-k_i}(\partial a_i)$$

.
We denote it $b_i$.

The static situation is trival. If all the static points of the streaklines are measure of zero, we ignore all of these points. We choose all the volumes which does not include any point which is static. If any volume is not canonical, there are 2 kinds of cases. (1) the streak surface $\Psi_{t_i+k_i}^{-k_i}(\partial a_i)$ $\cup$ the time surface $\phi_{t_i+k_i}^{-k_i}(a_i)$ is not $\emptyset$. (2) the time surface $\phi_{t_i+k_i}^{-k_i}(a_i)$ $\cup$ the neiberhood $a_i$ is not $\emptyset$.

For the first case, if it happens, then for one point x in $b_i$, the streakline of x either self-intersect or intersect with other streaklines, which is a contradiction to the way we choose the neighberhood of each point. For the second case, if it happens, because we choose x in volume is not static, so it must either self-intersect or intersect with other streaklines, which also shows a contradiction. Therefore, every volumes we choose are canonical. Moreover, we can prevent $b_j$ from intersecting with other point's streakline seeded at time $t_i$ by reducing time interval $k_i$ because of the theorem 3.2. Two cases are shown in Figure 2 and Figure 3.

Thus, for the lemma 3.4, we know for any $b_i$, the identity has been proved.

Because of the finite covering theorem, we can find finite covering $b_i (i = 1, 2, \cdots, n)$ such that every point with time in DV $(\boldsymbol{\phi_{t_0+s}^{-s}(\gamma^o)}, t_0 + s) \subset \cup_{i=1}^n b_i$. Each of those coverings is canonical DV, which shows the lemma 3.4 can be applied to those canonical volumes.

If we use the map to show that covering, those coverings like some cuboids which sides parallel the corresponding sides of $\partial DV$, and the union of those cuboids are also cuboids. They covers the whole DV. And we must note that even if the streaklines may intersect with each other, the image of those covering's maps will never self-intersect because of the content above.

It just like the figure (4) shows.

Owing to inclusion-exclusion principle, DV $\Psi_{t_0+k}^{-k}(\gamma^o)$ can be splitted to the association of those covering $b_i$, that is

$$\Psi_{t_0+k}^{-k}(\gamma^o) = \cup_{i=1}^n b_i - \cup_{1 \le i < j \le n} b_i \cap b_j + \cup_{1 \le s < i < j \le n} b_i \cap b_j \cap b_s - \cdots + (-1)^n b_1 \cap b_2 \cap \cdots b_n \quad (3.6)$$

Because the number of covering are finite, the mold of the set $\{b_i \cap b_j\}, \{b_i \cap b_j \cap b_s\} \cdots$ are also finite.

We can state that any meeting of $b_j$ are conincal volumes. We prove the situation $b_1 \cap b_2 \cap \cdots \cap b_s$, and other meetings are similar.

Without a loss of generation, denote $t_1 \le t_2 \le \cdots t_s$. Because $b_i$ are combined by many streaklines, if ($\boldsymbol{x}$,t) is in $b_1 \cap b_2 \cap \cdots \cap b_s$, then there must be points $s_i \in a_i (i = 1, 2, \ldots, s)$ and time interval $k_i' \in (0, k_i)$ such that

$$\boldsymbol{x} = \phi_{t_i+k_i'}^{-k_i'}(s_i), \forall i = 1, 2, \ldots, s$$

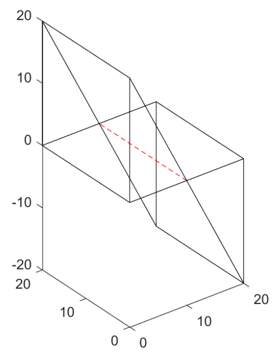


Figure 2: $\gamma^o$ intersects with time surface

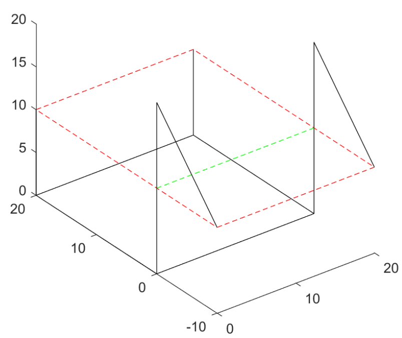


Figure 3: streak surface intersects with time surface

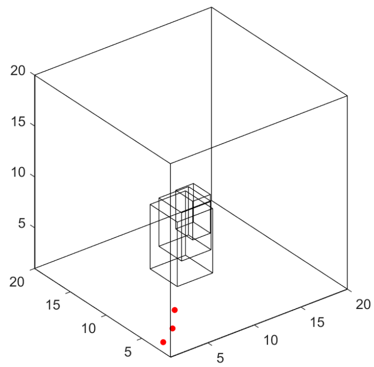


Figure 4: coverings:the big one is the preimage of DV, and the small ones are the preimage chosen coverings, and finite coverings theorem ensure that they will cover all the cuboid. If we discuss the meeting of those coverings, it is the meeting of those cuboids, and still a cuboid as the picture shows. It satisfies the form of DV.

In other word, $\boldsymbol{x}$ must be on the streaklines of $b_i$, that is the streaklines of $b_i$ must intersect or the same. If there are 2 innitial time $t_i$ and $t_j$ such that $t_i = t_j$, then $s_i = s_j$ because if $s_i \neq s_j$, the streaklines would not intersect owing to the way we choose the time interval. If any innitial time $t_i$ and $t_j$ do not equal, then we denote that $t_i < t_j$, we find $\phi_{t_j}^{t_i - t_j}(s_j)$, then like the above mentioned discussion, it must be the point $s_i$. So the streaklines must be the same one.

So $b_1 \cap b_2 \cap \cdots \cap b_s$ must be a DV. We find the image of $b_i$ when $t = t_1$, denote it as $image_i$, and $min\{k_1 + t_1 - t_s, k_2 + t_2 - t_s, \cdots, k_s\}$ to $t'$ then we will have

$$b_1 \cap b_2 \cap \cdots \cap b_s = \begin{cases} \Psi_{t_s+t'}^{-t'}(\cap_{i=1}^{s} image_i) \quad t' > 0 \\ \emptyset \quad t' \leq 0 \end{cases} \tag{3.7}$$

Because for any $b_i$ the streaklines of $a_i$ are all packed nicely, the streaklines of $b_1 \cap b_2 \cap \cdots \cap b_s$ are also packed nicely. Like the case above mentioned, the boundary of this meeting is also canonical.

Let $D^{\pm}$ denote the winding number positive or negative of the point in canonical DVs $b_i$, $D_2^{\pm}$ denote the winding number positive or negative of the point in canonical DVs $b_i \cap b_j$, and so is

$D_s^{\pm}(s=3,4,\cdots,n)$, then equation 3.5 can become like this

$$\begin{aligned}
&\forall k>0, \int_{t_0}^{t_0+k}\int_{\gamma^o} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}\\
\iff &\int_{\cup_{i=1}^n b_i - \cup_{1\le i<j\le n} b_i\cap b_j + \cup_{1\le s<i<j\le n} b_i\cap\cdots b_n} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}\\
\iff &\sum_{i=1}^n\int_{b_i} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt - \sum_{1\le i<j\le n}\int_{b_i\cap b_j} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt + \cdots = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}\\
\iff &\sum_{i=1}^n\int_{D^+} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt - \sum_{i=1}^n\int_{D^-} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt - \sum_{1\le i<j\le n}\int_{b_i\cap b_j} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt + \cdots = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}\\
\iff &\int_{D_1^+} f(\boldsymbol{x},t_0)d\boldsymbol{x}t - \int_{D_1^-} f(\boldsymbol{x},t_0)d\boldsymbol{x}t - \sum_{1\le i<j\le n}\int_{b_i\cap b_j} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt + \cdots = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}\\
\iff &\sum_{i=1}^n(\int_{D_i^+} f(\boldsymbol{x},t_0)d\boldsymbol{x}t - \int_{D_i^-} f(\boldsymbol{x},t_0)d\boldsymbol{x}t) = \sum_{n\in Z} n\int_{D^n_{\gamma^o}(t_0,k)} f(\boldsymbol{x},t_0)d\boldsymbol{x}
\end{aligned}$$

The second and third equation is from the resolution of the streaklines. The fourth equation is from the lemma 3.4. The fifth equation is from the canonic volumes and the lemma 3.4. And then it combines to the last equation owing to the identity of the flux number and winding number.

So if there is not any static point in $\Psi_{t_0+k}^{-k}(\gamma^o)$, we gain such a conclusion.

If there are some static points in $\Psi_{t_0+k}^{-k}(\gamma^o)$, we discuss it. If the measure of static point is zero, we ignore all the static points then we can prove it like the previous method. If the measure of the static point is bigger than zero, we can take out a open set $\alpha$, which measure is the same with the static point in $\Phi_{t_0+k}^{-k}(\gamma^o)$, such that the static point is all in $\alpha$. And we will have

$$\int_{\alpha} f\boldsymbol{u}\cdot\boldsymbol{n}_{\gamma^o} dsdt = 0,$$

and the flux number of the point in $\alpha$ would not change, so it does not affect the other formula. In conclusion, we can use the method aboved mentioned to the volume $\Psi_{t_0+k}^{-k}(\gamma^o)\setminus\alpha$. $\square$