1. Prove that the following are equivalent for a ring R:

a) R has an identity element;

b) whenever $R \triangleleft S$ for a ring S (possibly without an identity), then R is a ring direct summand of S (i. e. there is an ideal $T \triangleleft S$ such that $R \cap T = \{0\}$ and $R + T = S$).

Solution. a) $\Rightarrow$ b). Consider the set $T = \mathrm{Ann}_S R = \{s \in S \mid sR = 0\}$. Since $R \triangleleft S$, we get that $T \triangleleft S$, furthermore $R \cap T = \{0\}$, since R has an identity element e, hence no element of R annihilates R. Finally, we can observe that $s = se + (s - se)$ for each $s \in S$, and here $se \in R$ and $s - se \in T$. This means that S is the direct sum of the ideals R and T, as required. — b) $\Rightarrow$ a) We know that any ring R can be embedded as an ideal into a ring S with identity. If R is also a direct summand of S, then the projection of 1_S in R is going to be an identity element in R.

2. Give an example of a non-commutative ring R for which the multiplicative group of invertible elements is commutative.

Solution. Take $R = K\langle x, y\rangle$, the free algebra over a (commutative field) K. Here the invertible elements are only the non-zero elements of K and the multiplication in K is commutative. – **Second solution:** Take $R = T_2(\mathbb{Z}_2)$, the set of upper triangular matrices over $\mathbb{Z}_2$. The multiplication is non-comutative (for example $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$, but $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$), on the other hand there are only two invertible elements, hence they form an abelian group.

3. Let us adopt a convention that if M is a left R-module then $\mathrm{End}_R(M)$ acts on M from the right. With this notation in mind show that if R is a ring (with identity) then the endomorphism ring of R as a module over itself is isomorphic to R. What happens to this statement if we write both the scalars and the endomorphisms from the left?

Solution. (Observe that $(r)\varphi$ will stand for the image of r under the action of φ since we apply endomorphisms from the right.) Right multiplication by elements of R always gives a left-module homomorphism of ${}_RR$, and this gives a mapping of R into the endomorphism ring $\mathrm{End}({}_RR)$. Since $1 \in R$, this mapping is injective. On the other hand it is easy to show that any endomorphism φ acts as right multiplication by $\varphi(1)$: namely $(r)\varphi = (r \cdot 1)\varphi = r \cdot (1)\varphi$. This gives a bijection $R \to \mathrm{End}({}_RR)$ and clearly this is a ring homomorphism; in particular right multiplication by rs is right multilplication by r followed by right multiplication by s. – If we let the endomorphisms act from the left, then the ring of endomorphisms is R^{op}.

4. How many ring-endomorphisms are there for the ring $\mathbb{R}$? And for $\mathbb{C}$?

Solution. $\mathbb{R}$ has only two ring-endomorphisms: the identity and the zero map. It is easy to show that if a map is non-zero then 1 maps to 1: we have $1 \cdot 1 = 1$ hence $\varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1) = \varphi(1)$ for any ring-endomorphism φ, hence $\varphi(1)$ is the root of the polynomial $x^2 - x$. Thus $\varphi(1) = 1$ or $\varphi(1) = 0$. This implies that for a non-zero endomorphism the elements of $\mathbb{Q}$ are also fixed. Since any endomorphism maps positive numbers to positive numbers – they can be characterized by the fact that they have a square root –, this means that any endomorphism preserves the ordering. Hence all real numbers must be kept fixed. – For $\mathbb{C}$, take a transcendence basis S of $\mathbb{C}$ over $\mathbb{Q}$. Clearly S has cardinality $c = 2^{\aleph_0}$. Then the uniqueness of the algebraic closure ensures that any automorphism of $\mathbb{Q}(S)$ can be extended to an automorphism of $\mathbb{C}$. Since we have 2^c bijections of S to itself (prove it!) which can be extended to an automorphism of $\mathbb{C}$, we get 2^c automorphisms of $\mathbb{C}$. Since this is the same as the cardinality of all possible maps from $\mathbb{C}$ to $\mathbb{C}$, we get that the number of endomorphisms of $\mathbb{C}$ is 2^c. (Note that not all endomorphisms are automorphisms although non-zero endomorphisms must be injective. For example any injective but not surjective map $S \to S$ gives rise to such an endomorphism. Thus there are many subfields of $\mathbb{C}$ isomorphic to $\mathbb{C}$.)

5. Let a, b be elements of a ring R. Show that if $1 - ab$ is left invertible then $1 - ba$ is also left invertible.

Solution. If $1-ab$ is left invertible, then $R(1-ab) = R$, hence $Ra = R(1-ab)a = Ra(1-ba) \subseteq R(1-ba)$, thus the left ideal $R(1-ba)$ contains $1-ba$ and ba hence also their sum $(1-ba)+ba = 1$. This means that $1-ba$ is left invertible. (Note that the argument above also gives an easy way of describing the left inverse of $1-ba$ explicitly.) – **A hint for a second solution.** From analysis we "know" that $\dfrac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots$. Thus $\dfrac{1}{1-ba} = 1+ba+(ba)^2+\cdots+(ba)^n+\cdots = 1+b(1+ab+(ab)^2+\cdots+(ab)^n+\cdots)a = 1+b\dfrac{1}{1-ab}a$.

6. Show that if an element $a \in R$ has at least 2 right inverses then it has infinitely many right inverses.

Solution. Let b_0 be a fixed right inverse of a. Note that from the assumption it follows that a has no left inverse. Consider now the set S of all right inverses to a and define the following map from S to itself: $b \mapsto ba - 1 + b_0$. Then it is clear that we indeed get a map into S, furthermore b_0 is not in the image of the map. On the other hand the map is injective. Hence S must be infinite.

7. Show that the free K-algebra generated by a countably infinite set, $K\langle x_1, x_2, \ldots\rangle$ is isomorphic to a subalgebra of $K\langle x, y\rangle$.

Solution. Take the subalgebra A generated by the elements $z_i = xy^i \in K\langle x, y\rangle$. Then one can see easily that different monomials in z_i's convert into different monomials in x and y (i. e. one can trace back the monomial in z_i's from a monomial in x and y) and these monomials are K-independent. Hence $A \simeq K\langle z_1, z_2, \ldots\rangle$, i. e. the z_i's are free generators.

8. Let V be a vector space with basis $\{e_1, \ldots, e_n\}$. Give a basis for the exterior algebra $\bigwedge(V)$. Show that it is not true that the multiplication in $\bigwedge(V)$ is anticommutative.

Solution. $T(V) = \langle e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mid 1 \le i_1, \ldots i_k \le n\rangle$, hence the image of these basis elements (denoted by $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$) is going to generate the exterior algebra. Since $\bigwedge(V)$ is obtained from $T(V)$ by factoring out with elements $v \otimes v$, we get that the image of $(e_i + e_j) \otimes (e_i + e_j) = (e_i \otimes e_i) + (e_i \otimes e_j) + (e_j \otimes e_i) + (e_j \otimes e_j)$ is zero, thus $e_i \wedge e_j = -e_j \wedge e_i$. Thus $\bigwedge(V) = \langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < i_2 < \ldots < i_k \le n\rangle$. We still have to show that this is a basis. One can check that by taking the vector space spanned by these (formal) elements and defining a multiplication in a natural way, taking into the account that the order of the basis elements in a product can be exchanged using the anticommutativity rule, we get an associative algebra A which is clearly the image of $T(V)$ and this mapping factors through the natural map $T(V) \to \bigwedge(V)$. This shows the independence of our generating set. – The anticommutativity rule applies only to elements of v: we have $(*)$: $u \wedge v = -v \wedge u$ for $u, v \in V$. On the other hand elements of even degree will commute: $(e_1 \wedge e_2) \wedge (e_3 \wedge e_4) = (e_3 \wedge e_4) \wedge (e_1 \wedge e_2)$ since we have to make four changes of type $(*)$ to obtain the right hand side from the left hand side. If V is of dimension at least 4, the above elements are non-zero, thus the multiplication in general is not anticommutative.

9. a) Let k be an arbitrary ring. Characterize the set of invertible elements in $k\llbracket x \rrbracket$, the ring of formal power series over k.

b) Show that if k is a field, then the ring of formal Laurent series, $k(\!(x)\!)$ is also a field.

Solution. a) The invertible elements are those power series $f = f_0 + f_1x + f_2x^2 + \cdots$ for which the constant term f_0 is a unit in k. To show this, observe first that in $k\llbracket x \rrbracket$ the constant term of a product is the product of the constant terms. Thus if f is invertible in $k\llbracket x \rrbracket$ then f_0 has to be invertible in k. On the other hand solving $fg = 1$ in $k\llbracket x \rrbracket$ amounts to solving the equations $f_0g_0 = 1$, $f_0g_1 + f_1g_0 = 0$, $f_0g_2 + f_1g_1 + f_2g_0 = 0$, $\ldots$. Since f_0 is invertible, we can solve these equations recursively. b) If k is a field, any nonzero element in $k(\!(x)\!)$ can be multiplied by a suitable power of x (possibly with negative exponent) so that by a) we get an invertible element in $k\llbracket x \rrbracket$. This shows that every such element is invertible in $k\llbracket x \rrbracket$.

10. Let R be the ring of local germs of real continuous functions defined in a neighborhood of 0. Describe the invertible elements and show that R contains a unique maximal (proper) ideal.

Solution. A germ of a function f is invertible if and only if $f(0) \neq 0$: in this case $f(x) \neq 0$ in a neighborhood of 0, hence in this neighborhood $1/f$ is defined and the product of the two germs is equal to the germ of the constant 1 function, while if $f(0) = 0$ then no inverse can exist. From this we also get that the non-invertible elements form an ideal in R; clearly this ideal is the unique maximal ideal in (the commutative ring) R.

11. Let R be the ring of additive endomorphisms of $K[y]$. Let $D \in R$ be the differentiation with respect to y and Y the (left) multiplication by y. Let S be the K-subalgebra of R generated by D and Y.

a) Describe the elements of S.

b) Show that if $\operatorname{char} K = 0$ then S is isomorphic to the Weyl-algebra $A_1(K)$.

Solution. a) The elements of S are of the form $\sum p(Y)D^i$, where p is a polynomial in $K[y]$: clearly, all these elements must belong to S, on the other hand they form a subalgebra. In order to show the closure under multiplication one has to use the relation $DY = YD + 1$: using the Leibniz rule for differentiation, we get that for arbitrary $f(y) \in K[y]$ we have: $(DY)(f(y)) = (yf(y))' = f(y)+yf'(y) = f(y)+(YD)(f(y))$. Thus we get that $DY-YD = 1$ in R. Hence every product of type DY can be replaced by a product YD and he identity function. b) The previous identity $DY - YD = 1$ implies that for the unique ring epimorphism $\Phi : K\langle x, y\rangle \to R$, mapping x to D and y to Y we have $xy-yx-1 \in \operatorname{Ker}\Phi$. Thus Φ factors through $\bar{\Phi} : K\langle x, y\rangle/(xy-yx-1) = A_1(K) \to R$. Dimension counting on subspaces spanned by $\left\{x^iy^j \mid 0 \leq i, j \leq n\right\}$ and $\left\{D^iY^j \mid 0 \leq i, j \leq n\right\}$ shows that $\bar{\Phi}$ is an isomorphism.

12. a) Let $R = R_1 \oplus \cdots \oplus R_n$ (i. e. R is a ring direct sum of some two-sided ideals). Determine the left, right and two-sided ideals of R. What can one say about the structure of modules over R?

b) Let $B_1, \ldots, B_n$ be left ideals (resp. two sided ideals) in the ring R. Show that R is the module theoretic (resp. ring theoretic) direct sum of $B_1 \oplus \cdots \oplus B_n$ if and only if there exist idempotent elements (resp. central idempotents) $e_1, \ldots, e_n$ for which $e_ie_j = 0$ whenever $i \neq j$, $1 = e_1 + \cdots e_n$ and $B_i = Re_i$.

Solution. a) Let I be a (left, right, two-sided) ideal in R, and let I_i be the image of I under the natural projection $\pi_i : R \to R_i$. It is easy to see that I_i is also a (left, right, two-sided) ideal in R_i. Moreover besides the obvious containment $I \subseteq I_1 \oplus I_2 \oplus \cdots \oplus I_n$ we also have $I_1 \oplus I_2 \oplus \cdots \oplus I_n \subseteq I$, since multiplication of an element $r \in R$ by the identity element e_i of R_i shows that $\pi_i(r) = re_i \in I$, thus $I_i \subseteq I$. Hence a (left, right, two-sided) ideal of a ring direct sum is always a direct sum of similar substructures of the components. Obviously the converse also holds. Finally, the modules over R are direct sums of modules over R_i, each of which is annihilated by the other ring direct summands of R. – b) The existence of such idempotents clearly implies the existence of the corresponding direct decompositions. (Note that if e_i is central, then Re_i automatically becomes a two-sided ideal.) Conversely, assume that R is a direct sum as a left R-module of the left ideals B_i (or as a ring of the two-sided ideals B_i). Then 1 can be written uniquely as $1 = e_1 + e_2 + \cdots + e_n$ with e_i in B_i. From $r = r \cdot 1 = re_1 + re_2 + \cdots + re_n$ and $re_i \in B_i$ we get that $B_i = Re_i$. Furthermore, $e_i = e_ie_1 + e_ie_2 + \cdots + e_ie_i + \cdots + e_ie_n$ and the uniqueness of the decomposition gives $e_i^2 = e_i$ and $e_ie_j = 0$ for $i \neq j$. Finally, if B_i are two-sided ideals, then e_i will play the role of the identity element of B_i hence it commutes with the elements of B_i, furthermore it clearly commutes with elements of B_j for $j \neq i$. Hence each e_i is central.

13. A module is Noetherian (resp. Artinian) if there is no strictly increasing (resp. decreasing) infinite chain of submodules. A ring is left Noetherian (left Artinian) if it is Noetherian (Artinian) as a left module over itself.

a) Show that if $M \leq N$, then N is Noetherian (resp. Artinian) if and only if both M and N/M are Noetherian (Artinian).

b) Find examples among Abelian groups which would have the first, the second, none or both of these properties. (*) Can you do the same thing with rings?

Solution. a) It is clear that if N satisfies one of the chain conditions then the same condition is satisfied for any submodule or homomorphic image of N. Let us now assume that $M \leq N$ and both M an N/M satisfy, say, the ascending chain condition. Consider now an ascending chain of submodules of N: $N_1 \leq N_2 \leq \ldots \leq N_k \leq \ldots$ and define the corresponding chains in M and N/M, consisting of the modules $N_i \cap M$ and $(N_i + M)/M$. By assumption, both chains must stabilize after a finite number of steps. But if for some n we have $M_n \cap M = M_{n+1} \cap M$ and $(M_n + M)/M = (M_{n+1} + M)/M$, then we must have $M_n = M_{n+1}$. To show this, suppose that $a \in M_{n+1}$. Then $a + N \in (M_{n+1} + M)/M = (M_n + M)/M$, hence there is $b \in M_n$ such that $a + N = b + N$. Then $a - b \in N$, and $M_n \subseteq M_{n+1}$ implies that $a - b \in M_{n+1} \cap N$. But $M_{n+1} \cap N = M_n \cap N$, hence $a - b \in M_n$ also holds, implying that $a = b + (a - b) \in M_n$. Thus $M_n = M_{n+1}$, and this proves that N satisfies the ascending chain condition. b) ${}_{\mathbb{Z}}\mathbb{Z}$ is Noetherian, but not Artinian; ${}_{\mathbb{Z}}\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian; ${}_{\mathbb{Z}}\mathbb{Q}$ does not satisfy any of these conditions and ${}_{\mathbb{Z}}\mathbb{Z}_n$ satisfies both. Let us now turn to the case of rings. The ring of integers $\mathbb{Z}$ is left Noetherian but not left Artinian, $M_n(\mathbb{R})$ satisfies both chain conditions while $\mathbb{R}[x_1, \ldots, x_n, \ldots]$ satisifes none of the chain conditions. It will be shown later (see the theorem of Hopkins) that any left Artinian ring is also left Noetherian. (Note that the latter statement is not true without assuming that the rings have identity elements since for example $\mathbb{Z}_\infty$ as a zero-ring will be left artinian but not left Noetherian.)

14. a) Show that the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian.

b) Show that the ring $S = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ is both right noetherian and right artinian but neither left noetherian nor left artinian.

Solution. a) R is not left noetherian as the following infinite increasing sequence of left ideals shows:y

$$\begin{pmatrix} 0 & \mathbb{Z}\cdot 1 \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & \mathbb{Z}\cdot 1/2 \\ 0 & 0 \end{pmatrix} \subseteq \cdots \subseteq \begin{pmatrix} 0 & \mathbb{Z}\cdot 1/2^n \\ 0 & 0 \end{pmatrix} \subseteq \cdots$$

On the other hand, to show that R is right noetherian, we shall use the fact (proved in 13.a)) that if a module M has a submodule N such that both N and M/N are Noetherian then M is also Noetherian. Consider the following sequence of submodules of R_R:

$$0 \le \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

Here the corresponding quotient modules have the same submodule structure as $\mathbb{Q}_\mathbb{Q}$, $\mathbb{Z}_\mathbb{Z}$ and $\mathbb{Q}_\mathbb{Q}$ which are all noetherian. – b) We will show first that S is neither left artinian nor left noetherian. To this end consider the left ideal $\begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$. It is clear that the S-submodules will correspond to $\mathbb{Q}$-subspaces of $\mathbb{R}$. Since $\dim_\mathbb{Q} \mathbb{R} = \infty$, we get both decreasing and increasing infinite chains of submodules. On the other hand for the right submodule structure, like in the previous example, we give the following "filtration" (i. e. chain of submodules):

$$0 \le \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$$

Here the submodules of the consecutive quotient modules will correspond to the submodules of $\mathbb{R}_\mathbb{R}$, $\mathbb{Q}_\mathbb{Q}$ and $\mathbb{R}_\mathbb{R}$. Hence S is both noetherian and artinian from the right side.

15. Let $\Gamma = (V, E)$ be an oriented graph. Give necessary and sufficient conditions so that:

a) the path algebra $K\Gamma$ has an identity;
b) the path algebra $K\Gamma$ is finite dimensional over K;
c) the path algebra $K\Gamma$ is semisimple (i. e. left artinian with no nilpotent ideals);
d) indecomposable as an algebra, i. e. there is no notnontrivial decomposition of $K\Gamma$ into the direct sum of ideals.

Solution. a) $K\Gamma$ has an identity element if and only if Γ has finitely many vertices. In this case $1_{K\Gamma} = e_1 + e_2 + \ldots + e_n$ where e_i is the path of length zero, starting and ending at vertex 0. First of all it is easy to see that this element indeed acts as an identity element in the path algebra. Secondly, it is easy to see that the product of two elements in $K\Gamma$ whose supports are disjoint is zero. (Note that by support we mean the set of vertices belonging to some paths in the element.) Note that the support of elements in $K\Gamma$ is always finite thus there is no identity element in $K\Gamma$ if Γ has infinitely many vertices. – b) $K\Gamma$ is finite dimensional if and only if Γ is finite and there are no oriented cycles in Γ. – c) If $K\Gamma$ is infinite dimensional then we will show that it cannot be left artinian. First, if $K\Gamma$ contains paths of arbitrary length then the ideals B_i generated by paths of length at least i will form a strictly decreasing infinite chain. Thus assume that the length of paths is bounded. Then either the number of arrows or the number of vertices in Γ must be infinite. If the number of vertices is infinite then by taking an infinite strictly decreasing sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of subsets of vertices we get an infinite strictly decreasing sequence of left ideals L_i consisting of linear combinations of those paths which end in a vertex belonging to I_i. Clearly these left ideals must be distinct since the idempotents corresponding to a vertex will belong to J_i if and only if the vertex belongs to I_i. Thus we may assume that Γ contains infinitely many arrows. Taking a similar strictly decreasing sequence of subsets of arrows we can take the left ideals generated by the corresponding arrows. Once more we get a strictly decreasing infinite sequence of left ideals. Thus if $K\Gamma$ is left artinian then $K\Gamma$ must be finite dimensional. Take now the ideal J generated by arrows: this is the set of linear combinations of paths of non-zero length. Since the length of paths is bounded, say, by n, we get that $J^{n+1} = 0$. The semisimplicity assumption implies that $J = 0$. Hence Γ must be a finite set of isolated points. Clearly, any such graph gives rise to a semisimple path algebra which is isomorphic to a direct sum of finite number of copies of the base field K. – d) If Γ is not connected then $\Gamma = \Gamma_1 \cup \Gamma_2$ with Γ_1 and Γ_2 being disjoint, non-empty and not connected by arrows. Then it is easy to see that $K\Gamma = K\Gamma_1 \oplus K\Gamma_2$. Thus Γ must be connected. The converse implication that if Γ is connected then $K\Gamma$ is indecomposable is similarly easy.

16. Is it true that in a path algebra the only idempotent elements are the basis elements corresponding to the vertices (and their sums)?

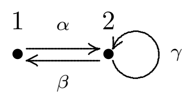
Solution. No: take the graph with two vertices (1 and 2) and one arrow α from 1 to 2. Then $(e_1 + \alpha)^2 = e_1 + \alpha$ is an idempotent but not of the given kind.

17. Take the path algebra corresponding to an oriented chain with n vertices. Denote by e_i the path of 0 length corresponding to vertex i.

a) Find the dimension of the path algebra $K\Gamma$.
b) Find the dimension of the right, left and twosided ideal genarated by the element e_2. Do the same for the path $2 \to \cdots \to (n-1)$.
c) Show that $K\Gamma$ as a left module over itself is the direct sum of left ideals generated by the elements e_i.

Solution. a) There are n paths of length 0, $n-1$ paths of length 1, etc. and 1 path of length $n = 1$. Hence $\dim K\Gamma = \dfrac{n(n+1)}{2}$. b) The left ideal $K\Gamma e_2$ is spanned by those paths which end at vertex 2, the right ideal generated by e_2 is spanned by those paths which start at vertex 2, and the ideal generated by e_2 is spanned by those paths which go through the vertex 2. Assuming that there are arrows α_i from i to $i+1$, we get that $K\Gamma e_1 = \langle e_2, \alpha_1 \rangle$, $e_2 K\Gamma = \langle e_2, \alpha_2, \alpha_2\alpha_3, \ldots, \alpha_2\alpha_3 \cdots \alpha_{n-1} \rangle$ and $(e_2) = \langle \alpha_1, \alpha_1\alpha_2, \ldots, \alpha_1\alpha_2 \cdots \alpha_{n-1}, e_2\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_2\alpha_3 \cdots \alpha_{n-1} \rangle$. Hence the required dimensions are 2, $n-1$ and $2(n-1)$. Similarly, the dimensions of the left, right and two-sided ideals generated by the path between 2 and $n-1$ are 2, 2 and 4. c) Use the characterization in Problem 8.

18. Take the following graph G:



and let I be the ideal in KG generated by the elements $\alpha\gamma$ and $\beta\alpha - \gamma^2$. Determine the dimension of the path algebra KG/I, and give the multiplicatin table of the algebra.

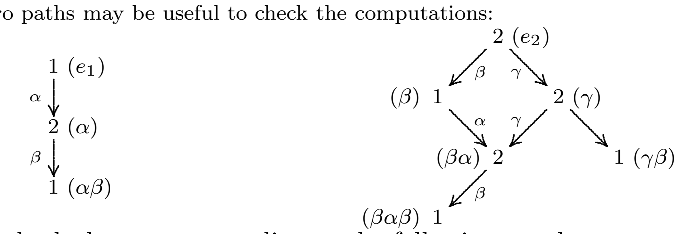
Solution. One can check that the form of the generating admissible relations will imply that we shall have a basis in $A = KG/I$ formed of non-zero paths (more precisely of images of paths) and it will be a *multiplicative basis* in the sense that the product of any two basis elements will always result in a basis element or 0. So the list of non-zero paths is the following:

$$\text{paths starting at 1: } e_1,\ \alpha,\ \alpha\beta$$
$$\text{paths starting at 2: } e_2,\ \beta,\ \gamma,\ \beta\alpha,\ \gamma\beta,\ \beta\alpha\beta$$

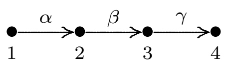
The fact that for example $\gamma\beta\alpha = 0$ in A follows from $\gamma\beta\alpha = \gamma\gamma^2 = \gamma^2\gamma = \beta\alpha\gamma = 0 \cdot \gamma = 0$. Thus $\dim A = 9$ and the multiplication table on the basis elements is the following:

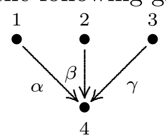
	e_1	α	$\alpha\beta$	e_2	β	γ	$\beta\alpha$	$\gamma\beta$	$\beta\alpha\beta$
e_1	e_1	α	$\alpha\beta$	0	0	0	0	0	0
α	0	0	0	α	$\alpha\beta$	0	0	0	0
$\alpha\beta$	$\alpha\beta$	0	0	0	0	0	0	0	0
e_2	0	0	0	e_2	β	γ	$\beta\alpha$	$\gamma\beta$	$\beta\alpha\beta$
β	β	$\beta\alpha$	$\beta\alpha\beta$	0	0	0	0	0	0
γ	0	0	0	γ	$\gamma\beta$	$\beta\alpha$	0	$\beta\alpha\beta$	0
$\beta\alpha$	0	0	0	$\beta\alpha$	$\beta\alpha\beta$	0	0	0	0
$\gamma\beta$	$\gamma\beta$	0	0	0	0	0	0	0	0
$\beta\alpha\beta$	$\beta\alpha\beta$	0	0	0	0	0	0	0	0

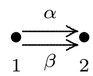
The following diagram of non-zero paths may be useful to check the computations:



19. Write as matrix rings the path algebras corresponding to the following graphs:

Γ_1: 

Γ_2: 

Γ_3: 

Solution. $K\Gamma_1 \simeq \begin{pmatrix} K & K & K & K \\ 0 & K & K & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}$; $\quad K\Gamma_2 \simeq \begin{pmatrix} K & 0 & 0 & K \\ 0 & K & 0 & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}$; $\quad K\Gamma_3 \simeq \begin{pmatrix} K & K \oplus K \\ 0 & K \end{pmatrix}$.

20. Write, if possible, as path algebras (possibly modulo some relations) the following algebras:
a) $A_1 = K[x]$; b) $A_2 = K\langle x, y\rangle$; c*) $A_3 = K\mathbb{Z}_3$.

Solution. a) and b) The corresponding graphs for A_1 and A_2 are Γ_1 with one vertex and one loop, and Γ_2 with one vertex and two loops. Then $A_i \simeq K\Gamma_i$ for $i = 1, 2$. c) The case of A_3 is more difficult: it will depend on the base field K. Observe first that if we take Γ with one vertex and one loop denoted by α, then $K\Gamma/(\alpha^3 - 1) \simeq A_3$. However the ideal $(\alpha^3 - 1)$ is not admissible (since not all summands of the generating relations are paths of length at least 2) thus we have to find a different representation. c$_1$) When $\operatorname{char} K = 3$, the group algebra $K\mathbb{Z}_3$ is isomorphic to $A = K\Gamma_3/(\alpha^3)$, where Γ_3 is a graph with one vertex and one loop, denoted by α: if $\mathbb{Z}_3 = \langle a\rangle$, then we can define an algebra homomorphism $\varphi : K\Gamma_3 \to K\mathbb{Z}_3$ given by $e_1 \mapsto 1_{K\mathbb{Z}_3}$, $\alpha \mapsto (1 - a)$. It is clear that $\varphi(\alpha^2) = 1 + a + a^2$ and $\operatorname{Ker}(\varphi) = (\alpha^3)$, giving the required isomorphism. c$_2$) For the non-modular case (i. e. when $\operatorname{char} K \neq 3$) here is a sketch of the solution. $1 \in K\mathbb{Z}_3$ always decomposes into the sum of two idempotents: $f_1 = \frac{1}{3}(1 + a + a^2)$ and $f_2 = 1 - f_1 = \frac{1}{3}(2 - a - a^2)$. Note that $f_1 f_2 = 0$, i. e. these idempotent elements are called *orthogonal*. c$_{21}$) If K contains all third roots of unity, then f_2 will still further decompose into a sum of two orthogonal idempotents, f_2' and f_2''. These idempotents will give an algebra decomposition of A_3 into a direct sum of three one-dimensional K-algebras: $A_3 = K\mathbb{Z}_3 = A_3 f_1 \oplus A_3 f_2' \oplus A_3 f_2'' \simeq K \oplus K \oplus K$. Hence in this case A_3 is isomorphic to the path algebra over the graph having three vertices and no arrows. c$_{22}$) Finally, when K does not contain the primitive third roots of unity then $A_3 f_2$ is indecomposable two dimensional (the action of a is irreducible on $A_3 f_2$, i. e. there are no nontrivial a-invariant subspaces). In this case the graph corresponding to A_3 could contain only two vertices but it could have no arrows because there are no nilpotent elements in $K\mathbb{Z}_3$. Hence in this case $K\mathbb{Z}_3$ cannot be written as a path algebra modulo an admissible ideal.

21. There are n knights sitting around King Arthur's Round Table. Each of them has a coin in front of him. They play the following game. In every round each of them checks the coin in front of his neighbour to the right: if it is head then he flips his own coin; if it is tail the he leaves his own coin unchanged. They keep repeating this process until no one flips his coin. Determine those numbers n for which the game ends after a finite number of steps for each initial configuration of coins.

Solution. Each distribution of coins can be represented as a sequence $(a_{n-1}, \ldots, a_0) \in \mathbb{F}_2^n$ where 1 corresponds to head and 0 to tail, moreover the order follows the pattern that a_i is to the right of a_{i+1} and a_{n-1} is to the right of a_0. Thus we may also associate to this sequence a polynomial in $\mathbb{F}_2[x]$, or even better, an element of the factor ring $\mathbb{F}_2[x]/(x^n + 1)$ (reflecting the fact that a_0 "acts" as a_n. Under these circumstances the flipping rule corresponds to multiplication by $x + 1$. The game ends for each initial distibution if repeated multiplications by $x + 1$ take the whole factor ring to 0. This is equivalent to the requirement that $(x + 1)^k = 0$ in the factor ring for some k. The latter condition means that $x^n + 1$ divides $(x + 1)^k$, and the unique factorization property of $\mathbb{F}_2[x]$ implies that $x^n + 1 = (x + 1)^n$. This is true if and only if $n = 2^k$. Observe that if a game ends in a finite number of steps then it has to end in at most n steps.