1. Show that the converse of Schur's lemma does not hold.

Solution. Consider the additive group of $\mathbb{Q}$ as a module over $\mathbb{Z}$. Then $\mathbb{Q}$ is clearly not simple but $\operatorname{End}_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, since each endomorphism φ is determined by the image $\varphi(1) = \alpha \in \mathbb{Q}$ and it is also clear that multiplication by $\alpha \in \mathbb{Q}$ gives an endomorphism.

2. Let V be a vector space over a field K. Show that $R = \operatorname{End}(V_K)$ is left primitive but not necessarily simple. Describe the ideal structure of R.

Solution. Clearly V as a left R-module is faithful and simple. Hence R is left primitive. If $\dim_K V$ is finite, then R is simple, so let us consider the infinite dimensional case. It is easy to show that if κ is an arbitrary infinite cardinality then the endomorphisms of rank less than κ form and ideal I_κ. Furthermore if $f \in R$ is of rank λ then any $g \in R$ of rank not greater than λ will belong to the ideal generated by f. Hence the ideals are all of this form. Thus the ideals of R form a chain and they are in one-to-one correspondence with infinite cardinalities κ such that $\kappa \le (\dim V)^+$, the correspondence being given by $\kappa \mapsto I_\kappa$.

3. Show that if $R = \operatorname{End}(V_K)$ as above then R has a minimal left ideal and conclude that every simple faithful left R-module is isomorphic to ${}_RV$.

Solution. Let $\dim V = \kappa$ and $\mathcal{B} = \{\mathbf{b}_\lambda \mid \lambda < \kappa\}$ be a basis for V. Then $I = \operatorname{Ann}_R \mathcal{B} \setminus \{\mathbf{b}_0\}$ is a left ideal in R and it is clearly minimal. Namely, let $0 \ne f \in I$ and $g \in I$ be arbitrary elements. Then $f(\mathbf{b}_0) = \mathbf{v} \ne 0$, hence there is $r \in R$ for which $r(\mathbf{v}) = g(\mathbf{b}_0)$. With this r we have $rf = g$, hence $Rf = I$. This implies that I is minimal. Since any primitive ring is prime, a statement from the lecture implies that every faithul simple R-module is isomorphic, hence they must be isomorphic to ${}_RV$ (which is faithful and simple).

4. Let R be a left primitive ring and $0 \ne e \in R$ an idempotent element. Show that $S = eRe$ is also left primitive.

Solution. Let V be a faithful simple R-module. We will show that $U = eV$ is a faithful simple S-module. Clearly $ere \cdot ev = erev \in U$, hence we get an S-structure on U. Furthermore, if $ereU = 0$ then $ereeV = ereV = 0$, hence ere annihilates V, implying that $ere = 0$. Finally, if $u = ev \ne 0$ for some $u \in U$, then $Ru = V$ by simplicity of V. But then $Su = eReu = eReev = eRev = eRu = eV = U$ hence U is a simpe S-module. Thus S is also left primitive.

5. Show that for K a field of characteristic 0 the Weyl-algebra $A_1(K) = K\langle x, y\rangle/(xy - yx - 1)$ is left primitive.

Solution. Let $V = K[t]$ and let us consider the ring homomorphism $\Phi : K\langle x, y\rangle \to \operatorname{End}_{\mathbb{Z}} V$ defined by $x \mapsto (p(t) \mapsto p'(t))$ and $y \mapsto (p(t) \mapsto tp(t)$. Since $K\langle x, y\rangle$ is freely generated by x and y, such a homomorphism exists. A straightforward calculation shows that $\Phi(xy - yx)t^i = (i + 1)t^i - it^i = t^i$, thus $\Phi(xy - yx - 1) = 0$. Thus Φ factors through $A_1(K)$, hence V becomes an $A_1(K)$-module. A relatively easy calculation shows that V is a simple $A_1(K)$-module. Namely, for arbitrary nonzero $p(t) = a_0 + \cdots + a_nt^n$ we have $x^n \cdot p(t) = a_nn!$ and $y^mx^n \cdot p(t) = a_nn!t^m$. Thus since the characteristic of K is 0, for $q(t) = b_0 + \cdots + b_kt^k$ we define $r_t = \frac{b_t}{a_nn!}y^tx^n$ and $r = r_0 + \cdots + r_k$. Then $rp(t) = q(t)$, showing that V is simple as an $A_1(K)$-module. A somewhat lengthier calculation shows that V is faithful; the details are omitted. *Remark:* Basically we have shown that via the mapping Φ the ring $A_1(K)$ can be thought of as the ring of differential operators on $K[t]$, i. e. $\left\{\sum_{i=0}^n p_i(t)D^i \mid p_i(t) \in K[t]\right\}$, where D is the derivation on $K[t]$, and this ring acts densely on $K[t]$.

6. Decide whether the following implications are true:

a) A ring R is left primitive if and only if the full matrix ring $M_n(R)$ is left primitve.
b) A ring R is prime if and only if the full matrix ring $M_n(R)$ is prime.

Solution. a) If $M_n(R)$ is primitive then Problem 4 implies that $E_{11}M_n(R)E_{11} \simeq R$ is also primitive. Conversely, assume that R is left primitive with V a simple faithful R-module. Then one can show that V^n as a set of column vectors is a faithful simple $M_n(R)$-module. b) Observe first that the ideals $\mathcal{I}$ of $M_n(R)$ are precisely the subsets of the form $M_n(I)$ for some $I \lhd R$. A standard matrix multiplication argument shows that the projection $\pi_{ij} : \mathcal{I} \to R$, with $\pi_{ij}(A) = A_{ij}$ maps $\mathcal{I}$ onto the same ideal $I \lhd R$ for all i, j, furthermore $\mathcal{J}$ is the direct sum (as Abelian group) of these projections, hence it is of the form $M_n(I)$. Conversely, the sets $M_n(I)$ are ideals in $M_n(R)$. Now, for ideals $I_1, I_2 \lhd R$ and the corresponding ideals $\mathcal{I}_1, \mathcal{I}_2 \lhd M_n(R)$ we have that $I_1 \cdot I_2 = 0$ if and only if $\mathcal{I}_1 \cdot \mathcal{I}_2 = 0$. Thus R is prime if and only if $M_n(R)$ is prime.

7. a) Suppose the path algebra $K\Gamma$ is finite dimensional. Give a precise condition for $K\Gamma$ to be primitive (prime, resp.).
b) Show that $K\Gamma$ (without the assumption on the dimension) is prime if and only if for each pair of vertices i, j in Γ there is an (oriented) path from i to j.

Solution. a) If $K\Gamma$ is primitive then it is prime. Thus we will show first that if $K\Gamma$ is prime then Γ has one vertex and no arrows. Since $K\Gamma$ is finite dimensional, the arrows generate a nilpotent ideal, contradicting the primeness of $K\Gamma$. Hence there are no arrows in $K\Gamma$. Next, if Γ contains at least two vertices then the ideals generated by the corresponding idempotents are disjoint hence their product is zero contradicting the primeness of $K\Gamma$. Thus Γ contains only one vertex. Conversely, if Γ has one vertex and no arrows then $K\Gamma$ is a field and thus it is primitive and hence also prime. b) If $K\Gamma$ is prime then $e_iK\Gamma e_j \ne 0$, hence there is an oriented path from i to j for any vertices $i, j \in \Gamma$. Conversely, suppose that the latter condition holds. We want to show that $K\Gamma$ is prime, i. e. that $aK\Gamma b \ne 0$ for any pair of elements $a \ne 0 \ne b$ of $K\Gamma$. Multiplying a and b by suitable idempotents, we may assume that a is a linear combination of different paths a_α, each starting at h and ending at i; let a_{α_0} be a path of maximal length among the summands. Similarly, we may assume that b is a linear combination of different paths b_β, each starting at j and ending at k; let b_{β_0} be a path of maximal length among the summands. Let c be an oriented path from i to j. Then in the product acb there will be a summand $a_{\alpha_0}cb_{\beta_0}$ different from all other summands. Thus the product cannot be zero. This shows that in this case $K\Gamma$ is prime.

8. Show that the fact that $R \subseteq \mathrm{End}(V_D)$ is 1-transitive does not imply that R is dense in $\mathrm{End}(V_D)$ (although it follows that V is a simple faithful R-module). (*Hint:* Construct an example where $\mathrm{End}({}_RV)$ is strictly larger than D.)

Solution. Take $\mathbb{C}$ as an $\mathbb{R}$-vector-space. Clearly, $\mathbb{C}$ acts on $\mathbb{C}$ faithfully and transitively, on the other hand the action of $\mathbb{C}$ on $\mathbb{C}_{\mathbb{R}}$ is not 2-transitive. (Note that this does not contradict the density theorem since $\mathrm{End}_{\mathbb{C}}\, \mathbb{C} = \mathbb{C} \supset \mathbb{R}$.)

9. Prove that if R is primitive then the centre $Z(R)$ is an integral domain. Conversely, show that for any integral domain S there is a primitive ring R with $Z(R) \simeq S$.

Solution. If $a, b \in Z(R)$ would be nonzero elements for which $ab = 0$, then for the nonzero ideals $A = Ra$ and $B = Rb$ and the faithful simple R-module S we would get $RaS = S$ and $RbS = S$ but then $0 = abS = abRRS = RaRbS = RaS = S$, a contradiction. Hence $ab \neq 0$. For the converse let S be an integral domain and Q the field of quotients of S. Let V be an infinite dimensional vector space over Q and define R to be the subring of $\mathrm{End}(V_Q)$ as follows:

$$R = \left\{ f = g + s \cdot \mathrm{id}_V \in \mathrm{End}(V_Q) \mid g \text{ is of finite rank, } s \in S \right\}$$

Then one can check easily that R is dense in $\mathrm{End}(V_Q)$, moreover $Z(R) \simeq S$.

10. Show that in Wedderburn's theorem (which states that a simple left artinian left primitive ring is isomorphic to $M_n(D)$ for some divsion ring D and $n \in \mathbb{N}$) the division ring D and the positive integer n are uniquely determined by R.

Solution. We have to show that if $M_n(D) \simeq M_k(\Delta)$ then $D \simeq \Delta$ and $n = k$. Suppose φ gives an isomorphism between the two matrix rings and let $e = E_{11} \in M_n(D)$ be the idempotent element containing 1 in the upper left corner and 0's elsewhere. Then e generates a minimal left ideal in $M_n(D)$, hence $f = \varphi(e)$ must be an idempotent, generating a minimal left ideal in $M_k(\Delta)$. By a change of basis, i. e. an inner automorphism of $M_k(\Delta)$, if necessary, we may assume that f is an element where in the upper left corner we have an identity matrix of rank r and 0's elsewhere. By the minimality of $M_k(\Delta)f$, we get that $r = 1$. Thus $D \simeq eM_n(D)e \simeq fM_k(\Delta)f \simeq \Delta$. A dimension argument shows that $n = k$.

11. Observe the following chain of implications:

$$R \text{ is simple} \Rightarrow R \text{ is primitive} \Rightarrow R \text{ is prime}$$

Show that none of the implications can be reversed but if we assume that R is left Artinian then both implications can be replaced by equivalences.

Solution. $\mathbb{Z}$ is prime but not primitive and $\mathrm{End}_K V$ is primitive but it is not simple if V is infinite dimensional. Hence the reverse implications are false. On the other hand if R is left artinian and prime then it contains a minimal left ideal hence it is primitive by a theorem proved in class. Furthermore, if R is primitive and left Artinian then we have seen that $A \simeq M_n(D)$ for some division ring D and $n \in \mathbb{N}$, hence it is simple.

12. Suppose $R \subseteq \mathrm{End}(V_D)$ is a dense subring. Show that the *socle* of R as a left module (i. e. the left ideal generated by all minimal left ideals) consists of all elements of R which have finite rank.

Solution. Let I be the set of all endomorphisms in R which are of finite rank. It is clear that I is an ideal in $\mathrm{End}(V_D)$. We will show first that $I \subseteq \mathrm{Soc}(R)$, the socle of R. We claim that if $\mathrm{rank}(r) = 1$ for an element $r \in R$, then Rr is a minimal left ideal in R. To this end we have to show that if $sr \neq 0$ for some $s \in R$, then $Rsr = Rr$, i. e. that $r \in Rsr$. But $sr \neq 0$ implies that there is an element $v \in V$ such that $sr(v) \neq 0$. Then the density of R implies that there is $t \in R$ such that $tsr(v) = r(v)$. We claim that $tsr = r$, i. e. $tsr(w) = r(w)$ for any $w \in V$. But if $r(w) \neq 0$ then $r(w) = r(v)\lambda$ for some $\lambda \in D$ hence $tsr(w) = tsr(v)\lambda = r(v)\lambda = r(w)$. This implies that Rr is a minimal left ideal in R, hence $Rr \leq \mathrm{Soc}\, R$. Now let us take an $r \in I$ with $\mathrm{rank}(r) = k$. This means that $r(V) = \langle v_1, \ldots, v_k \rangle$ for some D-independent elements. The density of R implies that there are elements $a_1, \ldots a_k \in R$ such that $a_i(v_j) = \delta_{ij} v_j$. This implies that $r = (\sum_{i=1}^k a_i) r = \sum_{i=1}^k (a_i r)$ and $\mathrm{rank}(a_i r) = 1$. Hence r is a sum of rank 1 elements in R, thus it is in $\mathrm{Soc}\, R$. We have shown that $I \subseteq \mathrm{Soc}\, R$. To prove the opposite direction, i. e. that $\mathrm{Soc}\, R \subseteq I$, it is enough to show that any minimal left ideal $0 \neq L = Rr$ is generated by an element of rank 1. Suppose $\mathrm{rank}\, r \geq 2$. This means that we can find elements $v_1, v_2 \in V$ for which $r(v_1)$ and $r(v_2)$ are D-independent. Density of R implies that there is an element $t \in R$ for which $tr(v_1) \neq 0$ an $tr(v_2) = 0$. But in this case $tr \in Rr$ for which all elements of Rtr will contain v_2 in their kernels, implying that $r \notin Rtr$. This contradicts the minimality of Rr. Thus $\mathrm{Soc}\, R \subseteq I$, as required.

13. a) Show that if R is a prime ring and $L \leq {}_RR$ is a minimal left ideal then L is generated by an idempotent element.

b) Show that the left and right socle of a prime ring coincide.

Solution. a) Let R be a prime ring and $0 \neq L = Rr$ be a minimal left ideal. Since R is prime, $L^2 \neq 0$, hence $La \neq 0$ for some $a \in L$. Since La is a left ideal, the minimality of L implies that $La = L$. Thus there is an element $e \in L$ such that $ea = a$. Hence $(e^2 - e)a = 0$. But the annihilator of a in L is a left ideal of R, strictly contained in L (because $ea \neq 0$). Hence this annihilator is 0, thus $e^2 = e$. So we get an idempotent element in $L = Rr$ and by minimality of L we get that $Rr = Re$ and thus $r = re$. (Observe that $\mathrm{End}_R(Re) \simeq eRe$ and by Schur's lemma eRe is a division ring.) b) We want to show that if Rr is a minimal left ideal in a prime ring R then rR is also a minimal right ideal in R. Thus let $0 \neq s \in rR$; we need to show that $r \in sR$. But $s \in rR$ means $s = rt$ for some $t \in R$. Hence $s = rt = ret$. Note that R being a prime ring implies that $sRs \neq 0$ so there is an element $u \in R$ such that $sus = returet \neq 0$. But the $reture \neq 0$, and since eRe is a division ring, there exists an element $q \in R$ such that $etureeqe = e$. Thus $r = re = retureqe = sureqe \in sR$. This proves that the left and right socle of a prime ring coincide.