
preparation (evaluate this cell to initialize)

general definition

definition for E_6

`norcharC[6, λ]` : computation of $C_{\lambda}^{(6)}$

`norcharC[3, λ]` : computation of $C_{\lambda}^{(3)}$ for $\lambda \in \{0, \lambda_1, \lambda_3, \lambda_4\}$

`norcharD[3, λ]` : computation of $D_{\lambda}^{(3)}$ for $\lambda \in \{0, \lambda_1, \lambda_3, \lambda_4\}$

proof of $\sum_{W \in W_{\lambda}} (-1)^{l(W)} e^{w(\rho)} E_{W;\lambda} = 0$ for $\lambda \in \{\lambda_2, \lambda_3, \lambda_4\}$

$\lambda = \lambda_2$

sum over $W_{1,3,5,6}$

```
Block[{ty = E, rk = 6, summand, targetweight, subgroup, sum},
  targetweight =  $\lambda[2]$ ;
  summand[w_] :=
    (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * Factor[pfE[w, targetweight]];
  subgroup = isotropysubgp[ty, rk][targetweight];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  Print[{"time elapsed", "sum"}];
  Timing[Simplify[sum]]
]
{size of the isotropy subgroup : , 24}
{time elapsed, sum}
{0., 0}
```

sum over $W_{1,5,6}$

```
Block[{ty = E, rk = 6, summand, targetweight, subgroup, sum},
  targetweight =  $\lambda[2]$ ;
  summand[w_] :=
    (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * Factor[pfE[w, targetweight]];
  subgroup = isotropysubgp[ty, rk][wt[0, 1, 1, 1, 0, 0]];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  Print[{"time elapsed", "sum"}];
  Timing[Simplify[sum]]
]
```

```
{size of the isotropy subgroup : , 8}
{time elapsed, sum}
{0., 0}
```

$$\lambda = \lambda_3$$

sum over $W_{1,2,4,5,6}$

```
Block[{ty = E, rk = 6, summand, targetweight, subgroup, sum},
  targetweight =  $\lambda[3]$ ;
  summand[w_] :=
    (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * Factor[pfE[w, targetweight]];
  subgroup = isotropysubgp[ty, rk][targetweight];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  Print[{"time elapsed", "sum"}];
  Timing[Simplify[sum]]
]
{size of the isotropy subgroup : , 72}
{time elapsed, sum}
{0., 0}
```

sum over $W_{1,4,5,6}$

```
Block[{ty = E, rk = 6, summand, targetweight, subgroup, sum},
  targetweight =  $\lambda[3]$ ;
  summand[w_] :=
    (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * Factor[pfE[w, targetweight]];
  subgroup = isotropysubgp[ty, rk][wt[0, 1, 1, 0, 0, 0]];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  Print[{"time elapsed", "sum"}];
  Timing[Simplify[sum]]
]
{size of the isotropy subgroup : , 24}
{time elapsed, sum}
{0., 0}
```

$\lambda = \lambda_4$

```
Block[{ty = E, rk = 6, summand, targetweight, subgroup, sum},
  targetweight =  $\lambda[4]$ ;
  summand[w_] :=
    (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * Factor[pfE[w, targetweight]];
  subgroup = isotropysubgp[ty, rk][targetweight];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  Print[{"time elapsed", "sum"}];
  Timing[Simplify[sum]]
]

{size of the isotropy subgroup : , 720}

{time elapsed, sum}

{70.9375, 0}
```

 $\lambda = \lambda_4$: more efficient check

```
Block[{ty = E, rk = 6, summand, WA5, WA2A2, sumoverWA2A2, sum, WA5WA2Acosets},
  summand[w_] := (-1)Length[w] exp[WeylR[ty, rk][w][rho]] * pfE[w,  $\lambda[4]$ ];
  WA5 = isotropysubgp[E, 6][wt[0, 0, 0, 0, 0, 1]];
  Print[{"size of the isotropy subgroup W' : ", WA5 // Length}];
  WA2A2 = Select[WA5, FreeQ[#, 3] &];
  Print[{"size of a subgroup W'' of W' : ", WA2A2 // Length}];
  sumoverWA2A2 = Total@Map[summand, WA2A2] // Simplify;
  WA5WA2Acosets = cosetReps[A, 5][{1, 2, 4, 5}];
  Print[{"size of W'/W'' : ", WA5WA2Acosets // Length}];
  Print[{"coset reps:", WA5WA2Acosets}];
  WA5WA2Acosets = {{}, {3, 2, 1, 4, 3, 2, 5, 4, 3}, {3}, {2, 1, 4, 3, 2, 5, 4, 3}, {2, 3},
    {1, 4, 3, 2, 5, 4, 3}, {4, 3}, {2, 1, 3, 2, 5, 4, 3}, {1, 2, 3}, {4, 3, 2, 5, 4, 3},
    {2, 4, 3}, {1, 3, 2, 5, 4, 3}, {5, 4, 3}, {2, 1, 3, 2, 4, 3}, {3, 2, 4, 3},
    {1, 2, 5, 4, 3}, {1, 2, 4, 3}, {3, 2, 5, 4, 3}, {2, 5, 4, 3}, {1, 3, 2, 4, 3}};
  Print[{"coset reps reordered:", WA5WA2Acosets}];
  sum = 0;
  Do[
    sum = Simplify[sum + (-1)Length[ww] (sumoverWA2A2 /. weyltorule[ty, rk][ww])];
    (* Print[sum]; *)
    , {ww, WA5WA2Acosets}
  ];
  sum
]
```

```

{size of the isotropy subgroup W' : , 720}
{size of a subgroup W'' of W' : , 36}
{size of W'/W'' : , 20}
{coset reps:, {{}, {3}, {2, 3}, {4, 3}, {1, 2, 3}, {2, 4, 3}, {5, 4, 3}, {3, 2, 4, 3},
  {1, 2, 4, 3}, {2, 5, 4, 3}, {3, 2, 5, 4, 3}, {1, 3, 2, 4, 3}, {1, 2, 5, 4, 3},
  {4, 3, 2, 5, 4, 3}, {1, 3, 2, 5, 4, 3}, {2, 1, 3, 2, 4, 3}, {1, 4, 3, 2, 5, 4, 3},
  {2, 1, 3, 2, 5, 4, 3}, {2, 1, 4, 3, 2, 5, 4, 3}, {3, 2, 1, 4, 3, 2, 5, 4, 3}}}
{coset reps reordered:, {{}, {3, 2, 1, 4, 3, 2, 5, 4, 3}, {3}, {2, 1, 4, 3, 2, 5, 4, 3}, {2, 3},
  {1, 4, 3, 2, 5, 4, 3}, {4, 3}, {2, 1, 3, 2, 5, 4, 3}, {1, 2, 3}, {4, 3, 2, 5, 4, 3},
  {2, 4, 3}, {1, 3, 2, 5, 4, 3}, {5, 4, 3}, {2, 1, 3, 2, 4, 3}, {3, 2, 4, 3},
  {1, 2, 5, 4, 3}, {1, 2, 4, 3}, {3, 2, 5, 4, 3}, {2, 5, 4, 3}, {1, 3, 2, 4, 3}}}
0

```

proof of $C_{\lambda}^{(3)} = D_{\lambda}^{(3)}$ for $\lambda \in \{\lambda_1, \lambda_3, \lambda_4\}$

$\lambda = \lambda_1$

```

Block[{C3, D3, la, cdlist, sum, timestart},
  Print["process initiated!"];
  timestart = Floor[AbsoluteTime[]];
  la =  $\lambda$ [1];
  Print["list of summands for  $C_{\lambda}$  : "];
  Print[norcharC3C6List[la]];
  C3 = (Weyldenom * norcharCList[3, la]);
  D3 = (-Weyldenom * norcharDList[3, la]);
  cdlist = Join@@{C3, D3};
  sum = Factor[Total[cdlist]];
  Print[{"time elapsed: ", Floor[AbsoluteTime[] - timestart]};
  sum
]
process initiated!
list of summands for  $C_{\lambda}$  :

$$\left\{ -\left( c[6, \text{wt}[0, 0, 0, 0, 0, 1]] c[6, \text{wt}[1, 0, 0, 0, 1, -1]] (-1 + x[2] x[3]^2 x[4] x[6]^2)^2 \right) / \right.$$


$$\left. \left( x[2] x[3]^2 x[4] x[6]^2 \right), -\frac{1}{x[2] x[3]^2 x[4]} \right.$$


$$c[6, \text{wt}[0, 0, 1, 0, 0, -1]] c[6, \text{wt}[1, 0, -1, 0, 1, 1]] (-1 + x[2] x[3]^2 x[4])^2,$$


$$-\frac{1}{x[2] x[4]} c[6, \text{wt}[0, 1, -1, 1, 0, 0]] c[6, \text{wt}[1, -1, 1, -1, 1, 0]] (-1 + x[2] x[4])^2,$$


$$\left. -\frac{1}{x[2] x[4]} c[6, \text{wt}[0, 1, 0, -1, 1, 0]] c[6, \text{wt}[1, -1, 0, 1, 0, 0]] (x[2] - x[4])^2 \right\}$$

{time elapsed: , 100}
0

```

$\lambda = \lambda_3$

```
Block[{C3, D3, la, cdlist, sum, timestart},
  Print["process initiated!"];
  timestart = Floor[AbsoluteTime[]];
  la =  $\lambda[3]$ ;
  Print["list of summands for  $C_\lambda$  : "];
  Print[norcharC3C6List[la]];
  C3 = (Weyldenom * norcharCList[3, la]);
  D3 = (-Weyldenom * norcharDList[3, la]);
  cdlist = Join@@{C3, D3};
  sum = Factor[Total[cdlist]];
  Print[{"time elapsed: ", Floor[AbsoluteTime[]] - timestart}];
  sum
]
```

process initiated!

list of summands for C_λ :

$$\left\{ -\frac{1}{x[6]} c[6, \text{wt}[0, 0, 0, 0, 0, 1]] c[6, \text{wt}[0, 0, 1, 0, 0, -1]] (-1 + x[6])^2 \right\}$$

{time elapsed: , 2}

0

$\lambda = \lambda_4$ (this uses the saved list of functions; put the data file in the same folder with this notebook)

```
Block[{summandlist, timestart, out, toexp},
  SetDirectory[FileNameJoin[{NotebookDirectory[]}]];
  summandlist = Last /@
    SortBy[ReadList[FileNameJoin[{Directory[], "E6_3_la4_poly_46.txt"}]], First];
  toexp[pairs_] := Times@@Map[#[[1]]#[[2]] &, pairs];
  timestart = Floor[AbsoluteTime[]];
  out = Expand[Total[toexp /@ summandlist]];
  Print[{"result: ", out}];
  Print[{"time elapsed: ", Floor[AbsoluteTime[]] - timestart}];
]
```

{result: , 0}

{time elapsed: , 17763}

```

Block[{summandlist, timestart, out, toexp},
  SetDirectory[FileNameJoin[{NotebookDirectory[]}]];
  summandlist = Last /@
    SortBy[ReadList[FileNameJoin[{Directory[], "E6_3_la4_ratio_46.txt"}]], First];
  toexp[pairs_] := Times @@ Map[#[[1]]^#[[2]] &, pairs];
  timestart = Floor[AbsoluteTime[]];
  out = Factor[Total[toexp /@ summandlist]];
  Print[{"result: ", out}];
  Print[{"time elapsed: ", Floor[AbsoluteTime[] - timestart]}];
]

{result: , 0}

{time elapsed: , 102609}

```

numerical check for $C_{\lambda}^{(3)} = D_{\lambda}^{(3)}$ for $\lambda \in \{\lambda_0\}$

$\lambda = \lambda_0$

```

Block[{C3, D3, la, rule, rk = 6},
  la =  $\lambda[0]$ ;
  rule = Inner[Rule, Array[x, rk], RandomInteger[{1, 1000}, rk], List];
  norcharC3C6List[la] // Print;
  C3 = norcharC[3, la] /. rule;
  D3 = norcharD[3, la] /. rule;
  C3 - D3
]

```

$$\begin{aligned}
& - \left((c[6, wt[0, 0, 0, 0, -1]] c[6, wt[0, 0, 0, 0, 1]] (-1 + x[1] x[2]^2 x[3]^3 x[4]^2 x[5] x[6]^2)^2 \right. \\
& \quad \left. (1 + x[1] x[2]^2 x[3]^3 x[4]^2 x[5] x[6]^2)^2 \right) / (x[1]^2 x[2]^4 x[3]^6 x[4]^4 x[5]^2 x[6]^4), \\
& - \left((c[6, wt[0, 0, -1, 0, 0, 1]] c[6, wt[0, 0, 1, 0, 0, -1]] (-1 + x[1] x[2]^2 x[3]^3 x[4]^2 x[5] x[6])^2 \right. \\
& \quad \left. (1 + x[1] x[2]^2 x[3]^3 x[4]^2 x[5] x[6])^2 \right) / (x[1]^2 x[2]^4 x[3]^6 x[4]^4 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[0, -1, 1, -1, 0, 0]] c[6, wt[0, 1, -1, 1, 0, 0]] \right. \\
& \quad \left. (-1 + x[1] x[2]^2 x[3]^2 x[4]^2 x[5] x[6])^2 (1 + x[1] x[2]^2 x[3]^2 x[4]^2 x[5] x[6])^2 \right) / \\
& \quad (x[1]^2 x[2]^4 x[3]^4 x[4]^4 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[-1, 1, 0, -1, 0, 0]] c[6, wt[1, -1, 0, 1, 0, 0]] (-1 + x[1] x[2] x[3]^2 x[4]^2 x[5] x[6])^2 \right. \\
& \quad \left. (1 + x[1] x[2] x[3]^2 x[4]^2 x[5] x[6])^2 \right) / (x[1]^2 x[2]^2 x[3]^4 x[4]^4 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[0, -1, 0, 1, -1, 0]] c[6, wt[0, 1, 0, -1, 1, 0]] (-1 + x[1] x[2]^2 x[3]^2 x[4] x[5] x[6])^2 \right. \\
& \quad \left. (1 + x[1] x[2]^2 x[3]^2 x[4] x[5] x[6])^2 \right) / (x[1]^2 x[2]^4 x[3]^4 x[4]^2 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[-1, 0, 0, 1, 0, 0]] c[6, wt[1, 0, 0, -1, 0, 0]] (-1 + x[2] x[3]^2 x[4]^2 x[5] x[6])^2 \right. \\
& \quad \left. (1 + x[2] x[3]^2 x[4]^2 x[5] x[6])^2 \right) / (x[2]^2 x[3]^4 x[4]^4 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[-1, 1, -1, 1, -1, 0]] c[6, wt[1, -1, 1, -1, 1, 0]] \right. \\
& \quad \left. (-1 + x[1] x[2] x[3]^2 x[4] x[5] x[6])^2 (1 + x[1] x[2] x[3]^2 x[4] x[5] x[6])^2 \right) / \\
& \quad (x[1]^2 x[2]^2 x[3]^4 x[4]^2 x[5]^2 x[6]^2), \\
& - \left((c[6, wt[0, -1, 0, 0, 1, 0]] c[6, wt[0, 1, 0, 0, -1, 0]] (-1 + x[1] x[2]^2 x[3]^2 x[4] x[6])^2 \right. \\
& \quad \left. (1 + x[1] x[2]^2 x[3]^2 x[4] x[6])^2 \right) / (x[1]^2 x[2]^4 x[3]^4 x[4]^2 x[6]^2), \\
& - \left((c[6, wt[-1, 0, 1, 0, -1, -1]] c[6, wt[1, 0, -1, 0, 1, 1]] (-1 + x[1] x[2] x[3] x[4] x[5] x[6])^2 \right. \\
& \quad \left. (1 + x[1] x[2] x[3] x[4] x[5] x[6])^2 \right) / (x[1]^2 x[2]^2 x[3]^2 x[4]^2 x[5]^2 x[6]^2),
\end{aligned}$$

$$\begin{aligned}
& - \left(\left(c[6, \text{wt}[-1, 0, 1, -1, 1, 0]] c[6, \text{wt}[1, 0, -1, 1, -1, 0]] (-1 + x[2] x[3]^2 x[4] x[5] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3]^2 x[4] x[5] x[6])^2 \right) / (x[2]^2 x[3]^4 x[4]^2 x[5]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, -1, 0, 1, 0]] c[6, \text{wt}[1, -1, 1, 0, -1, 0]] (-1 + x[1] x[2] x[3]^2 x[4] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3]^2 x[4] x[6])^2 \right) / (x[1]^2 x[2]^2 x[3]^4 x[4]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, -1, 0, 1, 1]] c[6, \text{wt}[1, -1, 1, 0, -1, -1]] (-1 + x[2] x[3] x[4] x[5] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3] x[4] x[5] x[6])^2 \right) / (x[2]^2 x[3]^2 x[4]^2 x[5]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, 1, -1, 1, -1]] c[6, \text{wt}[1, 0, -1, 1, -1, 1]] (-1 + x[1] x[2] x[3] x[4] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3] x[4] x[6])^2 \right) / (x[1]^2 x[2]^2 x[3]^2 x[4]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, 1, 0, -1, 0]] c[6, \text{wt}[1, 0, -1, 0, 1, 0]] (-1 + x[2] x[3]^2 x[4] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3]^2 x[4] x[6])^2 \right) / (x[2]^2 x[3]^4 x[4]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, 0, 0, -1, 1]] c[6, \text{wt}[1, 0, 0, 0, 1, -1]] (-1 + x[1] x[2] x[3] x[4] x[5])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3] x[4] x[5])^2 \right) / (x[1]^2 x[2]^2 x[3]^2 x[4]^2 x[5]^2) \right), \\
& - \left(\left(c[6, \text{wt}[0, -1, 0, 0, 1, 1]] c[6, \text{wt}[0, 1, 0, 0, -1, -1]] (-1 + x[3] x[4] x[5] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[3] x[4] x[5] x[6])^2 \right) / (x[3]^2 x[4]^2 x[5]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, 0, 1, 0, -1]] c[6, \text{wt}[1, 0, 0, -1, 0, 1]] (-1 + x[1] x[2] x[3] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3] x[6])^2 \right) / (x[1]^2 x[2]^2 x[3]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, -1, 1, -1, 1]] c[6, \text{wt}[1, -1, 1, -1, 1, -1]] (-1 + x[2] x[3] x[4] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3] x[4] x[6])^2 \right) / (x[2]^2 x[3]^2 x[4]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, 0, 0, 1, -1]] c[6, \text{wt}[1, -1, 0, 0, -1, 1]] (-1 + x[2] x[3] x[4] x[5])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3] x[4] x[5])^2 \right) / (x[2]^2 x[3]^2 x[4]^2 x[5]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, 0, -1, 1, 1]] c[6, \text{wt}[1, 0, 0, 1, -1, -1]] (-1 + x[1] x[2] x[3] x[4])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3] x[4])^2 \right) / (x[1]^2 x[2]^2 x[3]^2 x[4]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, 0, -1, 0, 1]] c[6, \text{wt}[1, -1, 0, 1, 0, -1]] (-1 + x[2] x[3] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3] x[6])^2 \right) / (x[2]^2 x[3]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[0, -1, 0, 1, -1, 1]] c[6, \text{wt}[0, 1, 0, -1, 1, -1]] (-1 + x[3] x[4] x[6])^2 \right. \right. \\
& \quad \left. \left. (1 + x[3] x[4] x[6])^2 \right) / (x[3]^2 x[4]^2 x[6]^2) \right), \\
& - \left(\left(c[6, \text{wt}[0, -1, 1, 0, 1, -1]] c[6, \text{wt}[0, 1, -1, 0, -1, 1]] (-1 + x[3] x[4] x[5])^2 \right. \right. \\
& \quad \left. \left. (1 + x[3] x[4] x[5])^2 \right) / (x[3]^2 x[4]^2 x[5]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 0, -1, 1, 0, 1]] c[6, \text{wt}[1, 0, 1, -1, 0, -1]] (-1 + x[1] x[2] x[3])^2 \right. \right. \\
& \quad \left. \left. (1 + x[1] x[2] x[3])^2 \right) / (x[1]^2 x[2]^2 x[3]^2) \right), \\
& - \left(\left(c[6, \text{wt}[-1, 1, 0, 1, -1, -1]] c[6, \text{wt}[1, -1, 0, -1, 1, 1]] (-1 + x[2] x[3] x[4])^2 \right. \right. \\
& \quad \left. \left. (1 + x[2] x[3] x[4])^2 \right) / (x[2]^2 x[3]^2 x[4]^2) \right), \\
& - \frac{1}{x[4]^2 x[5]^2} c[6, \text{wt}[0, 0, -1, 1, 1, 0]] c[6, \text{wt}[0, 0, 1, -1, -1, 0]] \\
& \quad (-1 + x[4] x[5])^2 (1 + x[4] x[5])^2, \\
& - \frac{1}{x[1]^2 x[2]^2} c[6, \text{wt}[-1, -1, 1, 0, 0, 0]] c[6, \text{wt}[1, 1, -1, 0, 0, 0]] \\
& \quad (-1 + x[1] x[2])^2 (1 + x[1] x[2])^2, \\
& - \frac{1}{x[3]^2 x[6]^2} c[6, \text{wt}[0, -1, 1, -1, 0, 1]] c[6, \text{wt}[0, 1, -1, 1, 0, -1]] \\
& \quad (-1 + x[3] x[6])^2 (1 + x[3] x[6])^2, \\
& - \frac{1}{x[2]^2 x[3]^2} c[6, \text{wt}[-1, 1, 1, -1, 0, -1]] c[6, \text{wt}[1, -1, -1, 1, 0, 1]] \\
& \quad (-1 + x[2] x[3])^2 (1 + x[2] x[3])^2, \\
& - \frac{1}{x[3]^2 x[4]^2} c[6, \text{wt}[0, -1, 1, 1, -1, -1]] c[6, \text{wt}[0, 1, -1, -1, 1, 1]] \\
& \quad (-1 + x[3] x[4])^2 (1 + x[3] x[4])^2,
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{x[1]^2} c[6, \text{wt}[-2, 1, 0, 0, 0, 0]] c[6, \text{wt}[2, -1, 0, 0, 0, 0]] (-1+x[1])^2 (1+x[1])^2, \\
& -\frac{1}{x[6]^2} c[6, \text{wt}[0, 0, -1, 0, 0, 2]] c[6, \text{wt}[0, 0, 1, 0, 0, -2]] (-1+x[6])^2 (1+x[6])^2, \\
& -\frac{1}{x[2]^2} c[6, \text{wt}[-1, 2, -1, 0, 0, 0]] c[6, \text{wt}[1, -2, 1, 0, 0, 0]] (-1+x[2])^2 (1+x[2])^2, \\
& -\frac{1}{x[5]^2} c[6, \text{wt}[0, 0, 0, -1, 2, 0]] c[6, \text{wt}[0, 0, 0, 1, -2, 0]] (-1+x[5])^2 (1+x[5])^2, \\
& -\frac{1}{x[4]^2} c[6, \text{wt}[0, 0, -1, 2, -1, 0]] c[6, \text{wt}[0, 0, 1, -2, 1, 0]] (-1+x[4])^2 (1+x[4])^2, \\
& -\frac{1}{x[3]^2} c[6, \text{wt}[0, -1, 2, -1, 0, -1]] c[6, \text{wt}[0, 1, -2, 1, 0, 1]] (-1+x[3])^2 (1+x[3])^2 \} \\
& 0
\end{aligned}$$

save list of rational functions for $C_{\lambda}^{(3)} = D_{\lambda}^{(3)}$ for $\lambda \in \{\lambda_4, \lambda_0\}$

functions for adding factored rational functions (run this cell for the following computation)

$$\lambda = \lambda_4$$

list of 802 rational functions (E6_3_la4_ratio_802.txt; ~900KB)

```
Block[{C3, D3, la, rk = 6, summand, C3firstelt, cdlist, file, stream, val,
  timestart, numtogo, commonfacC, commonfacD, commonfac, output = {}},
  la =  $\lambda[4]$ ;
  Print["C3 = list of summands for  $C_\lambda^{(3)}$ , obtained from the following
    by replacing  $c[6, \lambda] = C_\lambda^{(6)}$  by the corresponding summands : "];
  (* C3=norcharCList[3,la]; *)
  C3 = Flatten[norcharC3C6List[la] /.
    {c[6, wt[0, 0, 0, 0, 0, 0]]  $\rightarrow$  norcharCList[6, wt[0, 0, 0, 0, 0, 0]], c -> norcharC}];
  C3 = Factor[C3];
  Print["length of C3 : ", Length[C3]];
  Print["D3 = list of summands for  $D_\lambda^{(3)}$ "];
  D3 = norcharDList[3, la];
  Print["length of D3 : ", Length[D3]];
  (* make polynomials from rational functions by multiplying a common factor *)
  commonfac = Weyldenom;
  C3 = commonfac * C3;
  D3 = commonfac * D3;
  cdlist = Join@@{C3, -D3};
  file = FileNameJoin[{Directory[], "E6_3_la4_ratio_802.txt"}];
  timestart = AbsoluteTime[];
  numtogo = Length[cdlist];
  (* Print[{"step number", "time elapsed", "size of the expression"}]; *)
  Print["saving file..."];
  Do[
    val = {ss, FactorList@Factor[cdlist[[ss]]]};
    stream = OpenAppend[file];
    Write[stream, val];
    Close[stream];
    AppendTo[output, val];
    ,
    {ss, 1, numtogo}
  ];
  Print[{"time elapsed", Floor[AbsoluteTime[] - timestart]}];
]
```

C3 = list of summands for $C_\lambda^{(3)}$, obtained from the
 following by replacing $c[6, \lambda] = C_\lambda^{(6)}$ by the corresponding summands :
 length of C3 : 82
 D3 = list of summands for $D_\lambda^{(3)}$
 length of D3 : 720
 saving file...
 {time elapsed, 23}

802 rational functions -> 46 rational functions (E6_3_la4_ratio_46.txt;

~600KB)

```
Block[{v3la4ratio802, file, stream, line, goodpartitions802, timestart},
  v3la4ratio802 =
    Last /@ ReadList[FileNameJoin[{Directory[], "E6_3_la4_ratio_802.txt"}]];
  goodpartitions802 = {{1, 72}, {2, 71}, {3, 70}, {4, 68},
    {5, 69}, {6, 65}, {7, 66}, {8, 67}, {9, 64}, {10, 62},
    {11, 63}, {12, 60}, {13, 61}, {14, 59}, {15, 58}, {16, 56},
    {17, 57}, {18, 55}, {19, 53}, {20, 54},
    {21, 51}, {22, 52}, {23, 49}, {24, 50}, {25, 48}, {26, 46},
    {27, 47}, {28, 45}, {29, 43}, {30, 44},
    {31, 39}, {32, 41}, {33, 38}, {34, 42}, {35, 40}, {36, 37},
    {73, 83, 84, 85, 87, 88, 89, 91, 92, 93, 95, 96, 101, 102, 103, 105, 106, 111, 112, 114,
    115, 120, 121, 130, 133, 134, 139, 140, 149, 155, 156, 165, 185, 186, 195, 219, 260,
    625, 666, 690, 699, 700, 720, 729, 730, 736, 745, 746, 751, 752, 755, 764,
    765, 770, 771, 773, 774, 779, 780, 782, 783, 784, 789, 790, 792, 793, 794, 796,
    797, 798, 800, 801, 802}, {74, 86, 90, 94, 99, 100, 104, 109, 110, 113, 118,
    119, 129, 131, 132, 137, 138, 148, 150, 153, 154, 164, 166, 179, 183, 184, 194,
    196, 209, 218, 220, 233, 259, 261, 274, 305, 354, 531, 580, 611, 624, 626, 652,
    665, 667, 676, 689, 691, 701, 702, 706, 719, 721, 731, 732, 735, 737, 747,
    748, 753, 754, 756, 766, 767, 772, 775, 776, 781, 785, 786, 791, 795, 799},
    {75, 97, 107, 116, 124, 128, 135, 143, 147, 151, 159, 163, 177, 180, 181,
    189, 193, 207, 210, 213, 217, 231, 234, 250, 254, 258, 272, 275, 291,
    303, 306, 322, 352, 355, 371, 406, 428, 458, 480, 515, 532, 535, 564, 581,
    584, 595, 612, 615, 628, 630, 636, 653, 656, 669, 671, 677, 680, 693, 695,
    703, 707, 710, 723, 725, 733, 739, 741, 749, 758, 760, 768, 777, 787},
    {76, 98, 108, 117, 125, 127, 136, 144, 146, 152, 160, 162, 175, 178, 182,
    190, 192, 205, 208, 214, 216, 229, 232, 249, 255, 257, 270, 273, 290,
    301, 304, 321, 350, 353, 370, 405, 427, 457, 479, 514, 530, 533, 563, 579,
    582, 594, 610, 613, 627, 631, 635, 651, 654, 668, 672, 675, 678, 692, 696,
    704, 705, 708, 722, 726, 734, 738, 742, 750, 757, 761, 769, 778, 788},
    {77, 122, 141, 157, 168, 176, 187, 198, 206, 211, 222, 230, 247, 251, 252,
    263, 271, 288, 292, 294, 302, 319, 323, 330, 341, 343, 351, 368, 372,
    379, 390, 403, 407, 414, 425, 432, 433, 455, 459, 466, 477, 484, 485, 501,
    512, 519, 520, 536, 537, 550, 561, 568, 569, 585, 586, 599, 600, 616, 617,
    629, 640, 641, 657, 658, 670, 681, 682, 694, 711, 712, 724, 740, 759},
    {78, 123, 142, 158, 171, 172, 188, 201, 202, 212, 225, 226, 246, 248, 253,
    266, 267, 287, 289, 297, 298, 318, 320, 325, 340, 346, 347, 367, 369,
    374, 389, 402, 404, 409, 424, 429, 431, 454, 456, 461, 476, 481, 483, 496,
    511, 516, 518, 538, 539, 545, 560, 565, 567, 587, 588, 596, 598, 618, 619,
    632, 637, 639, 659, 660, 673, 683, 684, 697, 713, 714, 727, 743, 762},
    {79, 126, 145, 161, 173, 174, 191, 203, 204, 215, 227, 228, 244, 245, 256,
    268, 269, 285, 286, 299, 300, 316, 317, 324, 335, 348, 349, 365, 366,
    373, 384, 400, 401, 408, 419, 426, 430, 452, 453, 460, 471, 478, 482, 495,
    506, 513, 517, 534, 542, 544, 555, 562, 566, 583, 591, 593, 597, 614, 622,
    633, 634, 638, 655, 663, 674, 679, 687, 698, 709, 717, 728, 744, 763},
    {80, 169, 199, 223, 236, 238, 241, 264, 277, 279, 282, 295, 308, 310, 313,
    326, 332, 333, 339, 344, 357, 359, 362, 375, 381, 382, 388, 392, 394,
    397, 410, 416, 417, 423, 436, 439, 441, 444, 446, 449, 462, 468, 469, 475,
    488, 491, 493, 497, 503, 504, 510, 523, 526, 528, 541, 546, 552, 553, 559,
```

```

572, 575, 577, 590, 603, 606, 608, 621, 644, 647, 649, 662, 686, 716},
{81, 167, 197, 221, 237, 239, 242, 262, 278, 280, 283, 293, 309, 311, 314,
329, 331, 337, 338, 342, 358, 360, 363, 378, 380, 386, 387, 393, 395,
398, 413, 415, 421, 422, 434, 437, 442, 445, 447, 450, 465, 467, 473, 474,
486, 489, 494, 500, 502, 508, 509, 521, 524, 529, 540, 549, 551, 557, 558,
570, 573, 578, 589, 601, 604, 609, 620, 642, 645, 650, 661, 685, 715},
{82, 170, 200, 224, 235, 240, 243, 265, 276, 281, 284, 296, 307, 312, 315,
327, 328, 334, 336, 345, 356, 361, 364, 376, 377, 383, 385, 391, 396,
399, 411, 412, 418, 420, 435, 438, 440, 443, 448, 451, 463, 464, 470, 472,
487, 490, 492, 498, 499, 505, 507, 522, 525, 527, 543, 547, 548, 554, 556,
571, 574, 576, 592, 602, 605, 607, 623, 643, 646, 648, 664, 688, 718}
};
file = "E6_3_la4_ratio_46.txt";
timestart = Floor[AbsoluteTime[]];
Do[
(* line = {idx,
sumofratiolist[v3la4ratio802,randompermute[goodpartitions802[[idx]]]]]; *)
line = {idx, sumoftworationalfunc[v3la4ratio802[[goodpartitions802[[idx]][[1]]]],
v3la4ratio802[[goodpartitions802[[idx]][[2]]]]];
stream = OpenAppend[file];
Write[stream, line];
Close[stream],
{idx, 1, 36}
];
ParallelDo[
Print[
{"working on :", idx, "time elapsed : ", Floor[AbsoluteTime[]] - timestart}];
(* line = {idx,sumofratiolist[v3la4ratio802,
randompermute[goodpartitions802[[idx]]]]]; *)
line = {idx, FactorList@Total[toexp /@ Part[v3la4ratio802,
goodpartitions802[[idx]]]]];
stream = OpenAppend[file];
Write[stream, line];
Close[stream],
{idx, 37, 46}
];
];
{working on :, 37, time elapsed : , 20}
{working on :, 39, time elapsed : , 20}
{working on :, 41, time elapsed : , 20}
{working on :, 42, time elapsed : , 20}
{working on :, 43, time elapsed : , 94}
{working on :, 40, time elapsed : , 94}
{working on :, 38, time elapsed : , 97}
{working on :, 44, time elapsed : , 145}
{working on :, 45, time elapsed : , 176}
{working on :, 46, time elapsed : , 177}

```

46 rational functions -> 46 polynomials (E6_3_la4_poly_46.txt; ~600KB)

```
Block[{v3la4ratio46, v3la4poly46, denoms,
  commonfac, summandlist, file, stream, line, timestart},
  timestart = AbsoluteTime[];
  v3la4ratio46 = Last /@
    SortBy[ReadList[FileNameJoin[{Directory[], "E6_3_la4_ratio_46.txt"}]], First];
  denoms = Table[toexp@ ({#[[1]], -#[[2]]} & /@
    Select[v3la4ratio46[[jj]], #[[2]] < 0 &]), {jj, 1, Length[v3la4ratio46]}];
  commonfac = JoinFactorList[{
    Table[{x[jj], -Min[exponent[#, x[jj]] & /@ v3la4ratio46]}, {jj, 1, 6}],
    FactorList@ (PolynomialLCM@@denoms)
  }];
  v3la4poly46 = Table[JoinFactorList[{commonfac, ss}], {ss, v3la4ratio46}];
  summandlist = v3la4poly46;
  file = "E6_3_la4_poly_46.txt";
  Do[
    line = {idx, summandlist[[idx]]};
    stream = OpenAppend[file]; Write[stream, line]; Close[stream],
    {idx, 1, Length[summandlist]}
  ];
  Print[{"time elapsed:", Floor[AbsoluteTime[] - timestart]}];
];
{time elapsed:, 1}
```

$$\lambda = \lambda_0$$

list of 36+51480 rational functions (E6_3_la0_ratio.txt; ~ 12MB)

```
Block[{C3, D3, la, rk = 6, summand, C3firstelt, cdlist, file, stream, val,
  timestart, numtogo, commonfacC, commonfacD, commonfac, output = {}},
  la =  $\lambda[0]$ ;
  Print["C3 = list of summands for  $C_{\lambda}^{(3)}$ , obtained from the following
    by replacing  $c[6, \lambda] = C_{\lambda}^{(6)}$  by the corresponding summands : "];
  C3 = norcharCList[3, la];
  Print["length of C3 : ", Length[C3]];
  Print["D3 = list of summands for  $D_{\lambda}^{(3)}$ "];
  D3 = norcharDList[3, la];
  Print["length of D3 : ", Length[D3]];
  commonfac = Weyldenom;
  C3 = commonfac * C3;
  D3 = commonfac * D3;
  cdlist = Join@@{C3, -D3};
  file = FileNameJoin[{Directory[], "E6_3_la0_ratio.txt"}];
  stream = OpenAppend[file];
  timestart = AbsoluteTime[];
  numtogo = Length[cdlist];
  (* Print[
    {"step number", "time elapsed", "size of the expression", "polynomial in  $x_i$ ?"}]; *)
  Print["saving file..."];
  Do[
    val = {ss, FactorList[cdlist[[ss]]]};
    Write[stream, val];
    ,
    {ss, 1, numtogo}
  ];
  Close[stream];
  Print[{"time elapsed:", Floor[AbsoluteTime[] - timestart]}];
]
```

C3 = list of summands for $C_{\lambda}^{(3)}$, obtained from the
 following by replacing $c[6, \lambda] = C_{\lambda}^{(6)}$ by the corresponding summands :
 length of C3 : 36
 D3 = list of summands for $D_{\lambda}^{(3)}$
 length of D3 : 51480
 saving file...
 {time elapsed:, 146}

numerical check for saved polynomials

```
Block[{v3la0ratio, rule, rk = 6},
  v3la0ratio =
    Last /@ SortBy[ReadList[FileNameJoin[{Directory[], "E6_3_la0_ratio.txt"}]], First];
  rule = Inner[Rule, Array[x, rk], RandomInteger[{1, 1000}, rk], List];
  Total[(toexp /@ v3la0ratio) /. rule]
]
0
```