preparation (evaluate this cell to initialize)

general definition

definition for F_4

norcharC[1,λ] : computation of $C_\lambda^{(1)}$

norcharC[2,λ] : computation of $C_\lambda^{(2)}$ for $\lambda \in \{0, \lambda_1, \lambda_3, \lambda_4\}$

norcharD[2,λ] : computation of $D_\lambda^{(2)}$ for $\lambda \in \{0, \lambda_1, \lambda_3, \lambda_4\}$

proof of $\sum_{w \in W_\lambda} (-1)^{l(w)} e^{w(\rho)} E_{w;\lambda} = 0$ for $\lambda \in \{\lambda_2, \lambda_3, \lambda_4\}$ (Proposition 4.1)

$\lambda = \lambda_2 \ (= 2\,\omega_3)$

```
Block[{ty = F, rk = 4, summand, subgroup, sum},
  summand[w_] := (-1)^Length[w] exp[WeylR[ty, rk][w][rho]] * pfE[w, λ[2]];
  (* subgroup=Select[weylgroup[ty,rk],FreeQ[#,3]&];
  Print[subgroup]; *)
  subgroup = {{}, {1}, {2}, {2, 1}, {1, 2}, {1, 2, 1}} ;
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  sum
] // Simplify
```

```
{size of the isotropy subgroup : , 6}
```

```
0
```

$\lambda = \lambda_3 \ (= \omega_2)$

```
Block[{ty = F, rk = 4, summand, subgroup, sum},
  summand[w_] := (-1)^Length[w] exp[WeylR[ty, rk][w][rho]] * pfE[w, λ[3]];
  (* subgroup=Select[weylgroup[ty,rk],FreeQ[#,2]&]; *)
  subgroup = {{}, {1}, {3}, {3, 1}};
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  sum
] // Simplify
```

```
{size of the isotropy subgroup : , 4}
```

```
0
```

$\lambda = \lambda_4$ ($= \omega_1$)

```
Block[{ty = F, rk = 4, summand, subgroup, sum},
  summand[w_] := (-1)^Length[w] exp[WeylR[ty, rk][w][rho]] * pfE[w, λ[4]];
  subgroup = Select[weylgroup[ty, rk], FreeQ[#, 1] &];
  Print[{"size of the isotropy subgroup : ", subgroup // Length}];
  sum = Total@Map[summand, subgroup];
  sum
] // Simplify
```

```
{size of the isotropy subgroup : , 48}
```

0

$\lambda = \lambda_4$: more efficient check

```
Block[{ty = F, rk = 4, summand, WC3, WC2, sumoverWC2, sum, WC3C2cosets},
 summand[w_] := (-1)^Length[w] exp[WeylR[ty, rk][w][rho]] * pfE[w, λ[4]];
 WC3 = Select[weylgroup[ty, rk], FreeQ[#, 1] &];
 WC2 = Select[WC3, FreeQ[#, 3] &];
 sumoverWC2 = Total@Map[summand, WC2] // Simplify;
 WC3C2cosets = {{}, {3, 2, 3, 4, 3, 2, 3}, {3}, {2, 3, 4, 3, 2, 3}, {4, 3}, {2, 3, 2, 4, 3},
   {2, 3}, {3, 4, 3, 2, 3}, {3, 2, 3}, {4, 3, 2, 3}, {2, 4, 3}, {3, 2, 4, 3}};
 sum = 0;
 Do[
  sum = Simplify[sum + (-1)^Length[ww] (sumoverWC2 /. weyltorule[ty, rk][ww])];
  (* Print[sum]; *)
  , {ww, WC3C2cosets}
 ];
 sum
]
```

0

proof of $C_\lambda^{(2)} = D_\lambda^{(2)}$ for $\lambda \in \{0, \lambda_1, \lambda_3, \lambda_4\}$

$\lambda = \lambda_1$($= 2\,\omega_4$)

```
Block[{C2, D2, Subscript, la, sum},
 la = λ[1];
 norcharC2C1[la] // Print;
 C2 = norcharC[2, la];
 D2 = norcharD[2, la];
 sum = C2 - D2;
 Print[{"time elapsed", "sum"}];
 Timing[Simplify[sum]]
]
```

```
c[1, wt[0, -1, 2, 0]] c[1, wt[0, 1, -2, 2]] (1 - 1/x[3]^2) +
 c[1, wt[-1, 1, 0, 0]] c[1, wt[1, -1, 0, 2]] (1 - 1/(x[2]^2 x[3]^2)) +
 c[1, wt[-1, 0, 0, 2]] c[1, wt[1, 0, 0, 0]] (1 - 1/(x[1]^2 x[2]^2 x[3]^2)) +
 c[1, wt[0, -1, 2, 0]] c[1, wt[0, 1, -2, 2]] (1 - x[3]^2) +
 c[1, wt[-1, 1, 0, 0]] c[1, wt[1, -1, 0, 2]] (1 - x[2]^2 x[3]^2) +
 c[1, wt[-1, 0, 0, 2]] c[1, wt[1, 0, 0, 0]] (1 - x[1]^2 x[2]^2 x[3]^2)
{time elapsed, sum}
{1.08319, 0}
```

$\lambda = \lambda_3 \; (= \omega_2)$

```
Block[{C2, D2, Subscript, la, sum},
 la = λ[3];
 norcharC2C1[la] // Print;
 C2 = norcharC[2, la];
 D2 = norcharD[2, la];
 sum = C2 - D2;
 Print[{"time elapsed", "sum"}];
 Timing[Simplify[sum]]
]
```

```
c[1, wt[-1, 1, 0, 0]] c[1, wt[1, 0, 0, 0]] (1 - 1/x[1]) +
 c[1, wt[-1, 1, 0, 0]] c[1, wt[1, 0, 0, 0]] (1 - x[1])
{time elapsed, sum}
{0.076456, 0}
```

$\lambda = \lambda_4 \; (= \omega_1)$

```
Block[{C2, D2, Subscript, la, sum},
 la = λ[4];
 norcharC2C1[la] // Print;
 C2 = norcharC[2, la];
 D2 = norcharD[2, la];
 sum = C2 - D2;
 Print[{"time elapsed", "sum"}];
 Timing[Simplify[sum]]
]
```

$$c[1, wt[0, 1, -2, 2]]\, c[1, wt[1, -1, 2, -2]] \left(1 - \frac{1}{x[2]\, x[4]^2}\right) +$$
$$c[1, wt[0, 1, 0, -2]]\, c[1, wt[1, -1, 0, 2]] \left(1 - \frac{x[2]}{x[4]^2}\right) +$$
$$c[1, wt[-1, 1, 0, 0]]\, c[1, wt[2, -1, 0, 0]] \left(1 - \frac{1}{x[2]^3\, x[3]^4\, x[4]^2}\right) +$$
$$c[1, wt[0, 0, 0, 0]]\, c[1, wt[1, 0, 0, 0]] \left(1 - \frac{1}{x[1]^2\, x[2]^3\, x[3]^4\, x[4]^2}\right) +$$
$$c[1, wt[0, -1, 2, 0]]\, c[1, wt[1, 1, -2, 0]] \left(1 - \frac{1}{x[2]\, x[3]^4\, x[4]^2}\right) +$$
$$c[1, wt[0, 1, 0, -2]]\, c[1, wt[1, -1, 0, 2]] \left(1 - \frac{x[4]^2}{x[2]}\right) +$$
$$c[1, wt[0, 1, -2, 2]]\, c[1, wt[1, -1, 2, -2]] \left(1 - x[2]\, x[4]^2\right) +$$
$$c[1, wt[0, -1, 2, 0]]\, c[1, wt[1, 1, -2, 0]] \left(1 - x[2]\, x[3]^4\, x[4]^2\right) +$$
$$c[1, wt[-1, 1, 0, 0]]\, c[1, wt[2, -1, 0, 0]] \left(1 - x[2]^3\, x[3]^4\, x[4]^2\right) +$$
$$c[1, wt[0, 0, 0, 0]]\, c[1, wt[1, 0, 0, 0]] \left(1 - x[1]^2\, x[2]^3\, x[3]^4\, x[4]^2\right)$$

{time elapsed, sum}

{18.0381, 0}

$\lambda = 0$

definition :
 norcharCList = list of summands in Weyldenom*norcharC
 norcharDList = list of summands in Weyldenom*norcharD (equation 4.10)

```
norcharCList[2, wt[0, 0, 0, 0]] = Block[{ty = F, rk = 4, c},
   Map[Factor, Weyldenom * (List @@ (norcharC2C1[wt[0, 0, 0, 0]]) /. {c -> norcharC})]
  ];
norcharDList[2, wt[0, 0, 0, 0]] =
  Block[{ty = F, rk = 4, summand, subgroup134, cosets, sumoversubgroup134},
   summand[w_] := (-1)^Length[w] * exp[WeylR[ty, rk][w][rho]] * pfD[w, wt[0, 0, 0, 0]];
   subgroup134 = Select[weylgroup[ty, rk], FreeQ[#, 2] &];
   sumoversubgroup134 = (Total@Map[summand, subgroup134]) ;
   cosets =
    Map[movetochamberList[ty, rk][#][[1]] &, WeylOrbit[ty, rk][wt[0, 1, 0, 0]]];
   Factor /@ Map[(-1)^Length[#] * sumoversubgroup134 /. weyltorule[ty, rk][#] &, cosets]
];
(* numerical check by specialization*)
(*
-2412883260336065492616853902229261226552068298521570889 25/235362790369165882420894514010166548594626760011899597463278;
{
 Weyldenom*norcharC[2,wt[0,0,0,0]]/. {x[1]→2,x[2]→3,x[3]→5,x[4]→7},
 Weyldenom*norcharD[2,wt[0,0,0,0]]/. {x[1]→2,x[2]→3,x[3]→5,x[4]→7},
 Total[norcharCList[2,wt[0,0,0,0]]/. {x[1]→2,x[2]→3,x[3]→5,x[4]→7}],
 Total[norcharDList[2,wt[0,0,0,0]]/.{x[1]→2,x[2]→3,x[3]→5,x[4]→7}]
}
*)
```

proof 1: sum of 120 rational functions = 0

```
(* this function is to group a list of rational functions
 in x_i by its denominator with x_i specialized by numbers *)
maxprimefactor[xx_, num_List: {2, 3, 5, 7}] := Block[{temp, rule},
  rule = Inner[Rule, Array[x, 4], num, List];
  temp = xx /. rule;
  temp = Denominator@temp;
  temp = FactorInteger@temp;
  temp = (First /@ temp);
  Max@temp
 ]
Block[{differencetemp, dlist, timestart, numtogo, num, repeatGather},
 Label["begin"]; Print["process initiated!"];
 {num, repeatGather} = {{3, 7, 2, 5}, 3};
 dlist = norcharDList[2, wt[0, 0, 0, 0]];
 Do[
  dlist = Map[Total, GatherBy[dlist, maxprimefactor[#, num] &]];
  , {repeatGather}
 ];
 dlist = SortBy[dlist, maxprimefactor[#, num] &] // Reverse;
 dlist = Map[Total[dlist[[#]]] &, {{1}, {2, 3, 4}, {5}, {6},
    {7, 8, 9}, {10}, {11}, {12, 13, 14, 15}, {16, 17, 18, 19, 20, 21, 22}}];
 timestart = AbsoluteTime[];
 numtogo = Length[dlist];
 differencetemp = Factor[Total[norcharCList[2, wt[0, 0, 0, 0]]]];
 Print[{"step number", "time elapsed", "size of the expression"}];
 Do[
  differencetemp = Factor[differencetemp - dlist[[s]]];
  Print[{ToString[s] <> "/" <> ToString[numtogo],
    Floor[AbsoluteTime[] - timestart], LeafCount[differencetemp]}],
  {s, 1, numtogo}
 ];
 differencetemp
]
```

```
process initiated!
{step number, time elapsed, size of the expression}
{1/9, 64, 5 610 568}
{2/9, 241, 11 298 124}
{3/9, 409, 10 849 511}
{4/9, 477, 11 839 177}
{5/9, 843, 13 922 778}
{6/9, 1149, 12 291 439}
{7/9, 1227, 11 109 440}
{8/9, 1374, 3 039 509}
{9/9, 1385, 1}
0
```

proof 2 : sum of 120 polynomials = 0

```
v2la0poly = Block[{v2la0ratio, commonfac, denoms},
  v2la0ratio = Join[norcharCList[2, wt[0, 0, 0, 0]], -norcharDList[2, wt[0, 0, 0, 0]]];
  commonfac = 1/(x[1] x[2]^2 x[3]^3 x[4]) (-1 + x[1])^2 (-1 + x[2])^2 (-1 + x[1] x[2])^2
    (x[1] - x[3]) (-1 + x[3]) (1 + x[3]) (-1 + x[1] x[3]) (-1 + x[2] x[3])
    (1 + x[2] x[3]) (-1 + x[1] x[2] x[3]) (1 + x[1] x[2] x[3]) (-1 + x[1] x[2]^2 x[3])
    (-1 + x[2] x[3]^2)^2 (-1 + x[1] x[2] x[3]^2)^2 (-1 + x[1] x[2]^2 x[3]^2)^2
    (-1 + x[1] x[2]^2 x[3]^3) (x[1] - x[4]) (x[2] - x[4]) (x[1] x[2] - x[4])
    (x[3] - x[4]) (x[2] x[3] - x[4]) (x[1] x[2] x[3] - x[4]) (-1 + x[4]) (1 + x[4])
    (-1 + x[1] x[4]) (-1 + x[2] x[4]) (-1 + x[1] x[2] x[4]) (x[1] - x[3] x[4])
    (-1 + x[3] x[4]) (1 + x[3] x[4]) (-1 + x[1] x[3] x[4]) (-1 + x[2] x[3] x[4])
    (1 + x[2] x[3] x[4]) (-1 + x[1] x[2] x[3] x[4]) (1 + x[1] x[2] x[3] x[4])
    (-1 + x[1] x[2]^2 x[3] x[4]) (-1 + x[3]^2 x[4]) (-1 + x[2] x[3]^2 x[4])
    (1 + x[2] x[3]^2 x[4]) (-1 + x[1] x[2] x[3]^2 x[4]) (1 + x[1] x[2] x[3]^2 x[4])
    (-1 + x[2]^2 x[3]^2 x[4]) (-1 + x[1] x[2]^2 x[3]^2 x[4]) (1 + x[1] x[2]^2 x[3]^2 x[4])
    (-1 + x[1]^2 x[2]^2 x[3]^2 x[4]) (-1 + x[2] x[3]^3 x[4]) (-1 + x[1] x[2] x[3]^3 x[4])
    (-1 + x[2]^2 x[3]^3 x[4]) (-1 + x[1] x[2]^2 x[3]^3 x[4]) (1 + x[1] x[2]^2 x[3]^3 x[4])
    (-1 + x[1]^2 x[2]^2 x[3]^3 x[4]) (-1 + x[1] x[2]^3 x[3]^3 x[4]) (-1 + x[1]^2 x[2]^3 x[3]^3 x[4])
    (-1 + x[1] x[2]^2 x[3]^4 x[4]) (-1 + x[1] x[2]^3 x[3]^4 x[4]) (-1 + x[1]^2 x[2]^3 x[3]^4 x[4])
    (-1 + x[3] x[4]^2) (-1 + x[2] x[3] x[4]^2) (-1 + x[1] x[2] x[3] x[4]^2)
    (-1 + x[2] x[3]^2 x[4]^2)^2 (-1 + x[1] x[2] x[3]^2 x[4]^2)^2 (-1 + x[1] x[2]^2 x[3]^2 x[4]^2)^2
    (-1 + x[2] x[3]^3 x[4]^2) (-1 + x[1] x[2] x[3]^3 x[4]^2) (-1 + x[2]^2 x[3]^3 x[4]^2)
    (-1 + x[1] x[2]^2 x[3]^3 x[4]^2) (1 + x[1] x[2]^2 x[3]^3 x[4]^2) (-1 + x[1]^2 x[2]^2 x[3]^3 x[4]^2)
    (-1 + x[1] x[2]^3 x[3]^3 x[4]^2) (-1 + x[1]^2 x[2]^3 x[3]^3 x[4]^2)
    (-1 + x[1] x[2]^2 x[3]^4 x[4]^2)^2 (-1 + x[1] x[2]^3 x[3]^4 x[4]^2)^2
    (-1 + x[1]^2 x[2]^3 x[3]^4 x[4]^2)^2 (-1 + x[1] x[2]^3 x[3]^5 x[4]^2)
    (-1 + x[1]^2 x[2]^3 x[3]^5 x[4]^2) (-1 + x[1]^2 x[2]^4 x[3]^5 x[4]^2)
    (-1 + x[1] x[2]^2 x[3]^3 x[4]^3) (-1 + x[1] x[2]^2 x[3]^4 x[4]^3)
    (-1 + x[1] x[2]^3 x[3]^4 x[4]^3) (-1 + x[1]^2 x[2]^3 x[3]^4 x[4]^3)
    (-1 + x[1] x[2]^3 x[3]^5 x[4]^3) (-1 + x[1]^2 x[2]^3 x[3]^5 x[4]^3)
    (-1 + x[1]^2 x[2]^4 x[3]^5 x[4]^3) (-1 + x[1]^2 x[2]^4 x[3]^6 x[4]^3);
  Factor[commonfac * v2la0ratio]
 ];
```

```
(* check if these are all polynomials *)
And @@ Map[PolynomialQ[#, Array[x, 4]] &, v2la0poly]
Block[{timestart},
 timestart = Floor[AbsoluteTime[]];
 Print[Expand[Total[v2la0poly]]];
 Print[{"time elapsed", Floor[AbsoluteTime[]] - timestart}]
]
```

True

0

{time elapsed, 1389}

proof 3 : sum of 120 rational functions = 0

```
Block[{timestart, v2la0ratio},
 timestart = Floor[AbsoluteTime[]];
 v2la0ratio = Join[norcharCList[2, wt[0, 0, 0, 0]], -norcharDList[2, wt[0, 0, 0, 0]]];
 Print[Factor[Total[v2la0ratio]]];
 Print[{"time elapsed", Floor[AbsoluteTime[]] - timestart}]
]
```

0

{time elapsed, 503}