Lecture 1 Statement of Selberg trace formula

1.1 Laplacian on a Riemannian manifold

undergraduate differential geometry

parametrized surface S: $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in D$, where D is a domain in $\mathbb{R}^2$

first fundamental form $E\,du^2 + 2\,F\,du\,dv + G\,dv^2$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v$$

arclength of parametrized curve $(u(t), v(t))$, $a \le t \le b$:

$$\int_a^b \sqrt{E\,u'(t)^2 + 2F\,u'(t)v'(t) + G\,v'(t)^2}\,dt$$

surface area

$$\iint_D \sqrt{EG - F^2}\,du\,dv$$

main theme : express interesting quantities about S in terms of E, F, G (e.g. Gaussian curvature)

Riemannian manifold (M, g) : smooth manifold M equipped with a positive-definite inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ on the tangent space T_pM at each point $p \in M$.

In local coordinates, $(x^1, \ldots, x^n) : U \subset M \to \mathbb{R}^n$, the vectors

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

form a basis of T_pM. g is determined by n^2 functions

$$g_{ij}(x^1(p), \ldots, x^n(p)) := g_p\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \right)$$

g is often specified by $ds^2 = \sum_{j,k} g_{jk} dx^j dx^k$, line element

$dV = \sqrt{\det(g)} dx^1 \ldots dx^n$: volume element

Laplace-Beltrami operator (Laplacian) Δ on M : operator taking functions into functions

$$\Delta = -\sum_{j,k} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g}\, g^{jk} \frac{\partial}{\partial x^k} \right)$$

where g^{jk} entries of the inverse of the matrix (g_{jk}), and $g = \det(g_{jk})$.

Assume M is compact, connected and orientable.

Δ has non-negative discrete eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty,$$

with corresponding eigenfunctions

$$\Delta \phi_i = \lambda_i \phi_i$$

which form an orthonormal basis of $L^2(M)$.

Example 1.1 (circle)**.** Laplacian on $S^1 = \mathbb{R}/\mathbb{Z}$: $\Delta = -\frac{d^2}{dx^2}$
eigenfunctions $\varphi_m(x) = \mathrm{e}^{2\pi \mathrm{i} m x}$, $m = 0, \pm 1, \pm 2, \ldots$
eigenvalues $4\pi^2 m^2$

Example 1.2 (unit sphere)**.** $\mathbf{r}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$
Laplacian:

$$-\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

eigenfunctions : spherical harmonics $f = Y_l^m$ for $l = 0, 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots, \pm l$, where

$$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{(2l+1)}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i} m\phi}$$

and P_l^m associated Legendre function of the first kind.
eigenvalues: $\lambda = l(l+1)$ with multiplicity $2l+1$

Example 1.3 (flat torus)**.** flat torus T = quotient of $\mathbb{R}^n$ by any lattice Λ
lattice : set of all integral linear combinations of a basis of $\mathbb{R}^n$
$f(x) = e^{2\pi i\langle\xi, x\rangle}$, $\xi \in \mathbb{R}^n$ is well-defined on T exactly when $\langle\xi, x\rangle \in \mathbb{Z}$ for all $x \in \Lambda$.
Those ξ form a lattice $\Lambda^\vee$, called the dual lattice of Λ.
eigenfunctions : $e^{2\pi i\langle\xi, x\rangle}$ for $\xi \in \Lambda^\vee$ with eigenvalue $4\pi^2|\xi|^2$.
Milnor (1964) : there are non-isomorphic isospectral tori of dimension 16; there two lattices whose number of points having a given norm is always the same

In general, almost always impossible to find explicit eigenvalues and eigenfunctions
Selberg trace formula for compact hyperbolic surfaces : model for other general trace formulas; relates eigenvalues of the Laplacian and length spectrum of geodesics

1.2 Hyperbolic plane

two models of hyperbolic plane :
two models : unit disk $\mathbb{D} = \{z : |z| < 1\}$ and upper-half plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$
line element ds, volume element $d\mu$, distance $d(z, z')$ between z, z':

	ds^2	$d\mu$	$\cosh d(z, z')$
$\mathbb{D}$	$\dfrac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$	$\dfrac{4\,dx\,dy}{(1-x^2-y^2)^2}$	$1 + \dfrac{2\|z-z'\|^2}{(1-\|z\|)^2(1-\|z'\|)^2}$
$\mathbb{H}$	$\dfrac{dx^2+dy^2}{y^2}$	$\dfrac{dx\,dy}{y^2}$	$1 + \dfrac{\|z-z'\|^2}{2\operatorname{Im} z \operatorname{Im} z'}$

Exercise 1.4. Laplacian takes the following form:

	$-\Delta$
$\mathbb{D}$	$\dfrac{(1-x^2-y^2)^2}{4}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$
$\mathbb{H}$	$y^2\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$

$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ acts on $\mathbb{H}$:

$$g : \mathbb{H} \to \mathbb{H}, \qquad z \mapsto gz := \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

An element of $\mathrm{PSL}(2,\mathbb{R})$ is an isometry of $\mathbb{H}$.

- $K = \mathrm{Stab}_i = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \theta \in \mathbb{R} \right\}$.
- $A = \mathrm{Stab}_{0,\infty} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda > 0 \right\}$
- $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}$

Let $g \in \mathrm{PSL}(2,\mathbb{R})$ with $g \neq \mathrm{Id}$.

1. $|\operatorname{tr}(g)| < 2$ iff g is conjugated to an element of K iff g fixes a single point in $\mathbb{H}$.
2. $|\operatorname{tr}(g)| = 2$ iff g is conjugated to an element of N iff g fixes a single point in $\partial\mathbb{H}$.
3. $|\operatorname{tr}(g)| > 2$, iff g is conjugated to an element of A iff g fixes two points in $\partial\mathbb{H}$.

length of g :

$$\ell(g) := \inf_{z \in \mathbb{H}} d(gz, z).$$

$l(g) > 0$ only for hyperbolic g and is given by

$$\ell(g) = 2\operatorname{arccosh}(|\operatorname{tr}(g)|/2).$$

1.3 Selberg trace formula

Let F be a compact Riemann surface of genus $g \geq 2$.

Uniformization theorem : F is conformally equivalent to $\Gamma\backslash\mathbb{H}$, where Γ is discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Each element $\gamma \in \Gamma - \{I\}$ is hyperbolic since Γ is torsion-free (and so does not contain any elliptic elements) and cocompact (and so does not contain any parabolic elements);

metric on $\mathbb{H}$ induces metric on F, and so Laplacian makes sense.

Exercise 1.5. For a hyperbolic $P \in \Gamma$, the centralizer $Z(P) = \{g \in \Gamma : gP = Pg\}$ is an infinite cyclic group.

There exists unique generator P_0 of $Z(P)$ such that $P = P_0^n$ for $n \in \mathbb{Z}_{\geq 0}$.

Theorem 1.6 (Selberg (195?))**.** *Let h be an analytic function on $|\operatorname{Im}(r)| \leq \frac{1}{2} + \delta$ such that*

$$h(-r) = h(r) \quad \text{and} \quad |h(r)| \leq A[1+|r|]^{-2-\delta} \quad (A > 0, \delta > 0).$$

Then

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\operatorname{area}(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr + \sum_{\{P\}} \frac{\ell(P_0)}{e^{\ell(P)/2} - e^{-\ell(P)/2}} g(\ell(P)),$$

where the sum is over all conjugacy classes of hyperbolic elements; $\{P\}$ denotes the conjugacy class containing P; $g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r) e^{-iru} dr$.

The sums and integrals are all absolutely convergent.

The sum can be rewritten as

$$\sum_{\{P_0\}} \sum_{n=1}^{\infty} \frac{\ell(P_0)}{2\sinh[n\ell(P_0)/2]} g(n\ell(P_0))$$

where the sum is over all conjugacy classes of primitive hyperbolic elements.

Lecture 2 Applications

2.1 Spectrum of the Bolza surface

Use the disk model.

The Bolza surface is defined as the quotient

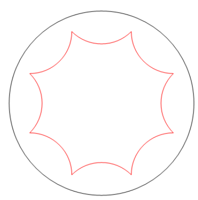
$$\mathcal{M} = G\backslash\mathbb{D}$$

where G is subgroup of $\mathrm{SU}(1,1)/\{\pm 1\}$, generated by

$$g_k = \begin{bmatrix} \xi^2 & e^{ik\pi/4}\sqrt{2}\xi \\ e^{-ik\pi/4}\sqrt{2}\xi & \xi^2 \end{bmatrix}, \text{ where } \xi = \sqrt{1+\sqrt{2}}.$$

g_k and g_{k+4} are inverses of each other

We can the regular octagon as a fundamental domain.



This is a compact Riemann surface of genus 2.

The translations g_k all have the same length

$$\ell\left(g_k\right) = 2\operatorname{arccosh}(1+\sqrt{2}) \approx 3.05714, k = 0, 1, \ldots, 7$$

Fact: for any hyperbolic $P \in G$, $\ell(P)$ is of the form $2\operatorname{arccosh}(m+n\sqrt{2})$ for some $m, n \in \mathbb{Z}_{>0}$.

We apply the trace formula.

Choose any $\epsilon > 0$ and define

$$h_z(r) = \exp\left[-\left(z-r\right)^2/\epsilon^2\right] + \exp\left[-\left(z+r\right)^2/\epsilon^2\right].$$

For fixed r, $h_z(r)$ is sum of two Gaussians around r as a function of z; ϵ standard deviation

fourier transform of h_z (as a function of r):

$$g_z(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_z(r) e^{-iru} dr = \frac{\epsilon}{\sqrt{\pi}} \cos(zu) \exp\left[-\frac{\epsilon^2}{4}u^2\right]$$

spectral side:

$$\sum_{n=0}^{\infty} h_z(r_n)$$

As a function of $z \in \mathbb{R}$, it has peaks around r_n.

geometric side: Consider the multiset $\{\ell(P_0) : \{P_0\}\}$ of lengths of conj. classes. of primitive hyperbolic elements.

Order its elements $0 < l_1 < l_2 < \dots$ and let g_n be the multiplicity of l_n.

$$\int_{-\infty}^{\infty} r \tanh(\pi r)\, h_z(r)\, dr + \frac{\epsilon}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{g_n l_n}{\sinh(k l_n/2)} \cos(z k l_n) \exp\left[-\frac{\epsilon^2}{4}(k l_n)^2\right]$$

By evaluating RHS for many z, we can plot it as a graph of z.

From all words of length ≤ 11, we find 206796230 primitive hyperbolic conjugacy classes; need 2.5 GB to save words; See `https://github.com/chlee-0/bolza`.

2.2 Weyl's law

Let $F = \Gamma\backslash\mathbb{H}$, $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ as before.

Let

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}.$$

Weyl's law:

$$N(\lambda) \sim \frac{\mathrm{Area}(F)}{4\pi}\lambda, \qquad \lambda \to \infty.$$

2.3 Prime geodesic theorem

Let $\pi(x)$ be the number of prime closed geodesics γ such that $e^{\ell(\gamma)} \leqslant x$.

Prime geodesic theorem:

$$\pi(x) \sim \frac{x}{\log(x)}, \qquad x \to \infty$$

Lecture 3 Sketch of proof

Assume $F = \Gamma\backslash\mathbb{H}$ so that F is a compact Riemann surface of genus ≥ 2.

$\mathfrak{F}$: compact fundamental domain of Γ (one can take this as a geodesic polygon)

inner product on $L^2(\Gamma\backslash\mathbb{H})$:

$$(f_1, f_2) = \int_{\mathfrak{F}} f_1(z)\overline{f_2(z)} d\mu(z),$$

where

$$d\mu(z) = \frac{dx\,dy}{y^2}$$

Recall

$$\Delta u = y^2(u_{xx} + u_{yy})$$

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$
$$\Delta\varphi_n = \lambda_n \varphi_n$$

$$L^2(F) = \oplus_{n=0}^{\infty} \mathbb{C}\varphi_n$$

We can assume that φ_n is real-valued.

For a careful treatment of analytical issues, see Spectral Theory and the Trace Formula by Bump (`http://sporadic.stanford.edu/bump/match/trace.pdf`).

3.1 point-pair invariant and integral operator

Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a smooth function with compact support. Define $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ by

$$k(z, w) = \Phi\left[\frac{|z-w|^2}{\operatorname{Im}(z)\operatorname{Im}(w)}\right].$$

The function $k(z, w)$ is called a point-pair invariant.

Define an integral operator L with kernel k:

$$Lf(z) = \int_{\mathbb{H}} k(z, w) f(w) d\mu(w)$$

Fact: An eigenfunction $f : \mathbb{H} \to \mathbb{C}$ of Δ is also an eigenfunction of L. In particular, if $\Delta f = \lambda f$, then

$$\int_{\mathbb{H}} k(z, w) f(w) d\mu(w) = h(r) f(z)$$

where $\lambda = \frac{1}{4} + r^2$ and h is the Selberg/Harish-Chandra transform of k defined by

$$Q(x) = \int_x^{\infty} \frac{\Phi(t)}{\sqrt{t-x}} dt,\ x \ge 0$$
$$g(u) = Q\left(e^u + e^{-u} - 2\right),\ u \in \mathbb{R}.$$
$$h(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du.$$

Then g and h are even functions; g has compact support and h decays faster than any polynomial (.

Define automorphic kernel

$$K(z, w) \stackrel{\text{def}}{=} \sum_{T \in \Gamma} k(Tz, w) \quad \text{for } (z, w) \in \mathbb{H} \times \mathbb{H}$$

and restrict the domain of integral operator L to functions in $L^2(\Gamma\backslash\mathbb{H})$.

Compute the trace of L two different ways.

First,

$$L\varphi_n = h(r_n)\varphi_n$$

implies $\operatorname{tr}(L) = \sum_{n=0}^{\infty} h(r_n)$.

3.2 spectral expansion of kernel

Claim:

$$K(z,w) = \sum_{n=0}^{\infty} h(r_n)\varphi_n(z)\varphi_n(w)$$

Proof. Let $G(z) = K(z,w)$ for w fixed. Since $G \in C^\infty(\Gamma\backslash\mathbb{H})$, it follows that $G(z) = \sum c_n\varphi_n(z)$, where

$$c_n = (G,\varphi_n) = \int_{\mathbb{H}} k(z,w)\varphi_n(z)\, d\mu(z).$$

The integral is

$$(L\varphi_n)(w) = h(r_n)\varphi_n(w).$$

□

From $K(z,z) = \sum_{n=0}^{\infty} h(r_n)\varphi_n(z)\varphi_n(z)$

$$\int_{\mathfrak{F}} K(z,z)d\mu(z) = \sum_{n=0}^{\infty} h(r_n).$$

3.3 geometric side

The integral can be written as a sum over the conjugacy classes:

$$\begin{aligned}\int_{\mathfrak{F}} K(z,z)d\mu(z) &= \sum_{T\in\Gamma}\int_{\mathfrak{F}} k(Tz,z)d\mu(z)\\ &= \sum_{\{P\}}\sum_{T\in\{P\}}\int_{\mathfrak{F}} k(Tz,z)d\mu(z)\end{aligned}$$

The inner sum can be rewritten as a single integral: note that $T = \tau^{-1}P\tau$ for unique $\tau \in Z(P)\backslash\Gamma$.

$$\begin{aligned}\sum_{T\in\{P\}}\int_{\mathfrak{F}} k(Tz,z)d\mu(z) &= \sum_{\tau\in Z(P)\backslash\Gamma}\int_{\mathfrak{F}} k\left(\tau^{-1}P\tau z, z\right) d\mu(z)\\ &= \sum_{\tau\in Z(P)\backslash\Gamma}\int_{\mathfrak{F}} k(P\tau z,\tau z)d\mu(z)\\ &= \sum_{\tau\in Z(P)\backslash\Gamma}\int_{\tau(\mathfrak{F})} k(Pw,w)d\mu(w)\\ &= \int_{FD|Z(P)|} k(Pw,w)d\mu(w)\end{aligned}$$

where $FD[Z(P)]$ denotes a fundamental domain for $Z(P)$.

P identity :

$$\int_{\mathfrak{F}} k(w,w)d\mu(w) = \int_{\mathfrak{F}} \Phi(0)d\mu(w) = \text{area}(F)\Phi(0) = \frac{\text{area}(F)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)dr.$$

The final integral allows to remove Φ in the statement.

P hyperbolic:

Let $P = P_0^k$ for P_0 primitive and $k \in \mathbb{Z}_{\geq 0}$.

Let $\lambda_0 = e^{\ell(P_0)}$ and $\lambda = e^{\ell(P)}$.

Inside $\mathrm{PSL}(2,\mathbb{R})$, P_0 is conjugate to $Q_0(z) = \lambda_0 z$ and we can replace the integral:

$$\int_{FD|Z(P)|} k(Pw,w)d\mu(w) = \int_{FD|\langle Q_0\rangle} k(Qw,w)d\mu(w).$$

$$\int_{FD|Z(P)|} k(Pw,w)d\mu(w) = \frac{\ln\lambda_0}{\lambda^{1/2}-\lambda^{-1/2}}g(\ln\lambda)$$

This proves a weaker version of Selbert trace formula with the assumption that g has compact support and h is its inverse Fourier transform From here, one can use an approximation argument to upgrade this to the version stated before.

Lecture 4 Advanced topics

4.1 Selbert trace formula for PSL(2,Z)

$\Gamma = \mathrm{PSL}(2,\mathbb{Z})$

$\Gamma\backslash\mathbb{H}$ is no longer compact, and the spectrum has a continuous part

$$K(z,w) = \sum_j h(r_j)\, u_j(z)\overline{u_j(w)} + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)E\left(z,\frac{1}{2}+ir\right)\overline{E\left(w,\frac{1}{2}+ir\right)}\,dr$$

$E(z,s)$ is the Eisenstein series

Geometric side : parabolic, elliptic conjugacy classes

parabolic conj. class: power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

elliptic conj. class: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (order 2), $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ (order 3)

$$\begin{aligned}
\sum_{j=0}^{\infty} h(r_j) =& \frac{1}{12}\int_{-\infty}^{+\infty} rh(r)\tanh(\pi r)\,dr \\
&+ \frac{1}{4}\int_{-\infty}^{\infty}\frac{h(r)\,dr}{\cosh(\pi r)} + \frac{2\sqrt{3}}{9}\int_{-\infty}^{\infty} h(r)\frac{\cosh(\pi r/3)}{\cosh(\pi r)}\,dr \\
&+ \sum_{\{P\}} \frac{\ell(P_0)}{e^{\ell(P)/2}-e^{-\ell(P)/2}}g(\ell(P)) \\
&+ g(0)\log(\pi/2) + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\left[\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right]dr
\end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ with } p \text{ prime and } k \in \mathbb{Z}_{>0} \\ 0 & \text{otherwise} \end{cases}$$

4.2 Jacquet-Langlands correspondence

Let F be a field and let $a, b \in F^\times$. The quaternion algebra $D_{a,b}(F)$ is the ring

$$\{x_0 + x_1 i + x_2 j + x_3 k \mid x_0, \dots, x_3 \in F\}$$

with multiplication

$$i^2 = a, j^2 = b, ij = k = -ji.$$

Example 4.1. $D_{-1,-1}(\mathbb{R})$: Hamilton's quaternions.

The conjugate of α is

$$\bar{\alpha} = x_0 - x_1 i - x_2 j - x_3 k,$$

and the reduced norm of α is $\mathrm{N_{red}}(\alpha) := \alpha\bar{\alpha} = \bar{\alpha}\alpha$; trace $\mathrm{Tr}(\alpha) = \alpha + \bar{\alpha}$.

A quaternion algebra is a division algebra if every non-zero element α admits an inverse (iff $\mathrm{N_{red}}(\alpha) \neq 0$)

A subring $\mathcal{O}$ of $D_{a,b}(\mathbb{Q})$ is an order when $1 \in \mathcal{O}$ and $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank 4, i.e.,

$$\mathcal{O} = \{x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \mid x_1, \dots, x_4 \in \mathbb{Z}\}$$

where (e_1, e_2, e_3, e_4) is a basis of A over $\mathbb{Q}$.

The discriminant of an order $\mathcal{O} = \mathbb{Z}[e_1, e_2, e_3, e_4]$ is defined to be:

$$d(\mathcal{O}) = \left|\det\left[\mathrm{Tr}\left(e_i e_j\right)\right]_{1 \le i,j \le 4}\right|.$$

This is of the form r^2 for a positive integer r.

Fact : Every order is contained in a maximal order, i.e., an order which is not strictly contained in any other one.

Example 4.2. Assume

$$\begin{cases} ab > 1 \\ a \equiv 1 (\mathrm{mod} 4),\ b \text{ odd} \\ \left(\frac{b}{p}\right) = -1 \text{ for every prime } p \text{ dividing } a \\ \left(\frac{a}{p}\right) = -1 \text{ for every prime } p \text{ dividing } b. \end{cases}$$

$D_{a,b}(\mathbb{Q})$ is a division algebra and

$$\mathcal{O} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{1+i}{2} + \mathbb{Z} \cdot j + \mathbb{Z} \cdot \frac{j+k}{2}$$

is a maximal order, and $d(\mathcal{O}) = (ab)^2$.

Fix two positive integers a, b, relative prime and square-free.

Let $D_{a,b}(\mathbb{R})^1 := \{g \in D_{a,b}(\mathbb{R}) \mid \mathrm{N_{red}}(g) = 1\}$.

There exists an isomorphism $\Phi : D_{a,b}(\mathbb{R})^1 \to \mathrm{SL}(2, \mathbb{R})$.

Let $\mathcal{O}$ be an order in $D_{a,b}(\mathbb{Q})$ and $\mathcal{O}^1 := \mathcal{O} \cap D_{a,b}(\mathbb{R})^1$.

Fact : $\Gamma_{\mathcal{O}} = \Phi(\mathcal{O}^1)$ is cocompact (i.e. $\Gamma_{\mathcal{O}} \backslash \mathbb{H}$ is compact) iff $D_{a,b}(\mathbb{Q})$ is a division algebra iff $(0, 0, 0)$ is the unique solution in integers of the Diophantine equation $x^2 - ay^2 - bz^2 = 0$.

Theorem 4.3. *Let $\mathcal{O}$ be a maximal order in a division algebra $D_{a,b}(\mathbb{Q})$ with $d(\mathcal{O}) = r^2$. Then the set of non-zero eigenvalues for $\Gamma_{\mathcal{O}}\backslash\mathbb{H}$, counted with multiplicity, coincides with the set of eigenvalues associated with primitive Maass forms for the group $\Gamma_0(r)\backslash\mathbb{H}$,*

$$\Gamma_0(N) = \left\{ \gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

This is a special case of the Jacquet-Langlands correspondence.

Lecture 5 Exercises

Exercise 5.1. Compute the Laplacian Δ for $\mathbb{D}$ and $\mathbb{H}$.

	ds^2	$-\Delta$
$\mathbb{D}$	$\dfrac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$	$\dfrac{(1-x^2-y^2)^2}{4}\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$
$\mathbb{H}$	$\dfrac{dx^2+dy^2}{y^2}$	$y^2\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$

Exercise 5.2. Consider the following subgroups of $\mathrm{SL}(2,\mathbb{R})$:

- $K=\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \theta\in\mathbb{R}\right\}$.
- $A=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \lambda>0\right\}$
- $N=\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, t\in\mathbb{R}\right\}$

For any $g\in\mathrm{SL}(2,\mathbb{R})$ there exists a unique $(k,a,n)\in K\times A\times N$ such that $g=kan$.

Exercise 5.3. Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. For a hyperbolic $P\in\Gamma$, the centralizer $Z(P)=\{g\in\Gamma : gP=Pg\}$ is an infinite cyclic group.

Exercise 5.4. The video at `https://www.youtube.com/watch?v=ajDx_HCMIBg` is intended to visualize the action of two hyperbolic elements g_0 and $g_0g_3g_4$ on the unit disk, where

$$g_k=\begin{bmatrix}\xi^2 & e^{ik\pi/4}\sqrt{2}\xi\\ e^{-ik\pi/4}\sqrt{2}\xi & \xi^2\end{bmatrix},\qquad \xi=\sqrt{1+\sqrt{2}}.$$

Explain the computations required to produce it.

Exercise 5.5. Let $F=\Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. A geodesic of F is obtained as the image under the canonical projection of a geodesic of $\mathbb{H}$. A closed geodesic on F is the projection of a geodesic of $\mathbb{H}$ preserved by a non-trivial element $\gamma\in\Gamma$. Two constant speed parametrizations $\alpha,\alpha' : S^1=\mathbb{R}/\mathbb{Z}\to F$ of a closed geodesic are equivalent if $\alpha'(t)=\alpha(t+c)$ for some constant c. An oriented closed geodesic is an equivalence class of closed parametrized geodesics. Then there is a bijection between the set of conjugacy classes of hyperbolic elements in Γ and the set of oriented closed geodesics on F.

The video at `https://www.youtube.com/watch?v=06pv6X8gaQQ` shows an oriented prime closed geodesic on the Bolza surface. What is the corresponding primitive hyperbolic conjugacy class? Find a representative.

Exercise 5.6 (optional)**.** Let $F=\Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface of genus $g\geq 2$. Check that $\mathrm{area}(F)=4\pi(g-1)$.

Exercise 5.7. Derive Weyl's law:

$$N(\lambda)\sim\frac{\mathrm{area}(F)}{4\pi}\lambda,\qquad \lambda\to\infty,$$

where

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}.$$

Exercise 5.8 (optional)**.** Prove that

$$\Phi(0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)dr.$$

Exercise 5.9. Let $P_0(z) = \lambda_0 z$, $\lambda_0 > 1$ and $P(z) = \lambda z$ with $\lambda = \lambda_0^n, n \in \mathbb{Z}_{>0}$.

1. The fundamental domain for the cyclic group $\langle P_0 \rangle$ is $\{z \in \mathbb{H} : 1 < y < \lambda_0\}$.
2. Show that
$$\int_{[1 \leq \mathrm{Im}(z) \leq \lambda_0]} k(\lambda z, z)d\mu(z) = \frac{\ln \lambda_0}{\lambda^{1/2} - \lambda^{-1/2}} g(\ln \lambda).$$