Lecture 5 Exercises

Exercise 5.1. Compute the Laplacian Δ for $\mathbb{D}$ and $\mathbb{H}$.

	ds^2	$-\Delta$
$\mathbb{D}$	$\dfrac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$	$\dfrac{(1-x^2-y^2)^2}{4}\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$
$\mathbb{H}$	$\dfrac{dx^2+dy^2}{y^2}$	$y^2\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$

Exercise 5.2. Consider the following subgroups of $\mathrm{SL}(2,\mathbb{R})$:

- $K=\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \theta\in\mathbb{R}\right\}$.
- $A=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \lambda>0\right\}$
- $N=\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, t\in\mathbb{R}\right\}$

For any $g\in\mathrm{SL}(2,\mathbb{R})$ there exists a unique $(k,a,n)\in K\times A\times N$ such that $g=kan$.

Exercise 5.3. Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. For a hyperbolic $P\in\Gamma$, the centralizer $Z(P)=\{g\in\Gamma : gP=Pg\}$ is an infinite cyclic group.

Exercise 5.4. The video at `https://www.youtube.com/watch?v=ajDx_HCMIBg` is intended to visualize the action of two hyperbolic elements g_0 and $g_0g_3g_4$ on the unit disk, where

$$g_k=\begin{bmatrix}\xi^2 & e^{ik\pi/4}\sqrt{2}\xi\\ e^{-ik\pi/4}\sqrt{2}\xi & \xi^2\end{bmatrix},\qquad \xi=\sqrt{1+\sqrt{2}}.$$

Explain the computations required to produce it.

Exercise 5.5. Let $F=\Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. A geodesic of F is obtained as the image under the canonical projection of a geodesic of $\mathbb{H}$. A closed geodesic on F is the projection of a geodesic of $\mathbb{H}$ preserved by a non-trivial element $\gamma\in\Gamma$. Two constant speed parametrizations $\alpha,\alpha' : S^1=\mathbb{R}/\mathbb{Z}\to F$ of a closed geodesic are equivalent if $\alpha'(t)=\alpha(t+c)$ for some constant c. An oriented closed geodesic is an equivalence class of closed parametrized geodesics. Then there is a bijection between the set of conjugacy classes of hyperbolic elements in Γ and the set of oriented closed geodesics on F.

The video at `https://www.youtube.com/watch?v=06pv6X8gaQQ` shows an oriented prime closed geodesic on the Bolza surface. What is the corresponding primitive hyperbolic conjugacy class? Find a representative.

Exercise 5.6 (optional)**.** Let $F=\Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface of genus $g\geq 2$. Check that $\mathrm{area}(F)=4\pi(g-1)$.

Exercise 5.7. Derive Weyl's law:

$$N(\lambda)\sim\frac{\mathrm{area}(F)}{4\pi}\lambda,\qquad \lambda\to\infty,$$

where

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}.$$

Exercise 5.8 (optional)**.** Prove that

$$\Phi(0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr.$$

Exercise 5.9. Let $P_0(z) = \lambda_0 z$, $\lambda_0 > 1$ and $P(z) = \lambda z$ with $\lambda = \lambda_0^n, n \in \mathbb{Z}_{>0}$.

1. The fundamental domain for the cyclic group $\langle P_0 \rangle$ is $\{z \in \mathbb{H} : 1 < y < \lambda_0\}$.

2. Show that

$$\int_{[1 \leq \mathrm{Im}(z) \leq \lambda_0]} k(\lambda z, z) d\mu(z) = \frac{\ln \lambda_0}{\lambda^{1/2} - \lambda^{-1/2}} g(\ln \lambda).$$