
code

data (pre-computed)

```

In[*]:= type = "E7";
(*Define basis elements as symbols*)
basisElements = {V0, V1, V2, V3, V4, V5};
(*Define multiplication rules*)
multiplicationRules = {
  (V0 * V0) => V0,
  (V0 * V1) => V1,
  (V0 * V2) => V2,
  (V0 * V3) => V3,
  (V0 * V4) => V4,
  (V0 * V5) => V5,
  (V1 * V1) => V5 + V4 + V2 + V0,
  (V1 * V2) => V3 + V1,
  (V1 * V3) => V4 + V2,
  (V1 * V4) => V3 + V1,
  (V1 * V5) => V1,
  (V2 * V2) => V4 + V0,
  (V2 * V3) => V1,
  (V2 * V4) => V5 + V2,
  (V2 * V5) => V4,
  (V3 * V3) => V5 + V0,
  (V3 * V4) => V1,
  (V3 * V5) => V3,
  (V4 * V4) => V4 + V0,
  (V4 * V5) => V2,
  (V5 * V5) => V0};
KRvariableToVerlindeImage = {
  x1 -> ringElement[V0 + V2],
  x2 -> ringElement[V0],
  x3 -> ringElement[V0 + 2 V2 + V4],
  x4 -> ringElement[V0],
  x5 -> ringElement[2 V0 + 3 V2 + 3 V4 + V5],
  x6 -> ringElement[V0],
  x7 -> ringElement[V1 + V3],
  x8 -> ringElement[V5],
  x9 -> ringElement[2 V1 + 2 V3],
  x10 -> ringElement[V5],
  x11 -> ringElement[V0 + V2 + V4],
  x12 -> ringElement[V0],
  x13 -> ringElement[V1],
  x14 -> ringElement[V5],
  x31 -> ringElement[V1],
  x41 -> ringElement[V1 + V3],
  x43 -> ringElement[V0 + V2],
  x44 -> ringElement[V0 + 2 V2 + V4],
  x45 -> ringElement[2 V0 + 3 V2 + 3 V4 + V5],
  x53 -> ringElement[V0 + V2 + V4],
  x59 -> ringElement[2 V1 + 2 V3]
};

```

import exchange relations

```

In[ ]:= extractxindex[var_] := Module[{numberStr}, Which[
  ListQ[var], extractxindex /@ var,
  True, numberStr = StringDrop[SymbolName[var], 1];
  ToExpression[numberStr]]]
xSort[vars_List] := SortBy[vars, extractxindex]
swapIfNotOrdered[triple_List] := Module[{second = triple[[2]], third = triple[[3]]},
  If[OrderedQ[{second, third}], ReplacePart[triple, {2 → third, 3 → second}], triple]]
filePath = FileNameJoin[{NotebookDirectory[], "exchange_relations_" <> type <> ".txt"}];
fileLines = ReadList[filePath, String];
convertedLines = DeleteDuplicates[StringReplace[fileLines, {"[" → "{", "]" → "}"}]];
exchangerelations = ToExpression /@ convertedLines;
exchangerelations =
  Map[{xSort[#[[1]]], xSort[#[[2]]], xSort[#[[3]]]} &, exchangerelations];
exchangerelations = SortBy[exchangerelations, extractxindex[#[[1]]] &];
exchangerelations = Map[swapIfNotOrdered, exchangerelations];
exchangerelations = DeleteDuplicates[exchangerelations];
(* display sample exchange relations *)
exchangerelations[[1 ;; 10]]
(* display cluster variables *)
clustervariables = Flatten[exchangerelations] // DeleteDuplicates // xSort

Out[ ]:= {{x1, x19}, {x2, x7, x13}, {x18}}, {{x1, x20}, {x2, x13}, {x40}},
  {{x1, x21}, {x2, x7}, {x60}}, {{x1, x23}, {x2, x13, x69}, {x18, x28}},
  {{x1, x24}, {x2, x13, x27}, {x71}}, {{x1, x25}, {x2, x9}, {x28}},
  {{x1, x26}, {x28, x60}, {x2, x69}}, {{x1, x29}, {x2, x11, x27}, {x33}},
  {{x1, x30}, {x2, x27}, {x65}}, {{x1, x32}, {x2, x27, x37}, {x33, x60}}

Out[ ]:= {x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14, x15, x16, x17,
  x18, x19, x20, x21, x22, x23, x24, x25, x26, x27, x28, x29, x30, x31, x32,
  x33, x34, x35, x36, x37, x38, x39, x40, x41, x42, x43, x44, x45, x46, x47,
  x48, x49, x50, x51, x52, x53, x54, x55, x56, x57, x58, x59, x60, x61, x62,
  x63, x64, x65, x66, x67, x68, x69, x70, x71, x72, x73, x74, x75, x76, x77}

```

code for Verlinde ring

```

In[ ]:= rankofVerlinde = Length[basisElements];

simplifyRingElement[expr_] := Module[{result},
  result = Total[Coefficient[expr, #] * # & /@basisElements];
  result]

(*Define a function for addition of ring elements*)
ringAdd[element1_, element2_] := element1 + element2
(*Define a function for multiplication of ring elements*)
ringMultiply[element1_, element2_] :=
  Expand[Expand[(element1 element2)] /. multiplicationRules]

(*Define the ringElement class*)
ringElement /: Plus[ringElement[e1_], ringElement[e2_]] :=
  ringElement[simplifyRingElement[ringAdd[e1, e2]]];
ringElement /: Times[ringElement[e1_], ringElement[e2_]] :=
  ringElement[simplifyRingElement[ringMultiply[e1, e2]]];
ringElement /: Times[scalar_, ringElement[e_]] /;
  FreeQ[scalar, Alternatives @@ basisElements] :=
  ringElement[simplifyRingElement[scalar * e]];
(*Subtraction rule using addition and scalar multiplication*)
ringElement /: Subtract[ringElement[e1_], ringElement[e2_]] :=
  ringElement[simplifyRingElement[ringAdd[e1, -1 * e2]]];

unknownindex = Complement[Range[1, Length[clustervariables]],
  Map[extractxindex, First /@ KRvariableToVerlindeImage]];

variableToVerlindeImage = Union[
  KRvariableToVerlindeImage,
  Table[ToExpression["x" <> ToString[index]] ->
    ringElement[Sum[c[index, k] * Evaluate[Symbol["V" <> ToString[k]]],
      {k, 0, rankofVerlinde - 1}]], {index, unknownindex}];
unknowns = Flatten@Table[Table[c[index, k], {k, 0, rankofVerlinde - 1}],
  {index, unknownindex}];

(* convert expression in cluster variables to Verlinde element *)
convertToRingElements[expr_] := expr /. variableToVerlindeImage
(*Function to transform {{x1,x17},x2 x3+x4} to x1*x17-(x2*x3+x4)*)
exchangeVerlinde[{{a_, b_}, X_, Y_}] :=
  convertToRingElements[a] * convertToRingElements[b] -
  (Times @@ convertToRingElements[X]) - (Times @@ convertToRingElements[Y])
exponents[expr_] := Module[{terms, totalDegree},
  terms = If[Head[Expand[expr]] === Plus, List @@ Expand[expr], {Expand[expr]}];
  totalDegree = Map[Total[Cases[Factor[#], c[_ , _]^n_ .> n, {0, 1}]] &, terms];
  totalDegree
]
extractCoefficients[ringElement[expr_]] := Coefficient[expr, #] & /@basisElements

```

exemple of exchange relations and corresponding

equations for Verlinde coefficients

```
In[ ]:= unknowns // Length
```

```
Out[ ]:= 336
```

```
In[ ]:= exchangerelations[[1]]
```

```
Out[ ]:= {{x1, x19}, {x2, x7, x13}, {x18}}
```

```
In[ ]:= Map[# == 0 &, extractCoefficients@exchangeVerlinde[{{x1, x19}, {x2, x7, x13}, {x18}}]] //  
TableForm
```

```
Out[ ]//TableForm=
```

```
-1 - c[18, 0] + c[19, 0] + c[19, 2] == 0  
-c[18, 1] + 2 c[19, 1] + c[19, 3] == 0  
-2 - c[18, 2] + c[19, 0] + c[19, 2] + c[19, 4] == 0  
-c[18, 3] + c[19, 1] + c[19, 3] == 0  
-2 - c[18, 4] + c[19, 2] + c[19, 4] + c[19, 5] == 0  
-1 - c[18, 5] + c[19, 4] + c[19, 5] == 0
```

```
In[*]:= Flatten[Table[extractCoefficients@exchangeVerlinde[exchangerelations[[index]]],
  {index, 10}]] // TableForm
```

```
Out[*]//TableForm=
```

```
-1 - c[18, 0] + c[19, 0] + c[19, 2]
-c[18, 1] + 2 c[19, 1] + c[19, 3]
-2 - c[18, 2] + c[19, 0] + c[19, 2] + c[19, 4]
-c[18, 3] + c[19, 1] + c[19, 3]
-2 - c[18, 4] + c[19, 2] + c[19, 4] + c[19, 5]
-1 - c[18, 5] + c[19, 4] + c[19, 5]
c[20, 0] + c[20, 2] - c[40, 0]
-1 + 2 c[20, 1] + c[20, 3] - c[40, 1]
c[20, 0] + c[20, 2] + c[20, 4] - c[40, 2]
c[20, 1] + c[20, 3] - c[40, 3]
c[20, 2] + c[20, 4] + c[20, 5] - c[40, 4]
c[20, 4] + c[20, 5] - c[40, 5]
c[21, 0] + c[21, 2] - c[60, 0]
-1 + 2 c[21, 1] + c[21, 3] - c[60, 1]
c[21, 0] + c[21, 2] + c[21, 4] - c[60, 2]
-1 + c[21, 1] + c[21, 3] - c[60, 3]
c[21, 2] + c[21, 4] + c[21, 5] - c[60, 4]
c[21, 4] + c[21, 5] - c[60, 5]
c[23, 0] + c[23, 2] - c[18, 0] × c[28, 0] - c[18, 1] × c[28, 1] - c[18, 2] × c[28, 2] - c[18, 3] × c
2 c[23, 1] + c[23, 3] - c[18, 1] × c[28, 0] - c[18, 0] × c[28, 1] - c[18, 2] × c[28, 1] - c[18, 4] ×
c[23, 0] + c[23, 2] + c[23, 4] - c[18, 2] × c[28, 0] - c[18, 1] × c[28, 1] - c[18, 3] × c[28, 1] - c
c[23, 1] + c[23, 3] - c[18, 3] × c[28, 0] - c[18, 2] × c[28, 1] - c[18, 4] × c[28, 1] - c[18, 1] × c
c[23, 2] + c[23, 4] + c[23, 5] - c[18, 4] × c[28, 0] - c[18, 1] × c[28, 1] - c[18, 3] × c[28, 1] - c
c[23, 4] + c[23, 5] - c[18, 5] × c[28, 0] - c[18, 1] × c[28, 1] - c[18, 4] × c[28, 2] - c[18, 3] × c
c[24, 0] + c[24, 2] - c[27, 1] - c[71, 0]
2 c[24, 1] + c[24, 3] - c[27, 0] - c[27, 2] - c[27, 4] - c[27, 5] - c[71, 1]
c[24, 0] + c[24, 2] + c[24, 4] - c[27, 1] - c[27, 3] - c[71, 2]
c[24, 1] + c[24, 3] - c[27, 2] - c[27, 4] - c[71, 3]
c[24, 2] + c[24, 4] + c[24, 5] - c[27, 1] - c[27, 3] - c[71, 4]
c[24, 4] + c[24, 5] - c[27, 1] - c[71, 5]
c[25, 0] + c[25, 2] - c[28, 0]
-2 + 2 c[25, 1] + c[25, 3] - c[28, 1]
c[25, 0] + c[25, 2] + c[25, 4] - c[28, 2]
-2 + c[25, 1] + c[25, 3] - c[28, 3]
c[25, 2] + c[25, 4] + c[25, 5] - c[28, 4]
c[25, 4] + c[25, 5] - c[28, 5]
c[26, 0] + c[26, 2] - c[28, 0] × c[60, 0] - c[28, 1] × c[60, 1] - c[28, 2] × c[60, 2] - c[28, 3] × c
2 c[26, 1] + c[26, 3] - c[28, 1] × c[60, 0] - c[28, 0] × c[60, 1] - c[28, 2] × c[60, 1] - c[28, 4] ×
c[26, 0] + c[26, 2] + c[26, 4] - c[28, 2] × c[60, 0] - c[28, 1] × c[60, 1] - c[28, 3] × c[60, 1] - c
c[26, 1] + c[26, 3] - c[28, 3] × c[60, 0] - c[28, 2] × c[60, 1] - c[28, 4] × c[60, 1] - c[28, 1] × c
c[26, 2] + c[26, 4] + c[26, 5] - c[28, 4] × c[60, 0] - c[28, 1] × c[60, 1] - c[28, 3] × c[60, 1] - c
c[26, 4] + c[26, 5] - c[28, 5] × c[60, 0] - c[28, 1] × c[60, 1] - c[28, 4] × c[60, 2] - c[28, 3] × c
-c[27, 0] - c[27, 2] - c[27, 4] + c[29, 0] + c[29, 2] - c[33, 0]
-3 c[27, 1] - 2 c[27, 3] + 2 c[29, 1] + c[29, 3] - c[33, 1]
-c[27, 0] - 2 c[27, 2] - c[27, 4] - c[27, 5] + c[29, 0] + c[29, 2] + c[29, 4] - c[33, 2]
-2 c[27, 1] - c[27, 3] + c[29, 1] + c[29, 3] - c[33, 3]
-c[27, 0] - c[27, 2] - 2 c[27, 4] - c[27, 5] + c[29, 2] + c[29, 4] + c[29, 5] - c[33, 4]
-c[27, 2] - c[27, 4] - c[27, 5] + c[29, 4] + c[29, 5] - c[33, 5]
-c[27, 0] + c[30, 0] + c[30, 2] - c[65, 0]
-c[27, 1] + 2 c[30, 1] + c[30, 3] - c[65, 1]
-c[27, 2] + c[30, 0] + c[30, 2] + c[30, 4] - c[65, 2]
-c[27, 3] + c[30, 1] + c[30, 3] - c[65, 3]
-c[27, 4] + c[30, 2] + c[30, 4] + c[30, 5] - c[65, 4]
-c[27, 5] + c[30, 4] + c[30, 5] - c[65, 5]
c[32, 0] + c[32, 2] - c[27, 0] × c[37, 0] - c[27, 1] × c[37, 1] - c[27, 2] × c[37, 2] - c[27, 3] × c
2 c[32, 1] + c[32, 3] - c[27, 1] × c[37, 0] - c[27, 0] × c[37, 1] - c[27, 2] × c[37, 1] - c[27, 4] ×
c[32, 0] + c[32, 2] + c[32, 4] - c[27, 2] × c[37, 0] - c[27, 1] × c[37, 1] - c[27, 3] × c[37, 1] - c
c[32, 1] + c[32, 3] - c[27, 3] × c[37, 0] - c[27, 2] × c[37, 1] - c[27, 4] × c[37, 1] - c[27, 1] × c
c[32, 2] + c[32, 4] + c[32, 5] - c[27, 4] × c[37, 0] - c[27, 1] × c[37, 1] - c[27, 3] × c[37, 1] - c
c[32, 4] + c[32, 5] - c[27, 5] × c[37, 0] - c[27, 1] × c[37, 1] - c[27, 4] × c[37, 2] - c[27, 3] × c
```

Solving equations for Verlinde coefficients

set of all equations among coefficients $\{c[j,k]\}$

```
In[ ]:= coeffList = DeleteDuplicates@
  Flatten[Table[(extractCoefficients@exchangeVerlinde[exchangerelations[[index]]],
    {index, 1, Length[exchangerelations]}]];
coeffList = SortBy[coeffList, Max[exponents[#]] &];
coeffList[[1 ;; 50]]

Out[ ]:= {0, -c[15, 0], 1 - c[15, 1], -c[15, 2], -c[15, 3],
  -2 + 2 c[15, 1] + c[15, 3], -2 + 2 c[15, 1] + 2 c[15, 3], -3 + 3 c[15, 1] + 2 c[15, 3],
  -4 + 4 c[15, 1] + 2 c[15, 3], -c[15, 4], c[15, 0] + c[15, 2] + c[15, 4], -c[15, 5],
  c[15, 2] + c[15, 4] + c[15, 5], c[15, 0] + 2 c[15, 2] + c[15, 4] + c[15, 5],
  c[15, 0] + c[15, 2] + 2 c[15, 4] + c[15, 5], 2 c[15, 0] + 2 c[15, 2] + 2 c[15, 4] + 2 c[15, 5],
  2 c[15, 0] + 4 c[15, 2] + 4 c[15, 4] + 2 c[15, 5], -c[16, 0], -c[16, 1], -c[15, 0] + c[16, 1],
  -c[15, 5] + c[16, 1], 1 - c[16, 2], -c[16, 3], -c[15, 2] + c[16, 1] + c[16, 3],
  -c[15, 4] + c[16, 1] + c[16, 3], -c[15, 2] - c[15, 4] + 2 c[16, 1] + c[16, 3],
  -c[15, 0] - c[15, 2] - c[15, 4] - c[15, 5] + 3 c[16, 1] + 2 c[16, 3], 1 - c[16, 4],
  -2 - c[15, 3] + c[16, 2] + c[16, 4], -1 - c[15, 1] + c[16, 0] + c[16, 2] + c[16, 4],
  1 - c[16, 5], -2 - c[15, 1] + c[16, 2] + c[16, 4] + c[16, 5],
  -2 - c[15, 1] + c[16, 0] + c[16, 2] + c[16, 4] + c[16, 5],
  -3 - c[15, 1] - c[15, 3] + c[16, 0] + 2 c[16, 2] + c[16, 4] + c[16, 5],
  -3 - c[15, 1] - c[15, 3] + c[16, 0] + c[16, 2] + 2 c[16, 4] + c[16, 5],
  1 - c[17, 0], -c[17, 1], 1 - c[17, 2], -c[17, 3], 3 c[17, 1] + c[17, 3],
  4 c[17, 1] + 3 c[17, 3], 6 c[17, 1] + 3 c[17, 3], 9 c[17, 1] + 6 c[17, 3],
  -c[17, 4], -3 + c[17, 0] + 2 c[17, 2] + c[17, 4], -c[17, 5],
  -1 + c[17, 2] + 2 c[17, 4] + c[17, 5], -4 + 2 c[17, 0] + 2 c[17, 2] + 2 c[17, 4] + c[17, 5],
  -5 + 2 c[17, 0] + 3 c[17, 2] + 3 c[17, 4] + c[17, 5],
  -3 + c[17, 0] + 2 c[17, 2] + 2 c[17, 4] + 2 c[17, 5]}
```

first set of linear equations

```
In[ ]:= Block[{knownsols, newSols, numEquations, equations, solution, currentIndex, batchSize},
  knownsols = {};
  Print["number of known variables before:", Length@knownsols];
  equations = coeffList;
  equations = equations /. knownsols;
  equations = DeleteDuplicates@ Select[equations, Max[exponents[#]] == 1 &];
  Print["number of equations used:", Length[equations]];
  equations = Map[# == 0 &, equations];
  equations = Join[equations, knownsols /. Rule -> Equal];
  solution = Solve[equations, unknowns];
  If[Length[solution] == 1,
    newSols = Select[solution[[1]], NumericQ[#[[2]]] &];
    knownsols = Union[knownsols, newSols];
    Print["number of known variables after: ", {Length[#], #} &@knownsols];
  ]
]

number of known variables before:0
number of equations used:1276
```

... Solve: Equations may not give solutions for all "solve" variables.

... Solve: Equations may not give solutions for all "solve" variables.

number of known variables after:

```
{288, {c[15, 0] → 0, c[15, 1] → 1, c[15, 2] → 0, c[15, 3] → 0, c[15, 4] → 0, c[15, 5] → 0,
c[16, 0] → 0, c[16, 1] → 0, c[16, 2] → 1, c[16, 3] → 0, c[16, 4] → 1, c[16, 5] → 1, c[17, 0] → 1,
c[17, 1] → 0, c[17, 2] → 1, c[17, 3] → 0, c[17, 4] → 0, c[17, 5] → 0, c[18, 0] → 1, c[18, 1] → 0,
c[18, 2] → 1, c[18, 3] → 0, c[18, 4] → 0, c[18, 5] → 0, c[19, 0] → 1, c[19, 1] → 0, c[19, 2] → 1,
c[19, 3] → 0, c[19, 4] → 1, c[19, 5] → 0, c[20, 0] → 0, c[20, 1] → 1, c[20, 2] → 0, c[20, 3] → 0,
c[20, 4] → 0, c[20, 5] → 0, c[21, 0] → 0, c[21, 1] → 1, c[21, 2] → 0, c[21, 3] → 0, c[21, 4] → 0,
c[21, 5] → 0, c[22, 0] → 0, c[22, 1] → 1, c[22, 2] → 0, c[22, 3] → 0, c[22, 4] → 0, c[22, 5] → 0,
c[23, 0] → 0, c[23, 1] → 2, c[23, 2] → 0, c[23, 3] → 2, c[23, 4] → 0, c[23, 5] → 0, c[24, 0] → 0,
c[24, 1] → 1, c[24, 2] → 0, c[24, 3] → 1, c[24, 4] → 0, c[24, 5] → 0, c[25, 0] → 0, c[25, 1] → 1,
c[25, 2] → 0, c[25, 3] → 1, c[25, 4] → 0, c[25, 5] → 0, c[26, 0] → 1, c[26, 1] → 0, c[26, 2] → 1,
c[26, 3] → 0, c[26, 4] → 1, c[26, 5] → 0, c[27, 0] → 1, c[27, 1] → 0, c[27, 2] → 1, c[27, 3] → 0,
c[27, 4] → 0, c[27, 5] → 0, c[28, 0] → 0, c[28, 1] → 1, c[28, 2] → 0, c[28, 3] → 0, c[28, 4] → 0,
c[28, 5] → 0, c[29, 0] → 1, c[29, 1] → 0, c[29, 2] → 2, c[29, 3] → 0, c[29, 4] → 1, c[29, 5] → 0,
c[30, 0] → 1, c[30, 1] → 0, c[30, 2] → 1, c[30, 3] → 0, c[30, 4] → 0, c[30, 5] → 0, c[33, 0] → 1,
c[33, 1] → 0, c[33, 2] → 1, c[33, 3] → 0, c[33, 4] → 1, c[33, 5] → 0, c[34, 0] → 1, c[34, 1] → 0,
c[34, 2] → 1, c[34, 3] → 0, c[34, 4] → 0, c[34, 5] → 0, c[35, 0] → 0, c[35, 1] → 1, c[35, 2] → 0,
c[35, 3] → 0, c[35, 4] → 0, c[35, 5] → 0, c[36, 0] → 0, c[36, 1] → 1, c[36, 2] → 0, c[36, 3] → 1,
c[36, 4] → 0, c[36, 5] → 0, c[37, 0] → 0, c[37, 1] → 1, c[37, 2] → 0, c[37, 3] → 1, c[37, 4] → 0,
c[37, 5] → 0, c[38, 0] → 1, c[38, 1] → 0, c[38, 2] → 2, c[38, 3] → 0, c[38, 4] → 1, c[38, 5] → 0,
c[39, 0] → 1, c[39, 1] → 0, c[39, 2] → 1, c[39, 3] → 0, c[39, 4] → 0, c[39, 5] → 0, c[40, 0] → 0,
c[40, 1] → 1, c[40, 2] → 0, c[40, 3] → 1, c[40, 4] → 0, c[40, 5] → 0, c[42, 0] → 0, c[42, 1] → 2,
c[42, 2] → 0, c[42, 3] → 2, c[42, 4] → 0, c[42, 5] → 0, c[46, 0] → 0, c[46, 1] → 1, c[46, 2] → 0,
c[46, 3] → 0, c[46, 4] → 0, c[46, 5] → 0, c[47, 0] → 1, c[47, 1] → 0, c[47, 2] → 1, c[47, 3] → 0,
c[47, 4] → 1, c[47, 5] → 0, c[48, 0] → 0, c[48, 1] → 1, c[48, 2] → 0, c[48, 3] → 1, c[48, 4] → 0,
c[48, 5] → 0, c[49, 0] → 1, c[49, 1] → 0, c[49, 2] → 1, c[49, 3] → 0, c[49, 4] → 0, c[49, 5] → 0,
c[50, 0] → 0, c[50, 1] → 2, c[50, 2] → 0, c[50, 3] → 2, c[50, 4] → 0, c[50, 5] → 0, c[51, 0] → 1,
c[51, 1] → 0, c[51, 2] → 2, c[51, 3] → 0, c[51, 4] → 1, c[51, 5] → 0, c[52, 0] → 2, c[52, 1] → 0,
c[52, 2] → 3, c[52, 3] → 0, c[52, 4] → 3, c[52, 5] → 1, c[56, 0] → 1, c[56, 1] → 0, c[56, 2] → 1,
c[56, 3] → 0, c[56, 4] → 1, c[56, 5] → 0, c[58, 0] → 1, c[58, 1] → 0, c[58, 2] → 1, c[58, 3] → 0,
c[58, 4] → 0, c[58, 5] → 0, c[60, 0] → 0, c[60, 1] → 1, c[60, 2] → 0, c[60, 3] → 0, c[60, 4] → 0,
c[60, 5] → 0, c[61, 0] → 1, c[61, 1] → 0, c[61, 2] → 1, c[61, 3] → 0, c[61, 4] → 1, c[61, 5] → 0,
c[62, 0] → 1, c[62, 1] → 0, c[62, 2] → 2, c[62, 3] → 0, c[62, 4] → 1, c[62, 5] → 0, c[64, 0] → 2,
c[64, 1] → 0, c[64, 2] → 3, c[64, 3] → 0, c[64, 4] → 3, c[64, 5] → 1, c[65, 0] → 1, c[65, 1] → 0,
c[65, 2] → 1, c[65, 3] → 0, c[65, 4] → 1, c[65, 5] → 0, c[66, 0] → 0, c[66, 1] → 2, c[66, 2] → 0,
c[66, 3] → 2, c[66, 4] → 0, c[66, 5] → 0, c[67, 0] → 1, c[67, 1] → 0, c[67, 2] → 2, c[67, 3] → 0,
c[67, 4] → 1, c[67, 5] → 0, c[68, 0] → 0, c[68, 1] → 1, c[68, 2] → 0, c[68, 3] → 1, c[68, 4] → 0,
c[68, 5] → 0, c[69, 0] → 1, c[69, 1] → 0, c[69, 2] → 2, c[69, 3] → 0, c[69, 4] → 1, c[69, 5] → 0,
c[70, 0] → 1, c[70, 1] → 0, c[70, 2] → 2, c[70, 3] → 0, c[70, 4] → 1, c[70, 5] → 0,
c[71, 0] → 0, c[71, 1] → 1, c[71, 2] → 0, c[71, 3] → 1, c[71, 4] → 0, c[71, 5] → 0,
c[74, 0] → 0, c[74, 1] → 2, c[74, 2] → 0, c[74, 3] → 2, c[74, 4] → 0, c[74, 5] → 0,
c[75, 0] → 2, c[75, 1] → 0, c[75, 2] → 3, c[75, 3] → 0, c[75, 4] → 3, c[75, 5] → 1,
c[77, 0] → 2, c[77, 1] → 0, c[77, 2] → 3, c[77, 3] → 0, c[77, 4] → 3, c[77, 5] → 1}}
```

second set of linear equations

```
In[ ]:= Block[{knownsols, newSols, numEquations,
equations, solution, currentIndex, batchSize},
knownsols = {c[15, 0] → 0, c[15, 1] → 1, c[15, 2] → 0, c[15, 3] → 0, c[15, 4] → 0,
c[15, 5] → 0, c[16, 0] → 0, c[16, 1] → 0, c[16, 2] → 1, c[16, 3] → 0,
c[16, 4] → 1, c[16, 5] → 1, c[17, 0] → 1, c[17, 1] → 0, c[17, 2] → 1,
c[17, 3] → 0, c[17, 4] → 0, c[17, 5] → 0, c[18, 0] → 1, c[18, 1] → 0,
```



```

c[18, 2] → 1, c[18, 3] → 0, c[18, 4] → 0, c[18, 5] → 0, c[19, 0] → 1,
c[19, 1] → 0, c[19, 2] → 1, c[19, 3] → 0, c[19, 4] → 1, c[19, 5] → 0, c[20, 0] → 0,
c[20, 1] → 1, c[20, 2] → 0, c[20, 3] → 0, c[20, 4] → 0, c[20, 5] → 0, c[21, 0] → 0,
c[21, 1] → 1, c[21, 2] → 0, c[21, 3] → 0, c[21, 4] → 0, c[21, 5] → 0, c[22, 0] → 0,
c[22, 1] → 1, c[22, 2] → 0, c[22, 3] → 0, c[22, 4] → 0, c[22, 5] → 0, c[23, 0] → 0,
c[23, 1] → 2, c[23, 2] → 0, c[23, 3] → 2, c[23, 4] → 0, c[23, 5] → 0, c[24, 0] → 0,
c[24, 1] → 1, c[24, 2] → 0, c[24, 3] → 1, c[24, 4] → 0, c[24, 5] → 0, c[25, 0] → 0,
c[25, 1] → 1, c[25, 2] → 0, c[25, 3] → 1, c[25, 4] → 0, c[25, 5] → 0, c[26, 0] → 1,
c[26, 1] → 0, c[26, 2] → 1, c[26, 3] → 0, c[26, 4] → 1, c[26, 5] → 0, c[27, 0] → 1,
c[27, 1] → 0, c[27, 2] → 1, c[27, 3] → 0, c[27, 4] → 0, c[27, 5] → 0, c[28, 0] → 0,
c[28, 1] → 1, c[28, 2] → 0, c[28, 3] → 0, c[28, 4] → 0, c[28, 5] → 0, c[29, 0] → 1,
c[29, 1] → 0, c[29, 2] → 2, c[29, 3] → 0, c[29, 4] → 1, c[29, 5] → 0, c[30, 0] → 1,
c[30, 1] → 0, c[30, 2] → 1, c[30, 3] → 0, c[30, 4] → 0, c[30, 5] → 0, c[33, 0] → 1,
c[33, 1] → 0, c[33, 2] → 1, c[33, 3] → 0, c[33, 4] → 1, c[33, 5] → 0, c[34, 0] → 1,
c[34, 1] → 0, c[34, 2] → 1, c[34, 3] → 0, c[34, 4] → 0, c[34, 5] → 0, c[35, 0] → 0,
c[35, 1] → 1, c[35, 2] → 0, c[35, 3] → 0, c[35, 4] → 0, c[35, 5] → 0, c[36, 0] → 0,
c[36, 1] → 1, c[36, 2] → 0, c[36, 3] → 1, c[36, 4] → 0, c[36, 5] → 0, c[37, 0] → 0,
c[37, 1] → 1, c[37, 2] → 0, c[37, 3] → 1, c[37, 4] → 0, c[37, 5] → 0, c[38, 0] → 1,
c[38, 1] → 0, c[38, 2] → 2, c[38, 3] → 0, c[38, 4] → 1, c[38, 5] → 0, c[39, 0] → 1,
c[39, 1] → 0, c[39, 2] → 1, c[39, 3] → 0, c[39, 4] → 0, c[39, 5] → 0, c[40, 0] → 0,
c[40, 1] → 1, c[40, 2] → 0, c[40, 3] → 1, c[40, 4] → 0, c[40, 5] → 0, c[42, 0] → 0,
c[42, 1] → 2, c[42, 2] → 0, c[42, 3] → 2, c[42, 4] → 0, c[42, 5] → 0, c[46, 0] → 0,
c[46, 1] → 1, c[46, 2] → 0, c[46, 3] → 0, c[46, 4] → 0, c[46, 5] → 0, c[47, 0] → 1,
c[47, 1] → 0, c[47, 2] → 1, c[47, 3] → 0, c[47, 4] → 1, c[47, 5] → 0, c[48, 0] → 0,
c[48, 1] → 1, c[48, 2] → 0, c[48, 3] → 1, c[48, 4] → 0, c[48, 5] → 0, c[49, 0] → 1,
c[49, 1] → 0, c[49, 2] → 1, c[49, 3] → 0, c[49, 4] → 0, c[49, 5] → 0, c[50, 0] → 0,
c[50, 1] → 2, c[50, 2] → 0, c[50, 3] → 2, c[50, 4] → 0, c[50, 5] → 0, c[51, 0] → 1,
c[51, 1] → 0, c[51, 2] → 2, c[51, 3] → 0, c[51, 4] → 1, c[51, 5] → 0, c[52, 0] → 2,
c[52, 1] → 0, c[52, 2] → 3, c[52, 3] → 0, c[52, 4] → 3, c[52, 5] → 1, c[56, 0] → 1,
c[56, 1] → 0, c[56, 2] → 1, c[56, 3] → 0, c[56, 4] → 1, c[56, 5] → 0, c[58, 0] → 1,
c[58, 1] → 0, c[58, 2] → 1, c[58, 3] → 0, c[58, 4] → 0, c[58, 5] → 0, c[60, 0] → 0,
c[60, 1] → 1, c[60, 2] → 0, c[60, 3] → 0, c[60, 4] → 0, c[60, 5] → 0, c[61, 0] → 1,
c[61, 1] → 0, c[61, 2] → 1, c[61, 3] → 0, c[61, 4] → 1, c[61, 5] → 0, c[62, 0] → 1,
c[62, 1] → 0, c[62, 2] → 2, c[62, 3] → 0, c[62, 4] → 1, c[62, 5] → 0, c[64, 0] → 2,
c[64, 1] → 0, c[64, 2] → 3, c[64, 3] → 0, c[64, 4] → 3, c[64, 5] → 1, c[65, 0] → 1,
c[65, 1] → 0, c[65, 2] → 1, c[65, 3] → 0, c[65, 4] → 1, c[65, 5] → 0, c[66, 0] → 0,
c[66, 1] → 2, c[66, 2] → 0, c[66, 3] → 2, c[66, 4] → 0, c[66, 5] → 0, c[67, 0] → 1,
c[67, 1] → 0, c[67, 2] → 2, c[67, 3] → 0, c[67, 4] → 1, c[67, 5] → 0, c[68, 0] → 0,
c[68, 1] → 1, c[68, 2] → 0, c[68, 3] → 1, c[68, 4] → 0, c[68, 5] → 0, c[69, 0] → 1,
c[69, 1] → 0, c[69, 2] → 2, c[69, 3] → 0, c[69, 4] → 1, c[69, 5] → 0, c[70, 0] → 1,
c[70, 1] → 0, c[70, 2] → 2, c[70, 3] → 0, c[70, 4] → 1, c[70, 5] → 0, c[71, 0] → 0,
c[71, 1] → 1, c[71, 2] → 0, c[71, 3] → 1, c[71, 4] → 0, c[71, 5] → 0, c[74, 0] → 0,
c[74, 1] → 2, c[74, 2] → 0, c[74, 3] → 2, c[74, 4] → 0, c[74, 5] → 0, c[75, 0] → 2,
c[75, 1] → 0, c[75, 2] → 3, c[75, 3] → 0, c[75, 4] → 3, c[75, 5] → 1, c[77, 0] → 2,
c[77, 1] → 0, c[77, 2] → 3, c[77, 3] → 0, c[77, 4] → 3, c[77, 5] → 1};
Print["number of known variables before:", Length@knownsols];
equations = coeffList;
equations = equations /. knownsols;
equations = DeleteDuplicates@ Select[equations, Max[exponents[#]] == 1 &];
Print["number of equations used:", Length[equations]];
equations = Map[# == 0 &, equations];
equations = Join[equations, knownsols /. Rule -> Equal];
solution = Solve[equations, unknowns];
If[Length[solution] == 1,
  newSols = Select[solution[[1]], NumericQ[#[[2]]] &];

```

```
knownsols = Union[knownsols, newSols];  
Print["number of known variables after: ", {Length[#], #} &@knownsols];  
]  
]  
number of known variables before:288  
number of equations used:662
```


[illegible]

[illegible]

final output

```
ln[6]:= {x[extractindex[#]], # /. variableToVerlindeImage /. solution /.  
        ringElement -> Identity} & /@ clustervariables  
% //  
TableForm
```



```

Out[ ]:= { {x[1], V0 + V2}, {x[2], V0}, {x[3], V0 + 2 V2 + V4}, {x[4], V0},
  {x[5], 2 V0 + 3 V2 + 3 V4 + V5}, {x[6], V0}, {x[7], V1 + V3}, {x[8], V5},
  {x[9], 2 V1 + 2 V3}, {x[10], V5}, {x[11], V0 + V2 + V4}, {x[12], V0}, {x[13], V1},
  {x[14], V5}, {x[15], V1}, {x[16], V2 + V4 + V5}, {x[17], V0 + V2}, {x[18], V0 + V2},
  {x[19], V0 + V2 + V4}, {x[20], V1}, {x[21], V1}, {x[22], V1}, {x[23], 2 V1 + 2 V3},
  {x[24], V1 + V3}, {x[25], V1 + V3}, {x[26], V0 + V2 + V4}, {x[27], V0 + V2}, {x[28], V1},
  {x[29], V0 + 2 V2 + V4}, {x[30], V0 + V2}, {x[31], V1}, {x[32], 2 V1 + 2 V3},
  {x[33], V0 + V2 + V4}, {x[34], V0 + V2}, {x[35], V1}, {x[36], V1 + V3},
  {x[37], V1 + V3}, {x[38], V0 + 2 V2 + V4}, {x[39], V0 + V2}, {x[40], V1 + V3},
  {x[41], V1 + V3}, {x[42], 2 V1 + 2 V3}, {x[43], V0 + V2}, {x[44], V0 + 2 V2 + V4},
  {x[45], 2 V0 + 3 V2 + 3 V4 + V5}, {x[46], V1}, {x[47], V0 + V2 + V4}, {x[48], V1 + V3},
  {x[49], V0 + V2}, {x[50], 2 V1 + 2 V3}, {x[51], V0 + 2 V2 + V4}, {x[52], 2 V0 + 3 V2 + 3 V4 + V5},
  {x[53], V0 + V2 + V4}, {x[54], 2 V1 + 2 V3}, {x[55], 2 V0 + 3 V2 + 3 V4 + V5},
  {x[56], V0 + V2 + V4}, {x[57], V0 + 2 V2 + V4}, {x[58], V0 + V2}, {x[59], 2 V1 + 2 V3},
  {x[60], V1}, {x[61], V0 + V2 + V4}, {x[62], V0 + 2 V2 + V4}, {x[63], 2 V0 + 3 V2 + 3 V4 + V5},
  {x[64], 2 V0 + 3 V2 + 3 V4 + V5}, {x[65], V0 + V2 + V4}, {x[66], 2 V1 + 2 V3},
  {x[67], V0 + 2 V2 + V4}, {x[68], V1 + V3}, {x[69], V0 + 2 V2 + V4}, {x[70], V0 + 2 V2 + V4},
  {x[71], V1 + V3}, {x[72], 2 V1 + 2 V3}, {x[73], 2 V0 + 3 V2 + 3 V4 + V5}, {x[74], 2 V1 + 2 V3},
  {x[75], 2 V0 + 3 V2 + 3 V4 + V5}, {x[76], 2 V0 + 3 V2 + 3 V4 + V5}, {x[77], 2 V0 + 3 V2 + 3 V4 + V5} }

```

Out[]//TableForm=

x[1]	V0 + V2
x[2]	V0
x[3]	V0 + 2 V2 + V4
x[4]	V0
x[5]	2 V0 + 3 V2 + 3 V4 + V5
x[6]	V0
x[7]	V1 + V3
x[8]	V5
x[9]	2 V1 + 2 V3
x[10]	V5
x[11]	V0 + V2 + V4
x[12]	V0
x[13]	V1
x[14]	V5
x[15]	V1
x[16]	V2 + V4 + V5
x[17]	V0 + V2
x[18]	V0 + V2
x[19]	V0 + V2 + V4
x[20]	V1
x[21]	V1
x[22]	V1
x[23]	2 V1 + 2 V3
x[24]	V1 + V3
x[25]	V1 + V3
x[26]	V0 + V2 + V4
x[27]	V0 + V2
x[28]	V1
x[29]	V0 + 2 V2 + V4
x[30]	V0 + V2
x[31]	V1
x[32]	2 V1 + 2 V3
x[33]	V0 + V2 + V4
x[34]	V0 + V2
x[35]	V1
x[36]	V1 + V3
x[37]	V1 + V3
x[38]	V0 + 2 V2 + V4
x[39]	V0 + V2
x[40]	V1 + V3
x[41]	V1 + V3

```

x[42] 2 V1 + 2 V3
x[43] V0 + V2
x[44] V0 + 2 V2 + V4
x[45] 2 V0 + 3 V2 + 3 V4 + V5
x[46] V1
x[47] V0 + V2 + V4
x[48] V1 + V3
x[49] V0 + V2
x[50] 2 V1 + 2 V3
x[51] V0 + 2 V2 + V4
x[52] 2 V0 + 3 V2 + 3 V4 + V5
x[53] V0 + V2 + V4
x[54] 2 V1 + 2 V3
x[55] 2 V0 + 3 V2 + 3 V4 + V5
x[56] V0 + V2 + V4
x[57] V0 + 2 V2 + V4
x[58] V0 + V2
x[59] 2 V1 + 2 V3
x[60] V1
x[61] V0 + V2 + V4
x[62] V0 + 2 V2 + V4
x[63] 2 V0 + 3 V2 + 3 V4 + V5
x[64] 2 V0 + 3 V2 + 3 V4 + V5
x[65] V0 + V2 + V4
x[66] 2 V1 + 2 V3
x[67] V0 + 2 V2 + V4
x[68] V1 + V3
x[69] V0 + 2 V2 + V4
x[70] V0 + 2 V2 + V4
x[71] V1 + V3
x[72] 2 V1 + 2 V3
x[73] 2 V0 + 3 V2 + 3 V4 + V5
x[74] 2 V1 + 2 V3
x[75] 2 V0 + 3 V2 + 3 V4 + V5
x[76] 2 V0 + 3 V2 + 3 V4 + V5
x[77] 2 V0 + 3 V2 + 3 V4 + V5

```