When Smoothness is Not Enough: Toward Exact Quantification and Optimization of the Price of Anarchy

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The design of incentives that promote efficient user behaviours in competitive settings hinges on our ability to accurately evaluate the performance of emergent system outcomes. In these settings, the most popular performance metric is the price of anarchy, which is the ratio between the worst emergent behaviour and the system optimum. Although the study of the price of anarchy is widespread, obtaining exact bounds even for a specified class of games remains a challenge. The widely studied smoothness framework (Roughgarden 2015) is capable of providing universal price of anarchy bounds for a large class of games, but fails to provide exact bounds when the system-level objective is not necessarily aligned with social welfare.

Given this limitation, we introduce a generalization of the smoothness framework which alleviates this issue. Based on this framework, we provide a tractable mechanism for computing the exact price of anarchy for a class of games wherein the sum over agents' local costs is not necessarily equal to the social cost (e.g., congestion games with incentives). Furthermore, we demonstrate how this mechanism extends naturally to a methodology for deriving incentive structures that optimize the price of anarchy. We conclude the paper by applying our techniques to the problem of rebate design in atomic congestion games. Interestingly, our findings suggest that there is a significant gap between the performance guarantees and efficient incentive structures in discrete environments when compared to their continuous flow counterparts.*

Key words: game theory, multiagent systems, smoothness, price of anarchy

1. Introduction

The operation of next-generation shared infrastructure will be distributed either by nature or by design. As smartphones have become ubiquitous, and applications like Waze, Uber and Nest have entered the mainstream, individual end users are now able to access detailed system-level information when making their daily decisions. This has led to a momentous shift in the operation of shared technological infrastructure like road-traffic networks, cloud computing and the power grid, where the local behaviours and interactions of the individual decision makers are increasingly

* For the interested reader, the authors provide a software package, available in both MATLAB® and Python, that implements the techniques described in this manuscript at https://github.com/rahul-chandan/resalloc-poa.

dictating the system-wide performance. As a particular example, consider the setting of drivers on a road-traffic network. Using detailed traffic network information provided free-of-charge by navigation apps on their smartphones, modern drivers can make route selections that strike the perfect balance between travel time, tolls, etc. Counterintuitively, many studies suggest that the proliferation of navigation applications has actually contributed to *increased* congestion and *unmanageable* traffic patterns (Cabannes et al. 2019, Macfarlane 2019). In order to quantify and optimize the system-wide performance of shared technological infrastructure, societal planners require improved methodologies to model and characterize the behaviour of local decision makers.

In any of the aforementioned settings, the best achievable performance would be attained if a central coordinator could dictate the choices of the individual decision making entities across the system. However, in the systems we discussed above, the decision making is intrinsically distributed, resulting in an inevitable loss in performance. Given a behavioural model of a system's decision makers, game theory provides various metrics for measuring the loss in system performance caused by the distribution of decision making. The most popular of these metrics is termed the *price of anarchy* (Koutsoupias and Papadimitriou 1999), which is defined as the worst-case ratio of a system's performance under distributed decision making and under central coordination.

The price of anarchy has been used to quantify the efficiency of distributed decision making in various settings, including congestion of road-traffic networks (Roughgarden and Tardos 2002), signal integrity in wireless networks (Zhou et al. 2018), and distributed resource allocation (Gkatzelis et al. 2016), while the design of incentives that optimize this metric has been studied more recently, c.f., Cominetti et al. (2009), Gairing (2009), Roughgarden (2015), Paccagnan et al. (2019a). Although real-world problem settings often consist of finite, indivisible (atomic) decision makers, the majority of existing works study approximate settings in which decision makers are modelled as divisible (nonatomic) entities (Roughgarden and Tardos 2002, Correa et al. 2004, Maillé and Stier-Moses 2009, Colini-Baldeschi et al. 2020). Among the works that *do* consider atomic decision makers, the majority provide *ad hoc* price of anarchy guarantees that apply only within a restricted class of problems, and the prevailing framework for obtaining universal price of anarchy bounds – termed *smoothness* (Roughgarden 2015) – is generally imprecise and still applies only within a restricted class of problems. An overarching framework that provides exact price of anarchy bounds has not yet been put forward for an important class of problems such as that introduced in the next section.

1.1. A motivating example

This section focuses on the problem of incentive rebate design in congestion games, c.f., Maillé and Stier-Moses (2009). Atomic congestion games (Rosenthal 1973) have been widely studied in the operations research literature as they capture the adverse effects of local decision making on system-wide performance. A representative set of examples includes inefficient routing in traffic and communication networks (Suri et al. 2007, Scarsini et al. 2018), distributed resource allocation (Harks and Miller 2011), and credit assignment in teams (Kleinberg and Oren 2011).

An atomic congestion game consists of a set of agents $N = \{1, 2, \ldots, n\}$ sharing the use of a common set of resources $\mathcal{R}$, where each resource $r \in \mathcal{R}$ is associated with a congestion function $c_r : \{1, \ldots, n\} \to \mathbb{R}$. The term $c_r(k)$ identifies the cost a agent experiences for selecting resource r given that there are $1 \leq k \leq n$ agents using resource r. Further, each agent $i \in N$ is associated with a given action set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ that meets its individual needs, and the collective action set is denoted by $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Given an admissible allocation of agents to resources $a = (a_1, a_2, \ldots, a_n) \in \mathcal{A}$, the system cost is measured by

$$C(a) = \sum_{r \in \mathcal{R}} |a|_r \cdot c_r(|a|_r), \tag{1}$$

where $|a|_r = |\{i \in N : r \in a_i\}|$ denotes the number of agents selecting resource r in the allocation a. The goal of the societal planner is to achieve an optimal allocation of agents to resources of the form $a^{\text{opt}} \in \arg\min_{a \in \mathcal{A}} C(a)$.

One of the fundamental challenges associated with the allocation of resources in such systems is the self-interested behaviour of the agents. That is, each agent $i \in N$ independently selects an action $a_i \in \mathcal{A}_i$ in response to a agent cost function of the form $J_i : \mathcal{A} \to \mathbb{R}$. One commonly studied class of agent cost functions is of the form

$$J_i(a) = \sum_{r \in a_i} c_r(|a|_r), \tag{2}$$

where an agent's cost is the cumulative cost accrued over the resources it uses. Regardless of the specific model of the agent cost functions $\{J_1, \ldots, J_n\}$, the emergent collective behaviour is typically assumed to be characterized by a (pure) Nash equilibrium of the game derived from the agent set N, collective action set $\mathcal{A}$, and agents' cost functions $\{J_1, \ldots, J_n\}$, i.e., an allocation $a^{\text{ne}} \in \mathcal{A}$ such that, for each agent $i \in N$, the action a_i^{ne} minimizes the corresponding agent cost function J_i when the other agents' actions are fixed. Accordingly, there has been significant research seeking to address the efficiency (or inefficiency) of Nash equilibria relative to the optimal allocation. Many of the past works in this area focus on the *price of anarchy* (Koutsoupias and Papadimitriou 1999), which is the worst case ratio $C(a^{\text{ne}})/C(a^{\text{opt}})$ across a well-defined family of games.

Given the inefficiency of Nash equilibria in many systems of interest, researchers have shifted their attention to the design of admissible incentive mechanisms for improving the efficiency of Nash equilibria. Specifically, we define an incentive mechanism for each resource $r \in \mathcal{R}$ through an incentive function $\tau_r : \{1, \ldots, n\} \to \mathbb{R}$, where $\tau_r(k)$ denotes the incentive imposed on each agent using resource r when there are $k \geq 1$ users. Given these incentives, the cost function associated with each agent $i \in N$ for any $a \in \mathcal{A}$ takes on the form

$$J_i(a) = \sum_{r \in a_i} c_r(|a|_r) + \tau_r(|a|_r), \tag{3}$$

which thereby influences the resulting Nash equilibria. It should be stressed that while the incentive mechanisms influence the agents' cost function, they do not alter the assessment of the system cost, which is still of the form $C(a) = \sum_{r \in \mathcal{R}} C_r(|a|_r)$. The central questions in the study of incentive mechanism design are the following:

– *Question #1* : Given incentive mechanisms $\{\tau_r\}_{r \in \mathcal{R}}$, what is the resulting price of anarchy?

– *Question #2* : What are the incentive mechanisms $\{\tau_r\}_{r \in \mathcal{R}}$ that optimize the price of anarchy, i.e., minimize the worst possible ratio $C(a^{\mathrm{ne}})/C(a^{\mathrm{opt}})$?

In Section 2.1, we show that the well-studied smoothness framework Roughgarden (2015) is unsuitable for the design of incentives, as it provides strictly inexact bounds on the price of anarchy when incentives are introduced to an atomic congestion game. More significantly, the price of anarchy bounds obtained using the smoothness framework do not apply if there exists $a \in \mathcal{A}$ such that $\sum_{i \in N} J_i(a) \leq C(a)$, which generally holds true in atomic congestion games with rebates. Our main contribution in this work is a generalized smoothness framework that provides exact price of anarchy bounds within the class of atomic congestion games with incentives and/or rebates. In Section 5, we apply our framework to the design of rebates, i.e., mechanisms $\{\tau_r\}_{r \in \mathcal{R}}$ operating under the constraint $\tau_r(k) \leq 0$ for all k and r, that minimize the price of anarchy. The work in Maillé and Stier-Moses (2009) focuses on this task in the nonatomic setting and presents several different positive results characterizing rebate mechanisms that ensure $C(a^{\mathrm{ne}}) = C(a^{\mathrm{opt}})$, i.e., the price of anarchy is 1. However, we demonstrate that these incentive do not provide similar guarantees in their atomic counterparts. More surprisingly, these optimal incentive mechanisms result in a higher price of anarchy than if no incentives were employed at all. Finally, leveraging the methodology put forward in Paccagnan et al. (2019a), we show how our novel framework can be used to compute optimal dynamic and fixed rebates in the atomic setting.

1.2. Our contributions

As we discussed in Section 1.1, the design of incentives that optimize the price of anarchy depends on our ability to accurately characterize the price of anarchy for a given set of incentives. This paper establishes a novel framework for obtaining exact, tractable price of anarchy bounds as follows:

Generalized Smoothness: It is important to highlight that current approaches for characterizing the price of anarchy are insufficient for computing the price of anarchy of a system comprised of a given set of congestion and incentive functions $\{c_r, \tau_r\}_{r\in\mathcal{R}}$. For example, the well-studied (λ,μ)-smoothness framework provided in Roughgarden (2015) suffers from the following drawbacks: (i) The analysis only yields a tight price of anarchy for the case when $\tau_r = 0$ for all r; (ii) The analysis does not cover the case when $\tau_r(k) \le 0$; and (iii) There are no computationally efficient methods for deriving the smoothness coefficients (λ,μ) that characterize the price of anarchy. These issues are discussed more formally in Section 2.1. Accordingly, in Section 2.3, we introduce a modified notion of smoothness, termed *generalized smoothness*, that alleviates the concerns highlighted above. Furthermore, we show that the generalized smoothness framework always provides better bounds on the price of anarchy and applies to a broader class of games than the original smoothness framework. Finally, we observe that bounds obtained using the generalized smoothness framework apply to broader notions of equilibrium, including *mixed Nash equilibrium* and *coarse-correlated equilibrium*. In fact, Theorem 2 proves that the generalized smoothness framework provides exact bounds on the *average coarse-correlated equilibrium* efficiency in any given game.

Characterizing the Price of Anarchy: Our second set of results focus on developing a tractable technique for evaluating the price of anarchy in generalized congestion games, a class of games that includes the atomic congestion games with incentives discussed in Section 1.1. In Theorem 3, we demonstrate that generalized smoothness actually yields a tight price of anarchy for *any* class of generalized congestion games, and provide a tractable and scalable linear program for computing the exact price of anarchy associated with such systems; addressing all of the issues discussed above. Further, in Corollary 1, we show that linear program for computing the price of anarchy can be modified to compute the incentives that minimize the price of anarchy in generalized congestion games.

Illustration of the Generalized Smoothness Framework: Lastly, in Section 5, we show that the results of the previous sections answer the questions posed in Section 1.1. In this respect, we first evaluate the performance of optimal rebates designed for the nonatomic setting in Maillé and Stier-Moses (2009). We show that atomic congestion games using these rebates have worse price of anarchy guarantees than the same games without incentive structures. Then, building upon recent results from Paccagnan et al. (2019a), we highlight that the problem of optimal incentive design in atomic congestion games that we introduced in Section 1.1 is equivalent to solving the linear program in Corollary 1.

1.3. Related literature

Although the terminology for price of anarchy and atomic congestion games was formalized in Koutsoupias and Papadimitriou (1999) and Rosenthal (1973), respectively, the design of incentive mechanisms that induce improved system-wide performance guarantees dates as far back as Pigou

(1920). A great deal of literature exists on the topic of incentive design in resource allocation settings, but the majority of these works focus on the nonatomic scenario we described above. Notable examples in this body of literature include Pigou (1920), Fleischer et al. (2004), Maillé and Stier-Moses (2009). The literature on price of anarchy guarantees in atomic settings is much more limited (c.f., Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005), Suri et al. (2007), Aland et al. (2011), Roughgarden (2015)). and the number of works that address the problem of incentive design in such settings is fewer still (c.f., Caragiannis et al. (2010), Gairing (2009), Harks and Miller (2011), Bilò and Vinci (2016), Paccagnan et al. (2019a)). This gap in results is likely due to the increased complexity in quantifying equilibrium performance when the uniqueness of equilibrium cannot be guaranteed, as is the case in these atomic settings. In this manuscript, we consider resource allocation problems within the atomic setting, and explicitly address the multiplicity of Nash equilibria.

The breakthrough in the analysis of the price of anarchy in atomic settings came with the introduction of smoothness-style arguments by Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005), in their independent studies of atomic congestion games with affine latency functions. The smoothness framework was then formalized in Roughgarden (2015). This approach has not only proven to be useful for characterizing the efficiency of broad classes of equilibria (Nisan et al. 2007, Caragiannis et al. 2015), it has also been applied more broadly in problems including learning (Foster et al. 2016) and mechanism design (Syrgkanis and Tardos 2013). The smoothness framework provides several advantages when deriving bounds on the price of anarchy: it does not require the explicit computation of equilibrium and optimal allocations of a game; it has been shown to be tight for some well studied families of games (e.g., congestion games); and, it consists of a standard set of inequalities that govern the price of anarchy bound. This greatly increases the analytical tractability of price of anarchy bounds. However, as we mentioned in Section 1.2, the original smoothness argument is limited in its applicability to the problem of incentive mechanism design in resource allocation problems because it does not provide exact bounds on the price of anarchy. The novel notion of *generalized smoothness* that we propose in this manuscript is tailored to resolve this weakness, while retaining all the strengths of the original smoothness argument.

The notion of generalized smoothness presented in this work is most similar to the style of argument used in Gairing (2009), Ramaswamy et al. (2017) to quantify the price of anarchy of covering problems. This work also builds upon the results in Paccagnan et al. (2019b), Chandan et al. (2019), where the authors develop a linear programming framework for characterizing and optimizing the efficiency of pure Nash equilibria in a well-studied class of resource allocation games. The notion of generalized smoothness, introduced here for the first time, permits a non-trivial extension of their framework: we are now able to construct linear programs for computing and optimizing the (average) coarse-correlated equilibrium efficiency, relative to a broader class of problems that includes the well-studied classes of congestion games. For an in-depth study on optimal incentive design within the class of atomic congestion games, we refer the interested reader to Paccagnan et al. (2019a).

While linear programming approaches for analyzing the price of anarchy have appeared in Nadav and Roughgarden (2010), Bilo (2012), the techniques introduced in both these works cannot be used to compute the price of anarchy of a class of games in a tractable fashion. This is because the complexity of the linear programs they propose grows exponentially in the number of agents and the size of the agents' action sets. In comparison, the linear programs derived in this work *are* tractable, and *can* be used to compute concrete bounds on the price of anarchy. Our gains in tractability come from formulating an upper bound on the price of anarchy, which we term *generalized price of anarchy*, that is exact for the class of generalized congestion games. The complexity of computing the generalized price of anarchy for a family of generalized congestion games only grows quadratically in the number of agents n, as it is a linear program with $\mathcal{O}(n^2)$ constraints and two decision variables.

1.4. Outline

Section 2 defines the class of games and the performance metrics that we consider throughout this paper, reviews the original notion of smoothness from Roughgarden (2015), defines the novel generalized smoothness argument, and presents our results demonstrating that generalized smoothness resolves the issues of the original smoothness argument. Section 3 refines our study to the class of generalized congestion games, and presents our results relating to the tightness of generalized smoothness, the computation of the price of anarchy, and the derivation of optimal incentives under this specialized game model. Section 4 presents analogous results for the welfare maximization problem setting without proof. Section 5 applies our theoretical results to the problem of optimal rebate design under the framework of atomic congestion games, and compares our findings with those of the nonatomic study in Maillé and Stier-Moses (2009). Section 6 includes our conclusions and a brief discussion on future work. All proofs omitted from the main body of the text are presented in the appendices, unless otherwise specified.

2. Generalized Smoothness in Cost Minimization Games

This section introduces the class of games and performance metrics used throughout this paper. We proceed to review the framework of (λ, μ)-smoothness from Roughgarden (2015), and highlight its inherent limitations. We then introduce a revised framework, termed generalized (λ, μ)-smoothness, that alleviates these limitations and offers a strict improvement over the efficiency guarantees provided by the original smoothness framework.

2.1. Cost minimization games

We consider the class of cost minimization problems in which there is a set of agents $N = \{1, \dots, n\}$, and where each agent $i \in N$ is associated with a given action set $\mathcal{A}_i$ and a cost function $J_i : \mathcal{A} \to \mathbb{R}$. The system cost induced by an allocation $a = (a_1, \dots, a_n) \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is measured by the function $C : \mathcal{A} \to \mathbb{R}_{>0}$, and the optimal allocation is of the form

$$a^{\text{opt}} \in \arg\min_{a \in \mathcal{A}} C(a). \tag{4}$$

We represent a cost minimization game as defined above as a tuple $G = (N, \mathcal{A}, C, \mathcal{J})$, where $\mathcal{J} = \{J_1, \dots, J_n\}$. Note that the example highlighted in Section 1.1 represents a special class of cost minimization games, where the agents' cost functions and the system cost are separable over a given set of shared resources.

The main focus of this work is on characterizing the degradation in system-wide performance resulting from local decision-making. To that end, we focus on the solution concept of (pure) Nash equilibrium as a model of the emergent behavior in such systems. A Nash equilibrium is defined as any allocation $a^{\text{ne}} \in \mathcal{A}$ such that

$$J_i(a^{\text{ne}}) \le J_i(a_i, a_{-i}^{\text{ne}}) \quad \forall a_i \in \mathcal{A}_i, \forall i \in N. \tag{5}$$

For a given game G, let $\text{NE}(G)$ denote the set of all allocations $a \in \mathcal{A}$ that satisfy (5). Under the assumption that the set $\text{NE}(G)$ is non-empty, we define the *price of anarchy* of the game G as

$$\text{PoA}(G) := \frac{\max_{a \in \text{NE}(G)} C(a)}{\min_{a \in \mathcal{A}} C(a)} \ge 1. \tag{6}$$

The price of anarchy represents the ratio between the costs of the worst-performing pure Nash equilibrium in the game G, and the optimal allocation. For a given class of cost minimization games $\mathcal{G}$, which may contain infinitely many game instances, we further define the price of anarchy as,

$$\text{PoA}(\mathcal{G}) := \sup_{G \in \mathcal{G}} \text{PoA}(G) \ge 1. \tag{7}$$

Note that a lower price of anarchy corresponds to an improvement in worst-case equilibrium performance, and that $\text{PoA}(\mathcal{G}) = 1$ implies that all pure Nash equilibria of all games $G \in \mathcal{G}$ are optimal.

2.2. Smoothness in cost minimization games

The framework of (λ, μ)-smoothness, introduced in Roughgarden (2015), is widely used in the existing literature aimed at characterizing the price of anarchy (7) over various classes of games. A cost minimization game G is termed (λ, μ)-smooth if the following two conditions are met:

(i) For all $a \in \mathcal{A}$ we have $\sum_{i=1}^{n} J_i(a) \ge C(a)$;

(ii) For all $a, a' \in \mathcal{A}$ we have

$$\sum_{i \in N} J_i(a_i', a_{-i}) \leq \lambda C(a') + \mu C(a). \tag{8}$$

If a game G is (λ, μ)-smooth, then the price of anarchy of game G is upper-bounded by

$$\text{PoA}(G) \leq \frac{\lambda}{1-\mu}.$$

Observe that if all the games in a class $\mathcal{G}$ are shown to be (λ, μ)-smooth, then the price of anarchy of the class $\text{PoA}(\mathcal{G})$ is also upper-bounded by $\lambda/(1-\mu)$. We refer to the best upper-bound obtainable using a smoothness argument on a given class of games $\mathcal{G}$ as the *robust price of anarchy*, i.e.,

$$\text{RPoA}(\mathcal{G}) := \inf_{\lambda > 0, \mu < 1} \left\{ \frac{\lambda}{1-\mu} \text{ s.t. (8) holds } \forall G \in \mathcal{G} \right\}. \tag{9}$$

Note that, the robust price of anarchy represents only an upper-bound on the price of anarchy, i.e., for any class of (λ, μ)-smooth games $\mathcal{G}$, $\text{PoA}(\mathcal{G}) \leq \text{RPoA}(\mathcal{G})$, and it could be that $\text{PoA}(\mathcal{G}) < \text{RPoA}(\mathcal{G})$.

2.3. Generalized smoothness in cost minimization games

In this section we provide a generalization of the smoothness framework, termed *generalized smoothness*. We will then proceed to show how this new framework provide tighter efficiency bounds and covers a broader spectrum of problem settings than the original smoothness framework defined in the previous section.

Definition 1 (Generalized smoothness). The cost minimization game G is (λ, μ)-generalized smooth if, for any two allocations $a, a' \in \mathcal{A}$, there exist $\lambda > 0$ and $\mu < 1$ satisfying,

$$\sum_{i=1}^{n} J_i(a_i', a_{-i}) - \sum_{i=1}^{n} J_i(a) + C(a) \leq \lambda C(a') + \mu C(a). \tag{10}$$

Note that we maintain the notation of (λ, μ) as in the original notion of smoothness for ease of comparison. In the specific case when $\sum_{i=1}^{n} J_i(a) = C(a)$ for all $a \in \mathcal{A}$, the condition (10) is equivalent to the smoothness condition in (8). As with (9), we define the *generalized price of anarchy* of a class of cost minimization games $\mathcal{G}$ as the best upper-bound obtainable using a generalized smoothness argument, i.e.,

$$\text{GPoA}(\mathcal{G}) := \inf_{\lambda > 0, \mu < 1} \left\{ \frac{\lambda}{1-\mu} \text{ s.t. (10) holds } \forall G \in \mathcal{G} \right\}. \tag{11}$$

Our first result follows immediately from Roughgarden (2015) in that a (λ, μ)-generalized smooth game inherits a price of anarchy upper-bound of $\lambda/(1-\mu)$. However, note that we no longer have the restriction that $\sum_{i=1}^{n} J_i(a) \geq C(a)$ for all $a \in \mathcal{A}$.

THEOREM 1. *The price of anarchy of a (λ, μ)-generalized smooth game G is upper-bounded as,*

$$\mathrm{PoA}(G) \le \frac{\lambda}{1-\mu}.$$

Proof. For all $a^{\mathrm{ne}} \in \mathrm{NE}(G)$ and $a^{\mathrm{opt}} \in \mathcal{A}$,

$$C(a^{\mathrm{ne}}) \le \sum_{i=1}^{n} J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) - \sum_{i=1}^{n} J_i(a^{\mathrm{ne}}) + C(a^{\mathrm{ne}}) \le \lambda C(a^{\mathrm{opt}}) + \mu C(a^{\mathrm{ne}}). \tag{12}$$

The first inequality holds by (5), and the second, by (10). Rearranging gives the result. □

This first result demonstrates that the generalized price of anarchy provides similar price of anarchy guarantees for a broader set of scenarios than the original notion of smoothness. The following proposition demonstrates that generalized smoothness outperforms the original smoothness framework in bounding the price of anarchy for scenarios where both are defined.

PROPOSITION 1. *Consider any cost minimization game G such that $\sum_{i=1}^{n} J_i(a) \ge C(a)$ for all $a \in \mathcal{A}$. The generalized price of anarchy is always a better upper-bound on the price of anarchy than the robust price of anarchy, i.e., for any (λ, μ)-smooth game G,*

$$\mathrm{RPoA}(G) \ge \mathrm{GPoA}(G) \ge \mathrm{PoA}(G).$$

Furthermore, if $\sum_{i=1}^{n} J_i(a) > C(a)$ for all $a \in \mathcal{A}$, then $\mathrm{RPoA}(G) > \mathrm{GPoA}(G)$.

Proof. Since the condition $\sum_{i=1}^{n} J_i(a) \ge C(a)$ for all $a \in \mathcal{A}$ implies that any pair of (λ, μ) satisfying (8) necessarily satisfies (10), we note that the generalized price of anarchy is at least as strict an upper-bound as the robust price of anarchy, in general, i.e. $\mathrm{RPoA}(G) \ge \mathrm{GPoA}(G) \ge \mathrm{PoA}(G)$.

Note that for any game $G = (N, \mathcal{A}, \mathcal{C}, \mathcal{J})$ with $\sum_{i=1}^{n} J_i(a) > C(a)$ for all $a \in \mathcal{A}$ there must exist a uniform scaling factor $0 < \gamma < 1$ such that $\sum_{i=1}^{n} \gamma J_i(a) \ge C(a)$, but for which the price of anarchy remains the same, i.e., for $G' = (N, \mathcal{A}, \mathcal{C}, \mathcal{J}')$ where $\mathcal{J}' = \{\gamma J_1, \ldots, \gamma J_n\}$, it holds that $\mathrm{PoA}(G') = \mathrm{PoA}(G)$. The price of anarchy remains the same despite the rescaling, because the inequalities in (5) are unaffected by a positive scaling factor (i.e., $\mathrm{NE}(G) = \mathrm{NE}(G')$), and because the optimal cost remains unchanged since the scaling does not impact the system cost. Further, one can verify from (8) that $\mathrm{RPoA}(G) > \mathrm{RPoA}(G')$, and thus $\mathrm{RPoA}(G) > \mathrm{RPoA}(G') \ge \mathrm{PoA}(G') = \mathrm{PoA}(G)$. Finally, we know that $\mathrm{GPoA}(G')$ is less than or equal to $\mathrm{RPoA}(G')$, and can verify from (10) that $\mathrm{GPoA}(G) = \mathrm{GPoA}(G')$. Thus,

$$\mathrm{RPoA}(G) > \mathrm{RPoA}(G') \ge \mathrm{GPoA}(G') = \mathrm{GPoA}(G) \ge \mathrm{PoA}(G). \quad \square$$

Generalized smoothness also provides additional benefits when compared to the original notion of smoothness as summarized by the following observations. These observations are stated without proof for brevity, but can easily be verified by the reader.

– *Observation #1*: The price of anarchy and generalized price of anarchy are shift-, and scale-invariant, i.e., for any given $\gamma > 0$ and $(\delta_1, \dots, \delta_n) \in \mathbb{R}^n$,

$$\mathrm{PoA}((N, \mathcal{A}, C, \{J_i\}_{i=1}^n)) = \mathrm{PoA}((N, \mathcal{A}, C, \{\gamma J_i + \delta_i\}_{i=1}^n)),$$
$$\mathrm{GPoA}((N, \mathcal{A}, C, \{J_i\}_{i=1}^n)) = \mathrm{GPoA}((N, \mathcal{A}, C, \{\gamma J_i + \delta_i\}_{i=1}^n)).$$

Neither of these properties hold for the robust price of anarchy.

– *Observation #2*: The robust price of anarchy is optimized by budget-balanced agent cost functions, i.e., $\sum_{i \in N} J_i(a) = C(a)$ for all $a \in \mathcal{A}$. In general, this does not hold for the price of anarchy and generalized price of anarchy.

2.4. Average coarse-correlated equilibria

Although conceptually simple, Nash equilibria can be intractable to find, or even nonexistent, in general. Fortunately, all performance bounds obtained in this work extend automatically to the more general class of *average coarse-correlated equilibria*, defined below.

Definition 2. For a given cost minimization game G, we define an average coarse-correlated equilibrium as a probability distribution over the set of actions $\sigma \in \Delta(\mathcal{A})$ satisfying, for all $a' \in \mathcal{A}$,

$$\mathbb{E}_{a \sim \sigma}\left[\sum_{i=1}^{N} J_i(a)\right] := \sum_{a \in \mathcal{A}}\left[\sigma_a \sum_{i=1}^{N} J_i(a)\right] \leq \sum_{a \in \mathcal{A}}\left[\sigma_a \sum_{i=1}^{N} J_i(a_i, a'_{-i})\right], \tag{13}$$

where $\sigma_a \in [0,1]$ is the probability associated with action $a \in \mathcal{A}$ in the distribution σ.

Observe that this definition of average coarse-correlated equilibrium is a generalization of its first definition in Nadav and Roughgarden 2010 for the case where $\sum_{i=1}^n J_i(a) \neq C(a)$. For games with $\sum_{i=1}^n J_i(a) = C(a)$ for all $a \in \mathcal{A}$, the two definitions are equivalent. Note that the set of average coarse-correlated equilibria of a game contains all of the game's pure Nash equilibria, mixed Nash equilibria, correlated equilibria and coarse-correlated equilibria (Roughgarden 2015). Here we show that the generalized price of anarchy tightly characterizes the average coarse-correlated equilibrium performance of any cost minimization game G, and, thus, of any class of cost minimization games $\mathcal{G}$. This result also implies that, for any (λ, μ)-generalized smooth game, $\lambda/(1-\mu)$ is an upper bound on the average coarse-correlated equilibrium performance.

Theorem 2. *Let G denote a cost-minimization game. The following holds:*

$$\frac{\max_{\sigma \in \mathrm{ACCE}(G)} \mathbb{E}_{a \sim \sigma}[C(a)]}{\min_{a \in \mathcal{A}} C(a)} = \mathrm{GPoA}(G).$$

where $\mathrm{ACCE}(G)$ *is the set of all average coarse-correlated equilibria of the game G.*

Proof. Without loss of generality, we assume that the optimal allocation $a^{\text{opt}} \in \mathcal{A}$ of the game G is known a priori. Observe that any pair of parameters $\lambda > 0$, $\mu < 1$ that solve (11) for $\mathcal{G} = \{G\}$ is also the solution to the following program:

$$\min_{\lambda>0,\mu<1} \quad \frac{\lambda}{1-\mu} \quad \text{s.t.} \quad \sum_{i=1}^{n} J_i(a_i^{\text{opt}}, a_{-i}) - \sum_{i=1}^{n} J_i(a) + C(a) \le \lambda C(a^{\text{opt}}) + \mu C(a) \quad \forall a \in \mathcal{A}.$$

Setting $u = \lambda/(1-\mu)$ and $v = 1/(1-\mu)$, this simplifies to the following linear program:

$$\min_{u>0,v>0} \quad u \quad \text{s.t.} \quad C(a) \ge uC(a^{\text{opt}}) - v\sum_{i=1}^{n}\left[J_i(a_i^{\text{opt}}, a_{-i}) - J_i(a)\right] \quad \forall a \in \mathcal{A}.$$

Taking the dual of the above linear program, and rescaling such that $C(a^{\text{opt}}) = 1$, we obtain the following equivalent linear program:

$$\max_{\sigma\in\Delta(\mathcal{A})} \quad \sum_{a\in\mathcal{A}} \sigma_a \frac{C(a)}{C(a^{\text{opt}})} \quad \text{s.t.} \quad \sum_{a\in\mathcal{A}} \sigma_a \sum_{i=1}^{n}\left[J_i(a_i^{\text{opt}}, a_{-i}) - J_i(a)\right] \ge 0, \quad \sum_{a\in\mathcal{A}} \sigma_a = 1, \quad \sigma_a \ge 0 \quad \forall a \in \mathcal{A},$$

where the rescaling is valid because $C(a^{\text{opt}}) > 0$ for the definition of price of anarchy. Observe that this last linear program computes the worst-case average coarse-correlated equilibrium efficiency subject to constraints enforcing the average coarse-correlated equilibrium condition (13). By strong duality of linear programming, we have shown that computing the worst-case average coarse-correlated equilibrium efficiency of the game G is equivalent to solving for GPoA(G). $\square$

3. Generalized Smoothness in Generalized Congestion Games

The previous section introduced the framework of generalized smoothness and showed that the resulting generalized price of anarchy provides improved bounds on the price of anarchy when compared with the robust price of anarchy put forward in Roughgarden (2015). The generalized price of anarchy for a given cost minimization game is defined as the solution of (11). In general, computing the optimal λ and μ governing (11) is a difficult problem regardless of whether we consider the original smoothness or generalized smoothness condition. It is important to highlight that there are as of yet no tractable techniques for computing these optimal smoothness coefficients even in the original smoothness framework, and that all existing works using smoothness arguments, c.f., Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005), Gairing (2009), Aland et al. (2011), derive the smoothness coefficients *ad hoc* for their particular class of problem instances. In this section, we show that the optimal parameters λ and μ for a generalization of the well-studied class of congestion games can be computed as solutions of a tractable linear program. Furthermore, we demonstrate that the generalized price of anarchy tightly characterizes the price of anarchy for this important class of games.

3.1. Generalized congestion games

In this section, we consider a generalization of the congestion game framework that consists of a set of agents $N=\{1,\dots,n\}$ and a finite set of resources $\mathcal{R}$. For a given allocation $a\in\mathcal{A}$, the system cost and agent cost functions have the following separable structure:

$$J_i(a_i,a_{-i})=\sum_{r\in a_i}F_r(|a|_r), \tag{14}$$

$$C(a)=\sum_{r\in\mathcal{R}}C_r(|a|_r), \tag{15}$$

where $C_r:\{0,1,\dots,n\}\to\mathbb{R}_{\geq 0}$ and $F_r:\{1,\dots,n\}\to\mathbb{R}$ define the system cost and agent cost generating functions, respectively. This generalized congestion game framework covers many of the existing models studied in the game theoretic literature, including congestion games (Rosenthal 1973).

Example 1 (Congestion Games). In congestion games, each resource $r\in\mathcal{R}$ is associated with a congestion function $c_r:\{1,\dots,n\}\to\mathbb{R}_{\geq 0}$. For this class of games, the agent cost generating and system cost functions take on a specific form: $F_r(k)=c_r(k)$ and $C_r(k)=k\cdot c_r(k)$ for any $k\geq 1$. Note that $C_r(k)=k\cdot F_r(k)$ for this case, hence the definitions of smoothness and generalized smoothness coincide.

Example 2 (Congestion Games with Incentives). When incentives are introduced into the congestion game setup, each resource $r\in\mathcal{R}$ is also associated with an incentive function $\tau_r:\{1,\dots,n\}\to\mathbb{R}_{\geq 0}$. For this class of games, the agent cost generating and system cost functions take on the form where $F_r(k)=c_r(k)+\tau_r(k)$ and $C_r(k)=k\cdot c_r(k)$ for any $k\geq 1$. Note that $C_r(k)\neq k\cdot F_r(k)$ for this case, hence the framework of generalized smoothness provides a strictly closer characterization of the price of anarchy than original smoothness by Proposition 1.

In this manuscript, we consider generalized congestion games in which each $r\in\mathcal{R}$ has function pair $\{C_r,F_r\}$ that can be represented as linear combinations of basis function pairs $C^j:\{0,1,\dots,n\}\to\mathbb{R}_{\geq 0}$, $F^j:\{1,\dots,n\}\to\mathbb{R}$, $j=1,\dots,m$, with nonnegative coefficients. Formally, for each $r\in\mathcal{R}$, there exist coefficients $\alpha_1,\dots,\alpha_m\geq 0$ such that $C_r(k)=\sum_{j=1}^m\alpha_j\cdot C^j(k)$ and $F_r(k)=\sum_{j=1}^m\alpha_j\cdot F^j(k)$, for $k=1,\dots,n$. For the remainder of this section, we denote with $\mathcal{G}$ the set of all generalized congestion games that can be constructed with a maximum of n agents, arbitrary resource set $\mathcal{R}$ and arbitrary joint action set $\mathcal{A}$, where each $r\in\mathcal{R}$ is associated with functions $\{C_r,F_r\}$ constructed as above.[1] As we will see, the basis function pairs $\{C^j,F^j\}$, $j=1,\dots,m$, are the driving elements in the computation of the price of anarchy over the set of games $\mathcal{G}$.

[1] In certain cases, one might be interested in characterizing the price of anarchy for the set of all generalized congestion games with a maximum of n agents, where each $r\in\mathcal{R}$ has $\{C_r,F_r\}$ that belongs to some countable set of function pairs $\mathcal{Z}$. Using similar arguments as in the proof of Roughgarden (2015), Theorem 5.8, one can show that the price of anarchy of this reduced set of games must be equal to the price of anarchy of the set of generalized congestion games $\mathcal{G}$ as defined above, with basis function pairs $\mathcal{Z}$.

The following example demonstrates how a limited set of basis function pairs can actually model a diverse set of problem settings.

Example 3 (Affine and Polynomial Congestion Games). One commonly studied class of congestion games is affine congestion games, where each resource $r \in \mathcal{R}$ is associated with a agent cost generating function $F_r(k) = a_r \cdot k + b_r$ and system cost function $C_r(k) = k \cdot (a_r \cdot k + b_r) = k \cdot F_r(k)$ for any $k \geq 1$, where $a_r, b_r \geq 0$. Observe that the all admissible function pairs $\{C_r, F_r\}$ can be represented as linear combinations of the basis function pairs $\{\{C^1, F^1\}, \{C^2, F^2\}\}$ where $\{C^1(k), F^1(k)\} = \{k, 1\}$ (case where $a_r = 0$ and $b_r = 1$) and $\{C^2(k), F^2(k)\} = \{k^2, k\}$ (case where $a_r = 1$ and $b_r = 0$). Similarly, the function pairs $\{C_r, F_r\}$ of any polynomial congestion game of degree $d \geq 1$, i.e., where each resource is associated with a agent cost function of the form $F_r(k) = \sum_{j=0}^{d} \alpha_j \cdot k^j$ such that $\alpha_0, \ldots, \alpha_d \geq 0$ and $C_r(k) = k \cdot F_r(k)$, can be represented as linear combinations of a set of $d+1$ basis function pairs in the same fashion as in the affine case.

3.2. Tight price of anarchy for generalized congestion games

Our goal in this section is to characterize the price of anarchy for a given set of generalized congestion games $\mathcal{G}$. In the previous section, we demonstrated that the generalized price of anarchy always provides improved bounds when compared to the robust price of anarchy. With our next result, we show that the generalized price of anarchy achieves a tight bound on the price of anarchy for any set of generalized congestion games, and that the price of anarchy associated with a set of generalized congestion games $\mathcal{G}$ can be characterized by means of a tractable linear program that scales linearly in its complexity with the size of the set of basis function pairs, m, and quadratically with the maximum number of agents n. Throughout, we define $C^j(0) = F^j(0) = F^j(n+1) = 0$, $j = 1, \ldots, m$, for ease of notation, and without loss of generality. Additionally, $\mathcal{I}_{\mathcal{R}}(n)$ is defined as the set of all triplets $(x, y, z) \in \{0, 1, \ldots, n\}^3$ that satisfy: (i) $1 \leq x + y - z \leq n$ and $z \leq \min\{x, y\}$; and, (ii) $x + y - z = n$ or $(x - z)(y - z)z = 0$.

Theorem 3. *Let $\mathcal{G}$ denote the set of generalized congestion games as defined above, with basis function pairs $\{C^j, F^j\}$, $j = 1, \ldots, m$, defined for n agents. The following statements are true:*

(i) The price of anarchy and the generalized price of anarchy satisfy

$$\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G}).$$

(ii) Let ρ_{opt} be the value of the following linear program:

$$\begin{aligned} \rho_{\mathrm{opt}} = \max_{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}} \quad & \rho \\ s.t. \quad & C^j(y) - \rho C^j(x) + \nu[(x-z)F^j(x) - (y-z)F^j(x+1)] \geq 0, \\ & \qquad \forall j = 1, \ldots, m, \quad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned} \tag{16}$$

Then, it holds that $\mathrm{PoA}(\mathcal{G}) = 1/\rho_{\mathrm{opt}}$.

Table 1 The price of anarchy in polynomial congestion games.

d	λ	μ	PoA
1	1.67	0.333	2.50
2	6.05	0.368	9.58
3	17.89	0.569	41.54
4	141.19	0.472	267.64
5	569.11	0.624	1513.57
6	2320.00	0.812	12345.20

Exact price of anarchy and optimal smoothness parameters for polynomial congestion games with $d=1,\dots,6$ and $n=100$. These values were all derived using the tractable linear program (16), and match the asymptotic results from Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005), Aland et al. (2011).

In the following discussion, we highlight several important implications of Theorem 3:

First, although the authors of Roughgarden (2015) demonstrated that $\mathrm{RPoA}(\mathcal{G})=\mathrm{PoA}(\mathcal{G})$ when all $\{C_r,F_r\}$ satisfy $C_r(k)=k\cdot F_r(k)$ for all k, we have shown in Proposition 1 that $\mathrm{RPoA}(\mathcal{G})\neq\mathrm{PoA}(\mathcal{G})$ when this condition is not satisfied, e.g., congestion games with incentives. Theorem 3 effectively shows that the canonical results in Roughgarden (2015) extend to all generalized congestion games through the framework of generalized smoothness. Further, the proof of Theorem 3 (i) guarantees that every set of generalized congestion games $\mathcal{G}$ contains a "small problem instance" (i.e., a game with resource set $\mathcal{R}$ such that $|\mathcal{R}|\leq 2n$) with price of anarchy that matches the worst-case price of anarchy bound. Finally, we have shown that the exact price of anarchy of the set $\mathcal{G}$ can be computed as the solution to the linear progam (16) that has at most $m\times|\mathcal{I}_{\mathcal{R}}(n)|=m\times(2n^2+1)$ constraints and is both exact and tractable (i.e., it can be solved efficiently for moderate n and m).

We conclude this section by revisiting the class of congestion games with polynomial cost functions as described in Example 3. Table 1 provides the price of anarchy for such games ($d=1,\dots,6$ and $n=100$) as well as the corresponding optimal values of λ and μ derived using the linear program in Theorem 3.[2] It is important to highlight that the price of anarchy computed for each set of games matches the asymptotic result from Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005), Aland et al. (2011). However, while these previous works characterize the price of anarchy following a series of *ad hoc* analytical arguments, we reproduce their results via a straightforward application of the tractable linear program (16).

3.3. Optimizing the price of anarchy

The previous section focused on how to characterize the price of anarchy in any set of generalized congestion games. In this section, we shift our focus to the design of the agents' cost generating functions in order to optimize the price of anarchy. That is, given a set of system cost functions

[2] Although the computation of the optimal generalized smoothness parameters λ and μ is hidden in the linear program (16), they can be recovered from the solutions of the linear program as $\lambda=1/\nu_{\mathrm{opt}}$ and $\mu=1-\rho_{\mathrm{opt}}/\nu_{\mathrm{opt}}$. See step b) of the proof of Theorem 3 (i) for the technical details.

$C^1,\dots,C^m$, what is the corresponding set of agent cost generating functions $F^1,\dots,F^m$ that minimizes the price of anarchy $\mathrm{PoA}(\mathcal{G})$? Recall from the introduction that this line of questioning is relevant to the problem of incentive design given in Section 1.1, where the price of anarchy was the performance bound of interest.

In the following corollary, we provide a tractable and scalable methodology for computing the set of agent cost generating functions that minimizes the price of anarchy.

Corollary 1. *Let $C^1,\dots,C^m$ denote a set of system cost functions defined for n agents, and let $(F^j_{\mathrm{opt}},\rho^j_{\mathrm{opt}})$, $j=1,\dots,m$, be solutions to the following m linear programs:*

$$\begin{aligned}(F^j_{\mathrm{opt}},\rho^j_{\mathrm{opt}}) \in \operatorname*{arg\,max}_{F\in\mathbb{R}^n,\rho\in\mathbb{R}} \quad & \rho \\ \text{s.t.} \quad & C^j(y)-\rho C^j(x)+(x-z)F(x)-(y-z)F(x+1)\ge 0, \quad \forall (x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).\end{aligned} \tag{17}$$

Then the agent cost generating functions $F^1_{\mathrm{opt}},\dots,F^m_{\mathrm{opt}}$ minimize the price of anarchy, and the price of anarchy corresponding to the basis function pairs $\{C^j,F^j_{\mathrm{opt}}\}$, $j=1,\dots,m$, satisfies

$$\mathrm{PoA}(\mathcal{G})=\max_{j\in\{1,\dots,m\}}\frac{1}{\rho^j_{\mathrm{opt}}}.$$

Corollary 1 states that we can derive agent cost generating functions $F^1_{\mathrm{opt}},\dots,F^m_{\mathrm{opt}}$ that minimize the price of anarchy by solving m independent linear progams, where each F^j_{opt} can be derived using only information about the corresponding system cost function C^j. Accordingly, the price of anarchy of this optimized system corresponds to the worst price of anarchy associated with any single characteric cost system, i.e.,

$$\mathrm{PoA}(\mathcal{G})=\max_{j\in\{1,\dots,m\}}\mathrm{PoA}(\mathcal{G}^j),$$

where $\mathcal{G}^j\subseteq\mathcal{G}$ represents the set of generalized congestion games induced by n and the basis function pair $\{C^j,F^j\}$. Observe that this statement is not true in general for an arbitrary set of basis function pairs, i.e., there exists a set of basis function pairs $\{C^j,F^j\}$, $j=1,\dots,m$, such that[3]

$$\mathrm{PoA}(\mathcal{G})>\max_{j\in\{1,\dots,m\}}\mathrm{PoA}(\mathcal{G}^j).$$

However, when we restrict our attention to optimal F^j_{opt} for each C^j, the above strict inequality holds with equality. This constitutes the key observation in the proof of Corollary 1.

Recall our discussion from Sections 1.1 and 1.2 on the importance of deriving tractable methodologies within the atomic setting for characterizing the price of anarchy corresponding to a given

[3] To see this, consider the set of generalized congestion games $\mathcal{G}$ induced by $n=3$, and $\{\{C^1,F^1\},\{C^2,F^2\}\}$, where $\{C^1(k),F^1(k)\}=\{k^2,k\}$ and $\{C^2,F^2\}=\{k,k\}$ for all $k=1,\dots,n$. Using the linear program (16), we get $\mathrm{PoA}(\mathcal{G}^1))=2.5$, $\mathrm{PoA}(\mathcal{G}^2))=2.0$, and $\mathrm{PoA}(\mathcal{G})=2.6$. For this particular choice of $\mathcal{G}$, observe that $\mathrm{PoA}(\mathcal{G})>\max_{j\in\{1,\dots,m\}}\mathrm{PoA}(\mathcal{G}^j)$.

set of incentives, and computing incentives that minimize the price of anarchy. In this respect, the results in Theorem 3 demonstrate that the generalized smoothness inequalities are the inequalities that govern the price of anarchy in generalized congestion games, and that the price of anarchy for a given set of generalized congestion games can be characterized using a tractable linear program. Furthermore, the result in Corollary 1 represents a tractable approach to computing a set of agent cost generating functions that minimize the price of anarchy in atomic scenarios that fall under the model of generalized congestion games.

The main result in Paccagnan et al. (2019a) is a tractable methodology for computing optimal incentives in atomic congestion games, where the incentives $\tau^j_{\rm opt}$, $j=1,\dots,m$, are constructed from the solutions $(F^j_{\rm opt}, \rho^j_{\rm opt})$, $j=1,\dots,m$, of the linear program (17) in Corollary 1 of this paper. In Section 5, we compare the price of anarchy guarantees achieved using optimal incentives from a study in the nonatomic setting (Maillé and Stier-Moses 2009) with those achieved by a modified version of the methodology in Paccagnan et al. (2019a) that constructs rebates, i.e., incentives that satisfy $\tau_r(k)\le 0$ for all k and r. We observe that using optimal incentives from the nonatomic setting can actually result in worse price of anarchy guarantees than not using incentives at all. In contrast, the price of anarchy achieved using our framework is significantly lower than the price of anarchy of the congestion games without incentives.

4. Generalized Smoothness in Welfare Maximization Games

Although the primary focus of this paper is on cost minimization settings, many of the results that we obtain can be analogously derived for welfare maximization problems. A welfare maximization problem consists of a set $N=\{1,\dots,n\}$ of agents, where each agent $i\in N$ is associated with a finite action set $\mathcal{A}_i$. The global objective is to *maximize* the system's welfare, which is measured by the welfare function $W:\mathcal{A}\to\mathbb{R}_{>0}$, i.e. we wish to find the allocation $a^{\rm opt}\in\mathcal{A}$, such that $a^{\rm opt}\in\arg\max_{a\in\mathcal{A}} W(a)$. As with cost minimization problems, we consider a game-theoretic model where each agent i is associated with a utility function $U_i:\mathcal{A}\to\mathbb{R}$, which it uses to evaluate its own actions against the collective actions of the other agents a_{-i}. We represent a *welfare maximization game* as a tuple $G=(N,\mathcal{A},W,\{U_i\})$.

Given a welfare maximization game G, a pure Nash equilibrium is defined as an allocation $a^{\rm ne}\in\mathcal{A}$ such that $U_i(a^{\rm ne})\ge U_i(a_i,a^{\rm ne}_{-i})$ for all $a\in\mathcal{A}_i$, and all $i\in N$. The price of anarchy in welfare maximization games is defined similarly to (6) and (7), [4]

$$\mathrm{PoA}(G):=\frac{\max_{a\in\mathcal{A}} W(a)}{\min_{a\in\mathrm{NE}(G)} W(a)}\ge 1,\quad \mathrm{PoA}(\mathcal{G}):=\sup_{G\in\mathcal{G}}\mathrm{PoA}(G)\ge 1,$$

[4] For consistency with the previous sections, we opt to define the price of anarchy in welfare maximization games as the ratio between the welfare at the optimal allocation and the system welfare at the worst performing Nash equilibrium, in contrast with previous works, c.f. Gairing 2009, Roughgarden 2015. This is achieved by inverting the ratio, i.e., defining the price of anarchy as the worst-case ratio between the welfare at optimum, and the welfare at the equilibria in $\mathrm{NE}(G)$. By adopting this formalism, we retain the overall objective of *minimizing* the system's price of anarchy.

where a lower value of the price of anarchy corresponds to an improvement in performance.

4.1. Generalized smoothness in welfare maximization games

We begin with the definition of generalized smoothness in welfare maximization games and then provide the analogue of Theorem 1.

DEFINITION 3. The welfare maximization game G is (λ,μ)-generalized smooth if, for any two allocations $a,a'\in\mathcal{A}$, there exist $\lambda>0$ and $\mu>-1$ satisfying,

$$\sum_{i=1}^{n} U_i(a_i',a_{-i}) - \sum_{i=1}^{n} U_i(a) + W(a) \geq \lambda W(a') - \mu W(a). \tag{18}$$

THEOREM 4. *The price of anarchy of a (λ,μ)-generalized smooth welfare maximization game G is upper-bounded as,*

$$\mathrm{PoA}(G) \leq \frac{1+\mu}{\lambda}.$$

We define the generalized price of anarchy of a set of welfare maximization games $\mathcal{G}$ as

$$\mathrm{GPoA}(\mathcal{G}) := \inf_{\lambda>0,\mu>-1} \left\{ \frac{1+\mu}{\lambda} \text{ s.t. (18) holds } \forall G\in\mathcal{G} \right\}. \tag{19}$$

As with cost minimization games, all efficiency guarantees also extend to average coarse correlated equilibria (as in Theorem 2) and there are also provable advantages of generalized smoothness over the original smoothness framework in terms of characterizing price of anarchy bounds (as in Proposition 1). We do not explicitly state these parallel results or provide the proofs to avoid redundancy.

4.2. Generalized smoothness in local welfare maximization games

In this section we introduce a special class of welfare maximization games, called *local welfare maximization games*, which are analogous to the class of generalized congestion games. Games in this class feature a set of agents $N=\{1,\dots,n\}$ and a finite set of resources $\mathcal{R}$. The system welfare and agent utility functions are defined as

$$W(a) = \sum_{r\in\mathcal{R}} W_r(|a|_r),$$
$$U_i(a_i,a_{-i}) = \sum_{r\in a_i} F_r(|a|_r),$$

where, for each $r\in\mathcal{R}$, $W_r:\{0,1,\dots,n\}\to\mathbb{R}_{\geq 0}$ and $F_r:\{1,\dots,n\}\to\mathbb{R}_{\geq 0}$ are the system welfare and agent utility generating functions, respectively. For the remainder of this section, given basis function pairs $\{W^j,F^j\}$, $j=1,\dots,m$, we define the set of local welfare maximization games $\mathcal{G}$ in the same fashion as for generalized congestion games given in Section 3. Local welfare maximization games can be employed to model several problems of interest including those consider in Gairing (2009), Kleinberg and Oren (2011), Barman et al. (2019).

The following theorem provides the analagous results derived for generalized congestion games to the domain of local welfare maximization games. As before, we define $W^j(0) = F^j(0) = F^j(n+1) = 0$, for $j = 1, \dots, m$, for ease of notation. Theorem 5 is stated without proof as the reasoning follows almost identically to the proofs of Theorem 3 and Corollary 1.

THEOREM 5. *Let* $\{W^j, F^j\}$, $j = 1, \dots, m$, *denote a set of basis function pairs defined for* n *agents. The following statements are true:*

(i) The price of anarchy and the generalized price of anarchy satisfy

$$\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G}).$$

(ii) Let ρ_{opt} *be the value of the following linear program:*

$$\begin{aligned} \rho_{\mathrm{opt}} = \min_{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}} \quad & \rho \\ s.t. \quad & W^j(y) - \rho W^j(x) + \nu \left[(x - z) F^j(x) - (y - z) F^j(x+1) \right] \leq 0 \\ & \forall j = 1, \dots, m, \quad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n), \end{aligned} \tag{20}$$

Then, it holds that $\mathrm{PoA}(\mathcal{G}) = \rho_{\mathrm{opt}}$.

(iii) Let the parameters $(F^j_{\mathrm{opt}}, \rho^j_{\mathrm{opt}})$, $j = 1, \dots, m$, *be solutions to the following* m *linear programs:*

$$\begin{aligned} (F^j_{\mathrm{opt}}, \rho^j_{\mathrm{opt}}) \in \operatorname*{arg\,min}_{F \in \mathbb{R}^n, \rho \in \mathbb{R}} \quad & \rho \\ s.t. \quad & W^j(y) - \rho W^j(x) + (x - z) F(x) - (y - z) F(x+1) \leq 0, \quad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned}$$

Then the agent utility generating functions $F^1_{\mathrm{opt}}, \dots, F^m_{\mathrm{opt}}$ *minimize the price of anarchy, and the price of anarchy corresponding to the basis function pairs* $\{W^j, F^j_{\mathrm{opt}}\}$, $j = 1, \dots, m$, *satisfies*

$$\mathrm{PoA}(\mathcal{G}) = \max_{j \in \{1, \dots, m\}} \rho^j_{\mathrm{opt}}.$$

5. Optimal Rebates in Atomic Congestion Games

As we have highlighted previously, the congestion game model has widely been used to capture the deterioration in performance of multiagent systems within a variety of real-world problem settings. While populations of agents are finite and indivisible (atomic) in the majority of real-world applications, most works consider an alternative, "large population" scenario, where agents are treated as divisible (nonatomic) entities. The advantages of using this nonatomic approximation often include guarantees on the uniqueness of the resulting equilibria, i.e., Wardrop equilibria. In comparison, the atomic congestion game model often admits a multiplicity of equilibria which complicates the underlying analysis.

In this section, we revisit the incentive design problem given in Section 1.1 to study the efficacy of our approach. First, we investigate whether incentives designed in nonatomic environments offer similar performance guarantees when they are applied to more realistic atomic environments. In this case study, we actually show that the answer to this question is a resounding “no” as the following analysis demonstrates. Then, building on the main result in Paccagnan et al. (2019a), we show that the linear program (17) provides a tractable mechanism for deriving rebate incentives that optimize the price of anarchy in atomic congestion games.

5.1. Nonatomic approximation

This section focuses on the special class of generalized congestion games introduced in Section 3 where we restrict our attention to resources with polynomial congestion functions, i.e., for a given polynomial degree $d \geq 1$, each resource $r \in \mathcal{R}$ is associate with a congestion function of the form $c_r(k) = \sum_{j=0}^{d} \alpha_j \cdot k^j$ for all $k \in \{1, \dots, n\}$, where $\alpha_j \geq 0$ for all $j \in \{0, 1, \dots, d\}$. In this particular study, we further restrict our attention to the problem of rebate design (i.e., $\tau_r(k) \leq 0$ for all k and r), as in Maillé and Stier-Moses (2009). Our aim is to improve equilibrium efficiency by using the rebates to align agents' local costs with the overall system objective.

Here we examine the validity of the nonatomic approximation by assessing the worst-case equilibrium efficiency achieved in an atomic congestion game when using a rebate that has been shown to be optimal under the nonatomic assumption. In Maillé and Stier-Moses (2009), the authors show that, when the optimal allocation $a^{\text{opt}} \in \mathcal{A}$ is known, the following fixed rebate guarantees optimal equilibrium efficiency in nonatomic congestion games:

$$\tau_r(k) = -c_r(|a^{\text{opt}}|_r). \tag{21}$$

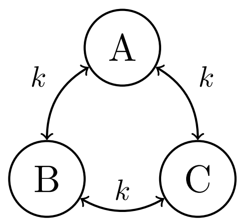
We demonstrate that implementing this rebate derived under the nonatomic assumption is worse than doing nothing in the atomic congestion game setting.

Observe that the affine congestion game in Fig. 1 has $c_r(k) = k$ and $|a^{\text{opt}}|_r = 1$ for all $r \in \mathcal{R}$. Therefore, the corresponding rebate (21) is $\tau_r(k) = -1$, and $c_r(k) + \tau_r(k) = k - 1$. Furthermore, it holds that the allocation $a^{\text{ne}} = (a_1^{\text{ne}}, a_2^{\text{ne}}, a_3^{\text{ne}})$ is a pure Nash Equilibrium, as

$$J_i(a^{\text{ne}}) = (2-1) + (2-1) = 2 = (3-1) = J_i(a_i^{\text{opt}}, a_{-i}^{\text{ne}}), \quad \forall i = 1, 2, 3.$$

The system costs of allocations a^{opt} and a^{ne} are $C(a^{\text{opt}}) = 1 + 1 + 1 = 3$ and $C(a^{\text{ne}}) = 4 + 4 + 4 = 12$, respectively. Thus, while the fixed rebate from Maillé and Stier-Moses (2009) guaranteed optimal equilibrium efficiency in *nonatomic* affine congestion games (i.e., $C(a^{\text{ne}})/C(a^{\text{opt}}) = 1$ always holds), the worst case equilibrium efficiency within *atomic* affine congestion games is at least

Figure 1 Three-agent affine congestion game with three resources.



	a_i^{ne}	a_i^{opt}
$i=1$	A $\to$ B $\to$ C	A $\to$ C
$i=2$	B $\to$ C $\to$ A	B $\to$ A
$i=3$	C $\to$ A $\to$ B	C $\to$ B

Note. The three resources (i.e. the edges in the graph on the left) have congestion functions $c_r(k)=k$ for all k, and the agent action sets are $\mathcal{A}_i = \{a_i^{\mathrm{ne}}, a_i^{\mathrm{opt}}\}$, for each agent $i=1,2,3$. The agents' actions are illustrated in the table on the right, where each agent $i=1,2,3$ has corresponding origin and destination (i.e., agent 1 has origin A, and destination C). Observe that each agent $i=1,2,3$ selects two of the resources in a_i^{ne}, and the one remaining resource in a_i^{opt}. One can verify that the allocation $a^{\mathrm{opt}}=(a_1^{\mathrm{opt}},a_2^{\mathrm{opt}},a_3^{\mathrm{opt}})$ is optimal, with system cost $C(a^{\mathrm{opt}})=1+1+1=3$.

$C(a^{\mathrm{ne}})/C(a^{\mathrm{opt}})=4$. In comparison, in the absence of incentive mechanisms (i.e., $\tau_r(j)=0$ for all k and r), the worst-case equilibrium efficiency in atomic affine congestion games is at most 2.5 (Awerbuch et al. 2005, Christodoulou and Koutsoupias 2005).

We observe from Table 2 that the price of anarchy in all polynomial congestion games with degree $d=1,\dots,6$ is worse under the fixed rebate from Maillé and Stier-Moses (2009) than in the absence of incentives. The values in column 3 of Table 2 were computed using the linear program (16) for $n=100$, and basis function pairs $\{C^j,F^j\}$, $j=0,1,\dots,d$, where $C^j(k)=k^{j+1}$ and $F^j(k)=k^j-y^j$ for any given triplet $(x,y,z)\in\mathcal{I}_{\mathcal{R}}(n)$. Thus, for polynomial congestion games of degree $d\geq 1$, for each $j\in\{0,1,\dots,d\}$, the constraints of the linear program in this setting are

$$y^{j+1}-\rho x^{j+1}+\nu[(x-z)\cdot x^j-(y-z)\cdot(x+1)^j+(y-x)\cdot y^j]\geq 0,\quad \forall (x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).$$

5.2. Designing rebates without network knowledge

In addition to modelling the agents as infinitely divisible flows, the authors of Maillé and Stier-Moses (2009) assume that the the optimal allocation $a^{\mathrm{opt}}\in\mathcal{A}$ is known *a priori*. This leads to several immediate issues in the implementability of their rebate, including: (i) the computation of the optimal allocation $a^{\mathrm{opt}}\in\mathcal{A}$ is combinatorial (i.e., NP-hard), and thus intractable for problem settings of moderate size; and, (ii) access to complete network knowledge is often an unrealistic assumption due to a number of possible factors, including the imposed cost overheads and communication requirements. Regardless, a framework that does not rely on prior network knowledge for designing rebates is certainly more widely applicable. In this respect, the linear programming methodology put forward in this manuscript can be used to derive optimal rebates for atomic congestion games without complete network knowledge.

Table 2 The price of anarchy in polynomial congestion games under rebate-style incentives.

d	No Incentives	Optimal Rebates from Nonatomic Approx.	Dynamic Rebates (Our Setting)	Fixed Rebates (Our Setting)
1	2.50	4.00	2.01	2.50
2	9.58	11.45	5.10	9.58
3	41.54	44.89	15.55	41.54
4	267.64	271.50	55.45	267.64
5	1513.57	1519.51	220.40	1513.57
6	12345.20	12359.65	967.53	12345.20

Price of anarchy guarantees for polynomial congestion games with degree $d=1,\ldots,6$ and $n=100$, under the influence of various rebate-style incentive schemes. The second column contains the asymptotic results for the price of anarchy without incentives (Awerbuch et al. 2005, Christodoulou and Koutsoupias 2005, Aland et al. 2011). The third column lists lower bounds on the price of anarchy for polynomial congestion games with atomic agents, under the influence of optimal rebates designed for the nonatomic problem formulation in Maillé and Stier-Moses (2009). These values were obtained using the tractable linear program (16), as detailed in Section 5.1. In the fourth and fifth columns, we report price of anarchy guarantees for the optimal dynamic rebates and optimal fixed rebates in the atomic congestion game setting, respectively. These values were computed using the tractable linear program (17), as we discuss in Section 5.2.

To that end, let $\{C^j, F^j\}$, $j=1,\ldots,m$, denote the set of basis function pairs corresponding to a class of congestion games. Recall from Example 3 that, in the case of polynomial congestion games, the function pairs $\{C_r, F_r\}$ satisfy the following for each k:

$$C_r(k) = \sum_{j=0}^{d} \alpha_j k^{j+1},$$

$$F_r(k) = \left(\sum_{j=0}^{d} \alpha_j k^j\right) + \tau_r(k).$$

As in Maillé and Stier-Moses (2009), we bound the incentive on each resource such that it is nonpositive and its magnitude cannot exceed that of the congestion function of that resource. Formally, we constrain the admissible incentive functions as $-c_r(k) \le \tau_r(k) \le 0$ for all k and r.

Recall from Proposition 1 that the original smoothness argument provides strictly loose bounds on the price of anarchy in general, and is therefore unsuitable for the characterization or optimization of the price of anarchy in generalized congestion games. More significantly, the robust price of anarchy is undefined for rebates (i.e., $\tau_r(k) < 0$ for any k), and is always minimized for the incentives $\tau_r(k) = 0$ for all k and r. In contrast, we show below that the cost generating functions that minimize the generalized price of anarchy in a given set of generalized congestion games can be used to construct incentive rebates that minimize the price of anarchy of the corresponding class of congestion games.

Optimal dynamic rebates. The main result in Paccagnan et al. (2019a) is a tractable methodology for constructing optimal dynamic incentives for atomic congestion games without complete network knowledge. In particular, the authors show that, for any class of atomic congestion games induced by basis congestion functions $c^1,\ldots,c^m$, an optimal incentive function τ_r is a linear map of the congestion function c_r, i.e., for the set of all congestion games in which each $r\in\mathcal{R}$ has congestion function $c_r(k) = \alpha_1 c^1(k) + \cdots + \alpha_m c^m(k)$ for all k such that $\alpha_1,\ldots,\alpha_m \ge 0$, there exist basis incentive functions $\tau^1_{\text{opt}},\ldots,\tau^m_{\text{opt}}$ such that the incentive function $\tau_r(k)=\alpha_1\tau^1_{\text{opt}}(k)+\cdots+\alpha_m\tau^m_{\text{opt}}(k)$ for all k minimizes the price of anarchy. The authors go on to demonstrate how the functions τ^j_{opt}, $j=1,\ldots,m$, can be constructed from the solutions $(F^j_{\text{opt}},\rho^j_{\text{opt}})$, $j=1,\ldots,m$, obtained using the linear program (17) in Corollary 1 of this paper. In the discussion below, we detail how the methodology provided in Paccagnan et al. (2019a) can be modified to construct optimal dynamic rebates from the solutions to the linear program (17).

Consider the set of atomic congestion games induced by basis congestion functions $c^1,\ldots,c^m$ defined for n agents. We assume that every game G in this set has agent set satisfying $|N|\le n$, and that, for each $r\in\mathcal{R}$, the congestion function $c_r:\{1,\ldots,n\}\to\mathbb{R}$ takes on the form

$$c_r(k)=\alpha_1c^1(k)+\cdots+\alpha_mc^m(k),$$

where $\alpha_j\ge 0$ for all $j=1,\ldots,m$. For a resource r with congestion function c_r corresponding to coefficients $\alpha_1,\ldots,\alpha_m$, a rebate function τ_r that minimizes the price of anarchy of the class of congestion games can be constructed as

$$\tau_r(k)=\sum_{j=1}^m\alpha_j\left[\frac{F^j_{\text{opt}}(k)}{n}-c^j(k)\right]. \tag{22}$$

where, for each $j=1,\ldots,m$, the function F^j_{opt} is a solution to the following linear program,

$$\begin{aligned}(F^j_{\text{opt}},\rho^j_{\text{opt}})\in \underset{F\in\mathbb{R}^n,\rho\in\mathbb{R}}{\arg\max}\quad & \rho\\ \text{s.t.}\quad & c^j(y)y-\rho c^j(x)x+(x-z)F(x)-(y-z)F(x+1)\ge 0,\quad \forall (x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).\end{aligned} \tag{23}$$

Observe that the above rebate design methodology only depends on the maximum number of agents and the set of basis congestion functions. Further, one can verify that the rebate functions satisfy $-c_r(k)\le\tau_r(k)\le 0$ for all k and r, as desired. We report the resulting price of anarchy guarantees for polynomial congestion games of degree $d=1,\ldots,6$ and $n=100$ in Column 4 of Table 2. The achieved price of anarchy guarantees are identical to the optimal incentives for the class of polynomial congestion games (c.f., Paccagnan et al. (2019a)), and are significantly lower than those achieved without rebates and the fixed rebates from Maillé and Stier-Moses (2009).

Optimal fixed rebates. In certain scenarios, we might additionally constrain the rebates such that $\tau_r(k)=\tau_r(1)$, where $-c_r(k)\le\tau_r(1)\le 0$ for all k and r. For constructing optimal fixed incentives, Paccagnan et al. provide an alternate methodology based on a modified version of the linear program (17) in Corollary 1 of this work. In the following, we show to use their methodology to compute optimal fixed rebates:

Consider the set of atomic congestion games induced by basis congestion functions $c^1,\ldots,c^m$ defined for n agents. For a resource r with congestion function c_r corresponding to coefficients

$\alpha_1,\ldots,\alpha_m \geq 0$, a fixed rebate function τ_r that minimizes the price of anarchy of the class of congestion games can be constructed as

$$\tau_r(k) = \sum_{j=1}^{m} \alpha_j \frac{\sigma_{\text{fix}}^j}{\nu_{\text{fix}}}, \quad k = 1,\ldots,n, \tag{24}$$

where the tuple $(\nu_{\text{fix}}, \rho_{\text{fix}}, \sigma_{\text{fix}})$ and $\sigma_{\text{fix}} = (\sigma_{\text{fix}}^1, \ldots, \sigma_{\text{fix}}^m)$ is a solution to the following linear program,

$$\begin{aligned}
(\nu_{\text{fix}}, \rho_{\text{fix}}, \sigma_{\text{fix}}) \in \underset{\nu\in\mathbb{R}_{\geq 0}, \rho\in\mathbb{R}, \sigma\in\mathbb{R}^m}{\arg\max} \quad & \rho \\
\text{s.t.} \quad & c^j(y)y - \rho c^j(x)x + \nu\left[c^j(x)x - c^j(x+1)y\right] + \sigma^j(x-y) \geq 0, \\
& \qquad\qquad \forall j = 1,\ldots,m, \quad \forall x+y \leq n, \\
& c^j(y)y - \rho c^j(x)x + \nu\left[c^j(x)(n-y) - c^j(x+1)(n-x)\right] + \sigma^j(x-y) \geq 0, \\
& \qquad\qquad \forall j = 1,\ldots,m, \quad \forall x+y > n, \\
& -\nu c^j(k) \leq \sigma^j \leq 0, \qquad \forall j = 1,\ldots,m, \quad \forall k = 1,\ldots,n.
\end{aligned} \tag{25}$$

Observe that the definition in (24) and the final constraint in (25) ensure that the rebate is fixed and admissible, i.e., $\tau_r(k) = \tau_r(1)$ and $-c_r(k) \leq \tau_r(1) \leq 0$ for all k and r. In the case of polynomial congestion games of degreed $d = 1,\ldots,6$, the optimal fixed rebates are $\tau_r(k) = 0$ for all k and r. This is consistent with the literature, as the optimal fixed incentive mechanisms for polynomial congestion games are positive (Caragiannis et al. 2010, Paccagnan et al. 2019a). We report the price of anarchy for optimal fixed rebates in polynomial congestion games in Column 5 of Table 2.

6. Conclusion

In this work, we sought to improve our capabilities in computing and optimizing the price of anarchy. The end goal was to use our new insights to inform the design of incentive mechanisms in scenarios where decision making is distributed. Toward this end, we first introduced the generalized smoothness framework, which we showed is more widely applicable, and provides more precise price of anarchy bounds when compared to the original smoothness approach of Roughgarden 2015. Next, we proved that by using a generalized smoothness argument one obtains the exact price of anarchy for the class of generalized congestion/local welfare maximization games, which does not hold true for the original smoothness argument. Finally, we showed that the problems of computing the price of anarchy and optimizing the price of anarchy over the agents' cost/utility generating functions can be posed as tractable linear programs. We showcased the strength and breadth of our approach by analyzing the design of rebates for congestion games. After showing that incentives derived under the continuous flow approximation in Maillé and Stier-Moses (2009) do not offer the same performance guarantees in the atomic setting, we provided a tractable linear programming-based methodology based on our novel smoothness approach for the design of optimal dynamic and fixed rebates in atomic congestion games. While we illustrated our framework by applying it to the problem of rebate design in atomic congestion games, it can be further applied to many relevant and well-studied classes of problems. We note that the price of anarchy represents but one of many metrics for measuring algorithm performance. Future work should be devoted to analyzing the potential losses in performance with respect to other metrics when designing algorithms with the best achievable price of anarchy.

Appendix A: Proof of Theorem 3

Proof of Theorem 3 (i). We split the proof into the following two steps, in step a) we construct an upper-bound on the generalized price of anarchy $\mathrm{GPoA}(\mathcal{G})$, and in step b) we derive game instances with price of anarchy that matches this upper-bound, hence proving $\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G})$.

Step a): For any game G, we denote an optimal allocation as a^{opt}, and a worst-performing Nash equilibrium as a^{ne}, i.e. $a^{\mathrm{ne}} \in \mathrm{NE}(G)$ such that $\mathrm{PoA}(G) = C(a^{\mathrm{ne}})/C(a^{\mathrm{opt}})$. For every resource $r \in \mathcal{R}$, let $x_r = |a^{\mathrm{ne}}|_r$, and $y_r = |a^{\mathrm{opt}}|_r$. We define z_r as the number of agents that select resource r in both a^{ne} and a^{opt}, i.e. $z_r := |\{i \in N : r \in a_i^{\mathrm{ne}}\} \cap \{i \in N : r \in a_i^{\mathrm{opt}}\}|$. We observe that using the above definitions of (x_r, y_r, z_r) for all $r \in \mathcal{R}$, it follows for any game $G \in \mathcal{G}$ that

$$\sum_{i=1}^{n} J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) = \sum_{r\in\mathcal{R}} (y_r - z_r) F_r(x_r + 1) + z_r F_r(x_r).$$

Informally, if a agent $i \in N$ selects a given resource $r \in \mathcal{R}$ in both a_i^{ne} and a_i^{opt}, then by deviating from a_i^{ne} to a_i^{opt}, the agent does not add to the load on r, i.e., $|a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}|_r = |a^{\mathrm{ne}}|_r = x_r$. However, if $r \in a_i^{\mathrm{opt}}$ and $r \notin a_i^{\mathrm{ne}}$, then $|a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}|_r = |a^{\mathrm{ne}}|_r + 1 = x_r + 1$. Thus, the left-hand side of the generalized smoothness condition (10) can be written as

$$\begin{aligned}
&\sum_{i=1}^{n} J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) - \sum_{i=1}^{n} J_i(a^{\mathrm{ne}}) + C(a^{\mathrm{ne}}) \\
&= \sum_{r\in\mathcal{R}} (y_r - z_r) F_r(x_r + 1) + z_r F_r(x_r) - \sum_{r\in\mathcal{R}} x_r F_r(x_r) + \sum_{r\in\mathcal{R}} C_r(x_r) \\
&= \sum_{r\in\mathcal{R}} [(z_r - x_r) F_r(x_r) + (y_r - z_r) F_r(x_r + 1) + F_r(x_r)].
\end{aligned} \tag{26}$$

Note that for any game $G \in \mathcal{G}$, it must hold that $z_r \le \min\{x_r, y_r\}$, and $1 \le x_r + y_r - z_r \le n$. We define the set of triplets $\mathcal{I}(n) \subseteq \{0, 1, \dots, n\}^3$ as

$$\mathcal{I}(n) := \{(x, y, z) \in \mathbb{N}^3 \text{ s.t. } 1 \le x + y - z \le n \text{ and } z \le \min\{x, y\}\},$$

and $\gamma(\mathcal{G})$ as the value of the following fractional program:

$$\begin{aligned}
\gamma(\mathcal{G}) := \inf_{\lambda > 0, \mu < 1} \quad & \frac{\lambda}{1 - \mu} \\
\text{s.t.} \quad & (z - x) F^j(x) + (y - z) F^j(x + 1) + C^j(x) \le \lambda C^j(y) + \mu C^j(x), \\
& \forall j = 1, \dots, m, \quad \forall (x, y, z) \in \mathcal{I}(n).
\end{aligned} \tag{27}$$

Observe that, for any pair (λ,μ) in the feasible set of the above fractional program, all games $G\in\mathcal{G}$ are (λ,μ)-generalized smooth. This is because, according to the definition of the set $\mathcal{G}$ and the reasoning in (26), the generalized smoothness condition can be expressed as a weighted sum over the left-hand side expressions of a subset of the $|\mathcal{I}(n)|$ constraints in (27). Thus, $\gamma(\mathcal{G})$ must represent an upper bound on $\mathrm{GPoA}(\mathcal{G})$, i.e., $\gamma(\mathcal{G})\geq \mathrm{GPoA}(\mathcal{G})$.

To conclude step a) of the proof, we show that it is sufficient to define $\gamma(\mathcal{G})$ in (27) over the reduced set of constraints corresponding to $j\in\{1,\dots,m\}$ and triplets in $\mathcal{I}_{\mathcal{R}}(n)\subseteq\mathcal{I}(n)$, where

$$\mathcal{I}_{\mathcal{R}}(n):=\{(x,y,z)\in\mathcal{I}\text{ s.t. }x+y-z=n\text{ or }(x-z)(y-z)z=0\}.$$

For each $j\in\{1,\dots,m\}$ and any $(x,y,z)\in\mathcal{I}(n)$, observe that the constraint in (27) is equivalent to $yF^j(x+1)-xF^j(x)+z[F^j(x)-F^j(x+1)]\leq\lambda C^j(y)+(\mu-1)C^j(x)$. If $F^j(x+1)\geq F^j(x)$, the strictest condition on λ and μ corresponds to the lowest value of z. Thus, $z=\max\{0,x+y-n\}$, and either $(x-z)(y-z)z=0$ or $x+y-z=n$. Otherwise, if $F^j(x+1)<F^j(x)$, then the largest value of z is strictest, i.e., $z=\min\{x,y\}$ and $(x-z)(y-z)z=0$.

Step b): In order to derive the game instances with price of anarchy matching $\gamma(\mathcal{G})$, it is convenient to perform the following change of variables: $\nu(\lambda,\mu):=1/\lambda$ and $\rho(\lambda,\mu):=(1-\mu)/\lambda$. For ease of notation, we will refer to the new variables simply as ν and ρ, respectively, i.e., $\nu=\nu(\lambda,\mu)$ and $\rho=\rho(\lambda,\mu)$. For each $j\in\{1,\dots,m\}$ and each $(x,y,z)\in\mathcal{I}_{\mathcal{R}}(n)$, it is straightforward to verify that the constraints in (27) can be rewritten in terms of ν and ρ as

$$C^j(y)-\rho C^j(x)+\nu[(x-z)F^j(x)-(y-z)F^j(x+1)]\geq 0.$$

Thus, the value $\gamma(\mathcal{G})$ must be equal to $1/\rho_{\text{opt}}$, where ρ_{opt} is the value of the following linear program:

$$\begin{aligned}\rho_{\text{opt}}=\max_{\nu\in\mathbb{R}_{\geq0},\rho\in\mathbb{R}}\quad &\rho\\ \text{s.t.}\quad &C^j(y)-\rho C^j(x)+\nu[(x-z)F^j(x)-(y-z)F^j(x+1)]\geq 0,\\ &\qquad\qquad\forall j=1,\dots,m,\quad\forall(x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).\end{aligned}\tag{28}$$

It is important to note here that while $\gamma(\mathcal{G})$ is the infimum of a fractional program (in, e.g., (27)), the value ρ_{opt} can be computed as a maximum because the feasible set is bounded and closed. Firstly, since $\gamma(\mathcal{G})$ is an upper-bound on the price of anarchy, its inverse (i.e., ρ) must be in the bounded and closed interval between 0 and 1. Additionally, one can verify that ν is not only bounded from below by 0, but also from above by the quantity

$$\begin{aligned}\bar{\nu}:=\min_{j\in\{1,\dots,m\}}\quad\min_{(x,y,z)\in\mathcal{I}_{\mathcal{R}}(n)}\quad&\frac{C^j(y)}{(y-z)F^j(x+1)-(x-z)F^j(x)}\\ \text{s.t.}\quad&(x-z)F^j(x)-(y-z)F^j(x+1)<0\text{ and }C^j(x)=0,\end{aligned}\tag{29}$$

which comes from the constraints in (28) corresponding to triplets $(x,y,z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $C^j(x)=0$ and $(x-z)F^j(x)-(y-z)F^j(x+1)<0$. Such a value must exist, as we assume $C^j(0)=0$. One can verify that any $j \in \{1,\dots,m\}$ and $(x,y,z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $C^j(x)=0$ and $(x-z)F^j(x)-(y-z)F^j(x+1) \geq 0$ correspond to constraints that are satisfied trivially in (28) since $\nu \geq 0$, by definition, and $C^j(y) \geq 0$ for all $y=0,1,\dots,n$, by assumption.

We denote with $\mathcal{H}^j_{(x,y,z)}$ the halfplane of (ν,ρ) values that satisfy the constraint corresponding to $j \in \{1,\dots,m\}$ and $(x,y,z) \in \mathcal{I}_{\mathcal{R}}(n)$, i.e.,

$$\mathcal{H}^j_{(x,y,z)} := \left\{ (\nu,\rho) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \text{ s.t. } \rho \leq \frac{C^j(y)}{C^j(x)} + \frac{1}{C^j(x)} \nu \left[(x-z)F^j(x) - (y-z)F^j(x+1) \right] \right\}.$$

The set of feasible (ν,ρ) is the intersection of these $m \times |\mathcal{I}_{\mathcal{R}}(n)|$ halfplanes. Since the objective is to maximize ρ, any solution $(\nu_{\text{opt}},\rho_{\text{opt}})$ to the linear program (28) must be on the (upper) boundary of the feasible set. We argue below that a solution $(\nu_{\text{opt}},\rho_{\text{opt}})$ can only exist in one of the three following scenarios: (1) at the intersection of two halfplanes' boundaries, where one halfplane has boundary line with positive slope, and the other has boundary line with nonpositive slope; (2) on a halfplane boundary line with positive slope at $\nu_{\text{opt}}=\bar{\nu}$; or (3) at $(\nu_{\text{opt}},\rho_{\text{opt}})=(0,0)$.

We denote with $\partial\mathcal{H}^j_{(x,y,z)}$ the boundary line of the halfplane $\mathcal{H}^j_{(x,y,z)}$, i.e.,

$$\partial\mathcal{H}^j_{(x,y,z)} := \left\{ (\nu,\rho) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \text{ s.t. } \rho = \frac{C^j(y)}{C^j(x)} + \frac{1}{C^j(x)} \nu \left[(x-z)F^j(x) - (y-z)F^j(x+1) \right] \right\}.$$

Observe that the boundary lines of halfplanes corresponding to the choice $y=z=0$ have ρ-intercept equal to zero and slope $xF^j(x)/C^j(x)$. If $F^j(x) \leq 0$ for any $j \in \{1,\dots,m\}$ and $x \in \{1,\dots,n\}$, then an optimal pair (ν,ρ) is trivially at the origin, i.e., $(\nu_{\text{opt}},\rho_{\text{opt}})=(0,0)$ (i.e., scenario (3) above). Note that the ρ-intercept of any halfplane boundary cannot be below 0, as we only consider cost functions such that $C^j(k) \geq 0$ for all k and all j. Otherwise, the maximum value of ρ occurs at the intersection of a boundary line with positive slope and a boundary line with nonpositive slope (i.e., scenario (1) above), or on a boundary line with positive slope at $\nu=\bar{\nu}$ (i.e., scenario (2) above). We illustrate this reasoning in Fig. 2.

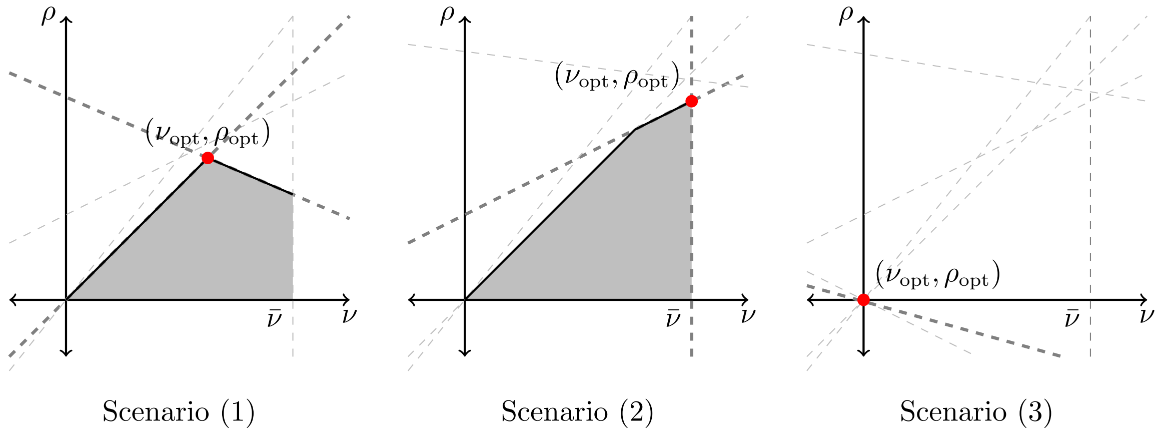
Observe that for scenarios (1) and (2), the pair $(\nu_{\text{opt}},\rho_{\text{opt}})$ is at the intersection of two boundary lines, which we denote as $\partial\mathcal{H}^j_{(x,y,z)}$ and $\partial\mathcal{H}^{j'}_{(x',y',z')}$. The parameters $j,j' \in \{1,\dots,m\}$ and $(x,y,z),(x',y',z') \in \mathcal{I}_{\mathcal{R}}(n)$ satisfy the following:

$$\begin{aligned} \nu_{\text{opt}}[(x-z)F^j(x)-(y-z)F^j(x+1)] &= \rho_{\text{opt}}C^j(x)-C^j(y), \\ \nu_{\text{opt}}[(x'-z')F^{j'}(x')-(y'-z')F^{j'}(x'+1)] &= \rho_{\text{opt}}C^{j'}(x')-C^{j'}(y'), \end{aligned} \tag{30}$$

because $(\nu_{\text{opt}},\rho_{\text{opt}})$ is on both boundary lines. Further, there exists $\eta \in [0,1]$ such that

$$\eta \left[(x-z)F^j(x)-(y-z)F^j(x+1) \right] + (1-\eta) \left[(x'-z')F^{j'}(x')-(y'-z')F^{j'}(x'+1) \right] = 0. \tag{31}$$

Figure 2 The three different scenarios in which optimal solutions $(\nu_{\text{opt}}, \rho_{\text{opt}})$ to (28) can exist.



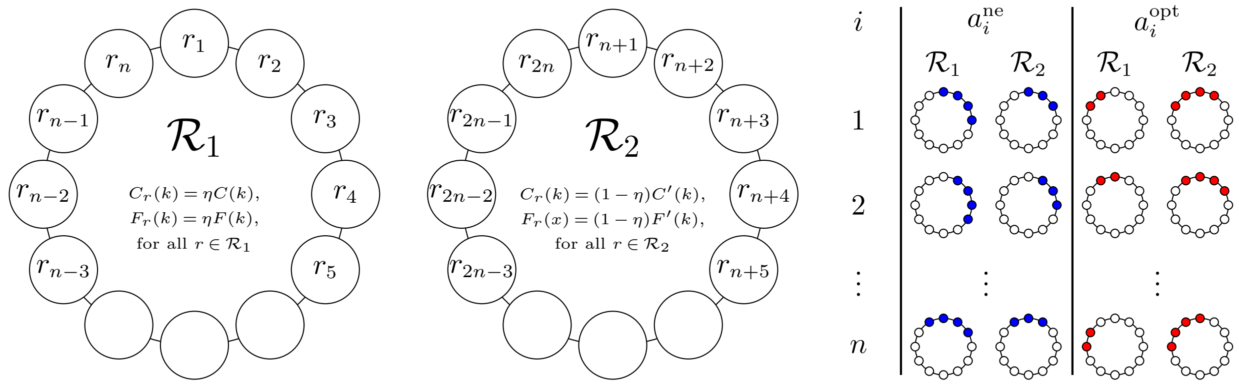
Note. We illustrate the reasoning behind each of the three scenarios for optimal solutions $(\nu_{\text{opt}}, \rho_{\text{opt}})$ to the linear program (28). Since the objective of (28) is to maximize ρ, the optimal values will be at the (upper) boundary of the feasible set, illustrated with a solid, bolded line in each of the examples above. Additionally, the optimal solution $(\nu_{\text{opt}}, \rho_{\text{opt}})$ is marked by a solid, red dot in the illustrations above. In scenario (1), on the left, $(\nu_{\text{opt}}, \rho_{\text{opt}})$ lie on the intersection of a boundary line with positive slope and a boundary line with nonpositive slope. In scenario (2), centre, $(\nu_{\text{opt}}, \rho_{\text{opt}})$ lie on the intersection of a boundary line with positive slope at $\nu = \bar{\nu}$, which is defined in (29). In scenario (3), on the right, there exists a halfplane boundary line with nonpositive slope and ρ-intercept equal to zero, and so $(\nu_{\text{opt}}, \rho_{\text{opt}}) = (0, 0)$. Using the parameters corresponding to the halfplanes on which the pair $(\nu_{\text{opt}}, \rho_{\text{opt}})$ lays, we can construct game instances $G \in \mathcal{G}$ with $\text{PoA}(G) = 1/\rho_{\text{opt}}$ in each of these three scenarios.

(31) holds in scenario (1) because one of the boundary lines has positive slope, i.e., $(x-z)F^j(x) - (y-z)F^j(x+1) > 0$, while the other has nonpositive slope, and in scenario (3) because one boundary line has positive slope while the other is the vertical line $\nu = \bar{\nu}$ which corresponds to a particular choice of $j \in \{1, \ldots, m\}$ and $(x, y, z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $(x-z)F^j(x) - (y-z)F^j(x+1) < 0$ by (29).

Next, for the parameters $j, j' \in \{1, \ldots, m\}$, $(x, y, z), (x', y', z') \in \mathcal{I}_{\mathcal{R}}(n)$, and $\eta \in [0, 1]$ obtained above, we construct a game instance $G \in \mathcal{G}$ such that $\text{PoA}(G) = 1/\rho_{\text{opt}}$.

Let $\mathcal{R}_1 = \{r_1, \ldots, r_n\}$ and $\mathcal{R}_2 = \{r_{n+1}, \ldots, r_{2n}\}$ denote two disjoint cycles of resources. Every resource $r \in \mathcal{R}_1$ has cost function $C_r(k) = \eta C^j(k)$, and agent cost generating function $F_r(k) = \eta F^j(k)$ for all k. Meanwhile, every $r \in \mathcal{R}_2$ has cost function $C_r(k) = (1-\eta)C^{j'}(k)$, and agent cost generating function $F_r(k) = (1-\eta)F^{j'}(k)$ for all k. We define the agent set $N = \{1, \ldots, n\}$, where each agent $i \in N$ has action set $\mathcal{A}_i = \{a_i^{\text{ne}}, a_i^{\text{opt}}\}$. In action a_i^{ne}, the agent i selects x consecutive resources in $\mathcal{R}_1$ starting with r_i, i.e. $\{r_i, r_{(i \bmod n)+1}, \ldots, r_{((i+x-2) \bmod n)+1}\}$, and x' consecutive resources in $\mathcal{R}_2$ starting with resource r_{n+i}. In a_i^{opt}, agent i selects y consecutive resources in $\mathcal{R}_1$ ending with resource $r_{((i+z-2) \bmod n)+1}$, i.e. $\{r_{((i+z-y-1) \bmod n)+1}, \ldots, r_{((i+z-2) \bmod n)+1}\}$, and y' consecutive resources in $\mathcal{R}_2$ ending with resource $r_{n+((i+z'-2) \bmod n)+1}$. We provide an illustration of this

Figure 3 The game instance construction G consisting of n players, and two disjoint cycles $\mathcal{R}_1$ and $\mathcal{R}_2$, as described in the proof of Theorem 3 (i), Part ii).



Note. Consider the set of games $\mathcal{G}(n,\mathcal{Z})$, where n is the maximum number of players and $\mathcal{Z}$ is a set of permissible characteristic cost functions, and suppose that $(\nu_{\mathrm{opt}}, \rho_{\mathrm{opt}})$ satisfy the conditions of scenarios (1) or (3). Further, suppose that the parameters for which (30) and (31) hold are $C, F, C', F' \in \mathcal{Z}$, $(x,y,z) = (4,2,0), (x',y',z') = (3,4,2) \in \mathcal{I}_{\mathcal{R}}(n)$ and $\eta \in [0,1]$. In the above figure, we illustrate the game G' that can be approximated by a game $G \in \mathcal{G}(n,\mathcal{Z})$ such that $\mathrm{PoA}(G) = \mathrm{PoA}(G') = 1/\rho_{\mathrm{opt}}$ according to the reasoning for constructing game instances in scenarios (1) and (3). Observe that each resource $r \in \mathcal{R}_1$ has $C_r(k) = \eta C(k)$, and $F_r(k) = \eta F(k)$, whereas each resource $r \in \mathcal{R}_2$ has $C_r(k) = (1-\eta)C'(k)$, and $F_r(x) = (1-\eta)F'(k)$, for all $k \in \{1,\dots,n\}$. Each player $i \in N$ has two actions a_i^{ne} and a_i^{opt}, as defined in the table on the right. Observe that, by the definitions of the players' actions, every resource in $\mathcal{R}_1$ is selected by 4 players in the allocation $a^{\mathrm{ne}} = (a_1^{\mathrm{ne}},\dots,a_n^{\mathrm{ne}})$, and 3 players in $a^{\mathrm{opt}} = (a_1^{\mathrm{opt}},\dots,a_n^{\mathrm{opt}})$, where no player $i \in N$ has a common resource between its actions a_i^{ne} and a_i^{opt}, i.e., $x_r = 4 = x$, $y_r = 3 = y$, and $z_r = 0 = z$ for all $r \in \mathcal{R}_1$. Similarly, $x_r = 3 = x'$, $y_r = 4 = y'$, and $z_r = 2 = z'$, for each resource $r \in \mathcal{R}_2$.

game construction in Fig. 3. Observe that $a^{\mathrm{ne}} = (a_1^{\mathrm{ne}},\dots,a_n^{\mathrm{ne}})$ satisfies the conditions for a Nash equilibrium,

$$\begin{aligned} J_i(a^{\mathrm{ne}}) &= \eta x F^j(x) + (1-\eta)x'F^{j'}(x') \\ &= \eta[zF^j(x) + (y-z)F^j(x+1)] + (1-\eta)[z'F^{j'}(x') + (y'-z')F^{j'}(x'+1)] = J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}), \end{aligned}$$

which holds by (31). Then, by the above equality and (30),

$$\begin{aligned} 0 &= \sum_{i=1}^n J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) - \sum_{i=1}^n J_i(a^{\mathrm{ne}}) \\ &= \frac{1}{\nu_{\mathrm{opt}}}\left[n\cdot\eta\left[\rho_{\mathrm{opt}}C^j(x) - C^j(y)\right] + n\cdot(1-\eta)\left[\rho_{\mathrm{opt}}C^{j'}(x') + C^{j'}(y')\right]\right] \\ &= \frac{1}{\nu_{\mathrm{opt}}}\left[\rho_{\mathrm{opt}}C(a^{\mathrm{ne}}) - C(a^{\mathrm{opt}})\right], \end{aligned}$$

where $a^{\mathrm{opt}} = (a_1^{\mathrm{opt}},\dots,a_n^{\mathrm{opt}})$. Thus, $\mathrm{PoA}(G) = 1/\rho_{\mathrm{opt}}$.

For scenario (3), observe that $\rho_{\mathrm{opt}} = 0$, and so $1/\rho_{\mathrm{opt}}$ is unbounded. Recall that, in this scenario, there exist $j \in \{1,\dots,m\}$ and $x \in \{1,\dots,n\}$ such that $F^j(x) \le 0$. We use the basis function pair $\{C^j, F^j\}$ to construct a game G with unbounded price of anarchy. Consider a game instance with x agents and resource set $\mathcal{R} = \{r_1, r_2\}$, where $x \in \{1, \ldots, n\}$ is the value that minimizes the function $F(x)$, i.e., $F^j(x) = \min_{k \in \{1,\ldots,n\}} F^j(k) \leq 0$. Every agent $i \in \{1, \ldots, x\}$ has action set $\mathcal{A}_i = \{\{r_1\}, \{r_2\}\}$. The resource r_1 has cost function $C_r(k) = \eta C^j(k)$ and agent cost generating function $F_r(k) = \eta F^j(k)$ for all k. Similarly, the resource r_2 has cost function $C_r(k) = (1-\eta)C^j(k)$ and agent cost generating function $F_r(k) = (1-\eta)F(k)$. It is straightforward to verify that, for η approaching 0 from above, the allocation in which all agents select r_1 is an equilibrium and the price of anarchy is unbounded. $\square$

Proof of Theorem 3 (ii) In the proof of Theorem 3 (i), we show that $\text{GPoA}(\mathcal{G}) \leq 1/\rho_{\text{opt}}$, where $1/\rho_{\text{opt}}$ is the solution to (16). We also demonstrate that $1/\rho_{\text{opt}} \leq \text{PoA}(\mathcal{G})$. $\square$

Appendix B: Proof of Corollary 1.

The following lemma is used in the proof of Corollary 1, but is stated here more generally.

LEMMA 1. *Let $\mathcal{G}$ denote the set of generalized congestion games with at most n agents and basis cost function pairs $\{C^j, F^j\}$, $j = 1, \ldots, m$. There exist scaling parameters $\beta_j > 0$, $j = 1, \ldots, m$ such that,*

$$\text{PoA}(\mathcal{G}') = \max_{j \in \{1,\ldots,m\}} \text{PoA}(\mathcal{G}^j),$$

where we define $\mathcal{G}'$ as the set of generalized congestion games induced by n and basis function pairs $\{C^j, \beta_j \cdot F^j\}$, $j = 1, \ldots, m$, and $\mathcal{G}^j \subseteq \mathcal{G}$ represents the set of generalized congestion games induced by n and the basis function pair $\{C^j, F^j\}$.

Proof. For each $j \in \{1, \ldots, m\}$, we denote by $(\nu^j_{\text{opt}}, \rho^j_{\text{opt}})$ the solution to (16) for the class of games $\mathcal{G}^j$. Note that a positive scaling of the agent cost generating function F^j does not affect the equilibrium conditions of the games in $\mathcal{G}^j$. In particular, for any $j \in \{1, \ldots, m\}$, the set of generalized congestion games $\mathcal{G}'^j$ induced by n and the basis function pair $\{C^j, \beta_j \cdot F^j\}$ for given $\beta_j > 0$ satisfies $\text{PoA}(\mathcal{G}'^j) = \text{PoA}(\mathcal{G}^j)$.

Since, for each $j \in \{1, \ldots, m\}$, any worst-case game in $\mathcal{G}'^j$ is also a member of the set of games $\mathcal{G}'$, it holds that $\text{PoA}(\mathcal{G}') \geq \max_{j \in \{1,\ldots,m\}} \text{PoA}(\mathcal{G}^j)$. If there exists $j \in \{1, \ldots, m\}$ such that $\nu^j_{\text{opt}} = 0$, recall from the proof of Theorem 3 (i) that $\text{PoA}(\mathcal{G}^j)$ is unbounded, and so is $\text{PoA}(\mathcal{G}')$ for $\beta_j = 1$, $j = 1, \ldots, m$. Otherwise, we select $\beta_j = \nu^j_{\text{opt}}$, $j = 1, \ldots, m$, and define $\hat{\rho} := \min_{j \in \{1,\ldots,m\}} \rho^j_{\text{opt}}$. By construction, $(\rho, \nu) = (\hat{\rho}, 1)$ satisfies all the constraints in (16) for $\mathcal{G}'$. Thus, $\text{PoA}(\mathcal{G}') \leq 1/\hat{\rho} = \max_{j \in \{1,\ldots,m\}} \text{PoA}(\mathcal{G}^j)$, which concludes the proof. $\square$

Proof of Corollary 1. For each $j \in \{1, \ldots, m\}$, the function F^j_{opt} maximizes ρ^j_{opt} by the following reasoning, borrowed from Chandan et al. 2019, Thm. 3. For each basis system cost function C^j, we wish to find the function F^j_{opt} that maximizes ρ in (16). Such a function is guaranteed to exist by Paccagnan et al. 2019b, Lem. 5, and finding one is equivalent to finding the solution to

$$\begin{aligned}(F^j_{\text{opt}}, \nu^j_{\text{opt}}, \rho^j_{\text{opt}}) \in &\mathop{\arg\max}_{F\in\mathbb{R}^n, \nu\in\mathbb{R}_{\geq 0}, \rho\in\mathbb{R}} \quad \rho \\ &\text{s.t.} \quad C^j(y) - \rho C^j(x) + \nu[(x-z)F(x) - (y-z)F(x+1)] \geq 0, \quad \forall (x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).\end{aligned}$$

To avoid having to solve a nonlinear program, we combine the decision variables ν and F in $\tilde{F}(k) := \nu F(k)$ to get

$$\begin{aligned}(\tilde{F}^j_{\text{opt}}, \tilde{\rho}^j_{\text{opt}}) \in &\mathop{\arg\max}_{F\in\mathbb{R}^n, \rho\in\mathbb{R}} \quad \rho \\ &\text{s.t.} \quad C^j(y) - \rho C^j(x) + (x-z)F(x) - (y-z)F(x+1) \geq 0, \quad \forall (x,y,z)\in\mathcal{I}_{\mathcal{R}}(n).\end{aligned}$$

Note that $\tilde{F}^j_{\text{opt}} \in \mathbb{R}^n$ must be feasible as $\tilde{F}^j_{\text{opt}}(k) = \nu^j_{\text{opt}} F^j_{\text{opt}}(k)$, and we know that $F^j_{\text{opt}} \in \mathbb{R}^n$ exists. Further, $\tilde{\rho}^j_{\text{opt}} = \rho^j_{\text{opt}}$, as equilibrium conditions are invariant to scaling of F.

For the set of generalized congestion games $\mathcal{G}$ induced by n and basis function pairs $\{C^j, F^j_{\text{opt}}\}$, $j = 1,\dots,m$, and the set of games $\mathcal{G}^j$ induced by n and the basis function pair $\{C^j, F^j_{\text{opt}}\}$, it must hold that $\text{PoA}(\mathcal{G}) \geq \max_{j\in\{1,\dots,m\}} \text{PoA}(\mathcal{G}^j)$. We conclude by proving that the converse also holds, i.e.,

$$\text{PoA}(\mathcal{G}) \leq \max_{j\in\{1,\dots,m\}} \text{PoA}(\mathcal{G}^j).$$

This is true by the construction of F^j_{opt} in Chandan et al. 2019, Thm. 3, and because, in the proof of Lemma 1, we show that $\beta_j = \nu^j_{\text{opt}}$ for all $j = 1,\dots,m$ is a set of scaling parameters that satisfies the above inequality with equality. $\square$

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