The contributions of A. N. Kolmogorov to the theory of turbulence

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Abstract

Two of the papers published by Kolmogorov in 1941 are generally considered to be the origin of modern turbulence theory, including the concepts of scale similarity and of a universal inertial cascade. His third important paper, in 1962, although later superseded, was in the same way the origin of the modern investigations on intermittency. This note summarizes the history of turbulence theory before Kolmogorov, his contributions in these three papers, and his influence on the present understanding of turbulence in fluids.

Resumen

Dos de los artículos publicados por Kolmogorov en 1941 son considerados generalmente como el origen de la teoría moderna de la turbulencia. Estos artículos incluyen los conceptos de semejanza de escala y de una cascada universal inercial. Su tercer artículo importante sobre el tema, publicado en 1962, es igualmente el origen de las investigaciones modernas sobre intermitencia. En esta nota se resume la historia de las teorías sobre la turbulencia antes de Kolmogorov, sus aportaciones en estos tres artículos, y su influencia sobre la comprensión actual del movimiento turbulento de los fluidos.

Introduction

A. N. Kolmogorov wrote very few papers on turbulence theory [17-22], most of them in the decade from 1940 to 1950. In all, they contain less than thirty pages, but their influence on the field has been so profound that it is a source of permanent wonder to fluid mechanicians to learn that Kolmogorov is known primarily as a statistician by most people.

Most of Kolmogorov's turbulence papers have suffered the usual fate of old research, which is to be superseded by later work and to fall out of the citation lists. Reference [20] for example is an early attempt to formulate a turbulence model, on which there is a huge and much better later literature. Reference [21] is an interesting application of turbulence theory to rain formation, which remains current but which is now known to be only one of several equally-important factors in that process. References [18] and [22] were later found to be incorrect for different reasons, although we will see below that the second one stimulated enough discussion to be considered a classic.

References [17] and [19] are on the other hand universally acknowledged to be the origin of modern turbulence theory, so much so that between the years 2000 and 2003, sixty years after their publication, [17] was still cited 451 times, and [19] was cited 100 times. This note briefly traces the history of those two papers, and discusses how they fit into the later development of turbulence theory.

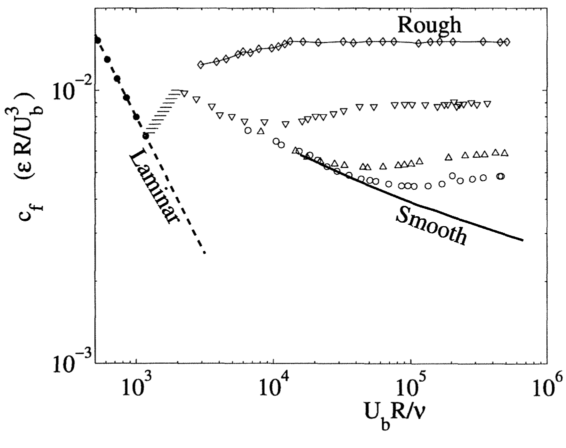
Turbulence before Kolmogorov

Turbulence was not a new subject when [17] was published in 1941. Mankind had probably known that fluids do not flow smoothly at large scales since it became aware of rivers and clouds, and the first quantitative observations of turbulence had been made in the middle of the nineteenth century.

At that time the pressure drop in water pipes was a problem in practical hydraulics which, together with the drag on moving bodies, had been the subject of experimental and theoretical investigations for at least 150 years. It was known that it had two components, one linear and the other approximately quadratic in the fluid velocity, and that only the first one depended on the viscosity of the fluid. In 1854 Hagen and Darcy [4; 8] published independent careful measurements in large pipes. They both noted that the quadratic component was associated with disordered motion in the fluid, and that it became the dominant contribution when the pipes were large and the flow fast enough (see figure 1). They speculated that the increased drag was due to the energy spent in creating the velocity fluctuations.

A little later Boussinesq [2] published a long paper clearly distinguishing between laminar and turbulent flows, and introduced many of the themes that later became associated with turbulence research, such as an enhanced eddy viscosity and the idea that turbulent flows are too complicated for a deterministic description. An account of this early period of hydrodynamics can be found in [34].

Figure 1: Friction coefficient, $c_f = -R\, \partial_x p \;/\; \rho U_b^2$, in pipes as a function of the Reynolds number [26]



The transition between laminar and turbulent flows was clarified by Reynolds [30], who studied experimentally the process by which the flow became disordered. He realized that the criterion for the transition could only depend on a dimensionless group incorporating the viscosity ν of the fluid, and introduced what later became known as the «Reynolds number»,

$$Re = \frac{UR}{\nu}, \tag{1}$$

where U and R are characteristic velocity and length scales. In his second important paper on the subject [31] he used for the first time the decomposition of the flow into mean and fluctuating parts, and stated the turbulence problem as the computation of the mean flow. This restricted view was hugely influential, but it soon became clear that it was inherently incomplete. Consider a flow satisfying the incompressible Navier-Stokes equations

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \rho^{-1} \nabla p = \boldsymbol{f} + \nu \nabla^2 \boldsymbol{v}, \tag{2}$$

and

$$\nabla \cdot \boldsymbol{v} = 0, \tag{3}$$

where $\boldsymbol{v}$ is the velocity, ν is the density, and p is the pressure. The flow is forced by $\boldsymbol{f}\,(\boldsymbol{x};\, t)$, which is assumed to be smooth in some large scale L_f. Boldfaced quantities are vectors, and the corresponding roman symbols denote their scalar magnitudes.

Consider a suitable averaging operation $\langle\rangle$, such as taking the mean over many equivalent experiments, and apply it to equation (2). We obtain for the mean velocity $\boldsymbol{V} = \langle \boldsymbol{v} \rangle$ the equation

$$\partial_t \boldsymbol{V} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \rho^{-1} \nabla P = \boldsymbol{F} + \nabla \cdot \langle -\boldsymbol{v}' \otimes \boldsymbol{v}' \rangle + \nu \nabla^2 \boldsymbol{V}, \tag{4}$$

that has the same structure as (2) but which contains an extra term in the righthand side depending on the average of the product of the fluctuating velocity $v' = v - V$. This «Reynolds stress» tensor cannot be calculated from the averaged equations, and every attempt to write equations for it fails. The evolution equations for the binary products contains triple products; those for the triple products contain fourth order ones, and the hierarchy continues unclosed forever.

The next forty years were dominated by attempts to estimate the Reynolds stresses from quantities related to the mean flow which, perhaps inevitably, soon became driven by analogies with the kinetic theory of gases. The most common approximation was to assume that the Reynolds-stress tensor was proportional to the symmetric part of the velocity gradient, with a scalar proportionality coefficient which had the dimensions of a viscosity. We have seen that such an «eddy» viscosity had been proposed fifty years earlier by Boussinesq, but it was now made quantitative by Prandtl [29; 35], who expressed it as the product of a velocity scale, of the order of the velocity fluctuations, and of a «mixing» length which played the role of the mean free path of kinetic theory. Both scales were illdefined but, through a mixture of sound physical reasoning and empirical tuning, this approach led to the successful approximate computation of many flows of practical interest, and even to basic theoretical results such as the logarithmic behaviour of the mean turbulent velocity near walls [12; 35]. Much of the modern engineering turbulence practice is essentially independent of later theoretical developments, and can be traced to this approach.

The turbulence problem

Turbulence thus first appeared as a technologically important but ultimately unsurprising problem in mechanics. It is after all to be expected that an unsteady, fluctuating, chaotic flow should be difficult to compute, and that it could only be approximately predicted with the analytic tools of the nineteenth century. To understand why turbulence came to be viewed within a few years as «the chief outstanding difficulty of [hydrodynamics]» [24] we need to consider the balance of the mean kinetic energy of the flow.

Contracting (2) with $\boldsymbol{v}$ we obtain

$$\partial_t K + \nabla\Phi = \boldsymbol{f}\cdot\boldsymbol{v} - \nu|\nabla\boldsymbol{v}|^2 \tag{5}$$

where $K = v^2 = 2$ is the kinetic energy per unit mass, Φ is a spatial energy flux, and the two terms in the right-hand side are the work done by the forcing and the energy dissipation per unit mass due to the viscosity.

In a statistically homogeneous turbulent flow it is convenient to average (5) by integrating it over a domain D large enough for the boundary and unsteady terms to be either negligible or easily computable. Under those conditions the energy introduced through the boundary and by the forcing must cancel the dissipation

$$\varepsilon = \nu\langle|\nabla\boldsymbol{v}|^2\rangle = \frac{\nu}{D}\int_D |\nabla\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x}^3. \tag{6}$$

In the case of pipes the only energy input is the work of the pressure gradient on the volumetric flux, and equation (6) takes the form

$$\varepsilon = -\frac{U_b}{\rho}\,\partial_x p = \frac{U_b^3 c_f}{R}, \tag{7}$$

where U_b is the mean velocity in the pipe, R is the pipe radius, and the friction coefficient, $c_f = -R\partial_x p = \rho U_b^2$, is a dimensionless expression of the pressure gradient. These two equations were seen as a serious problem in the nineteenth century, because it follows from figure 1 that the right-hand side of (7) tends to some nonzero constant when $\nu \to 0$, while it is clear from (6) that the left-hand side is proportional to ν. The only way to resolve that inconsistency is for the integral in the right-hand side of (6), which is the meansquare magnitude of the velocity gradient, to depend on the viscosity and to become infinite in the infinite-Reynolds number limit. Although we would say today that the velocity field becomes fractal as $Re \to \infty$, the possibility of such a non-differentiable function was at the time a major challenge for a field which even today is called the «mechanics of continuous media».

Stochastic fields and fractals

The first construction of a continuous non-differentiable object was probably the one presented by Weirstrass in 1872 to the Prussian Academy of Sciences [41], and an increasing number of such «irregular» sets appeared through the end of the nineteenth century and the beginning of the twentieth. By 1910 such objects were well known. The Koch snowflakes, which are now used in many textbooks as elementary examples of fractals, date from 1904 [16]. Such objects were at first considered to be mathematical oddities with no physical application, but already Boltzmann noted in 1898 that non-differentiable functions were natural constructs in statistical mechanics [cited in 36, page 8]. In 1906 Perrin, who would later develop the theory of Brownian motion, published what is essentially a modern statement on the importance of fractal geometry in nature [28]. A compilation of many of these early works can be found in [5], and an entertaining historical outlook is contained in [36]. The modern understanding of fractals in nature, including the coining of the word, is due to Mandelbrot [25].

Kolmogorov was well aware of the work in this field, and later made contributions to it [23], but in 1941 fractals had not yet made a major impact in mechanics. The exception was Richardson [33], who had proposed in 1926 a model for turbulence in terms of a hierarchy of eddies of different sizes, with smaller eddies feeding on larger ones and being fed upon by even smaller ones, eventually reaching a smallest size where viscosity smooths the flow. He was a meteorologist who had proposed a few years earlier the even more revolutionary concept that atmospheric turbulence could be computed numerically [32]. He wrote a program to do so, using a team of human «computers», and it was presumably while organizing those computations that he was led to the consideration of the multiscale nature of turbulence.

His suggestion was not adopted immediately by the community, who was preoccupied at the moment with identifying a single turbulence scale which could be used in Prandtltype mixinglength models. This was the driving force behind the next historically important attempt to clarify turbulence. Taylor [38] and von Kármán [13] moved away from the model of turbulent flows as a collection of interacting «fluid particles», and began to treat the velocity as a continuous stochastic field whose statistical properties could be described in terms of structure functions and power spectra. The theory of stochastic fields was not well-developed at the time, and many of the techniques used today were introduced, or in some cases rediscovered, by the practitioners of this school. They established the equivalence of the alternative descriptions of the flow in terms of correlations and of spectra, wrote the corresponding formulas, and defined the «integral» flow scale as an integral of the velocity correlation function. They also introduced the concept of statistically isotropic and homogeneous turbulence as a canonical case on which statistical analysis could be applied without the complications of real flows.

The contributions of Kolmogorov to turbulence can be considered as the culmination of the early period of this statistical school of thought.

The turbulent energy cascade

To understand the resolution by Kolmogorov of the problem of the energy dissipation we have to introduce the concept of an «eddy», which is a patch of fluid conceptually separated from the rest of the flow, with a size r and a characteristic internal velocity difference u_r (figure 2). There are two time scales in the evolution of an eddy. In the «inertial» time

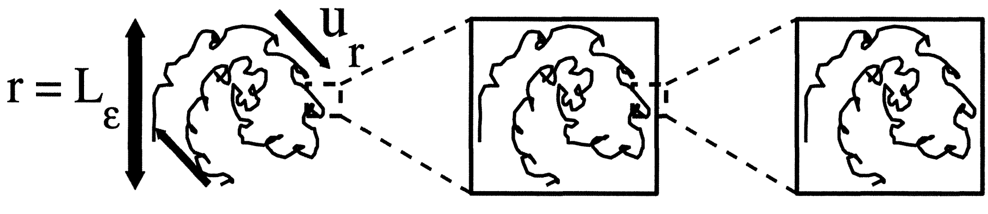
$T_I \sim r\ /\ u_r$ the eddy deforms and loses its individuality. In the «viscous» time $T\nu \sim r^2\ /\ \nu$ its internal velocity differences are smoothed by viscosity. The ratio of the two scales is a Reynolds number

$$Re_r = \frac{T_\nu}{T_I} = \frac{u_r r}{\nu}. \tag{8}$$

When $Re_r \gg 1$ the inertial time is much shorter than the time of viscous decay, and the eddy deforms into something else before it thas a chance of being appreciably damped by viscosity. When $Re_r \gg 1$ the opposite is true.

The first of Kolmogorov's insights was to recover Richardson's idea that eddies could be arranged in a hierarchy of sizes, along which they «decayed» into one another (figure 2). He also remarked that the smaller members of this hierarchy are so far removed from the forces that maintain turbulence at the largest scales that they should eventually become independent of the mechanism, as well as statistically homogeneous and isotropic. He thus reintroduced the simplifying concept of universal isotropic turbulence as a natural property of the small turbulent structures, rather than as an «ad-hoc» particular case. Since the concept of an eddy is not well defined, it is difficult to know which was Kolmogorov's idea of how this decay should be visualized. The closest modern concept is a representation of the flow in terms of simple wavelets, partially localized both in scale and in collocation spaces [6; 39]. As time passes this representation evolves, and the intensive properties of the eddies, such as their energy, are transferred from one scale of the representation into another.

Figure 2: The inertial energy cascade is formed by a hierarchy of inviscid eddies. Because in the absence of viscosity, there is no intrinsic length scale. Each part of a larger eddy is a smaller eddy, statistically similar to its «parent». Energy passes on average from larger to smaller eddies, and the equilibrium mean energy transfer rate has to be the same at all scales.



Kolmogorov noted next that the largest eddies in a turbulent flow, whose characteristic scales are defined as the «integral» length and velocity L_∞ and u_L, have Reynolds numbers of the order of that of the flow, which is always large in turbulence. Viscosity is therefore unimportant in their evolution, and they do not dissipate energy. Their energy has to pass to other scales of the representation once the eddies deform. The energy flux by unit mass from one scale to another can be estimated as the kinetic energy divided by the deformation time,

$$\varepsilon \sim \frac{u_L^2}{T_I} = \frac{u_L^3}{L_\varepsilon}. \tag{9}$$

If the flow is in equilibrium, this transfer rate has to be constant for all the scales in which viscosity is not important, giving the relation between the velocity and the length scales throughout the inviscid cascade,

$$u_r \sim (\varepsilon r)^{1/3}. \tag{10}$$

Each eddy can deform into several smaller ones or merge with others to grow in scale, but in three-dimensional turbulence the energy is transferred in the average from larger to smaller scales [39]. It is easy to see from (10) that this process decreases the Reynolds number of the eddies until, at the point in which

$$Re_r = \frac{u_r r}{\nu} \approx 1, \tag{11}$$

viscosity becomes important and the energy is dissipated. The «Kolmogorov» length and velocity scales of these viscous eddies can be computed from (10) and (11) as,

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \quad u_\eta = (\nu\varepsilon)^{1/4}. \tag{12}$$

It follows from the respective definitions that the ratio between the largest and the smallest scales in the flow is

$$L_\varepsilon/\eta = Re_L^{3/4}, \tag{13}$$

so that the effect of increasing the Reynolds number is not to decrease the rate of energy dissipation, but just to lengthen the range of scales across which the energy has to cascade before reaching the dissipative limit.

We have seen that the effect of the forcing is lost for scales much smaller than the integral length, $r \ll L\varepsilon$. The assumption of strict self-similarity then requires that the flow statistics in that range should only be function of r / η. For scales which are also $r \gg \eta$, viscosity is not important, and statistics can not even depend on η. There is in this «inertial» range of scales no natural length or velocity scale to characterize the eddies, and the only available quantities are r itself and the energy transfer rate ε. Otherwise all the inertial eddies are equivalent, independently of how the turbulence has been forced, or of its Reynolds number Re_L.

For the purpose of comparing with experiments, the universal scaling of the inertial range is usually expressed in terms of the second-order structure function of the velocity difference between two points, $\boldsymbol{\delta v} = \boldsymbol{v(x+r) - v(x)}$, defined as

$$S_2 = \langle \delta v_{\parallel}^2 \rangle = C(\varepsilon r)^{2/3}. \tag{14}$$

where the subindex in $\delta v_{\parallel}^2$ refers to the component of δv parallel to r, the average is taken over x, and C is a universal constant. Equivalently, the one-dimensional energy spectrum of v is

$$E(k) = C_2 \varepsilon^{2/3} k^{-5/3}, \quad \text{where} \quad C_2 = \frac{\Gamma(2/3)}{\pi\sqrt{3}} C. \tag{15}$$

It is important to stress that equations (14) and (15) were *predictions* when they were first written. Their experimental confirmation, shown in figure 3, came almost a decade later, and marked the first clear triumph of turbulence theory.

The Kolmogorov 4/5 equation

While the similarity arguments leading to (14) are suggestive, Kolmogorov felt that they had to be connected directly with the equations of motion, and devoted his second important paper to that subject [19]. He started from the Kármán-Howarth equation, that had been written a few years before [14] and which is essentially an energy equation for δv. Assuming statistical homogeneity, it follows from the Navier–Stokes equations (2) and (3) that

$$\partial_t Q + \frac{1}{4}\nabla_r \cdot \left[\langle |\boldsymbol{\delta v}|^2 \boldsymbol{\delta v} \rangle + \frac{4}{3}\varepsilon \boldsymbol{r} \right] = -\langle \boldsymbol{v} \cdot F_2 \rangle + \nu \nabla_r^2 Q, \qquad (16)$$

Figure 3: (a) Longitudinal second-order structure functions, and (b) one-dimensional longitudinal energy spectra, for a variety of turbulent flows at different Reynolds numbers, normalized in Kolmogorov units. The heavy dashed lines are equations (14) and (15) with $C = 1.37$, as implied by equation (22)

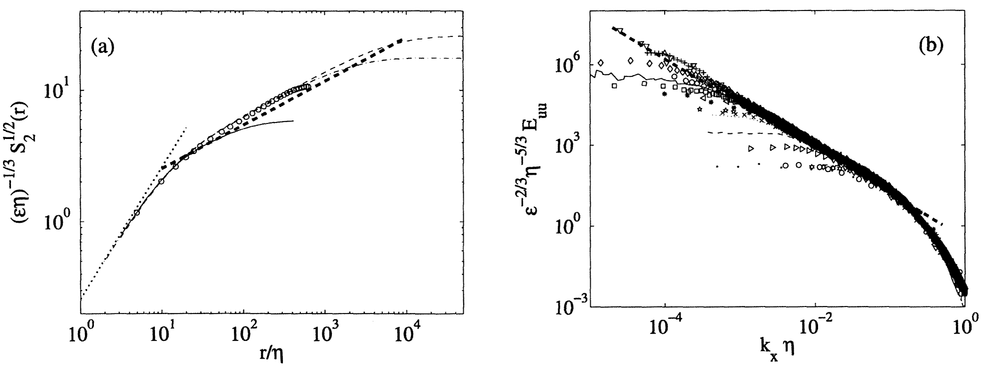
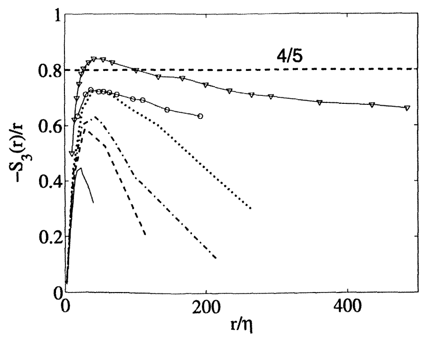


Figure 4: Longitudinal third-order structure functions for several flows with $Re_L = 65 - 3300$. S_3 increases with the Reynolds number. The dotted horizontal line is equation (20)



where the averaging is done over $\boldsymbol{x}$, and all the quantities are therefore functions only of the three-dimensional vector separation $\boldsymbol{r}$. The operator ∇_r is defined in the separation space $\boldsymbol{r}$, and $Q = \langle |\boldsymbol{\delta v}| \rangle / 2$ is representative of the kinetic energy of the velocity fluctuations smaller than r. The forcing term contains the combination

$$F_2 = \frac{\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r}) - 2\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{x}-\boldsymbol{r})}{2}, \tag{17}$$

which is $O(r^2/L_f^2)$, and therefore small if r is small enough. It is then clear that both the forcing and the viscous terms are small in the same inertial range in which (14) is valid, and that, if the flow is in statistical equilibrium, the divergence term has to cancel.

In the case of isotropic turbulence, Kolmogorov showed that the triple product inside the divergence can be written in terms of the third-order structure function of the velocity differences $S_3(r) = \langle \delta v_{||}^3 \rangle$, as

$$\langle |\boldsymbol{\delta v}|^2 \boldsymbol{\delta v} \rangle = \frac{1}{3}\left(\frac{\partial S_3}{\partial r} + \frac{4S_3}{r}\right)\boldsymbol{r}, \tag{18}$$

so that the equilibrium condition is

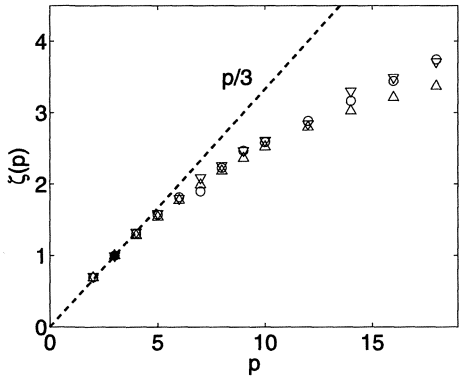
$$\frac{\partial S_3}{\partial r} + \frac{4S_3}{r} + 4\varepsilon = 0, \tag{19}$$

and

$$S_3 = -\frac{4}{5}\varepsilon r. \tag{20}$$

This is to this day one of the few equations without adjustable parameters that have been derived in turbulence. It is checked against experiments in figure 4, and it again fits them in the limit $Re_L \gg 1$.

Figure 5: Scaling exponent of the p-th order structure functions for several flows and Reynolds numbers. The dashed line would be the result of the strict similarity hypothesis [adapted from 9].



Intermittency corrections

Kolmogorov used (20) to provide an alternative derivation for (14), by *assuming* that the skewness $\sigma_3 = S_3/S_2^{3/2}$ of the velocity increment was independent of r within the inertial range, in which case

$$S_2 = \left(\frac{-4\varepsilon r}{5\sigma_3}\right)^{2/3}, \tag{21}$$

and the Kolmogorov constant C is

$$C = \left(\frac{-4}{5\sigma_3}\right)^{2/3}. \tag{22}$$

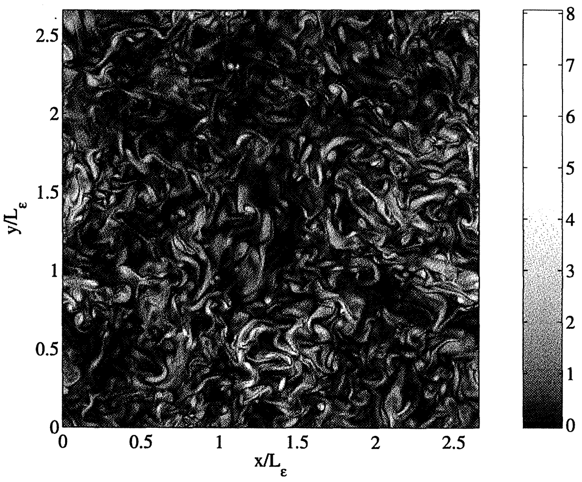
Experimentally $\sigma_3 \approx -0.5$, implying $C \approx 1.37$. This is the value used in figure 3 and agrees reasonably well with experiments.

The argument leading from (20) to (14) is consistent with strict similarity assumption that all the statistics in the inertial range, including the normalized probability density tunetions of the velocity differences, are universal. Unfortunately, that assumption was later shown to disagree with experiments. An argument similar to the one above, applied to the structure function of order p, would predict that

$$\langle \delta v_{\parallel}^3 \rangle \sim (\varepsilon r)^{\zeta(p)}, \quad \text{where} \quad \zeta(p) = p/3. \tag{23}$$

This is not satisfied experimentally, as shown in figure 5. Highorder scaling exponents are substantially lower than (23), and it is not known to this day which is their asymptotic behaviour when $p \gg 1$. Because the effect is larger for the structure functions with higher p, which sample the extreme tails of the probability distributions, this implies that there are rare intermittent events whose velocity differences are much stronger than those predicted by strict similarity, and that this effect increases with decreasing distance. Similar disagreements were soon also found in the probability densities of the velocity gradients [1].

Figure 6: Vorticity magnitude in a section of numerically simulated turbulence at $Re_L = 1880$. The smallest visible features are about 4η [11].



Kolmogorov tried to explain these observations in his final important paper on turbulence [22], which was published in 1962. He proposed that the similarity argument should be modified to use a «local» dissipation averaged over a ball of radius r, instead of the overall mean dissipation ε. He also gave arguments to show that the probability distribution of the local dissipation should be log-normal, and computed the resulting scaling exponents $\zeta(p)$ of the structure functions. That prediction was eventually also found to disagree with experiments in the large-p limit, and the log-normality argument was challenged on the grounds that it relied on the application of the central-limit theorem to extreme events [27]

However, even if flawed in detail, Kolmogorov's 1962 paper had an immense influence, and intermittency continues to be an active research field which has generated some of the most beautiful theoretical results in turbulence theory. Reviews can be found in [9; 10; 37], and the book by Frisch [7] devotes several chapters to it. In modern notation the question is that, even if the cascade argument and equation (20) show that the mean singularity of the velocity field $\delta v \sim r^{1/3}$ in the zero-viscosity limit, they say nothing about the geometry of the support of that singularity; i.e., whether it is concentrated in a very-singular small set, or whether it is distributed uniformly over the whole flow. Kolmogorov original model corresponds to the latter alternative.

A recent development is that numerical simulation has allowed us to compute some turbulent flows in detail, not just as statistical objects (see figure 6). The intermittent events are now known to be elongated vortices with radii of the order of the Kolmogorov scale, which survive longer than could be expected from dimensional arguments because they are essentially stable objects, which «turn around» many times before breaking down [11; 40].

The legacy of Kolmogorov

Turbulence has undoubtedly progressed a lot since the time of Kolmogorov. Theoretical tools and, above all, experimental techniques and numerical simulations, have allowed us to study turbulence in a detail that would have been unthinkable in 1941. Direct numerical simulations, which solve the Navier-Stokes equations exactly, except for the approximations implicit in numerical analysis, have in particular allowed us to view turbulence as a far more deterministic phenomenon than what is implied by a statistical description.

Even if there is clearly a lot left to do, many turbulent flows can now be considered as essentially understood, and it is common for example to control some of them, such as mixing layers and jets, to increase mixing or to decrease noise.

Under those circumstances it may be asked whether anything is left of the contributions of Kolmogorov to the field. We saw above that engineering predictions methods appeared before Kolmogorov, and that they do not use too much of the statistical machinery developed in the 1940's.

That statement is deceptive. The key contributions of Kolmogorov, in particular the ideas of a turbulent cascade and of the multiscale nature of turbulence, and the identification of the energy transfer rate as the key parameter controlling the inertial range, are at the root of any argument dealing with turbulence today. The name of the most common engineering turbulence model ($k - \varepsilon$) clearly shows how much engineering practice is indebted to him.

Turbulence theory is similarly unthinkable without the cascade argument. Perhaps the most important contribution of Kolmogorov's 1941 papers was to take away from turbulence the «mysterious» character of the infinite-Reynolds number limit. We have understood since then that energy is pumped into the large scales and transmitted to viscous dissipation as a fractal. Those papers also showed how to «construct» that fractal, and how to compute its similarity exponent. They explained clearly for the first time that it is not important how energy is dissipated but how it is transferred, and that, if it is allowed to cascade far enough, dissipation will eventually take over at some sufficiently small scale.

Only when turbulence had been reduced in this way to a regular physical phenomenon did other approaches become possible. All the later work on intermittency, for example, has been essentially an enrichment of the cascade argument. A modern and very fruitful approach to turbulence has been its study in terms of quasi-deterministic structures, mostly for the largest flow scales [3; 15]. That has been the driving force behind the control work mentioned above, but it has only been possible because the relation of the large scales with the dissipation was by then well understood.

In many ways, everything we know today about turbulent flows rests on Kolmogorov's papers of 1941.

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