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Magnetic body- and surface-forces on electrically conducting fluids

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It is well known that when an electrically conducting fluid sweeps across the lines of force of a magnetic field a magnetic body-force is produced which acts at each point within a considered closed surface in the fluid and that the total force exerted on this body of fluid is the same as that due to two surface distributions, one of which is a certain tensile surface force per unit area directed along the lines of force, the other being in the nature of a certain hydrostatic pressure. In this paper it is further shown that the total couple exerted on the considered body of fluid is the same in both cases. This additional property would seem to reinforce the dynamical equivalence of the two models and the mathematical analyses that are made in works on magneto-fluid dynamics in which the magnetic body force distribution per unit volume is replaced by the equivalent 'Maxwell stress' system of surface forces. Throughout, the treatment employs the methods of Cartesian tensor analysis. These methods are vividly illustrated in the analysis of the problem.

1. Introduction

Temple [1, pp. 22, 23] considers the motion of a body of fluid particles of volume v bounded by a closed surface S which always contains the same particles so that the mass of the fluid body is constant. He shows that if each point of S is subjected to a hydrostatic pressure p (always directed along the inward normal to S), then the net effect of this surface distribution is dynamically equivalent to the action of a body-force per unit volume of $-\operatorname{grad} p$ at each point of v in the sense that both distributions produce the same resultant force and the same resultant couple on the fluid body. The general motion of any body, solid or fluid, is dependent on the resultant force and on the resultant couple.

The object of this paper is to show that when an electrically conducting fluid cuts the lines of force of a magnetic field in its motion the magnetic body-force produced is equivalent in Temple's sense to the well-known Maxwell distributions involving two kinds of surface forces. The equality of the resultant forces for the two modes is well known (see, e.g. [2, p. 32]), but establishment of the equality of the couples exerted on the fluid body seems not to have been undertaken.

2. Magnetic stress analysis in a moving conducting fluid

We follow throughout the motion of a body of conducting fluid in the sense defined by Temple [1, p. 3], that is, we follow the motion of a volume v of fluid enclosed within a closed surface S and always containing the same number of fluid particles, so that the total mass of fluid within S is constant.

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When the magnetic induction at any point P of v or S is $\mathbf{B}$ and the magnetic permeability is μ, the magnetic body force per unit volume at P is known to be [2, p. 32] $(\operatorname{curl}\mathbf{B})\wedge\mathbf{B}/\mu$. This gives rise to a total force $\mathbf{F}$ on the fluid body where

$$\mathbf{F}=\frac{1}{\mu}\int_v (\nabla\wedge\mathbf{B})\wedge\mathbf{B}\,dv \tag{1}$$

and to a total couple whose moment about O is

$$\mathbf{G}=\frac{1}{\mu}\int_v \mathbf{r}\wedge\{(\nabla\wedge\mathbf{B})\wedge\mathbf{B}\}\,dv \tag{2}$$

We express both $\mathbf{F}$ and $\mathbf{G}$ as the sum of two surface integrals taken over S and from them derive surface distributions over S that are equivalent to the volume distribution within v. In this treatment the methods of Cartesian tensor analysis are used in which the Cartesian coordinates of a point P are specified by x_i in a rectangular frame of axes OX_i $(i=1, 2, 3)$. The unit vectors in the directions $\overline{\mathrm{OX}}_i$ are denoted by $\mathbf{e}_i$ $(i=1, 2, 3)$ and the component of a vector $\mathbf{F}$ in direction $\overline{\mathrm{OX}}_i$ will be denoted by F_i. Needless to say the summation convention is employed throughout.

In tensor form (1) may be written

$$F_i=\frac{1}{\mu}\int_v \varepsilon_{ijk}(\nabla\wedge\mathbf{B})_j B_k\,dv \tag{3}$$

where ε_{ijk} is the alternating symbol. Also the jth component of curl $\mathbf{B}$ is

$$(\nabla\wedge\mathbf{B})_j=\varepsilon_{jpq}\frac{\partial}{\partial x_p}B_q=\varepsilon_{jpq}B_{q,p}$$

Thus (3) becomes

$$F_i=\frac{1}{\mu}\int_v \varepsilon_{ijk}\varepsilon_{jpq}B_{q,p}B_k\,dv \tag{4}$$

Now

$$\begin{aligned}
&\varepsilon_{ijk}\varepsilon_{jpq}B_{q,p}B_k\\
&=\varepsilon_{jki}\varepsilon_{jpq}B_{q,p}B_k\\
&=(\delta_{kp}\delta_{iq}-\delta_{kq}\delta_{ip})B_{q,p}B_k\\
&=B_{i,k}B_k-B_{k,i}B_k\\
&=\frac{\partial}{\partial x_k}(B_iB_k)-B_iB_{k,k}-\frac{\partial}{\partial x_i}(\tfrac{1}{2}B_k^2)\\
&=\frac{\partial}{\partial x_k}(B_iB_k)-\frac{\partial}{\partial x_i}(\tfrac{1}{2}\mathbf{B}^2)
\end{aligned}$$

using the Maxwell equation $B_{k,k}=\operatorname{div}\mathbf{B}=0$ and also $\mathbf{B}^2=B_k^2$.
Thus (4) becomes

$$F_i=\frac{1}{\mu}\int_v\frac{\partial}{\partial x_k}(B_iB_k)\,dv-\frac{1}{2\mu}\int_v\frac{\partial}{\partial x_i}(\mathbf{B}^2)\,dv \tag{5}$$

On using tensor forms for the Gauss divergence theorem we obtain from (5)

$$F_i = \frac{1}{\mu}\int_S n_k (B_i B_k)\, dS - \frac{1}{2\mu}\int_S n_i \mathbf{B}^2\, dS \tag{6}$$

where $\mathbf{n} = n_i\, \mathbf{e}_i$ is the unit vector normal to a surface element δS of S and drawn outwards from v. Since $n_k B_k = \mathbf{n}\cdot\mathbf{B}$, equation (6) gives at once

$$\mathbf{F} = \frac{1}{\mu}\int_S (\mathbf{n}\cdot\mathbf{B})\mathbf{B}\, dS - \frac{1}{2\mu}\int_S \mathbf{B}^2 \mathbf{n}\, dS \tag{7}$$

Equation (7) expresses the total force $\mathbf{F}$ as being due to the following two surface distributions acting on each element δS of S:

(i) a tensile force $(\mathbf{n}\cdot\mathbf{B})\ \mathbf{B}/\mu$ per unit area along the $\mathbf{B}$ lines;
(ii) a hydrostatic pressure $\mathbf{B}^2/(2\mu)$ directed along the inward normal vector $-\mathbf{n}$ to δS.

The question which now arises, and which seems not to have been dealt with previously, is whether the compound surface distributions (i), (ii) (the 'Maxwell stress'), which are equivalent to the stated magnetic body force per unit volume so far as the estimation of $\mathbf{F}$ is concerned, also give the same estimates of $\mathbf{G}$. We show that in fact they do so.

In tensor form equation (2) may be written

$$G_i = \frac{1}{\mu}\int_v \varepsilon_{ijk} x_j \{(\nabla\wedge\mathbf{B})\wedge\mathbf{B}\}_k\, dv \tag{8}$$

where $\mathbf{r} = x_i \mathbf{e}_i$ is the position vector of either a volume element δv of v or a surface element δS of S. Now

$$\begin{aligned}\{(\nabla\wedge\mathbf{B})\wedge\mathbf{B}\}_k &= \varepsilon_{kpq}(\nabla\wedge\mathbf{B})_p B_q \\ &= \frac{\partial}{\partial x_q}(B_k B_q) - \frac{\partial}{\partial x_k}(\tfrac{1}{2}\mathbf{B}^2)\end{aligned}$$

using an adaptation of the previous analysis. Thus equation (8) becomes

$$G_i = \frac{1}{\mu}\int_v \varepsilon_{ijk} x_j \frac{\partial}{\partial x_q}(B_k B_q)\, dv - \frac{1}{2\mu}\int_v \varepsilon_{ijk} x_j \frac{\partial}{\partial x_k}(\mathbf{B}^2)\, dv \tag{9}$$

Now

$$x_j \frac{\partial}{\partial x_q}(B_k B_q) = \frac{\partial}{\partial x_q}(x_j B_k B_q) - \delta_{jq} B_k B_q$$

$$x_j \frac{\partial}{\partial x_k}(\mathbf{B}^2) = \frac{\partial}{\partial x_k}(x_j \mathbf{B}^2) - \delta_{jk}\, \mathbf{B}^2$$

Thus (9) becomes

$$G_i = \frac{1}{\mu}\int_v \varepsilon_{ijk} \frac{\partial}{\partial x_q}(x_j B_k B_q)\, dv - \frac{1}{2\mu}\int_v \varepsilon_{ijk} \frac{\partial}{\partial x_k}(x_j \mathbf{B}^2)\, dv \tag{10}$$

when use is made of the simplifications

$$\varepsilon_{ijk}\delta_{jq}B_kB_q=\varepsilon_{ijk}B_kB_j=0$$
$$\varepsilon_{ijk}\delta_{jk}\mathbf{B}^2=\varepsilon_{ijj}\mathbf{B}^2=0$$

since $\varepsilon_{ijj}=0$. Converting the volume integrals of equation (10) into equivalent surface integrals by means of the Gauss divergence theorem, we obtain

$$\begin{aligned}G_i&=\frac{1}{\mu}\int_S \varepsilon_{ijk}n_q(x_jB_kB_q)\,dS-\frac{1}{2\mu}\int_S \varepsilon_{ijk}\,n_k(x_j\,\mathbf{B}^2)\,dS\\&=\frac{1}{\mu}\int_S \varepsilon_{ijk}x_jB_k(\mathbf{n}\cdot\mathbf{B})\,dS-\frac{1}{2\mu}\int_S \varepsilon_{ijk}x_jn_k\mathbf{B}^2\,dS\end{aligned}$$

This leads at once to the vectorial form

$$\mathbf{G}=\frac{1}{\mu}\int_S(\mathbf{r}\wedge\mathbf{B})(\mathbf{n}\cdot\mathbf{B})\,dS-\frac{1}{2\mu}\int_S(\mathbf{r}\wedge\mathbf{n})\mathbf{B}^2\,dS \tag{11}$$

The first integral in (11) is the moment about O of the tensile force (i): the second integral is the moment about O of the hydrostatic pressure (ii).

We have thus shown that the magnetic body force $(\nabla\wedge\mathbf{B})\wedge\mathbf{B}/\mu$ per unit volume at each point P of v is equivalent to the two surface distributions (i), (ii) in the senses that both give the same total force $\mathbf{F}$ and the same total couple $\mathbf{G}$ on the fluid body and so, for mathematical analysis of the conducting fluid motion, either distribution may be used.

In conclusion it would appear that in general body forces do not satisfy the kind of conditions of the above investigation. Let $\mathbf{f}$ be the body force per unit volume. It is shown in [3, p. 74] that any such vector may be expressed in the form

$$\mathbf{f}=\nabla\phi+\nabla\wedge\mathbf{H}$$

where ϕ is a suitable scalar function and $\mathbf{H}$ a suitable vector function (Helmholtz's theorem). This gives for the total body force on the fluid body

$$\begin{aligned}\mathbf{F}&=\int_v\nabla\phi\,dv+\int_v\nabla\wedge\mathbf{H}\,dv\\&=\int_S\mathbf{n}\phi\,dS+\int_S\mathbf{n}\wedge\mathbf{H}\,dS\end{aligned}$$

which produces appropriate equivalent surface force distributions. However, the total couple on the fluid body is

$$\mathbf{G}=\int_v\mathbf{r}\wedge\nabla\phi\,dv+\int_v\mathbf{r}\wedge(\nabla\wedge\mathbf{H})\,dv$$

For the first integral it is easy to show that $\mathbf{r}\wedge\nabla\phi=-\nabla\wedge(\phi\mathbf{r})$ and then to establish

$$\int_v\mathbf{r}\wedge\nabla\phi\,dv=\int_S\mathbf{r}\wedge(\phi\mathbf{n})\,dS$$

in accordance with the expected pattern. But no similar reduction seems to be possible for the second integral.

References

[1] TEMPLE, G., 1958, *An Introduction to Fluid Dynamics* (Clarendon Press).
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[3] WEATHERBURN, C. E., 1957, *Advanced Vector Analysis* (G. Bell & Sons).