THE RHEOLOGY AND FLOW OF VISCOPLASTIC MATERIALS

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1. INTRODUCTION

During the past several decades the emphasis in rheology and continuum mechanics has been on one-phase materials, with particular attention to polymer solutions and polymer melts. Slurries, pastes, and suspensions, frequently encountered in industrial problems, have received less attention than they deserve. Many of these materials have a "yield stress", a critical value of stress below which they do not flow; they are sometimes called "viscoplastic materials". In Table I we list some of the materials that fall into this category, along with information about research papers in which they have been studied.

We begin in § 2 by giving some of the standard empirical expressions for the stress tensor of viscoplastic materials; for the sake of simplicity this discussion is restricted to shear flows. In § 3 we show how the models have to be written for more complex flows, and the von Mises-Hencky yield criterion is introduced. Then in § 4 we give a summary of the elementary solutions of flow problems for the Bingham model. This is followed in § 5 by a similar presentation for heat transfer problems. Then in § 6 a summary is given of solutions for models other than the Bingham model as well as references where numerical solutions may be found.

It is apparent that an enormous amount of work has been done on viscoplastic materials. As far as we know a summary of the type offered here has not appeared elsewhere, although there have been more limited reviews within various fields of application /2, 42, 80, 97, 111, 148, 136/. Recent textbooks and monographs on rheology and continuum mechanics have devoted little or no space to the subject. One has to go back to the books by Reiner /141, 143/, Prager /137/, Ulbrecht and Mitschka /190/, Fredrickson /49/, Skelland /171/, and Wilkinson /209/, to find any substantial discussions of materials with a yield stress. Anyone wishing to do further work on the Bingham model or other models would do well to read the pertinent chapters in these books. Also attention should be paid to the series of eight carefully written publications by Oldroyd /119–126/ which include the formulation of viscous flow problems, boundary-layer theory for slow motions, and illustrations of methods of solution of non-trivial flow problems, both without and with time dependence. We also call attention to the book, in Russian, by Ogibalov and Mirzadzhanzade /115/ which is an extensive treatise on the research done by scientists in the U.S.S.R. on the unsteady-state flow of viscoplastic materials. Also the book, in Japanese, by Tomita /189/ contains some useful material on the Bingham and Casson models along with comparisons with experimental data.

TABLE I

Materials with Yield Stresses

Material	Constitutive Equation	Experiment	Author(s)	Reference
clay/water	Bingham	tube flow	Heywood and Richardson	67
	Bingham	tube flow, settling	Thomas	183
	Bingham	tube flow	Thomas	184
	Bingham, Crowley-Kitzes	tube flow	Thomas	185
	Bingham	Couette	Firth and Hunter	45-47
A	Bingham	tube flow, Couette	Umeya	191
	Bingham	Couette	Weymann, et al.	205
	Bingham	Couette	Weymann	204
	Bingham	tube flow, Couette	Caldwell and Babbitt	24
	Bingham	drag on bodies	Brookes and Whitmore	21
	Bingham	tube flow	Dobrychenko, et al.	39
		Ostwald viscometer	Bingham and Durham	14

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
	Bingham	tube flow	Bingham	12
	Bingham	drag on bodies, Couette	Pazwash and Robertson	133
clay/methanol		Ostwald viscometer	Bingham and Durham	14
cement/clay / water	Bingham, Shul'man	tube flow, Couette	Bukhman, et al.	23
drilling mud	Bingham	Couette	Van Olphen	196
sewage sludge	Bingham	tube flow, Couette	Caldwell and Babbitt	24
iron oxide/ ethylene glycol	Casson	cone and plate	Smith and Bruce	176
thorium dioxide/ methanol	Bingham	tube flow, settling	Thomas	183

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
metal oxides/	Bingham	tube flow	Thomas	184
water	Bingham	tube flow, w/wo heat transfer	Thomas	182
	Bingham	tube flow, settling	Thomas	183
	Bingham, Crowley-Kitzes	tube flow	Thomas	185
	Bingham	tube flow	Thomas	186
	Bingham	Couette	Firth and Hunter	45-47
	Crowley-Kitzes	tube flow	Murdock and Kearsey	114
	Bingham	tube flow, Couette	Umeya	191
silica or silicate/	Bingham	Couette	Firth and Hunter	45-47
Newtonian liquid	Herschel-Bulkley	Couette	Ferraro	44
	Bingham	tube flow, Couette	Umeya	191
	Herschel-Bulkley	tube, Couette, cone and plate	Vinogradov, et al.	199

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
	Casson	tube flow	Walawender, et al.	202
sulphur/water	Bingham	tube flow	Govier and Charles	54
barite/water	Bingham		Earnshaw and Spronson	41
graphite/water	Bingham	tube flow	Thomas	184
	Bingham	tube flow, settling	Thomas	183
		Ostwald viscometer	Bingham and Durham	14
carbon black/oil	Bingham	tube flow	Thomas	186
	Bingham	Couette	Firth and Hunter	45-47
coal/Newtonian	Bingham	tube flow	Sacks, et al.	149
liquid	Bingham	Couette	Munro, et al.	113

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
	Bingham, Casson, Herschel-Bulkley	Couette	Darby and Rogers	36
grease (oil/soap)	Bingham	tube flow	Mahncke and Tabor	90
	Herschel-Bulkley	tube, Couette, cone and plate	Vinogradov, et al.	199
	Bingham	tube, penetrometer	Singleterry and Stone	170
water/benzene	Bingham	Couette	Firth and Hunter	45-47
tritolyl phosphate/castor oil	Casson, Herschel-Bulkley	(published data)	Scott Blair	159
xanthan gum/water		cone and plate	Whitcomb, et al.	207
		cone and plate	Whitcomb	206
PVC/organic liquids		cone and plate	Willey and Macosko	210

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
polymer latex/	Bingham	Couette	Firth and Hunter	45-47
Newtonian liquid	Casson	parallel plates	Krieger and Eguiluz	81
	Bingham	Couette	Hunter and Frayne	71
inorganic solid/	modified Casson	Couette	Matsumoto, et al.	100
polymer/solvent	modified Casson	cone and plate	Onogi, et al.	128
inorganic solid/	Casson	tube flow, Couette	Kataoka, et al.	75
polymer	Plastic-viscoelastic		White and Tanaka	208
		tube, Couette, cone and plate, parallel plates	Vinogradov, et al.	200
		tube, cone and plate	Chapman and Lee	27
glass/glycerol	Bingham	Couette	Van Kao, et al.	194, 195

Table I Cont/..

Material	Constitutive Equation	Experiment	Author(s)	Reference
glass/polymer	Bingham	Couette	Van Kao, et al.	194,195
		cone and plate	Kataoka, et al.	74
rubber/benzene	Herschel-Bulkley	Ostwald and Bingham viscometers	Herschel and Bulkley	65
acrylic rubber/ poly(ethyl acrylate)	Bingham	tube, bicone and plates	Rosen and Rodriguez	147
styrene-divinyl-benzene copolymer/	Casson	Couette	Onogi, et al.	127
polystyrene/DEP	Casson	cone and plate	Kataoka, et al.	74
paint	Bingham	tube flow	Bingham	15
printing ink	Casson	cone and plate	Casson	25

Table I Cont/..

Material	Constitutive	Experiment	Author(s)	Reference
plastic rocket propellant	Bingham	squeezing flow	Dukes	40
meat extract	Bingham		Denn	38
fermentation broth	Bingham	parallel plates	Richards	145
			Deindoerfer and Nest	37
butter			Kawanari, et al.	76
orange juice concentrate	gener. Casson, Herschel-Bulkley	tube flow	Mizrahi and Firstenberg	108
sweet potato puree	Herschel-Bulkley	Couette	Rao, et al.	139
mayonnaise B	Herschel-Bulkley	cone and plate	Tiu and Boger	187

Table I Cont/..

Material	Constitutive	Experiment	Author(s)	Reference
apple sauce C	Casson, Herschel-Bulkley	Couette	Charm	28
tomato sauce	Bingham	tube flow, stokes flow	Ansley and Smith	5
tomato puree	Casson, Herschel-Bulkley	Couette	Charm	28
blood	Casson	tube flow	Walawender, et al.	202
	Casson	tube flow, Couette	Merrill, et al.	102
	Casson	Couette	Merrill, et al.	103
	Casson, Herschel-Bulkley	(published data)	Scott Blair	159
	Casson	Couette, cone and plate	Charm and Kurland	29
	Casson	tube, cone and plate, parallel plates, sedimentation, conicylinder	Stoltz and Larcan	179

Notes: A: Sodium silicate added B: To account for time dependent effect, $\tau_{yx} = \lambda(t)\,[\tau_0 + k_0 |\dot{\gamma}_{yx}|^n]$

C: In food literature, the Herschel-Bulkley model is often labelled Charm's model.

2. USEFUL EMPIRICISMS FOR THE STRESS IN SHEAR FLOWS

For structurally simple fluids (gases and liquids of low molecular weight) the relation between shear stress and velocity gradient in a shear flow $v_x = v_x(y)$ (see Fig. 1) is the Newtonian expression:

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \tag{2.1}$$

in which μ is the viscosity of the fluid. We use the sign convention adopted elsewhere /17-19, 69/ that τ_{yx} is the force in the positive x-direction on a plane perpendicular to the y-direction, the force being transmitted by the material below the plane to the material above the plane.

For pastes, slurries, and suspensions a number of empirical relations have been suggested to replace Eq. (2.1). The three most widely used are the Bingham, Casson, and Herschel-Bulkley equations.

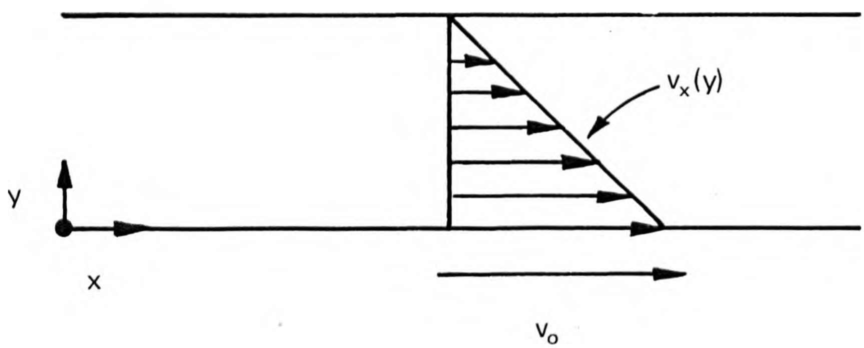

Fig. 1. Flow between two parallel plates. Here the velocity gradient dv_x/dy is negative, and the shear stress τ_{yx} is positive.

a. *Bingham Model:*

The simplest and most widely used model is the two-parameter model proposed by Bingham /12, 13, 19 (p. 11)/:

$$\tau_{yx} = \pm\ \tau_0 - \mu_0 \frac{dv_x}{dy} \qquad |\tau_{yx}| \geqslant \tau_0 \tag{2.2}$$

$$dv_x/dy = 0 \qquad |\tau_{yx}| \leqslant \tau_0 \tag{2.3}$$

Here τ_0 is the "yield stress" and μ_0 is the "plastic viscosity"; the positive sign in Eq. (2.2) is to be used when τ_{yx} is positive, and the negative sign is used when τ_{yx} is negative. (Actually the Bingham model is a special case of a much older model proposed by Schwedoff /141, 143, 156/: $\tau_{yx} + \lambda_0 \partial\tau_{yx}/\partial t = \pm\tau_0 - \mu_0(dv_x/dy)$, in which λ_0 is a time constant.) Several dimensionless groups are defined using a characteristic length D, a characteristic velocity V, and the parameters in the Bingham model: the Reynolds number, $Re = DV\rho/\mu_0$; the Bingham number, $Bi = \tau_0 D/\mu_0 V$; and the Hedström number, $He = \tau_0 D^2 \rho/\mu_0^2$.

b. Casson Model:

Another two-constant model is that proposed by Casson /25/ for pigment-oil suspensions:

$$\sqrt{\pm\tau_{yx}} = \sqrt{\tau_0} + \sqrt{\mu_0}\,\sqrt{\mp(dv_x/dy)} \qquad |\tau_{yx}| \geqslant \tau_0 \qquad (2.4)$$

$$dv_x/dy = 0 \qquad |\tau_{yx}| \leqslant \tau_0 \qquad (2.5)$$

in which the upper signs are to be used when τ_{yx} is positive, and the lower signs when τ_{yx} is negative. This model has often been used for describing blood /29, 85, 86, 103, 116, 159, 160, 179/. The dimensionless group $\tau_0 D/\mu_0 V$ that appears when solving problems for equation 2.4 is called the Casson number, Ca.

c. Herschel-Bulkley Model:

A simple generalization of the Bingham model is the three-constant Herschel-Bulkley /65, 66/ equation:

$$\tau_{yx} = \pm\tau_0 - m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} \qquad |\tau_{yx}| \geqslant \tau_0 \qquad (2.6)$$

$$dv_x/dy = 0 \qquad |\tau_{yx}| \leqslant \tau_0 \qquad (2.7)$$

The plus sign is to be used when $\tau_{yx} > 0$, and the minus sign when $\tau_{yx} < 0$.

The three models above are sketched in Figure 2. Note that the symbol τ_0 is used in all three models for the yield stress. The symbol μ_0 in Eqs. (2.2) and (2.4) have different meanings. We have tried to be particularly careful to write the stress expressions in such a way that they may be applied when the

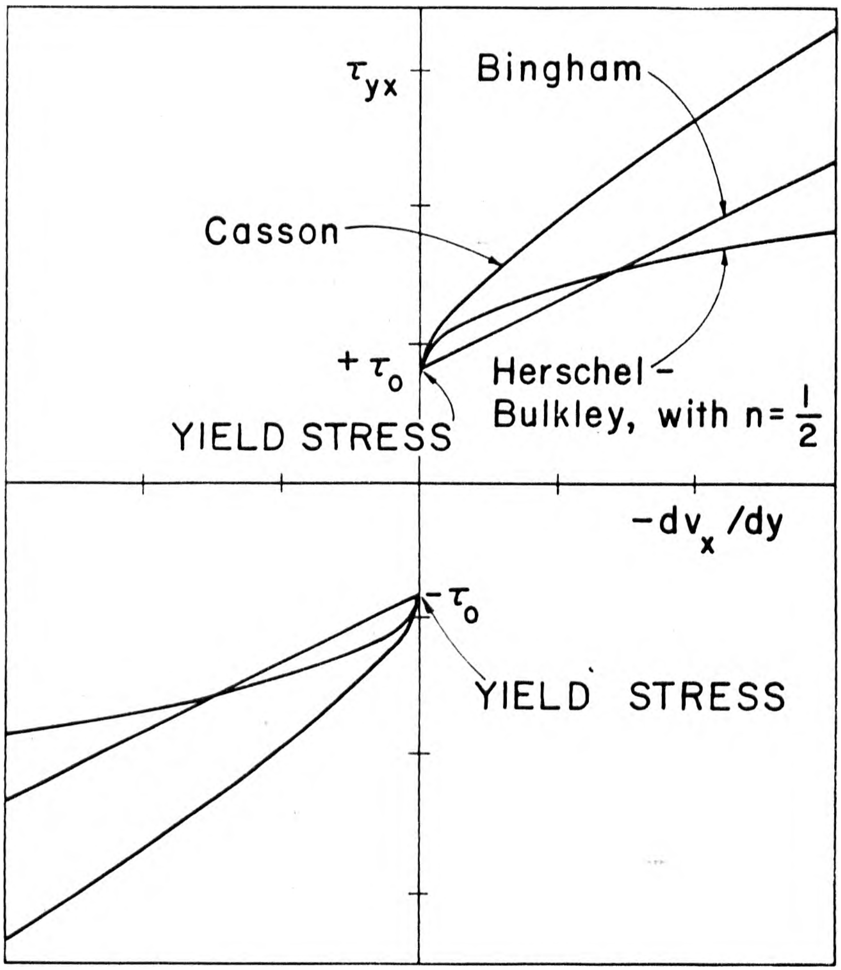

Fig. 2. Sketch showing the general shapes of the empirical expressions.

velocity gradients and stresses are either positive or negative. This is important when in a single flow problem there are regions of positive and negative velocity gradients (for example, in the axial flow in an annulus).

Other empirical models containing a yield stress have been proposed: the modified Casson equation /100/, the generalized Casson equation /108/, the Crowley-Kitzes model /34/, the Shul'man equation /177/. They are all more complicated than the three displayed above.

To give some idea as to the extent to which the empirical models are successful in portraying steady-state shear flow data, we give some data comparisons in Figures 3, 4, and 5 for the Bingham, Casson, and Herschel-Bulkley models. It appears that these empiricisms serve reasonably well for describing the steady-state shear behavior of some real materials.

Many efforts have been made to relate the constants in the models to parameters which describe the structure of the suspension or paste. Quantities such as the diameter of the suspended particles, the number density of the particles, and the viscosity of the suspending liquid clearly play an important role. In Table II we list some of the publications in which this kind of information is available.

3. THE STRESS TENSOR FOR ARBITRARY FLOW FIELDS

In the foregoing section we presented three empirical models containing a yield stress, but they were given specifically for the simple shear-flow field $v_x = v_x(y)$ of Fig. 1. If one wishes to solve problems in three-dimensional flows and flow problems with time dependence, it is necessary to set up the problems in greater generality. We illustrate the procedure for the Bingham model. We restrict the discussion to incompressible materials.

The equations of change for incompressible fluids are /17, 19/:

Continuity: $$(\nabla \cdot \underline{v}) = 0 \tag{3.1}$$

Motion: $$\rho \frac{D\underline{v}}{Dt} = -\nabla p - [\nabla \cdot \underline{\underline{\tau}}] + \rho \underline{g} \tag{3.2}$$

Energy: $$\rho \hat{C}_p \frac{DT}{Dt} = -(\nabla \cdot \underline{q}) - (\underline{\underline{\tau}} : \nabla \underline{v}) \tag{3.3}$$

Here ρ and $\hat{C}_p$ are the fluid density and heat capacity of the material; p, $\underline{v}$, and T are the pressure, velocity and temperature fields respectively. The symbol $\underline{g}$ stands for the gravitational acceleration. The j^{th} component of $[\nabla \cdot \underline{\underline{\tau}}]$ is $\Sigma_i (\partial/\partial x_i) \tau_{ij}$, and $(\underline{\underline{\tau}} : \nabla \underline{v}) = \Sigma_i \Sigma_j \tau_{ij} (\partial/\partial x_j) v_i$; the notation D/Dt is used for the "substantial derivative" $\partial/\partial t + (\underline{v} \cdot \nabla)$. These equations cannot be

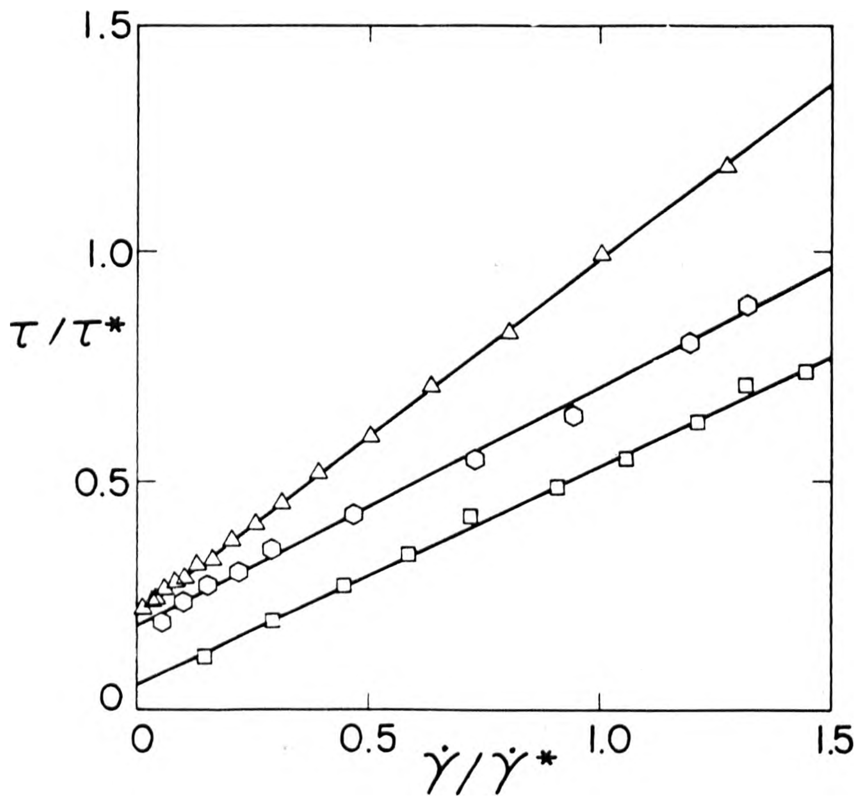

Fig. 3. Comparison of experimental data with the Bingham model

	$\dot{\gamma}^*$	τ^*	τ_0	μ_0
△ Meat extract /38/	6.97 s^{-1}	83.1 N/m^2	17.8 N/M^2	9.29 Pa.s
○ Kaolin/water /184/ ϕ = 0.25	1200	200	36.1	0.00883
□ Glass bead/polyethylene /74/ ϕ = .192	0.05	900	46.7	8780

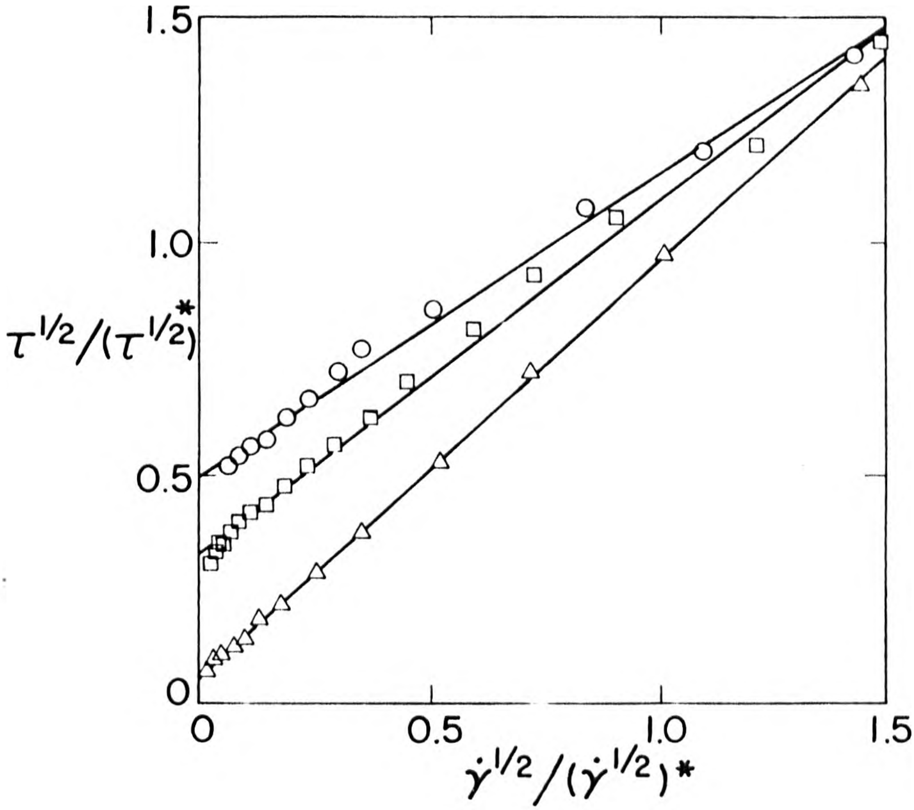

Fig. 4. Comparison of experimental data with the Casson model

		$(\dot{\gamma}^{1/2})^*$	$(\tau^{1/2})^*$	τ_0	μ_0
○	9% Bentonite/water with 0.5% surfactant /23/	12.0 $s^{-1/2}$	5.50 $N^{1/2}$/M	7.56 N/m^2	0.09 Pa.s
△	1% TiO_2/20% polystyrene diethyl phthalate (by wt.) /128/	6.5	16.0	0.801	4.97
□	18% Fe_2O_3/ethylene glycol /176/	25.5	5.25	2.89	0.0253

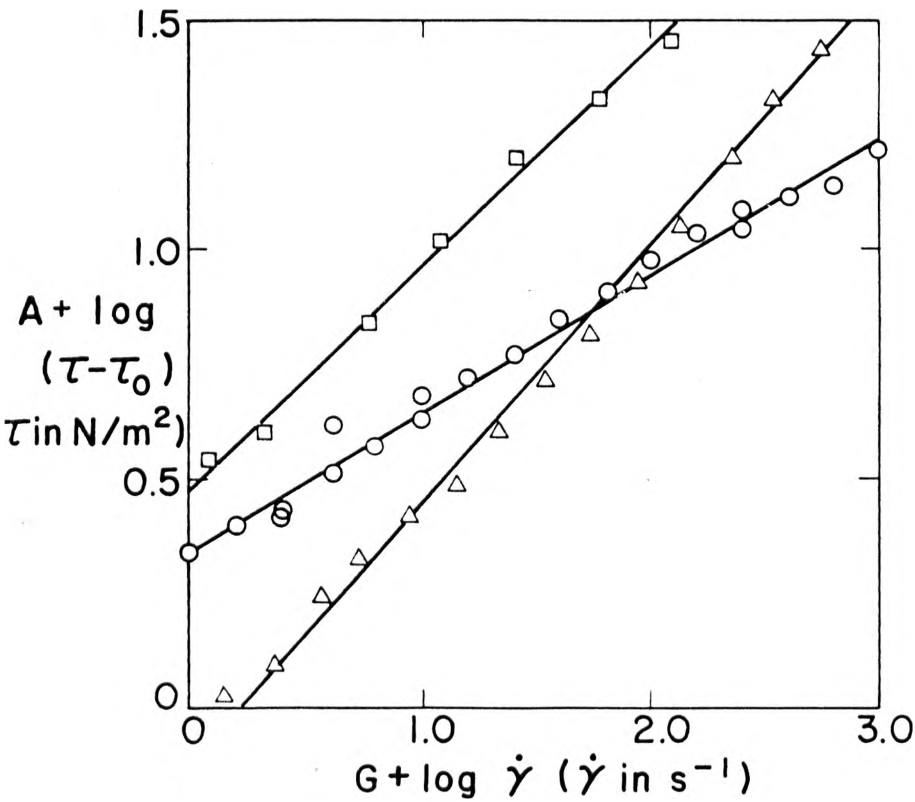

Fig. 5. Comparison of experimental data with the Herschel-Bulkley model

		τ_0	m	n	A	G
△ Fe_2O_3/ethylene glycol ϕ = 0.18	/176/	2.16 N/m^2	0.749 Pa.s^n	0.570	0	0
□ Carbon black/ polyisobutylene ϕ = 0.13	/200/	2045	47,400	0.486	-2.5	3.5
○ Xanthan gum/H_2O 10,000 ppm (by wt.)	/206/	1.15	2.17	0.305	0	0

TABLE II

Equations for Model Parameters

Reference	Author(s)	Type	Equations[A]	Explanation
101	Medvedev	Bingham	$\tau_0 = (0.195\phi - 0.102)\ \sigma/D_p$	
203	Walski	Bingham	$\tau_0 = \tau_a Y^b$ *or* $\tau_0 = a(Y - L_a)$	τ_a, b or a, L_a are constants. Y = depth in pool of sludge behind a weir
204	Weymann	Bingham	$\tau_0 = K\ N\ F_c(N,T)$	K is a constant; F_c = cohesive force between particles in a floc
205	Weymann, et al.	Bingham	$\tau_0 = F_c(N) N^{2/3}/2.88$; $\mu_0 = \mu_s \exp\left[\dfrac{0.52(a/b)(c/\rho)}{1 - \lambda(c/\rho)}\right]$	λ is a crowding factor. (a/b) = aspect ratio of plate-like particle; $\gg 1$
39	Dobrychenko, et al.	Bingham	$\tau_0 = k_1 (k_2\phi)^{3.0}$; $\mu_0 = \mu_s \exp(k_2\phi)$	k_1, k_2 are constants.

Table II Cont/..

Reference	Author(s)	Type	Equations[A]	Explanation
194,195	Van Kao, et al.	Bingham	$\tau_0 = 2.51\sigma V^{0.324}\phi^2/(\pi r^2)$	V = volume of bridging liquid in flocs
36	Darby and Rogers	Bingham, Casson	$\tau_0 = Y\exp(18.3\phi + b_0\phi_{30})$ $\mu_0 = V\exp(18.3\phi + b_1\phi_{30})$	Y, b_0, V, b, are constants ϕ_{30} = volume fraction of fines ($D_p < 30\mu m$)
183-186	Thomas	Bingham	$\tau_0 = A\psi_1\phi^3/D_p^2$ $\mu_0 = \mu_s\exp[2.5 + (B/\sqrt{D_p})\psi_2]\phi$	A, B are constants $\psi_1 = \exp 0.7[S/S_0)-1]$ $\psi_2 = \sqrt{(S/S_0)}$ (S/S_0) = ratio of surface area of platelike particle to that of equivalent sphere
45-47	Firth and Hunter	Bingham	$\tau_0 = \phi^2 C_{FP}\mu_s(r_e^2/r^2)/\pi$	r_e = radius of curvature of particle $C_{FP} = \phi_F/\phi$ ϕ_F = volume fraction of flocs

Table II Cont/..

Reference	Author(s)	Type	Equations[A]	Explanation
193	Van de Ven and Hunter	Bingham	$\tau_0 = \lambda\mu_s(Da^2)(\delta/r^3)\phi(a_0\beta C_{FP}\phi+\gamma^*)$	β, γ^* are theoretical constants δ = bond stretching distance in floc (Da^2) = (shear rate . floc size) a_0 = orthokinetic capture frequency λ = shielding factor
71	Hunter and Frayne	Bingham	$\tau_0 = \lambda'(Da^2)(\delta/r^3)\mu_s\phi^2 C_{FP}$	λ' = theoretical parameter (= $\lambda a_0 \beta$ above)
74	Kataoka, et al.	Casson	$\tau_0 = k_1 \exp(k_2\phi)$	k_1, k_2 are constants
176	Smith and Bruce	Casson	$\tau_0 = k'(\phi - \phi^0)^3$	k', ϕ^0 are constants
103	Merrill, et al.	Casson	$\tau_0 = \left\{[a\beta/(a\alpha-1)]\left[\left(\frac{1}{1-H}\right)^{(a\alpha-1)/2} - 1\right]\right\}^{1/2}$ $\mu_0 = \mu_s/(1-H)^{(a\alpha-1)}$	α, β are length parameters H = hematocrit a = orientation parameter

Table II Cont/..

Reference	Author(s)	Type	Equations[A]	Explanation
127	Onogi, et al.	modif. Casson	$\tau^{1/2} = \tau_0^{1/2} + k\left(\frac{\eta_a}{\eta_0}\right)^{1/2} \dot{\gamma}^{1/2}$ $\tau_0 = K\,we^{aw}$	K, a, k are constants η_a, η_0 are apparent and zero-shear rate viscosity of medium
185	Thomas	Crowley-Kitzes	$\dot{\gamma} = \frac{\tau}{\mu_s}\left[\frac{1.2 - 2\phi(k\,\tau^{-0.2}+1)^3}{1.2 + \phi(k\,\tau^{-0.2}+1)^3}\right]$ $k = A/\sqrt{D_p}$	A is a constant

Definition of symbols:

μ_s = Viscosity of the continuous medium
ϕ = Volume fraction of particles
c = Weight concentration of particles
w = Weight fraction of particles
N = Number density of particles
D_p = Particle diameter
σ = Interfacial stress
ρ = Density

Notes:

A: We list here only those equations that apply specifically to the appropriate constitutive equation (Bingham, etc.) Some of these authors /26, 74, 75, 193/ and others /176/ have also used equations for the Bingham or Casson viscosity that were originally developed for systems with no yield stress.

solved until expressions are inserted for the fluxes. For the heat flux $\underline{q}$ we use Fourier's law:

$$\underline{q} = -k\nabla T \tag{3.4}$$

where k is the thermal conductivity, and for Newtonian liquids we use

$$\underline{\underline{\tau}} = -\mu\dot{\underline{\underline{\gamma}}} = -\mu[\nabla\underline{v} + (\nabla\underline{v})^{\dagger}] \tag{3.5}$$

for the stress tensor, where μ is the viscosity. In Eq. (3.5) we have introduced the symbol $\dot{\underline{\underline{\gamma}}}$ for the rate-of-deformation tensor.

In what follows we make use of the "magnitude" of the tensors $\underline{\underline{\tau}}$ and $\dot{\underline{\underline{\gamma}}}$ defined as follows:

$$\tau = \pm\sqrt{\tfrac{1}{2}(\underline{\underline{\tau}}:\underline{\underline{\tau}})} \quad \text{and} \quad \dot{\gamma} = \pm\sqrt{\tfrac{1}{2}(\dot{\underline{\underline{\gamma}}}:\dot{\underline{\underline{\gamma}}})}$$

in which the + or – sign is so chosen that τ and $\dot{\gamma}$ are positive.

If we now wish to use the Bingham model, we replace Eq. (3.5) by /17 (p. 212), 19 (p. 103), 119–126/:

$$\begin{cases} \underline{\underline{\tau}} = -\left(\mu_0 + \dfrac{\tau_0}{\dot{\gamma}}\right)\dot{\underline{\underline{\gamma}}} & \text{for} \quad \tau \geqslant \tau_0 \qquad (3.7) \\ \dot{\underline{\underline{\gamma}}} = 0 & \text{for} \quad \tau \leqslant \tau_0 \qquad (3.8) \end{cases}$$

This is the appropriate generalization of Eqs. (2.2) and (2.3). When the Bingham model is written in this way, it appears that in the "plastic flow region" ($\tau \geqslant \tau_0$), the material behaves as a fluid with a non-Newtonian viscosity $\eta(\dot{\gamma}) = \mu_0 + (\tau_0/\dot{\gamma})$.

For some purposes it is convenient to write the Bingham model in such a form that the non-Newtonian viscosity is given as a function of the magnitude of the stress. To do this we form the double-dot product of Eq. (3.7) with itself to obtain

$$\left(\underline{\underline{\tau}}:\underline{\underline{\tau}}\right) = +\left(\mu + \frac{\tau_0}{\dot{\gamma}}\right)^2\left(\dot{\underline{\underline{\gamma}}}:\dot{\underline{\underline{\gamma}}}\right) \tag{3.9}$$

Multiplication of the equation by ½ then gives:

$$\tau^2 = \left(\mu_0 + \frac{\tau_0}{\dot{\gamma}}\right)^2 \dot{\gamma}^2 \tag{3.10}$$

or

$$\tau = \tau_0 + \mu_0 \dot{\gamma} \tag{3.11}$$

This then enables us to rewrite Eqs. (3.7) and (3.8) as /119-126, 129, 175/:

$$\begin{cases} \underline{\underline{\tau}} = -\left(\dfrac{\mu_0 \tau}{\tau - \tau_0}\right) \underline{\underline{\dot{\gamma}}} & \tau \geqslant \tau_0 \quad (3.12) \\ \underline{\underline{\dot{\gamma}}} = 0 & \tau \leqslant \tau_0 \quad (3.13) \end{cases}$$

When written in this form the non-Newtonian viscosity is given by $\eta(\tau) = \mu_0 \tau / (\tau - \tau_0)$.

The above expressions for the generalization of the Bingham material to complex flows was first set forth by Hohenemser and Prager /70/ and discussed more fully by Oldroyd /119-126/; the textbooks of Prager /137/ and Fredrickson /49/ also discuss this subject.

For the solution of Bingham model flow problems with boundaries of complicated shapes, it is useful to use a variational procedure. For flows in which the inertial terms can be neglected, Prager /138/ has given a minimum principle for the velocity field and a maximum principle for the stress field. The flow takes place in a volume V, and the surface is divided into two parts: S_v, on which the velocity is specified, and S_f, on which the force (i.e., $[\underline{n} \cdot \underline{\underline{\pi}}]$) is specified. Then Prager's principles for the Bingham model are stated as:

a. For continuous velocity fields with piecewise continuous first derivatives, which satisfy identically the equation of continuity and the boundary conditions on S_v, the actual velocity field *minimizes* the expression

$$H = \int_V \left(\mu_0 \dot{\gamma}^2 + 2\tau_0 \dot{\gamma}\right) dV + 2\int_{S_f} (\underline{n} \cdot [\underline{\underline{\pi}} \cdot \underline{v}])\, dS - 2\int_V (\underline{v} \cdot \underline{g})\, dV \tag{3.14}$$

except when $S_v = 0$. The $\underline{n}$ in the second integral is the outwardly directed unit normal at each surface element dS on S_f.

b. For continuous stress fields $\underline{\underline{\pi}} = p\underline{\underline{\delta}} + \underline{\underline{\tau}}$ with piecewise continuous first derivatives and which satisfy identically the equations of motion and the boundary conditions on S_f, the actual stress field *maximizes* the expression.

$$K = -2 \int_{S_v} (\underline{n} \cdot [\underline{\underline{\pi}} \cdot \underline{v}]) \, dS - \frac{1}{4\mu_0} \int_V \left[|\tau - \tau_0| - (\tau - \tau_0) \right]^2 dV \qquad (3.15)$$

except when $S_f = 0$.

For other discussions of variational principles for the Bingham model, see discussions by Ziegenhagen /214/, Oldroyd /119/, Johnson /72, 73/, and Haddow and Luming /58, 59/. To our knowledge variational principles have not been developed for other model fluids mentioned in §2.

The models in §2 and their generalizations to three-dimensional velocity fields in this section are all special cases of the "generalized Newtonian fluid"; that is, the stress tensor at the current time t is related to the rate-of-strain tensor at the current time t. Very little work has been done to extend the theory of materials with yield value to include time-dependent effects. We cite the work of Slibar and Pasley /174/ on the flow of thixotropic materials, and the work of White and Tanaka /208/ on "plastic-viscoelastic" constitutive equations as examples of attempts to include time-dependent effects.

4. SOME ANALYTICAL FLOW SOLUTIONS FOR THE BINGHAM MODEL

Since the Bingham model is the simplest and most widely used of the viscoplastic models, we summarize here the analytical solutions which are available for flow problems. Any of these solutions can be used to obtain the values of the model parameters μ_0 and τ_0 from appropriately designed experiments.

a. *Unidirectional Flow between Fixed Parallel Plates*

A Bingham material between two parallel plates of length L and width W is being driven in the +z-direction by a pressure gradient (see Fig. 6). The postulates that $v_z = v_z(x)$ and $p = p(z)$ allow the equation of motion to be simplified to:

$$0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} - \frac{d}{dx} \tau_{xz} \qquad (4.1)$$

(The symbol $\mathcal{P}$ is used as an abbreviation for p + ρgh where h is the distance upwards (i.e., opposed to gravity); the use of $\mathcal{P}$ allows us to solve problems in which the driving force is a pressure gradient or gravitation acceleration or a

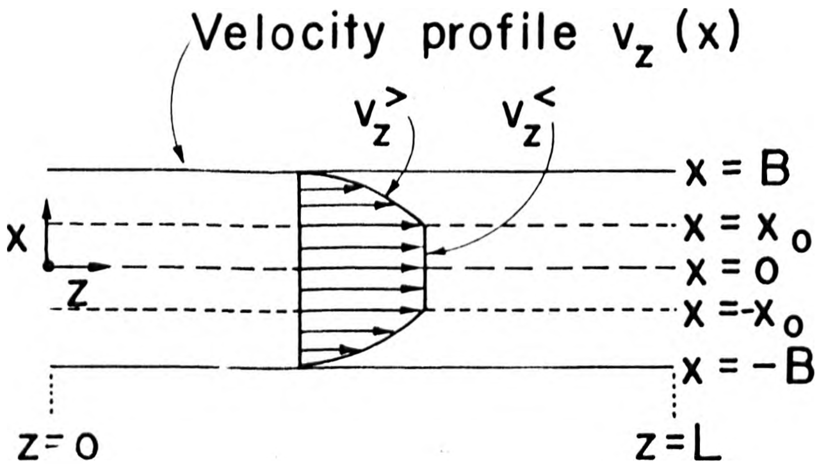

Fig. 6. Flow in a plane slit, showing the "plug flow region" $-x_0 \leqslant x \leqslant x_0$ and the "plastic flow regions" $x_0 \leqslant x \leqslant B$ and $-B \leqslant x \leqslant -x_0$.

combination of both.) Integration of Eq. (4.1) gives:

$$\tau_{xz} = \frac{P_0 - P_L}{L} x + C_1 \tag{4.2}$$

and the integration constant C_1 must be zero because of the symmetry of the problem. We now consider the region $0 \leqslant x \leqslant B$. The "yield surface", at which $\tau_{yx} = \tau_0$, is then located at $x = x_0$ where $x_0 = \tau_0 L/(P_0 - P_L)$. Inserting the appropriate form of Eq. (2.2) into Eq. (4.2) then gives

$$\tau_0 - \mu_0 \frac{dv_z^>}{dx} = \frac{P_0 - P_L}{L} x \quad (x_0 \leqslant x \leqslant B) \tag{4.3}$$

Integration of this equation gives:

$$v_z^> = \frac{(P_0 - P_L) B^2}{2\mu_0 L}\left[1 - \left(\frac{x}{B}\right)^2\right] - \frac{\tau_0 B}{\mu_0}\left[1 - \left(\frac{x}{B}\right)\right] \quad (x_0 \leqslant x \leqslant B) \tag{4.4}$$

$$v_z^< = \frac{(P_0 - P_L) B^2}{2\mu_0 L}\left[1 - \left(\frac{x_0}{B}\right)^2\right] - \frac{\tau_0 B}{\mu_0}\left[1 - \left(\frac{x_0}{B}\right)\right] \quad (0 \leqslant x \leqslant x_0) \tag{4.5}$$

The superscripts > and < emphasize that the expressions are for $x \geqslant x_0$ or $x \leqslant x_0$ respectively. The second of these expressions, for the "plug flow" region, results from the fact that $dv_z/dx = 0$ for $\tau_{xz} \leqslant \tau_0$.

The volume rate of flow is then:

$$Q = W \int_{-B}^{B} v_z dx = 2W \int_{0}^{B} v_z dx$$

$$= -2W \int_{x_0}^{B} x(dv_z^{>}/dx)dx$$

$$= \frac{2WB^2\tau_B}{\mu_0} \int_{\tau_0/\tau_B}^{1} \left[\left(\frac{x}{B}\right)^2 - \frac{\tau_0}{\tau_B}\left(\frac{x}{B}\right)\right] d\left(\frac{x}{B}\right) \tag{4.6}$$

in which $\tau_B = (P_0 - P_L)B/L$ is the wall shear stress. The integration by parts simplifies the integration and $dv_z^{>}/dx$ can be inserted from Eq. (4.3). One obtains then:

$$Q = \frac{2WB^2\tau_B}{3\mu_0} \left[1 - \frac{3}{2}\left(\frac{\tau_0}{\tau_B}\right) + \frac{1}{2}\left(\frac{\tau_0}{\tau_B}\right)^3\right] (\tau_B \geqslant \tau_0) \tag{4.7}$$

for the volume flow rate.

b. *Thickness of a Falling Film on an Inclined Surface*

The flow shown in Fig. 7 may be solved similarly to that for the slit flow. One replaces the slit half-width B by the film thickness δ, and by specializing the driving force $(P_0 - P_L)/L$ to $\rho g \cos\beta$. Then Eqs. (4.4) and (4.5) with proper modification of the symbols give the velocity distribution in the film. Equation (4.7), with B replaced by δ and τ_B given by $(\rho g \cos\beta)\delta$, is then a cubic equation for δ, the film thickness /16/.

c. *Axial Flow in a Circular Tube*

For flow in a circular tube of radius R and length L the wall shear stress is given by $\tau_R = (P_0 - P_L)R/2L$ and the yield surface is located at $r_0 = 2L\tau_0/(P_0 - P_L) = R(\tau_0/\tau_R)$. The velocity distribution is given as:

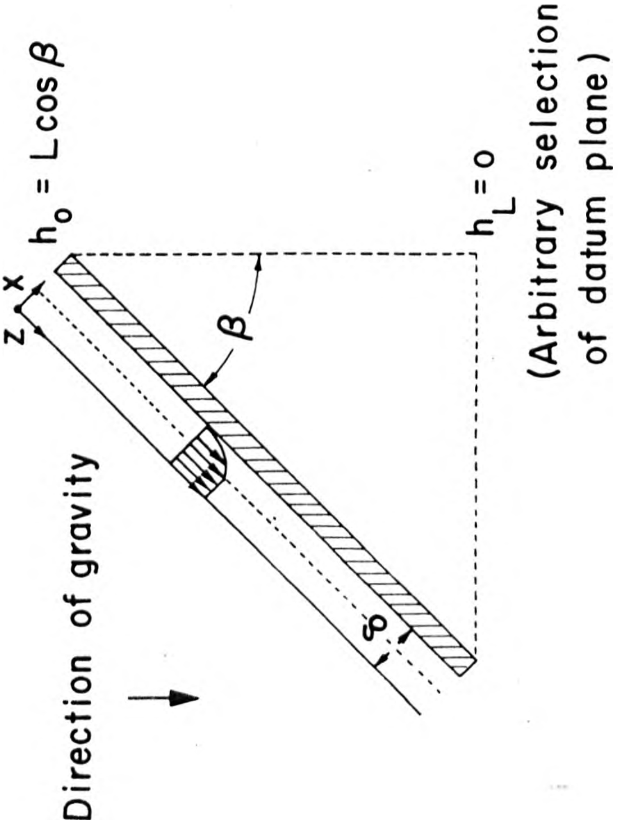

Fig. 7. A film of a viscoplastic material on an inclined surface with film thickness δ.

$$v_z^{>} = \frac{(P_0-P_L)R^2}{4\mu_0 L}\left[1-\left(\frac{r}{R}\right)^2\right] - \frac{\tau_0 R}{\mu_0}\left[1-\left(\frac{r}{R}\right)\right] \qquad r_0 \leqslant r \leqslant R \tag{4.8}$$

$$v_z^{<} = \frac{(P_0-P_L)R^2}{4\mu_0 L}\left(1-\frac{r_0}{R}\right)^2 \qquad 0 \leqslant r \leqslant r_0 \tag{4.9}$$

the second expression describing the plug flow region. The volume flow rate is then:

$$Q = \frac{\pi(P_0-P_L)R^4}{8\mu_0 L}\left[1-\frac{4}{3}\left(\frac{\tau_0}{\tau_R}\right)+\frac{1}{3}\left(\frac{\tau_0}{\tau_R}\right)^4\right] \tag{4.10}$$

which is known as the Buckingham-Reiner /19 (pp. 48-50), 22, 142/ equation.

d. Tangential Flow in a Concentric Annulus

For the system depicted in Fig. 8 the equation of motion in cylindrical coordinates becomes, for the postulates $v_r = 0$, $v_z = 0$, $v_\theta = v_\theta(r)$:

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\tau_{r\theta}\right) \tag{4.11}$$

Integration gives $\tau_{r\theta} = C_1/r^2$. But the torque applied to the outer cylinder is given by $T = -\tau_{r\theta}(r=R)\cdot 2\pi RL\cdot R$ (i.e., force per unit area $\times$ area $\times$ lever arm). Hence $C_1 = -T/2\pi L$ and

$$\tau_{r\theta} = -T/2\pi Lr^2 \tag{4.12}$$

The yield surface location is obtained from the relation $\tau = \tau_0$, where

$$\tau = \sqrt{\tfrac{1}{2}(\underline{\underline{\tau}}:\underline{\underline{\tau}})} = \sqrt{\tau_{r\theta}^2} = -\tau_{r\theta} \tag{4.13}$$

The minus sign has to be chosen so that τ is positive. Hence the yield surface is at

$$r_0 = \sqrt{T/2\pi\tau_0 L} \tag{4.14}$$

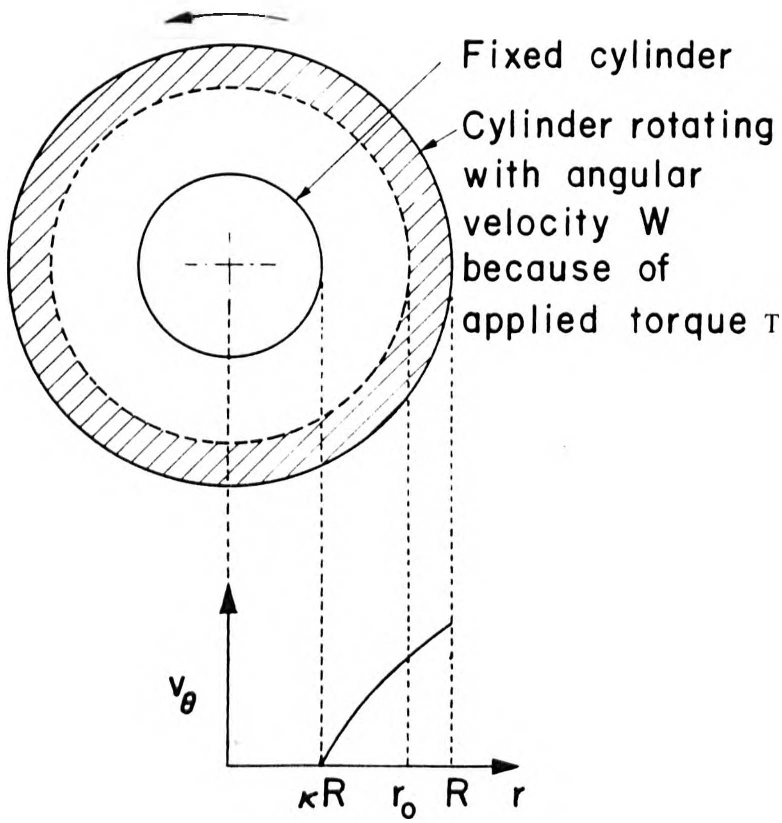

Fig. 8. Tangential annular flow with outer cylinder rotating. The surface $r = r_0$ is the yield surface, and for the shaded region, $r_0 \leqslant r \leqslant R$, according to Eq. (3.8), $\dot{\gamma}_{r\theta} = r d(v_\theta/r)/dr = 0$. Note that in this region it is not true that $dv_\theta/dr = 0$ /17 (pp. 218-221)/.

Note that (a) if $r_0 \leqslant \kappa R$, there is no plastic flow, (b) if $\kappa R < r_0 \leqslant R$, there is plastic flow for $\kappa R < r < r_0$ and solid like behavior for $r_0 \leqslant r \leqslant R$, and (c) if $r_0 \geqslant R$ there is plastic flow throughout. Then from Eq. (3.7) written in cylindrical coordinates

$$\tau_{r\theta} = -\tau_0 - \mu_0 r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) \tag{4.15}$$

For case (c) above the angular velocity distribution is

$$\frac{v_\theta}{r} = W + \frac{T}{4\pi L \mu_0 R^2}\left[1 - \left(\frac{R}{r}\right)^2\right] - \frac{\tau_0}{\mu_0} \ln \frac{r}{R} \tag{4.16}$$

and setting $v_\theta / r = 0$ for $r = \kappa R$ gives the relation between angular velocity and torque:

$$W = \frac{T}{4\pi L \mu_0 R^2}\left(\frac{1}{\kappa^2} - 1\right) + \frac{\tau_0}{\mu_0} \ln \kappa \tag{4.17}$$

known as the Reiner-Riwlin /17 (pp. 218-221), 144/ equation.

e. Axial Flow in a Horizontal Concentric Annulus with Inner Cylinder Moving Axially /121/

For this axial flow system, shown in Figure 9, the equation of motion is:

$$0 = -\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) \tag{4.18}$$

Integration gives

$$\tau_{rz} = C\mu_0 v_0 / r \tag{4.19}$$

in which C is a dimensionless positive constant of integration. Then from Eq. (3.7), $\tau_{rz} = \tau_0 - \mu_0 (dv_z/dr)$, so that:

$$\tau_0 - \mu_0 \frac{dv_z}{dr} = \frac{C\mu_0 v_0}{r} \tag{4.20}$$

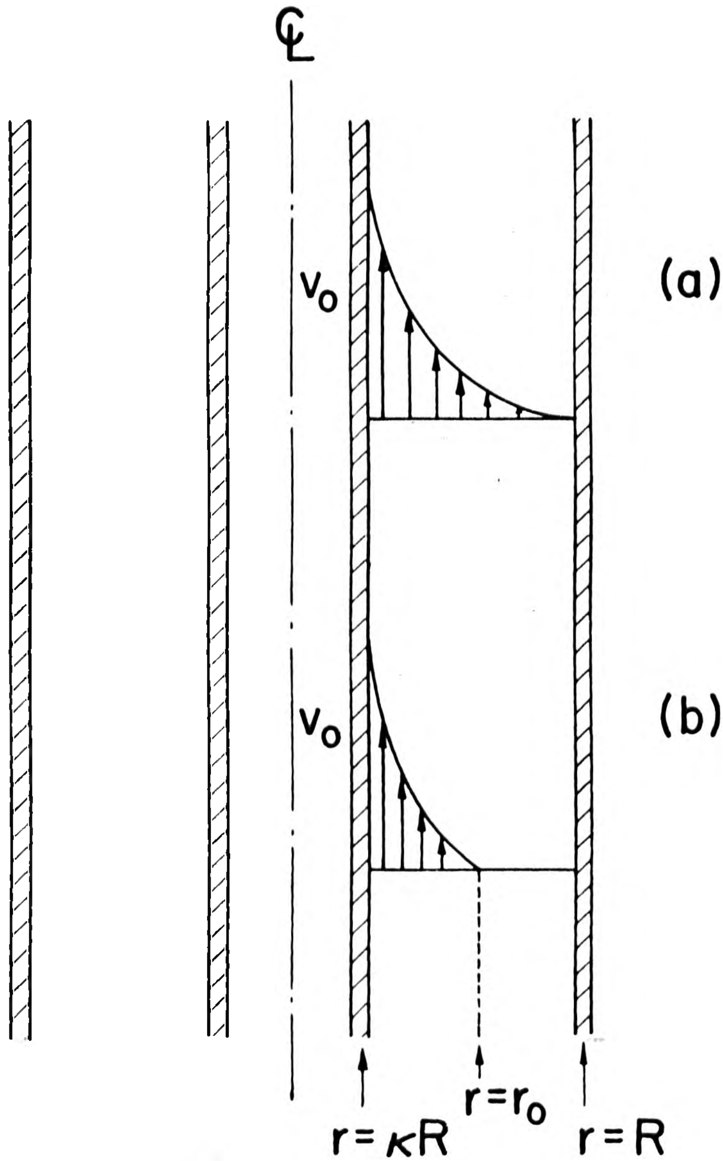

Fig. 9. Axial annular flow resulting from axial motion of the inner cylinder. Two types of flow patterns are possible: (a) all material flows, (b) material in region $r_0 \leqslant r \leqslant R$ is not in motion.

This may be integrated with the boundary condition that $v_z = v_0$ at $r = \kappa R$ to give

$$\frac{v_z}{v_0} = 1 - C \ln \frac{r}{\kappa R} + \frac{\tau_0}{\mu_0 v_0} (r - \kappa R) \tag{4.21}$$

We now distinguish two types of flow patterns:

a) All material is in a state of flow since τ_{rz} exceeds τ_0 for $\kappa R \leqslant r \leqslant R$; then C is determined by using the boundary condition that $v_z = 0$ at $r = R$ so that

$$\frac{v_z}{v_0} = 1 - \frac{\ln(r/\kappa R)}{\ln(1/\kappa)} + \frac{\tau_0}{\mu_0 v_0} \left[(r - \kappa R) - R(1-\kappa) \frac{\ln(r/\kappa R)}{\ln(1/\kappa)} \right] \tag{4.22}$$

b) Material in the region $r_0 \leqslant r \leqslant R$ is not in motion, since τ_{rz} does not exceed τ_0 for in that region; then C is determined by using the boundary condition that $v_z = 0$ at $r = r_0$, so that:

$$\frac{v_z}{v_0} = 1 - \frac{\ln(r/\kappa R)}{\ln(r_0/\kappa R)} + \frac{\tau_0}{\mu_0 v_0} \left[(r - \kappa R) - (r_0 - \kappa R) \frac{\ln(r/\kappa R)}{\ln(r_0/\kappa R)} \right] \tag{4.23}$$

where the yield surface is given by $r = r_0$ and r_0 is obtained from:

$$\frac{\tau_0 r_0}{\mu_0 v_0} \ln \frac{r_0}{\kappa R} = 1 + \frac{\tau_0 r_0}{\mu_0 v_0} \left(1 - \frac{\kappa R}{r_0} \right) \tag{4.24}$$

f. Axial Annular Flow between two Fixed Coaxial Cylinders

For the axial annular flow driven by a pressure difference and/or gravity (see Fig. 10), the equation of motion gives:

$$0 = + \frac{(P_0 - P_L)}{L} - \frac{1}{r} \frac{d}{dr} (r \tau_{rz}) \tag{4.25}$$

This equation may be integrated to give:

$$\tau_{rz} = \frac{(P_0 - P_L) R}{2L} \left[\left(\frac{r}{R} \right) - \lambda^2 \left(\frac{R}{r} \right) \right] \tag{4.26}$$

where $r = \lambda R$ is the surface at which the shear stress τ_{rz} is zero (see Fig. 10). The "plug-flow region" is bounded by $r = \lambda_- R$ and $r = \lambda_+ R$ which are de-

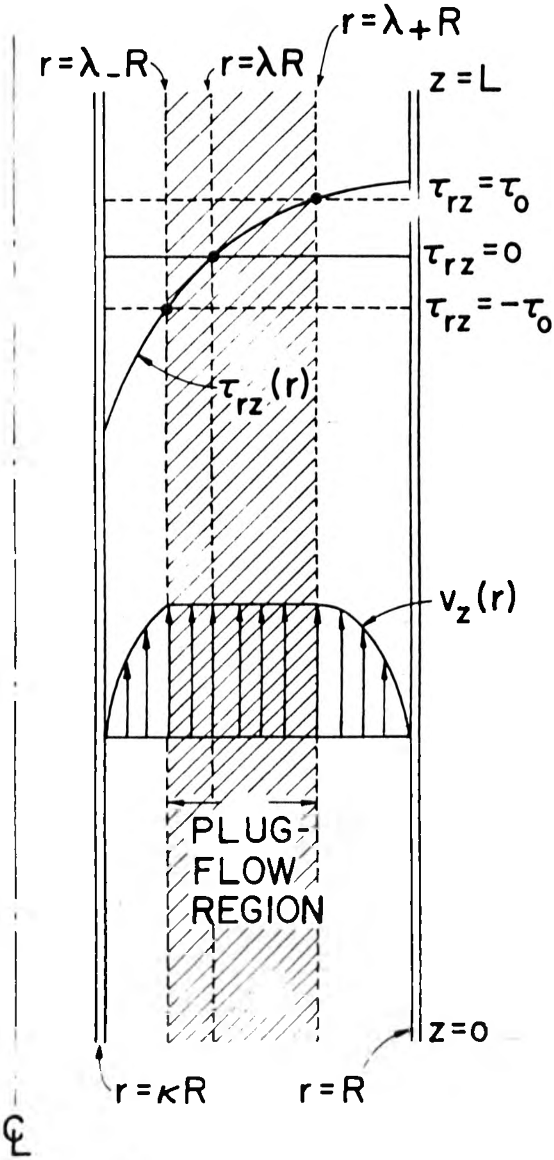

Fig.10. Axial annular flow in the space between the coaxial cylindrical surfaces at $r = \kappa R$ and $r = R$. The cylindrical surface $r = \lambda R$ is the surface of zero shear stress. The "plug flow" region is bounded by $r = \lambda_- R$ and $r = \lambda_+ R$. The pressure is p_0 at $z = 0$ and p_L at $z = L$.

fined as the surfaces for which τ_{rz} is equal to $-\tau_0$ and $+\tau_0$ respectively; this definition gives

$$\lambda_{\pm} - \frac{\lambda^2}{\lambda_{\pm}} = \pm \frac{2\tau_0 L}{(P_0 - P_L) R} \tag{4.27}$$

for λ_+ and λ_-. Note that

$$\lambda_+ - \lambda_- = 2\tau_0 L/(P_0 - P_L)R \tag{4.28}$$

and that $\lambda^2 = \lambda_+\lambda_-$.

The form of the Bingham model appropriate for this problem is:

$$\tau_{rz} = \pm\tau_0 - \mu_0 \frac{dv_z}{dr} \qquad |\tau_{rz}| \geqslant \tau_0 \tag{4.29}$$

Where the positive sign is used for $\lambda_+ R \leqslant r \leqslant R$, and the negative sign for $\kappa R \leqslant r \leqslant \lambda_- R$. Substitution of this into Eq. (4.26) and integration with respect to r gives the velocity distribution. If we use dimensionless quantities $\xi = r/R$, $T_0 = 2\tau_0 L/(P_0 - P_L)R$, and $\phi = 2\mu_0 L v_z/(P_0 - P_L)R^2$, then:

$$\phi_- = -T_0(\xi - \kappa) - \tfrac{1}{2}(\xi^2 - \kappa^2) + \lambda^2 \ln(\xi/\kappa) \qquad \kappa \leqslant \xi \leqslant \lambda \tag{4.30}$$

$$\phi_+ = -T_0(1 - \xi) + \tfrac{1}{2}(1 - \xi^2) + \lambda^2 \ln \xi \qquad \lambda_+ \leqslant \xi \leqslant 1 \tag{4.31}$$

and $\phi = \phi_-(\lambda_-) = \phi_+(\lambda_+)$ in the plug flow region. If we require that $\phi_-(\lambda_-) = \phi_+(\lambda_+)$ from Eqs. (4.30) and (4.31), and if, in addition, we use Eq. (4.28) we obtain the equation which gives λ_+ as a function of κ and T_0:

$$2\lambda_+(\lambda_+ - T_0) \ln \frac{\lambda_+ - T_0}{\lambda_+ \kappa} - 1 + (T_0 + \kappa)^2 + 2T_0(1 - \lambda_+) = 0 \tag{4.32}$$

The volume rate of flow can then be obtained /50, 175/ by integration of the velocity distribution:

$$Q = \frac{\pi R^4 (P_0 - P_L)}{8\mu_0 L} \Big[(1 - \kappa^4) - 2\lambda_+(\lambda_+ - T_0)(1 - \kappa^2) - \frac{4}{3}(1 + \kappa^3) T_0 + \frac{1}{3}(2\lambda_+ - T_0)^3 T_0 \Big] \tag{4.33}$$

Tables have been prepared for use with Eqs. (4.32) and (4.33) /50/ and some useful asymptotic expressions have been worked out /4/.

g. Radial Flow between Two Parallel Disks

No exact solution of this problem is available, but an approximate solution using the "lubrication approximation" has been obtained /35/. We can apply Eq. (4.7) locally (i.e., over some small region dr in the radial disk system of Fig. 11) by replacing W by $2\pi r$ and $(P_0 - P_L)/L$ by $-(dP/dr)$. This gives:

$$Q = \frac{2(2\pi r)B^2\tau_0}{3\mu_0}\left(-\frac{B}{\tau_0}\frac{dP}{dr}\right)\left[1 - \frac{3}{2}\left(-\frac{B}{\tau_0}\frac{dP}{dr}\right)^{-1} + \frac{1}{2}\left(\frac{B}{\tau_0}\frac{dP}{dr}\right)^{-3}\right] \tag{4.34}$$

This differential equation for P as a function of R has been integrated numerically, and a table of pressure drop *vs.* flow rate has been prepared /35/. Eq. (4.34) can be solved analytically for dP/dr to give /109/:

$$-\frac{B}{\tau_0}\frac{dP}{dr} = \left[\sqrt{\frac{8\alpha}{3}}\ \sin\left(\frac{1}{3}\arcsin\left(\frac{3}{2\alpha}\right)^{3/2}\right)\right]^{-1} \tag{4.35}$$

where $\alpha = (3/2) + (3\mu_0 Q/4\pi B^2\tau_0 r)$.

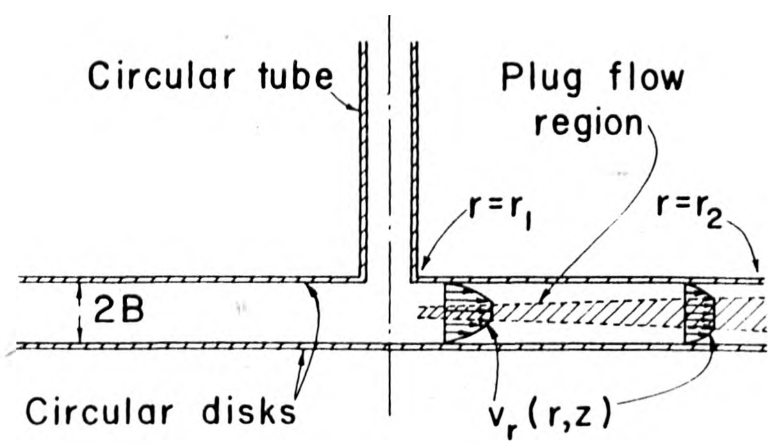

Fig.11. Radial flow between two circular disks.

5. SOME ANALYTICAL HEAT-TRANSFER SOLUTIONS FOR THE BINGHAM MODEL

Solution of heat-transfer problems requires the simultaneous solution of Eqs. (3.1), (3.2), and (3.3) along with the constitutive relation in Eqs. (3.7) and (3.8). In addition one needs to know the temperature dependence of the thermal conductivity k and the Bingham model parameters μ_0 and τ_0. Most analytical solutions to date have assumed these transport properties to be constants. We give here a brief discussion of several useful results that are available for simple problems.

a. *Viscous Heating in Flow between two Flat Plates*

Consider a viscoplastic fluid between two parallel plates. The plate at x = 0 is fixed and is maintained at a temperature T_0; the plate at x = b is moving with speed V in the positive z-direction and is maintained at a temperature T_b. Hence if the temperature dependence of the Bingham parameters is ignored, the velocity profile is $v_z = Vx/b$ and hence the viscous-heating source term in Eq. (3.3) is

$$
\begin{aligned}
S_v &= (-\underline{\underline{\tau}} : \nabla \underline{v}) = -\tau_{xz}(dv_z/dx) \\
&= -\left[-\tau_0 - \mu_0 \frac{dv_z}{dx}\right]\frac{dv_z}{dx} \\
&= (\tau_0 V/b) + \mu(V/b)^2 \qquad (5.1)
\end{aligned}
$$

Then Eq. (3.3) gives:

$$0 = +k\frac{d^2T}{dx^2} + S_v \qquad (5.2)$$

which has the solution (see Ref. 19, p. 278):

$$\frac{T - T_0}{T_b - T_0} = \frac{x}{b} + \frac{S_v b^2}{2k(T_b - T_0)}\left[\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right] \qquad (5.3)$$

The S_v term gives the deviation from a linear temperature profile that results from the viscous-dissipation term $(-\underline{\underline{\tau}}:\nabla\underline{v})$ in Eq. (3.3).

b. Flow in Circular Tubes

Two heat-transfer problems for tube flow are shown in Figure 12. For each of these problems it is possible to obtain asymptotic analytical solutions to the energy equation (Eq. (3.3)) for small z (the thermal entrance region) and for large z (the thermally fully-developed region). The procedures for doing this have been presented in illustrative examples in Reference 17, pp. 238-246, for non-Newtonian fluids with any viscosity function. Once the temperature profiles have been obtained it is relatively straightforward to get the Nusselt numbers. In Table III we summarize the Nusselt numbers obtained for the Bingham model. These results can be obtained by using entries

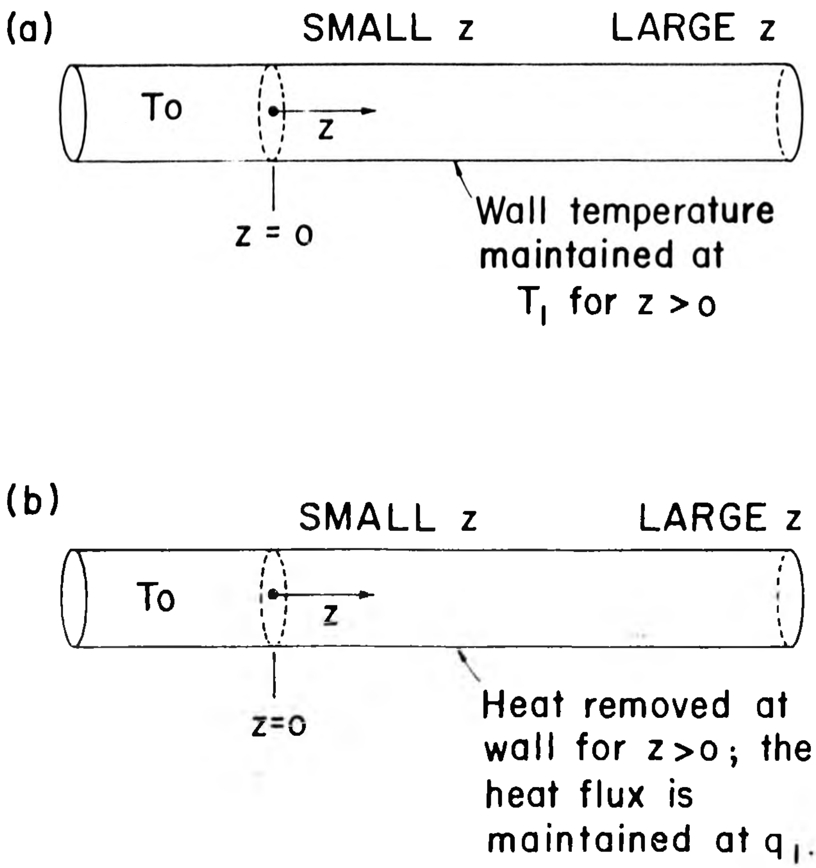

Fig.12. Heat transfer problems for tubes:
(a) the constant-wall-temperature problem;
(b) the constant-wall-heat-flux problem.

TABLE III

Nusselt Numbers for Bingham Fluids

(h = heat transfer coefficient; k = thermal conductivity)

Heat Transfer Problem		Constant Wall Temperature	Constant Wall Heat Flux
CIRCULAR TUBE* Radius R Diameter D Length L ($L \gg R$) Nu = hD/k $\xi_0 = \tau_0/\tau_R$ $\tau_R = \dfrac{(P_0 - P_L)R}{2L}$	small z	$\mathrm{Nu} = \dfrac{2}{9^{1/3}\Gamma(4/3)}\left[\dfrac{\tau_R R^3}{\mu_0 \alpha z}(1-\xi_0)\right]^{1/3}$	$\mathrm{Nu} = \dfrac{2\Gamma(2/3)}{9^{1/3}}\left[\dfrac{\tau_R R^3}{\mu_0 \alpha z}(1-\xi_0)\right]^{1/3}$
	large z	ξ_0: 1.00, 0.90, 0.80, 0.70, 0.60, 0.50 Nu: 5.80, 5.41, 5.07, 4.76, 4.49, 4.27 ξ_0: 0.40, 0.30, 0.20, 0.10, 0 Nu: 4.09, 3.94, 3.82, 3.73, 3.67	$\mathrm{Nu} = \dfrac{1680}{A}\left[9 - 24\xi_0 + 16\xi_0^{2} + 6\xi_0^{4} - 8\xi_0^{5} + \xi_0^{8}\right]$ $A = 3465 - 9792\xi_0 + 6944\xi_0^{2} - 3780\xi_0^{4} - 5600\xi_0^{5} - 1008\xi_0^{6} + 2016\xi_0^{7} + 195\xi_0^{8} - 840\xi_0^{8}\ln\xi_0$

* From Beek and Eggink /11/.

Table III Cont/..

Heat Transfer Problem		Constant Wall Temperature	Constant Wall Heat Flux
PLANE SPLIT ** Thickness 2B Width W Length L $(L \gg W \gg B)$ $Nu = 4Bh/k$ $\sigma = \tau_0/\tau_B$ $\tau_B = \frac{(P_0 - P_L)B}{L}$	small z	$Nu = \frac{4}{9^{1/3}\Gamma(4/3)}\left[\frac{\tau_B B^3}{\mu_0 \alpha z}(1-\sigma_0)\right]^{1/3}$	$Nu = \frac{4\Gamma(2/3)}{9^{1/3}}\left[\frac{\tau_B B^3}{\mu_0 \alpha z}(1-\sigma_0)\right]^{1/3}$
	large z	σ_0 1.00 0.90 0.80 0.70 0.60 0.50 Nu 9.86 9.54 9.22 8.92 8.64 8.38 σ_0 0.40 0.30 0.20 0.10 0 Nu 8.16 7.96 7.78 7.66 7.54	$Nu = \frac{280(\sigma_0 + 2)^2}{136 + 117\sigma_0 - 12\sigma_0^2 - 31\sigma_0^3}$

** Valstar and Beek /192/.

in Table 5.4-1 of Reference 17, or directly from the Dutch article by Beek and Eggink /11/ where the specific results for the Bingham model were worked out; Wissler and Schechter had calculated some of the entries for constant temperature wall and large z earlier /211/.

c. Flow in Plane Slits

Two heat-transfer problems for plane-slit flow are shown in Figure 13. These problems can be solved by the same procedures used for tube flow, and the resulting Nusselt number results appear in Table III. These results may be obtained by specializing the entries of Table 5.4-2 of Reference 17, or directly from the Dutch publication of Valstar and Beek /192/.

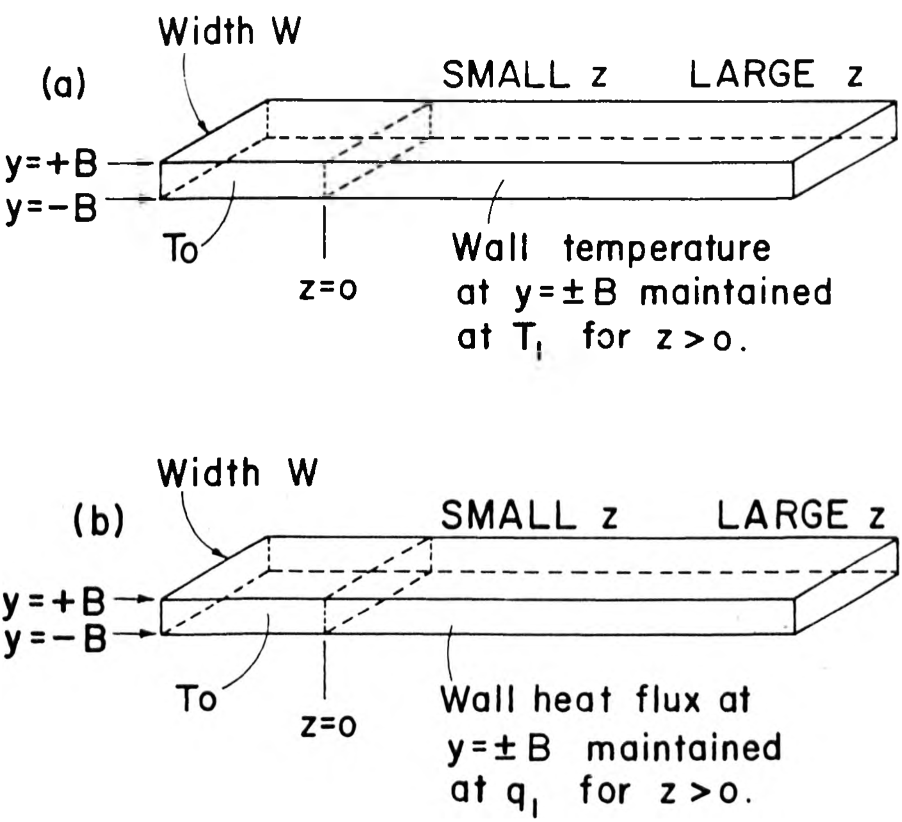

Fig.13. Heat-transfer problems for thin slits:
(a) the constant-wall-temperature problem;
(b) the constant-wall-heat-flux problem.

6. SUMMARY OF FLOW PROBLEMS

In §4 we summarized some of the most important results for flow calculations for the most widely used viscoplastic model, namely that of Bingham. We also illustrated the techniques needed for obtaining these elementary solutions. The solutions to many other problems and for more complicated rheological models have been published. We summarize these in Table IV (steady-state flow in enclosed regions), Table V (unsteady flows), Table VI (external flows), and Table VII (lubrication systems). In addition in Table VIII we give an abstract of the table of contents of the Russian book by Ogibalov and Mirzadzhanzade /115/, which contains an enormous amount of information not generally available elsewhere.

The publications cited in the tables deal with laminar flow. Some work has been done on laminar-turbulent transition and turbulent flow. Some of this work is summarized in a book by Skelland /17, Chapter 6/. For the Bingham model a successful data correlation procedure has been developed by Tomita /188, 189/ and Hanks /60, 61/, Hanks and Ricks /64/, and Hanks and Dadia /63/ have studied the laminar-turbulent transition. Other work on the transition-region has been done by Le Fur /84/ and by Chien /32/. Laminar flow stability in non-isothermal systems has been studied by Froishteter and Vinogradov /51/. The literature on mixing and agitation is rather sparse /171, Chapter 9/.

7. SUMMARY OF HEAT-TRANSFER PROBLEMS

In Tables IX and X we summarize the literature references for forced- and free-convection heat transfer respectively. In some of these publications the temperature dependence of the Bingham parameters μ_0 and τ_0 is accounted for. There seems to be no generally accepted formulas for the temperature dependence; the following examples illustrate what kinds of relations have been used:

Mamedov and Sattarov /96/:

$$\mu_0(T) = \mu_{0w}(T_w/T) \tag{7.1}$$

$$\tau_0(T) = \tau_{0w}(T_w/T) \tag{7.2}$$

Pervushin /134/:

$$\mu_0(T) = \mu_{0w} \exp -\beta_1 (T-T_w) \tag{7.3}$$

$$\tau_0(T) = \tau_{0w} \exp -\beta_2 (T-T_w) \tag{7.4}$$

TABLE IV

Steady Flows in Closed Regions

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Fully developed flow between parallel plates resulting from a pressure gradient	Bingham	Analytical solution; velocity distribution, relation between flow rate and pressure drop obtained	Slibar and Paslay (1957)	175
Fully developed flow between parallel plates resulting from a pressure gradient	Casson	Analytical solution; velocity profile, average velocity, flow rate expression obtained	Kooijman and Van Zanten (1972) Shah(1980)	79 160
Fully developed flow between parallel plates resulting from a pressure gradient	Herschel–Bulkley	Analytical solution, quasisolid core, velocity profile, flow rate obtained	Grinchik and Kim(1971)	56
Couette flow between parallel plates (pressure gradient and relative motion of plates)	Shul'man	Analytical method; core bounds, velocity profile obtained	Shul'man, Volchenok(1977)	167
Fully developed flow in a straight circular tube	Bingham	Analytical method; plug flow region, velocity profile, flow rate obtained	Buckingham Reiner	22 142

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Fully developed flow in a straight circular tube	Casson	Analytical method; velocity distribution, plug flow region, flow rate obtained	Oka (1965) Merrill, et al. (1965) Shah (1980) Kooijman and Van Zanten (1972) Lih (1975)	116 102 160 79 86
Fully developed flow in a straight circular tube	Herschel-Bulkley	Analytical method; velocity profile, radius of core obtained	Grinchik and Kim (1971)	56
Fully developed flow in a straight circular tube	Herschel-Bulkley	Analytical method; velocity profile, plug flow region, average velocity, flow rate obtained; compared with experimental data	Froishteter and Vinogradov (1980)	52
Fully developed flow in a straight circular tube	Briant	Analytical method; solution given in parametric form; comparison with experiment	LeFur, Martin (1967)	84
Fully developed flow in a slightly cuved tube	Bingham	Perturbation theory; approximate solution obtained; streamlines illustrated	Clegg. Power (1963)	33

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Fully developed flow in a noncircular tube (triangular and rectangular ducts)	Casson	Variational method; relation between flow rate and pressure drop given graphically	Batra and Koshy (1978)	8
Developing flow at entrance to a circular tube	Bingham	Variational method; velocity distribution, pressure drop, length of entrance region obtained, compared with experiment	Michiyoshi, Mizuno, Hoshiai (1965)	106
Developing flow at entrance to a circular tube	Bingham	Laplace transform; solution in general form obtained	Kim and Shul'man (1968)	77
Developing flow at entrance to a circular tube	Bingham	Momentum integral method; Campbell-Slattery method; velocity distribution, thickness of boundary layer, length of entrance region, plug flow radius, pressure drop obtained.	Chen, Fan, Hwang (1970)	30
Developing flow at entrance to a circular tube	Casson	Numerical method; velocity profile, center-line velocity, friction factor obtained	Shah, Soto(1974)	161

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Developing flow at entrance to a circular tube	Herschel-Bulkley	Numerical method; velocity, profile, centerline velocity, friction factor, length of entrance region obtained	Soto and Shah (1976)	178
Tube flow with non-uniform inlet velocity	Bingham	Numerical method; velocity, pressure, temperature distribution, length of entrance region obtained	Batra and Koshy (1978)	9
Fully developed axial flow in a concentric annulus	Bingham	Approximate solution	Van Olphen (1950)	196
Fully developed axial flow in a concentric annulus	Bingham	Analytical and graphical methods; velocity distribution, pressure-flow rate relation obtained (erroneous results)	Mori and Ototake (1953)	112
Fully developed axial flow in a concentric annulus	Bingham	Analytical method; expression for flow rate relation obtained	Laird (1957)	83
Fully developed axial flow in a concentric annulus	Bingham	Analytical method; formulas and graphs given for velocity distribution and volume flow rate	Slibar and Paslay (1957)	175

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Fully developed axial flow in a concentric annulus	Bingham	Analytical method; velocity distribution, dimensionless flow rate relation obtained; tabulated results; examples	Fredrickson and Bird (1958)	50
Fully developed axial flow in a concentric annulus	Bingham	Analytical method; velocity distribution, core velocity, flow rate relations obtained, approximate expressions for core bound given	Bostandzhiyan (1970)	20
Fully developed axial flow in a concentric annulus	Bingham	Simplification of Fredrickson-Bird's results; one-term expansion for velocity, core bound, flow rate obtained	Anshus(1974)	4
Fully developed axial flow in a concentric annulus	Casson	Numerical method; core bound, velocity, flow rate given in curves and table	Shul'man (1970) Shah (1980)	163 160
Fully developed axial flow in an eccentric annulus	Bingham	Finite difference method; flow rate presented as a series of dimensionless plots, compared with experiment	Guckes (1975)	57
Axial flow in eccentric annulus with cylinders in relative motion	Bingham	Analytical method; obtained velocity profiles and bounds on the plug flow region	Oldroyd (1947)	121

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Axial flow between confocal elliptic cylinders in relative motion	Bingham	Analytical method; obtained velocity profiles and bounds on the plug flow region	Oldroyd (1947)	122
Tangential flow in a concentric annulus	Bingham	Analytical method; velocity distribution plug flow region, external torque relation obtained	Reiner and Rivlin (1927) Slibar and Paslay (1958)	144 173
Helical flow in a concentric annulus	Bingham	Approximate solution; velocity distribution, core bounds obtained	Paslay and Slibar (1957)	130
Helical flow in a concentric annulus	Bingham	Analytical and numerical method; velocity variation, core given in curves	Mohan Rao (1965)	110
Entrance flow into a concentric annulus	Casson	Numerical method; developing velocity profile, friction factor given in curves	Liu and Shah (1975) Shah (1980)	87 160
Radial flow between parallel disks	Bingham	Analytical and numerical; used lubrication approximation; obtained velocity distribution and volume flow rate vs. pressure drop (tabulated results)	Dai and Bird (1981)	35

Table IV cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Steady tangential flow between two spheres in relative rotation	Bingham	Analytical method; velocity distribution, pressure distribution, critical Bingham number obtained.	Kumar (1964)	82
Steady tangential flow between two cones in relative rotation	Hencky-Il'yushin	Analytical and numerical method; velocity distribution obtained; compared with experimental data	Sugak (1966)	180
Steady tangential flow between two cones in relative rotation	Bingham	Analytical method; analytical velocity distribution obtained (only for linear viscosity law)	Chernyshov (1970)	31
Rotating sector bounded by two sides	Bingham	No core; no general solution obtained, solution only for special angle	Chernyshov (1970)	31

TABLE V
Unsteady Flows in Bounded Regions

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Flow between parallel plates with suddenly imposed pressure grad.	Bingham	Asymptotic solution for small time values	Safronik (1959)	150
Flow between parallel plates with suddenly imposed pressure gradient	Bingham	Laplace-Carson integral transform; system of functional equations for determining core and shear stress time-dependence obtained	Makarov (1970)	91
Flow between parallel plates with suddenly imposed pressure gradient	Bingham	Monte-Carlo statistical test method; position of interface versus time obtained for start up, damping gradient flow and Couette flow	Makarov, Pavlov, Simkhovich (1970)	92
Flow between parallel plates with suddenly imposed pressure grad.	Bingham	Analytical method, successive approximations; shear stress distribution, plug flow interface time-dependence obtained	Makarov, Zhelanova, Polozova (1971)	95
Flow between parallel plates with time-dependent shear stress imposed at wall	Bingham	Successive approximation; variation in position of boundary separating zones given in a figure	Makarov, Sal'nikov (1972)	93

Table V cont/..

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Flow between parallel plates with arbitrarily varying pressure grad.	Herschel-Bulkley	Successive approximation; transient behavior of flow shown in graphs	Makarov, Sal'nikov(1972)	94
Flow generated by an oscillating plate	Bingham	Analytical method; instantaneous velocity, shear stress obtained	Hanks(1974)	62
Pulsatile flow in a circular tube	Bingham	Multiviscous approximation numerical method; velocity distribution, plug size obtained	Ly and Bellet(1976)	88
Eccentric annular tangential flow	Bingham	Analytical and graphical results	Ratnawat(1968)	140
Plane channel	Bingham	Method of characteristics; velocity distribution, length of entrance, time for achieving a steady-state regime obtained	Shul'man, Zal'tsgendler(1979)	169
Squeezing flow between parallel plates	Bingham	Rough analytical solution; results have been questioned by subsequent workers	Scott(1931, 1935)	157, 158
Squeezing flow between parallel plates	Bingham	Analytical method; velocity distribution, pressure distribution relation between plate velocity and applied force obtained	Oka(1963) Oka and Ogawa(1964)	117 118

TABLE VI
External Flows

Flow System	Fluid Model	Method and Results	Author(year)	Reference
Film Flow	Bingham	Analytical method; velocity distribution, flow rate, layer thickness relations obtained	Paslay, Slibar (1958) Bird (1959)	131 16
Wavy film flow	Bingham	Successive approximation; velocity distribution, phase velocity of waves, wave number, mean layer thickness, critical plasticity parameter obtained	Shul'man, Baikov (1977)	164
Flow past flat plate	Bingham	Approximate solution using von Karman momentum balance	Skelland (1967)	171 (p.286)
Flow past sphere	Bingham	Variational and numerical method; stress distribution, velocity distribution, flow pattern, dimension of flow region, resistance coefficient obtained	Yoshioka, Adachi, and Ishimura (1971)	213
Flow past cylinder	Bingham	Variational method; stress distribution, velocity distribution, drag force, flow region obtained	Adachi, Yoshioka (1973)	1
Motion of gas bubble	Casson	Numerical method; variation of bubble radius, velocity and pressure at bubble wall with time obtained	Shima, Tsujino (1978)	162

TABLE VII
Lubrication Problems

Type of Bearing	Fluid Model	Method and Results	Author(year)	Reference
Slider bearing	Bingham	Analytical method; extent of core, load capacity , and frictional drag obtained; core region detected by visual observation	Milne(1954)	107
Journal bearing	Bingham	Analytical method; location of core, load capacity, moment of friction obtained	Batra (1965)	6
Roller bearing	Bingham	Analytical method; special core formed; frictional resistance, pressure distribution, load capacity expressions obtained	Sasaki, Mori, Okino (1960)	152–154
Roller bearing	Casson	Analytical method; flow criterion for no core formation; friction force, pressure distribution, load capacity expressions obtained	Batra, Mohan (1978)	10
Rotating bearing with radial leakage	Bingham	Approximate solution; load capacity, friction torque obtained; example given	Slibar, Paslay (1956)	172
Rotating circular step bearing	Bingham	Approximate solution; lifting force, moment of friction, condition for no-core formation obtained	Batra(1973)	7

TABLE VIII

Contents of the Monograph by Ogibalov and Mirzadzhanzade /115/

Because this book, published first in 1970 and then again in 1977, is not readily available in the West, we summarize its table of contents here:

Chapter 1. MODELS OF MEDIA AND DIFFERENTIAL RELATIONS

Some models for non Newtonian fluids and for the turbulent motion of non-Newtonian systems are given; the equations of motion and heat transfer are introduced.

Chapter 2. EXACT SOLUTION OF UNSTEADY PROBLEMS OF MOTION OF VISCOPLASTIC MEDIA

Solutions involving flow between parallel plates, in circular tubes, and near a rotating cylinder in an infinite medium are given; also magnetohydrodynamic problems, oscillatory motion, and Monte-Carlo methods are discussed.

Chapter 3. APPROXIMATE SOLUTIONS OF PROBLEMS OF UNSTEADY MOTION OF VISCOPLASTIC MEDIA

Approximate solutions for such problems as the impact of a viscoplastic core on a rigid obstruction, and a shearing wave in viscoplastic medium with a transverse magnetic field are described.

Chapter 4. UNSTEADY FLOW OF A VISCOPLASTIC MEDIUM IN TUBES AND CHANNELS

Unsteady motion in tubes and annuli are discussed, e.g. leakage through the gap between a cylinder and a piston, the start-up pressure in long pipelines, some problems of the Stefan type, and the hydrodynamic pressure on the walls of oil wells.

Chapter 5. SOME HEAT-TRANSFER PROBLEMS IN FLOWING VISCOPLASTIC MATERIALS

Solutions are given for various flow systems, including the thermal entrance region, viscous dissipation heating, temperature-dependent-viscosity effects, and time-dependent wall temperatures.

Chapter 6. UNSTEADY DIFFUSION IN VISCOPLASTIC MEDIA

Two topics are discussed here: the motion of the suspended particles, and the displacement of the viscoplastic medium in the structure regime in a circular tube.

Chapter 7. SOME PROBLEMS ON FILTRATION OF NON-NEWTONIAN SYSTEMS

Retardation phenomena, nonequilibrium effects, unsteady filtration, and moving boundaries are discussed.

No similar book appears to be available in English; the book is apparently primarily aimed at problems in the petroleum industry.

TABLE IX

Forced-Convection Heat Transfer Problems

Flow System	Characteristic of Heat Transfer	Fluid Model	Method and Results	Author(year)	Reference
Flow between parallel plates driven by a pressure gradient	Constant wall temp.; plastic viscosity and yield stress temp. dependent	Shul'man	Approximate solution; temperature profile given	Mamedov, Sattarov (1975)	96
Flow between parallel plates driven by a pressure gradient	Constant wall temp. and constant wall heat flux	Bingham	Analytical and numerical; formulas and tables for Nusselt number	Valstar and Beek (1963)	192
Flow between parallel plates, driven by a pressure gradient	Allowance for viscous dissipation, and discontinuous change in yield stress and plastic viscosity with temp.	Bingham	Qualitative analysis and numerical calculation	Shul'man Khusid Benderskaya (1979)	166
Flow between parallel plates, driven by a pressure gradient	Constant wall heat flux, taking into account viscous dissipation	Bingham	Analytical method; asymptotic Nusselt number obtained	Payvar (1973)	132
Flow between parallel plates, driven by a pressure gradient	Constant wall temp.; constant wall heat flux	Casson	Analytical method; temperature profiles, Nusselt numbers	Victor (1974) Shah (1980)	197 160

Table IX cont/..

Flow System	Characteristic of Heat Transfer	Fluid Model	Method and Result	Author(year)	Reference
Couette flow between parallel plates (upper plate moving, with pressure gradient)	Constant plate temp.	Shul'man	Analytical method; temp. profile, heat transfer rate obtained	Shul'man Volchenok (1977)	168
Couette flow between parallel plates (upper plate moving, with pressure gradient)	One plate constant temperature, other adiabatic	Shul'man	Analytical and numerical; temperature profile obtained	Volchenok Shul'man (1978)	201
Flow in a circular tube	Constant or variable wall temperature	Bingham	Analytical method; temp. profile, Nusselt formula obtained; solution is in error /69/	Hirai (1959)	68
Flow in a circular tube	Internal heat generation, thermally insulated wall	Bingham	Analytical and numerical method; temp. profile obtained	Schechter Wissler (1959)	155
Flow in a circular tube	Constant wall temp.; constant wall heat flux	Bingham	Analytical method; asymptotic Nu formulas derived	Beek Eggink (1962)	11

Table IX cont/..

Flow System	Characteristic of Heat Transfer	Fluid Model	Method and Result	Author(year)	Reference
Flow in a circular tube	Constant wall temp.; constant wall heat flux	Casson	Numerical method; temp. profile, Nu obtained; example given	Victor and Shah (1973) Shah (1980)	198 160
Flow in a circular tube	Constant wall temp.; constant wall heat flux; viscosity temp.-dependent	Herschel-Bulkley	Numerical method; velocity profile, Nu obtained	Forest Wilkinson (1973)	48
Flow in a circular tube	Constant wall temp.; yield stress and plastic viscosity temp. dependent (exponential law); consideration of dissipation	Bingham	Perturbation method and Numerical method; velocity profile, size of plug flow region, resistance coefficient obtained	Pervushin (1973)	134
Flow in a circular tube	Constant wall temp.; Viscous dissipation	Bingham	Analytical method; asymptotic Nu obtained	Payvar (1973)	132
Flow in a circular tube	Constant wall temp.;yield stress and viscosity temp. dependent; viscous dissipation	Bingham	Numerical method; temp. profile, Nu, velocity distribution, pressure gradient obtained; compared with theoretical solution and experimental data	Pervushin (1974)	135

Table IX cont/..

Flow System	Characteristic of Heat Transfer	Fluid Model	Method and Results	Author(year)	Reference
Flow in a circular tube	Constant wall temp., viscous dissipation	Shul'man	Analytical method;temp. profile and Nu obtained	Shul'man Gorislavets Rozhkov Uryadova (1970)	165
Flow in a circular tube	Uniform heat source; uniform wall heat flux	Bingham	Analytical method; velocity distribution, temp. distribution, Nu expression obtained	Michiyoshi (1962)	104
Entrance region to circular tube	Constant wall temp.	Bingham	Numerical method; velocity profile, temp. profile, Nu, entrance length obtained	Samant Marner (1971)	151
Entrance region to circular tube	Internal heat generation; thermally insulated wall, uniform wall heat flux, and uniform wall temp.	Bingham	Analytical and numerical methods; temp. profile, entrance length, Nu obtained	Michiyoshi Matsumoto Hozumi (1963)	105

TABLE X

Free-Convection Heat Transfer Problems

Flow System	Characteristic of Heat Transfer	Flow Model	Method and Result	Author(year)	Reference
Parallel plate	Two vertical plates at different temperature	Bingham	Analytical method; velocity profile, wall shear stresses given	Yang, Yeh (1965)	212
Parallel plate	Two horizontal plates heated from below; constant physical properties	Bingham	Analytical method; discussion of the convective stability	Gershuni Zhukhovitskii (1973)	53
Vertical plate	Constant wall temperature; constant properties except density	Bingham	Numerical method; velocity temp. profile, Nu, friction coefficient given	Kleppe Marner (1972)	78
Vertical plate	Constant wall temperature; constant properties except density	Bingham	Numerical method; study of effects of buoyancy on forced convection	Marner Kleppe (1976)	98
Closed square region	Heated from below; constant temp. at vertical plate; temp. changes linearly on horizontal segment	Williamson	Numerical method; Velocity,temp.,heat flux distribution, quasisolid zones given	Lyubimova (1976)	89

in which subscript w means "evaluated at the wall", and β_1 and β_2 are constants. Experimental confirmation of these equations is incomplete.

Of course many heat-transfer problems may be solved by analogy with mass-transfer problems. However, very little seems to have been done on mass-transfer problems /3, 43/.

8. CONCLUSIONS

a. The literature on materials with a yield stress is extensive and is spread out over many fields (engineering, lubrication, food-stuffs, paints, sewage sludge, etc.). There is no single source that one can consult for an authoritative compilation of experimental and theoretical information.

b. More needs to be done to relate the model parameters (such as μ_0 and τ_0 for the Bingham model) to structural details using fundamental hydrodynamic and electrostatic theories. An example of this kind of activity is the recent paper by Tanaka and White /181/, in which a cell theory is developed to explain the origin of the yield stress τ_0 in highly concentrated suspensions. It may be that photographic structure studies of the type begun by Graham /55/ will ultimately shed light on the structure and rheology of suspensions.

c. Another area that has not been explored is the use of computer simulation of particle motions in concentrated suspensions and pastes, making use of knowledge about the hydrodynamic and electrostatic interactions in the medium.

d. Although many efforts have been made to obtain empirical expressions for μ_0 and τ_0 in terms of particle shape and size for suspensions and pastes, there seems to have been no systematic effort to explore in a fundamental way the connection between particle shapes and constitutive equations. In the polymer solution of polymer melt fields striking advances have been made in the last two decades.

e. The Oldroyd papers cited in the introduction have really not been exploited. There remains much to be done in getting nontrivial solutions for viscoplastic flows and the Oldroyd papers should be used as the point of departure.

f. Very little seems to be known about the temperature dependence of model parameters such as μ_0 and τ_0.

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Notes added in proof:

a. Two significant reviews of microstructure and thixotropy of suspensions are J. Mewis and A.J.B. Spaull, *Adv. in Colloid and Interfacial Sci.*, **6**, 173-300 (1977), and J. Mewis, J. Non-Newtonian Fluid Mech., **6**, 1-20 (1979).

b. Yield-stress determination in blood has been discussed by G.R. Cokelet, Chapter 4 in "Biomechanics: Its Foundations and Objectives", (Eds: Y.C. Fung, N. Perrone, and M. Anliker), Prentice-Hall, Englewood Cliffs, N.J.

c. Constitutive equations for plastic viscoelastic fluids have been discussed by J.L. White, J. Non-Newtonian Fluid Mech., 8, 195-202 (1981).

d. Squeezing flow of Bingham fluids between two parallel disks has been re-examined by G.H. Covey and B.R. Stanmore, J. Non-Newtonian Fluid Mech., 8, 249-260 (1981).

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