AMA 563 Assignment 1

1. Suppose X follows an exponential distribution with pdf $f_X(x) = \lambda e^{-\lambda x}$ on $x > 0$, and zero elsewhere. Find the moment generating function of X. Find the expectation of $E\frac{1}{\sqrt{X}}$.

Solution: $f_X(x) = \lambda \cdot e^{-\lambda x}$ $\quad x \in (0, +\infty)$

MGF: $E(e^{tX}) = \int_0^\infty e^{tx} \cdot \lambda \cdot e^{-\lambda x}\, dx$

$$= \lambda \cdot \int_0^\infty e^{(t-\lambda)x}\, dx$$

$$= \lambda \cdot \frac{-1}{\lambda - t} \cdot e^{-(\lambda - t)x}\Big|_0^\infty$$

$$= \frac{\lambda}{\lambda - t}$$

$$E\left(\frac{1}{\sqrt{x}}\right) = \int_0^\infty \frac{1}{\sqrt{x}} \cdot \lambda \cdot e^{-\lambda x}\, dx$$

$$= \lambda^{\frac{3}{2}} \cdot \int_0^\infty (\lambda x)^{-\frac{1}{2}} \cdot e^{-\lambda x}\, dx$$

$$= \lambda^{\frac{1}{2}} \cdot \int_0^\infty (\lambda x)^{-\frac{1}{2}} \cdot e^{-\lambda x} \cdot d(\lambda x)$$

$$= \sqrt{\lambda \pi}$$

2. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\theta) = \frac{1}{\theta - 1}$ on $1 \le x \le \theta$ and zero elsewhere, with $\theta > 1$.

a. Find the method of moments estimator $\widehat{\theta}_{\mathsf{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta^2}_{\mathsf{MLE}}$ for θ^2.

c. Find the bias of $\widehat{\theta^2}_{\mathsf{MLE}}$.

Solution : a. $E(X) = \int_1^\theta x \cdot \frac{1}{\theta-1}\,dx = \frac{\theta+1}{2}$

$$\frac{\hat{\theta}_{MME}+1}{2} = \bar{X}$$

$$\hat{\theta}_{MME} = 2\bar{X} - 1$$

b. $X_{(n)} = \max(X_1, X_2, \cdots, X_n)$

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \begin{cases} \frac{1}{(\theta-1)^n}, & \theta \geq X_{(n)} \geq 1 \\ 0, & \text{elsewhere} \end{cases}$$

when $\theta = X_{(n)}$, $L(\theta)$ is maximized.

$$\widehat{\theta^2}_{MLE} = X_{(n)}^2$$

c. $X_1, X_2, \cdots X_n$ iid.

$$P(X_{(n)} < x) = P(X_1 < x, X_2 < x, \cdots, X_n < x)$$

$$= \int_1^x \frac{1}{\theta-1}\,dx \cdot \int_1^x \frac{1}{\theta-1}\,dx \cdots \int_1^x \frac{1}{\theta-1}\,dx$$

$$= \frac{(x-1)^n}{(\theta-1)^n}$$

$$f(x_{(n)}) = P'(X_{(n)} < x) = \frac{n(X-1)^{n-1}}{(\theta-1)^n}$$

$$E(X_{(n)}^2) = \int_1^\theta \frac{n}{(\theta-1)^n} \cdot x^2 \cdot (x-1)^{n-1}\,dx$$

$$= \frac{n}{(\theta-1)^n} \cdot \frac{(\theta-1)^n(\theta n \cdot (\theta + \theta n + 2) + 2)}{n \cdot (n+1) \cdot (n+2)} = \frac{\theta n \cdot (\theta n + \theta + 2) + 2}{(n+1)(n+2)}$$

$$\text{Bias}(\widehat{\theta^2}_{MLE}) = E(x_{(n)}^2) - \theta^2 = \frac{\theta n(\theta n + \theta + 2) + 2}{(n+1)(n+2)} - \theta^2$$

3. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$, $\theta > 0$, zero elsewhere.

a. Find the method of moments estimator $\widehat{\theta}_{\mathsf{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta}_{\mathsf{MLE}}$ for θ.

c. Compute the Fisher information $I(\theta)$.

$$\frac{\theta}{\theta+1}\left[(1-x)^{\theta-1}\cdot(\theta x+1) - (1-x)^{\theta-1}\cdot(1-x)\right]$$

$$\frac{\theta}{\theta+1}\cdot(1-x)^{\theta-1}\cdot(\theta x+1-1+x)$$

$$\frac{\theta}{\cancel{\theta+1}}\cdot(1-x)^{\theta-1}\cdot x\,\cancel{(\theta+1)}$$

Solution: a. $E(X) = \int_0^1 x\cdot\theta\cdot(1-x)^{\theta-1}\,dx$

$$= -\left.\frac{(1-x)^{\theta}(\theta x+1)}{\theta+1}\right|_0^1$$

$$= \frac{1}{\theta+1}$$

$$\frac{1}{\hat{\theta}_{MME}+1} = \overline{X} \qquad \hat{\theta}_{MME} = \frac{1}{\bar{x}} - 1$$

b. $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \cdot \prod_{i=1}^{n} (1-x_i)^{\theta-1}$

$l(\theta) = n\log\theta + (\theta-1)\cdot\sum_{i=1}^{n}\log(1-x_i)$

$l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-x_i)$

$l''(\theta) = -\frac{n}{\theta^2} < 0$

Let $l'(\theta) = 0$ $\qquad \hat{\theta}_{MLE} = -\frac{1}{\log(1-x)}$

c. $I(\theta)$

$\log f(x|\theta) = \log\,\theta(1-x)^{\theta-1} = \log\theta + (\theta-1)\cdot\log(1-x)$

$$\frac{\partial \log f(x|\theta)}{\partial\theta} = \frac{1}{\theta} + \log(1-x)$$

$$\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$$

$$I(\theta) = -E\left(\frac{\partial^2\log f(x|\theta)}{\partial\theta^2}\right) = -E\left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

4. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\theta) = -(\theta+\theta^2)(1+x)^{\theta-1}x$, $-1 < x < 0$, $\theta > 0$, zero elsewhere.

a. Find the method of moments estimator $\widehat{\theta}_{\text{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta}_{\text{MLE}}$ for θ.

c. Compute the minimum possible variance of all the unbiased estimators for θ^2.

Solution: a. $E(x) = \int_{-1}^{0} x \cdot -(\theta+\theta^2)\cdot(1+x)^{\theta-1} \cdot x \, dx$

$$= -(\theta^2+\theta)\cdot\frac{2\Gamma(\theta)}{\Gamma(\theta+3)}$$

$\hat{\theta}_{MME}$ s.t. $-(\hat{\theta}_{MME}^2 + \hat{\theta}_{MME})\cdot\dfrac{2\Gamma(\hat{\theta}_{MME})}{\Gamma(\hat{\theta}_{MME}+3)} = \overline{X}$

b. $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = -\prod_{i=1}^{n} \cdot (\theta^2+\theta)\cdot(1+x_i)^{\theta-1}\cdot x_i$

$$= (\theta^2+\theta)^n \cdot \prod_{i=1}^{n} \cdot (-x_i)\cdot(1+x_i)^{\theta-1}$$

$$l(\theta) = \log L(\theta) = n\cdot\log(\theta^2+\theta) + \sum_{i=1}^{n}\log(-x_i) + \sum_{i=1}^{n}(\theta-1)\cdot\log(1+x_i)$$

$$l'(\theta) = \frac{n\cdot(2\theta+1)}{\theta^2+\theta} + \sum_{i=1}^{n}\cdot\log(1+x_i)$$

$$= \frac{n}{\theta+1} + \frac{n}{\theta} + \sum_{i=1}^{n}\log(1+x_i)$$

$$l''(\theta) = -\frac{n}{(\theta+1)^2} - \frac{n}{\theta^2} < 0$$

$\therefore$ $\hat{\theta}_{MLE}$ s.t. $l'(\hat{\theta}_{MLE}) = 0$ $\hat{\theta}_{MLE} = -\dfrac{2n+\sum_{i=1}^{n}\log(1+x_i) + \sqrt{4n^2+4n\cdot\sum^{n}\log(1+x_i) - 7n^2\cdot\sum^{n}\log(1+x_i)}}{2\cdot\sum_{i=1}^{n}\log(1+x_i)}$

c. $\log f(x|\theta) = \log\theta + \log(\theta+1) + (\theta-1)\log(1+x) + \log(-x)$

$$\frac{\partial^2 \log f(x|\theta)}{\partial\theta} = -\frac{1}{\theta^2} - \frac{1}{(\theta+1)^2}$$

$$I(\theta) = -E\left(-\frac{1}{\theta^2} - \frac{1}{(\theta+1)^2}\right) = \frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}$$

$$\text{R-C Boundry} = \frac{(k'(\theta))^2}{n\cdot I(\theta)} = \frac{4\theta^2}{\frac{n}{\theta^2}+\frac{n}{(\theta+1)^2}} = \frac{4\theta^4(\theta+1)^2}{n(\theta+1)^2+n\theta^2}$$

5. Let $X_1, \dots, X_n$ be a random sample from $\mathsf{Poisson}(\lambda)$.

a. Find the maximum likelihood estimator $\widehat{\lambda}_{\mathsf{MLE}}$ for λ.

b. Compute the Fisher information $I(\lambda)$.

c. Compute the bias of $(\overline{X})^2$ as an estimator for λ^2.

d. Compute the minimum possible variance of all the unbiased estimators for $\frac{1}{\lambda}$.

Solution

a. $$L(\lambda) = \prod_{k=1}^{n} \cdot \frac{e^{-\lambda} \cdot \lambda^{k}}{k!}$$

$$l(\lambda) = \log L(\lambda) = -n \cdot \lambda + \frac{k(k+1)}{2} \log \lambda - \log \prod_{k=1}^{n} k!$$

$$l'(\lambda) = -n + \frac{k \cdot (k+1)}{2\lambda} \qquad l''(\lambda) = -\frac{k^2+k}{2\lambda^2} < 0$$

$$\hat{\lambda}_{MLE} = \frac{k^2+k}{2n}$$

b. $$\log f(x;\lambda) = \log \frac{e^{-\lambda} \cdot \lambda^k}{k!} = -\lambda + k \cdot \log \lambda - \log k!$$

$$\frac{\partial^2 \log f(x;\lambda)}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(-1 + \frac{k}{\lambda}\right) = -\frac{k}{\lambda^2}$$

$$I(\lambda) = -E\left(\frac{\partial^2 \log f(x;\lambda)}{\partial \lambda^2}\right) = -E\left(-\frac{k}{\lambda^2}\right) = \frac{1}{\lambda}$$

c. $$Var(\bar{X}) = E(\bar{X} - E\bar{X})^2 = E(\bar{X}^2) - (E\bar{X})^2$$

$$E(\bar{X}^2) = Var(\bar{X}) + (E\bar{X})^2$$

$$= Var\left(\frac{1}{n}\sum^{n} X_i\right) + \left(E \frac{1}{n}\sum^{n} X_i\right)^2$$

$$= \frac{1}{n^2} Var\left(\sum^{n} X_i\right) + \frac{1}{n^2}\left(E \sum^{n} X_i\right)^2$$

$$= \frac{n\lambda}{n^2} + \frac{\lambda^2}{n^2} = \frac{\lambda^2}{n^2} + \frac{\lambda}{n}$$

$$Bias(\bar{X}^2) = E(\bar{X}^2) - E(\lambda^2) = \frac{\lambda^2}{n^2} + \frac{\lambda}{n} - \lambda^2$$

d. R-C lower Bound, we have that:

from b. $I(\lambda) = \frac{1}{\lambda}$

R-C lower bound $= \frac{(k'(\lambda))^2}{n I(\lambda)} = \frac{\frac{1}{\lambda^4}}{n \cdot \frac{1}{\lambda}} = \frac{1}{n \lambda^3}$

The minimum possible $Var(\frac{1}{\lambda}) = \frac{1}{n \cdot \lambda^3}$