Generation and Exploitation of Counterexamples in Stochastic Models

Chloé Capon

Mathematics Department
University of Mons

UMONS

Faculté
des Sciences

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Motivations

- Reactive systems are systems that continuously interact with their environment.
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Given a formal model of the system and a specification, the goal is to check that the system satisfies the specification.

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Verification

Given a formal model of the system and a specification, the goal is to check that the system satisfies the specification.

Synthesis

Given a system to control trying to enforce some specification within an uncontrollable environment, it aims at the automated construction of provably-safe system controllers.

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- Synthesis algorithm permit to construct a suitable controller if one exists.
- Otherwise, they simply tell us that no such controller exists.

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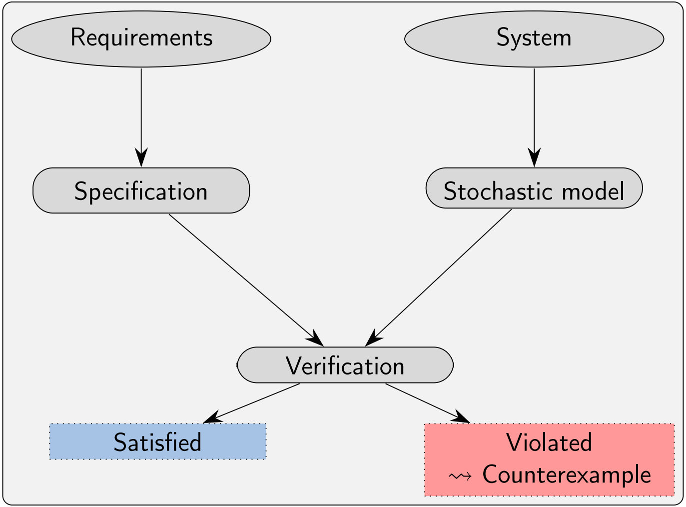
Idea

We need refinement mechanisms based on counterexamples that help practitioners understand:

1. why their attempt failed;
2. how they can patch the system – environment – specification triptych to make synthesis possible and obtain an adequate controller.

Verification

First, let us look at the verification of models containing only probabilistic transitions.

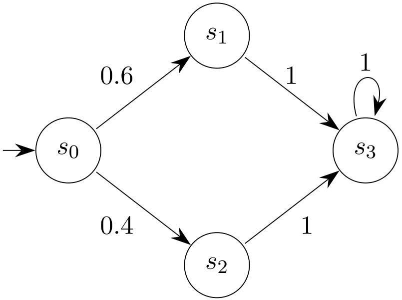


Markov Chains

Definition

A **Markov Chain** (MC) is a tuple $\mathcal{C} = (S, s_{\mathsf{init}}, \delta)$ where:

- S is a finite set of states;
- $s_{\mathsf{init}} \in S$ is an initial state;
- $\delta : S \times S \to [0, 1]$ is a transition probability function such that for all $s \in S$: $\sum_{s' \in S} \delta(s, s') \leq 1$.



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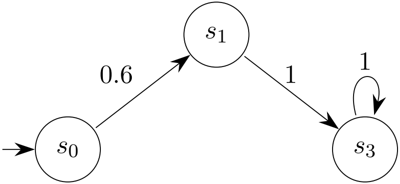

Reachability properties

What is the probability to reach a set of states T when starting in state s?

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Let us consider properties of the form: $\mathbb{P}_{\leq\lambda}(\Diamond T)$ for $\lambda \in \mathbb{Q} \cap [0, 1]$

Let $\mathcal{C} = (S, s_{\mathsf{init}}, \delta)$ be a Markov chain:

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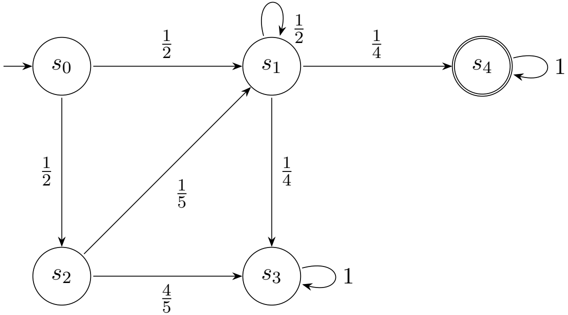

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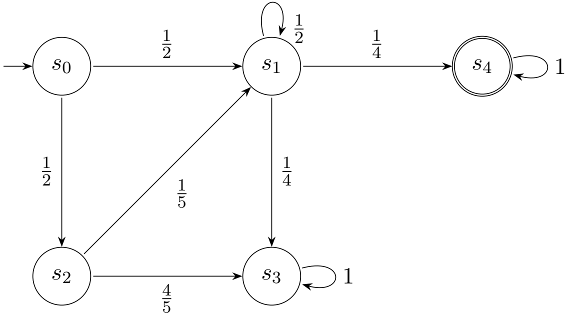

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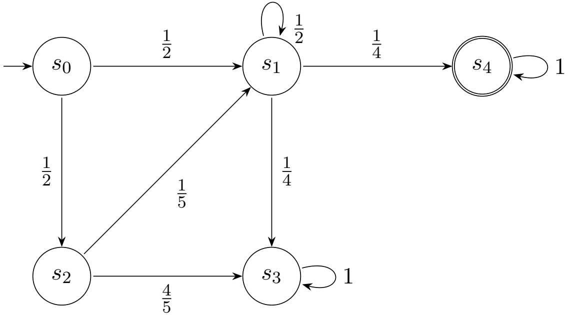

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$\rightsquigarrow \mathcal{C} \not\models \mathbb{P}_{\leq \frac{1}{5}}(\Diamond\{s_4\})$

Counterexamples for MCs

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A **counterexample** for $\mathbb{P}_{\leq\lambda}(\Diamond T)$ is a set of paths, each of them leading from the initial state to some target state in T, such that the probability of the path set is larger than λ.

Counterexamples for MCs

Different notions

Representations of counterexamples at different levels:

- At the level of paths:

The number of paths needed can be very large $\rightsquigarrow$ hard to understand and analyse.

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$\hookrightarrow$ Notion of critical subsystem.

Definition

A subsystem $\mathcal{C}'$ of $\mathcal{C}$ is critical for $\mathbb{P}_{\leq\lambda}(\Diamond T)$ if $\mathcal{C}' \not\models \mathbb{P}_{\leq\lambda}(\Diamond T)$.

Critical subsystems for MCs

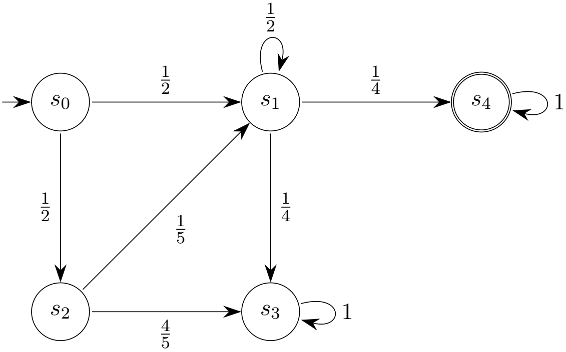
We look for counterexamples that are the most illustrative with regards to the violation of the property:

- A critical subsystem is minimal if it has a minimal set of states under all critical subsystems;
- A critical subsystem is a best critical subsystem if it has the largest probability to reach T among all minimal critical subsystems.

Critical subsystems for MCs

Example

Let's consider this MC and $\mathbb{P}_{\leq \frac{1}{5}}(\Diamond\{s_4\})$:



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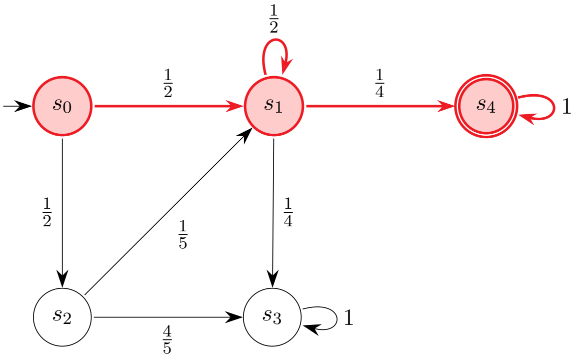

This subsystem is a critical subsystem, since the probability to reach $\{s_4\}$ from s_0 is $\frac{1}{4} > \frac{1}{5}$.

Verification

Adding non-determinism

Now, we consider models with probabilistic transitions and non-deterministic choices: used to describe transitions for which we do not know the exact probabilities.

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Verification

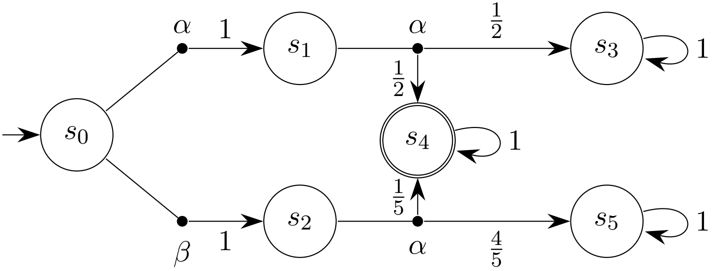
Verify that no matter how we resolve the non-determinism (i.e., no matter the strategy), the specification is verified.

Markov Decision Processes

Definition

A **Markov decision process** (MDP) is a tuple $\mathcal{M} = (S, s_{\mathsf{init}}, A, \delta)$ where:

- S is a finite set of states, s_{init} is an initial state, A is a set of actions;
- $\delta : S \times A \times S \to [0,1]$ is a transition probability function such that for all states $s \in S$ and actions $\alpha \in A(s)$: $\sum_{s' \in S} \delta(s, \alpha, s') \leq 1$.

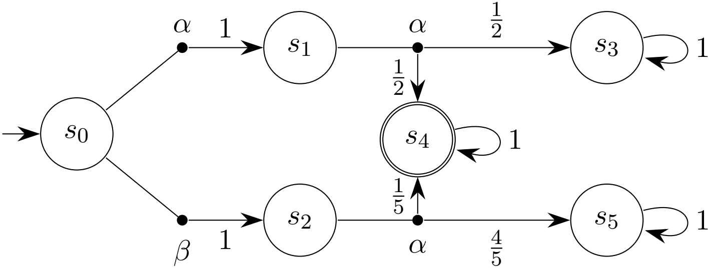


Markov Decision Processes

Strategies

Definition

A **strategy** for an MDP $\mathcal{M} = (S, s_{\text{init}}, A, \delta)$ is a function $\sigma : S \to A$. We denote the set of strategies on $\mathcal{M}$ by $\mathsf{Strat}(\mathcal{M})$.



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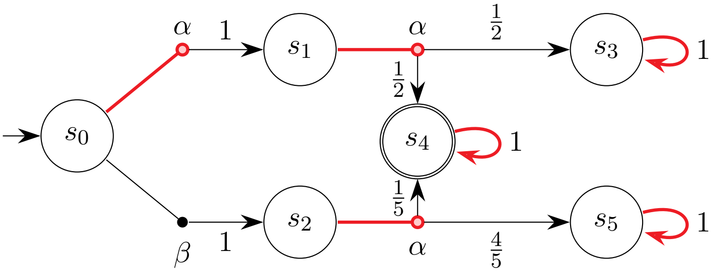

Universal reachability properties

For MDPs, $\mathcal{M} \models \mathbb{P}^{\forall}_{\leq\lambda}(\Diamond T)$ expresses that:

$$\forall \sigma \in \mathsf{Strat}(\mathcal{M}),\ \mathcal{M}^{\sigma} \models \mathbb{P}_{\leq\lambda}(\Diamond T).$$

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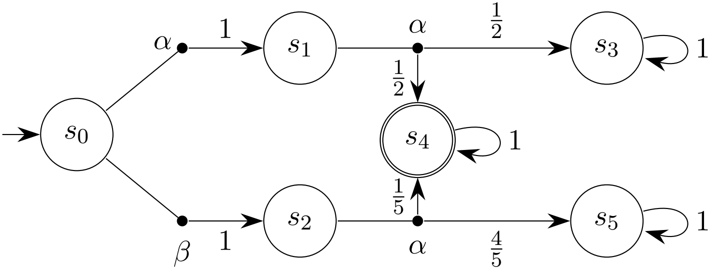
A counterexample consists in finding a strategy $\sigma \in \mathsf{Strat}(\mathcal{M})$ such that the MC induced by σ does not satisfy the corresponding reachability property and extract a critical subsystem.

Such strategies are called critical strategies.

Counterexamples for MDPs

Example

Let us consider the following MDP and $\mathbb{P}^{\forall}_{\leq \frac{2}{5}}(\Diamond\{s_4\})$.



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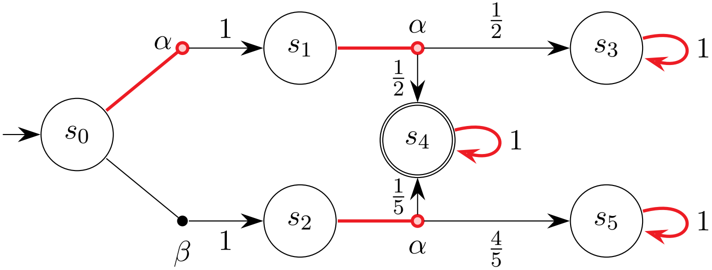

This strategy is critical.

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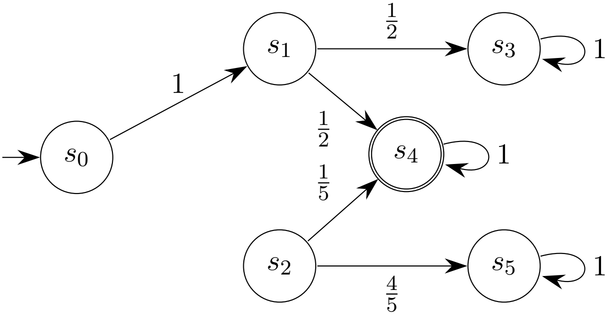

The probability to reach $\{s_4\}$ in the resulting MC is equal to $\frac{1}{2}$.

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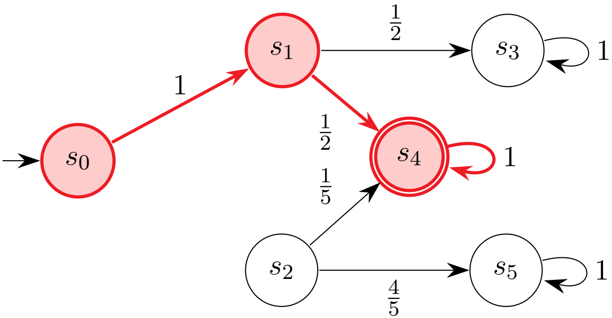

A best critical subsystem, where $\Pr(\Diamond\{s_4\}) = \frac{1}{2}$.

Generation of counterexamples

Existing works about counterexamples in stochastic models:

- In MCs[1] and MDPs[2]: path enumeration, critical subsystems;
- For omega-regular properties and for expected rewards[3];

All these works focus on the verification problem: proving the existence of a bad controller, but not the non-existence of a suitable one.

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What about synthesis?

In this case, the non-determinism is used to model actions of a system that can be controlled.
$\rightsquigarrow$ Find a controller (a suitable strategy) such that the induced MC satisfies the specification.

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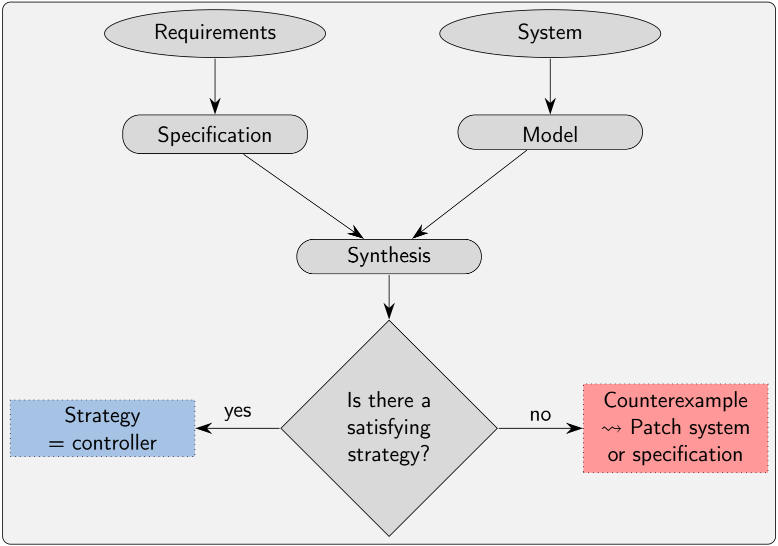
Synthesis


Requirements
System
Specification
Model
Synthesis
Is there a satisfying strategy?
yes
no
Strategy = controller
Counterexample ⇝ Patch system or specification

Generation of counterexamples

Synthesis

Verification

- Specification of the form $\mathbb{P}^{\forall}_{\leq\lambda}(\Diamond T)$: every strategy has a probability smaller than λ to reach T;
- If this specification is false, a counterexample needs to show a strategy that has a probability larger than λ to reach T.

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Synthesis

- Specification of the form $\mathbb{P}^{\exists}_{\leq\lambda}(\Diamond T)$: there exists a strategy with probability smaller than λ to reach T;
- If this specification is false, a counterexample needs to show that every strategy has a probability larger than λ to reach T.

Synthesis mindset

Critical subsystem

Let us consider an MDP $\mathcal{M} = (S, s_{\mathsf{init}}, A, \delta)$.

A **subsystem** $\mathcal{M}'$ of $\mathcal{M}$ is an MDP defined by a set of states $S' \subseteq S$ where for each state $s \in S'$, we have that every action of $A(s)$ is in $\mathcal{M}'$.

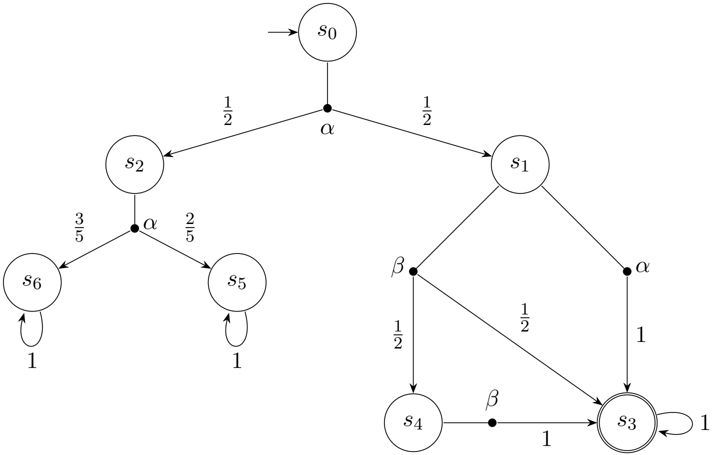
Definition

A subsystem of $\mathcal{M}$ is **critical** for $\mathbb{P}^{\exists}_{\leq\lambda}(\Diamond T)$ if for every strategy σ of $\mathcal{M}'$ we have that $\mathrm{Pr}^{\sigma}_{\mathcal{M}'}(\Diamond T) > \lambda$.

Synthesis mindset

Critical subsystem

Let's consider an MDP $\mathcal{M}$ and $\mathbb{P}^{\exists}_{\leq \frac{1}{5}}(\Diamond\{s_3\})$.



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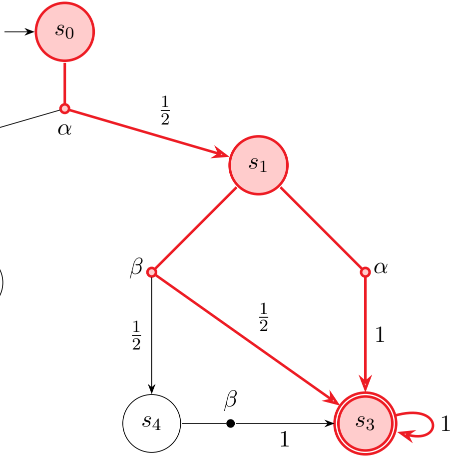

How to generate them?

- For verification: MILP formulations that compute best critical subsystems for MCs[4] and MDPs[5] have been implemented.
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- For synthesis: we developed an MILP formulation that computes best critical subsystems with the new mindset.

The idea of our approach

- Only consider a minimising strategy of the subsystem;
- If the probability of this strategy to reach T is already too large, then every strategy will have a probability to reach T that is too large.

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Exploitation of counterexamples

Two approaches that exploit the information given by counterexamples:

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[7] Ceska et al., "Counterexample-guided inductive synthesis for probabilistic systems".

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Two approaches that exploit the information given by counterexamples:

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$\rightsquigarrow$ It does not help when no suitable controller exists.

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Conclusion

- Overview of existing works on the generation of counterexamples for verification stochastic models;
- Introduction of a new notion of counterexamples for synthesis of MDPs.

Ongoing work

- Assess the performance of our method in practice;
- Use the new notion of counterexamples introduced in this talk for when the synthesis process fails.

Thank you for your attention!

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