Spin Transformation and Conformal Curvature Flow

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Conformal Transformations

- Surface in $\mathbb{R}^3$
- Rotate, Scale, ~~Shear~~

- Represent surface by $f : M \to \mathrm{Im}\,\mathbb{H}$
- Def: f and $\tilde{f}$ are spin equivalent if there is $\lambda \in \mathbb{H}$ at each point

$$d\tilde{f} = \bar{\lambda} df \lambda$$

Quaternions

- Rotation and scaling of $v \in \operatorname{Im}\mathbb{H}$:

$$\tilde{v} = \bar{q} v q$$

$$q = a(\cos(\theta/2) + \sin(\theta/2) u)$$

- Rotation by θ around axis u
- Scaling by a^2

What are *all* the conformal transformations?

- For arbitrary λ, the equation $d\tilde{f} = \bar{\lambda} df \lambda$ may have no solution

- To characterize all valid transformations λ, introduce integrability condition

$$(D - \rho)\lambda = 0$$

Dirac operator

mean curvature half-density

Integrability Condition

- On surfaces with trivial topology,

$d\tilde{f} = \bar{\lambda} df \lambda$ is exact if and only if $\bar{\lambda} df \lambda$ is closed

$$0 = d(\bar{\lambda} df \lambda) = \overline{\bar{\lambda} df \wedge d\lambda} - \bar{\lambda} df \wedge d\lambda.$$

This implies $\bar{\lambda} df \wedge d\lambda$ is purely real

$$D\lambda = -\frac{df \wedge d\lambda}{|df|^2} = \rho\lambda.$$

$$\rho : M \to \mathbb{R}$$

Dirac Operator

- $\psi : M \to \mathbb{H}$ can be decomposed as

$$\underbrace{\psi}_{\text{quaternion}} = \underbrace{a}_{\text{real}} + \underbrace{df(Y)}_{\text{tangent}} + \underbrace{bN}_{\text{normal}}$$

- Dirac operator

$$\mathcal{D}\psi = \begin{bmatrix} 0 & -\mathrm{curl} & 0 \\ \mathcal{J}\mathrm{grad} & -S & \mathrm{grad} \\ 0 & -\mathrm{div} & 2H \end{bmatrix} \begin{bmatrix} a \\ Y \\ b \end{bmatrix}$$

Mean Curvature Half-Density

- Mean curvature half-density = $H|df|$
- Scale-independent since H is inversely proportional to length

- For a spin transformation,

$$\tilde{H}|d\tilde{f}| = H|df| + \rho|df|$$

Solving Integrability Condition

- For any given scalar function $\rho : M \to \mathbb{R}$,
  Solve eigenvalue problem
  $$(D - \rho)\lambda = \gamma\lambda$$
  Get solution by adding a shift to ρ :
  $$(D - (\rho + \gamma))\lambda = 0$$

Discretization

- $f_i \in \mathrm{Im}\,\mathbb{H}$ and $\lambda_i \in \mathbb{H}$ live on vertices
- Discrete Dirac operator is sparse, rectangular

$$\mathsf{D} \in \mathbb{H}^{|F| \times |V|}$$

- Recall continuous version

$$D\lambda = -\frac{df \wedge d\lambda}{|df|^2} = \frac{d(df\,\lambda)}{|df|^2}$$

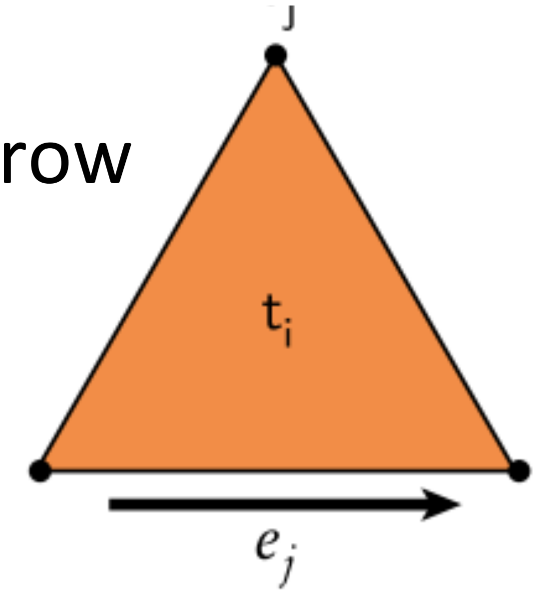
Discretization

- Integrate $D\lambda$ over a triangle

$$\frac{1}{\mathcal{A}_l}\int_{\sigma_l} D\lambda|df|^2 \;=\; -\frac{1}{2\mathcal{A}_l}(e_i\lambda_i + e_j\lambda_j + e_k\lambda_k)$$

- D is a matrix with 3 entries per row

$$\mathsf{D}_{ij} = -\frac{1}{2\mathcal{A}_i}e_j$$



Discretization

- Integrate ρ over triangles

$$\frac{1}{\mathcal{A}_i}\int_{\sigma_i}\rho\lambda|df|^2 = \rho_i\left(\frac{1}{3}\sum_{v_j\in\sigma_i}\lambda_j\right)$$

Rectangular Eigenvalue Problem

- $(D-\rho)\lambda = \gamma\lambda$ where $A = (D-\rho) \in \mathbb{H}^{|F|\times|V|}$
- In practice, interested in smallest eigenvalue – no need to worry about mixing eigenspaces
- Solve for

$$\underbrace{\mathsf{A}^*\mathsf{A}\lambda = \gamma \mathsf{M}_V \lambda}_{}$$

mass matrix

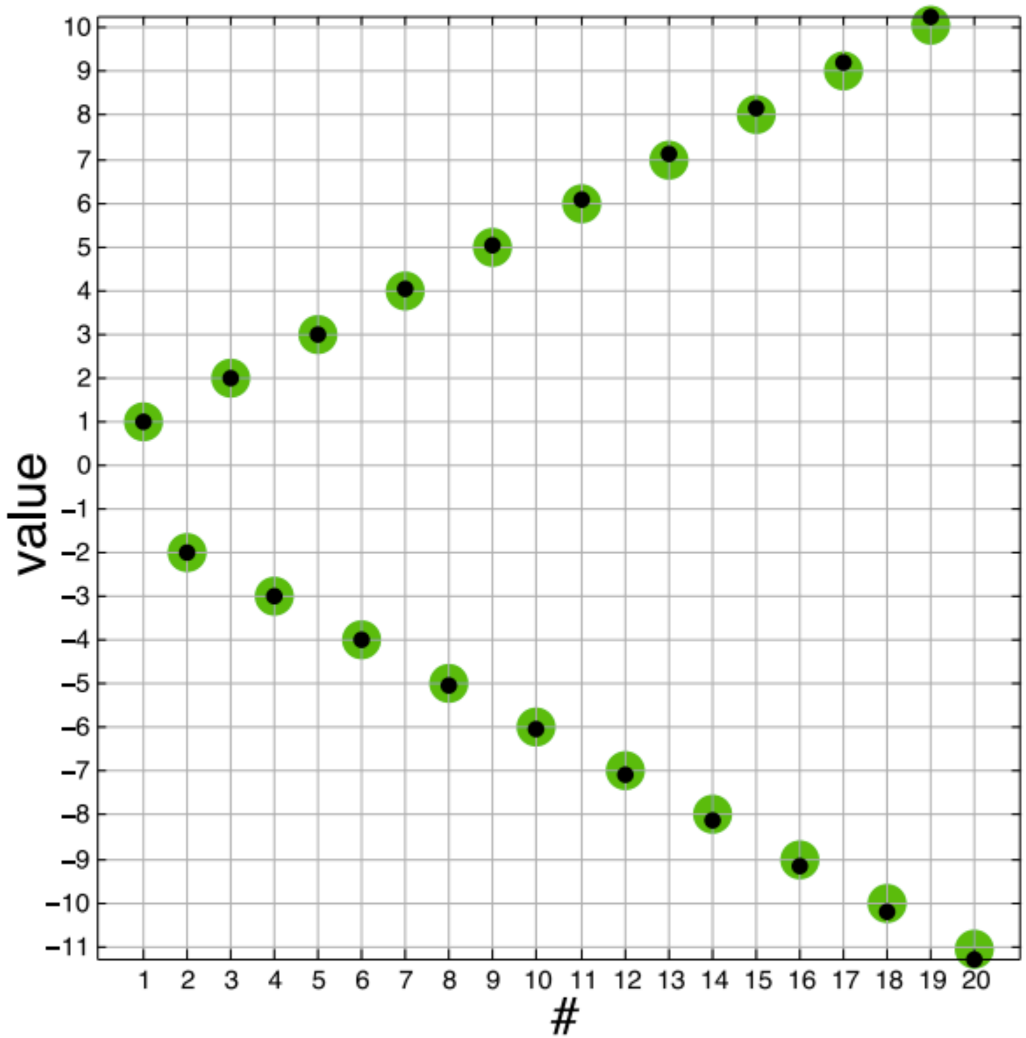


Figure 8: *Even on a coarse mesh of 8k triangles the spectrum of our discrete operator (black dots) matches the predicted spectrum (green dots) exceptionally well. Here we plot the first twenty distinct eigenvalues (out of 120 total) for the unit sphere.*

Solving for Vertex Positions

- Know $\lambda_i \in \mathbb{H}$ for each vertex

$$\tilde{e}_{ij} = \frac{1}{3}\bar{\lambda}_i e_{ij} \lambda_i + \frac{1}{6}\bar{\lambda}_i e_{ij} \lambda_j + \frac{1}{6}\bar{\lambda}_j e_{ij} \lambda_i + \frac{1}{3}\bar{\lambda}_j e_{ij} \lambda_j$$

- To solve for $d\tilde{f} = \tilde{e}$,

  Minimize residue $r = |d\tilde{f} - \tilde{e}|^2$

  Standard Poisson problem

$$\Delta \tilde{f} = \nabla \cdot \tilde{e}$$

Applications: Painting Curvature

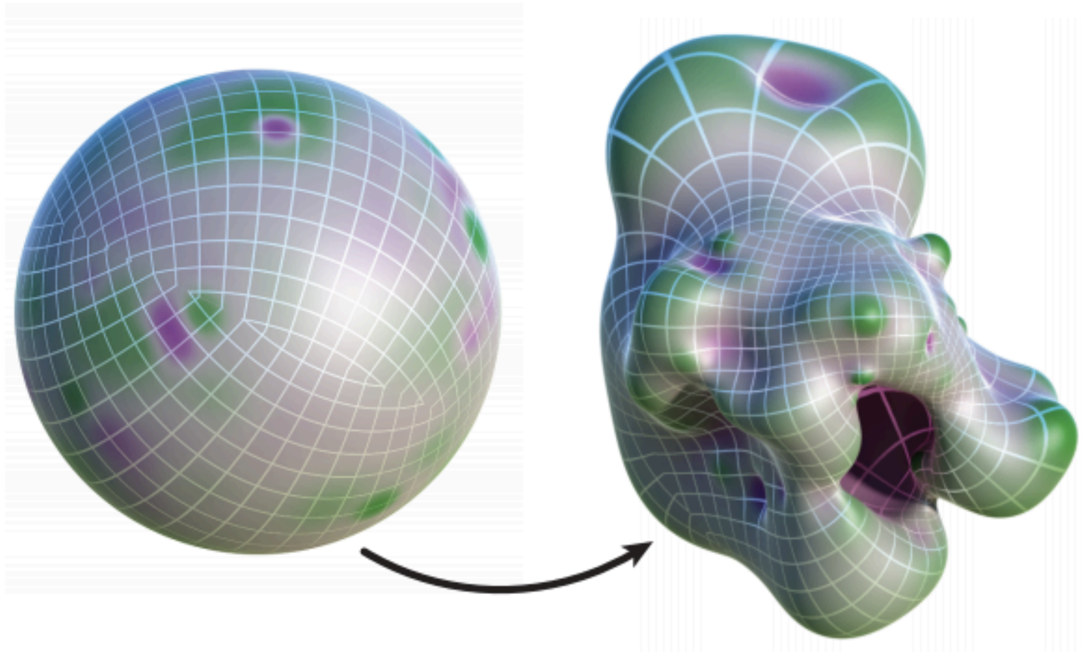


Figure 2: *Given desired change in curvature* (left) *we construct a new conformally equivalent surface* (right). *Green and purple indicate a positive and negative change in curvature, respectively.*

Application: Arbitrary Deformation

Find nearby spin transformations for arbitrary deformations:

- Get μ_i at each vertex by polar decomposition
- Find an approximate change in curvature

$$\rho_i = \frac{\mathrm{Re}((\overline{\mathsf{B}\mu})_i(\mathsf{D}\mu)_i)}{|(\mathsf{B}\mu)_i|^2}$$

- Solve for λ_i and $\tilde{f}$

Figure 14: *Giraffe attempts ballet: a textured mesh* (left) *can be modified arbitrarily and projected onto the nearest conformally equivalent surface* (middle, $\mathcal{Q}_{mean} = 1.015$), *preserving texture fidelity. We can also explicitly modify scale without disturbing texture – on the right we ask for a much larger head.*

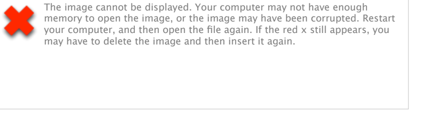
Conformal Curvature Flow

- Use mean curvature half-density $H|df|$ instead of position to represent surface
- Integrability condition
  (generalized to arbitrary topology, for both curves and surfaces)
- Benefits:
  - Easier to minimize Willmore energy
  - Large time steps
  - Naturally preserves mesh quality

Willmore Energy

$$E_W(f) := \int_M H^2 dA = \tfrac{1}{4}\langle\langle \Delta f, \Delta f \rangle\rangle = \tfrac{1}{4}\langle\langle \Delta^2 f, f \rangle\rangle$$

- If ignore dependence of Δ on f, can be approximated by

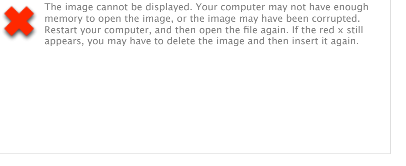


$$\dot{f} = -\tfrac{1}{2}\Delta^2 f$$

Non-linear PDE

- Instead,

$$E(u) := ||u||^2$$



generic curvature κ for curves or μ for surfaces

Integrability Constraints on Curves

- Must ensure the curvature remains integrable as it evolves according to the flow
  - Tangents must agree at endpoints

$$\langle\langle \dot{\kappa}, \mathbb{1} \rangle\rangle = \int_0^L \dot{\kappa} \, d\ell = 0$$

  - Endpoints must meet

$$\int_0^L \dot{\kappa} f \, d\ell = 0 \qquad \langle\langle \dot{\kappa}, f^x \rangle\rangle = \langle\langle \dot{\kappa}, f^y \rangle\rangle = 0$$

Linear constraints!

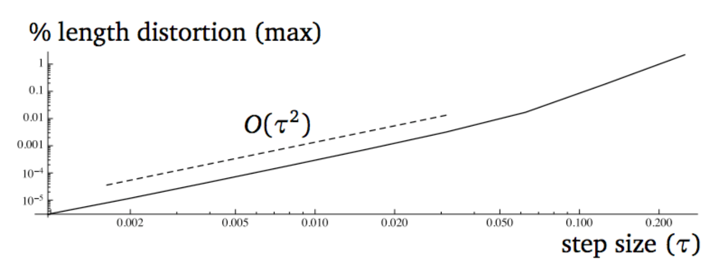
- Enforce constraints by projecting onto Gram-Schmidt basis

Recovering Position

- Length is preserved by construction (in continuous setting)

$$\tilde{T}(s) = (\cos\theta(s), \sin\theta(s))$$
$$\theta(s) = \theta_0 + \int_0^s \kappa \, d\ell$$
$$\tilde{f}(s) = \tilde{f}_0 + \int_0^s \tilde{T} \, d\ell$$

- In discrete setting, solve $\min_{\tilde{f}} |d\tilde{f} - \tilde{T}|^2$
- Small length distortion O(step size^2)



Algorithm for Curves

STEP	DESCRIPTION	CURVE
I.	Evaluate curvature.	$\kappa \leftarrow \frac{1}{2}\langle N, \Delta f\rangle$
II.	Pick a desired flow direction.	$\dot{\kappa} \leftarrow -\nabla E_C(\kappa)$
III.	Build a constraint basis.	GRAM-SCHMIDT $\{\mathbb{1}, f^x, f^y\}$
IV.	Project flow onto constraints.	$\dot{\kappa} \leftarrow \dot{\kappa} - \sum_i \langle\langle \dot{\kappa}, \hat{c}_i \rangle\rangle \hat{c}_i$
V.	Take an explicit Euler step.	$\kappa \leftarrow \kappa + \tau\dot{\kappa}$
VI.	Recover tangents.	$\tilde{T} \leftarrow$ INTEGRATE κ
VII.	Recover positions.	SOLVE $\Delta\tilde{f} = \nabla \cdot \tilde{T}$

Conformal Surface Flow

- $\rho : M \to \mathbb{R}$ describes a change in curvature

$$\dot{\rho} = -H$$

- Integrability constraint $(D - \rho)\lambda = \gamma\lambda$
- Solve for smallest eigenvalue to recover λ
- Recover tangents $\tilde{T} = \bar{\lambda} T \lambda$
- Recover positions $\Delta \tilde{f} = \nabla \cdot \tilde{T}$

Constraints on Non-trivial Topology

- Total curvature remains constant $\langle\langle \dot{\rho}, \mathbb{1} \rangle\rangle = 0$
- Exactness
  - When the surface is not simply connected, must ensure $\bar{\lambda} df \lambda$ has no harmonic component
  - Equivalently, for each of the $2g$ harmonic vector field basis v_i and each solution $DZ_i = v_i$,

$$\langle\langle \dot{\rho}, Z_i^x \rangle\rangle = \langle\langle \dot{\rho}, Z_i^y \rangle\rangle = \langle\langle \dot{\rho}, Z_i^z \rangle\rangle = 0$$

- Avoid inversion

$$\langle\langle \dot{\rho}, N^x \rangle\rangle = \langle\langle \dot{\rho}, N^y \rangle\rangle = \langle\langle \dot{\rho}, N^z \rangle\rangle = 0$$

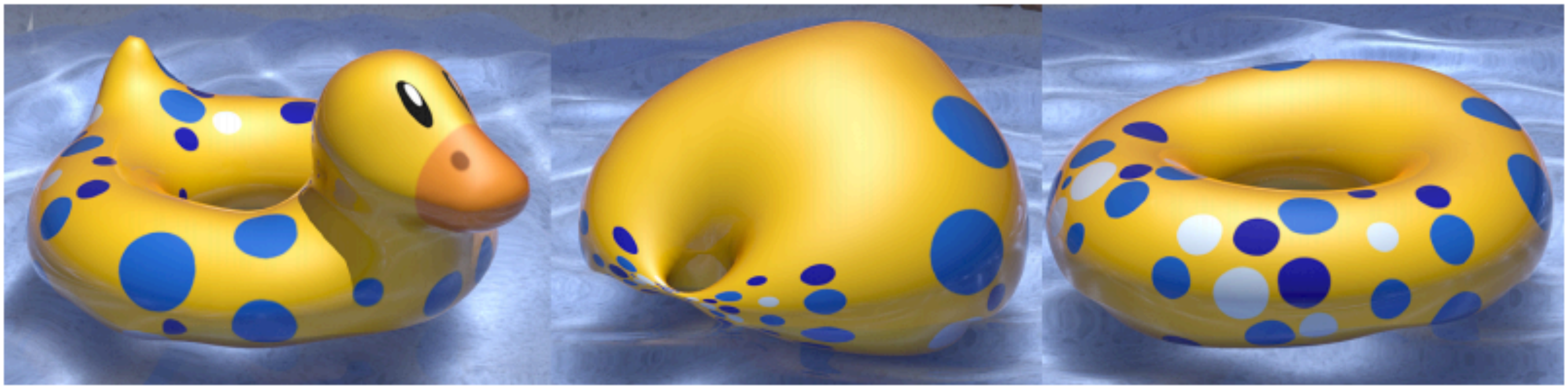


Figure 12: *Duck with nontrivial topology* (left). Center: *unconstrained flow yields distortion of both geometry and texture.* Right: *an exactness constraint prevents distortion.*

Figure 13: *Topologist's view a of a coffee cup.* Top: *Sphere inversions neither distort angles nor increase Willmore energy, but distort area in undesirable ways.* Bottom: *a simple linear constraint prevents unnecessary inversion, and improves area distortion by flowing toward a desired metric.*

Algorithm for Surfaces

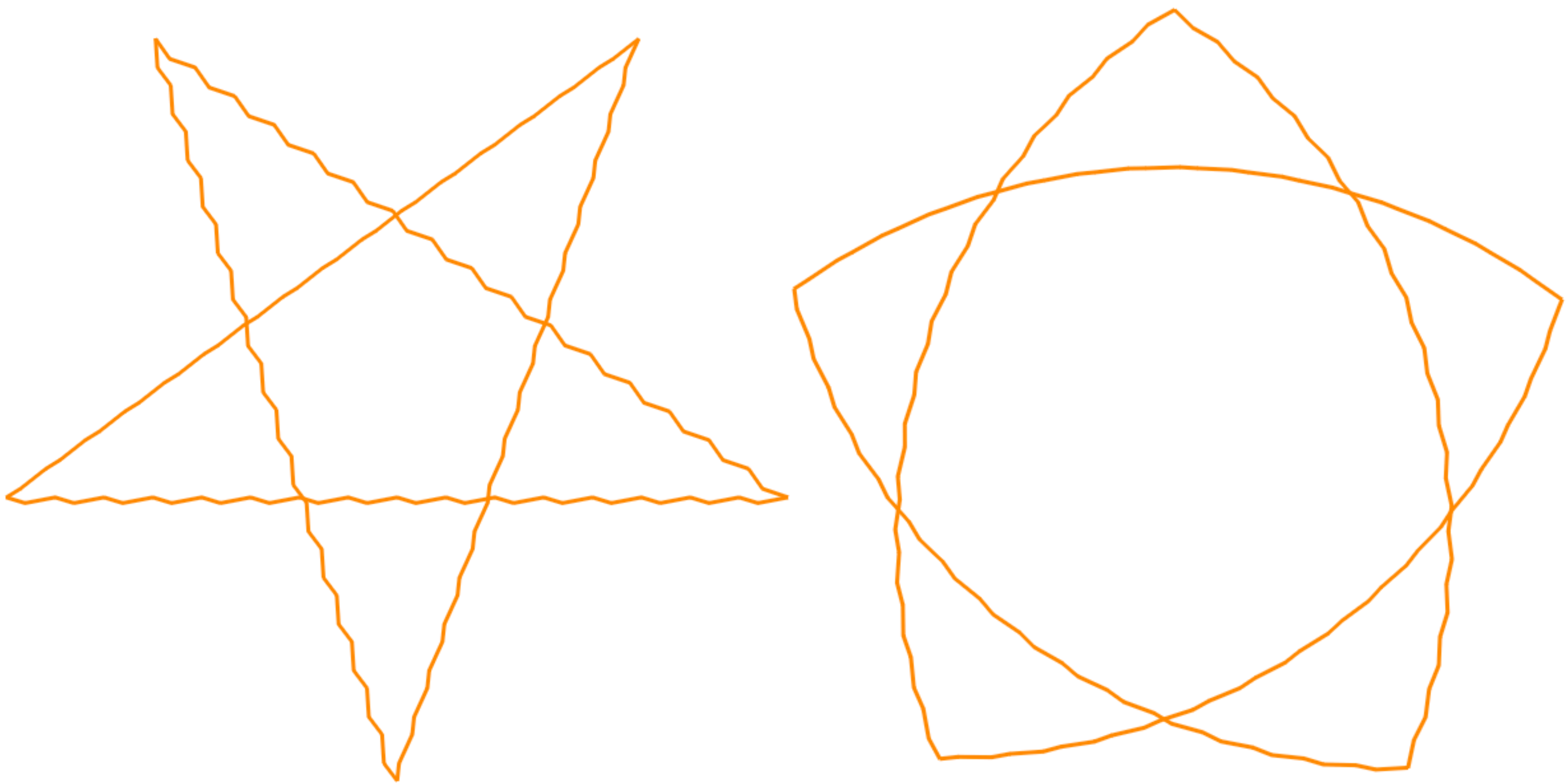
Step	Description	Surface
I.	Evaluate curvature.	$H \leftarrow \frac{1}{2}\langle N, \Delta f\rangle$
II.	Pick a desired flow direction.	$\dot{\rho} \leftarrow -\nabla E_W(\rho)$
III.	Build a constraint basis.	Gram-Schmidt $\{\mathbb{1}, N^{x,y,z}, Z_i^{x,y,z}\}$
IV.	Project flow onto constraints.	$\dot{\rho} \leftarrow \dot{\rho} - \sum_i \langle\langle \dot{\rho}, \hat{c}_i \rangle\rangle \hat{c}_i$
V.	Take an explicit Euler step.	$\rho \leftarrow \rho + \tau\dot{\rho}$
VI.	Recover tangents.	Solve $(D-\rho)\lambda = \gamma\lambda$, $\tilde{T} \leftarrow \bar{\lambda} T \lambda$
VII.	Recover positions.	Solve $\Delta\tilde{f} = \nabla \cdot \tilde{T}$

Curvature Filter

- $\dot{u} = -2u$ takes gradient w.r.t the $\mathscr{L}^2$ norm on curvature rather than position

- Large features shrink at the same rate as small bumps

- Apply high pass filter on flow direction



Conformal curvature flow without curvature filter

Spectrum of Discrete Laplacian

- Discrete Laplacian for 1D curve

$$K = \frac{1}{2}\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

- Real eigenvalues $0 \leq k_1 \leq \cdots \leq k_n \leq 2$

$$k_j = 1 - \cos(2\pi \lfloor j/2 \rfloor / n)$$

Spectrum of Discrete Laplacian

- Real eigenvalues $k_j = 1 - \cos(2\pi \lfloor j/2 \rfloor / n)$
- Eigenvectors

$$(u_j)_h = \begin{cases} \sqrt{1/n} & \text{if } j = 1 \\ \sqrt{2/n}\, \sin(2\pi h \lfloor j/2 \rfloor / n) & \text{if } j \text{ is even} \\ \sqrt{2/n}\, \cos(2\pi h \lfloor j/2 \rfloor / n) & \text{if } j \text{ is odd}\,. \end{cases}$$

- As k_j increases, u_j changes more rapidly

Spectrum of Discrete Laplacian

- More generally, weighted discrete Laplacian on surfaces $$\Delta x_i = \sum_{j \in i^\star} w_{ij} \left(x_j - x_i \right)$$
- Weight function

$$w_{ij} = \frac{\phi(v_i, v_j)}{\sum_{h \in i^\star} \phi(v_i, v_h)} \quad \phi(v_i, v_j) = \|v_i - v_j\|^\alpha$$

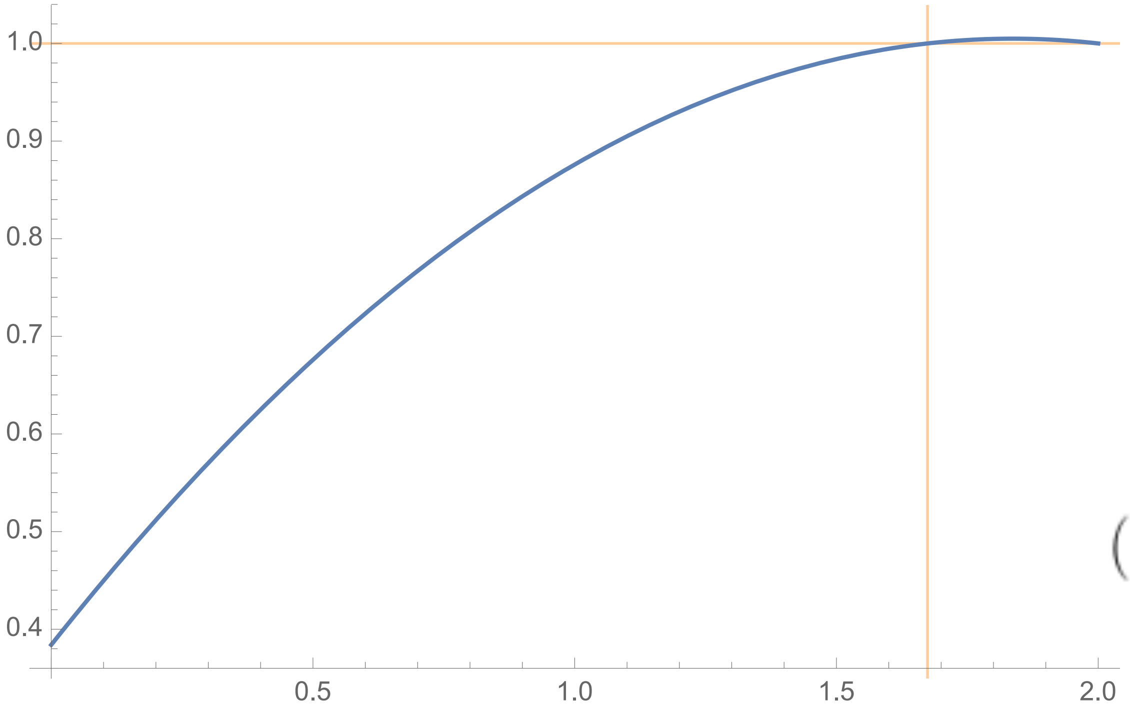
- Eigenvalues still satisfy

$$0 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq 2$$

Filter Design

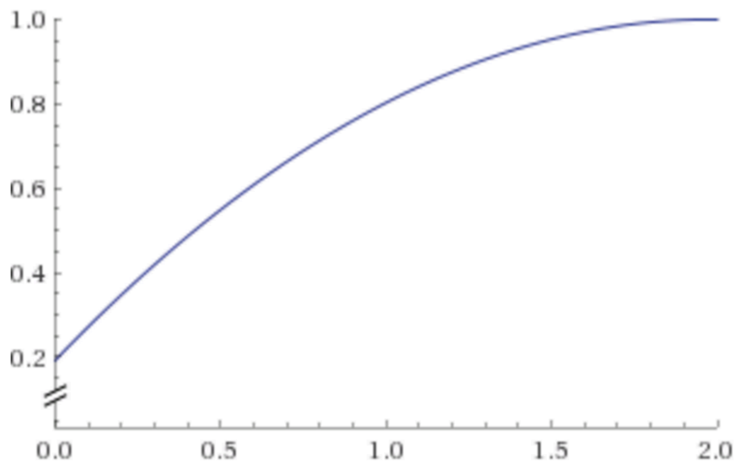
$$f(x) = (1 - \lambda(2 - x))(1 + \mu(2 - x))$$
$$\dot{\kappa} = (I - \lambda(2 - K))(I + \mu(2 - K))\dot{\kappa}$$

- Pass-band frequency $k_{PB} = 2 - \dfrac{1}{\lambda} + \dfrac{1}{\mu}$ $\quad f(k_{PB}) = 1$



$$(\lambda, \mu) = (0.4, 0.46)$$
$$k_{PB} = 1.674$$

Examples of High-Pass Filter




```
f[x_, lambda_, mu_] := (1 - lambda (2 - x)) (1 + mu (2 - x))
Plot[f[x, 0.45, 0.46]^20, {x, 0, 2}]
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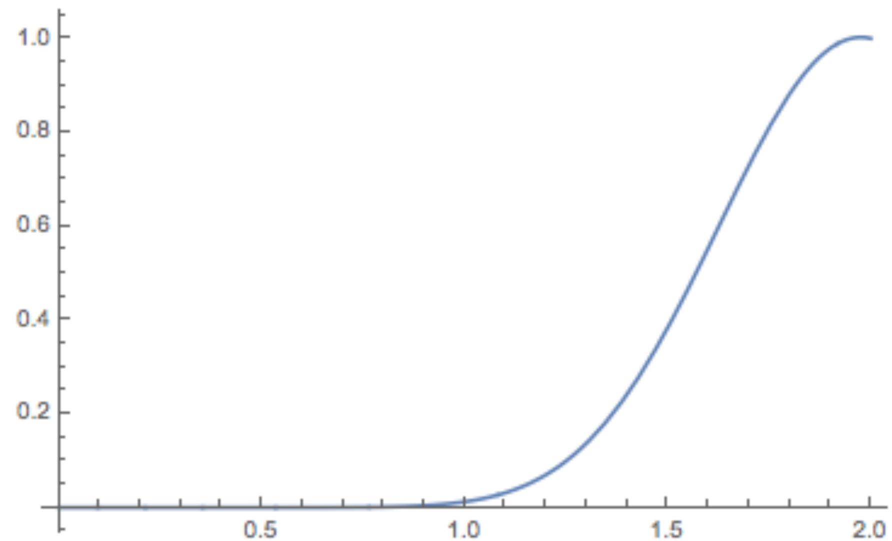


$$f(x) = (1 - \lambda(2 - x))(1 + \mu(2 - x))$$

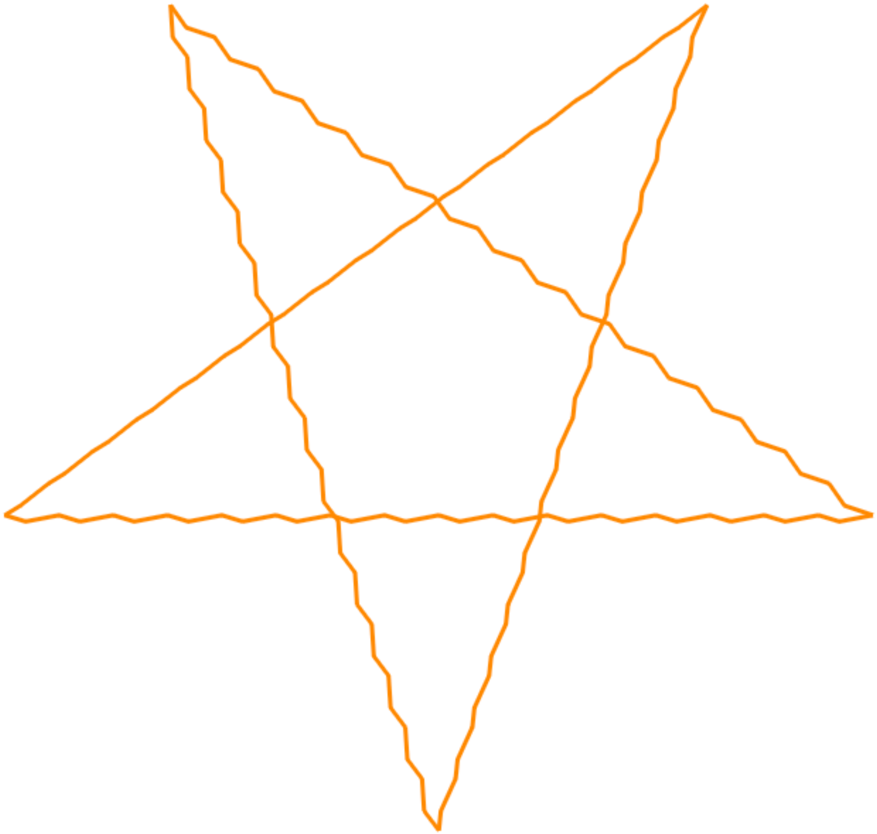
$$(\lambda, \mu) = (0.45, 0.46)$$

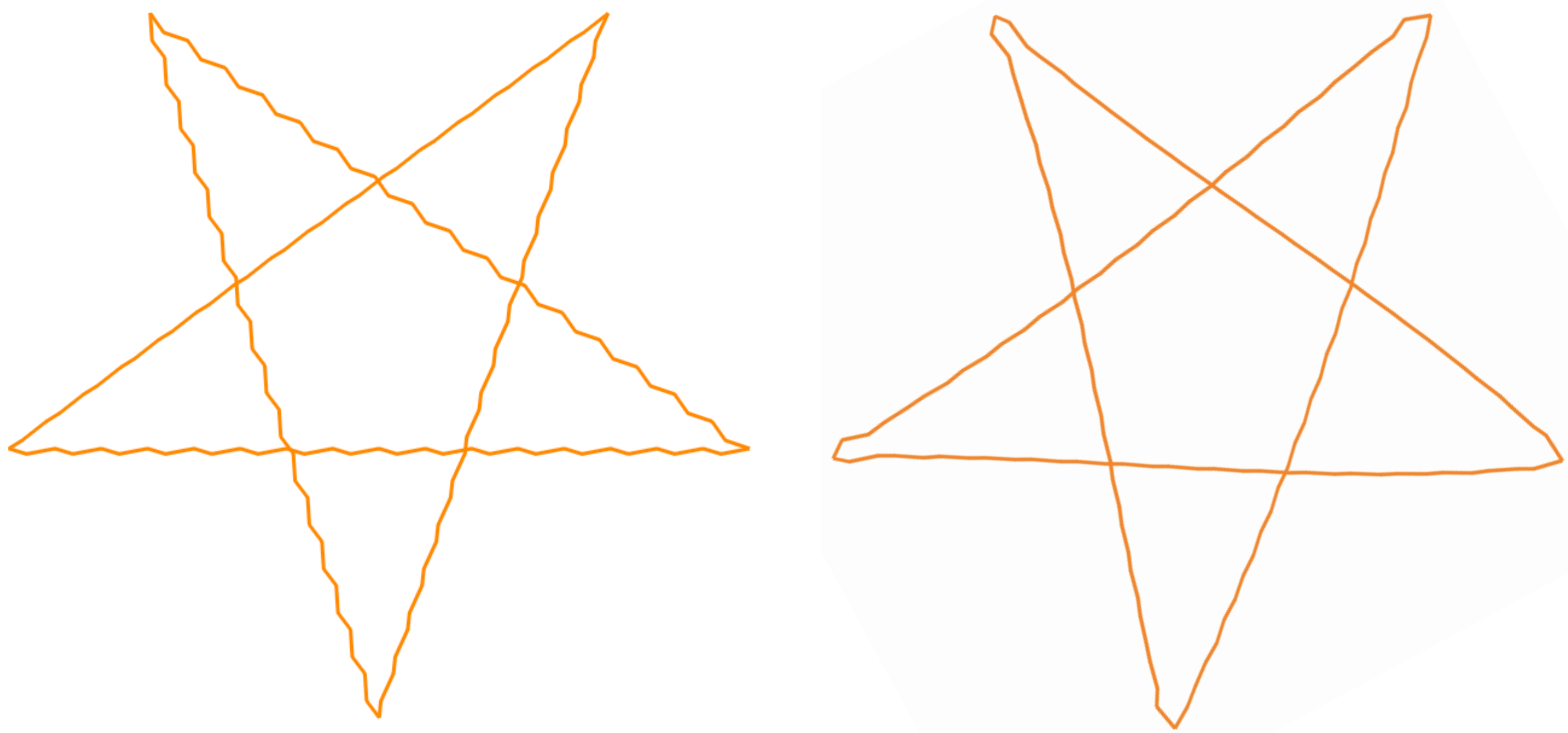
$$k_{PB} = 1.952$$

$$f(x)^N$$

$$N = 20$$

Star Example



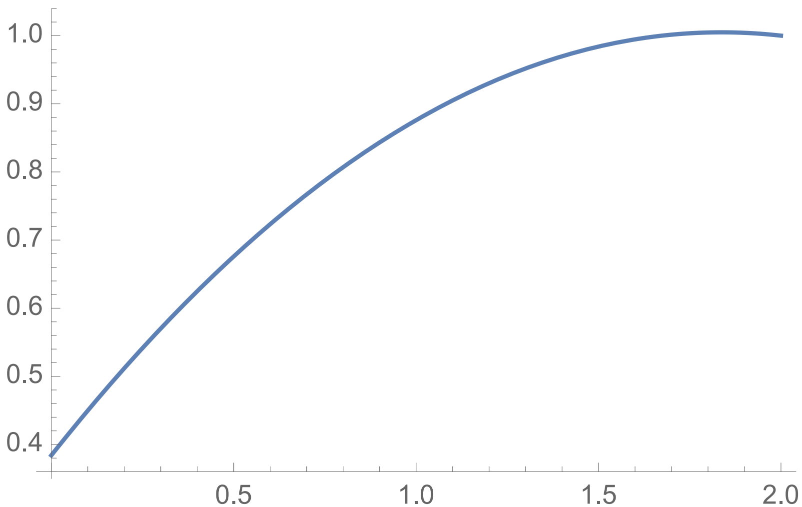


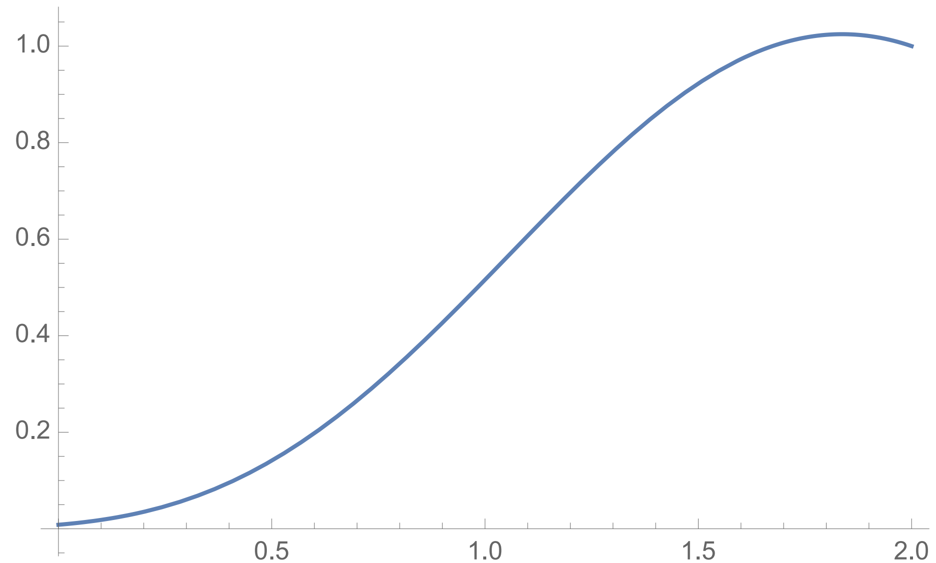
$$f(x) = (1 - \lambda(2 - x))(1 + \mu(2 - x))$$

$$(\lambda, \mu) = (0.45, 0.46)$$

$$f(x)^N \quad N = 20 \qquad k_{PB} = 1.952$$

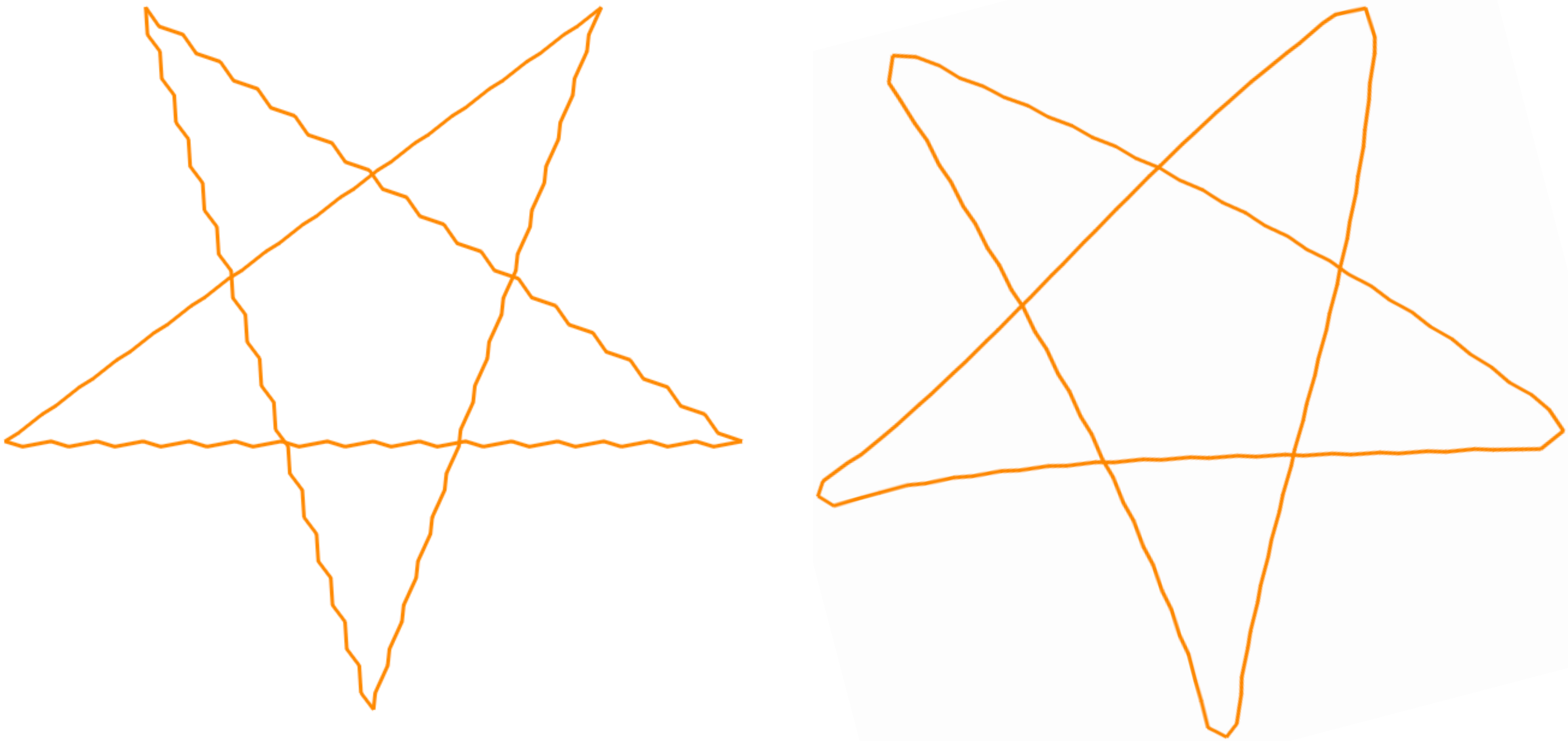
Examples of High-Pass Filter





$$f(x) = (1 - \lambda(2 - x))(1 + \mu(2 - x))$$
$$(\lambda, \mu) = (0.4, 0.46)$$
$$k_{PB} = 1.674$$

$$f(x)^N$$
$$N = 5$$

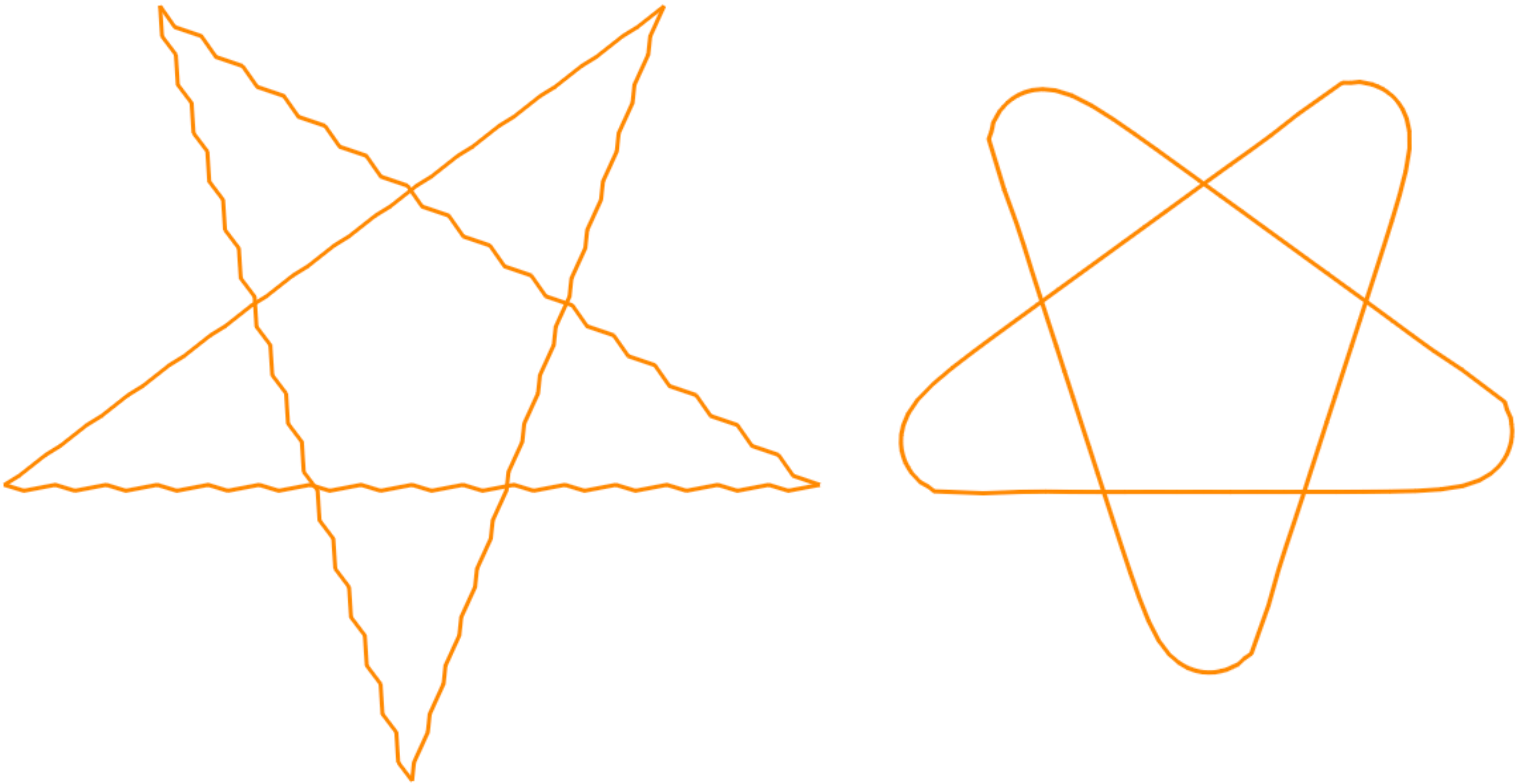


$$f(x) = (1 - \lambda(2-x))(1 + \mu(2-x))$$

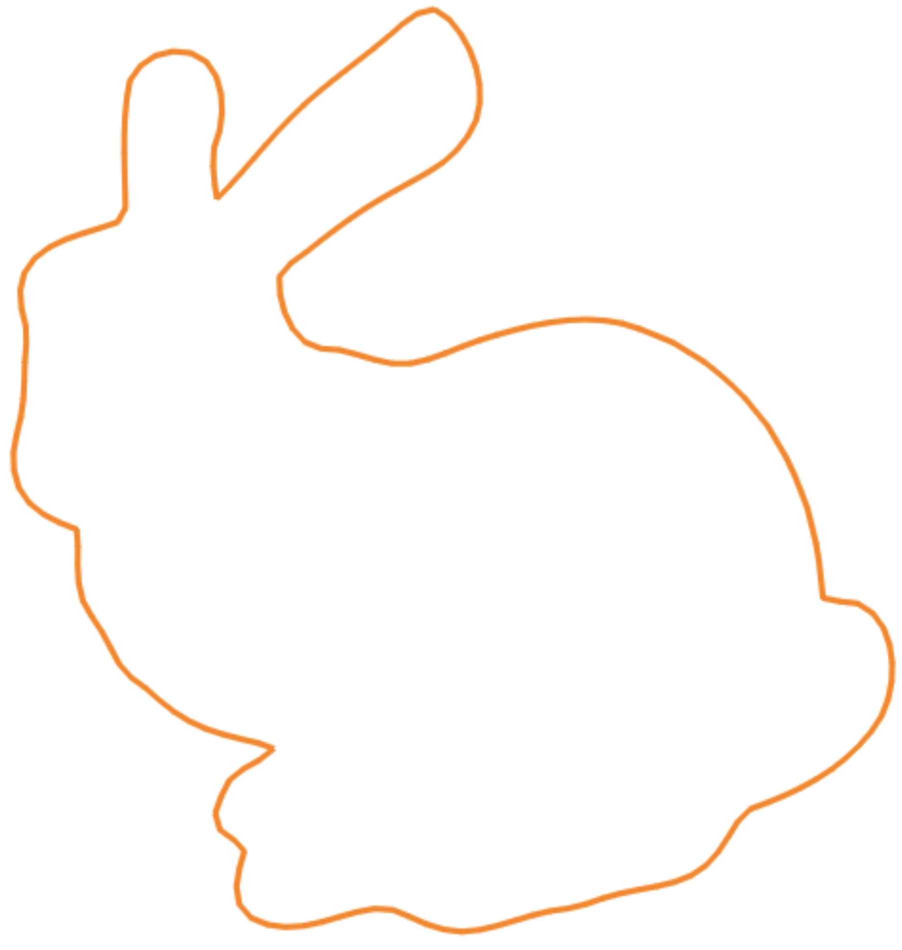
$$(\lambda, \mu) = (0.4, 0.46)$$

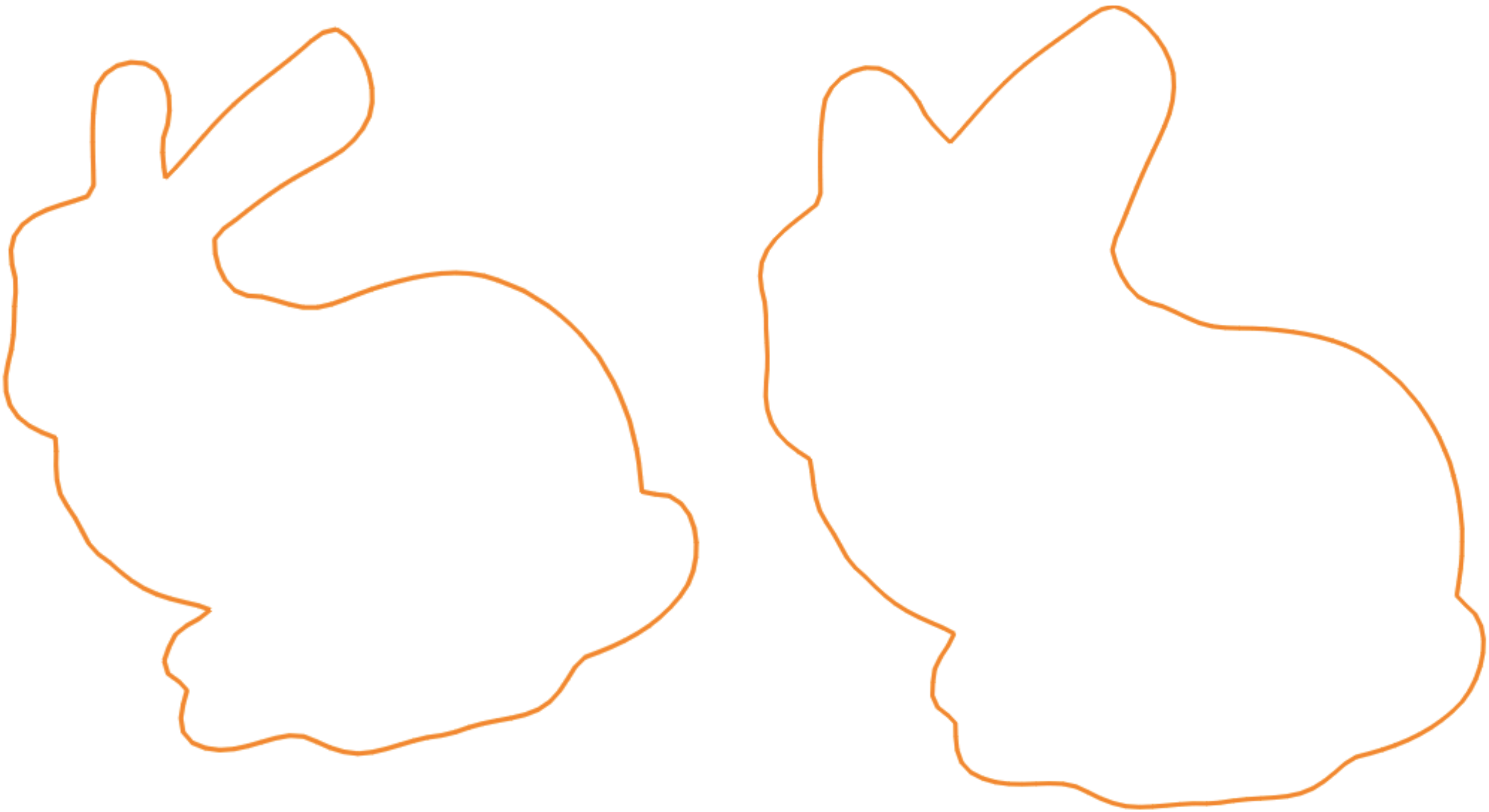
$$f(x)^N \quad N = 5 \qquad k_{PB} = 1.674$$

Comparison: Position-Based Willmore Flow

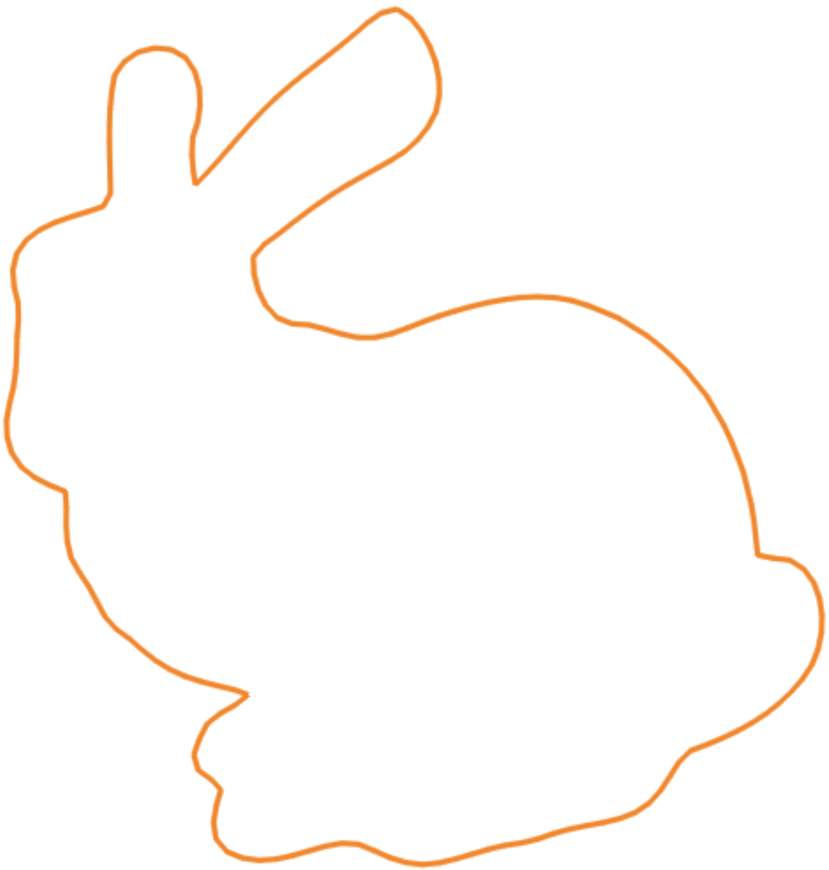


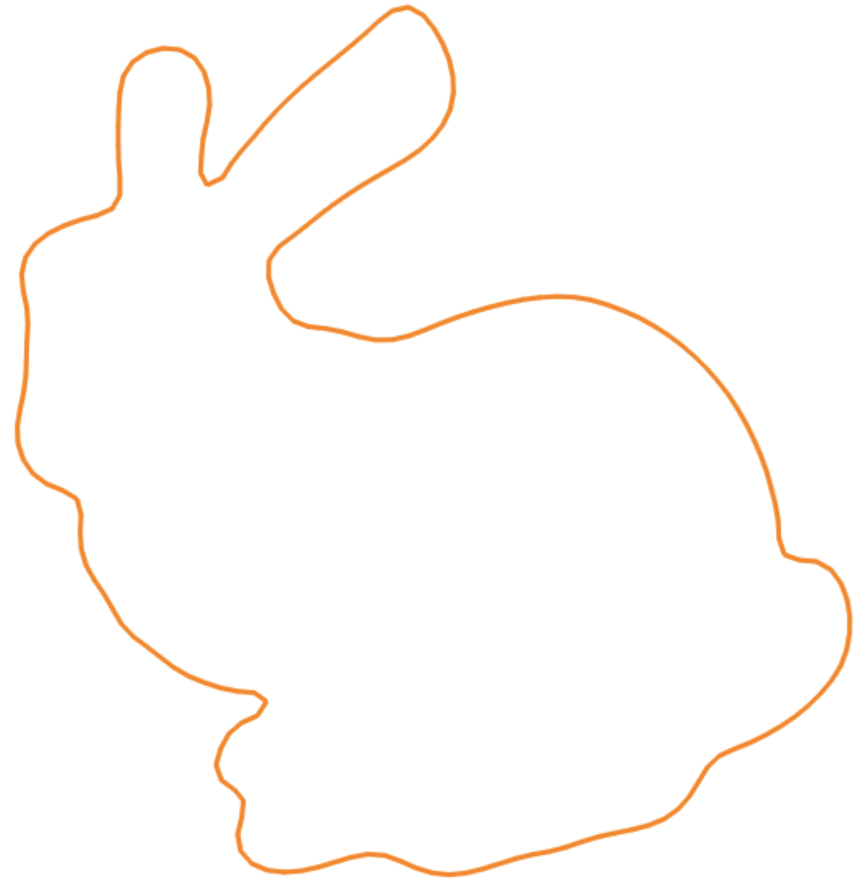
Bunny Example

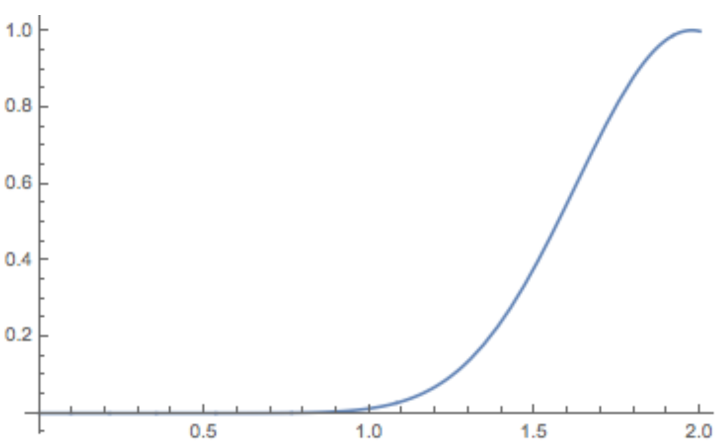




Conformal Curvature Flow without filter

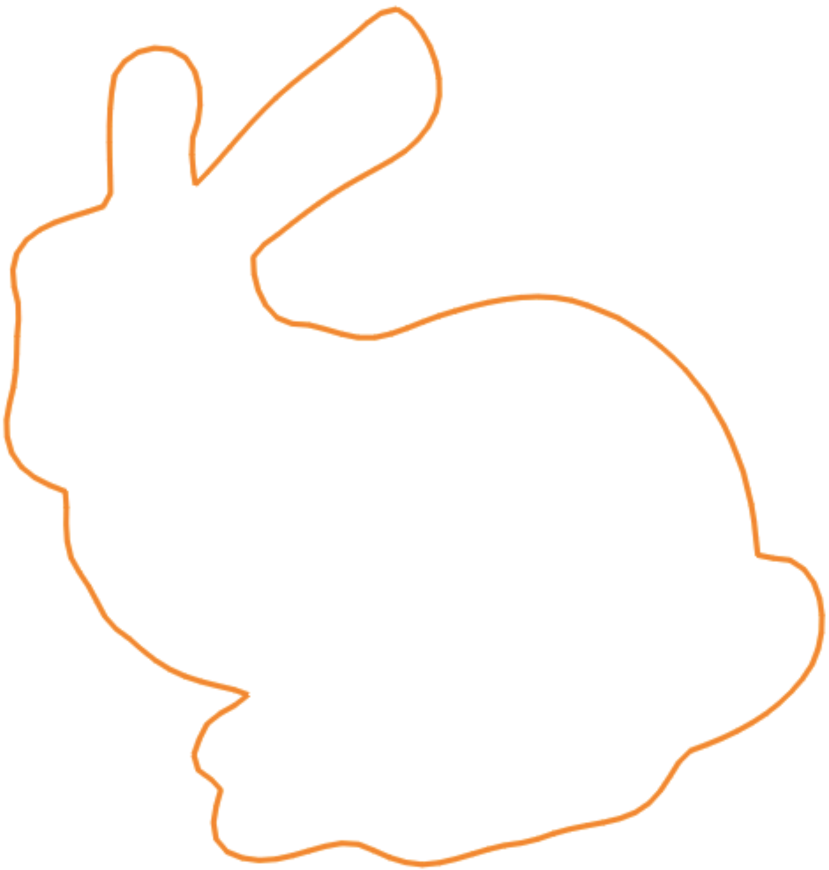


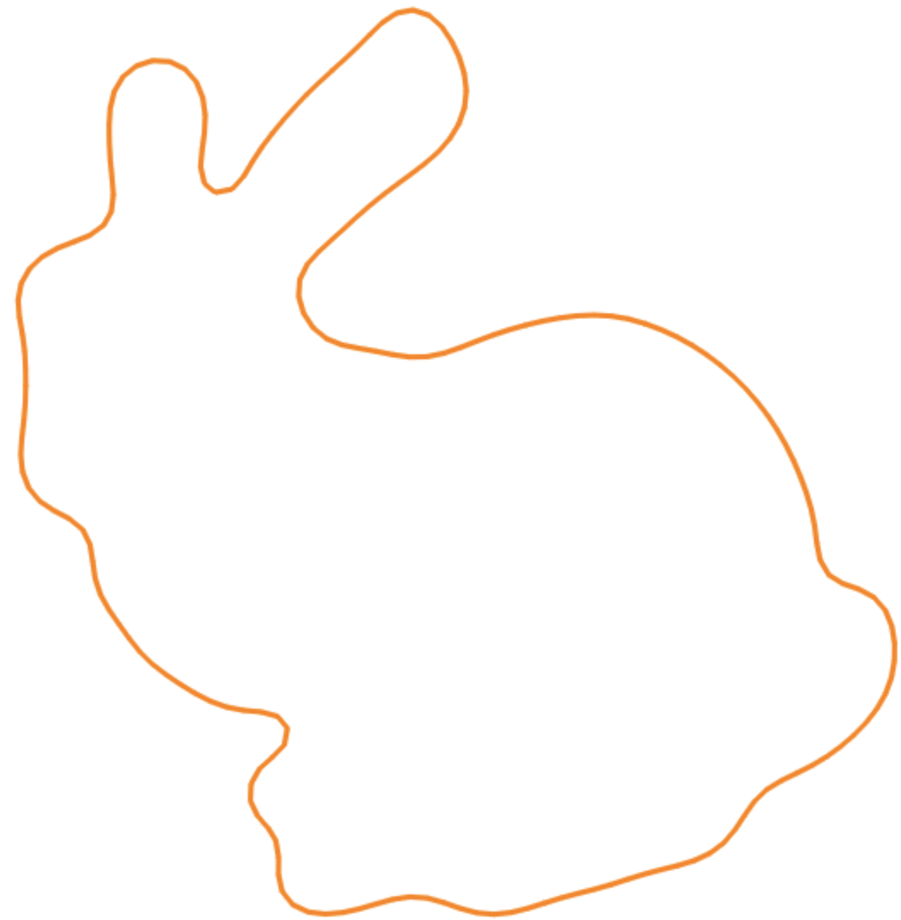


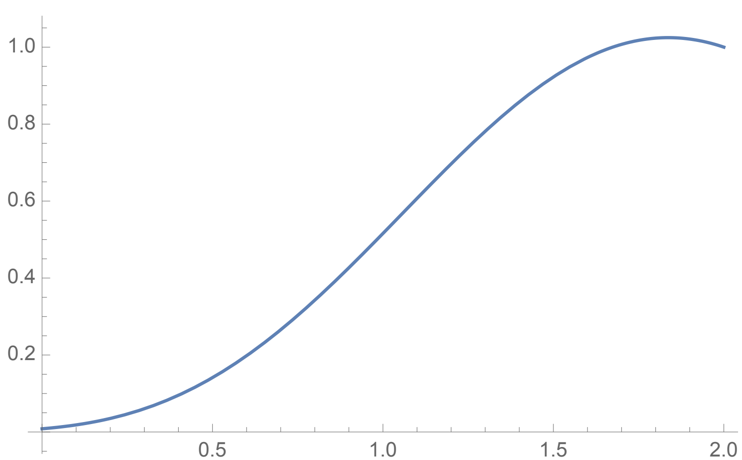


$(\lambda, \mu) = (0.45, 0.46) \qquad N = 20$

$k_{PB} = 1.952$







$$(\lambda, \mu) = (0.4, 0.46) \qquad N = 5$$

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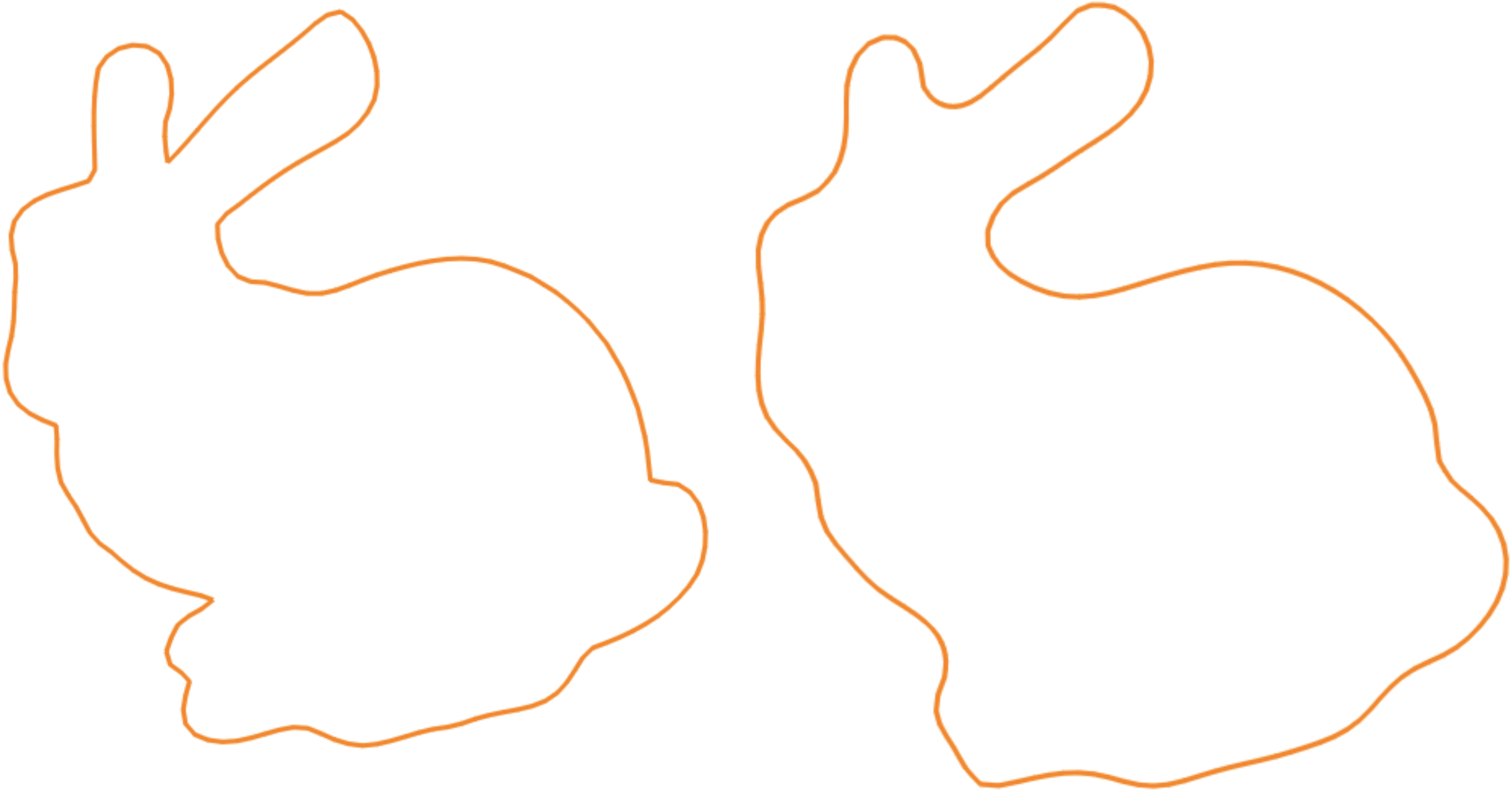

Position-based Willmore Flow

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