Some operations in 3D space: Basics

any point P1 has three coordinates: $[P1_x , P1_y , P1_z]$ relative to an origin at 0, 0, 0

any finite line has a start point, say A, and a distinct end point, say B: A---B

parametric representation of a line, $A + u(B - A)$, has already been mentioned

if $0 <= u <= 1$, a point on the line segment between A and B is described

if not, a point on the infinite line through A and B is described

the direction of A---B is a vector B-A, equivalent to a point: $[B_x-A_x , B_y-A_y , B_z-A_z]$

a *unit vector* V has magnitude (or length) 1: $V_x^2 + V_y^2 + V_z^2 = 1$

Some operations in 3D space: Dot Product, Cross Product

The so-called *dot product* of two 3D vectors (written V1•V2) is a scalar,
it is computed by summing pairwise products of individual components,
it can be generalised to two three or more dimensions

$V1•V2 = V1_x*V2_x + V1_y*V2_y +V1_z*V2_z$

magnitude of V mag(V) or ||V|| is $\sqrt{(V•V)}$

useful property: for non-zero V1,V2 V1•V2 = 0 iff V1 is perpendicular to V2

The so-called *cross product* of two 3D vectors (only) (written V1×V2) is a vector,
if V1 and V2 are parallel (even if opposite), ||V1×V2||=0,
otherwise V1×V2 is perpendicular to both V1 and V2: use 'xyzzy' mnemonic

$[V1_y*V2_z-V1_z*V2_y , V1_z*V2_x-V1_x*V2_z , V1_x*V2_y-V1_y*V2_x]$

right-hand-rule: if forefinger points alongV1, middle finger V2, then thumb V1×V2

Some applications of dot product, cross product

Dot:

- projection of a point P onto a vector, or onto a finite line A---B
- determine if two 3D vectors are perpendicular

Cross:

- determine if two 3d vectors are parallel
- determine if a convex polygon has a non-coplanar point to its "front" or "rear"
  - assuming say *CCW* (counter clockwise winding order) defines front
- determine if a convex polygon contains a coplanar point
- are three points A B C collinear? $\|(B-A) \times (C-A)\| = 0$ … several alternatives

Combination:

- are two planes coplanar?
- does a triangle intersect a plane, another triangle? …

3D Planar Intersection

Plane-Plane intersection

- Infinite planes are defined by
  - a normal (perpendicular) vector – preferably but not necessarily a unit vector
  - a distance between the plane and the coordinate system origin
    - find distance with dot product: (unit normal) • (any known point on the plane)
  - one may or may not insist that the distance is non-negative; if you have a negative distance you don't want then negate the distance and all the vector components
- two planes are parallel if cross product × of their normal vectors has zero length
- two infinite *parallel* planes intersect *iff* distances using unit normal vectors between plane and coordinate system origin are the same (if normal vectors have same direction) or opposite (if normal vectors have opposite directions)
- two infinite *non-parallel* planes always intersect (just like lines in 2D)

3D Planar Intersection

- Does a Triangle intersect a Plane?
  - this is an important question in building BSP Trees:
    - if a plane-parallel triangle is coplanar, it must be added to a pivot set
    - must a plane-tilted triangle be subdivided? if so, into two or into three?

- Does a Triangle intersect another Triangle?
  - this is an important question when using BSP Trees to decide if mesh objects intersect (collide)

Does **T**riangle T (with vertices A, B, C) intersect **P**lane P? [1/2]

T lies in some plane: one with the normal vector **TN** = (B-A) × (C-A)
(ie the cross product of the direction vectors of *any two* sides of the triangle)

- The unit vector in this direction (the unit normal) **TU** = TN / ||TN||

- The plane in which T lies has distance-to-origin **TD** = (A • TN) / ||TN||
  - (for example; although either B or C could substitute for A)

The plane P is defined by its normal vector and distance-to-origin
  - Call its unit normal **PU**
  - Call its distance-to-origin **PD**

Does **T**riangle T (with vertices A, B, C) intersect **P**lane P? [2/2]

T is parallel to P if TU = ± PU (ie $TU \times PU = [\,0\,,0\,,0\,]$)

- Then T is **coplanar with** (hence lies within the plane) P iff TU * PD = PU * TD

If T is not parallel to P,

- If any of A • PU - PD, B • PU - PD, C • PU - PD are of opposite sign,
  - then T **intersects** P
- Otherwise if any of the above are zero,
  - then T **touches** P at up to two of A, B, C

But in practice it is not that simple. Numerical errors are very likely to arise.
Never never never test floating-point numbers for equality, even against zero.
Treat numbers and differences "very close to zero" as if they actually are zero.

If **T**riangle (with A B C) does intersect **P**lane (with PU PD), where?

If coplanar, it does intersect – and all points within T are also within P

- Point A is on the plane P if its signed distance to the plane A • PU - PD ≈ 0,
  - otherwise *possibly* there is an intersection somewhere along A---B or A---C
    - A---B line if A • PU - PD is opposite sign to B • PU - PD (product is -ve)
    - A---C line if A • PU - PD is opposite sign to C • PU - PD (ditto)
- Similarly, the point B if B • PU - PD ≈ 0, else possibly along A---B or B---C
- Similarly, the point C if C • PU - PD ≈ 0, else possibly along A---C or B---C
- Taking A---B line as example, parametric equation of line is $A+u_{AB}*(B-A)$,
- Intersection is at the point defined by

  u_{AB} = A•PU-PD / ((A•PU-PD) - (B•PU-PD)) *as signs are opposite, difference is nonzero*
  (If $0 < u_{AB} < 1$ then the intersection is indeed **between** A and B, and it is one end-point of a line crossing and splitting the triangle)

Does Triangle T1 intersect Triangle T2? If so, where?

T1, T2 lie in planes P1, P2 (cross-products➔unit normals➔dot product with vertex)

If P1 and P2 are the same plane, use Separating Axis Theorem (from 2D)
Otherwise, ask whether *both* T1 intersects P2 *and* T2 intersects P1

- if not, no intersection/collision is possible
- but even if the triangles do intersect each others' planes, a *miss* is still possible!

Then calculate end-points of lines where T1 intersects P2, also where T2 intersects P1

- there will be four points, two from T1:P2 and two from T2:P1 end-points
- if one triangle touches the other's plane only at one vertex, there is a zero-length line
- if the lines are not both zero-length, they are necessarily collinear
- if lines T1:P2 and T2:P1 overlap, the triangles do intersect along that overlap
  - logic gets slightly tricky, it must handle vertex-in-common, vertex-on-edge, edge-in-common, edge-to-edge cases properly (all with lots of combinations)
  - see e.g. http://web.stanford.edu/class/cs277/resources/papers/Moller1997b.pdf

Does a line A---B intersect a line C---D? If so, where?

If neither A---B nor C---D is associated with a particular plane (such as a triangle gives)

B-A and D-C both produce vectors passing through the origin.
(B-A) × (D-C) defines a vector V perpendicular to each of them.
A•V = B•V and C•V = D•V are projections of A, B, C, D on to V

If A•V ≠ C•V then A---B and C---D are skew lines, not intersecting, otherwise

- $A+u_{AB}(B-A)$ and $C+u_{CD}(D-C)$ describe points on lines *through* A & B , C & D
- Test for collinearity, if that fails solve $u_{AB}(B-A)= C-A+u_{CD}(D-C)$
- by taking cross product of both sides with (D-C) making the u_{CD} term vanish ☺
- $u_{AB}(B-A)\times(D-C) = (C-A)\times(D-C)$
- as equal vectors have equal magnitudes, $u_{AB} = \|(C-A)\times(D-C)\| / \|(B-A)\times(D-C)\|$
- likewise, $u_{CD} = \|(B-A))\times(A-C)\| / \|(D-C)\times(B-A)\|$
- if $0\le u_{AB}\le 1$ and $0\le u_{CD}\le 1$ the finite lines A---B and C---D intersect at $A+u_{AB}(B-A)$