No.

Date.

(花了4h) Ex.2.5 | Ch.2 | P31 | Chloe | 2025.7.16

解：(a) 归一化波函数求A，$\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx=\int_{-\infty}^{+\infty}|A|^2[(\psi_1^*(x)+\psi_2^*(x))(\psi_1(x)+\psi_2(x)]dx$

由正交归一化，上式 $=2|A|^2=1$，$A^2=\frac{1}{2}$，$A=\frac{1}{\sqrt{2}}$

(b) 一维无限方势阱，由 $\Psi(x,0)=\frac{1}{\sqrt{2}}[\psi_1(x)+\psi_2(x)]$，知 $\Psi(x,t)=\frac{1}{\sqrt{2}}[\psi_1(x)e^{-iE_1t/\hbar}+\psi_2(x)e^{-iE_2t/\hbar}]$

其中 $E_1=\frac{\pi^2\hbar^2}{2ma^2}$，$E_2=\frac{4\pi^2\hbar^2}{2ma^2}$，令 $\omega=\frac{\pi^2\hbar}{2ma^2}$，则 $E_1=\hbar\omega$，$E_2=4\hbar\omega$，又 $\psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$

故 $\Psi(x,t)=\frac{1}{\sqrt{a}}[\sin(\frac{\pi x}{a})e^{-i\omega t}+\sin(\frac{2\pi x}{a})e^{-i4\omega t}]$

现在求 $|\Psi(x,t)|^2=\frac{1}{\sqrt{a}}[\sin(\frac{\pi x}{a})e^{i\omega t}+\sin(\frac{2\pi x}{a})e^{i4\omega t}]\cdot\frac{1}{\sqrt{a}}[\sin(\frac{\pi x}{a})e^{-i\omega t}+\sin(\frac{2\pi x}{a})e^{-i4\omega t}]$

$=\frac{1}{a}[\sin^2(\frac{\pi x}{a})+\sin^2(\frac{2\pi x}{a})+\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a})\cdot(e^{-i3\omega t}+e^{i3\omega t})]$ (由欧拉公式

$e^{i\alpha}+e^{-i\alpha}=2\cos\alpha$，$\alpha=3\omega t$) 故 $|\Psi(x,t)|^2=\frac{1}{a}[\sin^2(\frac{\pi x}{a})+\sin^2(\frac{2\pi x}{a})+2\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a})$

$\cdot\cos(3\omega t)]$

(c) $\langle x\rangle=\int_{-\infty}^{+\infty}\Psi^*(x,t)\,x\,\Psi(x,t)dx$

$=\frac{1}{a}\int_0^a x\cdot[\sin^2(\frac{\pi x}{a})+\sin^2(\frac{2\pi x}{a})+2\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a})\cdot\cos(3\omega t)]dx$

$=\underbrace{\frac{1}{a}\int_0^a x\sin^2(\frac{\pi x}{a})dx}_{(1)}+\underbrace{\frac{1}{a}\int_0^a x\cdot\sin^2(\frac{2\pi x}{a})dx}_{(2)}+\underbrace{\frac{2}{a}\cos(3\omega t)\int_0^a x\sin(\frac{\pi x}{a})\cdot\sin(\frac{2\pi x}{a})dx}_{(3)}$

其中，第(1)项 $=\frac{1}{2a}\int_0^a xdx-\frac{1}{2a}\int_0^a x\cos(\frac{2\pi x}{a})dx=\frac{a}{4}-\frac{1}{2a}[x\cdot\frac{a}{2\pi}\sin(\frac{2\pi x}{a})|_0^a-\int_0^a\frac{a}{2\pi}\sin(\frac{2\pi x}{a})dx]=\frac{a}{4}-0=\frac{a}{4}$

第(2)项 $=\frac{a}{4}$ (同上面的方法三角函数降幂公式 $\sin^2\alpha=\frac{1-\cos2\alpha}{2}$，分部积分2次)

第(3)项 $=\frac{\cos(3\omega t)}{a}[\int_0^a x\cos(\frac{\pi x}{a})dx-\int_0^a x\cos(\frac{3\pi x}{a})dx]$ (运用 $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$)

$=\frac{\cos(3\omega t)}{a}[(\frac{a^2}{\pi^2}\cos(\frac{\pi}{a}x)+\frac{ax}{\pi}\sin(\frac{\pi}{a}x))|_0^a-(\frac{a^2}{9\pi^2}\cos(\frac{3\pi x}{a})+\frac{ax}{3\pi}\sin(\frac{3\pi}{a}x))|_0^a]$

$=\frac{\cos(3\omega t)}{a}[\frac{a^2}{\pi^2}(\cos\pi-\cos0)-\frac{a^2}{9\pi^2}(\cos3\pi-\cos0)]$ (运用分部余弦积分公式)

$=\frac{\cos(3\omega t)}{a}\cdot(-\frac{2a^2}{\pi^2}+\frac{2a^2}{9\pi^2})=-\frac{16a}{9\pi^2}\cos(3\omega t)$

故 $\langle x\rangle=\frac{a}{4}+\frac{a}{4}-\frac{16a}{9\pi^2}\cos(3\omega t)=\frac{a}{2}-\frac{16a}{9\pi^2}\cos(3\omega t)$

振荡角频率为 3ω，振幅为 $\frac{16a}{9\pi^2}\approx0.18a<\frac{a}{2}$

(d) $\langle p\rangle=\int_{-\infty}^{+\infty}\Psi^*(x,t)(-i\hbar\frac{\partial}{\partial x})\Psi(x,t)dx$

又求 $\Psi(x,t)=\frac{1}{\sqrt{a}}[\frac{\pi}{a}\cos(\frac{\pi x}{a})e^{-i\omega t}+\frac{2\pi}{a}\cos(\frac{2\pi}{a}x)e^{-i4\omega t}]$

$\langle p\rangle=\frac{1}{\sqrt{a}}\cdot(-i\hbar)\int_{-\infty}^{+\infty}[\sin\frac{\pi x}{a}\cdot e^{i\omega t}+\sin(\frac{2\pi x}{a})\cdot e^{i4\omega t}]\cdot[\frac{\pi}{a}\cos\frac{\pi x}{a}\cdot e^{-i\omega t}+\frac{2\pi}{a}\cos\frac{2\pi x}{a}e^{-i4\omega t}]dx$

$=0$ (由正交归一性)

(e) $c_1=c_2=\frac{1}{\sqrt{2}}$，故 E_1、E_2 测几率为 $|\frac{1}{\sqrt{2}}|^2=\frac{1}{2}$

$\langle H\rangle=\int_{-\infty}^{+\infty}\Psi^*(x,t)H\Psi(x,t)dx=\frac{1}{2}(\int_{-\infty}^{+\infty}[\psi_1^*(x)e^{i\omega t}+\psi_2^*(x)e^{i4\omega t}]H[\psi_1(x)e^{-i\omega t}+\psi_2(x)e^{-i4\omega t}]dx$

$=\frac{1}{2}(E_1\int_{-\infty}^{+\infty}|\psi_1(x)|^2+E_2\int_{-\infty}^{+\infty}|\psi_2(x)|^2dx)=\frac{1}{2}(E_1+E_2)=\frac{5\pi^2\hbar^2}{4ma^2}$ $\langle H\rangle$ 介于 E_1 与 E_2 之间

(上式推导关键：$H\psi_n(x)=E\psi_n(x)$，运用正交性消除交叉项)