Ex.2.4 | Ch.2 | P31 | chloe | 2025.7.16

解：一维无限深势阱第n个定态的波函数 $\psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}),\ 0\le x\le a$

$\langle x\rangle=\int_{-\infty}^{+\infty}x|\psi(x)|^2dx=\int_0^a x\cdot\frac{2}{a}\cdot\sin^2(\frac{n\pi x}{a})dx$

$=\frac{2}{a}\int_0^a x\cdot\frac{1-\cos(2n\pi x/a)}{2}dx=\frac{1}{a}\int_0^a x\,dx-\frac{1}{a}\int_0^a x\cos(2n\pi x/a)dx$

$\frac{1}{a}\int_0^a x\,dx=\frac{1}{a}\cdot\frac{x^2}{2}\Big|_0^a=\frac{a}{2}$

$\frac{1}{a}\int_0^a x\cos(2n\pi x/a)dx=\frac{1}{a}\left[x\cdot\frac{a}{2n\pi}\sin(2n\pi x/a)\Big|_0^a-\int_0^a\frac{a}{2n\pi}\sin(\frac{2n\pi x}{a})dx\right]$

$=\frac{1}{a}\left[0+\frac{a^2}{(2n\pi)^2}\cos(\frac{2n\pi x}{a})\Big|_0^a\right]=0$

故 $\langle x\rangle=0$

$\langle x^2\rangle=\int_{-\infty}^{+\infty}x^2|\psi_n(x)|^2dx=\int_0^a x^2\cdot\frac{2}{a}\sin^2(\frac{n\pi x}{a})dx$

$=\frac{1}{a}\int_0^a x^2dx-\frac{1}{a}\int_0^a x^2\cos(\frac{2n\pi x}{a})dx$

$\frac{1}{a}\int_0^a x^2dx=\frac{1}{a}\cdot\frac{x^3}{3}\Big|_0^a=\frac{a^2}{3}$

$\frac{1}{a}\int_0^a x^2\cos(\frac{2n\pi x}{a})dx=\frac{1}{a}\left[x^2\frac{a}{2n\pi}\sin(\frac{2n\pi x}{a})\Big|_0^a-\int_0^a\frac{a}{n\pi}x\sin(\frac{2n\pi}{a})dx\right]$

$=\frac{1}{a}\int_0^a x^2\cos(\frac{2n\pi x}{a})dx=a^2/2n^2\pi^2$

故 $\langle x\rangle=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}$

$\langle p\rangle=\int_{-\infty}^{+\infty}\psi^*(x)(-i\hbar)\frac{d}{dx}\psi_n(x)dx=\int_0^a\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})(-i\hbar\frac{d}{dx})\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})dx$

$=\frac{2}{a}(-i\hbar)\int_0^a\sin\frac{n\pi x}{a}\cdot\cos(\frac{n\pi x}{a})\cdot\frac{n\pi}{a}dx=\frac{2}{a}(-i\hbar)\cdot\frac{n\pi}{a}\int_0^a\frac{1}{2}\sin(\frac{2n\pi x}{a})dx$

$=\frac{-i\hbar}{2a}\int_0^a\sin(\frac{2n\pi x}{a})d\frac{2n\pi x}{a}=0$

$\langle p^2\rangle=\int_{-\infty}^{+\infty}\psi_n^*(x)(-\hbar^2\cdot\frac{d^2}{dx^2})\psi_n(x)dx=\frac{2\hbar^2}{a}\cdot\frac{n^2\pi^2}{a^2}\int_0^a\sin^2(\frac{n\pi x}{a})dx$

$=\frac{n^2\pi^2\hbar^2}{a^2}$

$\sigma_x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}-\frac{a^2}{4}}=\sqrt{\frac{a^2}{12}-\frac{a^2}{2n^2\pi^2}}$

$\sigma_p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\frac{n^2\pi^2\hbar^2}{a^2}}=\frac{n\pi\hbar}{a}$

$\sigma_x\cdot\sigma_p=\hbar\sqrt{\frac{n^2\pi^2}{12}-\frac{1}{2}}$

当n=1时，$\sigma_x\cdot\sigma_p=\hbar\sqrt{\frac{\pi^2}{12}-\frac{1}{2}}\approx0.56\hbar>\frac{\hbar}{2}$ 不确定性原理成立

随着 $n\uparrow$，$\sigma_x\sigma_p\uparrow$，$n=1$ 时，最接近不确定性原理的极限