CPE/EE 695: Applied Machine Learning

Lecture 3 - 1: Linear Regression

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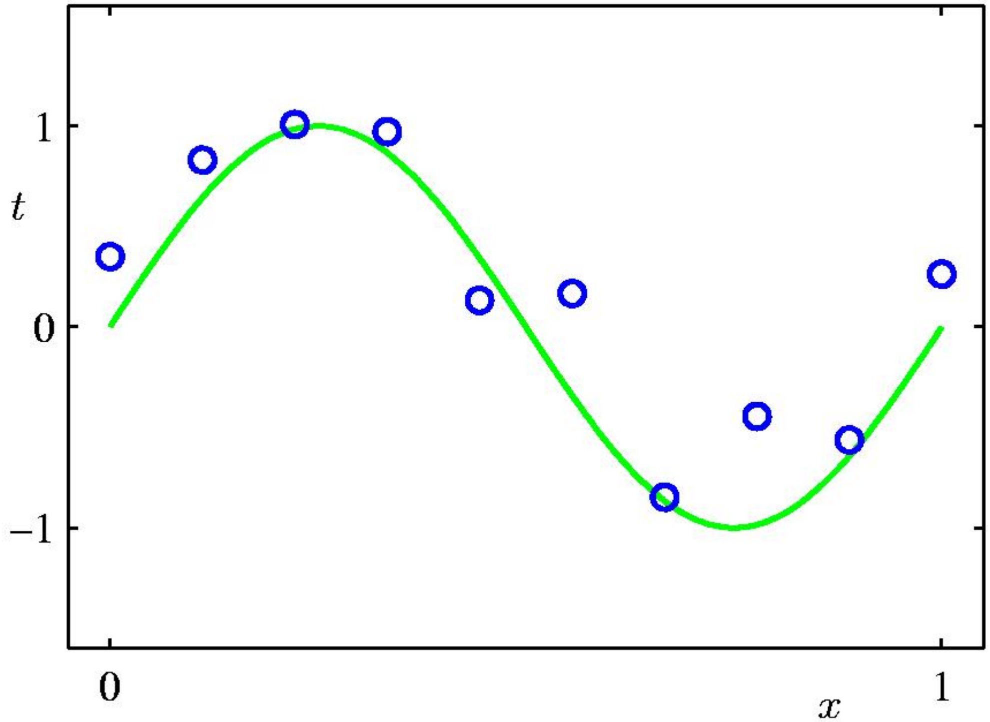
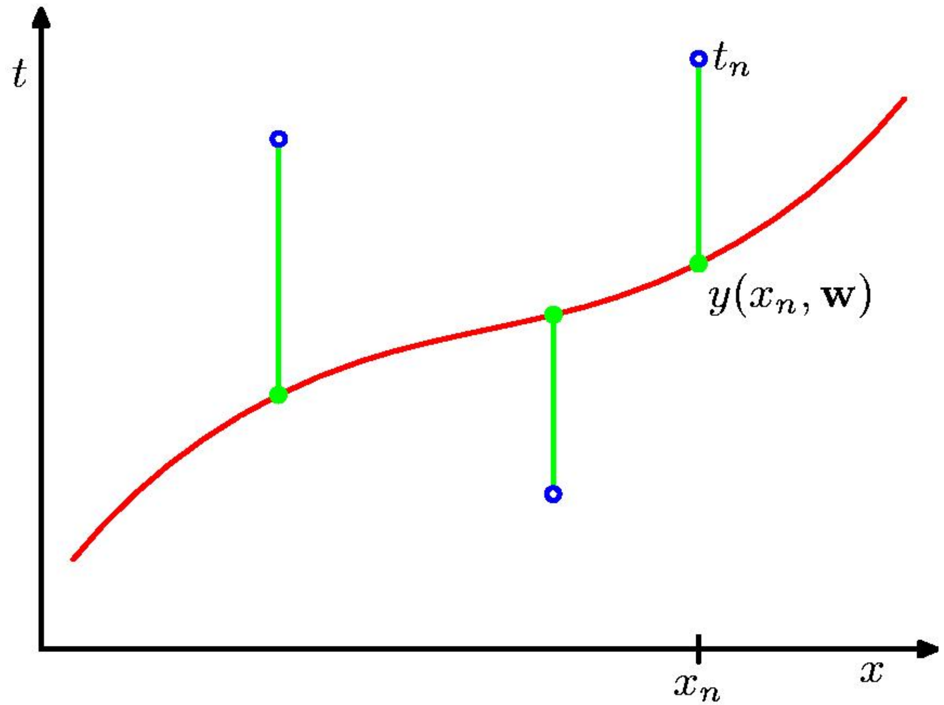
Polynomial Curve Fitting



$$y(x, \mathbf{w}) = w_0 + w_1 x + w_2 x^2 + \ldots + w_M x^M = \sum_{j=0}^{M} w_j x^j$$

Polynomial Regression

Sum-of-Squares Error Function



$$E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{y(x_n, \mathbf{w}) - t_n\}^2$$

Minimize E(w) for unknown w. (maximum likelihood)

Linear Regression

Linear model prediction:

$$\hat{y} = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

Linear Regression

Linear model prediction:

$$\hat{y} = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

Observed value of $\hat{y}$ would be:

$$y = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + \varepsilon$$

Linear Regression

Linear model prediction:

$$\hat{y} = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

Vectorized form:

$$\hat{y} = h_{\boldsymbol{w}}(\boldsymbol{x}) = \boldsymbol{w}^T \cdot \boldsymbol{x}$$

Linear Regression

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Vectorized form:

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Mean Square Error (MSE) cost function:

$$MSE(X, h_w) = \frac{1}{m} \sum_{i=1}^{m} (\boldsymbol{w}^T \cdot \boldsymbol{x}^{(\boldsymbol{i})} - \mathrm{y}^{(\mathrm{i})})^2$$

Linear Regression

Linear model prediction:

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Normal Equation (closed-form solution):

$$\widehat{w} = (X^T \cdot X)^{-1} \cdot X^T \cdot \boldsymbol{y}$$

$\widehat{w}$: the value of that minimizes the cost function

X : the training data set

$\boldsymbol{y}$: the vector of target values containing $\mathrm{y}^{(1)}$ to $\mathrm{y}^{(\mathrm{m})}$

Linear Regression

Linear model prediction:

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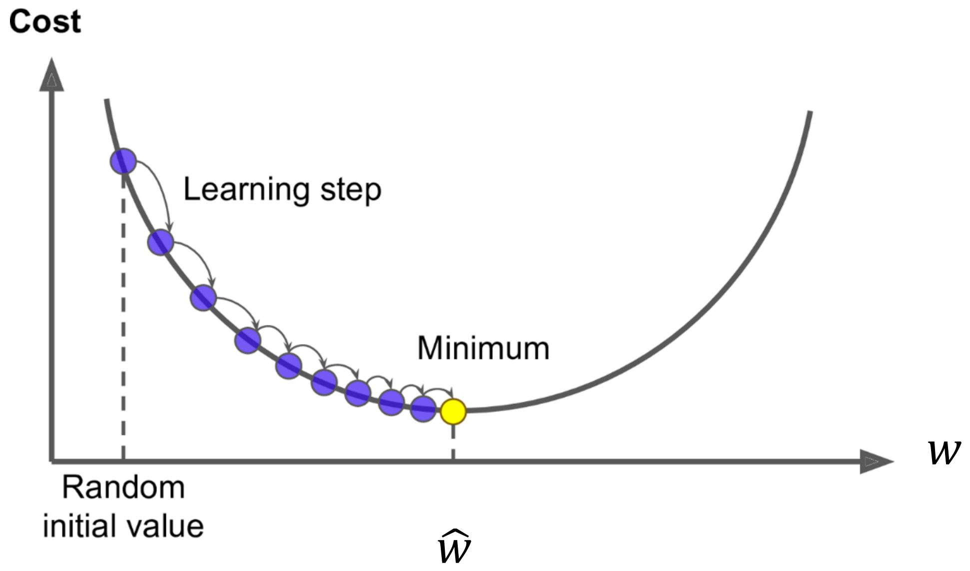
Complexity: $O(n^3)$

Linear Regression

Gradient Descent:



$$w^{(next\ step)} = w - \alpha \cdot \nabla_w MSE(w)$$

What is Grade Descent?

A first-order iterative optimization algorithm to find the minimum of a multivariable function F(**x**).

Rational:

If F(**x**) is differentiable around a point A, F decreases **fastest** from A in the direction of negative gradient of F(x) at A (i.e., **-**$\boldsymbol{\nabla F(A)}$). In other words, let

$$\boldsymbol{A_{n+1} = A_n - \alpha * \nabla F(A)}$$

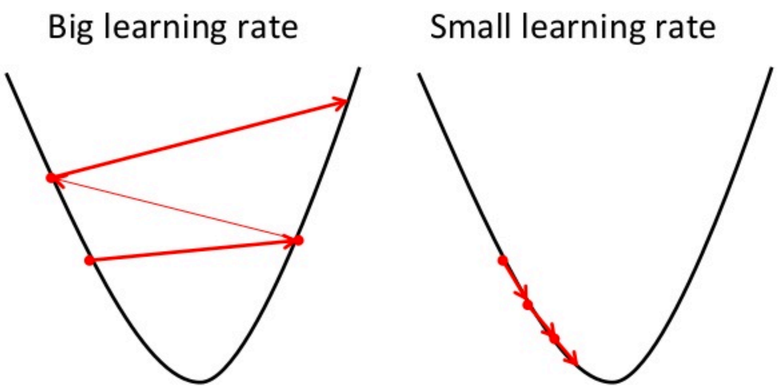
for small enough $\boldsymbol{\alpha}$, we have $\boldsymbol{F(A_{n+1}) \leq F(A_n)}$.

Through a set of such points $\boldsymbol{A_0, A_1}$, ..., it converges to a **local minimum**.

If function F(**x**) is **convex**, the local minimum is the **global minimum**.

Linear Regression

Gradient Descent:



Learning rate λ is very important

Linear Regression

Gradient Descent:

1) Batch Gradient Descent

   Using ALL data sets to calculate the gradient and update w:

   $$\frac{\partial}{\partial w_j} = \frac{2}{m}\sum_{i=1}^{m}\left(w^T \cdot x^{(i)} - y^{(i)}\right) {x_j}^{(i)}$$

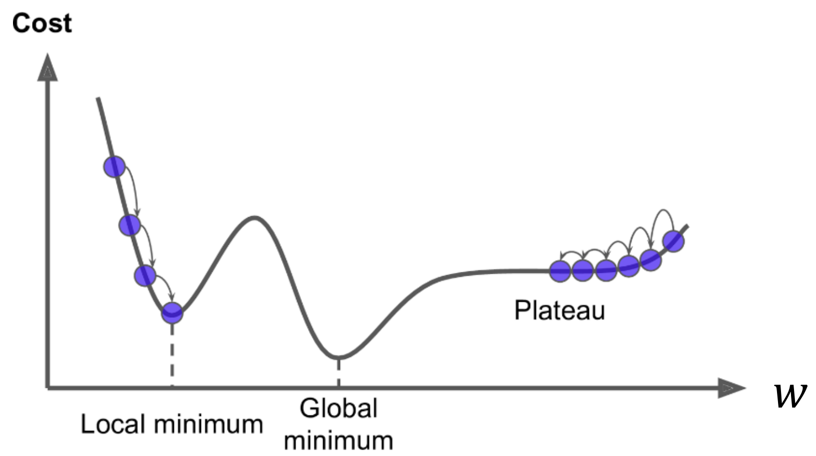
2) Stochastic Gradient Descent

   Using one RANDOM sample to calculate the gradient and update w.

3) Mini-batch Gradient Descent (using a small random set of samples)

Linear Regression

Gradient Descent:



Local minimum issue

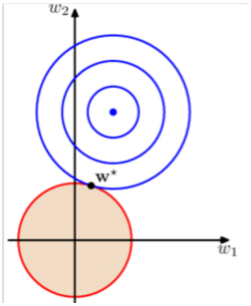
Batch Gradient Descent: converges fast, more likely to have local minimum
Stochastic Gradient Descent: converges slower, less likely to have local minimum
Mini-Batch Gradient Descent: in the middle of the two.

Regularized Linear Models

Ridge Regression

Cost function: $E(w) = MSE(w) + \frac{\alpha}{2}\sum_{i=1}^{m} {w_i}^2$

It has a closed-form solution $\mathbf{w} = \left(\lambda \mathbf{I} + \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{t}.$

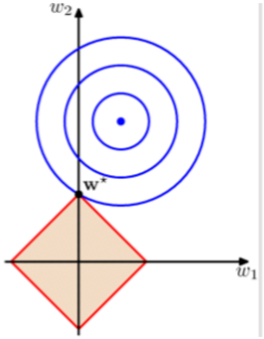


Lasso Regression

Cost function: $E(w) = MSE(w) + \alpha \sum_{i=1}^{m} |w_i|$

No closed-from solution for w;

But it tends to eliminate the weights of least important features.



Elastic Net

Cost function: $E(w) = MSE(w) + \lambda_1 \sum_{i=1}^{m} {w_i}^2 + \lambda_2 \sum_{i=1}^{m} |w_i|$

In the middle of Ridge and Lasso.

Usually is preferred over Lasso or Ridge.

Cost Functions

Mean Square Error (MSE):

$$MSE(X, h) = \frac{1}{m}\sum_{i=1}^{m}(h(\boldsymbol{x}^{(\boldsymbol{i})}) - \mathrm{y}^{(\mathrm{i})})^2$$

Root Mean Square Error (RMSE):

$$RMSE(X, h) = \sqrt{MSE}$$ (Euclidean norm)

Mean Absolute Error (MAE):

$$MAE(X, h) = \frac{1}{m}\sum_{i=1}^{m}\left|h\left(x^{(i)}\right) - y^{(i)}\right|$$ (Manhattan norm)

$\boldsymbol{l_k}$ ***norm*** of a vector v with n elements: $||v||_k = (|v_0|^k + \cdots + |v_n|^k)^{\frac{1}{k}}$

The higher the norm index, the more it focuses on large values and neglect small ones. Therefore, RMSE is more sensitive to outliers than MAE.

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