MEASURE THEORY

Volume 5

Part II

D.H.Fremlin

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MEASURE THEORY

Volume 5

Set-theoretic Measure Theory

Part II

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Contents

Chapter 54

Real-valued-measurable cardinals

Of the many questions in measure theory which involve non-trivial set theory, perhaps the first to have been recognised as fundamental is what I call the 'Banach-Ulam problem': is there a non-trivial measure space in which every set is measurable? In various forms, this question has arisen repeatedly in earlier volumes of this treatise (232Hc, 363S, 438A). The time has now come for an account of the developments of the last forty-five years.

The measure theory of this chapter will begin in §543; the first two sections deal with generalizations to wider contexts. If ν is a probability measure with domain $\mathcal{P}X$, its null ideal is ω_1-additive and ω_1-saturated in $\mathcal{P}X$. In §541 I look at ideals $\mathcal{I} \vartriangleleft \mathcal{P}X$ such that $\mathcal{I}$ is simultaneously κ-additive and κ-saturated for some κ; this is already enough to lead us to a version of the Keisler-Tarski theorem on normal ideals (541J), a great strengthening of Ulam's theorem on inaccessibility of real-valued-measurable cardinals (541Lc), a form of Ulam's dichotomy (541P), and some very striking infinitary combinatorics (541Q). In §542 I specialize to the case $\kappa = \omega_1$, still without calling on the special properties of null ideals, with more combinatorics (542E, 542I).

I have said many times in the course of this treatise that almost the first thing to ask about any measure is, what does its measure algebra look like? For an atomless probability measure with domain $\mathcal{P}X$, the Gitik-Shelah theorem (543E-543F) gives a great deal of information, associated with a tantalizing problem (543Z). §544 is devoted to the measure-theoretic consequences of assuming that there is some atomlessly-measurable cardinal, with results on repeated integration (544C, 544I, 544J), the null ideal of a normal witnessing probability (544E-544F) and regressive functions (544M).

I do not discuss consistency questions in this chapter (I will touch on some of them in Chapter 55). The ideas of §§541-544 would be in danger of becoming irrelevant if it turned out that there can be no two-valued-measurable cardinal. I have no real qualms about this. One of my reasons for confidence is the fact that very much stronger assumptions have been investigated without any hint of catastrophe. Two of these, the 'product measure extension axiom' and the 'normal measure axiom' are mentioned in §545.

One way of looking at the Gitik-Shelah theorem is to say that if X is a set and $\mathcal{I}$ is a proper σ-ideal of subsets of X, then $\mathcal{P}X/\mathcal{I}$ cannot be an atomless measurable algebra of small Maharam type. We can ask whether there are further theorems of this type provable in ZFC. Two such results are in §546: the 'Gitik-Shelah theorem for category' (546G, 546I), showing that $\mathcal{P}X/\mathcal{I}$ cannot be isomorphic to $\mathrm{RO}(\mathbb{R})$, and 546P-546Q, showing that some algebras of a type considered in §527 also cannot appear as power set σ-quotient algebras. Remarkably, this leads us to a striking fact about disjoint refinements of sequences of sets (547F).

541 Saturated ideals

If ν is a totally finite measure with domain $\mathcal{P}X$ and null ideal $\mathcal{N}(\nu)$, then its measure algebra $\mathcal{P}X/\mathcal{N}(\nu)$ is ccc, that is to say, $\mathrm{sat}(\mathcal{P}X/\mathcal{N}(\nu)) \le \omega_1$; while the additivity of $\mathcal{N}(\nu)$ is at least ω_1. It turns out that an ideal $\mathcal{I}$ of $\mathcal{P}X$ such that $\mathrm{sat}(\mathcal{P}X/\mathcal{I}) \le \mathrm{add}\,\mathcal{I}$ is either trivial or extraordinary. In this section I present a little of the theory of such ideals. To begin with, the quotient algebra has to be Dedekind complete (541B). Further elementary ideas are in 541C (based a method already used in §525) and 541D-541E. In a less expected direction, we have a useful fact concerning transversal numbers $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ (541F).

The most remarkable properties of saturated ideals arise because of their connexions with 'normal' ideals (541G). These ideals share the properties of non-stationary ideals (541H-541I). If $\mathcal{I}$ is an $(\mathrm{add}\,\mathcal{I})$-saturated ideal of $\mathcal{P}X$, we have a corresponding normal ideal on $\mathrm{add}\,\mathcal{I}$ (541J). Now there can be a κ-saturated normal ideal on κ only if there is a great complexity of cardinals below κ (541L).

The original expression of these ideas (Keisler & Tarski 64) concerned 'two-valued-measurable' cardinals, on which we have normal ultrafilters (541M). The dichotomy of Ulam 30 (438Ce-438Cf) reappears in the context of κ-saturated normal ideals (541P). For κ-saturated ideals, 'normality' implies some far-reaching extensions (541Q). Finally, I include a technical lemma concerning the covering numbers $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ (541S).

541A Definition If $\mathfrak{A}$ is a Boolean algebra, I is an ideal of $\mathfrak{A}$ and κ is a cardinal, I will say that I is **κ-saturated** in $\mathfrak{A}$ if $\kappa \ge \mathrm{sat}(\mathfrak{A}/I)$; that is, if for every family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}\setminus I$ there are distinct ξ, $\eta<\kappa$ such that $a_\xi \cap a_\eta \notin I$.

541B Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and I an ideal of $\mathfrak{A}$ which is $(\mathrm{add}\,I)^+$-saturated in $\mathfrak{A}$. Then the quotient algebra $\mathfrak{A}/I$ is Dedekind complete.

proof Take any $B \subseteq \mathfrak{A}/I$. Let C be the set of those $c \in \mathfrak{A}/I$ such that *either* $c \cap b = 0$ for every $b \in B$ *or* there is a $b \in B$ such that $c \subseteq b$. Then C is order-dense in $\mathfrak{A}/I$, so there is a partition of unity $D \subseteq C$ (313K). Enumerate D as $\langle d_\xi\rangle_{\xi<\kappa}$, where

$$\kappa=\#(D)<\operatorname{sat}(\mathfrak{A}/I)\le(\operatorname{add}I)^+.$$

For each $\xi<\kappa$, choose $a_\xi\in\mathfrak{A}$ such that $a_\xi^\bullet=d_\xi$. Now $\#(\xi)<\kappa\le\operatorname{add}I$, so $\sup_{\eta<\xi}a_\xi\cap a_\eta\in I$. Set $\tilde a_\xi=a_\xi\setminus\sup_{\eta<\xi}a_\eta$; then $\tilde a_\xi^\bullet=a_\xi^\bullet=d_\xi$. Set $L=\{\xi:\xi<\kappa,\ d_\xi\subseteq b$ for some $b\in B\}$ and $a=\sup_{\xi\in L}\tilde a_\xi$ in $\mathfrak{A}$.

? If $b\in B$ and $b\not\subseteq a^\bullet$, there must be a $\xi<\kappa$ such that $d_\xi\cap(b\setminus a^\bullet)\ne0$. But $d_\xi\in C$, so there must be a $b'\in B$ such that $d_\xi\subseteq b'$; accordingly $\xi\in L$, $\tilde a_\xi\subseteq a$ and $d_\xi\subseteq a^\bullet$. **X** Thus $a^\bullet$ is an upper bound of B in $\mathfrak{A}/I$.

? If there is an upper bound b^* of B such that $a^\bullet\not\subseteq b^*$, there must be a $\xi<\kappa$ such that $d_\xi\cap a^\bullet\setminus b^*\ne0$. As $d_\xi\not\subseteq b^*$, $d_\xi\not\subseteq b$ for every $b\in B$, and $\xi\notin L$. But this means that $\tilde a_\xi\cap\tilde a_\eta=0$ for every $\eta\in L$, so $\tilde a_\xi\cap a=0$ (313Ba) and $d_\xi\cap a^\bullet=0$. **X**

Thus $a^\bullet=\sup B$ in $\mathfrak{A}/I$; as B is arbitrary, $\mathfrak{A}/I$ is Dedekind complete.

541C Proposition Let X be a set, κ a regular infinite cardinal, Σ an algebra of subsets of X such that $\bigcup\mathcal{E}\in\Sigma$ whenever $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\kappa$, and $\mathcal{I}$ a κ-saturated κ-additive ideal of Σ.

(a) If $\mathcal{E}\subseteq\Sigma$ there is an $\mathcal{E}'\in[\mathcal{E}]^{<\kappa}$ such that $E\setminus\bigcup\mathcal{E}'\in\mathcal{I}$ for every $E\in\mathcal{E}$.

(b) If $\langle E_\xi\rangle_{\xi<\kappa}$ is any family in $\Sigma\setminus\mathcal{I}$, and $\theta<\kappa$ is a cardinal, then $\{x:x\in X,\ \#(\{\xi:x\in E_\xi\})\ge\theta\}$ includes a member of $\Sigma\setminus\mathcal{I}$ (and, in particular, is not empty).

(c) Suppose that no element of $\Sigma\setminus\mathcal{I}$ can be covered by κ members of $\mathcal{I}$. Then κ is a precaliber of $\Sigma/\mathcal{I}$.

proof Write $\mathfrak{A}$ for $\Sigma/\mathcal{I}$.

(a) Consider $A=\{E^\bullet:E\in\mathcal{E}\}\subseteq\mathfrak{A}$. By 514Db, there is an $\mathcal{E}'\in[\mathcal{E}]^{<\operatorname{sat}(\mathfrak{A})}$ such that $\{E^\bullet:E\in\mathcal{E}'\}$ has the same upper bounds as A. Now $\#(\mathcal{E}')<\operatorname{sat}(\mathfrak{A})\le\kappa$, so $F=\bigcup\mathcal{E}'$ belongs to Σ, and $F^\bullet$ must be an upper bound for A, that is, $E\setminus F\in\mathcal{I}$ for every $E\in\mathcal{E}$.

(b) For $\alpha\le\beta<\kappa$ set $F_{\alpha\beta}=\bigcup_{\alpha\le\xi<\beta}E_\xi\in\Sigma$. Then for every $\alpha<\kappa$ we have a $g(\alpha)<\kappa$ such that $g(\alpha)\ge\alpha$ and $E_\xi\setminus F_{\alpha,g(\alpha)}\in\mathcal{I}$ whenever $g(\alpha)\le\xi<\kappa$, by (a).

Define $\langle h(\alpha)\rangle_{\alpha<\kappa}$ by setting $h(0)=0$, $h(\alpha+1)=g(h(\alpha))$ for each $\alpha<\kappa$, and $h(\alpha)=\sup_{\beta<\alpha}h(\beta)$ for non-zero limit ordinals $\alpha<\kappa$. Set $G_\alpha=F_{h(\alpha),h(\alpha+1)}$ for each α. If $\beta<\alpha$, then

$$G_\alpha\setminus G_\beta=\bigcup\nolimits_{h(\alpha)\le\xi<h(\alpha+1)}E_\xi\setminus F_{h(\beta),g(h(\beta))}\in\mathcal{I}$$

because $\mathcal{I}$ is κ-additive. Consequently $G_\theta\setminus\bigcap_{\beta<\theta}G_\beta$ belongs to $\mathcal{I}$; but $G_\theta\supseteq E_{h(\theta)}\notin\mathcal{I}$, so $G=\bigcap_{\beta<\theta}G_\beta\in\Sigma\setminus\mathcal{I}$. But if $x\in G$ then $\{\xi:x\in E_\xi\}$ meets $[h(\beta),h(\beta+1)[$ for every $\beta<\theta$ and has cardinal at least θ.

(c) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family of non-zero elements in $\mathfrak{A}$. For each $\xi<\kappa$, choose $\tilde E_\xi\in\Sigma$ such that $\tilde E_\xi^\bullet=a_\xi$. Let $\mathcal{K}$ be the family of all those finite subsets K of κ such that $H_K=\bigcap_{\xi\in K}\tilde E_\xi\in\mathcal{I}$. Now set $E_\xi=\tilde E_\xi\setminus\bigcup\{H_K:K\in\mathcal{K},\ K\subseteq\xi\}$; then $E_\xi\in\Sigma\setminus\mathcal{I}$ and $E_\xi^\bullet=a_\xi$ for each $\xi<\kappa$.

Repeat the argument of (b). Once again we get a family $\langle G_\alpha\rangle_{\alpha<\kappa}$ in $\Sigma\setminus\mathcal{I}$ such that $G_\alpha\setminus G_\beta\in\mathcal{I}$ whenever $\beta\le\alpha<\kappa$. Now, applying (a) to $\langle X\setminus G_\alpha\rangle_{\alpha<\kappa}$, we have a $\gamma<\kappa$ such that $\bigcap_{\alpha<\gamma}G_\alpha\setminus G_\beta\in\mathcal{I}$ for every $\beta<\kappa$. On the other hand, $G_\gamma\setminus G_\alpha\in\mathcal{I}$ for every $\alpha<\gamma$, so in fact $G_\gamma\setminus G_\beta\in\mathcal{I}$ for every $\beta<\kappa$, while $G_\gamma\notin\mathcal{I}$. At this point, recall that we are now assuming that G_γ cannot be covered by $\bigcup_{\beta<\kappa}G_\gamma\setminus G_\beta$, and there is an $x\in\bigcap_{\beta<\kappa}G_\beta$.

As in (b), it follows that $\Gamma=\{\xi:\xi<\kappa,\ x\in E_\xi\}$ has cardinal κ. If $K\subseteq\Gamma$ is finite and not empty, take $\zeta\in\Gamma$ such that $K\subseteq\zeta$. Then

$$x\in E_\zeta\cap\bigcap\nolimits_{\xi\in K}\tilde E_\xi\subseteq E_\zeta\cap H_K;$$

it follows that $K\notin\mathcal{K}$ and $H_K\notin\mathcal{I}$, that is, that $\inf_{\xi\in K}a_\xi\ne0$ in $\mathfrak{A}$. So $\langle a_\xi\rangle_{\xi\in\Gamma}$ is centered; as $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of $\mathfrak{A}$.

541D Lemma Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $f:X\to[Y]^{<\kappa}$ there is an $M\in[Y]^{<\kappa}$ such that $\{x:f(x)\not\subseteq M\}\in\mathcal{I}$.

proof If $\kappa>\#(Y)$ this is trivial; suppose that $\kappa\le\#(Y)<\operatorname{add}\mathcal{I}$. For cardinals $\lambda<\kappa$ set $X_\lambda=\{x:\#(f(x))=\lambda\}$. If $A=\{\lambda:X_\lambda\notin\mathcal{I}\}$ then $\mathcal{I}$ is not $\#(A)$-saturated in $\mathcal{P}X$, so $\#(A)<\operatorname{cf}\kappa$ and $\theta=\sup A$ is less than κ. Set $X'=\{x:\#(f(x))\le\theta\}$; then $X\setminus X'$ is the union of at most $\lambda<\operatorname{add}\mathcal{I}$ members of $\mathcal{I}$, so belongs to $\mathcal{I}$.

For each $x\in X'$ let $\langle h_\xi(x)\rangle_{\xi<\theta}$ run over a set including $f(x)$. For each $\xi<\theta$,

$$Y_\xi=\{y:h_\xi^{-1}[\{y\}]\notin\mathcal{I}\}$$

has cardinal less than $\operatorname{cf}\kappa$, and because $\#(Y)<\operatorname{add}\mathcal{I}$, $h_\xi^{-1}[Y\setminus Y_\xi]\in\mathcal{I}$. Set $M=\bigcup_{\xi<\theta}Y_\xi\in[Y]^{<\kappa}$. (If κ is regular, M is the union of fewer than κ sets of size less than κ, so $\#(M)<\kappa$; if κ is not regular, then M is the union of fewer than κ sets of size at most $\operatorname{cf}\kappa$, so again $\#(M)<\kappa$.) Because $\theta<\operatorname{add}\mathcal{I}$,

$$\{x:f(x)\not\subseteq M\}\subseteq(X\setminus X')\cup\bigcup\nolimits_{\xi<\theta}h_\xi^{-1}[Y\setminus Y_\xi]\in\mathcal{I},$$

as required.

541E Corollary Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $g : X \to Y$ there is an $M \in [Y]^{<\kappa}$ such that $g^{-1}[Y \setminus M] \in \mathcal{I}$.

proof Apply 541D to the function $x \mapsto \{g(x)\}$. (In the trivial case $\kappa = 1$, $\mathcal{I} = \mathcal{P}X$.)

541F Lemma Let X and Y be sets, κ an uncountable regular cardinal, and $\mathcal{I}$ a proper κ-saturated κ-additive ideal of subsets of X. Then $\operatorname{Tr}_{\mathcal{I}}(X;Y)$ (definition: 5A1La) is attained, in the sense that there is a set $G \subseteq Y^X$ such that $\#(G) = \operatorname{Tr}_{\mathcal{I}}(X;Y)$ and $\{x : x \in X,\, g(x) = g'(x)\} \in \mathcal{I}$ for all distinct g, $g' \in G$.

proof It is enough to consider the case in which $Y = \lambda$ is a cardinal. Set $\theta = \operatorname{Tr}_{\mathcal{I}}(X;\lambda)$.

(a) If $\lambda^+ < \kappa$ then $\theta \le \lambda$. **P?** Suppose, if possible, that we have a family $\langle f_\xi\rangle_{\xi<\lambda^+}$ in λ^X such that $\{x : f_\xi(x) = f_\eta(x)\} \in \mathcal{I}$ whenever $\eta < \xi < \lambda^+$. Then λ is surely infinite, so λ^+ is uncountable and regular. For each $x \in X$ there is an $\alpha_x < \lambda^+$ such that $\{f_\xi(x) : \xi < \alpha_x\} = \{f_\xi(x) : \xi < \lambda^+\}$. Setting

$$F_\alpha = \{x : x \in X,\, \alpha_x = \alpha\} \subseteq \bigcup_{\eta<\alpha}\{x : f_\eta(x) = f_\alpha(x)\},$$

we see that $F_\alpha \in \mathcal{I}$ for each $\alpha < \lambda^+$. But $X = \bigcup_{\alpha<\lambda^+} F_\alpha$ and $\lambda^+ < \kappa$, so this is impossible. **XQ**

Since we can surely find a family $\langle f_\xi\rangle_{\xi<\lambda}$ in λ^X such that $f_\xi(x) \ne f_\eta(x)$ whenever $x \in X$ and $\eta < \xi < \lambda$, we have the result when $\lambda^+ < \kappa$.

(b) We may therefore suppose from now on that $\lambda^+ \ge \kappa$.

If $H \subseteq \lambda^X$ is such that

$$F = \{f : f \in \lambda^X,\, \{x : f(x) \le h(x)\} \in \mathcal{I} \text{ for every } h \in H\} \ne \emptyset,$$

then there is an $f_0 \in F$ such that

$$\{x : f(x) < f_0(x)\} \in \mathcal{I} \text{ for every } f \in F.$$

P? If not, choose a family $\langle f_\xi\rangle_{\xi<\kappa}$ in F inductively, as follows. f_0 is to be any member of F. Given f_ξ, there is an $f \in F$ such that $\{x : f(x) < f_\xi(x)\} \notin \mathcal{I}$; set $f_{\xi+1}(x) = \min(f(x), f_\xi(x))$ for every x; then $f_{\xi+1} \in F$. Given that $f_\eta \in F$ for every $\eta < \xi$, where $\xi < \kappa$ is a non-zero limit ordinal, set $f_\xi(x) = \min_{\eta<\xi} f_\eta(x)$ for each x; then for any $h \in H$ we shall have

$$\{x : f_\xi(x) \le h(x)\} = \bigcup_{\eta<\xi}\{x : f_\eta(x) \le h(x)\} \in \mathcal{I},$$

so $f_\xi \in F$ and the induction continues.

Now consider

$$E_\xi = \{x : f_{\xi+1}(x) < f_\xi(x)\} \in \mathcal{P}X \setminus \mathcal{I}$$

for $\xi < \kappa$. By 541Cb there is an $x \in X$ such that $A = \{\xi : x \in E_\xi\}$ is infinite. But if $\langle \xi(n)\rangle_{n\in\mathbb{N}}$ is any strictly increasing sequence in A, $\langle f_{\xi(n)}(x)\rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(c) Choose a family $\langle g_\xi\rangle_{\xi<\delta}$ in λ^X as follows. Given $\langle g_\eta\rangle_{\eta<\xi}$, set

$$F_\xi = \{f : f \in \lambda^X,\, \{x : f(x) \le g_\eta(x)\} \in \mathcal{I} \text{ for every } \eta < \xi\}.$$

If $F_\xi = \emptyset$, set $\delta = \xi$ and stop. If $F_\xi \ne \emptyset$ use (b) to find $g_\xi \in F_\xi$ such that $\{x : f(x) < g_\xi(x)\} \in \mathcal{I}$ for every $f \in F_\xi$, and continue. Note that for $\xi < \min(\lambda,\kappa)$, $\{x : g_\xi(x) \ne \xi\} \in \mathcal{I}$. **P** Induce on ξ. If $\xi < \min(\lambda,\kappa)$ and $\{x : g_\eta(x) \ne \eta\} \in \mathcal{I}$ for every $\eta < \xi$, then the constant function with value ξ belongs to F_ξ, so g_ξ is defined and $\{x : g_\xi(x) > \xi\} \in \mathcal{I}$. On the other hand, $\{x : g_\xi(x) = \eta\} \in \mathcal{I}$ for $\eta < \xi$; as $\xi < \operatorname{add}\mathcal{I}$, $\{x : g_\xi(x) < \xi\} \in \mathcal{I}$. **Q** Accordingly $\delta \ge \min(\lambda,\kappa)$.

(d) Because $g_\xi \in F_\xi$, $\{x : g_\xi(x) = g_\eta(x)\} \in \mathcal{I}$ whenever $\eta < \xi < \delta$, so $\#(\delta) \le \theta$. On the other hand, suppose that $F \subseteq \lambda^X$ is such that $\{x : f(x) = f'(x)\} \in \mathcal{I}$ for all distinct f, $f' \in F$. For each $f \in F$, set

$$\zeta'_f = \min\{\xi : \xi \le \delta,\, f \notin F_\xi\};$$

this must be defined because $F_\delta = \emptyset$. Also $F_0 = \lambda^X$ and $F_\xi = \bigcap_{\eta<\xi} F_\eta$ if $\xi \le \delta$ is a non-zero limit ordinal, so ζ'_f must be a successor ordinal; let ζ_f be its predecessor. We have $f \in F_{\zeta_f}$ and

$$\{x : f(x) < g_{\zeta_f}(x)\} \in \mathcal{I}, \quad \{x : f(x) \le g_{\zeta_f}(x)\} \notin \mathcal{I},$$

so that

$$E_f = \{x : f(x) = g_{\zeta_f}(x)\} \notin \mathcal{I}.$$

If f, f' are distinct members of F and $\zeta_f = \zeta_{f'}$, then $E_f \cap E_{f'} \in \mathcal{I}$. So

$$\{f : f \in F,\, \zeta_f = \zeta\}$$

must have cardinal less than κ for every $\zeta < \delta$.

If $\kappa = \lambda^+$, $\#(\{f : f \in F, \zeta_f = \zeta\}) \le \lambda$ for every $\zeta < \delta$, so $\#(F) \le \max(\delta, \lambda) = \delta$. On the other hand, if $\kappa \le \lambda$, then $\#(F) \le \max(\delta, \kappa) = \delta$. As F is arbitrary, $\theta = \delta$ and we may take $G = \{g_\xi : \xi < \delta\}$ as our witness that $\mathrm{Tr}_{\mathcal{I}}(X; \lambda)$ is attained.

541G Definition Let κ be a regular uncountable cardinal. A **normal ideal** on κ is a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$, including $[\kappa]^{<\kappa}$, such that

$$\{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} I_\eta\}$$

belongs to $\mathcal{I}$ for every family $\langle I_\xi \rangle_{\xi<\kappa}$ in $\mathcal{I}$. It is easy to check that a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ is normal iff the dual filter $\{\kappa \setminus I : I \in \mathcal{I}\}$ is normal in the sense of 4A1Ic.

541H Proposition Let κ be a regular uncountable cardinal and $\mathcal{I}$ a proper ideal of $\mathcal{P}\kappa$ including $[\kappa]^{<\kappa}$. Then the following are equiveridical:

(i) $\mathcal{I}$ is normal;

(ii) $\mathcal{I}$ is κ-additive and whenever $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive, then there is an $\alpha < \kappa$ such that $\{\xi : \xi \in S,\, f(\xi) \le \alpha\}$ is not in $\mathcal{I}$;

(iii) whenever $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive, then there is a $\beta < \kappa$ such that $\{\xi : \xi \in S,\, f(\xi) = \beta\}$ is not in $\mathcal{I}$.

proof (i)$\Rightarrow$(ii) Suppose that $\mathcal{I}$ is normal.

(α) (Cf. 4A1J.) Suppose that $\langle I_\eta \rangle_{\eta<\alpha}$ is a family in $\mathcal{I}$, where $0 < \alpha < \kappa$, and $I = \bigcup_{\eta<\alpha} I_\eta$. Then $I \setminus \alpha \subseteq \{\xi : \xi \in \bigcup_{\eta<\xi} I_\eta\}$ belongs to $\mathcal{I}$; as $\alpha \in \mathcal{I}$, $I \in \mathcal{I}$; as $\langle I_\eta \rangle_{\eta<\alpha}$ is arbitrary, $\mathcal{I}$ is κ-additive.

(β) Take S and f as in (ii). **?** If $I_\alpha = \{\xi : \xi \in S,\, f(\xi) \le \alpha\}$ belongs to $\mathcal{I}$ for every α, then $I = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\alpha<\xi} I_\alpha\}$ belongs to $\mathcal{I}$. But if $\xi \in S$ then $f(\xi) < \xi$ and $\xi \in I_{f(\xi)}$, so $S \subseteq I$. **X** As S and f are arbitrary, (ii) is true.

(ii)$\Rightarrow$(iii) Suppose (ii) is true and that S, f are as in (iii). By (ii), there is an $\alpha < \kappa$ such that $\{\xi : \xi \in S,\, f(\xi) \le \alpha\} \notin \mathcal{I}$. As $\mathcal{I}$ is κ-additive, there is a $\beta \le \alpha$ such that $\{\xi : \xi \in S,\, f(\xi) = \beta\} \notin \mathcal{I}$. As S and f are arbitrary, (iii) is true.

(iii)$\Rightarrow$(i) Now suppose that (iii) is true, and that $\langle I_\xi \rangle_{\xi<\kappa}$ is any family in $\mathcal{I}$; set $S = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} I_\eta\}$. Then we have a regressive function $f : S \to \kappa$ such that $\xi \in I_{f(\xi)}$ for every $\xi \in S$. Since $\{\xi : \xi \in S,\, f(\xi) = \beta\} \subseteq I_\beta \in \mathcal{I}$ for every $\beta < \kappa$, (iii) tells us that $S \in \mathcal{I}$. Since we are assuming that $\mathcal{I}$ is a proper ideal including $[\kappa]^{<\kappa}$, it is normal.

541I Lemma Let κ be a regular uncountable cardinal.

(a) The family of non-stationary subsets of κ is a normal ideal on κ, and is included in every normal ideal on κ.

(b) If $\mathcal{I}$ is a normal ideal on κ, and $\langle I_K \rangle_{K \in [\kappa]^{<\omega}}$ is any family in $\mathcal{I}$, then $\{\xi : \xi < \kappa,\, \xi \in \bigcup_{K \in [\xi]^{<\omega}} I_K\}$ belongs to $\mathcal{I}$.

proof (a) Let $\mathcal{I}$ be the family of non-stationary subsets of κ.

(i) Since a subset of κ is non-stationary iff it is disjoint from some closed cofinal set (4A1Ca), any subset of a non-stationary set is non-stationary. Because the intersection of two closed cofinal sets is again a closed cofinal set (4A1Bd), $\mathcal{I}$ is an ideal. Because $\kappa \setminus \xi$ is a closed cofinal set for any $\xi < \kappa$, and κ is regular, $[\kappa]^{<\kappa} \subseteq \mathcal{I}$.

Now suppose that $\langle I_\xi \rangle_{\xi<\kappa}$ is any family in $\mathcal{I}$, and that $A = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} I_\eta\}$. For each $\xi < \kappa$ let F_ξ be a closed cofinal subset of κ disjoint from I_ξ, and let F be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$; then F is a closed cofinal set (4A1B(c-ii)), and it is easy to check that F is disjoint from I, so $I \in \mathcal{I}$. Thus $\mathcal{I}$ is normal.

(ii) Let $\mathcal{J}$ be any normal ideal on κ. If $F \subseteq \kappa$ is a closed cofinal set containing 0, we have a regressive function $f : \kappa \setminus F \to F$ defined by setting $f(\xi) = \sup(F \cap \xi)$ for every $\xi \in \kappa \setminus F$. If $\alpha < \kappa$, $\{\xi : f(\xi) \le \alpha\}$ is bounded above by $\min(F \setminus \alpha)$ so belongs to $[\kappa]^{<\kappa} \subseteq \mathcal{J}$; by 541H(ii), $\kappa \setminus F$ must belong to $\mathcal{J}$. This works for any closed cofinal set containing 0; but as $\{0\}$ surely belongs to J, $\kappa \setminus F \in \mathcal{J}$ for every closed cofinal set F, that is, $\mathcal{I} \subseteq \mathcal{J}$.

(b) Set $J_\xi = \bigcup_{K \in [\xi+1]^{<\omega}} I_K$; because $\mathcal{I}$ is κ-additive, $J_\xi \in \mathcal{I}$ for each ξ. Now

$$\{\xi : \xi < \kappa,\, \xi \in \bigcup_{K \in [\xi]^{<\omega}} I_K\} = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} J_\eta\} \in \mathcal{I}$$

because $\mathcal{I}$ is normal.

541J Theorem (Solovay 71) Let X be a set and $\mathcal{J}$ an ideal of subsets of X. Suppose that $\mathrm{add}\,\mathcal{J} = \kappa > \omega$ and that $\mathcal{J}$ is λ-saturated in $\mathcal{P}X$, where $\lambda \le \kappa$. Then there are $Y \subseteq X$ and $g : Y \to \kappa$ such that $\{B : B \subseteq \kappa,\, g^{-1}[B] \in \mathcal{J}\}$ is a λ-saturated normal ideal on κ.

proof (Cf. 4A1K.) Let $\langle J_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{J}$ such that $Y=\bigcup_{\xi<\kappa}J_\xi\notin\mathcal{J}$. Let F be the set of functions $f:Y\to\kappa$ such that $f^{-1}[\alpha]\in\mathcal{J}$ for every $\alpha<\kappa$. Set $f_0(y)=\min\{\xi:y\in J_\xi\}$ for $y\in Y$; then $f_0\in F$. **P** If $\alpha<\kappa$, then $f^{-1}[\alpha]=\bigcup_{\xi<\alpha}J_\xi$ belongs to $\mathcal{J}$ because $\mathcal{J}$ is κ-additive. **Q**

The point is that there is a $g\in F$ such that $\{y:y\in Y,\ f(y)<g(y)\}\in\mathcal{J}$ for every $f\in F$. **P?** Otherwise, choose f_ξ, for $0<\xi<\kappa$, as follows. Given $f_\xi\in F$, where $\xi<\kappa$, there is an $f\in F$ such that $A_\xi=\{y:f(y)<f_\xi(y)\}\notin\mathcal{J}$; set $f_{\xi+1}(y)=\min(f(y),f_\xi(y))$ for every y. Then

$$f_{\xi+1}^{-1}[\alpha]=f^{-1}[\alpha]\cup f_\xi^{-1}[\alpha]\in\mathcal{J}$$

for every $\alpha<\kappa$, so $f_{\xi+1}\in F$. Given that $f_\eta\in F$ for every $\eta<\xi$, where $\xi<\kappa$ is a non-zero limit ordinal, set $f_\xi(y)=\min\{f_\eta(y):\eta<\xi\}$ for each $y\in Y$; then

$$f_\xi^{-1}[\alpha]=\bigcup_{\eta<\xi}f_\eta^{-1}[\alpha]\in\mathcal{J}$$

for every $\alpha<\kappa$, because $\#(\xi)<\kappa=\operatorname{add}\mathcal{J}$.

This construction ensures that $\langle f_\xi(y)\rangle_{\xi<\kappa}$ is non-increasing for every y, and that $\{y:f_{\xi+1}(y)<f_\xi(y)\}=A_\xi\notin\mathcal{J}$ for every $\xi<\kappa$. But as $\mathcal{J}$ is κ-saturated in $\mathcal{P}X$, there must be a point y belonging to infinitely many A_ξ (541Cb), so that there is a strictly decreasing sequence in $\{f_\xi(y):\xi<\kappa\}$, which is impossible. **XQ**

Now consider $\mathcal{I}=\{B:B\subseteq\kappa,\ g^{-1}[B]\in\mathcal{J}\}$. Because $\mathcal{J}$ is λ-saturated in $\mathcal{P}X$, $\mathcal{I}$ is λ-saturated in $\mathcal{P}\kappa$. **P** If $\langle B_\xi\rangle_{\xi<\lambda}$ is a family in $\mathcal{P}\kappa\setminus\mathcal{I}$, then $\langle g^{-1}[B_\xi]\rangle_{\xi<\lambda}$ is a family in $\mathcal{P}X\setminus\mathcal{J}$, so there are distinct ξ, $\eta<\lambda$ such that $g^{-1}[B_\xi\cap B_\eta]=g^{-1}[B_\xi]\cap g^{-1}[B_\eta]$ does not belong to $\mathcal{J}$, and $B_\xi\cap B_\eta$ does not belong to $\mathcal{I}$. **Q** Next, $\mathcal{I}$ is normal. **P** Of course $\kappa=\operatorname{add}\mathcal{J}$ is regular (513C(a-i)), and we are supposing that it is uncountable. If $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $h:S\to\kappa$ is regressive, set $f(y)=hg(y)$ if $y\in g^{-1}[S]$, $g(y)$ otherwise. Then $\{y:f(y)<g(y)\}=g^{-1}[S]\notin\mathcal{J}$, so $f\notin F$ and there is an $\alpha<\kappa$ such that $f^{-1}[\alpha]\notin\mathcal{J}$. But

$$f^{-1}[\alpha]\subseteq g^{-1}[\alpha]\cup\bigcup_{\beta<\alpha}g^{-1}[h^{-1}[\{\beta\}]];$$

as $\alpha<\operatorname{add}\mathcal{J}$, there is a $\beta<\alpha$ such that $g^{-1}[h^{-1}[\{\beta\}]]\notin\mathcal{J}$ and $h^{-1}[\{\beta\}]\notin\mathcal{I}$. As h is arbitrary, $\mathcal{I}$ is normal (541H). **Q**

541K Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ which is κ'-saturated in $\mathcal{P}\kappa$, where $\kappa'\le\kappa$.

(a) If $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $f:S\to\kappa$ is regressive, then there is a set $A\in[\kappa]^{<\kappa'}$ such that $S\setminus f^{-1}[A]\in\mathcal{I}$; consequently there is an $\alpha\le\kappa$ such that $\{\xi:\xi\in S,\ f(\xi)\ge\alpha\}\in\mathcal{I}$.

(b) If $\lambda<\kappa$, then $\{\xi:\xi<\kappa,\ \operatorname{cf}\xi\le\lambda\}\in\mathcal{I}$.

(c) If for each $\xi<\kappa$ we are given a relatively closed set $C_\xi\subseteq\xi$ which is cofinal with ξ, then

$$C=\{\alpha:\alpha<\kappa,\ \{\xi:\alpha\notin C_\xi\}\in\mathcal{I}\}$$

is a cofinal closed set in κ.

proof (a) Choose $\langle S_\eta\rangle_{\eta\le\gamma}$ and $\langle\alpha_\eta\rangle_{\eta<\gamma}$ inductively, as follows. $S_0=S$. If $S_\eta\in\mathcal{I}$, set $\gamma=\eta$ and stop. Otherwise, $f\restriction S_\eta$ is regressive, so (because $\mathcal{I}$ is normal) there is an $\alpha_\eta<\kappa$ such that $\{\xi:\xi\in S_\eta,\ f(\xi)=\alpha_\eta\}\notin\mathcal{I}$ (541H(iii)). Set $S_{\eta+1}=\{\xi:\xi\in S_\eta,\ f(\xi)\ne\alpha_\eta\}$. Given $\langle S_\zeta\rangle_{\zeta<\eta}$ for a non-zero limit ordinal η, set $S_\eta=\bigcap_{\zeta<\eta}S_\zeta$. Now $\langle S_\eta\setminus S_{\eta+1}\rangle_{\eta<\gamma}$ is a disjoint family in $\mathcal{P}\kappa\setminus\mathcal{I}$, so $\#(\gamma)<\kappa'$ and $A=\{\alpha_\eta:\eta<\gamma\}\in[\kappa]^{<\kappa'}$, while $S\setminus f^{-1}[A]=S_\gamma$ belongs to $\mathcal{I}$. Setting $\alpha=\sup A+1$, $\alpha<\kappa$ (because κ is regular) and $\{\xi:\xi\in S,\ f(\xi)\ge\alpha\}$ belongs to $\mathcal{I}$.

(b) ? Otherwise, set $S=\{\xi:0<\xi<\kappa,\ \operatorname{cf}\xi\le\lambda\}$ and for $\xi\in S$ choose a cofinal set $A_\xi\subseteq\xi$ with $\#(A_\xi)\le\lambda$. Let $\langle f_\eta\rangle_{\eta<\lambda}$ be a family of functions defined on S such that $A_\xi=\{f_\eta(\xi):\eta<\lambda\}$ for each $\xi\in S$. By (a), we have for each $\eta<\lambda$ an $\alpha_\eta<\kappa$ such that $B_\eta=\{\xi:\xi\in S,\ f_\eta(\xi)\ge\alpha_\eta\}\in\mathcal{I}$. Set $\alpha=\sup_{\eta<\lambda}\alpha_\eta<\kappa$; as $\lambda<\operatorname{add}\mathcal{I}$, there is a $\xi\in S\setminus\bigcup_{\eta<\lambda}B_\eta$ such that $\xi>\alpha$. But now $A_\xi\subseteq\alpha$ is not cofinal with ξ. **X**

(c) For $\alpha<\kappa$, $0<\xi<\kappa$ set

$$f_\alpha(\xi)=\min(C_\xi\setminus\alpha)\text{ if }\xi>\alpha,$$
$$=0\text{ otherwise.}$$

Then f_α is regressive, so by (a) there is a $\zeta_\alpha<\kappa$ such that $\kappa\setminus f_\alpha^{-1}[\zeta_\alpha]\in\mathcal{I}$, that is, $\{\xi:C_\xi\cap\zeta_\alpha\setminus\alpha=\emptyset\}\in\mathcal{I}$. Set $\tilde C=\{\alpha:\alpha<\kappa,\ \zeta_\beta<\alpha\text{ for every }\beta<\alpha\}$; then $\tilde C$ is cofinal with κ. If $\alpha\in\tilde C$, then

$$\{\xi:\xi>\alpha,\ \alpha\notin C_\xi\}\subseteq\{\xi:C_\xi\cap\alpha\text{ is not cofinal with }\alpha\}$$
$$\subseteq\{\xi:C_\xi\cap\zeta_\beta\setminus\beta=\emptyset\text{ for some }\beta<\alpha\}$$

is the union of fewer than κ members of $\mathcal{I}$, so belongs to $\mathcal{I}$, and $\alpha\in C$. Thus C is cofinal with κ. If $\alpha<\kappa$ and $\alpha=\sup(C\cap\alpha)$, then

$$\{\xi : \xi > \alpha,\, \alpha \notin C_\xi\} \subseteq \{\xi : \beta \notin C_\xi \text{ for some } \beta \in C \cap \alpha\}$$

is again the union of fewer than κ members of $\mathcal{I}$, so $\alpha \in C$. Thus C is closed.

541L Theorem Let κ be an uncountable cardinal such that there is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons.

(a) There is a κ-saturated normal ideal on κ.

(b) κ is weakly inaccessible.

(c) The set of weakly inaccessible cardinals less than κ is stationary in κ.

proof (a) Let $\mathcal{J}$ be a κ-saturated κ-additive ideal of $\mathcal{P}\kappa$. The additivity of $\mathcal{J}$ must be exactly κ, so 541J tells us that there is a κ-saturated normal ideal $\mathcal{I}$ on κ.

(b) Of course $\kappa = \operatorname{add}\mathcal{J} = \operatorname{add}\mathcal{I}$ is regular. **?** Suppose, if possible, that $\kappa = \lambda^+$ is a successor cardinal. For each $\alpha < \kappa$ let $\phi_\alpha : \alpha \to \lambda$ be an injection. For $\beta < \kappa$ and $\xi < \lambda$ set $A_{\beta\xi} = \{\alpha : \beta < \alpha < \kappa,\, \phi_\alpha(\beta) = \xi\}$. Then $\bigcup_{\xi<\lambda} A_{\beta\xi} = \kappa \setminus (\beta + 1) \notin \mathcal{J}$, so there is a $\xi_\beta < \lambda$ such that $A_{\beta,\xi_\beta} \notin \mathcal{J}$. Now there must be an $\eta < \lambda$ such that $B = \{\beta : \beta < \kappa,\, \xi_\beta = \eta\}$ has cardinal κ. But in this case $\langle A_{\beta\eta}\rangle_{\beta\in B}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{J}$, and $\mathcal{J}$ is not κ-saturated in $\mathcal{P}\kappa$. **X**

Thus κ is a regular uncountable limit cardinal, i.e., is weakly inaccessible.

(c) Write C for the set of cardinals less than κ, R for the set of regular infinite cardinals less than κ and L for the set of limit cardinals less than κ.

(i) $\kappa \setminus R \in \mathcal{I}$. **P?** Otherwise, $A = (\kappa \setminus R) \setminus \{0, 1\} \notin \mathcal{I}$. For $\xi \in A$, set $f(\xi) = \operatorname{cf}\xi$; then $f : A \to \kappa$ is regressive. Because $\mathcal{I}$ is normal, there must be a $\delta < \kappa$ such that $B = \{\xi : \xi < \kappa,\, \operatorname{cf}\xi = \delta\} \notin \mathcal{I}$. For each $\xi \in B$, let $\langle g_\eta(\xi)\rangle_{\eta<\delta}$ enumerate a cofinal subset of ξ. If $\eta < \delta$, then $g_\eta : B \to \kappa$ is regressive, so by 541Ka there is a $\gamma_\eta < \kappa$ such that $J_\eta = \{\xi : \xi \in B,\, g_\eta(\xi) \ge \gamma_\eta\} \in \mathcal{I}$. Set $\gamma = \sup_{\eta<\delta} \gamma_\eta$; as κ is regular, $\gamma < \kappa$; while $B \setminus (\gamma + 1) \subseteq \bigcup_{\eta<\delta} J_\eta$ belongs to $\mathcal{I}$, which is impossible. **XQ**

(ii) $R \setminus L \in \mathcal{I}$. **P** We have a regressive function $f : R \setminus L \to \kappa$ defined by setting $f(\lambda^+) = \lambda$ for every infinite cardinal $\lambda < \kappa$. Now $f^{-1}[\{\xi\}]$ is empty or a singleton for every ξ, so always belongs to $\mathcal{I}$; because $\mathcal{I}$ is normal, $R \setminus L \in \mathcal{I}$. **Q**

(iii) Accordingly the set $R \cap L$ of weakly inaccessible cardinals less than κ cannot belong to $\mathcal{I}$ and must be stationary, by 541Ia.

541M Definition (a) A regular uncountable cardinal κ is **two-valued-measurable** (often just **measurable**) if there is a proper κ-additive 2-saturated ideal of $\mathcal{P}\kappa$ containing singletons.

Of course a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ is 2-saturated iff it is maximal, that is, the dual filter $\{\kappa \setminus I : I \in \mathcal{I}\}$ is an ultrafilter; thus κ is two-valued-measurable iff there is a non-principal κ-complete ultrafilter on κ. From 541J we see also that if κ is two-valued-measurable then there is a normal maximal ideal of $\mathcal{P}\kappa$, that is, there is a normal ultrafilter on κ, as considered in §4A1.

(b) An uncountable cardinal κ is **weakly compact** if for every $S \subseteq [\kappa]^2$ there is a $D \in [\kappa]^\kappa$ such that $[D]^2$ is either included in S or disjoint from S.

541N Theorem (a) A two-valued-measurable cardinal is weakly compact.

(b) A weakly compact cardinal is strongly inaccessible.

proof (a) If κ is a two-valued-measurable cardinal, there is a non-principal normal ultrafilter on κ, so 4A1L tells us that κ is weakly compact.

(b) Let κ be a weakly compact cardinal.

(i) Set $\lambda = \operatorname{cf}\kappa$; let $A \in [\kappa]^\lambda$ be a cofinal subset of κ, and $\langle\alpha_\zeta\rangle_{\zeta<\lambda}$ the increasing enumeration of A. For $\xi < \kappa$ set $f(\xi) = \min\{\zeta : \xi < \alpha_\zeta\}$; now set $S = \{I : I \in [\kappa]^2,\, f \text{ is constant on } I\}$. If $D \in [\kappa]^\kappa$, take any $\xi \in D$; then there is an $\eta \in D \setminus \alpha_{f(\xi)}$, so f is not constant on $\{\xi, \eta\}$ and $[D]^2 \not\subseteq S$. There must therefore be a $D \in [\kappa]^2$ such that $[D]^2 \cap S = \emptyset$. But in this case f is injective on D, so $\lambda \ge \#(f[D]) = \kappa$ and $\operatorname{cf}\kappa = \kappa$.

Thus κ is regular.

(ii) ? Suppose, if possible, that κ is not strongly inaccessible. Then there is a least cardinal $\lambda < \kappa$ such that $2^\lambda \ge \kappa$; let $\phi : \kappa \to \mathcal{P}\lambda$ be an injective function. Set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa,\, \min(\phi(\xi)\triangle\phi(\eta)) \in \phi(\eta)\}.$$

If $D \in [\kappa]^\kappa$, then there are ξ, η, $\zeta \in D$ such that $\xi < \eta < \zeta$ and just one of $\{\xi,\eta\}$, $\{\eta,\zeta\}$ belongs to S. **P** Set $B = \{\phi(\xi) \cap \gamma : \xi \in D,\ \gamma < \lambda\}$. Then

$$\#(B) \le \#(\bigcup_{\gamma<\lambda} \mathcal{P}\gamma) \le \max(\lambda, \sup_{\gamma<\lambda} 2^\gamma) < \kappa$$

because κ is regular, $\lambda < \kappa$ and $2^\gamma < \kappa$ for every $\gamma < \lambda$. So there must be an $\eta \in D$ such that $B = \{\phi(\xi) \cap \gamma : \xi \in D \cap \eta,$ $\gamma < \lambda\}$. Take $\zeta \in D$ such that $\zeta > \eta$, set $\gamma = \min(\phi(\eta) \triangle \phi(\zeta))$ and take $\xi \in D \cap \eta$ such that $\phi(\xi) \cap (\gamma+1) = \phi(\zeta) \cap (\gamma+1)$. Now $\gamma = \min(\phi(\xi) \triangle \phi(\eta))$, so

$$\{\xi,\eta\} \in S \iff \gamma \in \phi(\eta) \iff \gamma \notin \phi(\zeta) \iff \{\eta,\zeta\} \notin S. \textbf{ Q}$$

But this means that $[D]^2$ is neither included in S nor disjoint from S; which is supposed to be impossible. **X**

Thus κ is strongly inaccessible.

541O Lemma Let X be a set and $\mathcal{I}$ a proper ideal of subsets of X such that $\mathcal{P}X/\mathcal{I}$ is atomless. If $\mathcal{I}$ is λ-saturated and κ-additive, with $\lambda \le \kappa$, then $\kappa \le \operatorname{cov} \mathcal{I} \le \sup_{\theta<\lambda} 2^\theta$.

proof We may take it that $\lambda = \operatorname{sat}(\mathcal{P}X/\mathcal{I})$. If $\lambda > \kappa$ the result is trivial because $\mathcal{I}$ contains singletons. So suppose that $\lambda \le \kappa$. For each $A \in \mathcal{P}X \setminus \mathcal{I}$ choose $A' \subseteq A$ such that neither A' nor $A \setminus A'$ belongs to $\mathcal{I}$; this is possible because $\mathcal{P}X/\mathcal{I}$ is atomless. Define $\langle \mathcal{A}_\xi \rangle_{\xi<\lambda}$ inductively, as follows. $\mathcal{A}_0 = \{X\}$. Given that $\mathcal{A}_\xi \subseteq \mathcal{P}X \setminus \mathcal{I}$, then set $\mathcal{A}_{\xi+1} = \{A' : A \in \mathcal{A}_\xi\} \cup \{A \setminus A' : A \in \mathcal{A}_\xi\}$. For a non-zero limit ordinal $\xi < \lambda$, set $E_\xi = \bigcap_{\eta<\xi} \bigcup \mathcal{A}_\eta$; for $x \in E_\xi$ set $C_{\xi x} = \bigcap\{A : x \in A \in \bigcup_{\eta<\xi} \mathcal{A}_\eta\}$; set $\mathcal{A}_\xi = \{C_{\xi x} : \zeta \in E_\xi\} \setminus \mathcal{I}$, and continue. Observe that this construction ensures that each $\mathcal{A}_\xi$ is disjoint, and that if $\eta \le \xi$ and $A \in \mathcal{A}_\xi$ then there is a $B \in \mathcal{A}_\eta$ such that $A \subseteq B$.

If $x \in X$, then $\alpha_x = \{\xi : \xi < \lambda,\ x \in \bigcup \mathcal{A}_\xi\}$ is an initial segment of λ, so is an ordinal less than or equal to λ. In fact $\alpha_x < \lambda$. **P** For each $\xi < \alpha_x$ take $A_\xi \in \mathcal{A}_\xi$ such that $x \in A_\xi$, and let B_ξ be either A'_ξ or $A_\xi \setminus A'_\xi$ and such that $x \notin B_\xi$. Then $\langle B_\xi \rangle_{\xi<\alpha_x}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{I}$ so has cardinal less than λ. **Q**

Of course each α_x is a non-zero limit ordinal, because $\bigcup \mathcal{A}_\xi = \bigcup \mathcal{A}_{\xi+1}$ for each ξ. Now set $\mathcal{A} = \bigcup_{\xi<\lambda} \mathcal{A}_\xi$; then $\#(\mathcal{A}) \le \lambda$. Next, for any $x \in X$, $\mathcal{B}_x = \{A : A \in \mathcal{A},\ x \in A\}$ has cardinal less than λ and $C_x = \bigcap \mathcal{B}_x$ belongs to $\mathcal{I}$ and contains x. So $\mathcal{C} = \{C_x : x \in X\}$ has cardinal at most $\#([\lambda]^{<\lambda}) = \sup_{\theta<\lambda} 2^\theta$ (because $\lambda = \operatorname{sat}(\mathcal{P}X/\mathcal{I})$ is regular, by 514Da), and $\mathcal{C} \subseteq \mathcal{I}$ covers X, so

$$\kappa \le \operatorname{add} \mathcal{I} \le \operatorname{cov} \mathcal{I} \le \#(\mathcal{C}) \le \sup_{\theta<\lambda} 2^\theta.$$

541P Theorem (Tarski 45, Solovay 71) Suppose that κ is a regular uncountable cardinal with a proper λ-saturated κ-additive ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons, where $\lambda \le \kappa$. Set $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Then

either $\kappa \le \sup_{\theta<\lambda} 2^\theta$ and $\mathfrak{A}$ is atomless

or κ is two-valued-measurable and $\mathfrak{A}$ is purely atomic.

proof (a) Let us begin by noting that $\mathcal{I}$ is λ-saturated iff $\lambda \ge \operatorname{sat}(\mathfrak{A})$; so it will be enough to prove the result when $\lambda = \operatorname{sat}(\mathfrak{A})$, in which case λ is either finite or regular and uncountable (514Da).

(b) Suppose that $\mathfrak{A}$ is atomless. By 541O, $\kappa \le \sup_{\theta<\lambda} 2^\theta$. So in this case we have the first alternative of the dichotomy.

(c) Before continuing with an analysis of atoms in $\mathfrak{A}$, I draw out some further features of the structure discussed in 541O. We find that if $\mathfrak{A}$ is atomless then κ is not weakly compact. **P** Construct $\langle \mathcal{A}_\xi \rangle_{\xi<\lambda}$, $\mathcal{A}$ and $\mathcal{C}$ as in 541O. Consider $\alpha^* = \sup\{\xi : \xi < \lambda,\ \mathcal{A}_\xi \ne \emptyset\}$.

case 1 If $\alpha^* < \kappa$, then $\#(\mathcal{A}) < \kappa$, because κ is regular and $\#(\mathcal{A}_\xi) < \lambda \le \kappa$ for every ξ. In this case $\kappa \le \#(\mathcal{C}) \le 2^{\#(\mathcal{A})}$ and κ is not strongly inaccessible, therefore not weakly compact, by 541Nb.

case 2 If $\alpha^* = \kappa$, then $\#(\mathcal{A}') = \kappa$, where $\mathcal{A}' = \bigcup_{\xi<\kappa} \mathcal{A}_{\xi+1}$. Note that each $D \in \mathcal{A}'$ has a companion $D^* \in \mathcal{A}'$ defined by saying that if $D \in \mathcal{A}_{\xi+1}$ then $D^* = D_0 \setminus D$ where D_0 is the unique member of $\mathcal{A}_\xi$ including D. Consider the relation $S = \{(D, D') : D, D' \in \mathcal{A}',\ D \cap D' = \emptyset\}$. Take any $\mathcal{D} \in [\mathcal{A}']^\kappa$. Then $[\mathcal{D}]^2 \not\subseteq S$, because $\mathcal{I}$ is κ-saturated. **?** If $[\mathcal{D}]^2 \cap S = \emptyset$, any two members of $\mathcal{D}$ meet. If D_1 and D_2 are distinct members of $\mathcal{D}$, then they cannot both belong to $\mathcal{A}_{\xi+1}$ for any ξ, so one must belong to $\mathcal{A}_{\eta+1}$ and the other to $\mathcal{A}_{\xi+1}$ where $\eta < \xi$; say $D_1 \in \mathcal{A}_{\eta+1}$ and $D_2 \in \mathcal{A}_{\xi+1}$. Now $D_2 \cup D_2^* \in \mathcal{A}_\xi$ meets D_1 and is therefore included in D_1; so $D_1^* \cap D_2^* = \emptyset$. Thus $\{D^* : D \in \mathcal{D}\}$ is a disjoint family in $\mathcal{A}$ of size κ, contrary to the hypothesis that $\mathcal{I}$ is κ-saturated. **X**

Thus if $\mathcal{D} \in [\mathcal{A}']^\kappa$, $[\mathcal{D}]^2$ is neither included in nor disjoint from S. Since $\#(\mathcal{A}') = \kappa$, this shows that κ cannot be weakly compact. **Q**

(d) Now suppose that $\mathfrak{A}$ has an atom a. Let $A \in \mathcal{P}\kappa \setminus \mathcal{I}$ be such that $A^\bullet = a$. Set $\mathcal{I}_A = \{I : I \subseteq \kappa,\ I \cap A \in \mathcal{I}\}$; then $\mathcal{I}_A$ is a κ-additive maximal ideal of κ containing singletons, so κ is two-valued-measurable. It follows that κ is weakly compact (541Na).

? Suppose, if possible, that $\mathfrak{A}$ is not purely atomic. Then there is a $C \in \mathcal{P}\kappa \setminus \mathcal{I}$ such that $\mathcal{P}\kappa/\mathcal{I}_C$ is atomless, where $\mathcal{I}_C = \{I : I \subseteq \kappa, I \cap C \in \mathcal{I}\}$. Also $\mathcal{I}_C$ is κ-additive and λ-saturated. But this is impossible, by (c). **X** Thus $\mathfrak{A}$ is purely atomic, and we have the second alternative of the dichotomy.

541Q Theorem Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Let $\theta < \kappa$ be a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$, and $f : [\kappa]^{<\omega} \to [\kappa]^{<\theta}$ any function. Then there are $C \in \mathcal{I}$ and $f^* : [\kappa \setminus C]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \in [\kappa \setminus C]^{<\omega}$ and $\eta < \kappa$.

proof (a) I show by induction on $n \in \mathbb{N}$ that if $g : [\kappa]^{\le n} \to [\kappa]^{<\theta}$ is a function then there are $A \in \mathcal{I}$ and $g^* : [\kappa \setminus A]^{\le n} \to [\kappa]^{<\theta}$ such that $g(I) \cap \eta \subseteq g^*(I \cap \eta)$ for every $I \in [\kappa \setminus A]^{\le n}$ and $\eta < \kappa$.

P If $n = 0$ this is trivial; take $A = \emptyset$, $g^*(\emptyset) = g(\emptyset)$. For the inductive step to $n+1$, given $g : [\kappa]^{\le n+1} \to [\kappa]^{<\theta}$, then for each $\xi < \kappa$ define $g_\xi : [\kappa]^{\le n} \to [\kappa]^{<\theta}$ by setting $g_\xi(J) = g(J \cup \{\xi\})$ for every $J \in [\kappa]^{\le n}$. Set

$$D = \{\xi : \xi < \kappa, \operatorname{cf}(\xi) \ge \theta\};$$

then $\kappa \setminus D \in \mathcal{I}$ (541Kb). For $\xi \in D$ and $J \in [\kappa]^{\le n}$ set $\zeta_{J\xi} = \sup(\xi \cap g_\xi(J)) < \xi$. Then for each $J \in [\kappa]^{\le n}$ the function $\xi \mapsto \zeta_{J\xi} : D \to \kappa$ is regressive, so there is a $\zeta^*_J < \kappa$ such that $\{\xi : \zeta_{J\xi} \ge \zeta^*_J\} \in \mathcal{I}$ (541Ka). Now $\operatorname{add}\mathcal{I} = \kappa$, by 541H, so 541D tells us that there is an $h(J) \in [\zeta^*_J]^{<\theta}$ such that $\{\xi : \xi \cap g_\xi(J) \not\subseteq h(J)\} \in \mathcal{I}$. By the inductive hypothesis, there are $B \in \mathcal{I}$ and $h^* : [\kappa \setminus B]^{\le n} \to [\kappa]^{<\theta}$ such that $h(J) \cap \eta \subseteq h^*(J \cap \eta)$ for every $J \in [\kappa \setminus B]^{\le n}$ and $\eta < \kappa$.

Try setting

$$A_J = \{\xi : \xi \cap g_\xi(J) \not\subseteq h(J)\} \text{ for } J \in [\kappa]^{\le n},$$

$$A = B \cup \{\xi : \xi \in \textstyle\bigcup_{J \in [\xi]^{\le n}} A_J\},$$

$$\begin{aligned} g^*(I) &= g(I) \text{ if } I \in [\kappa \setminus A]^{n+1}, \\ &= g(I) \cup h^*(I) \text{ if } I \in [\kappa \setminus A]^{\le n}. \end{aligned}$$

Then A_J always belongs to $\mathcal{I}$, by the choice of $h(J)$, so $A \in \mathcal{I}$, by 541Ib, while $g^*(I) \in [\kappa]^{<\theta}$ for every $I \in [\kappa \setminus A]^{\le n+1}$. Take $\eta < \kappa$ and $I \in [\kappa \setminus A]^{\le n+1}$. If $I \subseteq \eta$ then $g(I) \cap \eta \subseteq g^*(I) = g^*(I \cap \eta)$. Otherwise, set $\xi = \max I$ and $J = I \setminus \{\xi\}$. Then $\eta \le \xi \in \kappa \setminus A_J$, so

$$g(I) \cap \eta = g_\xi(J) \cap \xi \cap \eta \subseteq h(J) \cap \eta \subseteq h^*(J \cap \eta) = h^*(I \cap \eta) \subseteq g^*(I \cap \eta).$$

Thus the induction continues. **Q**

(b) Now applying (a) to $f{\restriction}[\kappa]^{\le n}$ we obtain sets $C_n \in \mathcal{I}$ and functions $f^*_n : [\kappa \setminus C_n]^{\le n} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*_n(I \cap \eta)$ whenever $I \in [\kappa \setminus C_n]^{\le n}$ and $\eta < \kappa$. Set $C = \bigcup_{n \in \mathbb{N}} C_n \in \mathcal{I}$ and $f^*(I) = \bigcup_{n \ge \#(I)} f^*_n(I)$ for each $I \in [\kappa \setminus C]^{<\omega}$. Because $\operatorname{cf}\theta > \omega$, $f(I) \in [\kappa]^{<\theta}$ for every I. If $I \in [\kappa \setminus C]^{<\omega}$ and $\eta < \kappa$, set $n = \#(I)$; then $I \in [\kappa \setminus C_n]^n$ so $f(I) \cap \eta \subseteq f^*_n(I \cap \eta) \subseteq f^*(I \cap \eta)$, as required.

541R Corollary Let κ be a regular uncountable cardinal, $\mathcal{I}$ a normal ideal on κ, and $\theta < \kappa$ a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$.

(a) If Y is a set of cardinal less than κ and $f : [\kappa]^{<\omega} \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $f(I) \subseteq M$ for every $I \in [\kappa \setminus C]^{<\omega}$.

(b) If Y is any set and $g : \kappa \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $g(\xi) \cap g(\eta) \subseteq M$ for all distinct $\xi, \eta \in \kappa \setminus C$.

proof (a) We may suppose that $Y \subseteq \kappa$. In this case, by 541Q, we have a $C_0 \in \mathcal{I}$ and an $f^* : [\kappa \setminus C_0]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \subseteq \kappa \setminus C_0$ is finite and $\eta < \kappa$. Let $\gamma < \kappa$ be such that $Y \subseteq \gamma$ and set $M = Y \cap f^*(\emptyset)$, $C = C_0 \cup \gamma$. Then $M \in [Y]^{<\theta}$, $C \in \mathcal{I}$ and if $I \in [\kappa \setminus C]^{<\omega}$ then

$$f(I) = f(I) \cap \gamma \subseteq Y \cap f^*(I \cap \gamma) \cap \gamma = M.$$

(b) Since $\bigcup_{\xi < \kappa} g(\xi)$ has cardinal at most κ, we may again suppose that $Y \subseteq \kappa$. Apply 541Q with $f(\{\xi\}) = g(\xi)$ for $\xi < \kappa$. Taking C and f^* from 541Q, set $M = Y \cap f^*(\emptyset)$. Set $F = \{\xi : \xi < \kappa,\ g(\eta) \subseteq \xi \text{ for every } \eta < \xi\}$; then F is a closed cofinal subset of κ (because $\theta \le \kappa$ and κ is regular), so $C' = C \cup (\kappa \setminus F) \in \mathcal{I}$ (541Ia). If ξ, η belong to $\kappa \setminus C' = F \setminus C$ and $\eta < \xi$, then

$$g(\xi) \cap g(\eta) \subseteq \xi \cap g(\xi) = \xi \cap f(\{\xi\}) \subseteq f^*(\xi \cap \{\xi\}) = f^*(\emptyset),$$

so $g(\xi) \cap g(\eta) \subseteq M$. Thus C' serves.

541S Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Suppose that γ and δ are cardinals such that $\omega \le \gamma < \delta < \kappa$, $\mathcal{I}$ is δ-saturated in $\mathcal{P}\kappa$, $2^\beta = 2^\gamma$ for $\gamma \le \beta < \delta$, but $2^\delta > 2^\gamma$. Then δ is regular and

$$2^\delta = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2).$$

proof By 5A1Eh, δ is regular. Of course

$$\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \le \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) \le \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2) \le \#([2^\gamma]^{\le\delta}) \le 2^\delta$$

(5A2D, 5A2Ea). For the reverse inequality, let $\mathcal{E} \subseteq [2^\gamma]^{<\kappa}$ be a set of cardinal $\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta)$ such that every member of $[2^\gamma]^{\le\delta}$ is covered by fewer than δ members of $\mathcal{E}$. For each ordinal $\xi < \delta$ let $\phi_\xi : \mathcal{P}\xi \to 2^\gamma$ be an injective function. For $A \subseteq \delta$ define $f_A : \delta \to 2^\gamma$ by

$$f_A(\xi) = \phi_\xi(A \cap \xi) \text{ for every } \xi < \delta.$$

Choose $E_A \in \mathcal{E}$ such that $f_A^{-1}[E_A]$ is cofinal with δ; such must exist because δ is regular and $f_A[\delta]$ can be covered by fewer than δ members of $\mathcal{E}$.

? If $2^\delta > \#(\mathcal{E})$ then there must be an $E \in \mathcal{E}$ and an $\mathcal{A} \subseteq \mathcal{P}\delta$ such that $\#(\mathcal{A}) = \kappa$ and $E_A = E$ for every $A \in \mathcal{A}$. For each pair A, B of distinct members of $\mathcal{A}$ set $\xi_{AB} = \min(A \triangle B) < \delta$. By 541R, there is a set $\mathcal{B} \subseteq \mathcal{A}$, of cardinal κ, such that $M = \{\xi_{AB} : A, B \in \mathcal{B}, A \ne B\}$ has cardinal less than δ. Set $\zeta = \sup M < \delta$. Next, for each $A \in \mathcal{B}$, take $\eta_A > \zeta$ such that $f_A(\eta_A) \in E$. Let $\eta < \delta$ be such that $\mathcal{C} = \{A : A \in \mathcal{B}, \eta_A = \eta\}$ has cardinal κ. Then we have a map

$$A \mapsto f_A(\eta) = \phi_\eta(A \cap \eta) : \mathcal{C} \to E$$

which is injective, because if A, B are distinct members of $\mathcal{C}$ then $\xi_{AB} \le \zeta < \eta$, so $A \cap \eta \ne B \cap \eta$. So $\#(E) \ge \kappa$; but $E \in \mathcal{E} \subseteq [2^\gamma]^{<\kappa}$. **X**

As $\mathcal{E}$ is arbitrary, $\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \ge 2^\delta$.

541X Basic exercises (a) Let κ be a regular infinite cardinal. Show that $\mathcal{P}\kappa/[\kappa]^{<\kappa}$ is not Dedekind complete, so $[\kappa]^{<\kappa}$ is not κ^+-saturated in $\mathcal{P}\kappa$. (*Hint*: construct a disjoint family $\langle A_\xi \rangle_{\xi<\kappa}$ in $[\kappa]^\kappa$; show that if $\#(A_\xi \setminus A) < \kappa$ for every ξ there is a $B \in [A]^\kappa$ such that $\#(B \cap A_\xi) < \kappa$ for every ξ.)

(b) Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Suppose there is a cardinal κ such that I is κ-additive and κ^+-saturated and $\mathfrak{A}$ is Dedekind κ^+-complete in the sense that $\sup A$ is defined in $\mathfrak{A}$ whenever $A \in [\mathfrak{A}]^{\le\kappa}$. Show that $\mathfrak{A}/I$ is Dedekind complete.

(c) Suppose that X and Y are sets and $\mathcal{I}$, $\mathcal{J}$ ideals of subsets of X, Y respectively. Suppose that κ is an infinite cardinal such that both $\mathcal{I}$ and $\mathcal{J}$ are κ-saturated and κ^+-additive. Show that $\mathcal{I} \ltimes \mathcal{J}$ (definition: 527Ba) is κ-saturated and κ^+-additive.

(d) Simplify the argument of 541D to give a direct proof of 541E in the case in which κ is regular.

>(e) Show that there is a two-valued-measurable cardinal iff there are a set I and a non-principal ω_1-complete ultrafilter on I.

>(f) Let κ be a two-valued-measurable cardinal and $\mathcal{I}$ a normal maximal ideal of $\mathcal{P}\kappa$. (i) Show that if $S \subseteq [\kappa]^{<\omega}$, there is a $C \in \mathcal{I}$ such that, for each $n \in \mathbb{N}$, $[\kappa \setminus C]^n$ is either disjoint from S or included in it. (ii) Show that if $\#(Y) < \kappa$, and $f : [\kappa]^{<\omega} \to Y$ is any function, then there is a $C \in \mathcal{I}$ such that f is constant on $[\kappa \setminus C]^n$ for each $n \in \mathbb{N}$.

(g) Let κ be a regular uncountable cardinal, $\mathcal{I}$ a κ-saturated normal ideal on κ and $f : [\kappa]^2 \to \kappa$ a function. Show that there are a $C \in \mathcal{I}$ and an $f^* : \kappa \to \kappa$ such that whenever $\eta \in \kappa \setminus C$ and $\xi \in \eta \setminus C$ then either $f(\{\xi, \eta\}) \ge \eta$ or $f(\{\xi, \eta\}) < f^*(\xi)$.

541Y Further exercises (a) Show that if κ is an uncountable regular cardinal and $S \subseteq \kappa$ is stationary, then S can be partitioned into κ stationary sets. (*Hint*: reduce to the case in which there is a κ-saturated normal ideal $\mathcal{I}$ of κ containing $\kappa \setminus S$. Define $f : S \to \kappa$ inductively by saying that

$$f(\xi) = \min(\bigcup\{\kappa \setminus f[C] : C \subseteq \xi \text{ is relatively closed and cofinal}\}.$$

Set $S_\gamma = f^{-1}[\{\gamma\}]$. Apply 541Kc with $C_\xi \cap S_\gamma = \emptyset$ for $\xi \in S_\gamma$ to show that $S_\gamma \in \mathcal{I}$ for every γ. Hence show that S_γ is stationary for every γ. See Solovay 71.)

(b) Show that if κ is two-valued-measurable and $\mathcal{F}$ is a normal ultrafilter on κ, then the set of weakly compact cardinals less than κ belongs to $\mathcal{F}$.

541 Notes and comments The ordinary principle of exhaustion (215A) can be regarded as an expression of ω_1-saturation (compare 316E and 514Db). In 541B-541E we have versions of results already given in special cases; but note that 541B, for instance, goes a step farther than the arguments offered in 314C and 316Fa can reach. 541Ca corresponds to 215B(iv); 541Cb-541Cc are associated with 516Q and 525D. In all this work you will probably find it helpful to use the words 'negligible' and 'conegligible' and 'almost everywhere', so that the conclusion of 541E becomes '$g(x) \in M$ a.e.(x)'. I don't use this language in the formal exposition because of the obvious danger of confusing a reader who is skimming through without reading introductions to sections very carefully; but in the principal applications I have in mind, $\mathcal{I}$ will indeed be a null ideal. The cardinals $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ will appear only occasionally in this book, but are of great importance in infinitary combinatorics generally. Note that the key step in the proof of 541F (part (b) of the proof) develops an idea from the proof of 541J.

For the special purposes of §438 I mentioned 'normal filters' in §4A1; I have now attempted an introduction to the general theory of normal filters and ideals. The central observation of Keisler & Tarski 64 was that if κ is uncountable then any κ-complete non-principal ultrafilter gives rise to a normal ultrafilter on κ. It was noticed very quickly that something similar happens if we have a κ-complete filter of conegligible sets in a totally finite measure space; the extension of the idea to general κ-additive κ-saturated ideals is in Solovay 71. In this chapter I speak oftener of ideals than of filters but the ideas are necessarily identical. Observe that the Pressing-Down Lemma (4A1Cc) is the special case of 541H(iii) when $\mathcal{I}$ is the ideal of non-stationary sets (541Ia).

541Lb here is a re-working of Ulam's theorem (438C, Ulam 30). The dramatic further step in 541Lc derives from Keisler & Tarski 64. The proof of 541Lc already makes it plain that much more can be said; for extensions of these ideas, see Fremlin 93, Levy 71 and Baumgartner Taylor & Wagon 77. In 541P we have an extension of Ulam's dichotomy (438C, 543B). 'Weak compactness' of a cardinal corresponds to Ramsey's theorem (4A1G); the idea was the basis of the proof of 451Q[1]. Here I treat it as a purely combinatorial concept, but its real importance is in model theory (Kanamori 03, §4).

541Q is a fairly strong version of one of the typical properties of saturated normal ideals. The simplest not-quite-trivial case is when we have a function $f : [\kappa]^2 \to \kappa$. In this case we find that if we discard an appropriate negligible set C then, for the remaining doubletons $I \in [\kappa \setminus C]^2$, $f(I)$ is either greater than or equal to $\max I$ or in a 'small' set determined by $\min I$. In this form, with the appropriate definition of 'small', it is enough for the ideal to be κ-saturated (541Xg). In the intended applications of 541Q, however, we shall be looking at functions $f : [\kappa]^{<\omega} \to [\kappa]^{\le\omega}$ and shall need to start from an ω_1-saturated ideal to obtain the full strength of the result.

Shelah's four-cardinal covering numbers $\mathrm{cov}_{\mathrm{Sh}}$ are not immediately digestible; in §5A2 I give the basic pcf theory linking them to cofinalities of products. 541S is here because it relies on a normal ideal being saturated.

Perhaps I have not yet sufficiently emphasized that there is a good reason why I have given no examples of normal ideals other than the non-stationary ideals, and no discussion of the saturation of those beyond Solovay's theorem 541Ya. We have in fact come to an area of mathematics which demands further acts of faith. I will continue, whenever possible, to express ideas as arguments in ordinary ZFC; but in most of the principal theorems the hypotheses will include assertions which can be satisfied only in rather special models of set theory. Moreover, these are special in a new sense. By and large, the assumptions used in the first three chapters of this volume (Martin's axiom, the continuum hypothesis, and so on) have been proved to be relatively consistent with ZFC (indeed, with ZF); that is, we know how to convert any proof in ZFC that '$\mathfrak{m} = \omega_1$' into a proof in ZF that '0 = 1'. The formal demonstration that this can be done is of course normally expressed in a framework reducible to Zermelo-Fraenkel set theory; but it is sufficiently compelling to be itself part of the material which must be encompassed by any formal system claiming to represent twenty-first century mathematics. In the present chapter, however, we are coming to results like 541P which have no content unless there can be non-trivial κ-saturated κ-additive ideals. And such objects are known to be strange in a different way from Souslin lines.

To describe this difference I turn to the simplest of the new propositions. Write ∃sic, ∃wic for the sentences 'there is a strongly inaccessible cardinal', 'there is a weakly inaccessible cardinal'. Of course ∃sic implies ∃wic, while 'there is a cardinal which is not measure-free' also implies ∃wic, by Ulam's theorem. We have no such implications in the other direction, but it is easy to adapt Gödel's argument for the relative consistency of GCH to show that if ZFC + ∃wic is consistent so is ZFC + GCH + ∃wic, while ZFC + GCH + ∃wic implies ∃sic. But we find also that there is a proof in ZFC + ∃sic that 'ZFC is consistent'. So if there were a proof in ZFC that 'if ZFC is consistent, then ZFC + ∃sic is consistent', there would be a proof in ZFC + ∃sic that 'ZFC + ∃sic is consistent'; by Gödel's incompleteness theorem, this would give us a proof that ZFC + ∃sic was in fact *inconsistent* (and therefore that ZFC and ZF are inconsistent).

The last paragraph is expressed crudely, in a language which blurs some essential distinctions; for a more careful account see Kunen 80, §IV.10. But what I am trying to say is that any theory involving inaccessible cardinals – and the theory of this chapter involves unthinkably many such cardinals – necessarily leads us to propositions which are not merely unprovable, but have high 'consistency strength'; we have long strings $\phi_0, \dots, \phi_n$ of statements such that (i) if ZFC + ϕ_{i+1} is consistent, so is ZFC + ϕ_i (ii) there can be no proof of the reverse unless ZF is inconsistent.

[1] Formerly 451P.

We do not (and in my view cannot) know for sure that ZF is consistent. It has now survived for a hundred years or so, which is empirical evidence of a sort. I do not suppose that the century of my own birth was also the century in which the structure of formal mathematics was determined for eternity; I hope and trust that mathematicians will come to look on ZFC as we now think of Euclidean geometry, as a glorious achievement and an enduring source of inspiration but inadequate for the expression of many of our deepest ideas. But (arguing from the weakest of historical analogies) I suggest that if and when a new paradox shakes the foundations of mathematics, it will be because some new Cantor has sought to extend apparently trustworthy methods to a totally new context. And I think that the mathematicians of that generation will stretch their ingenuity to the utmost to find a resolution of the paradox which is conservative, in that it retains as much as possible of their predecessors' ideas, subject perhaps to re-writing a good many proofs and tut-tutting over the naivety of essays such as this.

I think indeed (I am going a bit farther here) that they will be as reluctant to discard measurable cardinals as our forebears were to discard Cantor's cardinals. There is a flourishing theory of large cardinals in which very much stronger statements than 'there is a two-valued-measurable cardinal' have been explored in depth without catastrophe. (I mention a couple of these in §545; another is the Axiom of Determinacy in §567.) Occasionally a proof that there are no measurable cardinals is announced; but the last real fright was in 1976, and most of these arguments have easily been shown not to reach the claimed conclusion. My best guess is that measurable cardinals are safe. But even if I am wrong, and they are irreconcilable with ZFC as now formulated, it does not follow that ZFC will be kept and measurable cardinals discarded. It could equally happen that one of the axioms of ZF will be modified; or, at least, that a modified form will become a recognized option. This is a partisan view from somebody who has a substantial investment to protect. But if you wish to prove me wrong, I do not see how you can do so without giving part of your own life to the topic.

542 Quasi-measurable cardinals

As is to be expected, the results of §541 take especially dramatic forms when we look at ω_1-saturated σ-ideals. 542B-542C spell out the application of the most important ideas from §541 to this special case. In addition, we can us Shelah's pcf theory to give us some remarkable combinatorial results concerning cardinal arithmetic (542E) and cofinalities of partially ordered sets (542I-542J).

542A Definition A cardinal κ is **quasi-measurable** if κ is regular and uncountable and there is an ω_1-saturated normal ideal on κ.

542B Proposition If X is a set and $\mathcal{I}$ is a proper ω_1-saturated σ-ideal of $\mathcal{P}X$ containing singletons, then $\operatorname{add}\mathcal{I}$ is quasi-measurable.

proof This is immediate from the special case $\lambda = \omega_1$ of 541J.

542C Proposition If κ is a quasi-measurable cardinal, then κ is weakly inaccessible, the set of weakly inaccessible cardinals less than κ is stationary in κ, and either $\kappa \le \mathfrak{c}$ or κ is two-valued-measurable.

proof By 541L, κ is weakly inaccessible and the set of weakly inaccessible cardinals less than κ is stationary in κ. Let $\mathcal{I}$ be an ω_1-saturated normal ideal on κ and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Then 541P tells us that either $\mathfrak{A}$ is atomless and $\kappa \le \mathfrak{c}$ or $\mathfrak{A}$ is purely atomic and κ is two-valued-measurable.

542D Proposition Let κ be a quasi-measurable cardinal.

(a) Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family such that $\lambda < \kappa$ is a cardinal, every θ_ζ is a regular infinite cardinal and $\lambda < \theta_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) < \kappa$.

(b) If α and γ are cardinals less than κ, then $\Theta(\alpha,\gamma)$ (definition: 5A2Db) is less than κ.

(c) If α, β, γ and δ are cardinals, with $\alpha < \kappa$, $\gamma \le \beta$ and $\delta \ge \omega_1$, then $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ (definition: 5A2Da) is less than κ.

(d) $\Theta(\kappa,\kappa) = \kappa$.

proof Fix an ω_1-saturated normal ideal $\mathcal{I}$ on κ.

(a) ? Suppose, if possible, otherwise. Then λ is surely infinite. By 5A2Bc, there is an ultrafilter $\mathcal{F}$ on λ such that $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = \operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F})$, where the reduced product $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}$ is defined in 5A2A; by 5A2C, there is a family $\langle\theta'_\zeta\rangle_{\zeta<\lambda}$ of regular cardinals such that $\lambda < \theta'_\zeta \le \theta_\zeta$ for each ζ and $\operatorname{cf}(\prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}) = \kappa$. Let $\langle p_\xi\rangle_{\xi<\kappa}$ be a cofinal family in $P = \prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}$. For each $\xi < \kappa$ we can find $q_\xi \in P$ such that $q_\xi \not\le p_\eta$ for any $\eta \le \xi$; because P is upwards-directed, we can suppose that $q_\xi \ge p_\xi$, so that $\{q_\xi : \xi < \kappa\}$ also is cofinal with P. Choose $f_\xi \in \prod_{\zeta<\lambda}\theta'_\zeta$ such that $f^\bullet_\xi = q_\xi$ for each ξ.

For each $\zeta < \lambda$, $\theta'_\zeta < \kappa$, so there is a countable set $M_\zeta \subseteq \theta'_\zeta$ such that $I_\zeta = \kappa \setminus \{\xi : f_\xi(\zeta) \in M_\zeta\}$ belongs to $\mathcal{I}$ (541E). As

$$\omega \le \lambda < \theta'_\zeta = \operatorname{cf}\theta'_\zeta,$$

we can find $g(\zeta)$ such that $M_\zeta \subseteq g(\zeta) < \theta'_\zeta$. Consider $g^\bullet$ in $\prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}$. There is an $\eta < \kappa$ such that $g^\bullet \le p_\eta$, so that $f^\bullet_\xi \not\le g^\bullet$ for every $\xi \ge \eta$. On the other hand, $\eta \cup \bigcup_{\zeta<\lambda} I_\zeta$ belongs to $\mathcal{I}$, so there is a $\xi \ge \eta$ such that $f_\xi(\zeta) \in M_\zeta$ for every $\zeta < \lambda$; in which case $f_\xi \le g$, which is impossible. **X**

(b) ? Suppose, if possible, otherwise. Of course we can suppose that γ is infinite. For each $\xi < \kappa$ there must be a family $\langle\theta_{\xi\zeta}\rangle_{\zeta<\lambda_\xi}$ of regular cardinals less than α such that $\lambda_\xi < \gamma$, $\omega \le \lambda_\xi < \theta_{\xi\zeta}$ for every $\zeta < \lambda_\xi$ and $\operatorname{cf}(\prod_{\zeta<\lambda_\xi}\theta_{\xi\zeta}) \ge \xi$. Let λ be such that $A = \{\xi : \xi < \kappa,\ \lambda_\xi = \lambda\} \notin \mathcal{I}$. By 541Ra, applied to the function $I \mapsto \{\theta_{\xi\zeta} : \xi \in A \cap I,\ \zeta < \lambda_\xi\} : [\kappa]^{<\omega} \to [\alpha]^{<\lambda^+}$, there are $C \in \mathcal{I}$ and $M \in [\alpha]^{\le\lambda}$ such that $\theta_{\xi\zeta} \in M$ whenever $\xi \in A \setminus C$ and $\zeta < \lambda$. Let $\langle\theta_\zeta\rangle_{\zeta<\lambda'}$ enumerate $\{\theta : \theta \in M$ is a regular cardinal, $\theta > \lambda\}$. By (a), there is a cofinal set $F \subseteq \prod_{\zeta<\lambda'}\theta_\zeta$ with $\#(F) < \kappa$. Let $\xi \in A \setminus C$ be such that $\xi > \#(F)$. For each $f \in F$ define $g_f \in \prod_{\zeta<\lambda}\theta_{\xi\zeta}$ by setting

$$g_f(\zeta) = f(\zeta') \text{ whenever } \theta_{\xi\zeta} = \theta_{\zeta'}.$$

Then $\{g_f : f \in F\}$ is cofinal with $\prod_{\zeta<\lambda}\theta_{\xi\zeta}$, because if $h \in \prod_{\zeta<\lambda}\theta_{\xi\zeta}$ there is an $f \in F$ such that

$$f(\zeta') \ge \sup\{h(\zeta) : \zeta < \lambda,\ \theta_{\xi\zeta} = \theta_{\zeta'}\}$$

for every $\zeta' < \lambda'$, and in this case $h \le g_f$. So

$$\#(F) < \xi \le \operatorname{cf}(\textstyle\prod_{\zeta<\lambda}\theta_{\xi\zeta}) \le \#(F),$$

which is absurd. **X**

(c) This is trivial if any of the cardinals α, β or γ is finite; let us take it that they are all infinite. Then

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \max(\omega,\alpha,\Theta(\alpha,\gamma)) < \kappa$$

by 5A2D, 5A2G and (b) above.

(d) Because κ is an uncountable limit cardinal, $\kappa \le \Theta(\kappa,\kappa)$. (If $\omega \le \theta < \kappa$, then $\operatorname{cf}\theta^+ \le \Theta(\kappa,\kappa)$.) On the other hand, let $\lambda < \kappa$ be an infinite cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of infinite regular cardinals such that $\lambda < \theta_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\alpha = \sup_{\zeta<\kappa}\theta^+_\zeta < \kappa$ and

$$\operatorname{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta) \le \Theta(\alpha,\lambda^+) < \kappa.$$

As $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ is arbitrary, $\Theta(\kappa,\kappa) = \kappa$.

542E Theorem (Gitik & Shelah 93) If $\kappa \le \mathfrak{c}$ is a quasi-measurable cardinal, then

$$\{2^\gamma : \omega \le \gamma < \kappa\}$$

is finite.

proof ? Suppose, if possible, otherwise.

(a) Define a sequence $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ of cardinals by setting

$$\gamma_0 = \omega, \quad \gamma_{n+1} = \min\{\gamma : 2^\gamma > 2^{\gamma_n}\} \text{ for } n \in \mathbb{N}.$$

Then we are supposing that $\gamma_n < \kappa$ for every n, so by 541S, 5A2D, 5A2G and 5A2F γ_n is regular and

$$\begin{aligned}2^{\gamma_{n+1}} &= \operatorname*{cov}_{\mathrm{Sh}}(2^{\gamma_n},\kappa,\gamma^+_{n+1},\omega_1) \le \operatorname*{cov}_{\mathrm{Sh}}(2^{\gamma_n},\gamma^+_{n+1},\gamma^+_{n+1},\omega_1)\\ &\le \max(2^{\gamma_n},\Theta(2^{\gamma_n},\gamma^+_{n+1})) \le 2^{\gamma_{n+1}}\end{aligned}$$

for every $n \in \mathbb{N}$.

(b) Now $\Theta(2^{\gamma_n},\gamma) = \Theta(\mathfrak{c},\gamma)$ whenever $n \in \mathbb{N}$ and γ is a regular cardinal with $\gamma_n < \gamma < \kappa$. **P** Induce on n. For $n = 0$ we have $\mathfrak{c} = 2^{\gamma_0}$. For the inductive step to $n+1$, if γ is regular and $\gamma_{n+1} < \gamma < \kappa$, then $\mathfrak{c} \ge \kappa > \Theta(\gamma,\gamma)$ (542Db), so

$$\Theta(2^{\gamma_{n+1}},\gamma) = \Theta(\Theta(2^{\gamma_n},\gamma^+_{n+1}),\gamma)$$

(by (a))

$$\le \Theta(\Theta(2^{\gamma_n},\gamma),\gamma)$$

(because $\gamma \ge \gamma^+_{n+1}$ and Θ is order-preserving)

$$= \Theta(\Theta(\mathfrak{c}, \gamma), \gamma)$$

(by the inductive hypothesis)

$$\leq \Theta(\mathfrak{c}, \gamma)$$

(5A2H)

$$\leq \Theta(2^{\gamma_{n+1}}, \gamma)$$

(because $2^{\gamma_{n+1}} \geq \mathfrak{c}$). **Q** In particular,

$$2^{\gamma_{n+1}} = \Theta(2^{\gamma_n}, \gamma_{n+1}^+) = \Theta(\mathfrak{c}, \gamma_{n+1}^+)$$

for every $n \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, let λ_n be the least infinite cardinal such that $\Theta(\lambda_n, \gamma_n^+) > \mathfrak{c}$. Then $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ is non-increasing; also $\lambda_1 \leq \mathfrak{c}$, because

$$\Theta(\mathfrak{c}, \gamma_1^+) = 2^{\gamma_1} > \mathfrak{c},$$

so there are $n \geq 1$, $\lambda \leq \mathfrak{c}$ such that $\lambda_m = \lambda$ for every $m \geq n$. Now for $m \geq n$ we have

$$\mathfrak{c} < \Theta(\lambda, \gamma_m^+) \leq \max(\lambda, (\sup_{\lambda' < \lambda} \Theta(\lambda', \gamma_m^+))^{\operatorname{cf} \lambda})$$

(5A2I)

$$\leq \max(\lambda, \mathfrak{c}^{\operatorname{cf} \lambda}) = 2^{\operatorname{cf} \lambda}.$$

Also we still have $\lambda \geq \kappa > \Theta(\gamma_n^+, \gamma_n^+)$ because $\Theta(\lambda', \gamma_n^+) < \kappa \leq \mathfrak{c}$ for every $\lambda' < \kappa$. Using 5A2H again,

$$2^{\gamma_n} = \Theta(\mathfrak{c}, \gamma_n^+) \leq \Theta(\Theta(\lambda, \gamma_n^+), \gamma_n^+) \leq \Theta(\lambda, \gamma_n^+) \leq 2^{\operatorname{cf} \lambda};$$

consequently

$$2^{\gamma_n} < 2^{\gamma_{n+1}} \leq 2^{\operatorname{cf} \lambda}$$

and $\operatorname{cf} \lambda > \gamma_n$. But 5A2Ia now tells us that

$$\Theta(\lambda, \gamma_n^+) \leq \max(\lambda, \sup_{\lambda' < \lambda} \Theta(\lambda', \gamma_n^+)) \leq \mathfrak{c},$$

which is absurd. **X**

This contradiction proves the theorem.

542F Corollary Let $\kappa \leq \mathfrak{c}$ be a quasi-measurable cardinal.

(a) There is a regular infinite cardinal $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \leq \delta < \kappa$; that is, $\#([\kappa]^{<\kappa}) = 2^\gamma$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If γ is as in (a), then the cardinal power $\tau(\mathfrak{A})^\gamma$ is equal to 2^κ.

proof (a) Because κ is regular,

$$[\kappa]^{<\kappa} = \bigcup_{\xi < \kappa} \mathcal{P}\xi, \quad \#([\kappa]^{<\kappa}) = \max(\kappa, \sup_{\delta < \kappa} 2^\delta),$$

and the result is immediate from 542E.

(b) Of course $\tau(\mathfrak{A}) \leq \#(\mathfrak{A}) \leq \#(\mathcal{P}\kappa)$, so $\tau(\mathfrak{A})^\gamma \leq (2^\kappa)^\gamma = 2^\kappa$. In the other direction, we have an injective function $\phi_\xi : \mathcal{P}\xi \to \mathcal{P}\gamma$ for each $\xi < \kappa$. For $A \subseteq \kappa$ and $\eta < \gamma$ set

$$d_{A\eta} = \{\xi : \xi < \kappa, \eta \in \phi_\xi(A \cap \xi)\}^\bullet \in \mathfrak{A}.$$

If A, $B \subseteq \kappa$ are distinct then there is a $\zeta < \kappa$ such that $\phi_\xi(A \cap \xi) \neq \phi_\xi(B \cap \xi)$ for every $\xi \geq \zeta$, that is,

$$\bigcup_{\eta < \gamma} \{\xi : \eta \in \phi_\xi(A \cap \xi) \triangle \phi_\xi(B \cap \xi)\} \supseteq \kappa \setminus \zeta \notin \mathcal{I}.$$

Because $\mathcal{I}$ is κ-additive and $\gamma < \kappa$, there is an $\eta < \gamma$ such that $\{\xi : \eta \in \phi_\xi(A \cap \xi) \triangle \phi_\xi(B \cap \xi)\} \notin \mathcal{I}$, that is, $d_{A\eta} \neq d_{B\eta}$. Thus $A \mapsto \langle d_{A\eta} \rangle_{\eta < \gamma} : \mathcal{P}\kappa \to \mathfrak{A}^\gamma$ is injective, and $2^\kappa \leq \#(\mathfrak{A})^\gamma$. But $\mathfrak{A}$ is ccc, so $\#(\mathfrak{A}) \leq \max(4, \tau(\mathfrak{A})^\omega)$ (514De) and

$$2^\kappa \leq (\tau(\mathfrak{A})^\omega)^\gamma = \tau(\mathfrak{A})^\gamma.$$

542G Corollary Suppose that κ is a quasi-measurable cardinal.

(a) If $\kappa \leq \mathfrak{c} < \kappa^{(+\omega_1)}$ (notation: 5A1Ea), then $2^\lambda \leq \mathfrak{c}$ for every cardinal $\lambda < \kappa$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If $2^\lambda \le \mathfrak{c}$ for every cardinal $\lambda < \kappa$, then $\#(\mathfrak{A}) = 2^\kappa$.

proof (a) ? Otherwise, taking γ as in 542Fa, $2^\gamma > \mathfrak{c}$. Let $\gamma_1 \le \gamma$ be the first cardinal such that $2^{\gamma_1} > \mathfrak{c}$; note that, using 542E and 5A1Eh, we can be sure that γ_1 is regular. Next,

$$\mathrm{cov}_{\mathrm{Sh}}(\kappa, \kappa, \gamma_1^+, \gamma_1) \le \mathrm{cf}[\kappa]^{<\kappa} = \kappa,$$

by 5A2Ea. So $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \kappa, \gamma_1^+, \gamma_1) \le \alpha$ whenever $\kappa \le \alpha < \kappa^{(+\gamma_1)}$ (induce on α, using 5A2Eb). In particular, $\mathrm{cov}_{\mathrm{Sh}}(\mathfrak{c}, \kappa, \gamma_1^+, \gamma_1) \le \mathfrak{c}$. But 541S tells us that $\mathrm{cov}_{\mathrm{Sh}}(\mathfrak{c}, \kappa, \gamma_1^+, \gamma_1) = 2^{\gamma_1}$. **X**

(b) By 541B, $\mathfrak{A}$ is Dedekind complete; by 515L, $\#(\mathfrak{A})^\omega = \#(\mathfrak{A})$; so 542Fb gives the result.

542H Lemma Let κ be a quasi-measurable cardinal and $\langle \alpha_i \rangle_{i \in I}$ a countable family of ordinals less than κ and of cofinality at least ω_2. Then there is a set $F \subseteq P = \prod_{i \in I} \alpha_i$ such that

(i) F is cofinal with P;

(ii) if $\langle f_\xi \rangle_{\xi < \omega_1}$ is a non-decreasing family in F then $\sup_{\xi < \omega_1} f_\xi \in F$;

(iii) $\#(F) < \kappa$.

proof Note that $\mathrm{add}\, P = \min_{i \in I} \mathrm{cf}\, \alpha_i > \omega_1$ (I am passing over the trivial case $I = \emptyset$), so $\sup_{\xi < \omega_1} f_\xi$ is defined in P for every family $\langle f_\xi \rangle_{\xi < \omega_1}$ in P. We have

$$\mathrm{cf}\, P = \mathrm{cf}(\textstyle\prod_{i \in I} \mathrm{cf}\, \alpha_i) \le \Theta(\sup_{i \in I} (\mathrm{cf}\, \alpha_i)^+, \omega_1) < \kappa,$$

by 542Db. So we can find a cofinal set $F_0 \subseteq P$ of cardinal less than κ. Now for $0 < \zeta \le \omega_2$ define F_ζ by saying that

$$F_{\zeta+1} = \{\sup_{\xi < \omega_1} f_\xi : \langle f_\xi \rangle_{\xi < \omega_1} \text{ is a non-decreasing family in } F_\zeta\},$$

$$F_\zeta = \textstyle\bigcup_{\eta < \zeta} F_\eta \text{ for non-zero limit ordinals } \zeta \le \omega_2.$$

Then $\#(F_\zeta) < \kappa$ for every ζ. **P** Induce on ζ. For the inductive step to $\zeta + 1$, **?** suppose, if possible, that $\#(F_\zeta) < \kappa$ but $\#(F_{\zeta+1}) \ge \kappa$. Then there is a proper ω_1-saturated κ-additive $\mathcal{I} \lhd \mathcal{P}F_{\zeta+1}$ containing singletons. For each $h \in F_{\zeta+1}$ choose a non-decreasing family $\langle f_{h\xi} \rangle_{\xi < \omega_1}$ in F_ζ with supremum h. The set $h[I]$ of values of h is a countable subset of $Y = \bigcup_{i \in I} \alpha_i$, and $\#(Y) < \kappa$. By 541E, there is a set $H \subseteq F_{\zeta+1}$, of cardinal κ, such that $M = \bigcup_{h \in H} h[I]$ is countable. Now, for each $h \in H$, there is a $\gamma(h) < \omega_1$ such that whenever $i \in I$ and $\beta \in M$ then $h(i) > \beta$ iff $f_{h,\gamma(h)}(i) > \beta$. If g, $h \in H$ and $i \in I$ and $g(i) < h(i)$, then $f_{g,\gamma(g)}(i) \le g(i) < f_{h,\gamma(h)}(i)$, because $g(i) \in M$. Thus $h \mapsto f_{h,\gamma(h)} : H \to F_\zeta$ is injective; but $\#(F_\zeta) < \kappa = \#(H)$. **X**

Thus $\#(F_{\zeta+1}) < \kappa$ if $\#(F_\zeta) < \kappa$. At limit ordinals ζ the induction proceeds without difficulty because $\mathrm{cf}\, \kappa > \zeta$. **Q**

So $\#(F_{\omega_2}) < \kappa$ and we may take $F = F_{\omega_2}$.

542I Theorem (Shelah 96) Let κ be a quasi-measurable cardinal.

(a) For any cardinal θ, $\mathrm{cf}[\kappa]^{<\theta} \le \kappa$.

(b) For any cardinal $\lambda < \kappa$, and any θ, $\mathrm{cf}[\lambda]^{<\theta} < \kappa$.

proof (a)(i) Consider first the case $\theta = \omega_1$. Write G_1 for the set of ordinals less than κ of cofinality less than or equal to ω_1; for $\delta \in G_1$ let $\psi_\delta : \mathrm{cf}\, \delta \to \delta$ enumerate a cofinal subset of δ. Next, write G_2 for $\kappa \setminus G_1$, and for every countable set $A \subseteq G_2$ let $F(A) \subseteq \prod_{\alpha \in A} \alpha$ be a cofinal set, of cardinal less than κ, closed under suprema of non-decreasing families of length ω_1; such exists by 542H above.

(ii) It is worth observing at this point that if $\langle A_\zeta \rangle_{\zeta < \omega_1}$ is any family of countable subsets of G_2, $D = \bigcup_{\zeta < \omega_1} A_\zeta$, and $g \in \prod_{\alpha \in D} \alpha$, then there is an $f \in \prod_{\alpha \in D} \alpha$ such that $f \ge g$ and $f \restriction A_\zeta \in F(A_\zeta)$ for every $\zeta < \omega_1$. **P** Let $\langle \phi(\xi) \rangle_{\xi < \omega_1}$ run over ω_1 with cofinal repetitions. Choose a non-decreasing family $\langle f_\xi \rangle_{\xi < \omega_1}$ in $\prod_{\alpha \in D} \alpha$ in such a way that $f_0 = g$ and $f_{\xi+1} \restriction A_{\phi(\xi)} \in F(A_{\phi(\xi)})$ for every ξ; this is possible because $\mathrm{add}(\prod_{\alpha \in D}) \ge \omega_2$ and $F(A)$ is cofinal with $\prod_{\alpha \in A} \alpha$ for every A. Set $f = \sup_{\xi < \omega_1} f_\xi$; this works because every $F(A)$ is closed under suprema of non-decreasing families of length ω_1. **Q**

(iii) We can now find a family $\mathcal{A}$ of countable subsets of κ such that

(α) $\{\alpha\} \in \mathcal{A}$ for every $\alpha < \kappa$;

(β) whenever A, $A' \in \mathcal{A}$ and $\zeta < \omega_1$ then $A \cup A'$, $A \cap G_2$ and $\{\psi_\alpha(\xi) : \alpha \in A \cap G_1, \xi < \min(\zeta, \mathrm{cf}\, \alpha)\}$ all belong to $\mathcal{A}$;

(γ) whenever $A \in \mathcal{A} \cap [G_2]^{\le\omega}$ and $f \in F(A)$ then $f[A] \in \mathcal{A}$;

(δ) $\#(\mathcal{A}) \le \kappa$.

(iv) ? Suppose, if possible, that $\mathrm{cf}[\kappa]^{\le\omega} > \kappa$. Because $[\kappa]^{\le\omega} = \bigcup_{\lambda < \kappa} [\lambda]^{\le\omega}$, there is a cardinal $\lambda < \kappa$ such that $\mathrm{cf}[\lambda]^{\le\omega} > \kappa$. We can therefore choose inductively a family $\langle a_\xi \rangle_{\xi < \kappa}$ of countable subsets of λ such that

$$a_\xi \not\subseteq \bigcup_{\eta\in A\cap\xi} a_\eta$$

whenever $\xi<\kappa$ and $A\in\mathcal{A}$. By 541E, there is a set $W\subseteq\kappa$, of cardinal κ, such that $\bigcup_{\xi\in W} a_\xi$ is countable. Let $\delta<\kappa$ be such that $W\cap\delta$ is cofinal with δ and of order type ω_1; then $\delta\in G_1$.

(v) I choose a family $\langle A_{k\zeta}\rangle_{\zeta<\omega_1,k\in\mathbb{N}}$ in $\mathcal{A}$ as follows. Start by setting $A_{0\zeta}=\psi_\delta[\zeta]$ for every $\zeta<\omega_1$; then $A_{0\zeta}\in\mathcal{A}$ by (iii)(α-β). Given $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$, set $A'_{k\zeta}=A_{k\zeta}\cap G_2$ for each $\zeta<\omega_1$. For $\alpha\in D_k=\bigcup_{\zeta<\omega_1}A'_{k\zeta}$, set $g_k(\alpha)=\sup(\alpha\cap W\cap\delta)<\alpha$; choose $f_k\in\prod_{\alpha\in D_k}\alpha$ such that $g_k\le f_k$ and $f_k{\restriction}A'_{k\zeta}\in F(A'_{k\zeta})$ for every ζ; this is possible by (ii) above. Set

$$A_{k+1,\zeta}=A_{k\zeta}\cup f_k[A'_{k\zeta}]\cup\{\psi_\alpha(\xi):\alpha\in A_{k\zeta}\cap G_1,\,\xi<\min(\zeta,\operatorname{cf}\alpha)\}\in\mathcal{A}$$

for each $\zeta<\omega_1$, and continue. An easy induction on k shows that $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$ is non-decreasing for every k; also, every $A_{k\zeta}$ is a subset of δ.

(vi) Set $V_k=\bigcup_{\zeta<\omega_1}A_{k\zeta}$, $b_k=\bigcup\{a_\xi:\xi\in W\cap V_k\}$; then b_k is countable and there is a $\beta(k)<\omega_1$ such that $b_k=\bigcup\{a_\xi:\xi\in W\cap A_{k,\beta(k)}\}$. Now $\bigcup_{k\in\mathbb{N}}A_{k,\beta(k)}$ is a countable subset of δ, so there is a member γ of $W\cap\delta$ greater than its supremum. We have

$$a_\gamma\not\subseteq\bigcup\{a_\eta:\eta\in A_{k,\beta(k)}\},$$

so $a_\gamma\not\subseteq b_k$ and $\gamma\notin V_k$, for each k.

Set $V=\bigcup_{k\in\mathbb{N}}V_k$. We have just seen that $W\cap\delta\not\subseteq V$; set $\gamma_0=\min(W\cap\delta\setminus V)$. Because $V_0=\psi_\delta[\omega_1]$ is cofinal with δ, $V\setminus\gamma_0\ne\emptyset$; let γ_1 be its least member. Then $\gamma_1>\gamma_0$. There must be $k\in\mathbb{N}$ and $\zeta<\omega_1$ such that $\gamma_1\in A_{k\zeta}$. Observe that if $\alpha\in V\cap G_1$ then $V\cap\alpha$ is cofinal with α; but $V\cap\gamma_1\subseteq\gamma_0$, so $\gamma_1\notin G_1$ and $\gamma_1\in A'_{k\zeta}\subseteq D_k$. But now $f_k(\gamma_1)\in A_{k+1,\zeta}\subseteq V$ and $\gamma_0\le g_k(\gamma_1)\le f_k(\gamma_1)<\gamma_1$, so $\gamma_1\ne\min(V\setminus\gamma_0)$. **X**

(vii) This contradiction shows that $\operatorname{cf}[\kappa]^{\le\omega}\le\kappa$. Now consider $\operatorname{cf}[\kappa]^{\le\delta}$, where $\delta<\kappa$ is an infinite cardinal. Then

$$\operatorname*{cov}_{\mathrm{Sh}}(\kappa,\delta^+,\delta^+,\omega_1)=\max(\kappa,\sup_{\lambda<\kappa}\operatorname*{cov}_{\mathrm{Sh}}(\lambda,\delta^+,\delta^+,\omega_1))$$

(5A2Eb)

$$\le\kappa$$

by 542Dc. So there is a family $\mathcal{B}\subseteq[\kappa]^{\le\delta}$, of cardinal at most κ, such that every member of $[\kappa]^{\le\delta}$ is covered by a sequence in $\mathcal{B}$. But now $\operatorname{cf}[\mathcal{B}]^{\le\omega}\le\kappa$, so there is a family $\mathfrak{C}$ of countable subsets of $\mathcal{B}$ which is cofinal with $[\mathcal{B}]^{\le\omega}$ and of cardinal at most κ; setting $\mathcal{D}=\{\bigcup\mathcal{C}:\mathcal{C}\in\mathfrak{C}\}$, we have $\mathcal{D}$ cofinal with $[\kappa]^{\le\delta}$ and of cardinal at most κ. So $\operatorname{cf}[\kappa]^{\le\delta}\le\kappa$.

Finally, of course, $[\kappa]^{<\theta}=\bigcup_{\delta<\theta}[\kappa]^{\le\delta}$, so

$$\operatorname{cf}[\kappa]^{<\theta}\le\sup_{\delta<\theta}\operatorname{cf}[\kappa]^{\le\delta}\le\kappa$$

whenever $\theta\le\kappa$. For $\theta>\kappa$ we have $\operatorname{cf}[\kappa]^{<\theta}=1$, so $\operatorname{cf}[\kappa]^{<\theta}\le\kappa$ for every θ.

(b) If $\mathcal{A}$ is cofinal with $[\kappa]^{<\theta}$ then $\{A\cap\lambda:A\in\mathcal{A}\}$ is cofinal with $[\lambda]^{<\theta}$, so $\operatorname{cf}[\lambda]^{<\theta}\le\operatorname{cf}[\kappa]^\theta\le\kappa$, by (a).

? If $\operatorname{cf}[\lambda]^{<\theta}=\kappa$, there is a cofinal family $\langle A_\xi\rangle_{\xi<\kappa}$ in $[\lambda]^{<\theta}$ such that $A_\xi\not\subseteq A_\eta$ for any $\eta<\xi<\kappa$. Of course $\omega<\theta\le\lambda<\kappa=\operatorname{cf}\kappa$, so we may suppose that every A_ξ is infinite. So there is an infinite $\delta<\theta$ such that $E=\{\xi:\xi<\kappa,\,\#(A_\xi)=\delta\}$ has cardinal κ. Next, by 541E applied to a suitable ideal of subsets of E, there is a set $M\in[\lambda]^{\le\delta}$ such that $F=\{\xi:\xi\in E,\,A_\xi\subseteq M\}$ has cardinal κ. But now there must be an $\eta<\kappa$ such that $M\subseteq A_\eta$, and a $\xi\in F$ such that $\xi>\eta$; which is impossible. **X**

Thus $\operatorname{cf}[\lambda]^{<\theta}<\kappa$, as claimed.

542J Corollary Let κ be a quasi-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets such that $\lambda<\operatorname{add}P_\zeta\le\operatorname{cf}P_\zeta<\kappa$ for every $\zeta<\lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda}P_\zeta)<\kappa$.

proof For each $\zeta<\lambda$ let Q_ζ be a cofinal subset of P_ζ of cardinal less than κ. Set $P=\prod_{\zeta<\kappa}P_\zeta$, $Z=\bigcup_{\zeta<\lambda}Q_\zeta$; then $\#(Z)<\kappa$ so $\operatorname{cf}[Z]^{\le\lambda}<\kappa$, by 542Ib. Let $\mathcal{A}$ be a cofinal subset of $[Z]^{\le\lambda}$ with $\#(\mathcal{A})<\kappa$. For each $A\in\mathcal{A}$ choose $f_A\in P$ such that $f_A(\zeta)$ is an upper bound for $A\cap P_\zeta$ for every ζ; this is possible because $\operatorname{add}P_\zeta>\#(A)$. Set $F=\{f_A:A\in\mathcal{A}\}$.

If $g\in P$, then there is an $h\in\prod_{\zeta<\lambda}Q_\zeta$ such that $g\le h$. Now $h[\lambda]\in[Z]^{\le\lambda}$ so there is an $A\in\mathcal{A}$ such that $h[\lambda]\subseteq A$. In this case $h\le f_A$. Accordingly F is cofinal with P and $\operatorname{cf}P\le\#(F)<\kappa$, as required.

542X Basic exercises (a) Let κ be a quasi-measurable cardinal, and θ a cardinal such that $2\le\theta\le\kappa$. Show that $\operatorname{cf}[\kappa]^{<\theta}=\kappa$.

542Y Further exercises (a) Let X be a hereditarily weakly θ-refinable topological space such that there is no quasi-measurable cardinal less than or equal to the weight of X, and μ a totally finite Maharam submeasure on the Borel σ-algebra of X. (i) Show that μ is τ-subadditive in the sense that if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets in X with union H, then $\inf_{G\in\mathcal{G}}\mu(H\setminus G)=0$. (ii) Show that if X is Hausdorff and K-analytic, then the completion of μ is a Radon submeasure on X.

542 Notes and comments The arguments of this section have taken on a certain density, and I ought to explain what they are for. The cardinal arithmetic of 542E-542G is relevant to one of the most important questions in this chapter, to be treated in the next section: supposing that there is an extension of Lebesgue measure to a measure μ defined on all subsets of $\mathbb{R}$, what can we say about the Maharam type of μ? And 542I-542J will tell us something about the cofinalities of our favourite partially ordered sets under the same circumstances.

Let me draw your attention to a useful trick, used twice above. If κ is a quasi-measurable cardinal, and X is any set of cardinal at least κ, there is a non-trivial ω_1-saturated σ-ideal of subsets of X. This is the basis of the proof of 542H (taking $X=F_{\zeta+1}$ in the inductive step) and the final step in the proof of 542I (taking $X=E$). Exposed like this, the idea seems obvious. In the thickets of an argument it sometimes demands an imaginative jump.

543 The Gitik-Shelah theorem

I come now to the leading case at the centre of the work of the last two sections. If our ω_1-saturated σ-ideal of sets is the null ideal of a measure with domain $\mathcal{P}X$, it has some even more striking properties than those already discussed. I will go farther into these later in the chapter. But I will begin with what is known about one of the first questions I expect a reader of this book to ask: if $(X,\mathcal{P}X,\mu)$ is a probability space, what can, or must, its measure algebra be? There can, of course, be a purely atomic part; the interesting question relates to the atomless part, if any, always remembering that we need a special act of faith to believe that there can be an atomless case. Here we find that the Maharam type of an atomless probability defined on a power set must be greater than its additivity (543F), which must itself be 'large' (541L).

543A Definitions (a) A **real-valued-measurable cardinal** is an uncountable cardinal κ such that there is a κ-additive probability measure ν on κ, defined on every subset of κ, for which all singletons are negligible. In this context I will call ν a **witnessing probability**.

(b) If κ is a regular uncountable cardinal, a probability measure ν on κ with domain $\mathcal{P}\kappa$ is **normal** if its null ideal $\mathcal{N}(\nu)$ is normal. In this case, ν must be κ-additive (541H, 521Ad) and zero on singletons, so κ is real-valued-measurable, and I will say that ν is a **normal witnessing probability**.

(c) An **atomlessly-measurable cardinal** is a real-valued-measurable cardinal with an atomless witnessing probability.

543B Collecting ideas which have already appeared, some of them more than once, we have the following.

Proposition (a) Let $(X,\mathcal{P}X,\mu)$ be a totally finite measure space in which singletons are negligible and $\mu X>0$. Then $\kappa=\operatorname{add}\mu$ is real-valued-measurable, and there are a non-negligible $Y\subseteq X$ and a function $g:Y\to\kappa$ such that the normalized image measure $B\mapsto\frac{1}{\mu Y}\mu g^{-1}[B]$ is a normal witnessing probability on κ.

(b) Every real-valued-measurable cardinal is quasi-measurable (definition: 542A) and has a normal witnessing probability; in particular, every real-valued-measurable cardinal is uncountable and regular.

(c) If $\kappa\le\mathfrak{c}$ is a real-valued-measurable cardinal, then κ is atomlessly-measurable, and every witnessing probability on κ is atomless.

(d) If $\kappa>\mathfrak{c}$ is a real-valued-measurable cardinal, then κ is two-valued-measurable, and every witnessing probability on κ is purely atomic.

(e) A cardinal λ is measure-free (definition: 438A) iff there is no real-valued-measurable cardinal $\kappa\le\lambda$; $\mathfrak{c}$ is measure-free iff there is no atomlessly-measurable cardinal.

(f) Again suppose that $(X,\mathcal{P}X,\mu)$ is a totally finite measure space.

(i) If μ is purely atomic, $\operatorname{add}\mu$ is either ∞ or a two-valued-measurable cardinal.

(ii) If μ is not purely atomic, $\operatorname{add}\mu$ is atomlessly-measurable.

proof (a) By 521Ad, κ is the additivity of the null ideal $\mathcal{N}(\mu)$ of μ; because μ is σ-finite, $\mathcal{N}(\mu)$ is ω_1-saturated; and of course $\kappa\ge\omega_1$. By 541J, there are $Y\subseteq X$ and $g:Y\to\kappa$ such that $\mathcal{I}=\{B:B\subseteq\kappa,\ \mu g^{-1}[B]=0\}$ is a normal ideal on κ. In particular, $\kappa\notin\mathcal{I}$ and Y is non-negligible. Set $\nu B=\frac{1}{\mu Y}\mu g^{-1}[B]$ for $B\subseteq\kappa$; then ν is a probability measure with domain $\mathcal{P}\kappa$. Its null ideal $\mathcal{N}(\nu) = \mathcal{I}$ is normal, so it is a normal measure and witnesses that κ is real-valued-measurable.

(b) If κ is real-valued-measurable, then (a) tells us that any witnessing probability on κ can be used to define a normal witnessing probability ν say. Since κ is the additivity of a σ-ideal, it must be uncountable and regular; also $\mathcal{N}(\nu)$ is an ω_1-saturated normal ideal, so κ is quasi-measurable.

(c)-(d) Apply 541P. If κ is real-valued-measurable, it is a regular uncountable cardinal, and if ν is a witnessing probability on κ then $\mathcal{N}(\nu)$ is a proper ω_1-saturated κ-additive ideal of subsets of κ. Taking $\mathfrak{A}$ to be the measure algebra of ν, then 541P tells us that either $\mathfrak{A}$ is atomless and $\kappa \le \mathfrak{c}$, or $\mathfrak{A}$ is purely atomic and κ is two-valued-measurable, in which case κ is surely greater than $\mathfrak{c}$. Turning this round, if $\kappa \le \mathfrak{c}$ then $\mathfrak{A}$ and ν must be atomless and κ is atomlessly-measurable, while if $\kappa > \mathfrak{c}$ then $\mathfrak{A}$ and ν are purely atomic and κ is two-valued-measurable.

(e) If $\kappa \le \lambda$ is real-valued-measurable, then any witnessing probability on κ extends to a probability measure with domain $\mathcal{P}\lambda$ which is zero on singletons, so λ is not measure-free. If λ is not measure-free, let μ be a probability measure with domain $\mathcal{P}\lambda$ which is zero on singletons; then (a) tells us that $\operatorname{add}\mu$ is real-valued-measurable, and $\operatorname{add}\mu \le \lambda$ because $\lambda = \bigcup \mathcal{N}(\mu)$.

If there is an atomlessly-measurable cardinal κ, then κ is real-valued-measurable and there is an atomless witnessing probability on κ, and $\kappa \le \mathfrak{c}$, by (d). So in this case $\mathfrak{c}$ is not measure-free. On the other hand, if $\mathfrak{c}$ is not measure-free, there is a real-valued-measurable cardinal $\kappa \le \mathfrak{c}$, which is atomlessly-measurable, by (c).

(f)(i) If μ is purely atomic, and $\operatorname{add}\mu$ is not ∞, set $\kappa = \operatorname{add}\mu$ and let $\langle A_\xi\rangle_{\xi<\kappa}$ be a family of negligible sets in X with non-negligible union A. Let $E \subseteq A$ be an atom for μ. Repeating the construction of (a), but starting from the subspace $(E, \mathcal{P}E, \mu_E)$, we see that the normalized image measure constructed on $\kappa = \operatorname{add}\mu_E$ can take only the two values 0 and 1, so that its null ideal is 2-saturated and witnesses that κ is two-valued-measurable.

(ii) If μ is not purely atomic, E be the atomless part of X, so that μ_E is atomless and $\mu_{X\setminus E}$ is purely atomic. Singletons in E must be negligible, so (a) tells us that $\operatorname{add}\mu_E$ is a real-valued-measurable cardinal; also there is an inverse-measure-preserving function from E to $[0, \mu E]$ (343Cc), so E can be covered by $\mathfrak{c}$ negligible sets and $\operatorname{add}\mu_E \le \mathfrak{c}$ is atomlessly-measurable, by (c) here. Now (i) tells us that $\mathfrak{c} \le \operatorname{add}\mu_{X\setminus E}$, so $\operatorname{add}\mu = \min(\operatorname{add}\mu_E, \operatorname{add}\mu_{X\setminus E}) = \operatorname{add}\mu_E$ is atomlessly-measurable.

Remark 543Bc-543Bd are **Ulam's dichotomy**.

543C Theorem (see Kunen n70) Suppose that $(Y, \mathcal{P}Y, \nu)$ is a σ-finite measure space and that $(X, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite quasi-Radon measure space with $w(X) < \operatorname{add}\nu$. Let $f : X \times Y \to [0,\infty]$ be any function. Then

$$\overline{\int}\Bigl(\int f(x,y)\nu(dy)\Bigr)\mu(dx) \le \int\Bigl(\overline{\int} f(x,y)\mu(dx)\Bigr)\nu(dy).$$

proof Because μ is σ-finite and effectively locally finite, there is a sequence of open sets of finite measure with conegligible union in X. Since none of the integrals are changed by deleting a negligible subset of X, we may suppose that this conegligible union is X itself, so that μ is outer regular with respect to the open sets (412Wb). Set $\lambda = w(X) < \operatorname{add}\nu$; let $\langle G_\xi\rangle_{\xi<\lambda}$ enumerate a base for the topology of X. Fix $\epsilon > 0$. Because ν is σ-finite, we have a function $y \mapsto \epsilon_y : Y \to \,]0,\infty[$ such that $\int \epsilon_y \nu(dy) \le \epsilon$. For each $y \in Y$, let $h_y : X \to [0,\infty]$ be a lower semi-continuous function such that $f(x,y) \le h_y(x)$ for every $x \in X$ and

$$\int h_y(x)\mu(dx) \le \epsilon_y + \overline{\int} f(x,y)\mu(dx)$$

(412Wa). For $I \subseteq \lambda$, $x \in X$ and $y \in Y$, set

$$f_I(x,y) = \sup(\{0\} \cup \{s : \exists\, \xi \in I,\, x \in G_\xi,\, h_y(x') \ge s\ \forall\, x' \in G_\xi\}).$$

Then f_I is expressible as $\sup_{\xi\in I, s\in\mathbb{Q}^+} s\chi(G_\xi \times B_{\xi s})$, writing $\mathbb{Q}^+$ for the set of non-negative rational numbers and $B_{\xi s}$ for $\{y : h_y \ge s\chi G_\xi\}$. So f_I is $(\Sigma\widehat{\otimes}\mathcal{P}Y)$-measurable for all countable I, and for such I we shall have

$$\iint f_I(x,y)\mu(dx)\nu(dy) = \iint f_I(x,y)\nu(dy)\mu(dx),$$

by Fubini's theorem (252C). Next, for any $I \subseteq \lambda$, $x \mapsto f_I(x,y)$ is lower semi-continuous for each y, and

$$\sup_{I\in[\lambda]^{<\omega}} f_I(x,y) = h_y(x)$$

for all $x \in X$ and $y \in Y$, because each h_y is lower semi-continuous. So

$$\sup_{I\in[\lambda]^{<\omega}} \int f_I(x,y)\mu(dx) = \int h_y(x)\mu(dx)$$

for each $y \in Y$ (414Ba). Because $\lambda < \operatorname{add}\nu$, it follows that

$$\sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\mu(dx)\nu(dy) = \iint h_y(x)\mu(dx)\nu(dy)$$

(521B(d-i)). On the other hand, if we write

$$g_I(x) = \int f_I(x,y)\nu(dy)$$

for $x \in X$ and finite $I \subseteq \lambda$, then g_I also is lower semi-continuous. **P** If $x \in X$, set $J = \{\xi : \xi \in I,\ x \in G_\xi\}$ and $H = X \cap \bigcap_{\xi\in J} G_\xi$; then $f_I(x,y) \le f_I(x',y)$ whenever $x' \in H$ and $y \in Y$, so $g_I(x) \le g_I(x')$ for $x' \in H$, while $x \in \operatorname{int} H$. **Q** So $g = \sup_{I\in[\lambda]^{<\omega}} g_I$ is lower semi-continuous, and $\int g(x)\mu(dx) = \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)$. Also

$$g(x) = \sup_{I\in[\lambda]^{<\omega}} \int f_I(x,y)\nu(dy) = \int h_y(x)\nu(dy) \ge \int f(x,y)\nu(dy)$$

for every $x \in X$. So we have

$$\begin{aligned}\overline{\iint} f(x,y)\nu(dy)\mu(dx) &\le \int g(x)\mu(dx) = \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\nu(dy)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\mu(dx)\nu(dy)\\ &= \iint h_y(x)\mu(dx)\nu(dy) \le \epsilon + \int \overline{\int} f(x,y)\mu(dx)\nu(dy).\end{aligned}$$

As ϵ is arbitrary, we have the result.

543D Corollary Let κ be a real-valued-measurable cardinal, with witnessing probability ν, and take a totally finite quasi-Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$ with $w(X) < \kappa$.

(a) If $C \subseteq X \times \kappa$ then

$$\overline{\int} \nu C[\{x\}]\mu(dx) \le \int \mu^* C^{-1}[\{\xi\}]\nu(d\xi).$$

(b) If $A \subseteq X$ and $\#(A) \le \kappa$, then there is a $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu^*B = \mu^*A$.

(c) If $\langle C_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X \setminus \mathcal{N}(\mu)$ such that $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that $\mu^*(C_\xi \cap C_\eta) > 0$.

(d) If we have a family $\langle h_\xi\rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom} h_\xi$ is a non-negligible subset of X for each ξ and $\#(\bigcup_{\xi<\kappa} h_\xi) < \kappa$ (identifying each h_ξ with its graph), then there are distinct ξ, $\eta < \kappa$ such that

$$\mu^*\{x : x \in \operatorname{dom}(h_\xi) \cap \operatorname{dom}(h_\eta),\ h_\xi(x) = h_\eta(x)\} > 0.$$

proof (a) Apply 543C to $\chi C : X \times \kappa \to \mathbb{R}$.

(b)? Suppose, if possible, otherwise. Then surely $\#(A) = \kappa$; let $f : \kappa \to A$ be a bijection. Set

$$C = \{(f(\eta),\xi) : \eta \le \xi < \kappa\} \subseteq X \times \kappa.$$

If $x \in A$,

$$\nu C[\{x\}] = \nu\{\xi : f^{-1}(x) \le \xi < \kappa\} = 1,$$

so $\overline{\int} \nu C[\{x\}]\mu(dx) = \mu^*A$. If $\xi < \kappa$,

$$\mu^* C^{-1}[\{\xi\}] = \mu^*\{f(\eta) : \eta < \xi\} < \mu^*A,$$

so $\int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) < \mu^*A$. But this contradicts (a). **X**

(c) Let $\tilde\nu$ be the probability on $\kappa \times \kappa$ defined by writing

$$\tilde\nu A = \int \nu A[\{\xi\}]\,\nu(d\xi) \text{ for every } A \subseteq \kappa \times \kappa.$$

Then $\tilde\nu$ is κ-additive, by 521B(d-ii). Set

$$\begin{aligned}C &= \{(x,(\xi,\eta)) : \xi, \eta \text{ are distinct members of } \kappa,\ x \in C_\xi \cap C_\eta\}\\ &\subseteq X \times (\kappa \times \kappa).\end{aligned}$$

Set

$$E = \{x : x \in X,\ \nu\{\xi : x \in C_\xi\} = 0\}.$$

Because $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$,

$$\{\xi : E \cap C_\xi \ne \emptyset\} = \bigcup\{\{\xi : x \in C_\xi\} : x \in E \cap \textstyle\bigcup_{\eta<\kappa} C_\eta\}$$

is ν-negligible, and there is a $\xi < \kappa$ with $C_\xi \cap E = \emptyset$; thus $\mu^*(X \setminus E) > 0$. Now if $x \in X \setminus E$ then

$$\tilde{\nu}\{(\xi, \eta) : (x, (\xi, \eta)) \in C\} = (\nu\{\xi : x \in C_\xi\})^2 > 0.$$

So we have

$$\begin{aligned} 0 &< \overline{\int} \tilde{\nu}\{(\xi, \eta) : (x, (\xi, \eta)) \in C\}\mu(dx) \\ &\le \int \mu^*\{x : (x, (\xi, \eta)) \in C\}\tilde{\nu}(d(\xi, \eta)) \end{aligned}$$

by 543C, and there must be distinct ξ, $\eta < \kappa$ such that $\mu^*\{x : (x, (\xi, \eta)) \in C\} > 0$, as required.

(d) Set $Y = \bigcup_{\xi<\kappa} h_\xi[X]$. Give $X \times Y$ the measure $\tilde{\mu}$ and topology $\mathfrak{T}'$ defined as follows. The domain of $\tilde{\mu}$ is to be the family $\tilde{\Sigma}$ of subsets H of $X \times Y$ for which there are E, $E' \in \Sigma$ with $\mu(E' \setminus E) = 0$ and $E \times Y \subseteq H \subseteq E' \times Y$; and for such H, $\tilde{\mu}H$ is to be $\mu E = \mu E'$. The topology $\mathfrak{T}'$ is to be just the family $\{G \times Y : G \in \mathfrak{T}\}$. It is easy to check that $(X \times Y, \mathfrak{T}', \tilde{\Sigma}, \tilde{\mu})$ is a totally finite quasi-Radon measure space of weight less than κ, and that $\tilde{\mu}^* h_\xi = \mu^*(\operatorname{dom} h_\xi) > 0$ for each $\xi < \kappa$. So (c) gives the result.

543E The Gitik-Shelah theorem (Gitik & Shelah 89, Gitik & Shelah 93) Let κ be an atomlessly-measurable cardinal, with witnessing probability ν. Then the Maharam type of ν is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$.

proof (a) To begin with (down to the end of (g) below) let us suppose that ν is Maharam-type-homogeneous, with Maharam type λ; of course λ is infinite, because ν is atomless. Let $(\mathfrak{A}, \bar{\nu})$ be the measure algebra of ν, ν_λ the usual measure of $\{0,1\}^\lambda$ and $\mathfrak{B}_\lambda$ the measure algebra of ν_λ; then there is a measure-preserving isomorphism $\phi : \mathfrak{B}_\lambda \to \mathfrak{A}$. Because ν_λ is a compact measure (342Jd), there is a function $f : \kappa \to \{0,1\}^\lambda$ such that $\phi(E^\bullet) = f^{-1}[E]^\bullet$ whenever ν_λ measures E (343B).

(b)? Suppose, if possible, that $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$.

Set $\zeta = \max(\lambda^+, \kappa^+)$. Then we have an infinite cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover,

— if $\lambda < \operatorname{Tr}(\kappa)$ (definition: 5A1Lb), then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

— if $\lambda \ge \operatorname{Tr}(\kappa)$, then $g_\alpha \restriction \gamma = g_\beta \restriction \gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

P case 1 If $\lambda < \operatorname{Tr}(\kappa)$, then $\zeta \le \operatorname{Tr}(\kappa)$ (5A1Ma); as ζ is a successor cardinal, there is a family $\langle g_\alpha \rangle_{\alpha<\zeta}$ of functions from κ to κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ for all distinct α, $\beta < \zeta$. Set $S = \zeta \setminus \kappa$, so that S is a stationary set in ζ and $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$. We know that $\kappa \le \mathfrak{c}$; set $\delta = \omega$, so that $\delta < \kappa \le 2^\delta$ and g_α is a function from κ to 2^δ for every α.

case 2 Suppose that $\lambda \ge \operatorname{Tr}(\kappa)$. Then

$$\kappa < \operatorname{Tr}(\kappa) < \lambda^+ = \zeta \le \min(2^\kappa, \kappa^{(+\omega)}),$$

so there must be a cardinal $\delta < \kappa$ such that $2^\delta \ge \zeta$, by 5A1Mb. Because $\kappa < \lambda < \kappa^{(+\omega)}$, λ is regular, and of course $\lambda > \omega_1$. So 5A1O gives us the functions we need. **Q**

(c) Fix an injective function $h : 2^\delta \to \{0,1\}^\delta$. For $\alpha \in S$, $\iota < \delta$ set

$$U_{\alpha\iota} = \{\xi : \xi < \kappa, \, (hg_\alpha(\xi))(\iota) = 1\},$$

and choose a Baire set $H_{\alpha\iota} \subseteq \{0,1\}^\lambda$ such that $\phi^{-1}(U_{\alpha\iota}^\bullet) = H_{\alpha\iota}^\bullet$ in $\mathfrak{B}_\lambda$. Define $\tilde{g}_\alpha : \{0,1\}^\lambda \to \{0,1\}^\delta$ by setting

$$\begin{aligned} (\tilde{g}_\alpha(x))(\iota) &= 1 \text{ if } x \in H_{\alpha\iota}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then

$$\begin{aligned} \{\xi : \xi < \kappa, \, \tilde{g}_\alpha f(\xi) \ne hg_\alpha(\xi)\} &= \bigcup_{\iota<\delta} \{\xi : (\tilde{g}_\alpha f(\xi))(\iota) \ne (hg_\alpha(\xi))(\iota)\} \\ &= \bigcup_{\iota<\delta} U_{\alpha\iota} \triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{N}(\nu) \end{aligned}$$

because $\delta < \kappa = \operatorname{add} \mathcal{N}(\nu)$. Set $V_\alpha = \{\xi : \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi)\}$, so that $\nu V_\alpha = 1$, for each $\alpha \in S$.

(d) Because every $H_{\alpha\iota}$ is a Baire set, there is for each $\alpha \in S$ a set $I_\alpha \subseteq \lambda$ such that $\#(I_\alpha) \le \delta$ and $H_{\alpha\iota}$ is determined by coordinates in I_α for every $\iota < \delta$, that is, $\tilde{g}_\alpha(x) = \tilde{g}_\alpha(y)$ whenever x, $y \in \{0,1\}^\lambda$ and $x \restriction I_\alpha = y \restriction I_\alpha$. By 5A1J there is an $M \subseteq \lambda$ such that

$$S_1 = \{\alpha : \alpha \in S,\, I_\alpha \subseteq M\}$$

is stationary in ζ and $\mathrm{cf}(\#(M)) \le \delta$; because $\lambda < \kappa^{(+\omega)}$ and $\mathrm{cf}(\kappa) = \kappa > \delta$, $\#(M) < \kappa$. Set $\pi_M(z) = z\restriction M$ for $z \in \{0,1\}^\lambda$, and $f_M = \pi_M f$, so that $f_M : \kappa \to \{0,1\}^M$ is inverse-measure-preserving for ν and the usual measure ν_M of $\{0,1\}^M$. For $w \in \{0,1\}^M$ define $\psi(w) \in \{0,1\}^\lambda$ by setting

$$\begin{aligned}\psi(w)(\xi) &= w(\xi) \text{ if } \xi \in M,\\ &= 0 \text{ otherwise.}\end{aligned}$$

If we set

$$g^*_\alpha = \tilde g_\alpha \psi : \{0,1\}^M \to \{0,1\}^\delta,$$

then g^*_α is Baire measurable in each coordinate, while $\pi_M = \tilde g_\alpha$ for $\alpha \in S_1$.

(e) For each $\alpha \in S_1$, there is a $\theta_\alpha < \kappa$ such that $\mu^*_M(f_M[V_\alpha \cap \theta_\alpha]) = 1$. **P** Apply 543Db to $f_M[V_\alpha] \subseteq \{0,1\}^M$. There must be a set $B \subseteq f_M[V_\alpha]$ such that $\#(B) < \kappa$ and $\mu^*_M B = \mu^*_M(f_M[V_\alpha])$; because κ is regular, there is a $\theta_\alpha < \kappa$ such that $B \subseteq f_M[V_\alpha \cap \theta_\alpha]$. On the other hand, because f_M is inverse-measure-preserving, $\mu^*_M(f_M[V_\alpha]) \ge \nu V_\alpha = 1$. **Q**

Evidently we may take it that every θ_α is a non-zero limit ordinal.

(f) Because $\zeta = \mathrm{cf}\,\zeta > \kappa$, there is a $\theta < \kappa$ such that

$$S_2 = \{\alpha : \alpha \in S_1,\, \theta_\alpha = \theta\}$$

is stationary in ζ. Now there is a $Y \in [2^\delta]^{<\kappa}$ such that $S_3 = \{\alpha : \alpha \in S_2,\, g_\alpha[\theta] \subseteq Y\}$ is stationary in ζ.

P case 1 If $\lambda < \mathrm{Tr}(\kappa)$, then $g_\alpha[\theta]$ is a subset of κ, and is therefore bounded above in κ, for each α. Let $\theta' < \kappa$ be such that

$$S_3 = \{\alpha : \alpha \in S_2,\, g_\alpha[\theta] \subseteq \theta'\}$$

is stationary in ζ, and take $Y = \theta'$.

case 2 If $\lambda \ge \mathrm{Tr}(\kappa)$, then $g_\alpha(\theta) < \alpha$ for $\alpha \in S_2$; by the Pressing-Down Lemma there is a $\theta' < \zeta$ such that

$$S_2' = \{\alpha : \alpha \in S_2,\, g_\alpha(\theta) = \theta'\}$$

is stationary in ζ. Then $g_\alpha\restriction\theta = g_\beta\restriction\theta$ for all $\alpha, \beta \in S_2'$; take Y to be the common value of $g_\alpha[\theta]$ for $\alpha \in S_2'$. **Q**

(g) For each $\alpha \in S_3$, set

$$Q_\alpha = f_M[V_\alpha \cap \theta] = f_M[V_\alpha \cap \theta_\alpha],$$

so that $\mu^*_M Q_\alpha = 1$. If $y \in Q_\alpha$, take $\xi \in V_\alpha \cap \theta$ such that $f_M(\xi) = y$; then

$$g^*_\alpha(y) = g^*_\alpha \pi_M f(\xi) = \tilde g_\alpha f(\xi) = h g_\alpha(\xi) \in h[Y].$$

Thus $g^*_\alpha\restriction Q_\alpha \subseteq f_M[\theta] \times h[Y]$ for every $\alpha \in S_3$, and we can apply 543Dd to $X = \{0,1\}^M$, $\mu = \mu_M$ and the family $\langle g^*_\alpha\restriction Q_\alpha\rangle_{\alpha\in S'}$, where $S' \subseteq S_3$ is a set of cardinal κ, to see that there are distinct $\alpha, \beta \in S_3$ such that $\mu^*_M\{y : y \in Q_\alpha \cap Q_\beta,\, g^*_\alpha(y) = g^*_\beta(y)\} > 0$. Now, however, consider

$$E = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y) = g^*_\beta(y)\}.$$

Then $E = \bigcap_{\iota<\delta} E_\iota$, where

$$E_\iota = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y)(\iota) = g^*_\beta(y)(\iota)\}$$

is a Baire subset of $\{0,1\}^M$ for each $\iota < \delta$. Because $\delta < \kappa$,

$$\begin{aligned}\nu f_M^{-1}[E] &= \nu(\bigcap_{\iota<\delta} f_M^{-1}[E_\iota]) = \inf_{I\in[\delta]^{<\omega}} \nu(\bigcap_{\iota\in I} f_M^{-1}[E_\iota])\\ &= \inf_{I\in[\delta]^{<\omega}} \mu_M(\bigcap_{\iota\in I} E_\iota) \ge \mu^*_M E > 0.\end{aligned}$$

Consequently

$$\begin{aligned}0 < \nu f_M^{-1}[E] &= \nu\{\xi : g^*_\alpha \pi_M f(\xi) = g^*_\beta \pi_M f(\xi)\}\\ &= \nu\{\xi : \tilde g_\alpha f(\xi) = \tilde g_\beta f(\xi)\} = \nu\{\xi : \xi \in V_\alpha \cap V_\beta,\, \tilde g_\alpha f(\xi) = \tilde g_\beta f(\xi)\}\\ &= \nu\{\xi : h g_\alpha(\xi) = h g_\beta(\xi)\} = \nu\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}\end{aligned}$$

(because h is injective). But this is absurd, because in (b) above we chose g_α, g_β in such a way that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$ would be bounded in κ. **X**

(h) Thus the result is true for Maharam-type-homogeneous witnessing probabilities on κ. In general, if ν is any witnessing probability on κ, there is a non-negligible $A \subseteq \kappa$ such that the subspace measure on A is Maharam-type-homogeneous; setting $\nu' C = \frac{1}{\nu A}\nu(A \cap C)$ for $C \subseteq \kappa$, we obtain a Maharam-type-homogeneous witnessing probability ν'. Now the Maharam type of ν is at least as great as the Maharam type of ν', so is at least $\min(2^\kappa, \kappa^{(+\omega)})$, as required.

543F Theorem Let $(X, \mathcal{P}X, \mu)$ be an atomless semi-finite measure space. Write $\kappa = \operatorname{add}\mu$. Then the Maharam type of $(X, \mathcal{P}X, \mu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$, and in particular is greater than κ.

proof Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{N}(\mu)$ such that $E = \bigcup_{\xi<\kappa} E_\xi \notin \mathcal{N}(\mu)$. Let $F \subseteq E$ be a set of non-zero finite measure. Set $f(x) = \min\{\xi : x \in E_\xi\}$ for $x \in F$. Let μ_F be the subspace measure on F and $\mu' = (\mu F)^{-1}\mu_F$ the corresponding probability measure; of course $\operatorname{dom}\mu' = \mathcal{P}F$ and $\mu'(F \cap E_\xi) = 0$ for every $\xi < \kappa$. Note also that

$$\operatorname{add}\mu' = \operatorname{add}\mu_F \geq \operatorname{add}\mu \geq \kappa$$

(521Da). Let ν be the image measure $\mu' f^{-1}$, so that $\operatorname{dom}\nu = \mathcal{P}\kappa$ and ν is κ-additive (521Fb). Also $\nu\{\xi\} \leq \mu' E_\xi = 0$ for every ξ, so ν witnesses that κ is real-valued-measurable. Next, μ_F is atomless (214Ja), so μ' also is. There is therefore a function $g : F \to [0,1]$ which is inverse-measure-preserving for μ' and Lebesgue measure (343Cb), and F can be covered by $\mathfrak{c}$ negligible sets; accordingly $\operatorname{add}\mu' \leq \mathfrak{c}$ so $\kappa \leq \mathfrak{c}$ and ν must be atomless (543Bc).

Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{A}', \bar\mu')$ and $(\mathfrak{B}, \bar\nu)$ be the measure algebras of μ, μ' and ν respectively. Then $\mathfrak{A}'$ is isomorphic to a principal ideal of $\mathfrak{A}$ (322I), so $\tau(\mathfrak{A}) \geq \tau(\mathfrak{A}')$ (514Ed). Next, $f : F \to \kappa$ induces a measure-preserving Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}'$, so that $\tau(\mathfrak{A}') \geq \tau(\mathfrak{B})$ (332Tb). Now 543E tells us that

$$\min(\kappa^{(+\omega)}, 2^\kappa) \leq \tau(\mathfrak{B}) \leq \tau(\mathfrak{A}),$$

as required.

543G Corollary Let $(X, \mathcal{P}X, \nu)$ be an atomless probability space, and $\kappa = \operatorname{add}\nu$. Let (Z, Σ, μ) be a compact probability space with Maharam type $\lambda \leq \min(2^\kappa, \kappa^{(+\omega)})$ (e.g., $Z = \{0,1\}^\lambda$ with its usual measure). Then there is an inverse-measure-preserving function $f : X \to Z$.

proof Let $(\mathfrak{A}, \bar\mu)$ and $\mathfrak{B}, \bar\nu)$ be the measure algebras of μ, ν respectively. By 543F, the Maharam type of the subspace measure ν_C is at least λ whenever $C \subseteq X$ and $\nu C > 0$; that is, every non-zero principal ideal of $\mathfrak{B}$ has Maharam type at least λ. So there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (332P). Because μ is compact, this is represented by an inverse-measure-preserving function from X to Z (343B).

543H Corollary If κ is an atomlessly-measurable cardinal, and (Z, μ) is a compact probability space with Maharam type at most $\min(2^\kappa, \kappa^{(+\omega)})$, then there is an extension of μ to a κ-additive measure defined on $\mathcal{P}Z$.

proof Let ν be a witnessing probability on κ; by 543G, there is an inverse-measure-preserving function $f : X \to Z$; now the image measure νf^{-1} extends μ to $\mathcal{P}Z$.

543I Corollary If κ is an atomlessly-measurable cardinal, with witnessing probability ν, and $2^\kappa \leq \kappa^{(+\omega)}$, then $(\kappa, \mathcal{P}\kappa, \nu)$ is Maharam-type-homogeneous, with Maharam type 2^κ.

proof If $C \in \mathcal{P}\kappa \setminus \mathcal{N}(\nu)$, then the Maharam type of the subspace measure on C is at least 2^κ, by 543F; but also it cannot be greater than $\#(\mathcal{P}C) = 2^\kappa$.

543J Proposition Let κ be an atomlessly-measurable cardinal, ν a witnessing probability on κ, and $\mathfrak{A}$ the measure algebra of ν. Then

(a) there is a $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \leq \delta < \kappa$;

(b) the cardinal power $\tau(\mathfrak{A})^\gamma$ is 2^κ;

(c) if $\mathfrak{c} < \kappa^{(+\omega_1)}$, then $\#(\mathfrak{A}) = \tau(\mathfrak{A})^\omega = 2^\kappa$.

proof Use 542F-542G. Because κ is quasi-measurable and $\kappa \leq \mathfrak{c}$, 542Fa tells us that there is a γ as in (a); and now 542Fb and 515M tell us that

$$2^\kappa = \tau(\mathfrak{A})^\gamma = (\tau(\mathfrak{A})^\omega)^\gamma = \#(\mathfrak{A})^\gamma.$$

If $\mathfrak{c} < \kappa^{(+\omega_1)}$, then 542G tells us that $\tau(\mathfrak{A})^\omega = 2^\kappa$.

543K Proposition Let κ be an atomlessly-measurable cardinal. If there is a witnessing probability on κ with Maharam type λ, then there is a Maharam-type-homogeneous normal witnessing probability ν on κ with Maharam type at most λ.

proof Repeat the proof of 543Ba, with $X = \kappa$ and μ a witnessing probability on κ with Maharam type λ. Taking a non-negligible $Y \subseteq \kappa$ and $g : Y \to \kappa$ such that $\nu = \frac{1}{\mu Y}\mu_Y g^{-1}$ is normal, then g induces an embedding of the measure algebra of ν into a principal ideal of the measure algebra of μ, so the Maharam type of ν is at most λ. There is now an $E \in \mathcal{P}\kappa \setminus \mathcal{N}(\nu)$ such that the subspace measure ν_E is Maharam-type-homogeneous, and setting $\nu' A = \nu(A \cap E)/\nu E$ for $A \subseteq \kappa$ we obtain a Maharam-type-homogeneous probability measure ν' with Maharam type less than or equal to λ. Now ν' is again normal. **P** Let $\langle I_\xi \rangle_{\xi<\kappa}$ be any family in $\mathcal{N}(\nu')$, and set $I = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} I_\eta\}$. Then $I_\xi \cap E \in \mathcal{N}(\nu)$ for every ξ, so $I \cap E = \{\xi : \xi < \kappa,\, \xi \in \bigcup_{\eta<\xi} I_\eta \cap E\}$ is ν-negligible and I is ν'-negligible. **Q** So we have an appropriate normal witnessing probability.

543L Proposition Suppose that ν is a Maharam-type-homogeneous witnessing probability on an atomlessly-measurable cardinal κ with Maharam type λ. Then there is a Maharam-type-homogeneous witnessing probability ν' on κ with Maharam type at least $\mathrm{Tr}_{\mathcal{N}(\nu)}(\kappa;\lambda)$.

proof Let ν_1 be the κ-additive probability on $\kappa \times \kappa$ given by

$$\nu_1 C = \int \nu C[\{\xi\}]\nu(d\xi) \text{ for every } C \subseteq \kappa \times \kappa.$$

Set $\theta = \mathrm{Tr}_{\mathcal{N}(\nu)}(\kappa;\lambda)$. By 541F there is a family $F \subseteq \lambda^\kappa$ such that $\#(F) = \theta$ and $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}(\nu)$ for all distinct f, $g \in F$. Let $\langle E_\xi \rangle_{\xi<\lambda}$ be a ν-stochastically independent family of subsets of κ of ν-measure $\frac{1}{2}$. For each $f \in F$ set

$$C_f = \{(\xi, \eta) : \xi < \kappa,\, \eta \in E_{f(\xi)}\}.$$

Then for any non-empty finite subset I of F, $\nu(\bigcap_{f\in I} E_{f(\xi)}) = 2^{-\#(I)}$ for ν-almost every ξ, so that

$$\nu_1(\bigcap_{f\in I} C_f) = 2^{-\#(I)}.$$

Thus $\langle C_f \rangle_{f\in F}$ is stochastically independent for ν_1, and the Maharam type of the subspace measure $(\nu_1)_C$ is at least $\#(F) = \theta$ whenever $\nu_1 C > 0$. Once again, take $\nu_2 C = \nu_1(C \cap D)/\nu_1 D$ for some D for which $(\nu_1)_D$ is Maharam-type-homogeneous, to obtain a Maharam-type-homogeneous κ-additive probability ν_2 with Maharam type at least θ. Finally, of course, ν_2 can be copied onto a probability ν' on κ, as asked for.

543X Basic exercises (a) Let κ be an atomlessly-measurable cardinal. Show that the following are equiveridical: (i) every witnessing probability ν on κ is Maharam-type-homogeneous (ii) any two witnessing probabilities on κ have the same Maharam type.

(b) Let κ be an atomlessly-measurable cardinal. Show that the following are equiveridical: (i) every normal witnessing probability ν on κ is Maharam-type-homogeneous (ii) any two normal witnessing probabilities on κ have the same Maharam type.

(c) Suppose that $\mathfrak{c}$ is atomlessly-measurable. Show that there is a Maharam-type-homogeneous normal witnessing probability on $\mathfrak{c}$ with Maharam type $2^{\mathfrak{c}}$.

543Y Further exercises (a) Let ν be a witnessing probability on an atomlessly-measurable cardinal κ with Maharam type λ. Let F be the set of all functions $f \subseteq \kappa \times \lambda$ such that $\mathrm{dom}\, f \notin \mathcal{N}(\nu)$, and let θ be

$$\sup\{\#(F_0) : F_0 \subseteq F,\, \{\xi : \xi \in \mathrm{dom}\, f \cap \mathrm{dom}\, g,\, f(\xi) = g(\xi)\} \in \mathcal{N}(\nu) \text{ for all distinct } f,\, g \in F_0\}.$$

Show that there is a witnessing probability ν' on κ with Maharam type at least θ.

543Z Problems Let κ be an atomlessly-measurable cardinal.

(a) Must every witnessing probability ν on κ be Maharam-type-homogeneous? (See 555E.)

(b) Must every normal witnessing probability ν on κ be Maharam-type-homogeneous?

543 Notes and comments The results of 543I-543J leave a tantalizingly narrow gap; it seems possible that the Maharam type of a witnessing probability on an atomlessly-measurable cardinal κ is determined by κ (543Xa, 543Za). If so, there is at least a chance that there is a proof depending on no ideas more difficult than those above. To find a counter-example, however, we may need not only to make some strong assumptions about the potential existence of appropriate large cardinals, but also to find a new method of constructing models with atomlessly-measurable cardinals. Possibly we get a different question if we look at normal witnessing probabilities (543Zb). A positive answer to either part of 543Z would have implications for transversal numbers (543L, 543Ya).

544 Measure theory with an atomlessly-measurable cardinal

As is to be expected, a witnessing measure on a real-valued-measurable cardinal has some striking properties, especially if it is normal. What is less obvious is that the mere existence of such a cardinal can have implications for apparently unrelated questions in analysis. In 544J, for instance, we see that if there is any atomlessly-measurable cardinal then we have a version of Fubini's theorem, $\iint f(x,y)dxdy = \iint f(x,y)dydx$, for many functions f on $\mathbb{R}^2$ which are not jointly measurable. In this section I explore results of this kind. We find that, in the presence of an atomlessly-measurable cardinal, the covering number of the Lebesgue null ideal is large (544B) while its uniformity is small (544G-544H). There is a second inequality on repeated integrals (544C) to add to the one already given in 543C, and which tells us something about measure-precalibers (544D); I add a couple of variations (544I-544J). Next, I give a pair of theorems (544E-544F) on a measure-combinatorial property of the filter of conegligible sets of a normal witnessing measure. Revisiting the theory of Borel measures on metrizable spaces, discussed in §438 on the assumption that no real-valued-measurable cardinal was present, we find that there are some non-trivial arguments applicable to spaces with non-measure-free weight (544K-544L).

In §541 I briefly mentioned 'weakly compact' cardinals. Two-valued-measurable cardinals are always weakly compact; atomlessly-measurable cardinals never are; but atomlessly-measurable cardinals may or may not have a significant combinatorial property which can be regarded as a form of weak compactness (544M, 544Yc). Finally, I summarise what is known about the location of an atomlessly-measurable cardinal on Cichoń's diagram (544N).

544A Notation I repeat some of my notational conventions. For a measure μ, $\mathcal{N}(\mu)$ will be its null ideal. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra.

544B Proposition Let κ be an atomlessly-measurable cardinal. If (X,Σ,μ) is any locally compact (definition: 342Ad) semi-finite measure space with $\mu X > 0$, then $\operatorname{cov}\mathcal{N}(\mu) \ge \kappa$.

proof By 521Jb, applied to any Maharam-type-homogeneous subspace of X with non-zero finite measure, it is enough to show that $\operatorname{cov}\mathcal{N}_\lambda \ge \kappa$ for every λ; by 523F, we need look only at the case $\lambda = \kappa$. Fix on an atomless κ-additive probability ν with domain $\mathcal{P}\kappa$. By 543G there is an inverse-measure-preserving function $f : \kappa \to \{0,1\}^\kappa$. So $\operatorname{cov}\mathcal{N}_\kappa \ge \operatorname{cov}\mathcal{N}(\nu) = \kappa$, by 521Fa.

544C Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ; let (X,μ) be a compact probability space and $f : X\times\kappa \to [0,\infty[$ any function. Then

$$\underline{\int}\Bigl(\int f(x,\xi)\nu(d\xi)\Bigr)\mu(dx) \le \int\Bigl(\overline{\int} f(x,\xi)\mu(dx)\Bigr)\nu(d\xi).$$

proof ? Suppose, if possible, otherwise.

(a) We are supposing that there is a μ-integrable function $g : X \to \mathbb{R}$ such that $0 \le g(x) \le \int f(x,\xi)\nu(d\xi)$ for every $x \in X$ and

$$\int g(x)\,\mu(dx) > \int\overline{\int} f(x,\xi)\mu(dx)\nu(d\xi).$$

We can suppose that g is a simple function; express it as $\sum_{i=0}^n t_i\chi F_i$ where $F_0,\dots,F_n$ are disjoint non-empty measurable subsets of X; set $F = \bigcup_{i\le n} F_i$. For any $\xi < \kappa$,

$$\overline{\int} f(x,\xi)\mu(dx) \ge \overline{\int} f(x,\xi)\chi F(x)\mu(dx) = \sum_{i=0}^n \overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)$$

(133L[2]). So there must be some $i \le n$ such that

$$t_i\mu F_i > \int\overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)\nu(d\xi).$$

Set $Y = F_i$, $\mu_1 = (\mu F_i)^{-1}\mu\restriction\mathcal{P}F_i$, $t = t_i$; then (Y,μ_1) is a compact probability space (451Da) and

$$\int\overline{\int} f(y,\xi)\mu_1(dy)\nu(d\xi) = \frac{1}{\mu F_i}\int\overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)\nu(d\xi)$$

(135Id[2])

$$< t \le \inf_{y\in Y}\int f(y,\xi)\nu(d\xi).$$

[2]Later editions only.

(b) Let $(\mathfrak{A},\bar{\mu}_1)$ be the measure algebra of (Y,μ_1). Then there is a cardinal $\lambda\ge\kappa$ such that $(\mathfrak{A},\bar{\mu}_1)$ can be embedded in $(\mathfrak{B}_\lambda,\bar{\nu}_\lambda)$, the measure algebra of ν_λ. Because μ_1 is compact, there is an inverse-measure-preserving function $\phi:\{0,1\}^\lambda\to Y$ (343B). By 235A[2], $\overline{\int}f(\phi(z),\xi)\nu_\lambda(dz)\le\overline{\int}f(y,\xi)\mu_1(dy)$ for every ξ, so $\int\overline{\int}f(\phi(z),\xi)\nu_\lambda(dz)\nu(d\xi)<t$.

For each $\xi<\kappa$ choose a Baire measurable function $h_\xi:\{0,1\}^\lambda\to\mathbb{R}$ such that $f(\phi(z),\xi)\le h_\xi(z)$ for every $z\in\{0,1\}^\lambda$ and $\int h_\xi(z)\nu_\lambda(dz)=\overline{\int}f(\phi(z),\xi)\nu_\lambda(dz)$; we can do this because ν_λ is the completion of its restriction to the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\{0,1\}^\lambda)$ (see 4A3Of), so we can apply 133J(a-i) to the Baire measure $\nu_\lambda\restriction\mathcal{B}\mathfrak{a}(\{0,1\}^\lambda)$. For each ξ, there is a countable set $I_\xi\subseteq\lambda$ such that h_ξ is determined by coordinates in I_ξ, in the sense that $h_\xi(z)=h_\xi(z')$ whenever $z\restriction I_\xi=z'\restriction I_\xi$.

By 541Rb, there are $\Gamma\subseteq\kappa$ and a countable set $J\subseteq\lambda$ such that $\nu\Gamma=1$ and $I_\xi\cap I_\eta\subseteq J$ whenever ξ, η are distinct members of Γ.

(c) For $u\in\{0,1\}^J$ and $u'\in\{0,1\}^{\lambda\setminus J}$ write $u\cup u'$ for their common extension to a member of $\{0,1\}^\lambda$. Set

$$f_1(u,\xi)=\int h_\xi(u\cup u')\nu_{\lambda\setminus J}(du')$$

for $u\in\{0,1\}^J$ and $\xi<\kappa$. Then, applying Fubini's theorem to $\{0,1\}^\lambda\cong\{0,1\}^J\times\{0,1\}^{\lambda\setminus J}$, we have

$$\int f_1(u,\xi)\nu_J(du)=\int h_\xi(z)\nu_\lambda(dz),$$

so that

$$\iint f_1(u,\xi)\nu_J(du)\nu(d\xi)=\int\overline{\int}f(\phi(z),\xi)\nu_\lambda(dz)\nu(d\xi)<t,$$

and

$$\overline{\int}\int f_1(u,\xi)\nu(d\xi)\nu_J(du)<t$$

by Theorem 543C. Accordingly there is a $u\in\{0,1\}^J$ such that $\int f_1(u,\xi)\nu(d\xi)<t$.

(d) For each $\xi\in\Gamma$ take $v_\xi\in\{0,1\}^{\lambda\setminus J}$ such that $h_\xi(u\cup v_\xi)\le f_1(u,\xi)$. Let $w\in\{0,1\}^\lambda$ be such that

$$w\restriction J=u,\quad w\restriction I_\xi\setminus J=v_\xi\restriction I_\xi\setminus J\text{ for every }\xi\in\Gamma;$$

such a w exists because if ξ, $\eta\in\Gamma$ and $\xi\ne\eta$ then $I_\xi\cap I_\eta\subseteq J$. Now

$$f(\phi(w),\xi)\le h_\xi(w)=h_\xi(u\cup v_\xi)\le f_1(u,\xi)$$

for every $\xi\in\Gamma$, so

$$\int f(\phi(w),\xi)\nu(d\xi)\le\int f_1(u,\xi)\nu(d\xi)<t,$$

contradicting the last sentence of (a) above. **X**

This completes the proof.

544D Corollary If κ is an atomlessly-measurable cardinal and $\omega\le\lambda\le\kappa$, then λ is a measure-precaliber of every probability algebra.

proof If $\lambda<\kappa$ this is a corollary of 544B and 525K. If $\lambda=\kappa$, we can use 544C and 525D. For let $\langle E_\xi\rangle_{\xi<\kappa}$ be a non-decreasing family in $\mathcal{N}_\kappa$ with union E. Let ν be a normal witnessing probability on κ. Set

$$C=\{(x,\xi):\xi<\kappa,\,x\in E_\xi\}\subseteq\{0,1\}^\kappa\times\kappa.$$

Then

$$\underline{\int}\nu C[\{x\}]\nu_\kappa(dx)\ge\mu_*E,\qquad\int\nu_\kappa^*C^{-1}[\{\xi\}]\nu(d\xi)=0,$$

so 544C, applied to the characteristic function of C, tells us that $\mu_*E=0$; now 525Dc tells us that κ is a precaliber of $\mathfrak{B}_\kappa$, and therefore of every probability algebra, by 525Ja.

544E Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. If (X,μ) is a quasi-Radon probability space of weight strictly less than κ, and $f:[\kappa]^{<\omega}\to\mathcal{N}(\mu)$ is any function, then

$$\bigcap_{V\subseteq\kappa,\nu V=1}\bigcup_{I\in[V]^{<\omega}}f(I)\in\mathcal{N}(\mu).$$

proof Let $\mathcal{F}$ be the filter of ν-conegligible subsets of κ.

(a) I show by induction on $n\in\mathbb{N}$ that if $g:[\kappa]^{\le n}\to\mathcal{N}(\mu)$ is any function, then

$$E(g)=\bigcap_{V\in\mathcal{F}}\bigcup_{I\in[V]^{\le n}}g(I)\in\mathcal{N}(\mu).$$

P For $n=0$ this is trivial; $E(g)=g(\emptyset)\in\mathcal{N}(\mu)$. For the inductive step to $n+1$, given $g:[\kappa]^{\le n+1}\to\mathcal{N}(\mu)$, then for each $\xi<\kappa$ define $g_\xi:[\kappa]^{\le n}\to\mathcal{N}(\mu)$ by setting $g_\xi(I)=g(I\cup\{\xi\})$ for each $I\in[\kappa]^{\le n}$. By the inductive hypothesis, $E(g_\xi)\in\mathcal{N}(\mu)$. Set

$$C=\{(x,\xi):x\in E(g_\xi)\}\subseteq X\times\kappa.$$

Then

$$\int\mu^*C^{-1}[\{\xi\}]\nu(d\xi)=\int\mu^*E(g_\xi)\nu(d\xi)=0,$$

so by 543C

$$\overline{\int}\nu C[\{x\}]\mu(dx)=0,$$

and $\mu D=0$, where $D=g(\emptyset)\cup\{x:\nu C[\{x\}]>0\}$.

Take any $x\in X\setminus D$ and set $W=\kappa\setminus C[\{x\}]\in\mathcal{F}$. For each $\xi\in W$, $x\notin E(g_\xi)$, so there is a $V_\xi\in\mathcal{F}$ such that $\nu V_\xi=1$ and $x\notin g_\xi(I)$ for every $I\in[V_\xi]^{\le n}$. Set

$$V=W\cap\{\xi:\xi\in V_\eta\text{ for every }\eta<\xi\}.$$

Then $V\in\mathcal{F}$. If $I\in[V]^{\le n+1}$, either $I=\emptyset$ and $x\notin g(I)$, or there is a least element ξ of I; in the latter case, $\xi\in W$ and $J=I\setminus\{\xi\}\subseteq V_\xi$ and $x\notin g_\xi(J)=g(I)$. So $x\notin\bigcup\{g(I):I\in[V]^{\le n+1}\}$. As x is arbitrary, $E(g)\subseteq D\in\mathcal{N}(\mu)$ and the induction proceeeds. **Q**

(b) Now consider

$$G=\bigcup_{n\in\mathbb{N}}E(f\restriction[\kappa]^{\le n})\in\mathcal{N}(\mu).$$

If $x\in X\setminus G$ then for each $n\in\mathbb{N}$ there is a $V_n\in\mathcal{F}$ such that $x\notin\bigcup\{f(I):I\in[V_n]^{\le n}\}$. Set $V=\bigcap_{n\in\mathbb{N}}V_n\in\mathcal{F}$; then $x\notin\bigcup\{f(I):I\in[V]^{<\omega}\}$. As x is arbitrary,

$$E(f)\subseteq G\in\mathcal{N}(\mu),$$

as required.

544F Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal with a normal witnessing probability ν. If (X,μ) is a locally compact semi-finite measure space with $\mu X>0$ and $f:[\kappa]^{<\omega}\to\mathcal{N}(\mu)$ is a function, then there is a ν-conegligible $V\subseteq\kappa$ such that $\bigcup\{f(I):I\in[V]^{<\omega}\}\ne X$.

proof (a) Consider first the case $(X,\mu)=(\{0,1\}^\kappa,\nu_\kappa)$. For any $L\subseteq\kappa$ let $\pi_L:\{0,1\}^\kappa\to\{0,1\}^L$ be the restriction map. Let $\mathcal{F}$ be the conegligible filter on κ.

(i) For each $I\in[\kappa]^{<\omega}$, there is a countable set $g(I)\subseteq\kappa$ such that $\nu_{g(I)}(\pi_{g(I)}[f(I)])=0$ (254Od); enlarging $f(I)$ if necessary, we may suppose that $f(I)=\pi^{-1}_{g(I)}[\pi_{g(I)}[f(I)]]$. By 541Q there are a set $C\in\mathcal{F}$ and a function $h:[\kappa]^{<\omega}\to[\kappa]^{\le\omega}$ such that $g(I)\cap\eta\subseteq h(I\cap\eta)$ whenever $I\in[C]^{<\omega}$ and $\eta<\kappa$. Set

$$\Gamma=\{\gamma:\gamma<\kappa,\ h(I)\subseteq\gamma\text{ for every }I\in[\gamma]^{<\omega}\};$$

then Γ is a closed cofinal set in κ, because $\mathrm{cf}(\kappa)>\omega$. Let $\langle\gamma_\eta\rangle_{\eta\le\kappa}$ be the increasing enumeration of $\Gamma\cup\{0,\kappa\}$.

(ii) For $\eta<\kappa$, set $M(\eta)=\kappa\setminus\gamma_\eta$ and $L(\eta)=\gamma_{\eta+1}\setminus\gamma_\eta$; then $\nu_{M(\eta)}$ can be identified with the product measure $\nu_{L(\eta)}\times\nu_{M(\eta+1)}$. Choose $u_\eta\in\{0,1\}^{\gamma_\eta}$, $V_\eta\subseteq\kappa$ inductively, as follows. $u_0\in\{0,1\}^0$ is the empty function. Given u_η, then for each $I\in[\kappa]^{<\omega}$ set

$$f'_\eta(I)=\{v:v\in\{0,1\}^{L(\eta)},\ \nu_{M(\eta+1)}\{w:u_\eta\cup v\cup w\in f(I)\}>0\},$$

and

$$\begin{aligned}f_\eta(I)&=f'_\eta(I)\text{ if }\nu_{L(\eta)}(f'_\eta(I))=0,\\&=\emptyset\text{ otherwise.}\end{aligned}$$

By 544E, we can find for each $K\in[\gamma_{\eta+1}]^{<\omega}$ a set $E_{\eta K}\subseteq\{0,1\}^{L(\eta)}$ such that $\nu_{L(\eta)}E_{\eta K}=1$ and for every $v\in E_{\eta K}$ there is a set $V\in\mathcal{F}$ such that $v\notin f_\eta(K\cup J)$ for any $J\in[V]^{<\omega}$. Choose $v_\eta\in\bigcap\{E_{\eta K}:K\in[\gamma_{\eta+1}]^{<\omega}\}$ (using 544B); for $K\in[\gamma_{\eta+1}]^{<\omega}$ choose $V_{\eta K}\in\mathcal{F}$ such that $v_\eta\notin f_\eta(K\cup J)$ for any $J\in[V_{\eta K}]^{<\omega}$. Set $V_\eta=\bigcap\{V_{\eta K}:K\in[\gamma_{\eta+1}]^{<\omega}\}\in\mathcal{F}$ and $u_{\eta+1}=u_\eta\cup v_\eta\in\{0,1\}^{\gamma_{\eta+1}}$.

At limit ordinals η with $0<\eta\le\kappa$, set $u_\eta=\bigcup_{\xi<\eta}u_\xi\in\{0,1\}^{\gamma_\eta}$.

(iii) Now consider $u=u_\kappa\in\{0,1\}^\kappa$ and

$$V=\{\xi:\xi\in C,\ \xi\in V_\eta\text{ for every }\eta<\xi\}\in\mathcal{F}.$$

If $I \in [V]^{<\omega}$ then

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

for every $\eta < \kappa$. **P** Induce on η. For $\eta = 0$ this says just that $\nu_\kappa f(I) = 0$, which was our hypothesis on f. For the inductive step to $\eta + 1$, we have

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

by the inductive hypothesis, so Fubini's theorem tells us that

$$\nu_{L(\eta)}\{v : \nu_{M(\eta+1)}\{w : u \cup v \cup w \in f(I)\} > 0\} = 0,$$

that is, $\nu_{L(\eta)} f'_\eta(I) = 0$, so that $f_\eta(I) = f'_\eta(I)$. Now setting $K = I \cap \gamma_{\eta+1}$ and $J = I \setminus \gamma_{\eta+1}$, we see that $J \subseteq V_\eta$ (because of course $\eta < \gamma_{\eta+1}$), therefore $J \subseteq V_{\eta K}$ and $v_\eta \notin f_\eta(K \cup J) = f'_\eta(I)$; but this says just that

$$\nu_{M(\eta+1)}\{w : u_\eta \cup v_\eta \cup w \in f(I)\} = 0,$$

that is, that

$$\nu_{M(\eta+1)}\{w : u_{\eta+1} \cup w \in f(I)\} = 0,$$

so that the induction continues.

For the inductive step to a non-zero limit ordinal $\eta \le \kappa$, there is a non-zero $\zeta < \eta$ such that $I \cap \gamma_\eta \subseteq \gamma_\zeta$. Now

$$g(I) \cap \gamma_\eta \subseteq h(I \cap \gamma_\eta) = h(I \cap \gamma_\zeta) \subseteq \gamma_\zeta,$$

by the choice of Γ. As $f(I)$ is determined by coordinates in $g(I)$, this means that

$$\{w : w \in \{0,1\}^{M(\zeta)},\ u_\zeta \cup w \in f(I)\} = \{0,1\}^{\gamma_\eta \setminus \gamma_\zeta} \times \{w : w \in \{0,1\}^{M(\eta)},\ u_\eta \cup w \in f(I)\}.$$

By the inductive hypothesis,

$$\nu_{M(\zeta)}\{w : u_\zeta \cup w \in f(I)\} = 0,$$

so that

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

and the induction continues. **Q**

(iv) But now, given $I \in [V]^{<\omega}$, there is surely some $\eta < \kappa$ such that $g(I) \subseteq \gamma_\eta$, and in this case $\{w : u_\eta \cup w \in f(I)\}$ is either $\emptyset$ or $\{0,1\}^{M(\eta)}$. As it is $\nu_{M(\eta)}$-negligible it must be empty, and $u \notin f(I)$.

Thus we have a point $u \notin \bigcup\{f(I) : I \in [V]^{<\omega}\}$, as required.

(b) If (X, μ) is a compact probability space, we have a $\lambda \ge \kappa$ and an inverse-measure-preserving function $\phi : \{0,1\}^\lambda \to X$. For each $I \in [\kappa]^{<\omega}$ let $J_I \in [\lambda]^{\le\omega}$ be such that $\nu_{J_I}\pi_{J_I}[\phi^{-1}[f(I)]] = 0$, where here π_{J_I} is interpreted as a map from $\{0,1\}^\lambda$ to $\{0,1\}^{J_I}$; set $J = \kappa \cup \bigcup_{I \in [\kappa]^{<\omega}} J_I$. Let $q : \{0,1\}^J \to \{0,1\}^\lambda$ be any function such that $\pi_J q$ is the identity on $\{0,1\}^J$, and set $\psi = \phi q$. For any $I \in [\kappa]^{<\omega}$,

$$\psi^{-1}[f(I)] = q^{-1}[\phi^{-1}[f(I)]] \subseteq \pi_J[\phi^{-1}[f(I)]] \subseteq \pi_J[\pi_{J_I}^{-1}[\pi_{J_I}[\phi^{-1}[f(I)]]]]$$

is ν_J-negligible because $\pi_{J_I}^{-1}[\pi_{J_I}[\phi^{-1}[f(I)]]]$ is ν_λ-neglgible and determined by coordinates in J.

Because $\#(J) = \kappa$, (a) tells us that there are $u \in \{0,1\}^J$ and a conegligible $V \subseteq \kappa$ such that $u \notin \psi^{-1}[f(I)]$ for every $I \in [V]^{<\omega}$; in which case $x = \psi(u) \notin f(I)$ for every $I \in [V]^{<\omega}$ and $\bigcup\{f(I) : I \in [V]^{<\omega}\} \ne X$.

(c) For the general case, take a subset E of X with non-zero finite measure, and apply (b) to the function $I \mapsto E \cap f(I)$ and the normalized subspace measure $\dfrac{1}{\mu E}\nu_E$.

544G Proposition Let κ be an atomlessly-measurable cardinal and $\omega_1 \le \lambda < \kappa$. If (X, μ) is an atomless locally compact semi-finite measure space of Maharam type less than κ, and $\mu X > 0$, then there is a Sierpiński set $A \subseteq X$ of cardinal λ.

proof (a) To begin with, suppose that $X = \{0,1\}^\theta$ and $\mu = \nu_\theta$ where $\theta < \kappa$. Let ν be an atomless κ-additive probability defined on $\mathcal{P}\kappa$. By 543G there is a function $f : \kappa \to (\{0,1\}^\theta)^\lambda$ which is inverse-measure-preserving for ν and the usual measure ν_θ^λ of $(\{0,1\}^\theta)^\lambda$, which we may think of either as the power of ν_θ, or as the Radon power of ν_θ, or as a copy of $\nu_{\theta\times\lambda}$. For $\xi < \kappa$, set

$$A_\xi = \{f(\xi)(\eta) : \eta < \lambda\} \subseteq \{0,1\}^\theta.$$

? Suppose, if possible, that for every $\xi < \kappa$ there is a set $J_\xi \subseteq \lambda$ such that $\#(J_\xi) = \omega_1$ but $E_\xi = f(\xi)[J_\xi]$ is ν_θ-negligible. For each ξ choose a countable set $I_\xi \subseteq \theta$ such that $E'_\xi = \pi_{I_\xi}^{-1}[\pi_{I_\xi}[E_\xi]$ is ν_θ-negligible, writing $\pi_{I_\xi}(x) = x{\restriction}I_\xi$

for $x \in \{0,1\}^\theta$. By 541D, there is a countable $I \subseteq \theta$ such that $V = \{\xi : I_\xi \subseteq I\}$ is ν-conegligible. For $\xi \in V$ set $E^*_\xi = \pi_I[E_\xi] \subseteq \{0,1\}^I$, so that $\nu_I E^*_\xi = 0$. Fix a sequence $\langle U_m\rangle_{m\in\mathbb{N}}$ running over the open-and-closed subsets of $\{0,1\}^I$, and for each $\xi \in V$, $n \in \mathbb{N}$ choose an open set $G_{n\xi} \subseteq \{0,1\}^I$ such that $E^*_\xi \subseteq G_{n\xi}$ and $\nu_I(G_{n\xi}) \le 2^{-n}$. For $m, n \in \mathbb{N}$ set

$$D_{nm} = \{\xi : \xi \in V,\, U_m \subseteq G_{n\xi}\}.$$

For each $\alpha < \lambda$, set $f_\alpha(\xi) = \pi_I(f(\xi)(\alpha)) \in \{0,1\}^I$ for $\xi < \kappa$; then the functions f_α are all stochastically independent, in the sense that the σ-algebras $\Sigma_\alpha = \{f_\alpha^{-1}[H] : H \subseteq \{0,1\}^I$ is Borel$\}$ are independent. **P** Suppose that $\alpha_0, \dots, \alpha_n < \lambda$ are distinct and $H_0, \dots, H_n$ are Borel subsets of $\{0,1\}^I$. For each i, set

$$W_i = \{u : u \in (\{0,1\}^\theta)^\lambda,\, u(\alpha_i) \in \pi_I^{-1}[H_i]\}.$$

Then

$$\nu(\bigcap_{i\le n} f_{\alpha_i}^{-1}[H_i]) = \nu f^{-1}[\bigcap_{i\le n} W_i] = \nu_\theta^\lambda(\bigcap_{i\le n} W_i)$$
$$= \prod_{i\le n} \nu_\theta^\lambda W_i = \prod_{i\le n} \nu f_{\alpha_i}^{-1}[H_i]. \mathbf{Q}$$

By 272W[3], there is for each $\xi < \kappa$ an $\alpha(\xi) \in J_\xi$ such that $\Sigma_{\alpha(\xi)}$ is stochastically independent of the σ-algebra T generated by $\{D_{nm} : n, m \in \mathbb{N}\}$. Because $\lambda < \kappa$ and ν is κ-additive, there is a $\gamma < \lambda$ such that $B = \{\xi : \alpha(\xi) = \gamma\}$ has $\nu B > 0$. Take $n \in \mathbb{N}$ such that $\nu(B) > 2^{-n}$, and examine

$$C = \bigcup_{m\in\mathbb{N}}(D_{nm} \cap f_\gamma^{-1}[U_m]).$$

Then $\nu C = (\nu \times \nu_I)(\tilde C)$ where

$$\tilde C = \bigcup_{m\in\mathbb{N}}(D_{nm} \times U_m) \subseteq \kappa \times \{0,1\}^I$$

and $\nu \times \nu_I$ is the c.l.d. product measure on $\kappa \times \{0,1\}^I$. **P** Because T and Σ_γ are independent, and ν_I is the completion of its restriction to the Borel (or Baire) σ-algebra of $\{0,1\}^I$, the map $\xi \mapsto (\xi, f_\gamma(\xi)) : \kappa \to \kappa \times \{0,1\}^I$ is inverse-measure-preserving for ν and $(\nu{\restriction}\mathrm{T}) \times \nu_I$ (cf. 272J). The inverse of $\tilde C$ under this map is just C, so

$$\nu C = ((\nu{\restriction}\mathrm{T}) \times \nu_I)(\tilde C) = \int (\nu{\restriction}\mathrm{T})\tilde C^{-1}[\{u\}]\nu_I(du)$$
$$= \int \nu\tilde C^{-1}[\{u\}]\nu_I(du) = (\nu \times \nu_I)(\tilde C). \mathbf{Q}$$

But, for each $\xi < \kappa$, the vertical section $\tilde C[\{\xi\}]$ is just $\bigcup\{U_m : \xi \in D_{nm}\} = G_{n\xi}$, so

$$(\nu \times \nu_I)(\tilde C) = \int \nu_I(G_{n\xi})\nu(d\xi) \le 2^{-n}.$$

Accordingly $\nu C \le 2^{-n} < \nu B$ and there must be a $\xi \in B \cap V \setminus C$. But in this case $f(\xi)(\gamma) \in E_\xi$, because $\gamma = \alpha(\xi) \in J_\xi$, so $f_\gamma(\xi) = \pi_I(f(\xi)(\gamma)) \in E^*_\xi$. On the other hand, $f_\gamma(\xi) \notin G_{n\xi}$, because there is no m such that $f_\gamma(\xi) \in U_m \subseteq G_{n\xi}$; contrary to the choice of $G_{n\xi}$. **X**

So take some $\xi < \kappa$ such that $\nu^*_\theta(f(\xi)[J]) > 0$ for every uncountable $J \subseteq \lambda$. Evidently $f(\xi)$ is countable-to-one, so A_ξ must have cardinal λ, and will serve for A.

(b) Now suppose that (X, μ) is an atomless compact probability space with Maharam type $\theta < \kappa$. Then we have an inverse-measure-preserving map $h : \{0,1\}^\theta \to X$. Let $A \subseteq \{0,1\}^\theta$ be a Sierpiński set of cardinal λ; then $h[A]$ is a Sierpiński set of cardinal λ in X, by 537B(b-i).

(c) Finally, for the general case as stated, we can apply (b) to a normalized subspace measure, as usual.

544H Corollary Let κ be an atomlessly-measurable cardinal.

(a) $\operatorname{non}\mathcal{N}_\theta = \omega_1$ for $\omega \le \theta < \kappa$.

(b) $\operatorname{non}\mathcal{N}_\theta \le \kappa$ for $\theta \le \min(2^\kappa, \kappa^{(+\omega)})$.

(c) $\operatorname{non}\mathcal{N}_\theta < \theta$ for $\kappa < \theta < \kappa^{(+\omega)}$.

proof (a) Immediate from 544G.

(b) If ν is any witnessing probability on κ then we have an inverse-measure-preserving function $f : \kappa \to \{0,1\}^\theta$ (543G); now $f[\kappa]$ witnesses that $\operatorname{non}\mathcal{N}_\theta \le \kappa$.

(c) Induce on θ, using 523Id.

Remark There may be more to be said; see 544Zc.

[3]Later editions only.

544I The following is an elementary corollary of Theorem 543C.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite quasi-Radon measure space and $(Y,\mathcal{P}Y,\nu)$ a probability space; suppose that $w(X)<\operatorname{add}\nu$. Let $f:X\times Y\to\mathbb{R}$ be a bounded function such that all the sections $x\mapsto f(x,y):X\to\mathbb{R}$ are Σ-measurable. Then the repeated integrals $\iint f(x,y)\nu(dy)\mu(dx)$ and $\iint f(x,y)\mu(dx)\nu(dy)$ are defined and equal.

proof If $\mu X=0$ this is trivial; otherwise, re-scaling μ if necessary, we may suppose that $\mu X=1$. By 543C,

$$\overline{\int}\int f(x,y)\nu(dy)\mu(dx)\le\int\overline{\int}f(x,y)\mu(dx)\nu(dy)=\iint f(x,y)\mu(dx)\nu(dy).$$

Similarly

$$\overline{\int}\int(-f(x,y))\nu(dy)\mu(dx)\le\iint(-f(x,y))\mu(dx)\nu(dy),$$

so that

$$\underline{\int}\int f(x,y)\nu(dy)\mu(dx)\ge\iint f(x,y)\mu(dx)\nu(dy).$$

Putting these together we have the result.

544J Proposition (Zakrzewski 92) Let κ be an atomlessly-measurable cardinal and $(X,\mathfrak{T},\Sigma,\mu)$, $(Y,\mathfrak{S},\mathrm{T},\nu)$ Radon probability spaces both of weight less than κ; let $\mu\times\nu$ be the c.l.d. product measure on $X\times Y$, and Λ its domain. Let $f:X\times Y\to\mathbb{R}$ be a function such that all its horizontal and vertical sections

$$x\mapsto f(x,y^*):X\to\mathbb{R},\quad y\mapsto f(x^*,y):Y\to\mathbb{R}$$

are measurable. Then

(a) if f is bounded, the repeated integrals

$$\iint f(x,y)\mu(dx)\nu(dy),\quad\iint f(x,y)\nu(dy)\mu(dx)$$

exist and are equal;

(b) in any case, there is a Λ-measurable function $g:X\times Y\to\mathbb{R}$ such that all the sections $\{x:g(x,y^*)\ne f(x,y^*)\}$, $\{y:g(x^*,y)\ne f(x^*,y)\}$ are negligible.

proof (a) By 543H there is a κ-additive measure $\tilde\nu$ on Y, with domain $\mathcal{P}Y$, extending ν. Now 544I tells us, among other things, that the function

$$x\mapsto\int f(x,y)\nu(dy)=\int f(x,y)\tilde\nu(dy):X\to\mathbb{R}$$

is Σ-measurable. Similarly, $y\mapsto\int f(x,y)\mu(dx)$ is T-measurable. So returning to 544I we get

$$\begin{aligned}\iint f(x,y)\mu(dx)\nu(dy)&=\iint f(x,y)\mu(dx)\tilde\nu(dy)\\&=\iint f(x,y)\tilde\nu(dy)\mu(dx)=\iint f(x,y)\nu(dy)\mu(dx).\end{aligned}$$

(b)(i) Suppose first that f is bounded. By (a), we can define a measure θ on $X\times Y$ by saying that

$$\theta G=\int\nu G[\{x\}]\mu(dx)=\int\mu G^{-1}[\{y\}]\nu(dy)$$

whenever $G\subseteq X\times Y$ is such that $G[\{x\}]\in\mathrm{T}$ for almost every $x\in X$ and $G^{-1}[\{y\}]\in\Sigma$ for almost every $y\in Y$. This θ extends $\mu\times\nu$; so the Radon-Nikodým theorem (232G) tells us that there is a Λ-measurable function $h:X\times Y\to\mathbb{R}$ such that $\int_G f(x,y)\theta(dxdy)=\int_G h(x,y)\theta(dxdy)$ for every $G\in\Lambda$.

Let $\mathcal{U}$ be a base for the topology $\mathfrak{T}$, closed under finite intersections, with $\#(\mathcal{U})<\kappa$. For any $U\in\mathcal{U}$ consider

$$V_U=\{y:\int_U f(x,y)\mu(dx)>\int_U h(x,y)\mu(dx)\}.$$

The argument of (a) shows that $y\mapsto\int_U f(x,y)\mu(dx)$ is T-measurable, so $V_U\in\mathrm{T}$, and

$$\begin{aligned}\int_{V_U}\int_U f(x,y)\mu(dx)\nu(dy)&=\int_{U\times V_U}f(x,y)\theta(dxdy)\\&=\int_{U\times V_U}h(x,y)\theta(dxdy)=\int_{V_U}\int_U h(x,y)\mu(dx)\nu(dy),\end{aligned}$$

so $\nu V_U=0$. Similarly

$$\nu\{y:\int_U f(x,y)\mu(dx)<\int_U h(x,y)\mu(dx)\}=0.$$

Because $\#(\mathcal{U}) < \kappa$, and no non-negligible measurable set in Y can be covered by fewer than κ negligible sets (544B), we must have

$$\nu^*\{y : \int_U f(x,y)\mu(dx) = \int_U h(x,y)\mu(dx) \text{ for every } U \in \mathcal{U}\} = 1.$$

But because $\mathcal{U}$ is a base for the topology of X closed under finite intersections, we see that

$$\nu^*\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

(For each y such that $\int_U f(x,y)\mu(dx) = \int_U h(x,y)\mu(dx)$ for every $U \in \mathcal{U}$, apply 415H(iv) to the indefinite-integral measures over μ defined by the functions $x \mapsto f(x,y)$, $x \mapsto h(x,y)$; these are quasi-Radon by 415Ob.) Again using (a), we know that the the repeated integral $\iint |f(x,y) - h(x,y)|\mu(dx)\nu(dy)$ exists, and it must be 0. Thus

$$\nu\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

Similarly,

$$\mu\{x : f(x,y) = h(x,y) \text{ for } \nu\text{-almost every } y\} = 1.$$

But now, changing h on a set of the form $(E \times Y) \cup (X \times F)$ where $\mu E = \nu F = 0$, we can get a function g, still Λ-measurable, such that $\{(x,y) : f(x,y) \neq g(x,y)\}$ has all its horizontal and vertical sections negligible.

(ii) This deals with bounded f. But for general f we can look at the truncates $(x,y) \mapsto \operatorname{med}(-n, f(x,y), n)$ for each n to get a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of functions which will converge at an adequate number of points to provide a suitable g.

544K Proposition If X is a metrizable space and μ is a σ-finite Borel measure on X, then $\operatorname{add}\mathcal{N}(\mu) \geq \operatorname{add}\mathcal{N}_\omega$.

proof (a) If there is an atomlessly-measurable cardinal then

$$\operatorname{add}\mathcal{N}_\omega \leq \operatorname{non}\mathcal{N}_\omega = \omega_1$$

(544Ha), so the result is immediate. So let us henceforth suppose otherwise.

(b) Because there is a totally finite measure with the same domain and the same null ideal as μ (215B), we can suppose that μ itself is totally finite. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ and $\mathcal{U}$ a σ-disjoint base for the topology of X (4A2Lg); express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where each $\mathcal{U}_n$ is disjoint. For $\mathcal{V} \subseteq \mathcal{U}_n$ set $\nu_n\mathcal{V} = \mu(\bigcup\mathcal{V})$. Then ν_n is a totally finite measure with domain $\mathcal{P}\mathcal{U}_n$. Because there is no atomlessly-measurable cardinal, $\operatorname{add}\nu_n$ is either ∞ or a two-valued-measurable cardinal; in either case, ν_n is $\mathfrak{c}$-additive and purely atomic (438Ce, 543B).

(c) μ has countable Maharam type. **P** Because ν_n is purely atomic, there is a sequence $\langle\mathcal{U}_{ni}\rangle_{i\in\mathbb{N}}$ of subsets of $\mathcal{U}_n$ such that for every $\mathcal{V} \subseteq \mathcal{U}_n$ there is a $J \subseteq \mathbb{N}$ such that $\nu_n(\mathcal{V} \triangle \bigcup_{i\in J} \mathcal{U}_{ni}) = 0$. Set $W_{ni} = \bigcup\mathcal{U}_{ni}$ for each i. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{W_{ni}^\bullet : n, i \in \mathbb{N}\}$.

If $G \subseteq X$ is open, set $\mathcal{V}_n = \{U : U \in \mathcal{U}_n, U \subseteq G\}$ and $G_n = \bigcup\mathcal{V}_n$ for each n. Then we have $J_n \subseteq \mathbb{N}$ such that

$$0 = \nu_n(\mathcal{V}_n \triangle \bigcup_{i\in J_n} \mathcal{U}_{ni}) = \mu(G_n \triangle \bigcup_{i\in J_n} W_{ni}),$$

so $G_n^\bullet = \sup_{i\in J_n} W_{ni}^\bullet \in \mathfrak{B}$. Now $G = \bigcup_{n\in\mathbb{N}} G_n$ so $G^\bullet = \sup_{n\in\mathbb{N}} G_n^\bullet$ belongs to $\mathfrak{B}$.

The set $\Sigma = \{E : E \subseteq X \text{ is Borel}, E^\bullet \in \mathfrak{B}\}$ is a σ-algebra of subsets of X, and we have just seen that it contains every open set; so Σ is the whole Borel σ-algebra and $\mathfrak{A} = \mathfrak{B}$ has countable Maharam type. **Q**

(d) Next, if $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of open sets where $\kappa < \mathfrak{c}$, and $G = \bigcup_{\xi<\kappa} G_\xi$, then $G^\bullet = \sup_{\xi<\kappa} G_\xi^\bullet$ in $\mathfrak{A}$. **P** Look at $\mathcal{V}_{n\xi} = \{U : U \in \mathcal{U}_n, U \subseteq G_\xi\}$, $\mathcal{V}_n = \bigcup_{\xi<\kappa} \mathcal{V}_{n\xi}$ for each n. Because ν_n is $\mathfrak{c}$-additive,

$$\mu(\bigcup\mathcal{V}_n) = \nu\mathcal{V}_n = \sup_{J\subseteq\kappa \text{ is finite}} \nu(\bigcup_{\xi\in J} \mathcal{V}_{n\xi})$$

and there is a countable set $J_n \subseteq \kappa$ such that $\mu(\bigcup\mathcal{V}_n) = \mu(\bigcup_{\xi\in J_n} \bigcup\mathcal{V}_{n\xi})$. Now

$$G^\bullet = \sup_{n\in\mathbb{N}}(\bigcup\mathcal{V}_n)^\bullet = \sup_{n\in\mathbb{N}} \sup_{\xi\in J_n}(\bigcup\mathcal{V}_{n\xi})^\bullet \subseteq \sup_{\xi<\kappa} G_\xi^\bullet \subseteq G^\bullet. \textbf{ Q}$$

(e) Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{N}(\mu)$ where $\kappa < \operatorname{add}\mathcal{N}_\omega$. Because μ is inner regular with respect to the closed sets (412D), we can find closed sets $F_{\xi n} \subseteq X \setminus E_\xi$ such that $\mu F_{\xi n} \geq \mu X - 2^{-n}$ for $\xi < \kappa$ and $n \in \mathbb{N}$. By 524Mb and (c) above, $\operatorname{wdistr}(\mathfrak{A}) \geq \operatorname{add}\mathcal{N}_\omega$, so there is a countable $C \subseteq \mathfrak{A}$ such that $F_{\xi n}^\bullet = \sup\{c : c \in C, c \subseteq F_{\xi n}^\bullet\}$ for every $n \in \mathbb{N}$ and $\xi < \kappa$ (514K). Again because μ is inner regular with respect to the closed sets, there is a sequence$\langle F_m\rangle_{m\in\mathbb{N}}$ of closed sets such that whenever $C' \subseteq C$ is finite then $\bar\mu(\sup C') = \sup\{\mu F_m : m \in \mathbb{N}, F_m^\bullet \subseteq \sup C'\}$. Consequently

$$\mu F_{\xi n} = \sup\{\mu F_m : m \in \mathbb{N}, F_m^\bullet \subseteq F_{\xi n}^\bullet\}$$

for every $\xi < \kappa$ and $n \in \mathbb{N}$. Set

$$H_m = X \cap \bigcap\{F_{\xi n} : n \in \mathbb{N}, \xi < \kappa, F_m^\bullet \subseteq F_{\xi n}^\bullet\}.$$

Applying (d) to $\{X \setminus F_{\xi n} : F_m^\bullet \subseteq F_{\xi n}^\bullet\}$, we see that $H_m^\bullet \supseteq F_m^\bullet$, that is, $F_m \setminus H_m$ is negligible, for each m.

Each H_m is closed; let $f_m : X \to [0,1]$ be a continuous function such that $H_m = f_m^{-1}[\{0\}]$. Set $f(x) = \langle f_m(x)\rangle_{m\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ for $x \in X$, and let ν be the restriction of the image measure μf^{-1} to the Borel σ-algebra of $[0,1]^{\mathbb{N}}$. Then $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}_\omega$ (apply 522V(a-i) to the atomless part of ν). For each $\xi < \kappa$ and $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $F_m^\bullet \subseteq F_{\xi n}^\bullet$ and $\mu F_m \ge \mu F_{\xi n} - 2^{-n} \ge \mu X - 2^{-n+1}$; now $H_m \subseteq F_{\xi n}$ is disjoint from E_ξ and $\mu H_m \ge \mu X - 2^{-n+1}$. So $\{z : z \in [0,1]^{\mathbb{N}},\ z(m) > 0\}$ includes $f[E_\xi]$ and has measure at most 2^{-n+1}. Accordingly $f[E_\xi]$ is ν-negligible.

As $\operatorname{add}\mathcal{N}(\nu) > \kappa$, $\bigcup_{\xi<\kappa} f[E_\xi]$ is ν-negligible; as f is inverse-measure-preserving, $\bigcup_{\xi<\kappa} E_\xi$ is μ-negligible; as $\langle E_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}_\omega$.

544L Corollary Let X be a metrizable space.

(a) If $\mathcal{U}\mathfrak{n}$ is the σ-ideal of universally negligible subsets of X, then $\operatorname{add}\mathcal{U}\mathfrak{n} \ge \operatorname{add}\mathcal{N}_\omega$.

(b) If Σ_{um} is the σ-algebra of universally measurable subsets of X, then $\bigcup\mathcal{E} \in \Sigma_{\mathrm{um}}$ whenever $\mathcal{E} \subseteq \Sigma_{\mathrm{um}}$ and $\#(\mathcal{E}) < \operatorname{add}\mathcal{N}_\omega$.

proof (a) Let $\mathcal{E} \subseteq \mathcal{U}\mathfrak{n}$ be a set of cardinal less than $\operatorname{add}\mathcal{N}_\omega$, and μ a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then $\mathcal{E} \subseteq \mathcal{N}(\mu)$; by 544K, $\bigcup\mathcal{E} \in \mathcal{N}(\mu)$; as μ is arbitrary, $\bigcup\mathcal{E} \in \mathcal{U}\mathfrak{n}$; as $\mathcal{E}$ is arbitrary, $\operatorname{add}\mathcal{U}\mathfrak{n} \ge \operatorname{add}\mathcal{N}_\omega$.

(b) Let μ be a totally finite Borel measure on X and $\hat\mu$ its completion. By 521Ad and 544K,

$$\operatorname{add}\hat\mu = \operatorname{add}\mathcal{N}(\hat\mu) = \operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}_\omega > \#(\mathcal{E}).$$

Since $\hat\mu$ measures every member of $\mathcal{E}$, it also measures $\bigcup\mathcal{E}$ (521Aa); as μ is arbitrary, $\bigcup\mathcal{E} \in \Sigma_{\mathrm{um}}$.

544M Theorem Let κ be an atomlessly-measurable cardinal. Then the following are equiveridical:

(i) for every family $\langle f_\xi\rangle_{\xi<\kappa}$ of regressive functions defined on $\kappa \setminus \{0\}$ there is a family $\langle \alpha_\xi\rangle_{\xi<\kappa}$ in κ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{\alpha_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(ii) for every family $\langle f_\xi\rangle_{\xi<\kappa}$ in $\mathbb{N}^\kappa$ there is a family $\langle m_\xi\rangle_{\xi<\kappa}$ in $\mathbb{N}$ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{m_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(iii) $\operatorname{cov}\mathcal{N}_\kappa > \kappa$;

(iv) $\operatorname{cov}\mathcal{N}(\mu) > \kappa$ whenever (X,μ) is a locally compact semi-finite measure space and $\mu X > 0$.

proof Let ν be a normal witnessing probability on κ.

(i)⇒(ii) Given a family $\langle f_\xi\rangle_{\xi<\kappa}$ as in (ii), apply (i) to $\langle f'_\xi\rangle_{\xi<\kappa}$ where $f'_\xi(\eta) = 0$ if $0 < \eta < \omega$, $f_\xi(\eta)$ if $\omega \le \eta < \kappa$.

(ii)⇒(iii) Let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a family in $\mathcal{N}_\kappa$. For each $\alpha < \kappa$ let $\langle F_{\alpha n}\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of $\{0,1\}^\kappa \setminus A_\alpha$ such that $\nu_\kappa(\bigcup_{n\in\mathbb{N}} F_{\alpha n}) = 1$. By 543G there is a function $h : \kappa \to \{0,1\}^\kappa$ which is inverse-measure-preserving for ν and ν_κ. Set $H_\alpha = h^{-1}(\bigcup_{n\in\mathbb{N}} F_{\alpha n})$; then $\nu H_\alpha = 1$. Let H be the diagonal intersection of $\langle H_\alpha\rangle_{\alpha<\kappa}$, so that $\nu H = 1$. Let $\langle\gamma_\xi\rangle_{\xi<\kappa}$ be the increasing enumeration of H.

For α, $\xi < \kappa$ set

$$\begin{aligned} f_\alpha(\xi) &= n \text{ if } h(\gamma_\xi) \in F_{\alpha n}, \\ &= 0 \text{ if } \gamma_\xi \notin H_\alpha. \end{aligned}$$

Then there is a family $\langle m_\alpha\rangle_{\alpha<\kappa}$ in $\mathbb{N}$ such that $\mathcal{E} = \{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\alpha^{-1}[\{m_\alpha\}] : \alpha < \kappa\}$ has the finite intersection property. Let $\mathcal{F} \supseteq \mathcal{E}$ be an ultrafilter. For any $\alpha < \kappa$ we have $H \setminus H_\alpha \subseteq \alpha + 1$, so that $\{\xi : \gamma_\xi \notin H_\alpha\}$ is bounded above in κ and cannot belong to $\mathcal{F}$. Consequently $\{\xi : h(\gamma_\xi) \in F_{\alpha,m_\alpha}\} \in \mathcal{F}$. But this implies at once that $\langle F_{\alpha,m_\alpha}\rangle_{\alpha<\kappa}$ has the finite intersection property; because all the $F_{\alpha n}$ are compact, there is a $y \in \bigcap_{\alpha<\kappa} F_{\alpha,m_\alpha}$, and now $y \notin \bigcup_{\alpha<\kappa} A_\alpha$.

Because $\langle A_\alpha\rangle_{\alpha<\kappa}$ was arbitrary, $\operatorname{cov}\mathcal{N}_\kappa > \kappa$.

(iii)⇒(iv) As in 544B, this follows from 523F and 521Jb.

(iv)⇒(i) Let $(Z,\tilde\nu)$ be the Stone space of the measure algebra $\mathfrak{A}$ of ν; for $A \subseteq \kappa$ let A^* be the open-and-closed subset of Z corresponding to the image $A^\bullet$ of A in $\mathfrak{A}$.

Now let $\langle f_\xi\rangle_{\xi<\kappa}$ be a family of regressive functions defined on $\kappa \setminus \{0\}$. Because $\mathcal{N}(\nu)$ is normal and ω_1-saturated and f_ξ is regressive, there is for each $\xi < \kappa$ a countable set $D_\xi \subseteq \kappa$ such that $\nu f_\xi^{-1}[D_\xi] = 1$ (541Ka). For ξ, $\eta < \kappa$ set $A_{\xi\eta} = f_\xi^{-1}[\{\eta\}]$; then $\nu(\bigcup_{\eta\in D_\xi} A_{\xi\eta}) = 1$ so (because D_ξ is countable) $\sup_{\eta\in D_\xi} A_{\xi\eta}^\bullet = 1$ in $\mathfrak{A}$ and $\tilde\nu(\bigcup_{\eta\in D_\xi} A_{\xi\eta}^*) = 1$. Set $E_\xi = Z \setminus \bigcup_{\eta\in D_\xi} A_{\xi\eta}^* \in \mathcal{N}(\tilde\nu)$. By (iv), $Z \ne \bigcup_{\xi<\kappa} E_\xi$; take $z \in Z \setminus \bigcup_{\xi<\kappa} E_\xi$. Then for every $\xi < \kappa$ there must be an $\alpha_\xi \in D_\xi$ such that $z \in A^*_{\xi,\alpha_\xi}$. But this implies that $\{A^*_{\xi,\alpha(\xi)} : \xi < \kappa\}$ is a centered family of open subsets of Z. It follows that $\{A^\bullet_{\xi,\alpha_\xi} : \xi < \kappa\}$ is centered in $\mathfrak{A}$. Since $\nu\zeta = 0$ for every $\zeta < \kappa$, $\{A_{\xi,\alpha_\xi} : \xi < \kappa\} \cup \{\kappa \setminus \zeta : \zeta < \kappa\}$ must have the finite intersection property, as required.

544N Cichoń's diagram and other cardinals (a) Returning to the concerns of Chapter 52, we see from the results above that any atomlessly-measurable cardinal κ is necessarily connected with the structures there. By 544B, $\kappa \le \operatorname{cov}\mathcal{N}_\lambda$ for every λ; by 544G, $\operatorname{non}\mathcal{N}_\omega = \omega_1$, so all the cardinals on the bottom line of Cichoń's diagram (522B), and therefore the Martin numbers $\mathfrak{m}$, $\mathfrak{p}$ etc. (522S), must be ω_1, while all the cardinals on the top line must be at least κ. From 522Tb we see also that $\mathrm{FN}(\mathcal{P}\mathbb{N})$ must be at least κ. Concerning $\mathfrak{b}$ and $\mathfrak{d}$, the position is more complicated.

(b) If κ is an atomlessly-measurable cardinal, then $\mathfrak{b} < \kappa$. **P?** Otherwise, we can choose inductively a family $\langle f_\xi\rangle_{\xi<\kappa}$ in $\mathbb{N}^\mathbb{N}$ such that $\{n : f_\xi(n) \le f_\eta(n)\}$ is finite whenever $\eta < \xi < \kappa$. Let ν be a witnessing probability measure on κ. For m, $i \in \mathbb{N}$ set $D_{mi} = \{\xi : \xi < \kappa,\ f_\xi(m) = i\}$. Then

$$W = \{(\xi,\eta) : \eta < \xi < \kappa\} = \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\{(\xi,\eta) : f_\eta(m) < f_\xi(m)\}$$
$$= \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\bigcup_{i<j} D_{mi}\times D_{mj}$$

belongs to $\mathcal{P}\kappa\widehat{\otimes}\mathcal{P}\kappa$. But also

$$\int \nu W[\{\xi\}]\nu(d\xi) = 0 < 1 = \int \nu W^{-1}[\{\eta\}]\nu(d\eta),$$

so this contradicts Fubini's theorem. **XQ**

(c) If κ is an atomlessly-measurable cardinal, then $\operatorname{cf}\mathfrak{d} \ne \kappa$. **P?** Otherwise, let $A \subseteq \mathbb{N}^\mathbb{N}$ be a cofinal set of cardinal $\mathfrak{d}$, and express A as $\bigcup_{\xi<\kappa} A_\xi$ where $\langle A_\xi\rangle_{\xi<\kappa}$ is non-decreasing and $\#(A_\xi) < \mathfrak{d}$ for every $\xi < \kappa$. For each $\xi < \kappa$, we have an $f_\xi \in \mathbb{N}^\mathbb{N}$ such that $f_\xi \not\le g$ for any $g \in A_\xi$. Let ν be a witnessing probability on κ. Then for each $n \in \mathbb{N}$ we have an $h(n) \in \mathbb{N}$ such that $\nu\{\xi : f_\xi(n) \ge h(n)\} \le 2^{-n-2}$. This defines a function $h \in \mathbb{N}^\mathbb{N}$. There must be a $g \in A$ such that $h \le g$; let $\zeta < \kappa$ be such that $g \in A_\zeta$. The set $\{\xi : f_\xi \le h\}$ has measure at least $\frac{1}{2}$, so there is some $\xi \ge \zeta$ such that

$$f_\xi \le h \le g \in A_\zeta \subseteq A_\xi,$$

contrary to the choice of f_ξ. **XQ**

(d) As for the cardinals studied in §523, I have already noted that $\operatorname{cov}\mathcal{N}_\lambda \ge \kappa$ for any atomlessly-measurable cardinal κ and any λ, and we can say something about the possibility that $\operatorname{cov}\mathcal{N}_\lambda = \kappa$ (544M).

544X Basic exercises (a) Let κ be an atomlessly-measurable cardinal, and ν a witnessing probability on κ. Show that there is a set $C \subseteq \{0,1\}^\kappa \times \kappa$ such that $\nu_\kappa C^{-1}[\{\xi\}] = 0$ for every $\xi < \kappa$, but $\nu^*_\kappa\{x : \nu C[\{x\}] = 1\} = 1$.

(b) Suppose that κ is an atomlessly-measurable cardinal. Show that $\mathbb{R}^\lambda$ is strongly measure-compact for every $\lambda < \kappa$. (*Hint*: 533J.)

(c) Let κ be a two-valued-measurable cardinal, $\mathcal{I}$ a normal maximal ideal of $\mathcal{P}\kappa$, (X,μ) a quasi-Radon probability space of weight strictly less than κ, and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ a function. Show that there is a $V \in \mathcal{I}$ such that $\bigcup\{f(I) : I \in [\kappa\setminus V]^{<\omega}\}$ is μ-negligible. (*Hint*: 541Xf.)

(d) In 544F, show that if the magnitude of μ is less than κ then there is a ν-conegligible $V \subseteq \kappa$ such that $X \setminus \bigcup_{I\in[V]^{<\omega}} f(I)$ has full outer measure in X.

(e) Suppose that there is an atomlessly-measurable cardinal. Show that every Radon measure on a first-countable compact Hausdorff space is uniformly regular. (*Hint*: 533Hb.)

(f) Suppose that κ is an atomlessly-measurable cardinal and that $2^\kappa = \kappa^{(+n+1)}$. Show that $\operatorname{non}\mathcal{N}_{2^{(\kappa^+)}} \le \kappa^{(+n)}$. (*Hint*: 523Ie.)

(g) Suppose that κ is an atomlessly-measurable cardinal and that (X,ρ) is a metric space. Show that no subset of X with strong measure zero can have cardinal κ.

(h) Let (X,Σ,μ) be a σ-finite measure space such that every subset of X^2 is measured by the c.l.d. product measure $\mu\times\mu$. Show that there is a countable subset of X with full outer measure. (*Hint*: if singletons are negligible, consider a well-ordering of X as a subset of X^2.)

(i) Let κ be an atomlessly-measurable cardinal, and G a countable group of bijections from κ to itself. Show that there is a non-zero σ-finite atomless G-invariant measure with domain $\mathcal{P}\kappa$.

544Y Further exercises (a) Let κ be a real-valued-measurable cardinal with witnessing probability ν. Give κ its discrete topology, so that ν is a Borel measure and $\kappa^{\mathbb{N}}$ is metrizable. Let λ be the Borel measure on $\kappa^{\mathbb{N}}$ constructed from ν by the method of 434Ym. (i) Show that if κ is atomlessly-measurable then $\operatorname{add}\mathcal{N}(\lambda) = \omega_1$. (ii) Show that if κ is two-valued-measurable then $\operatorname{add}\mathcal{N}(\lambda) = \kappa$.

(b) Show that $\mathfrak{c}$ does not have the property of 544M(ii).

(c) Show that a cardinal κ is weakly compact iff it is strongly inaccessible and has the property (i) of 544M.

544Z Problems (a) In 543C, can we replace '$w(X) < \operatorname{add}\nu$' with '$\tau(\mu) < \operatorname{add}\nu$'? More concretely, suppose that (Z, λ) is the Stone space of $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$, κ is an atomlessly-measurable cardinal and ν a witnessing probability on κ, so that

$$\tau(\lambda) = \omega < \operatorname{add}\nu \le \mathfrak{c} = w(Z).$$

Let $C \subseteq \kappa \times Z$ be such that $\lambda C[\{\xi\}] = 0$ for every $\xi < \kappa$. By 544C, we know that $\{z : \nu C^{-1}[\{z\}] > 0\}$ has inner measure zero. But does it have to be negligible?

(b) Suppose that κ is an atomlessly-measurable cardinal. Must there be a Sierpiński set $A \subseteq \{0,1\}^\omega$ of cardinal κ? (See 552E.)

(c) Suppose that κ is an atomlessly-measurable cardinal. Can $\operatorname{non}\mathcal{N}_\kappa$ be greater than ω_1? What if $\kappa = \mathfrak{c}$? (See 552E.)

(d) Can there be an atomlessly-measurable cardinal less than $\mathfrak{d}$? (See the notes to §555.)

(e) Can there be an atomlessly-measurable cardinal less than or equal to $\operatorname{shr}\mathcal{N}_\omega$? (See 555Yd.)

(f) Suppose that there is an atomlessly-measurable cardinal. Does it follow that $\operatorname{cov}\mathcal{N}_\omega = \mathfrak{c}$? (See 552Gc.)

544 Notes and comments The vocabulary of this section ('locally compact semi-finite measure space', 'quasi-Radon probability space of weight less than κ', 'compact probability space with Maharam type less than κ') makes significant demands on the reader, especially the r who really wants to know only what happens to Lebesgue measure. But the formulations I have chosen are not there just on the off-chance that someone may wish to apply the results in unexpected contexts. I have tried to use the concepts established earlier in this treatise to signal the nature of the arguments used at each stage. Thus in 543C we had an argument which depended on topological ideas, and could work only on a space with a base which was small compared with the atomlessly-measurable cardinal in hand; in 544C, the argument depends on an inverse-measure-preserving function from some power $\{0,1\}^\lambda$, so requires a compact measure, but then finds a Δ-nebula with a countable root-cover J, so that 543C can be applied to the usual measure on $\{0,1\}^J$, irrespective of the size of λ. Similar, but to my mind rather deeper, ideas lead from 544E to 544F. In both cases, there is a price to be paid for moving to spaces X of arbitrary complexity; in one, an inequality $\overline{\int}\int \le \int\overline{\int}$ becomes the weaker $\underline{\int}\int \le \int\overline{\int}$; in the other, a negligible set turns into a set of inner measure zero (544Xd).

Another way to classify the results here is to ask which of them depend on the Gitik-Shelah theorem. The formulae in 544H betray such a dependence; but it seems that the Gitik-Shelah theorem is also needed for the full strength of 544B, 544G, 544J and 544M as written. Historically this is significant, because the idea behind 544G was worked out by K.PrikryW and R.M.Solovay before it was known for sure that a witnessing measure on an atomlessly-measurable cardinal could not have countable Maharam type. However 544B and 544J, for instance, can be proved for Lebesgue measure without using the Gitik-Shelah theorem.

In the next chapter I will present a description of measure theory in random real models. Those already familiar with random real forcing may recognise some of the theorems of this section (544G, 544N) as versions of characteristic results from this theory (552E, 552C).

544M is something different. It was recognised in the 1960s that some of the ways in which two-valued-measurable cardinals are astonishing is that they are 'weakly Π^1_1-indescribable' (and, moreover, have many weakly Π^1_1-indescribable cardinals below them; see Fremlin 93, 4K). I do not give the 'proper' definition of weak Π^1_1-indescribability, which relies on concepts from model theory; you may find it in Levy 71, Baumgartner Taylor & Wagon 77 or Fremlin 93; for our purposes here, the equivalent combinatorial definition in 544M(i) will I think suffice. For strongly inaccessible cardinals, it is the same thing as weak compactness (544Yc). Here I mention it only because it turns out to be related to one of the standard questions I have been asking in this volume (544M(iii)). Of course the arguments above beg the question, whether an atomlessly-measurable cardinal can be weakly Π^1_1-indescribable, especially in view of 544Yb; see Fremlin 93, 4R.

In 544K-544L I look at a question which seems to belong in Chapter 52, or perhaps with the corresponding result in Hausdorff measures (534Bb). But unless I am missing something, the facts here depend on the Gitik-Shelah theorem via 544Ha.

This section has a longer list of problems than most. In the last four sections I have tried to show something of the richness of the structures associated with any atomlessly-measurable cardinal; I remain quite uncertain how much more we can hope to glean from the combinatorial and measure-theoretic arguments available. The problems of this chapter mostly have a special status. They are of course vacuous unless we suppose that there is an atomlessly-measurable cardinal; but there is something else. There is a well-understood process, 'Solovay's method', for building models of set theory with atomlessly-measurable cardinals from models with two-valued-measurable cardinals (§555). In most cases, the problems have been solved for such models, and perhaps they should be regarded as challenges to develop new forcing techniques.

545 PMEA and NMA

One of the reasons for supposing that it is consistent to assume that there are measurable cardinals is that very much stronger axioms have been studied at length without any contradiction appearing. Here I mention two such axioms which have obvious consequences in measure theory.

545A Theorem The following are equiveridical:

(i) for every cardinal λ, there is a probability space $(X,\mathcal{P}X,\mu)$ with $\tau(\mu)\ge\lambda$ and $\operatorname{add}\mu\ge\mathfrak{c}$;

(ii) for every cardinal λ, there is an extension of the usual measure ν_λ on $\{0,1\}^\lambda$ to a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}(\{0,1\}^\lambda)$;

(iii) for every semi-finite locally compact measure space (X,Σ,μ) (definition: 342Ad), there is an extension of μ to a $\mathfrak{c}$-additive measure with domain $\mathcal{P}X$.

proof (i)$\Rightarrow$(ii) Assume (i). Let λ be a cardinal; of course (ii) is surely true for finite λ, so we may take it that $\lambda\ge\omega$. Let $(X,\mathcal{P}X,\mu)$ be a probability space with Maharam type at least λ^+ and with $\operatorname{add}\mu\ge\mathfrak{c}$. Taking $\mathfrak{A}$ to be the measure algebra of μ, there is an $a\in\mathfrak{A}$ such that the principal ideal $\mathfrak{A}_a$ it generates is homogeneous with Maharam type at least λ (332S). Let $E\in\mathcal{P}X$ be such that $E^\bullet=a$, so that the subspace measure μ_E is Maharam-type-homogeneous with Maharam type at least λ. Setting $\mu'A=\mu A/\mu E$ for $A\subseteq E$, $(E,\mathcal{P}E,\mu')$ is a Maharam-type-homogeneous probability space with Maharam type at least λ, and $\operatorname{add}\mu'\ge\operatorname{add}\mu\ge\mathfrak{c}$. By 343Ca, there is a function $f:E\to\{0,1\}^\lambda$ which is inverse-measure-preserving for μ' and ν_λ. Now the image measure $\nu=\mu'f^{-1}$ is a $\mathfrak{c}$-additive extension of ν_λ to $\mathcal{P}(\{0,1\}^\lambda)$.

(ii)$\Rightarrow$(iii) Assume (ii).

(α) Suppose that (X,Σ,μ) is a compact probability space. Set $\lambda=\max(\omega,\tau(\mu))$. Then there is an inverse-measure-preserving function $f:\{0,1\}^\lambda\to X$ (343Cd[4]). If ν is a $\mathfrak{c}$-additive extension of ν_λ to $\mathcal{P}(\{0,1\}^\lambda)$, then νf^{-1} is a $\mathfrak{c}$-additive extension of μ to $\mathcal{P}X$.

(β) Let (X,Σ,μ) be any semi-finite locally compact measure space. Set $\Sigma^{f+}=\{E:E\in\Sigma,\ 0<\mu E<\infty\}$ and let $\mathcal{E}\subseteq\Sigma^{f+}$ be maximal subject to $E\cap F$ being negligible for all distinct E, $F\in\Sigma$. If $H\in\Sigma$ and $\mu H<\infty$, then $\mathcal{H}=\{E:E\in\mathcal{E},\ \mu(E\cap H)>0\}$ is countable and $E\setminus\bigcup\mathcal{H}$ is negligible, so $\mu H=\sum_{E\in\mathcal{E}}\mu(E\cap H)$; because μ is semi-finite, $\mu H=\sum_{E\in\mathcal{E}}\mu(E\cap H)$ for every $H\in\Sigma$.

For each $E\in\mathcal{E}$, the subspace measure μ_E is compact; applying (α) to a normalization of μ_E, we have an extension μ'_E of μ_E to a $\mathfrak{c}$-additive measure with domain $\mathcal{P}E$. Set $\mu'A=\sum_{E\in\mathcal{E}}\mu'_E(A\cap E)$ for $A\subseteq X$; then $\mu':\mathcal{P}X\to[0,\infty]$ is a $\mathfrak{c}$-additive measure extending μ.

(iii)$\Rightarrow$(ii) and **(ii)$\Rightarrow$(i)** are trivial.

545B Definition PMEA (the '**product measure extension axiom**') is the assertion that the statements (i)-(iii) of 545A are true.

545C Proposition If PMEA is true, then $\mathfrak{c}$ is atomlessly-measurable.

proof By 545A(ii) we have an extension of the usual measure on $\{0,1\}^\omega$ to a $\mathfrak{c}$-additive measure μ with domain $\mathcal{P}(\{0,1\}^\omega)$. Since μ is zero on singletons, $\operatorname{add}\mu=\mathfrak{c}$ exactly, so 543Ba and 543Bc tell us that $\mathfrak{c}$ is real-valued-measurable, therefore atomlessly-measurable.

[4] Later editions only.

545D Definition NMA (the '**normal measure axiom**') is the statement

For every set I there is a $\mathfrak{c}$-additive probability measure ν on $S=[I]^{<\mathfrak{c}}$, with domain $\mathcal{P}S$, such that

(α) $\nu\{s : i \in s \in S\} = 1$ for every $i \in I$,

(β) if $A \subseteq S$, $\nu A > 0$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then there is an $i \in I$ such that $\nu\{s : s \in A,\, f(s) = i\} > 0$.

545E Proposition NMA implies PMEA.

proof Assume NMA. Let λ be any cardinal. Let κ be a regular infinite cardinal greater than the cardinal power λ^ω, and ν a $\mathfrak{c}$-additive probability on $[\kappa]^{<\mathfrak{c}}$ as in 545D. For $\xi < \kappa$ define $f_\xi : [\kappa]^{<\mathfrak{c}} \to \mathfrak{c}$ by setting $f_\xi(s) = \operatorname{otp}(s \cap \xi)$ for every $s \in [\kappa]^{<\mathfrak{c}}$. Then if $\xi < \eta < \kappa$ we have $f_\xi(s) < f_\eta(s)$ whenever $\xi \in s$, that is, for ν-almost every s.

Let $g : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be any injection. For $\xi < \kappa$ and $n \in \mathbb{N}$ let $a_{\xi n}$ be the equivalence class $\{s : n \in g(f_\xi(s))\}^\bullet$ in the measure algebra $\mathfrak{A}$ of ν. If $\xi < \eta < \kappa$ then $g(f_\xi(s)) \ne g(f_\eta(s))$ for ν-almost every s, so $\sup_{n\in\mathbb{N}} a_{\xi n} \triangle a_{\eta n} = 1$ in $\mathfrak{A}$ and there is an $n \in \mathbb{N}$ such that $a_{\xi n} \ne a_{\eta n}$. Accordingly $\#(\mathfrak{A})^\omega \ge \kappa > \lambda^\omega$ and $\#(\mathfrak{A}) > \lambda^\omega$. As $\mathfrak{A} \le \max(4, \tau(\mathfrak{A})^\omega)$ (4A1O/514De), $\tau(\mathfrak{A}) > \lambda$. So ν witnesses that 545A(i) is true of λ.

545F Proposition Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that whenever $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ there is a subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$, including s, with a strictly positive countably additive functional. Then there is a strictly positive countably additive functional on $\mathfrak{A}$.

Remark For the definition and elementary properties of countably additive functionals on arbitrary Boolean algebras, see §326.

proof Of course we can suppose that $\mathfrak{A} \ne \{0\}$. Let ν be a $\mathfrak{c}$-additive probability on $S = [\mathfrak{A}]^{<\mathfrak{c}}$ as in 545D. For each $s \in S$, let $\mathfrak{B}_s$ be a subalgebra of $\mathfrak{A}$ including s with a strictly positive countably additive functional μ_s. Normalizing μ_s if necessary, we may suppose that $\mu_s 1 = 1$. Now, for $a \in \mathfrak{A}$, set $\mu(a) = \int \mu_s(a)\nu(ds)$; because $a \in s \subseteq \mathfrak{B}_s = \operatorname{dom}\mu_s$ for ν-almost every s, the integral is well-defined. Because every μ_s is additive, so is μ; because every μ_s is strictly positive, so is μ. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty} \mu_s(a_n) = 0$ whenever $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ contains every a_n, that is, for ν-almost every s; so $\lim_{n\to\infty} \mu a_n = 0$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is countably additive (326Ga).

545G Corollary Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that every $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ is included in a subalgebra of $\mathfrak{A}$ which is, in itself, a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

proof (a) Because $\mathfrak{c}$ is atomlessly-measurable, it is surely greater than ω_1 (419G/438Cd/542C). So a family in $\mathfrak{A} \setminus \{0\}$ of cardinal ω_1 lies within some measurable subalgebra of $\mathfrak{A}$ and cannot be disjoint. Thus $\mathfrak{A}$ is ccc.

(b) If $A \subseteq \mathfrak{A}$, set

$$D_1 = \{d : d \in \mathfrak{A},\, d \subseteq a \text{ for some } a \in A\}, \quad D_2 = \{d : d \in \mathfrak{A},\, d \cap a = 0 \text{ for every } a \in A\}.$$

Then $D_1 \cup D_2$ is order-dense in $\mathfrak{A}$ so includes a partition D of unity in $\mathfrak{A}$. By (a), D is countable, so lies within a measurable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Now $D \cap D_1$ has a supremum b in $\mathfrak{B}$ which is disjoint from every member of $D \cap D_2$. But this means that b is the supremum of A in $\mathfrak{A}$. As A is arbitrary, $\mathfrak{A}$ is Dedekind complete.

(c) By 545F, $\mathfrak{A}$ has a strictly positive countably additive functional μ; but now $(\mathfrak{A}, \mu)$ is a totally finite measure algebra.

545X Basic exercises (a) Suppose that I is a set, and that ν is a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}([I]^{<\mathfrak{c}})$ satisfying the conditions of 545D. Suppose that $A \subseteq [I]^{<\mathfrak{c}}$ and $f : A \to I$ are such that $f(s) \in s$ for every $s \in A$. Show that there is a countable set $D \subseteq I$ such that $f(s) \in D$ for ν-almost every $s \in A$.

545Y Further exercises (a) Suppose that I is a set, and that ν is a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}S$, where $S = [I]^{<\mathfrak{c}}$, satisfying the conditions of 545D. Suppose that $f : [I]^{<\omega} \to S$ is any function. Show that $\nu\{s : s \in S,\, f(J) \subseteq s \text{ for every } J \in [s]^{<\omega}\} = 1$.

(b) Suppose that NMA is true. Show that $\square_\lambda$ is false for every $\lambda \ge \mathfrak{c}$. (Cf. 555Ye below.)

545 Notes and comments I have given the sketchiest of accounts here. The main interest of PMEA and NMA has so far been in their remarkable consequences in general topology and (for NMA) its associated reflection principles; see Fremlin 93 and the references there. 545G is such a reflection principle. Note that the measurable subalgebras declared to exist need not be regularly embedded in the given algebra. For K.Prikry's theorem that it is consistent to assume NMA if it is consistent to suppose that there is a supercompact cardinal, see 555P below.

546 Power set σ-quotient algebras

One way of interpreting the Gitik-Shelah theorem (543E) is to say that it shows that 'simple' atomless probability algebras cannot be of the form $\mathcal{P}X/\mathcal{N}(\mu)$. Similarly, the results of §541-§542 show that any ccc Boolean algebra expressible as the quotient of a power set by a non-trivial σ-ideal involves us in dramatic complexities, though it is not clear that these must appear in the quotient algebra itself. In this section I give two further results of M.Gitik and S.Shelah showing that certain algebras cannot appear in this way. I try to present the ideas in a form which leads naturally to some outstanding questions (546Z).

546A (a) Definition A **power set σ-quotient algebra** is a Boolean algebra which is isomorphic to an algebra of the form $\mathcal{P}T/\mathcal{J}$ where T is a set and $\mathcal{J}$ is a σ-ideal of subsets of T.

(b) I recall some notation which I will use in this section. If X is a topological space, $\mathcal{B}(X)$ will be its Borel σ-algebra, $\mathcal{B}\mathfrak{a}(X)$ its Baire σ-algebra, $\widehat{\mathcal{B}}(X)$ its Baire-property algebra, and $\mathcal{M}(X)$ the σ-ideal of meager sets in X. If (X, Σ, μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ.

546B Lemma (a) Any power set σ-quotient algebra is Dedekind σ-complete.

(b) If $\mathfrak{A}$ is a power set σ-quotient algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective sequentially order-continuous Boolean homomorphism, then $\mathfrak{B}$ is a power set σ-quotient algebra. In particular, any principal ideal of a power set σ-quotient algebra is a power set σ-quotient algebra.

(c) The simple product of any family of power set σ-quotient algebras is a power set σ-quotient algebra.

(d) If a non-zero atomless measurable algebra $\mathfrak{A}$ is a power set σ-quotient algebra, there is an atomlessly-measurable cardinal κ such that $\tau(\mathfrak{A}) \geq \min(\kappa^{(+\omega)}, 2^\kappa)$.

proof (a) 314C.

(b) Observe that by 313Qb a Boolean algebra $\mathfrak{A}$ is a power set σ-quotient algebra iff there are a set T and a surjective sequentially order-continuous Boolean homomorphism from $\mathcal{P}T$ onto $\mathcal{A}$. It follows immediately that an image of such an algebra under a sequentially order-continuous homomorphism is again a power set σ-quotient algebra. And of course a principal ideal of $\mathfrak{A}$ is an image of $\mathfrak{A}$ under a homomorphism $a \mapsto a \cap c$ which is actually order-continuous.

(c) If $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a family of power set σ-quotient algebras, then for each $i \in I$ we have a set T_i and a surjective sequentially order-continuous Boolean homomorphism $\phi_i : \mathcal{P}T_i \to \mathfrak{A}_i$. We can arrange that the T_i are disjoint; now $A \mapsto \langle \phi_i(A \cap T_i) \rangle_{i \in I} : \mathcal{P}T \to \prod_{i \in I} \mathfrak{A}_i$ is a surjective sequentially order-continuous Boolean homomorphism, so $\prod_{i \in I} \mathfrak{A}_i$ is a power set σ-quotient algebra.

(d) Let $\bar{\mu}$ be a measure on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{\mu})$ is a probability measure. Let T be a set and $\phi : \mathcal{P}T \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism. Then we have a probability measure $\nu = \bar{\mu}\phi$ with domain $\mathcal{P}T$, and the measure algebra of ν is isomorphic to $\mathfrak{A}$. As ν is atomless, $\nu\{t\} = 0$ for every $t \in T$. So $\kappa = \operatorname{add} \nu$ is real-valued-measurable (543Ba); since $\operatorname{add} \nu \leq \operatorname{cov} \mathcal{N}(\nu) \leq \mathfrak{c}$ (521G), κ is atomlessly-measurable (543Bc). And $\tau(\mathfrak{A}) \geq \max(\kappa^{(+\omega)}, 2^\kappa)$ by 543F.

546C An elementary construction will be used more than once below, and it may be less distracting if it is spelt out here.

Lemma Suppose that I is a set and $\mathcal{I}$ a σ-ideal of subsets of $X = \{0,1\}^I$ which is generated by $\mathcal{I} \cap \mathcal{B}\mathfrak{a}(X)$. Suppose that Σ is a σ-algebra of subsets of X such that

$$\mathcal{B}\mathfrak{a}(X) \subseteq \Sigma \subseteq \{E \triangle A : E \in \mathcal{B}\mathfrak{a}(X),\, A \in \mathcal{I}\},$$

and set $\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{I}$.

(a) If T is a set, $\mathcal{J}$ a σ-algebra of subsets of T and $\phi : \mathfrak{A} \to \mathcal{P}T/\mathcal{J}$ is a sequentially order-continuous Boolean homomorphism, there is a function $f : T \to X$ such that $f^{-1}[E]^\bullet = \phi(E^\bullet)$ for every $E \in \Sigma$.

(b) Now suppose that ϕ is injective. Set $\mathcal{I}^* = \{A : A \subseteq X,\ f^{-1}[A] \in \mathcal{J}\}$. Then $\mathcal{I}^*$ is a σ-ideal of subsets of X including $\mathcal{I}$, and $\Sigma \cap \mathcal{I}^* = \Sigma \cap \mathcal{I}$.

(c) If ϕ is an isomorphism then for every $A \subseteq X$ there is an $E \in \Sigma$ such that $A \triangle E \in \mathcal{I}^*$. So we have an isomorphism between $\mathfrak{A}$ and $\mathcal{P}X/\mathcal{I}^*$ obtained by mapping $E^\bullet$ (interpreted in $\Sigma/\Sigma \cap \mathcal{I}$) into $E^\bullet$ (interpreted in $\mathcal{P}X/\mathcal{I}^*$) for every $E \in \Sigma$.

(d) If ϕ is an isomorphism and $X = \bigcup \mathcal{I} \notin \mathcal{I}$, set $\kappa = \operatorname{add} \mathcal{I}^*$. Then there is a κ-additive ideal $\mathcal{J}^*$ of subsets of κ, containing singletons, such that $\mathcal{P}\kappa/\mathcal{J}^*$ is isomorphic to a σ-subalgebra of a non-zero principal ideal of $\mathfrak{A}$.

(e) If, moreover, $\mathfrak{A}$ is atomless and ccc, then $\kappa \leq \mathfrak{c}$ and $\mathcal{P}\kappa/\mathcal{J}^*$ is atomless; and we can arrange that $\mathcal{J}^*$ should be a normal ideal.

proof Recall that $\mathcal{B}\mathfrak{a}(X)$ is just the σ-algebra generated by $\{E_i : i \in I\}$ where $E_i = \{x : x \in X,\, x(i) = 1\}$ for each $i \in I$ (4A3N).

(a) For each $i \in I$, let $F_i \subseteq T$ be such that $F_i^\bullet = \phi E_i^\bullet$ in $\mathcal{P}T/\mathcal{J}$. Define $f : T \to X$ by setting $f(t)(i) = 1$ if $t \in F_i$, 0 if $t \in X \setminus F_i$. Then $\Sigma_0 = \{E : E \in \Sigma,\, f^{-1}[E]^\bullet = \phi E^\bullet$ is a σ-ideal of subsets of X containing every E_i and therefore including $\mathcal{B}\mathfrak{a}(X)$. If $A \in \mathcal{I}$, there is an $E \in \mathcal{B}\mathfrak{a}(X) \cap \mathcal{I}$ including A, so

$$f^{-1}[A]^\bullet \subseteq f^{-1}[E]^\bullet = \phi E^\bullet = 0$$

and $f^{-1}[A] \in \mathcal{J}$. If E is any member of Σ, there is an $E_0 \in \mathcal{B}\mathfrak{a}(X)$ such that $E \triangle E_0 \in \mathcal{I}$, so that $f^{-1}[E] \triangle f^{-1}[E_0] \in \mathcal{J}$ and

$$f^{-1}[E]^\bullet = f^{-1}[E_0]^\bullet = \phi E_0^\bullet = \phi E^\bullet.$$

(b) Of course $\mathcal{I}^*$ is a σ-ideal of subsets of X, and I have already noted that $\mathcal{I} \subseteq \mathcal{I}^*$. If ϕ is injective then, for $E \in \Sigma$,

$$\begin{aligned} E \in \mathcal{I}^* &\iff f^{-1}[E] \in \mathcal{J} \iff f^{-1}[E]^\bullet = 0 \\ &\iff \phi E^\bullet = 0 \iff E^\bullet = 0 \iff E \in \mathcal{I}. \end{aligned}$$

So $\Sigma \cap \mathcal{I}^* = \Sigma \cap \mathcal{I}$.

(c) If $A \subseteq X$ is any set, consider $a = \phi^{-1}(f^{-1}[A]^\bullet) \in \mathfrak{A}$. Let $E \in \Sigma$ be such that $E^\bullet$ (interpreted in $\Sigma/\Sigma \cap \mathcal{I}$) is equal to a. Then $f^{-1}[E]^\bullet = f^{-1}[A]^\bullet$, that is, $f^{-1}[E \triangle A] \in \mathcal{J}$ and $E \triangle A \in \mathcal{I}^*$.

Because $\Sigma \cap \mathcal{I}^* = \Sigma \cap \mathcal{I}$, the proposed assignment gives us an injective Boolean homomorphism $\psi : \mathfrak{A} \to \mathcal{P}X/\mathcal{I}^*$; and we have just seen that every subset of X is equivalent, mod $\mathcal{I}^*$, so some member of Σ. This shows that ψ is surjective, therefore an isomorphism, as claimed.

(d) If $X = \bigcup \mathcal{I}$ then $X = \bigcup \mathcal{I}^*$ because $\mathcal{I} \subseteq \mathcal{I}^*$; if $X \notin \mathcal{I}$ then $X \notin \mathcal{I}^*$ because $X \in \Sigma$. So $\kappa = \operatorname{add} \mathcal{I}^*$ is not ∞. Let $\langle A_\xi \rangle_{\xi < \kappa}$ be a family in $\mathcal{I}^*$ with union $A \notin \mathcal{I}^*$. Define $g : A \to \kappa$ by setting $g(x) = \min\{\xi : x \in A_\xi\}$ for $x \in A$. Set

$$\mathcal{J}^* = \{B : B \subseteq \kappa,\, g^{-1}[B] \in \mathcal{I}^*\} = \{B : B \subseteq \kappa,\, (gf)^{-1}[B] \in \mathcal{J}\}.$$

Then $\mathcal{J}^*$ is a proper κ-additive ideal of subsets of κ containing singletons. The map $B \mapsto g^{-1}[B] : \mathcal{P}\kappa \to \mathcal{P}A$ induces an injective sequentially order-continuous Boolean homomorphism from $\mathcal{P}\kappa/\mathcal{J}^*$ to the principal ideal of $\mathcal{P}X/\mathcal{I}^* \cong \mathfrak{A}$ generated by $A^\bullet$, so we have a sequentially order-continuous embedding of $\mathcal{P}\kappa/\mathcal{J}^*$ into a principal ideal of $\mathfrak{A}$, necessarily non-zero. By 314F(b-i), the image of $\mathcal{P}\kappa/\mathcal{J}^*$ is a σ-subalgebra of the principal ideal.

(e) If $\mathfrak{A} \cong \mathcal{P}X/\mathcal{I}^*$ is atomless and ccc, then 541O tells us that $\kappa \le \mathfrak{c}$. So $\mathcal{P}\kappa/\mathcal{J}^*$ will be atomless, by 541P.

Returning to the argument of (d), if $\mathfrak{A}$ is ccc we have the option of using 541J to give us a function $g : A \to \kappa$ such that $\mathcal{J}^* = \{B : B \subseteq \kappa,\, g^{-1}[B] \in \mathcal{I}^*\}$ is a normal ideal, and then proceeding as before.

546D In 527M I introduced 'harmless' algebras. Here we need to know a little about harmless power set σ-quotient algebras.

Lemma Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Then $\mathfrak{A} = \mathcal{P}(\kappa \times \kappa)/\mathcal{I} \ltimes \mathcal{I}$ (definition: 527Ba) is harmless.

proof (a) If $\kappa = \omega_1$ then $\mathcal{I}$ is of the form $\{I : A \cap I = \emptyset\}$ for some countable $A \subseteq \kappa$. **P** Set $A = \{\xi : \xi < \omega_1,\, \{\xi\} \notin \mathcal{I}\}$. Because $\mathcal{P}\omega_1/\mathcal{I}$ is harmless, it is ccc, so A is countable. Set $\mathcal{J} = \mathcal{I} \cap \mathcal{P}(\omega_1 \setminus A)$, so that $\mathcal{J}$ is an σ-ideal of subsets of $\omega_1 \setminus A$ containing singletons, and $\mathcal{P}(\omega_1 \setminus A)/\mathcal{J}$ can be identified with the principal ideal of $\mathcal{P}\omega_1/\mathcal{I}$ generated by $(\omega_1 \setminus A)^\bullet$, so is ccc. Since ω_1 is certainly not weakly inaccessible, $\mathcal{J}$ is not a proper ideal, by 541L, and $\omega_1 \setminus A \in \mathcal{I}$. It follows that $\mathcal{I} = \mathcal{P}(\omega_1 \setminus A)$, as stated. **Q**

In this case, $\mathfrak{A} \cong \mathcal{P}(A \times A)$ has a countable π-base and is harmless, by 527Nd. So let us suppose from now on that $\kappa > \omega_1$.

(b) By 527Bb, $\mathcal{I} \ltimes \mathcal{I}$ is κ-additive; in particular, it is a σ-ideal. Next, it is ω_1-saturated. **P** Let $\langle V_\alpha \rangle_{\alpha < \omega_1}$ be a disjoint family in $\mathcal{P}(\kappa \times \kappa)$. For each $\xi < \kappa$, there is an $\alpha_\xi < \omega_1$ such that $V_\alpha[\{\xi\}] \in \mathcal{I}$ for every $\alpha \ge \alpha_\xi$. Because $\mathcal{I}$ is ω_2-additive and ω_1-saturated, there is an $\alpha^* < \omega_1$ such that $\{\xi : \alpha_\xi > \alpha^*\} \in \mathcal{I}$ (541E). But now $V_\alpha \in \mathcal{I} \ltimes \mathcal{I}$ for every $\alpha > \alpha^*$. **Q**

Consequently $\mathfrak{A}$ is Dedekind complete (541B).

(c) Let $\mathfrak{C}$ be an order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type; let $\langle C_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of $\kappa \times \kappa$ such that $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $\{C_n^\bullet : n \in \mathbb{N}\}$. For each $\xi < \kappa$, let $\mathfrak{B}_\xi$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{C_n[\{\xi\}]^\bullet : n \in \mathbb{N}\}$. By 527Nb, $\mathfrak{B}_\xi$ has countable π-weight; let $\mathcal{E}_\xi$ be a countable subset of $\mathcal{P}\kappa$ such that $\{E^\bullet : E \in \mathcal{E}_\xi\}$ is an order-dense subset of $\mathfrak{B}_\xi$; let $\langle E_{\xi n} \rangle_{n \in \mathbb{N}}$ run over $\mathcal{E}_\xi$. We can of course suppose that $C_n[\{\xi\}] \in \mathcal{E}_\xi$ and that $E_{\xi,2n} = C_n[\{\xi\}]$ for each n. For $n \in \mathbb{N}$ set $E_n = \{(\xi, \eta) : \xi < \kappa,\, \eta \in E_{\xi n}\}$;

we have $E_{2n}=C_n$ for each n. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{\{\xi:E_{\xi m}\cap E_{\xi n}\in\mathcal{I}\}^\bullet:m,n\in\mathbb{N}\}$, and $\mathcal{F}$ a countable subset of $\mathcal{P}\kappa$, containing κ, such that $\{F^\bullet:F\in\mathcal{F}\}$ is an order-dense set in $\mathfrak{B}$. Let $\mathcal{A}$ be the countable set $\{\emptyset\}\cup\{E_m\cap(F\times\kappa):m\in\mathbb{N},F\in\mathcal{F}\}$.

(d) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{A}$, there is a sequence $\langle\hat{A}_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}}\hat{A}_n^\bullet$ is the complement of $\sup_{n\in\mathbb{N}}A_n^\bullet$ in $\mathfrak{A}$. **P** Express A_n as $E_{m_n}\cap(F_n\times\kappa)$ where $m_n\in\mathbb{N}$ and $F_n\in\mathcal{F}$ for each n. Set $W=\bigcup_{n\in\mathbb{N}}A_n$. If $\xi<\kappa$ then $W[\{\xi\}]=\bigcup_{n\in\mathbb{N},\xi\in F_n}E_{\xi,m_n}$, so $W[\{\xi\}]^\bullet\in\mathfrak{B}_\xi$; set $J_\xi=\{j:W[\{\xi\}]\cap E_{\xi j}\in\mathcal{I}\}$. If $G_j=\{\xi:\xi<\kappa,j\in J_\xi\}$ for each j and $\hat{W}=\bigcup_{j\in\mathbb{N}}E_j\cap(G_j\times\kappa)$, $W[\{\xi\}]^\bullet$ and $\hat{W}[\{\xi\}]^\bullet$ are complementary elements of $\mathfrak{B}_\xi$ for each ξ, so $W^\bullet$ and $\hat{W}^\bullet$ are complementary elements of $\mathfrak{A}$.

Now

$$G_j=\{\xi:W[\{\xi\}]\cap E_{\xi j}\in\mathcal{I}\}=\bigcap_{n\in\mathbb{N}}\{\xi:\xi\notin F_n\text{ or }E_{\xi m_n}\cap E_{\xi j}\in\mathcal{I}\},$$

so $G_j^\bullet\in\mathfrak{B}$ and there is an $\mathcal{F}_j\subseteq\mathcal{F}$ such that $G_j^\bullet=\sup_{F\in\mathcal{F}_j}F^\bullet$. Taking $\langle\hat{A}_n\rangle_{n\in\mathbb{N}}$ to run over $\{\emptyset\}\cup\{E_j\cap(F\times\kappa):j\in\mathbb{N},F\in\mathcal{F}_j\}$, we get a sequence in $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}}\hat{A}_n^\bullet=\hat{W}^\bullet$, as required. **Q**

(e) It follows that if we take $\mathfrak{D}$ to be the set of those $a\in\mathfrak{A}$ expressible in the form $\sup_{n\in\mathbb{N}}A_n^\bullet$ for some sequence in $\mathcal{A}$, the complement of an element of $\mathfrak{D}$ belongs to $\mathfrak{D}$; as $\mathfrak{D}$ is certainly closed under countable suprema, it is a σ-subalgebra of $\mathfrak{A}$, therefore order-closed, because $\mathfrak{A}$ is ccc. And $\{A_n^\bullet:n\in\mathbb{N}\}$ witnesses that $\pi(\mathfrak{D})\le\omega$.

As $C_n=E_{2n}\cap(\kappa\times\kappa)\in\mathcal{A}$ for each n, $\mathfrak{C}\subseteq\mathfrak{D}$. So $\omega\ge\pi(\mathfrak{D})\ge\pi(\mathfrak{C})$, by 514Eb.

As $\mathfrak{C}$ is an arbitrary countably generated order-closed subalgebra of $\mathfrak{A}$, $\mathfrak{A}$ is harmless, by 527Nb in the other direction.

546E I wish to follow the lines of the argument in 543C-543E to prove a similar theorem in which 'measure' is replaced by 'category'. The lemma just proved corresponds to the definition of $\tilde{\nu}$ in part (c) of the proof of 543D. The next result will play the role previously taken by 543C.

Proposition Suppose that κ is an uncountable regular cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Let X be a ccc topological space of π-weight less than κ. Then $\mathcal{M}(X)\rtimes\mathcal{I}\subseteq\mathcal{M}(X)\ltimes\mathcal{I}$.

proof (a) Take $C\in\mathcal{M}(X)\rtimes\mathcal{I}$. Set $A=\{\xi:\xi<\kappa,C^{-1}[\{\xi\}]\notin\mathcal{M}(X)\}$ and $B=\{x:x\in X,C[\{x\}]\notin\mathcal{I}\}$, so that $A\in\mathcal{I}$ and I need to show that $B\in\mathcal{M}(X)$. For $\xi\in\kappa\setminus A$, let $\langle F_{\xi n}\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets in X with union $C^{-1}[\{\xi\}]$; for $\xi\in A$ set $F_{\xi n}=\emptyset$ for every n. For each n, set $C_n=\{(x,\xi):\xi<\kappa,x\in F_{\xi n}\}$ and $B_n=\{x:C_n[\{x\}]\notin\mathcal{I}\}$, so that $B=\bigcup_{n\in\mathbb{N}}B_n$ and it will be enough to show that every B_n is meager. Fix $n\in\mathbb{N}$.

(b) Let $\langle G_\alpha\rangle_{\alpha<\pi(X)}$ enumerate a π-base for the topology of X, and for $\alpha<\pi(X)$ let D_α be the set of those $\xi<\kappa$ such that $G_\alpha\cap F_{\xi n}=\emptyset$. Then $W=\bigcup_{\alpha<\pi(X)}G_\alpha\times D_\alpha$ is disjoint from C_n. For each $\xi<\kappa$, set $I_\xi=\{\alpha:\alpha<\pi(X),\xi\in D_\alpha\}$; then $\bigcup_{\alpha\in I_\xi}G_\alpha$ is dense in X. Because X is ccc, there is a countable $J_\xi\subseteq I_\xi$ such that $\bigcup_{\alpha\in J_\xi}G_\alpha$ is dense (5A4Bd). Now $\mathcal{I}$ is ω_1-saturated and $\pi(X)<\operatorname{add}\mathcal{I}$, so there is a countable $I\subseteq\pi(X)$ such that $A'=\{\xi:\xi<\kappa,J_\xi\not\subseteq I\}$ belongs to $\mathcal{I}$ (541D).

(c) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{D_\alpha^\bullet:\alpha\in I\}$. Because $\mathcal{P}\kappa/\mathcal{I}$ is harmless, $\pi(\mathfrak{B})\le\omega$; let $\langle F_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathcal{P}\kappa$ such that $\{F_i^\bullet:i\in\mathbb{N}\}$ is order-dense in $\mathfrak{B}$. Let $\mathcal{E}$ be the countable subalgebra of $\mathcal{P}\kappa$ generated by $\{F_i:i\in\mathbb{N}\}\cup\{D_\alpha:\alpha\in I\}$, and set $V=\bigcup(\mathcal{E}\cap\mathcal{I})$, so that $V\in\mathcal{I}$. Give $Y=\kappa\setminus V$ the second-countable topology generated by $\{E\setminus V:E\in\mathcal{E}\}$.

If H is a dense open set in Y, then $\kappa\setminus H\in\mathcal{I}$. **P** Setting $\mathcal{E}'=\{E:E\in\mathcal{E},E\setminus V\subseteq H\}$, $H=(\bigcup\mathcal{E}')\setminus V$, so $H^\bullet=\sup_{E\in\mathcal{E}'}E^\bullet$ in $\mathcal{P}\kappa/\mathcal{I}$, and $H^\bullet\in\mathfrak{B}$. **?** If $\kappa\setminus H\notin\mathcal{I}$, then $H^\bullet\ne1$ and there is an $i\in\mathbb{N}$ such that $F_i^\bullet$ is non-zero and disjoint from $H^\bullet$. In this case, $F_i\cap E\in\mathcal{I}$ for every $E\in\mathcal{E}'$, so $F_i\setminus V$ is disjoint from H; but $F_i\setminus V$ is a non-empty open subset of Y. **XQ**

Consequently $\mathcal{M}(Y)\subseteq\mathcal{I}$.

(d) Set $W=\bigcup_{\alpha\in I}G_\alpha\times(D_\alpha\setminus V)$, so that W is an open set in $X\times Y$. Then W is dense. **P?** Otherwise, we have a non-empty open set $G\subseteq X$ and a non-empty open set $U\subseteq Y$ such that $I=I'\cup I''$, where $I'=\{\alpha:\alpha\in I,G\cap G_\alpha=\emptyset\}$ and $I''=\{\alpha:\alpha\in I,U\cap D_\alpha=\emptyset\}$. As U includes some non-empty set of the form $E\setminus V$ where $E\in\mathcal{E}$, $U\notin\mathcal{I}$. So there must be a $\xi\in U\setminus A'$. In this case, $J_\xi\subseteq I$ while $\bigcup_{\alpha\in J_\xi}G_\alpha$ is dense and meets G; there is therefore an $\alpha\in J_\xi\setminus I'$. But now $\alpha\in I_\xi$ so $\xi\in D_\alpha$, while also $\alpha\in I\setminus I'=I''$, so $U\cap D_\alpha=\emptyset$ and $\xi\notin D_\alpha$. **XQ**

(e) $W'=(X\times Y)\setminus W$ is therefore meager in $X\times Y$ and belongs to $\mathcal{M}(X)\ltimes\mathcal{M}(Y)$, by 527Db. If $x\in B_n$, then $C_n[\{x\}]\notin\mathcal{I}$; but $C_n[\{x\}]\subseteq W'[\{x\}]$, so $W'[\{x\}]\notin\mathcal{I}$ and $W'[\{x\}]\notin\mathcal{M}(Y)$. Accordingly

$$B_n\subseteq\{x:W'[\{x\}]\notin\mathcal{M}(Y)\}\in\mathcal{M}(X),$$

as required.

546F Corollary Suppose that X, κ and $\mathcal{I}$ are as in 546E, and that moreover $\bigcup\mathcal{I}=\kappa\notin\mathcal{I}$.

(a) Suppose that $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of subsets of X with union A. Then there is a $\zeta<\kappa$ such that $E\cap A_\zeta$ is non-meager whenever $E\subseteq X$ is a set with the Baire property and $E\cap A$ is not meager.

(b) If $\langle A_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X\setminus\mathcal{M}(X)$ such that $\#(\bigcup_{\xi<\kappa}A_\xi)<\kappa$, then there are distinct ξ, $\eta<\kappa$ such that $A_\xi\cap A_\eta\notin\mathcal{M}(X)$.

(c) If we have a family $\langle h_\xi\rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom}h_\xi$ is a non-meager subset of X for each ξ and $\#(\bigcup_{\xi<\kappa}h_\xi)<\kappa$ (identifying each h_ξ with its graph), then there are distinct ξ, $\eta<\kappa$ such that $\{x:h_\xi(x)$ and $h_\eta(x)$ are defined and equal$\}$ is non-meager.

proof (a) Let $\mathfrak{G}=\widehat{\mathcal{B}(X)}/\mathcal{M}(X)$ be the category algebra of X; for $B\subseteq X$ set $\psi(B)=\inf\{E^\bullet:B\subseteq E\in\widehat{\mathcal{B}(X)}\}$, as in 514Ie. Because $\mathfrak{G}$ is ccc (514Ja), there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ in κ such that $\sup_{n\in\mathbb{N}}\psi(A_{\xi_n})=\sup_{\xi<\kappa}\psi(A_\xi)$; setting $\zeta=\sup_{n\in\mathbb{N}}\xi_n$, we see that $\zeta<\kappa$ (because $\operatorname{cf}\kappa>\omega$) and that $\psi(A_\xi)\subseteq\psi(A_\zeta)$ for every $\xi<\kappa$.

? Suppose, if possible, that there is a set $E\in\widehat{\mathcal{B}(X)}$ such that $E\cap A_\zeta$ is meager but $E\cap A$ is non-meager. Replacing E by $E\setminus A_\zeta$ if necessary, we may suppose that $E\cap A_\zeta$ is empty. If $\xi<\kappa$, then

$$\psi(E\cap A_\xi)\subseteq E^\bullet\cap\psi(A_\zeta)\subseteq E^\bullet\cap(X\setminus E)^\bullet=0,$$

so $E\cap A_\xi$ is meager.

Define $f:E\cap A\to\kappa$ by setting $f(x)=\min\{\xi:x\in A_\xi\}$ for $x\in E$. Consider the set

$$C=\{(x,\xi):f(x)\le\xi<\kappa\}\subseteq(E\cap A)\times\kappa.$$

If $\xi<\kappa$, then

$$C^{-1}[\{\xi\}]=\{x:x\in E,\,f(x)\le\xi\}\subseteq E\cap A_\xi\in\mathcal{M}(X);$$

thus $C\in\mathcal{M}(X)\rtimes\mathcal{I}$. By 546E, $C\in\mathcal{M}(X)\ltimes\mathcal{I}$. As $E\cap A$ is not meager, there is an $x\in E\cap A$ such that $C[\{x\}]\in\mathcal{I}$. But $C[\{x\}]=\{\xi:f(x)\le\xi<\kappa\}\notin\mathcal{I}$. **X**

So ζ has the required property.

(b) Write $\mathcal{J}=\mathcal{I}\ltimes\mathcal{I}\triangleleft\mathcal{P}(\kappa\times\kappa)$. By 546D, $\mathcal{P}(\kappa\times\kappa)/\mathcal{J}$ is harmless. Set

$$W=\{(x,\xi,\eta):\xi,\,\eta<\kappa,\,\xi\ne\eta,\,x\in A_\xi\cap A_\eta\}.$$

? If $A_\xi\cap A_\eta\in\mathcal{M}(X)$ for all distinct ξ, $\eta<\kappa$, then W, regarded as a subset of $X\times(\kappa\times\kappa)$, belongs to $\mathcal{M}(X)\rtimes\mathcal{J}$; by 546E, $W\in\mathcal{M}(X)\ltimes\mathcal{J}$. For $x\in X$ set $C_x=\{\xi:\xi<\kappa,\,x\in A_\xi\}$. Then $W[\{x\}]=C_x^2\setminus\Delta$, where $\Delta=\{(\xi,\xi):\xi<\kappa\}$. So $W[\{x\}]\in\mathcal{J}$ iff $C_x\in\mathcal{I}$, and $E=\{x:C_x\notin\mathcal{I}\}$ is meager. Next, $A=\bigcup_{\xi<\kappa}A_\xi$ is supposed to have cardinal less than κ, so $\bigcup_{x\in A\setminus E}C_x\in\mathcal{I}$ and there is some $\zeta\in\kappa\setminus\bigcup_{x\in A\setminus E}C_x$. But in this case $A_\zeta\subseteq E$ is meager. **X** So we have the result.

(c)(i) For each $\xi<\kappa$, set $A_\xi=\operatorname{dom}h_\xi$ and let H_ξ be the regular open set in X such that $A_\xi\setminus H_\xi$ is meager and $G\cap H_\xi$ is empty whenever G is open and $G\cap A_\xi$ is meager (4A3Ra). Set $h'_\xi=h_\xi\restriction H_\xi$ and $Y=\bigcup_{\xi<\kappa}h'_\xi$; let $\pi_1:Y\to X$ be the first-coordinate projection. Give Y the topology $\mathfrak{S}=\{\pi_1^{-1}[G]:G\in\mathfrak{T}\}$, where $\mathfrak{T}$ is the topology of X.

(ii) If $\mathcal{U}$ is any π-base for $\mathfrak{T}$, then $\mathcal{V}=\{\pi_1^{-1}[U]:U\in\mathcal{U}\}$ is a π-base for $\mathfrak{S}$. **P** If $H\subseteq Y$ is open and not empty, take $G\in\mathfrak{T}$ such that $H=\pi_1^{-1}[G]$ and a $\xi<\kappa$ such that $H\cap h'_\xi\ne\emptyset$. Then $G\cap H_\xi\cap A_\xi=G\cap\operatorname{dom}h'_\xi$ is non-empty; by the choice of H_ξ, $G\cap H_\xi\cap A_\xi$ is non-meager. Set $\mathcal{U}'=\{U:U\in\mathcal{U},\,U\cap G\cap H_\xi\cap A_\xi=\emptyset\}$. Then $\bigcup\mathcal{U}'$ cannot be dense and there is a non-empty $U\in\mathcal{U}$ disjoint from $\bigcup\mathcal{U}'$. But now $U\cap G\ne\emptyset$, so there is a non-empty $U'\in\mathcal{U}$ with $U'\subseteq U\cap G$; in which case $V=\pi_1^{-1}[U']$ belongs to $\mathcal{V}$, is included in H and meets h'_ξ, so is not empty. As H is arbitrary, $\mathcal{V}$ is a π-base for $\mathfrak{S}$. **Q**

(iii) It follows at once that $\pi(Y)\le\pi(X)<\kappa$. We see also that if $A\subseteq X$ is nowhere dense, then $\{G:G\in\mathfrak{T},\,G\cap A=\emptyset\}$ is a π-base for $\mathfrak{T}$,

$$\{\pi_1^{-1}[G]:G\in\mathfrak{T},\,G\cap A=\emptyset\}=\{H:H\in\mathfrak{S},\,H\cap\pi_1^{-1}[A]=\emptyset\}$$

is a π-base for $\mathfrak{S}$ and $\pi_1^{-1}[A]$ is nowhere dense in Y. Accordingly $\pi_1^{-1}[A]\in\mathcal{M}(Y)$ for every $A\in\mathcal{M}(X)$.

(iv) If $B\subseteq Y$ is nowhere dense in Y then $\pi_1[B]$ is nowhere dense in X. **P** If $G\subseteq X$ is a non-empty open set, then either $\pi_1^{-1}[G]$ is empty and $G\cap\pi_1[B]=\emptyset$, or $\pi_1^{-1}[G]$ is a non-empty open subset of Y. In the latter case, $\pi_1^{-1}[G]\setminus\overline{B}$ must be of the form $\pi_1^{-1}[G']$ for some open set $G'\subseteq X$, and $G'\cap G$ is a non-empty open subset of G disjoint from $\pi_1[B]$. **Q** It follows at once that $\pi_1[B]\in\mathcal{M}(X)$ whenever $B\in\mathcal{M}(Y)$.

(v) Since $\pi_1[h'_\xi]=A_\xi\cap H_\xi$ is non-meager in X, h'_ξ is non-meager in Y, for every ξ. So (b) here tells us that there are distinct ξ, $\eta<\kappa$ such that $h'_\xi\cap h'_\eta$ is non-meager in Y. In this case, setting $A=\{x:h_\xi(x)$ and $h_\xi(y)$ are defined and equal$\}$, $\pi_1^{-1}[A]$ includes $h'_\xi\cap h'_\eta$ so is non-meager, and A is non-meager, by (iii).

546G The Gitik-Shelah theorem for Cohen algebras I come now to a companion result to the Gitik-Shelah Theorem in 543E. I follow the proof I gave before as closely as I can.

Theorem (Gitik & Shelah 89, Gitik & Shelah 93) Let κ be a regular uncountable cardinal and $\mathcal{I}$ a κ-additive ideal of subsets of κ such that $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$ is isomorphic to the category algebra $\mathfrak{G}_\lambda$ of $X = \{0,1\}^\lambda$ for some infinite cardinal λ. Then $\lambda \ge \min(\kappa^{(+\omega)}, 2^\kappa)$.

proof (a) Let $\phi : \mathfrak{G}_\lambda \to \mathfrak{A}$ be an isomorphism. As X is completely regular and ccc, $\mathcal{M}(X)$ is generated by $\mathcal{B}\mathfrak{a}(X) \cap \mathcal{M}(X)$ (5A4E(d-ii)), and $\mathfrak{G}_\lambda = \{H^\bullet : H \in \mathcal{B}\mathfrak{a}(X)\}$. By 546C, we have a function $f : \kappa \to X$ such that $f^{-1}[E]^\bullet = \phi E^\bullet$ in $\mathfrak{A}$ for every $E \in \widehat{\mathcal{B}(X)}$.

(b)? Suppose, if possible, that $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$.

Set $\zeta = \max(\lambda^+, \kappa^+)$. Then we have a cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha\rangle_{\alpha\in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover,

—— if $\lambda < \mathrm{Tr}(\kappa)$, then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

—— if $\lambda \ge \mathrm{Tr}(\kappa)$, then $g_\alpha\restriction\gamma = g_\beta\restriction\gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

P Copy the argument from part (b) of the proof of 543E. **Q**

(c) Fix an injective function $h : 2^\delta \to \{0,1\}^\delta$. For $\alpha \in S$ and $\iota < \delta$ set

$$U_{\alpha\iota} = \{\xi : \xi < \kappa,\ (hg_\alpha(\xi))(\iota) = 1\},$$

and let $H_{\alpha\iota} \in \mathcal{B}\mathfrak{a}(X)$ be such that $H^\bullet_{\alpha\iota} = \phi^{-1}(U^\bullet_{\alpha\iota})$ in $\mathfrak{G}_\lambda$; then $U_{\alpha\iota}\triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{I}$. Define $\tilde g_\alpha : X \to \{0,1\}^\delta$ by setting

$$\begin{aligned}(\tilde g_\alpha(x))(\iota) &= 1 \text{ if } x \in H_{\alpha\iota},\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then

$$\begin{aligned}\{\xi : \xi < \kappa,\ \tilde g_\alpha f(\xi) \ne hg_\alpha(\xi)\} &= \bigcup_{\iota<\delta}\{\xi : (\tilde g_\alpha f(\xi))(\iota) \ne (hg_\alpha(\xi))(\iota)\}\\ &= \bigcup_{\iota<\delta} U_{\alpha\iota}\triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{I}\end{aligned}$$

because $\delta < \kappa = \operatorname{add}\mathcal{I}$. Set $V_\alpha = \{\xi : \tilde g_\alpha f(\xi) = hg_\alpha(\xi)\}$, so that $\kappa \setminus V_\alpha \in \mathcal{I}$, for each $\alpha \in S$.

(d) Because every $H_{\alpha\iota}$ is determined by coordinates in a countable set, there is for each $\alpha \in S$ a set $I_\alpha \subseteq \lambda$ such that $\#(I_\alpha) \le \delta$ and $H_{\alpha\iota}$ is determined by coordinates in I_α for every $\iota < \delta$. By 5A1J there is an $M \subseteq \lambda$ such that $S_1 = \{\alpha : \alpha \in S,\ I_\alpha \subseteq M\}$ is stationary in ζ and $\mathrm{cf}(\#(M)) \le \delta$; because $\lambda < \kappa^{(+\omega)}$ and $\mathrm{cf}(\kappa) = \kappa > \delta$, $\#(M) < \kappa$. Set $\pi_M(z) = z\restriction M$ for $z \in X$, and $f_M = \pi_M f : \kappa \to \{0,1\}^M$.

If $E \subseteq \{0,1\}^M$ has the Baire property, then $\pi_M^{-1}[E]$ has the Baire property in X, and $\pi_M^{-1}[E]$ is meager iff E is (5A4E(c-iii), applied to $\{0,1\}^M \times \{0,1\}^{\lambda\setminus M}$). So $f_M^{-1}[E] \in \mathcal{I}$ iff E is meager.

(e) For each $\alpha \in S_1$, there is a $\theta_\alpha < \kappa$ such that $f_M[V_\alpha \cap \theta_\alpha]$ meets every non-empty open subset of $\{0,1\}^M$ in a non-meager set.

P Apply 546Fa to $f_M[V_\alpha] \subseteq \{0,1\}^M$. $\bigcup\mathcal{I} = \kappa$ because $\mathfrak{A}$ is atomless, and $\kappa \notin \mathcal{I}$ because $\mathfrak{A} \ne \{0\}$; while $\{0,1\}^M$ is certainly ccc, and has π-weight at most $\max(\omega, \#(M)) < \kappa$. There is therefore a set $B \subseteq f_M[V_\alpha]$ such that $\#(B) < \kappa$ and $E \cap B$ is non-meager whenever $E \subseteq \{0,1\}^M$ has the Baire property and $E \cap f_M[V_\alpha]$ is non-meager. Take $\theta_\alpha < \kappa$ such that $B \subseteq f_M[V_\alpha \cap \theta_\alpha]$. If $G \subseteq \{0,1\}^M$ is a non-empty open set, then

$$f_M^{-1}[G \setminus f_M[V_\alpha]] \subseteq \kappa \setminus V_\alpha \in \mathcal{I},$$

so either $G \setminus f_M[V_\alpha]$ is meager or it does not have the Baire property; in either case, $G \cap f_M[V_\alpha]$ is non-meager so $G \cap f_M[V_\alpha \cap \theta_\alpha] \supseteq G \cap B$ is non-meager. **Q**

Evidently we may take it that every θ_α is a non-zero limit ordinal.

(f) Because $\zeta = \mathrm{cf}\,\zeta > \kappa$, there is a $\theta < \kappa$ such that $S_2 = \{\alpha : \alpha \in S_1,\ \theta_\alpha = \theta\}$ is stationary in ζ. Now there is a $Y \in [2^\delta]^{<\kappa}$ such that $S_3 = \{\alpha : \alpha \in S_2,\ g_\alpha[\theta] \subseteq Y\}$ is stationary in ζ. **P** Use the argument of part (f) of the proof of 543E. **Q**

(g) For each $\alpha \in S_3$, set

$$Q_\alpha = f_M[V_\alpha \cap \theta] = f_M[V_\alpha \cap \theta_\alpha],$$

so that Q_α meets every non-empty open subset of $\{0,1\}^M$ in a non-meager set. Now $I_\alpha \subseteq M$, so we can express $\tilde g_\alpha$ as $g^*_\alpha\pi_M$, where $g^*_\alpha : \{0,1\}^M \to \{0,1\}^\delta$ is Baire measurable in each coordinate. If $y \in Q_\alpha$, take $\xi \in V_\alpha \cap \theta$ such that $f_M(\xi) = y$; then

$$g^*_\alpha(y) = g^*_\alpha \pi_M f(\xi) = \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi) \in h[Y].$$

Thus $g^*_\alpha \restriction Q_\alpha \subseteq f_M[\theta] \times h[Y]$ for every $\alpha \in S_3$, and we may apply 546Fc to $\{0,1\}^M$ and the family $\langle g^*_\alpha \restriction Q_\alpha \rangle_{\alpha \in S'}$, where $S' \subseteq S_3$ is a set of cardinal κ, to see that there are distinct α, $\beta \in S_3$ such that $\{y : y \in Q_\alpha \cap Q_\beta,\, g^*_\alpha(y) = g^*_\beta(y)\}$ is non-meager. Now, however, consider

$$E = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y) = g^*_\beta(y)\}.$$

Then $E = \bigcap_{\iota<\delta} E_\iota$, where

$$E_\iota = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y)(\iota) = g^*_\beta(y)(\iota)\}$$

is a Baire subset of $\{0,1\}^M$ for each $\iota < \delta$. Because $\delta < \kappa$ and $\mathcal{I}$ is κ-additive and ω_1-saturated,

$$\begin{aligned} f_M^{-1}[E]^\bullet &= (\bigcap_{\iota<\delta} f_M^{-1}[E_\iota])^\bullet = \inf_{\iota<\delta} f_M^{-1}[E_\iota]^\bullet \\ &= \inf_{\iota\in K} f_M^{-1}[E_\iota]^\bullet = f_M^{-1}[\bigcap_{\iota\in K} E_\iota]^\bullet \end{aligned}$$

for some countable $K \subseteq \delta$. In this case, $E' = \bigcap_{\iota\in K} E_\iota$ is a Baire set including E, and $f_M^{-1}[E' \setminus E] \in \mathcal{I}$; since E' includes the non-meager set $\{y : y \in Q_\alpha \cap Q_\beta,\, g^*_\alpha(y) = g^*_\beta(y)\}$, E' is non-meager and $f_M^{-1}[E'] \notin \mathcal{I}$, by (d) above; accordingly $f_M^{-1}[E] \notin \mathcal{I}$.

Consequently

$$\begin{aligned} \{\xi : g_\alpha(\xi) = g_\beta(\xi)\}^\bullet &= \{\xi : hg_\alpha(\xi) = hg_\beta(\xi)\}^\bullet \\ &= \{\xi : \xi \in V_\alpha \cap V_\beta,\, \tilde{g}_\alpha f(\xi) = \tilde{g}_\beta f(\xi)\}^\bullet \\ &= \{\xi : g^*_\alpha \pi_M f(\xi) = g^*_\beta \pi_M f(\xi)\}^\bullet = f_M^{-1}[E]^\bullet \neq 0 \end{aligned}$$

in $\mathfrak{A}$. But this is absurd, because in (b) above we chose g_α, g_β in such a way that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$ would be bounded in κ. **X**

So we have the required contradiction, and the theorem is true.

546H For the next step we need an elementary basic fact.

Lemma (a) A Boolean algebra $\mathfrak{A}$ is isomorphic to the category algebra $\mathfrak{G}_{\mathbb{N}}$ of $\{0,1\}^{\mathbb{N}}$ iff it is Dedekind complete, atomless, has countable π-weight and is not $\{0\}$.

(b) $\mathfrak{G}_{\mathbb{N}}$ is homogeneous.

(c) Every atomless order-closed subalgebra of $\mathfrak{G}_{\mathbb{N}}$ is isomorphic to $\mathfrak{G}_{\mathbb{N}}$.

proof (a)(i) All category algebras are Dedekind complete. The algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ is countable and atomless and isomorphic to an order-dense subalgebra of $\mathfrak{G}_{\mathbb{N}}$, so $\mathfrak{G}_{\mathbb{N}}$ is atomless and has countable π-weight.

(ii) If $\mathfrak{A}$ satisfies the conditions, let B be a countable order-dense subset of $\mathfrak{A}$ and $\mathfrak{B}$ the subalgebra of $\mathfrak{A}$ generated by B. Then $\mathfrak{B}$ is countable, atomless and not $\{0\}$, so is isomorphic to $\mathcal{E}$ (316M[5]). Now any isomorphism between $\mathfrak{B}$ and $\mathcal{E}$ extends to an isomorphism between their completions, which by 314Ub can be identified with $\mathfrak{A}$ and $\mathfrak{G}_{\mathbb{N}}$ respectively.

(b)-(c) All we have to observe is that any non-zero principal ideal of $\mathfrak{G}_{\mathbb{N}}$, and any atomless order-closed subalgebra of $\mathfrak{G}_{\mathbb{N}}$, satisfy the conditions of (a) (see 514E); or use 316P[6].

546I Corollary The category algebra $\mathfrak{G}_{\mathbb{N}}$ of $\{0,1\}^{\mathbb{N}}$ is not a power set σ-quotient algebra.

proof ? Suppose otherwise. By 546C there are an uncountable regular cardinal κ and a normal ideal $\mathcal{J}^*$ on κ such that $\mathcal{P}\kappa/\mathcal{J}^*$ is isomorphic to an atomless σ-subalgebra $\mathfrak{D}$ of a principal ideal $(\mathfrak{G}_{\mathbb{N}})_c$ of $\mathfrak{G}_{\mathbb{N}}$. As $\mathfrak{G}_{\mathbb{N}}$ is ccc and Dedekind complete, $\mathfrak{D}$ is order-closed in $(\mathfrak{G}_{\mathbb{N}})_c$, and is itself Dedekind complete. Also

$$\pi(\mathfrak{D}) \le \pi((\mathfrak{G}_{\mathbb{N}})_c) \le \pi(\mathfrak{G}_{\mathbb{N}}) = \omega$$

(514Eb). So $\mathfrak{D} \cong \mathfrak{G}_{\mathbb{N}}$; but by 546G this is impossible. **X**

546J I now embark on an investigation of algebras of the form $\mathfrak{A}\widehat{\otimes}\mathfrak{G}$, where $\mathfrak{A}$ is a measure algebra, $\mathfrak{G}$ is a category algebra and $\mathfrak{A}\widehat{\otimes}\mathfrak{G}$ is the Dedekind completion of the free product $\mathfrak{A} \otimes \mathfrak{G}$. The ideas here are based on Burke n96,

[5]Formerly 393F.
[6]Later editions only.

itself drawn from Gitik & Shelah 01. We have to begin with two lemmas which really refer to measure and category in $[0,1]^2$.

Lemma Let X be a set, Σ a σ-algebra of subsets of X, and Y a Polish space. If $V \in \Sigma\widehat{\otimes}\mathcal{B}(Y)$ and $V[\{x\}]$ is non-meager for every $x \in X$, there is a measurable function $f : X \to Y$ such that $(x, f(x)) \in V$ for every $x \in X$.

proof Let $\mathcal{W}$ be the family of sets expressible in the form $\bigcup_{n\in\mathbb{N}} E_n \times G_n$ where $E_n \in \Sigma$ and $G_n \subseteq Y$ is open for each n; let $\mathcal{W}^*$ be the set of those $W \in \mathcal{W}$ such that $W[\{x\}]$ is dense for every $x \in X$. Note that $W \cap W' \in \mathcal{W}$ for all W, $W' \in \mathcal{W}$. By 527I, there are W and $\langle W_n\rangle_{n\in\mathbb{N}}$ such that

$W \in \mathcal{W}$, $W_n \in \mathcal{W}^*$ for every n,

$(V\triangle W) \cap \bigcap_{n\in\mathbb{N}} W_n = \emptyset$.

Since every vertical section of V is non-meager, every vertical section of W is non-empty, while $W \cap \bigcap_{n\in\mathbb{N}} W_n \subseteq V$.

If Y is empty, so is X, and the result is trivial. Otherwise, proceed as follows. Fix a complete metric ρ on Y defining its topology. Set $V_0 = W$. Given that $V_n \in \mathcal{W}$ and $V_n[\{x\}]$ is non-empty for every $x \in X$, express V_n as $\bigcup_{i\in\mathbb{N}} E_{ni} \times G_{ni}$ where $E_{ni} \in \Sigma$ and $G_{ni} \subseteq Y$ is a non-empty open set for each $i \in \mathbb{N}$. Set $F_{ni} = E_{ni} \setminus \bigcup_{j<i} E_{nj}$ for each i, so that $\langle F_{ni}\rangle_{i\in\mathbb{N}}$ is disjoint and $\bigcup_{i\in\mathbb{N}} F_{ni} = \bigcup_{i\in\mathbb{N}} E_{ni} = X$. Next, for each $i \in \mathbb{N}$, let H_{ni} be a non-empty open set of diameter at most 2^{-n} such that $\overline{H}_{ni} \subseteq G_{ni}$, and y_{ni} a point of H_{ni}; then $V'_n = \bigcup_{i\in\mathbb{N}} F_{ni} \times H_{ni}$ belongs to $\mathcal{W}$, and all its vertical sections are non-empty. Define $f_n : X \to Y$ by saying that $f_n(x) = y_{ni}$ if $x \in F_{ni}$; then f_n is measurable. If we now take $V_{n+1} = V'_n \cap W_n$, V_{n+1} again belongs to $\mathcal{W}$, and its vertical sections are non-empty because the vertical sections of W_n are dense. Continue.

At the end of the induction, observe that, for each $x \in X$ and $m \le n \in \mathbb{N}$, $f_n(x) \in V_n[\{x\}] \subseteq V'_m[\{x\}]$, so $\rho(f_n(x), f_m(x)) \le \operatorname{diam} V'_m[\{x\}] \le 2^{-m}$. Accordingly $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and converges in Y to $f(x)$ say. Moreover,

$$f(x) \in \overline{V'_n[\{x\}]} \subseteq V_n[\{x\}] \subseteq W_n[\{x\}] \cap W[\{x\}]$$

for each n, so $(x, f(x)) \in W \cap \bigcap_{n\in\mathbb{N}} W_n \subseteq V$. By 418Ba, $f : X \to Y$ is measurable, so we have a suitable function.

546K Lemma Let (X, Σ, μ) be a probability space, Y a separable metrizable space and ν a topological probability measure on Y. Suppose we have a double sequence $\langle f_{kj}\rangle_{k,j\in\mathbb{N}}$ of measurable functions from X to Y such that $\{x : f_{ki}(x) = f_{kj}(x)\}$ is negligible whenever $k \in \mathbb{N}$ and $i \ne j$. Then for any $\epsilon > 0$ there is a Borel set $F \subseteq Y$ such that $\nu(Y \setminus F) \le \epsilon$ and $\bigcap_{j\in\mathbb{N}} f_{kj}^{-1}[F]$ is negligible for every $k \in \mathbb{N}$.

proof (a) Suppose that $\delta > 0$ and $k \in \mathbb{N}$. Then there is a Borel set $H \subseteq Y$ such that

$$\nu(Y \setminus H) \le \delta, \quad \mu(\textstyle\bigcap_{i\in\mathbb{N}} f_{ki}^{-1}[H]) \le \delta.$$

P Set $\eta = \frac{1}{3}\delta$ and take $m \in \mathbb{N}$ such that $(1-\eta)^m \le \eta$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of Y. For each n let $\mathcal{F}_n$ be the set of atoms in the algebra of subsets of Y generated by $\{U_i : i \le n\}$, and let E_n be the set of those $x \in X$ such that there are distinct i, $j < m$ such that $f_{ki}(x)$ and $f_{kj}(x)$ belong to the same member of $\mathcal{F}_n$. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of measurable sets. If $x \in \bigcap_{n\in\mathbb{N}} E_n$, then there must be $i < j < m$ such that $\{n : f_{ki}(x)$ and $f_{kj}(x)$ belong to the same member of $\mathcal{F}_n\}$ is infinite, in which case $f_{ki}(x) = f_{kj}(x)$; thus

$$\textstyle\bigcap_{n\in\mathbb{N}} E_n \subseteq \bigcup_{i<j<m}\{x : f_{ki}(x) = f_{kj}(x)\}$$

is negligible. There is therefore an $n \in \mathbb{N}$ such that $\mu E_n \le \eta$.

Give $\{0,1\}^{\mathcal{F}_n}$ the product measure λ in which each copy of $\{0,1\}$ is given the measure assigning measure η to $\{0\}$ and $1-\eta$ to $\{1\}$. For $w \in \{0,1\}^{\mathcal{F}_n}$ set

$$H_w = \textstyle\bigcup\{F : F \in \mathcal{F}_n,\ w(F) = 1\},$$

$$g(w) = \nu(Y \setminus H_w), \quad h(w) = \mu(\textstyle\bigcap_{i<m} f_{ki}^{-1}[H_w]).$$

Then

$$\textstyle\int g\,d\lambda = 1 - \sum_{F\in\mathcal{F}_n} \nu F \cdot \lambda\{w : w(F) = 1\} = 1 - (1-\eta)\sum_{F\in\mathcal{F}_n} \nu F = \eta.$$

Next, for each $x \in X \setminus E_n$,

$$\begin{aligned}\lambda\{w : x \in \bigcap_{i<m} f_{ki}^{-1}[H_w]\} &= \lambda\{w : w(F) = 1 \text{ whenever } i < m \text{ and } f_{ki}(x) \in F \in \mathcal{F}_n\}\\ &= (1-\eta)^m\end{aligned}$$

because, by the definition of E_n, all the $f_{ki}(x)$, for different $i < m$, belong to different members of $\mathcal{F}_n$. Since the set

$$\{(x, w) : w \in \{0,1\}^{\mathcal{F}_n},\ x \in \textstyle\bigcap_{i<m} f_{ki}^{-1}[H_w]\} = \bigcap_{i<m} \bigcup_{F\in\mathcal{F}_n} (f_{ki}^{-1}[F] \times \{w : w(F) = 1\})$$

belongs to $\Sigma\widehat{\otimes}\mathcal{B}(\{0,1\}^{\mathcal{F}_n})$, Fubini's theorem tells us that

$$\int h\,d\lambda = \int \lambda\{w : x \in \bigcap_{i<m} f_{ki}^{-1}[H_w]\}\mu(dx) \le \mu E_n + (1-\eta)^m \le 2\eta.$$

Accordingly $\int g + h\,d\lambda \le 3\eta$ and there is a $w \in \{0,1\}^{\mathcal{F}_n}$ such that $\max(g(w), h(w)) \le 3\eta = \delta$. Looking back at the definitions, we see that H_w will serve for H, with something to spare. **Q**

(b) Now all we need to do is take a double sequence $\langle\delta_{kl}\rangle_{k,l\in\mathbb{N}}$ of strictly positive numbers such that $\sum_{k,l\in\mathbb{N}}\delta_{kl} \le \epsilon$ and a corresponding family $\langle H_{kl}\rangle_{k,l\in\mathbb{N}}$ of Borel subsets of Y such that, for each k and l,

$$\nu(Y \setminus H_{kl}) \le \delta_{kl}, \quad \mu(\bigcap_{i\in\mathbb{N}} f_{ki}^{-1}[H_{kl}]) \le \delta_{kl},$$

and set $F = \bigcap_{k,l\in\mathbb{N}} H_{kl}$.

546L We come now to the central step, in which the results of the last two lemmas are adapted to give a key to the structure of the algebras $\mathfrak{A}\widehat{\otimes}\mathfrak{G}$ we are investigating.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra with countable π-weight, not $\{0\}$. Let $\mathfrak{C}$ be the Dedekind completion of $\mathfrak{A}\otimes\mathfrak{B}$, and write $\varepsilon_1 : \mathfrak{A} \to \mathfrak{A}\otimes\mathfrak{B} \subseteq \mathfrak{C}$, $\varepsilon_2 : \mathfrak{B} \to \mathfrak{A}\otimes\mathfrak{B} \subseteq \mathfrak{C}$ for the canonical order-continuous Boolean homomorphisms. For $c \in \mathfrak{C}$ write $\mathrm{upr}_1 c = \inf\{a : a \in \mathfrak{A},\ c \subseteq \varepsilon_1(a)\}$.

Let $\mathfrak{D}$ be an order-closed subalgebra of $\mathfrak{C}$ such that there is no non-zero principal ideal of $\mathfrak{D}$ which is measurable with Maharam type at most $\tau(\mathfrak{A})$.

(a) If $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ are non-zero, there is a $d \in \mathfrak{D}$ such that

$$a \cap \mathrm{upr}_1(\varepsilon_2(b) \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b) \setminus d) \ne 0.$$

(b) There is a countable set $D \subseteq \mathfrak{D}$ such that

$$\sup_{d\in D}(\mathrm{upr}_1(\varepsilon_2(b) \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b) \setminus d)) = 1$$

in $\mathfrak{A}$, for every $b \in \mathfrak{B} \setminus \{0\}$.

(c) For any $d \in \mathfrak{D}$, $b_0, \dots, b_m \in \mathfrak{B}$ and $\epsilon > 0$ there is a $d' \in \mathfrak{D}$ such that $d' \subseteq d$ and

$$\bar\mu(\mathrm{upr}_1(\varepsilon_2(b_k) \cap d')) \ge \bar\mu(\mathrm{upr}_1(\varepsilon_2(b_k) \cap d)) - \epsilon,$$

$$\bar\mu(\mathrm{upr}_1(\varepsilon_2(b_k) \cap d \setminus d')) \ge \bar\mu(\mathrm{upr}_1(\varepsilon_2(b_k) \cap d)) - \epsilon$$

for every $k \le m$.

(d) $\mathfrak{D}$ has a non-zero principal ideal with an atomless order-closed subalgebra with countable π-weight.

proof (a) ? Otherwise, set $\pi d = a \cap \mathrm{upr}_1(\varepsilon_2(b) \cap d)$ for $d \in \mathfrak{D}$. Then π is a function from $\mathfrak{D}$ to the principal ideal $\mathfrak{A}_a$ generated by a. Certainly $\pi(\sup D) = \sup \pi[D]$ for every non-empty $D \subseteq \mathfrak{D}$ (cf. 313Sb[7]), and $\pi 1 = 1$. The counter-hypothesis is that $\pi d \cap \pi(1 \setminus d) = 0$ for every $d \in \mathfrak{D}$; but this is enough to make π an order-continuous Boolean homomorphism (312H(iv), 313Lb). Setting

$$d_0 = \inf\{d : a \otimes b \subseteq d \in \mathfrak{D}\} = 1 \setminus \sup\{d : \pi d = 0\},$$

$\pi{\restriction}\mathfrak{D}_{d_0}$ is an injective order-continuous Boolean homomorphism from the principal ideal $\mathfrak{D}_{d_0}$ into $\mathfrak{A}_a$, so is an isomorphism between $\mathfrak{D}_{d_0}$ and a closed subalgebra of $\mathfrak{A}_a$ (314F(a-i)). But $\mathfrak{A}_a$ and all its closed subalgebras are measurable algebras with Maharam types at most $\tau(\mathfrak{A})$ (331Hc, 332Tb), so $\mathfrak{D}_{d_0}$ is a measurable algebra with Maharam type at most $\tau(\mathfrak{A})$, which is supposed to be impossible. **X**

(b) For each non-zero $b \in \mathfrak{B}$, (a) tells us that

$$\sup_{d\in\mathfrak{D}} \mathrm{upr}_1(\varepsilon_2(b) \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b) \setminus d) = 1.$$

Because $\mathfrak{A}$ is ccc, there is a countable $D_b \subseteq \mathfrak{D}$ such that

$$\sup_{d\in D_b} \mathrm{upr}_1(\varepsilon_2(b) \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b) \setminus d) = 1.$$

Let B be a countable order-dense subset of $\mathfrak{B} \setminus \{0\}$ and set $D = \bigcup_{b\in B} D_b$. If $b \in \mathfrak{B} \setminus \{0\}$ there is a $b' \in B$ such that $b' \subseteq b$, and in this case

$$\begin{aligned}&\sup_{d\in D} \mathrm{upr}_1(\varepsilon_2(b) \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b) \setminus d)\\ &\qquad\supseteq \sup_{d\in D_{b'}} \mathrm{upr}_1(\varepsilon_2(b') \cap d) \cap \mathrm{upr}_1(\varepsilon_2(b') \setminus d) = 1.\end{aligned}$$

[7]Formerly 314V.

(c) (The hard part.)

(i) Let (X,Σ,μ) be a probability space with measure algebra isomorphic to $(\mathfrak{A},\bar\mu)$ (321J); for $E\in\Sigma$ write $E^\bullet$ for the corresponding element of $\mathfrak{A}$. Next, let $\mathfrak{B}_0$ be a countable order-dense subalgebra of $\mathfrak{B}$, and Y the Stone space of $\mathfrak{B}_0$, so that Y is a compact metrizable space, and we can identify $\mathfrak{B}_0$ with the algebra of open-and-closed subsets of Y. In this case, $\mathfrak{B}$ is isomorphic to the regular open algebra of Y (see 314T) and can be identified with the category algebra of Y (514If). For $H\in\mathcal{B}(Y)$ let $H^\bullet$ be the corresponding member of $\mathfrak{B}$.

(ii) Write $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\mathcal{B}(Y))\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$. By 527O, we can identify $\mathfrak{C}$ with $(\Sigma\widehat{\otimes}\mathcal{B}(Y))/\mathcal{L}$, the canonical embeddings ε_1 and ε_2 corresponding to the maps $E^\bullet\mapsto(E\times Y)^\bullet$ and $F^\bullet\mapsto(X\times F)^\bullet$. Let $\mathcal{W}$ be the family of subsets of $X\times Y$ expressible in the form $\bigcup_{n\in\mathbb{N}}E_n\times G_n$ where $E_n\in\Sigma$ and $G_n\subseteq Y$ is open for every $n\in\mathbb{N}$. By 527I, every member of $\mathfrak{C}$ can be expressed as $W^\bullet$ for some $W\in\mathcal{W}$.

If $W\in\mathcal{W}$, then $\mathrm{upr}_1(W^\bullet)=\pi_1[W]^\bullet$, where $\pi_1(x,y)=x$ for $x\in X$ and $y\in Y$. **P** If E, $F\in\Sigma$ and $H\subseteq Y$ is a non-meager Borel set, then

$$\begin{aligned}(E\times H)^\bullet\subseteq\varepsilon_1(F^\bullet)&\iff(E\times H)\setminus(F\times Y)\in\mathcal{L}\iff(E\setminus F)\times H\in\mathcal{L}\\&\iff E\setminus F\text{ is negligible}\iff E^\bullet\subseteq F^\bullet.\end{aligned}$$

So $\mathrm{upr}_1((E\times H)^\bullet)=E^\bullet$ in $\mathfrak{A}$. Now if $W=\bigcup_{n\in\mathbb{N}}E_n\times G_n$, where $E_n\in\Sigma$ and G_n is open for each n, then

$$\begin{aligned}\mathrm{upr}_1(W^\bullet)&=\mathrm{upr}_1(\sup_{n\in\mathbb{N}}(E_n\times G_n)^\bullet)=\sup_{n\in\mathbb{N}}\mathrm{upr}_1((E_n\times G_n)^\bullet)\\&=\sup_{n\in\mathbb{N},G_n\ne\emptyset}E_n^\bullet=\pi_1[W]^\bullet.\ \mathbf{Q}\end{aligned}$$

(iii) Take a countable set $D\subseteq\mathfrak{D}$ as in (b) above, and let $\langle d_n\rangle_{n\in\mathbb{N}}$ run over D. For each $n\in\mathbb{N}$ choose V_n, $W_n\in\mathcal{W}$ such that $V_n^\bullet=d_n$ and $W_n^\bullet=1\setminus d_n$. Then

$$\begin{aligned}E_0^*&=\{x:x\in X,\ V_n[\{x\}]\cap W_n[\{x\}]\ne\emptyset\}\\&\qquad\cup\{x:x\in X,\ V_n[\{x\}]\cup W_n[\{x\}]\text{ is not dense}\}\\&=\{x:x\in X,\ V_n[\{x\}]\cap W_n[\{x\}]\text{ is not meager}\}\\&\qquad\cup\{x:x\in X,\ V_n[\{x\}]\cup W_n[\{x\}]\text{ is not comeager}\}\end{aligned}$$

is negligible. If $G\subseteq Y$ is a non-empty open set, then

$$\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times G)]\cap\pi_1[W_n\cap(X\times G)]\bigr)$$

is measurable and conegligible. **P** Set $b=(X\times G)^\bullet\in\mathfrak{B}$. Then

$$\begin{aligned}&\bigl(\bigcup_{n\in\mathbb{N}}(\pi_1[V_n\cap(X\times G)]\cap\pi_1[W_n\cap(X\times G)])\bigr)^\bullet\\&\qquad=\sup_{n\in\mathbb{N}}(\mathrm{upr}_1(\varepsilon_1(b)\cap d_n)\cap\mathrm{upr}_1(\varepsilon_2(b)\setminus d_n))=1.\ \mathbf{Q}\end{aligned}$$

In fact

$$X_0=\bigcap_{G\subseteq Y\text{ is open and not empty}}\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times G)]\cap\pi_1[W_n\cap(X\times G)]\bigr)$$

is measurable and conegligible. **P** Let $\langle U_m\rangle_{m\in\mathbb{N}}$ run over a base for the topology of Y consisting of non-empty sets. If $G\subseteq Y$ is open and not empty, then there is an $m\in\mathbb{N}$ such that $G\supseteq U_m$, so that

$$\begin{aligned}&\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times G)]\cap\pi_1[W_n\cap(X\times G)]\bigr)\\&\qquad\supseteq\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times U_m)]\cap\pi_1[W_n\cap(X\times U_m)]\bigr).\end{aligned}$$

This means that

$$X_0=\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times U_m)]\cap\pi_1[W_n\cap(X\times U_m)]\bigr)$$

is measurable and conegligible. **Q**

Set $X_1=X_0\setminus E_0^*$, so that X_1 also is measurable and conegligible.

(iv) Write Z for $\{0,1\}^{\mathbb{N}}$. Define $h:X\times Y\to Z$ by setting $h(x,y)(n)=1$ if $(x,y)\in V_n$, 0 otherwise. If $H\in\mathcal{B}(Z)$, then $h^{-1}[H]$ belongs to the σ-algebra generated by $\{W_n:n\in\mathbb{N}\}$, so $h^{-1}[H]^\bullet\in\mathfrak{D}$. If $W\in\mathcal{W}$ is such that $\pi_1[W]\supseteq X_1$, and $f:X\to Y$ is any measurable function, there is a $W'\in\mathcal{W}$ such that $W'\subseteq W$, $\pi_1[W]\supseteq X_1$ and $h(x,y)\ne h(x,f(x))$ whenever $x\in X_1$ and $(x,y)\in W'$. **P** Set

$$E_n=\pi_1[W\cap V_n]\cap\pi_1[W\cap W_n]$$

for each $n\in\mathbb{N}$. Then E_n is measurable (because $W\cap V_n$ and $W\cap W_n$ belong to $\mathcal{W}$) and for every $x\in X_1$ there must be some n such that

$$x\in\pi_1[V_n\cap(X\times W[\{x\}])]\cap\pi_1[W_n\cap(X\times W[\{x\}])],$$

in which case $x\in E_n$. Accordingly we can find a partition $\langle E'_n\rangle_{n\in\mathbb{N}}$ of X_1 into measurable sets such that $E'_n\subseteq E_n$ for each n. Next, set

$$F_n=\{x:x\in E_n,\,(x,f(x))\in W_n\}\text{ for each }n\in\mathbb{N},$$

$$\tilde W=\bigcup_{n\in\mathbb{N}}(((E'_n\cap F_n)\times Y)\cap V_n)\cup(((E'_n\setminus F_n)\times Y)\cap W_n),$$

$$W'=W\cap\tilde W.$$

Then $W'\in\mathcal{W}$. If $x\in X_1$, there is an $n\in\mathbb{N}$ such that $x\in E'_n$. In this case, $\tilde W[\{x\}]=V_n[\{x\}]\cup W_n[\{x\}]$ is dense, so meets $W[\{x\}]$, and $W'[\{x\}]$ is non-empty. If now $(x,y)\in W'$, then, because $V_n[\{x\}]\cap W_n[\{x\}]$ is empty,

$$\begin{aligned}h(x,f(x))(n)=1&\iff(x,f(x))\in V_n\iff x\notin F_n\\&\iff(x,y)\in W_n\iff h(x,y)(n)=0,\end{aligned}$$

so $h(x,f(x))\ne h(x,y)$. So W' serves. **Q**

(v) If $f:X\to Y$ is measurable, then $x\mapsto(x,f(x)):X\to X\times Y$ is $(\Sigma,\Sigma\widehat\otimes\mathcal{B}(Y))$-measurable, because $\{x:(x,f(x))\in E\times H\}$ belongs to Σ whenever $E\in\Sigma$ and $H\in\mathcal{B}(Y)$. So if we set $\tilde f(x)=h(x,f(x))$ for each x, $\tilde f:X\to Z$ is again measurable.

At this point, take the d and $b_0,\dots,b_m$ and ϵ of the statement of this part of the lemma. Let $W\in\mathcal{W}$ be such that $W^\bullet=d$. Write R for the family of measurable functions $f:X\to Y$ such that $(x,f(x))\in\bigcap_{n\in\mathbb{N}}V_n\cup W_n$ for every $x\in X_1$. For each $k\le m$, let $G_k\subseteq Y$ be the regular open set such that $G_k^\bullet=b_k$ in $\mathfrak{B}$, and write R_k for the set of functions $f\in R$ such that $f(x)\in G_k$ and $(x,f(x))\in W$ whenever $x\in\pi_1[W\cap(X_1\times G_k)]$.

(vi) Now, for each $k\le m$, there is a sequence $\langle f_{ki}\rangle_{i\in\mathbb{N}}$ in R_k such that $\tilde f_{ki}(x)\ne\tilde f_{kj}(x)$ whenever $i<j$ and $x\in X_1$. **P** Choose the sequence inductively. Given f_{ki} for $i<j$, set

$$W'_0=(W\cap(X\times G_k))\cup((X\setminus\pi_1[W\cap(X\times G_k)])\times Y).$$

Then W'_0 belongs to $\mathcal{W}$ and its vertical sections are non-empty. By (iv), we can choose $W'_1,\dots,W'_j$ such that, for each $i<j$, $W'_{i+1}\in\mathcal{W}$, $W'_{i+1}\subseteq W'_i$, $\pi_1[W'_{i+1}]\supseteq X_1$ and $h(x,y)\ne h(x,f_{ki}(x))$ whenever $x\in X_1$ and $(x,y)\in W'_{i+1}$. By 546J, we can find a measurable function $f_{kj}:X\to Y$ such that $(x,f_{kj}(x))\in W'_j\cap\bigcap_{n\in\mathbb{N}}(V_n\cup W_n)$ for every $x\in X_1$. Now, if $x\in X_1$ and $i<j$, $(x,f_{kj}(x))\in W'_{i+1}$ so

$$\tilde f_{kj}(x)=h(x,f_{kj}(x))\ne h(x,f_{ki}(x))=\tilde f_{ki}(x).$$

So this sequence works. **Q** For completeness, set $f_{ki}=f_{0i}$ if $k>m$ and $i\in\mathbb{N}$.

(vii) Let ν be the Borel probability measure on Z defined by saying

$$\nu H=\tfrac{1}{m+1}\sum_{k=0}^m\mu\tilde f_{k0}^{-1}[H]$$

for every $H\in\mathcal{B}(Z)$. By 546K there is a Borel set $H\subseteq Z$ such that

$$\nu(Z\setminus H)<\tfrac{\epsilon}{m+1},\quad\mu(\bigcap_{i\in\mathbb{N}}\tilde f_{ki}^{-1}[H])=0\text{ for every }k\le m.$$

Being a Borel probability measure on a compact metrizable space, ν must be inner regular with respect to the compact sets (433Ca); let $K\subseteq H$ be a compact set such that $\nu(Z\setminus K)\le\frac{\epsilon}{m+1}$. In this case, $\mu\tilde f_{k0}^{-1}[K]\ge1-\epsilon$ for every $k\le m$. Next, there is a non-increasing sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open-and-closed sets in Z with intersection K, so that $\bigcap_{n\in\mathbb{N}}\bigcap_{i\in\mathbb{N}}\tilde f_{ki}^{-1}[H_n]$ is negligible for each k, and there is a $p\in\mathbb{N}$ such that $\mu(\bigcap_{i\in\mathbb{N}}\tilde f_{ki}^{-1}[H_p])\le\epsilon$ for every $k\le m$. Let $r\in\mathbb{N}$ be such that H_p is determined by coordinates less than r. Set

$$V=(X_1\times Y)\cap\bigcap_{n\le r}(V_n\cup W_n)\cap h^{-1}[H_p],$$

$$\tilde V=(X_1\times Y)\cap\bigcap_{n\le r}(V_n\cup W_n)\setminus h^{-1}[H_p];$$

then V and $\tilde{V}$ both belong to $\mathcal{W}$, $V\cap\tilde{V}=\emptyset$, $V^\bullet$ and $\tilde{V}^\bullet$ both belong to $\mathfrak{D}$, and $V^\bullet\cup\tilde{V}^\bullet=1$ in $\mathfrak{D}$.

Set $d'=(W\cap V)^\bullet$, so that $d'\in\mathfrak{D}$ and $d'\subseteq d$.

(viii) For each $k\le m$,

$$(X\times G_k)\cap W\in\mathcal{W},\quad \varepsilon_2(b_k)\cap d=((X\times G_k)\cap W)^\bullet,$$

$$\bar{\mu}(\operatorname{upr}_1(\varepsilon_2(b_k)\cap d))=\mu\pi_1[W\cap(X\times G_k)]=\mu\pi_1[W\cap(X_1\times G_k)],$$

$$\bar{\mu}(\operatorname{upr}_1(\varepsilon_2(b_k)\cap d'))=\mu\pi_1[W\cap V\cap(X_1\times G_k)],$$

$$\bar{\mu}(\operatorname{upr}_1(\varepsilon_2(b_k)\cap d\setminus d'))=\mu\pi_1[W\cap\tilde{V}\cap(X\times G_k)].$$

$\pi_1[W\cap(X_1\times G_k)]\setminus\pi_1[W\cap V\cap(X_1\times G_k)]$ does not meet $\tilde{f}_{k0}^{-1}[K]$. **P** If $x\in\pi_1[W\cap(X_1\times G_k)]\setminus\pi_1[W\cap V\cap(X_1\times G_k)]$, then $(x,f_{k0}(x))\notin W\cap V\cap(X_1\times G_k)$; as $f_{k0}\in R_k$, $(x,f_{k0}(x))\in W\cap\bigcap_{n\in\mathbb{N}}(V_n\cup W_n)\cap(X_1\times G_k)$, so $(x,f_{k0}(x))\notin h^{-1}[H_p]$ and $\tilde{f}_{k0}(x)\notin H_p$. As $K\subseteq H_p$, $x\notin\tilde{f}_{k0}^{-1}[K]$. **Q** It follows that

$$\begin{aligned}\bar{\mu}(\operatorname{upr}_1(\epsilon_2(b_k)\cap d))&=\mu\pi_1[W\cap(X_1\times G_k)]\\&\le(1-\mu\tilde{f}_{k0}^{-1}[K])+\mu\pi_1[W\cap V\cap(X_1\times G_k)]\\&\le\bar{\mu}(\operatorname{upr}_1(\epsilon_2(b_k)\cap d'))+\epsilon.\end{aligned}$$

On the other side, $\pi_1[W\cap(X_1\times G_k)]\subseteq\pi_1[W\cap\tilde{V}\cap(X_1\times G_k)]\cup\bigcap_{i\in\mathbb{N}}\tilde{f}_{ki}^{-1}[H_p]$. **P** This time, if $x\in\pi_1[W\cap(X_1\times G_k)]\setminus\bigcap_{i\in\mathbb{N}}\tilde{f}_{ki}^{-1}[H_p]$, let $i\in\mathbb{N}$ be such that $h(x,f_{ki}(x))\notin H_p$. Since $f_{ki}\in R_k$, $(x,f_{ki}(x))\in W\cap(X_1\times Y)\cap\bigcap_{n\in\mathbb{N}}(V_n\cup W_n)\setminus h^{-1}[H_p]$ and $x\in\pi_1[W\cap\tilde{V}\cap(X_1\times G_k)]$. **Q** So we get

$$\begin{aligned}\bar{\mu}(\operatorname{upr}_1(\epsilon_2(b_k)\cap d))&\le\mu(\bigcap_{i\in\mathbb{N}}\tilde{f}_{ki}^{-1}[H_p])+\mu\pi_1[W\cap\tilde{V}\cap(X_1\times G_k)]\\&\le\bar{\mu}(\operatorname{upr}_1(\epsilon_2(b_k)\cap d\setminus d'))+\epsilon.\end{aligned}$$

This is true for every $k\le m$, so we have found an appropriate d'.

(d) Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over an order-dense subset of $\mathfrak{B}$, starting with $b_0=1$. Choose d_σ, for $\sigma\in S_2^*=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, as follows. $d_\emptyset=1$. Given $d_\sigma\in\mathfrak{D}$, where $\sigma\in\{0,1\}^n$, use (c) to find $d'_\sigma\in\mathfrak{D}$ such that $d'_\sigma\subseteq d_\sigma$ and

$$\bar{\mu}\bigl(\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k))\setminus(\operatorname{upr}_1(d'_\sigma\cap\varepsilon_2(b_k))\cap\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k)\setminus d'_\sigma))\bigr)\le 4^{-n-3}$$

whenever $k\le n$. Now, writing $\sigma^\frown\!<\!0\!>$ and $\sigma^\frown\!<\!1\!>$ for the two members of $\{0,1\}^{n+1}$ extending σ, set $d_{\sigma^\frown<0>}=d'_\sigma$ and $d_{\sigma^\frown<1>}=d_\sigma\setminus d'_\sigma$, and continue.

At the end of the construction,

$$a_0=\sup_{\sigma\in S_2^*,k\le\#(\sigma)}\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k))\setminus(\operatorname{upr}_1(d_{\sigma^\frown<0>}\cap\varepsilon_2(b_k))\cap\operatorname{upr}_1(d_{\sigma^\frown<1>}\cap\varepsilon_2(b_k)))$$

has measure at most

$$\textstyle\sum_{n=0}^\infty 2^n\sum_{k=0}^n 4^{-n-3}=\sum_{n=0}^\infty(n+1)2^{-n-3}=\frac{1}{2},$$

so $a=1\setminus a_0$ is non-zero. Now an easy induction on n shows that $a\cap\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k))=a\cap\operatorname{upr}_1(d_\tau\cap\varepsilon_2(b_k))$ whenever $k\le m\le n$, $\sigma\in\{0,1\}^m$, $\tau\in\{0,1\}^n$ and $\sigma\subseteq\tau$. In particular, $a\subseteq\operatorname{upr}_1(d_\sigma)$ for every $\sigma\in S_2^*$.

Set $d^*=\inf\{d:d\in\mathfrak{D},\,d\supseteq\varepsilon_1(a)\}$. Then $a\cap\operatorname{upr}_1(d^*\cap d)=a\cap\operatorname{upr}_1(d)$ for every $d\in\mathfrak{D}$; in particular, $a\subseteq\operatorname{upr}_1(d^*\cap d_\sigma)$ and $d^*\cap d_\sigma\ne0$ for every $\sigma\in S_2^*$. Writing $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $U_\sigma=\{z:\sigma\subseteq z\in Z\}$ for $\sigma\in S_2^*$, we have an injective Boolean homomorphism ϕ from $\mathcal{E}$ to the principal ideal $\mathfrak{D}_{d^*}$ defined by setting $\phi U_\sigma=d_\sigma\cap d^*$ for every $\sigma\in S_2^*$. Now ϕ is order-continuous. **P?** Otherwise, there is a set $T\subseteq S_2^*$ such that $\sup_{\tau\in T}U_\tau=1$ in $\mathcal{E}$, that is, $\bigcup_{\tau\in T}U_\tau$ is dense in Z, but $d^*\not\subseteq\sup_{\tau\in T}d_\tau$. In this case $\varepsilon_1(a)\not\subseteq\sup_{\tau\in T}d_\tau$, so there must be a non-zero $a'\subseteq a$ and a $k\in\mathbb{N}$ such that $\varepsilon_1(a')\cap\varepsilon_2(b_k)\cap d_\tau=0$ for every $\tau\in T$. Now, however, there is surely a $\sigma\in\{0,1\}^k$ such that $\varepsilon_1(a')\cap\varepsilon_2(b_k)\cap d_\sigma\ne0$, in which case there is a $\tau\in T$ such that $\tau\supseteq\sigma$, so that

$$\begin{aligned}0&=a'\cap\operatorname{upr}_1(d_\tau\cap\varepsilon_2(b_k))=a'\cap a\cap\operatorname{upr}_1(d_\tau\cap\varepsilon_2(b_k))\\&=a'\cap a\cap\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k))=a'\cap\operatorname{upr}_1(d_\sigma\cap\varepsilon_2(b_k))\ne0,\end{aligned}$$

which is absurd. **XQ**

Accordingly ϕ has an extension to an order-continuous embedding of the Dedekind completion $\mathfrak{G}$ of $\mathcal{E}$ into $\mathfrak{D}_{d^*}$ (314Tb), and the image of $\mathfrak{G}$ in $\mathfrak{D}_{d^*}$ is an atomless order-closed subalgebra of $\mathfrak{D}_{d^*}$ with countable π-weight.

546M Theorem Let $\mathfrak{A}$ be a measurable algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra with countable π-weight. Let $\mathfrak{C}$ be the Dedekind completion of $\mathfrak{A}\otimes\mathfrak{B}$ and $\mathfrak{D}$ an order-closed subalgebra of $\mathfrak{C}$. Then there is a $d^*\in\mathfrak{D}$ such that $\mathfrak{D}_{d^*}$ is a measurable algebra, with Maharam type at most $\max(\omega,\tau(\mathfrak{A}))$, and $\mathfrak{D}_{1\setminus d^*}$ has an atomless order-closed subalgebra of countable π-weight.

proof (a) Let D_0 be the set of those $d\in\mathfrak{D}$ such that $\mathfrak{D}_d$ is measurable with Maharam type at most $\tau(\mathfrak{A})$, and D_1 the set of those $d\in\mathfrak{D}$ such that $\mathfrak{D}_d$ has an atomless order-closed subalgebra of countable π-weight. Then $D_0\cup D_1$ is order-dense in $\mathfrak{D}$. **P** Take $d_0\in\mathfrak{D}\setminus\{0\}$. If there is a is non-zero $d\in D_0$ such that $d\subseteq d_0$, we can stop. Otherwise, there are non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ such that $c=a\otimes b\subseteq d_0$. Now the principal ideal $(\mathfrak{A}\otimes\mathfrak{B})_c$ is isomorphic to the free product $\mathfrak{A}_a\otimes\mathfrak{B}_b$, so the principal ideal $\mathfrak{C}_c$ can be identified with the Dedekind completion of $\mathfrak{A}_a\otimes\mathfrak{B}_b$. Consider $\mathfrak{D}_c=\{d\cap c:d\in\mathfrak{D}\}$; this is an order-closed subalgebra of $\mathfrak{C}_c$, and is isomorphic to the principal ideal of $\mathfrak{D}$ generated by $\operatorname{upr}(c,\mathfrak{D})=\inf\{d:c\subseteq d\in\mathfrak{D}\}\subseteq d_0$. In particular, none of the non-zero principal ideals of $\mathfrak{D}_c$ can be measurable algebras with Maharam type less than or equal to $\tau(\mathfrak{A})$.

Since $\mathfrak{A}_a$ is a non-zero measurable algebra, there is a functional $\bar\mu$ such that $(\mathfrak{A}_a,\bar\mu)$ is a probability algebra; while $\mathfrak{B}_b$ is Dedekind complete and has countable π-weight. So 546Ld tells us that there is a non-zero principal ideal $(\mathfrak{D}_c)_e$ of $\mathfrak{D}_c$ with an atomless order-closed subalgebra of countable π-weight. Copying this into $\mathfrak{D}_{\operatorname{upr}(c,\mathfrak{D})}$, we get a non-zero $d_1\subseteq d_0$ such that $\mathfrak{D}_{d_1}$ has an atomless order-closed subalgebra of countable π-weight, that is, $d_1\in D_1$. **Q**

(b) Accordingly there is a partition of unity $\langle d_i\rangle_{i\in I}$ in $\mathfrak{D}$ such that every d_i belongs to $D_0\cup D_1$. Set

$$J=\{i:i\in I,\,d_i\in D_0\},\quad K=I\setminus J,\quad d^*=\sup_{i\in J}d_i.$$

Because $\mathfrak{C}$ and $\mathfrak{D}$ are ccc (525Ub, 516U, 514Ea), I, J and K are countable. Now $\mathfrak{D}_{d^*}\cong\prod_{i\in J}\mathfrak{D}_{d_i}$ is measurable (391Ca, or otherwise), and its Maharam type is at most $\max(\omega,\sup_{i\in J}\tau(\mathfrak{D}_{d_i}))\le\max(\omega,\tau(\mathfrak{A}))$. If $K=\emptyset$ then $d^*=1$ and $\mathfrak{D}_{1\setminus d^*}=\{0\}$ is itself atomless and of countable π-weight. Otherwise, for each $i\in K$, let $\mathfrak{E}_i$ be an atomless order-closed subalgebra of $\mathfrak{D}_{d_i}$ with countable π-weight. Then $\mathfrak{E}=\prod_{i\in K}\mathfrak{E}_i$ is atomless and has countable π-weight and is an order-closed subalgebra of $\prod_{i\in K}\mathfrak{D}_{d_i}\cong\mathfrak{D}_{1\setminus d^*}$. So we have a decomposition as envisaged in the statement of the theorem.

546N Lemma Let $(X,\mathfrak{T},\Sigma,\mu)$ be a zero-dimensional quasi-Radon probability space and (Y,ρ) a complete separable metric space. On the space $C(X;Y)$ of continuous functions from X to Y we have a metric $\tilde\rho$ defined by saying that $\tilde\rho(f,g)=\sup_{x\in X}\min(1,\rho(f(x),g(x)))$ for f, $g\in C(X;Y)$. Now

$$\{g:g\in C(X;Y),\,\{x:(x,g(x))\in V\}\text{ is negligible}\}$$

is comeager in $C(X;Y)$ for every $V\in\mathcal{N}(\mu)\ltimes_{\Sigma\widehat{\otimes}\mathcal{B}(Y)}\mathcal{M}(Y)$.

proof (a) Let $\mathcal{W}$ be the family of sets expressible in the form $\bigcup_{n\in\mathbb{N}}E_n\times G_n$ where $E_n\in\Sigma$ and $G_n\subseteq Y$ is open for each n, and $\mathcal{W}^*$ the set of those $W\in\mathcal{W}$ such that all the vertical sections of W are dense in Y. If $W\in\mathcal{W}^*$ and $\epsilon>0$ then

$$Q=\{g:g\in C(X;Y),\,\mu\{x:x\in X,\,(x,g(x))\in W\}\ge 1-\epsilon\}$$

has dense interior in $C(X;Y)$. **P** Express W as $\bigcup_{n\in\mathbb{N}}E_n\times G_n$ where $E_n\in\Sigma$ and $G_n\subseteq Y$ is open for each n. Take any $g_0\in C(X;Y)$ and $\delta>0$, and let $\langle U_m\rangle_{m\in\mathbb{N}}$ run over a base for the topology of Y consisting of sets of diameter at most $\frac{1}{2}\delta$ (4A2Ob). For m, $n\in\mathbb{N}$ set $E_{nm}=E_n\cap g_0^{-1}[U_m]$, $G_{nm}=G_n\cap U_m$. If $x\in X$, there is an $m\in\mathbb{N}$ such that $g_0(x)\in U_m$; now $W[\{x\}]\cap U_m\ne\emptyset$, so there is an $n\in\mathbb{N}$ such that $x\in E_n$ and $G_n\cap U_m\ne\emptyset$; in which case $x\in E_{nm}$ and $G_n\cap U_m$ is non-empty. Thus $\bigcup_{(n,m)\in K}E_{nm}=X$, where $K=\{(n,m):G_n\cap U_m\ne\emptyset\}$. Of course every E_{nm} is measurable, because g_0 is continuous and $\mathfrak{T}\subseteq\Sigma$. Let $\langle E'_{nm}\rangle_{(n,m)\in K}$ be a disjoint family of measurable sets such that $E'_{nm}\subseteq E_{nm}$ for all $(n,m)\in K$ and $\bigcup_{(n,m)\in K}E'_{nm}=X$; let $J\subseteq K$ be a finite set such that $\mu(\bigcup_{(n,m)\in J}E'_{nm})\ge 1-\frac{1}{3}\epsilon$. Because μ is inner regular with respect to the closed sets, we can now find closed sets $F_{nm}\subseteq E'_{nm}$, for $(n,m)\in J$, such that $\mu(\bigcup_{(n,m)\in J}F_{nm})\ge 1-\frac{2}{3}\epsilon$.

For $(n,m)\in J$, set $B_{nm}=g_0^{-1}[U_m]\setminus\bigcup_{(i,j)\in J\setminus\{(n,m)\}}F_{ij}$, so that B_{nm} is open and includes F_{nm}. Each point of B_{nm} belongs to an open-and-closed set included in B_{nm}, because X is zero-dimensional. Because μ is τ-additive, there is an open-and-closed set $C_{nm}\subseteq B_{nm}$ such that $\mu(B_{nm}\setminus C_{nm})\le\frac{\epsilon}{3\#(J)+1}$. So if we set $C'_{nm}=C_{nm}\setminus\bigcup_{(i,j)\in J\setminus\{(n,m)\}}C_{ij}$, we shall again have an open-and-closed set such that $C'_{nm}\subseteq g_0^{-1}[U_m]$ and

$$\mu(F_{nm}\setminus C'_{nm})=\mu(F_{nm}\setminus C_{nm})\le\frac{\epsilon}{3\#(J)+1}$$

for $(n,m)\in J$; and $\mu(\bigcup_{(n,m)\in J}F_{nm}\cap C'_{nm})\ge 1-\epsilon$.

Now, for each $(n,m)\in J$, $G_n\cap U_m$ is non-empty; take $y_{nm}\in G_n\cap U_m$. Because $\langle C'_{nm}\rangle_{(n,m)\in J}$ is a finite disjoint family of open-and-closed subsets of X, we have a continuous $g_1:X\to Y$ defined by saying that $g_1(x)=y_{nm}$ whenever

$(n,m) \in J$ and $x \in C'_{nm}$, while $g_1(x) = g_0(x)$ for $x \in X \setminus \bigcup_{(n,m)\in J} C'_{nm}$. Note that if $(n,m) \in J$ and $x \in C'_{nm}$, then both $g_0(x)$ and y_{nm} belong to U_m, so $\rho(g_0(x), g_1(x)) \le \frac{1}{2}\delta$; accordingly $\tilde\rho(g_0, g_1) \le \frac{1}{2}\delta$.

Let $\eta \in \left]0, \min(1, \frac{1}{2}\delta)\right[$ be such that $y \in G_n$ whenever $(n,m) \in J$ and $\rho(y, y_{nm}) \le \eta$. Consider the non-empty open set $Q' = \{g : g \in C(X;Y),\ \tilde\rho(g, g_1) < \eta\}$. Then $\tilde\rho(g, g_0) < \delta$ for every $g \in Q'$. If $g \in Q'$, $(n,m) \in J$ and $x \in F_{nm} \cap C'_{nm}$, then $\rho(g(x), y_{nm}) < \eta$, so $g(x) \in G_n$; as $x \in E_n$, $(x, g(x)) \in W$. So $\{x : (x, g(x)) \in W\} = \bigcup_{n\in\mathbb{N}} E_n \cap g^{-1}[G_n]$ includes $\bigcup_{(n,m)\in J} F_{nm} \cap C'_{nm}$ has measure at least $1 - \epsilon$, and $g \in Q$.

This shows that $\operatorname{int} Q$ includes Q' and meets $\{g : \tilde\rho(g, g_0) < \delta\}$. As g_0 and δ are arbitrary, $\operatorname{int} Q$ is dense. **Q**

(b) Recall from 527I that every member of $\Sigma\widehat{\otimes}\mathcal{B}(Y)$ can be expressed in the form $W \triangle V$ where $W \in \mathcal{W}$ and $V \cap \bigcap_{n\in\mathbb{N}} W_n$ is empty for some sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}^*$. If $V \in \mathcal{N}(\mu) \ltimes_{\Sigma\widehat{\otimes}\mathcal{B}(Y)} \mathcal{M}(Y)$, we must have a $\tilde V \in (\mathcal{N}(\mu) \ltimes \mathcal{M}(Y)) \cap (\Sigma\widehat{\otimes}\mathcal{B}(Y))$ including V, and $W \in \mathcal{W}$ and a sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}^*$ such that $W \triangle \tilde V$ is disjoint from $\bigcap_{n\in\mathbb{N}} W_n$. For almost every $x \in X$, $W[\{x\}]$ is open and $\tilde V[\{x\}]$ is meager, so $W[\{x\}]$ has meager intersection with the comeager set $\bigcap_{n\in\mathbb{N}} W_n[\{x\}]$; because Y is a Baire space, $W[\{x\}] = \emptyset$. Thus $V[\{x\}] \cap \bigcap_{n\in\mathbb{N}} W_n[\{x\}] = \emptyset$ for almost every x. Now

$$Q_{mn} = \{g : g \in C(X;Y),\ \mu\{x : x \in X,\ (x, g(x)) \in W_m\} \ge 1 - 2^{-n}\}$$

is comeager for all m, $n \in \mathbb{N}$, by (a), while $\{g : g \in C(X;Y),\ \mu^*\{x : x \in X,\ (x, g(x)) \in V\} = 0\}$ includes $\bigcap_{m,n\in\mathbb{N}} Q_{mn}$, so is also comeager.

546O Lemma Suppose that Γ is an infinite set and $\mathcal{I}$ is a σ-ideal of subsets of $X \times Z$, where $X = \{0,1\}^\Gamma$ and $Z = \{0,1\}^{\mathbb{N}}$, such that

(i) writing ν_Γ for the usual measure on X, $\mathcal{I} \supseteq (\mathcal{N}(\nu_\Gamma) \ltimes \mathcal{M}(Z)) \cap \mathcal{B}\mathfrak{a}(X \times Z)$,

(ii) for every $W \subseteq X \times Z$ there is a $V \in \mathcal{B}\mathfrak{a}(X \times Z)$ such that $W \triangle V \in \mathcal{I}$,

(iii) $\operatorname{add} \mathcal{I} = \operatorname{non} \mathcal{M}(Z)$.

Then $\operatorname{cf}[\Gamma]^{\le\omega} > \operatorname{non} \mathcal{M}(Z)$.

proof ? Suppose, if possible, otherwise. Write κ for $\operatorname{non} \mathcal{M}(Z)$.

(a) Give $C(Z;Z)$ its topology of uniform convergence, so that it is a Polish space without isolated points (5A4I), and $\kappa = \operatorname{non} \mathcal{M}(C(Z;Z))$ (see 522Vb). Let $\langle g_\xi\rangle_{\xi<\kappa}$ enumerate a non-meager set in $C(Z;Z)$. Let $\langle I_\xi\rangle_{\xi<\kappa}$ run over a cofinal subset of $[\Gamma]^\omega$. For each $\xi < \kappa$, let $\theta_\xi : \omega \to I_\xi$ be a bijection, so that $x \mapsto x\theta_\xi$ is a continuous surjection from X onto Z, and for ξ, $\eta < \kappa$ write

$$h_{\xi\eta}(x) = g_\eta(x\theta_\xi)$$

for $x \in X$; this makes $h_{\xi\eta}$ a continuous function from X to Z.

For $\alpha < \kappa$ and $x \in X$,

$$M_{x\alpha} = \{h_{\xi\eta}(x) : \xi,\ \eta < \alpha\}$$

is meager in Z, because $\kappa = \operatorname{non} \mathcal{M}(Z)$. Let $\langle U_i\rangle_{i\in\mathbb{N}}$ enumerate a base for the topology of Z. For $x \in X$ and $\alpha < \kappa$ choose a sequence $\langle K(x,\alpha,n)\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that $\bigcup_{i\in K(x,\alpha,n)} U_i$ is dense in Z for each n and $M_{x\alpha} \cap \bigcap_{n\in\mathbb{N}} \bigcup_{i\in K(x,\alpha,n)} U_i = \emptyset$.

(b) Because $\operatorname{add} \mathcal{I} = \kappa$ there is a set $Y \subseteq X \times Z$ such that $Y \notin \mathcal{I}$ but there is a function $\psi : Y \to \kappa$ such that $\psi^{-1}[\alpha] \in \mathcal{I}$ for every $\alpha < \kappa$. Write

$$Q = \{(x, y, h_{\xi\eta}(x)) : (x,y) \in Y,\ \xi,\ \eta < \psi(x,y)\} \subseteq X \times Z \times Z.$$

Then $Q \in \mathcal{I} \ltimes \mathcal{M}(Z)$. Now there are $R \in \mathcal{B}\mathfrak{a}(X \times Z \times Z)$ and $A_0 \in \mathcal{I}$ such that $R[\{(x,y)\}] \in \mathcal{M}(Z)$ for every $x \in X$ and $y \in Z$, while $Q \subseteq R \cup (A_0 \times Z)$. **P** For n, $i \in \mathbb{N}$ set

$$C_{ni} = \{(x,y) : (x,y) \in Y,\ i \in K(x, \psi(x,y), n)\} \subseteq X \times Z.$$

Then

$$Q \cap \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} C_{ni} \times U_i = \emptyset.$$

Take $W_{ni} \in \mathcal{B}\mathfrak{a}(X \times Z)$ such that $C_{ni} \triangle W_{ni} \in \mathcal{I}$ for n, $i \in \mathbb{N}$, and set

$$R_0 = (X \times Z \times Z) \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} W_{ni} \times U_i,$$

$$A_0 = \bigcup_{n,i\in\mathbb{N}} C_{ni} \triangle W_{ni},$$

$$V = \{(x,y) : x \in X,\ y \in Z,\ R_0[\{(x,y)\}] \notin \mathcal{M}(Z)\},$$

$$R = R_0 \setminus (V \times Z).$$

Then $R_0 \in \mathcal{B}\mathfrak{a}(X \times Z \times Z)$ and $A_0 \in \mathcal{I}$. Next, $V \in \mathcal{B}\mathfrak{a}(X \times Z)$ (4A3Sa), so $R \in \mathcal{B}\mathfrak{a}(X \times Z \times Z)$, while $R[\{(x, y)\}]$ is meager for every $(x, y) \in X \times Z$. If $(x, y) \in Y \setminus A_0$, then

$$\begin{aligned} R_0[\{(x,y)\}] &= Z \setminus \bigcap_{n\in\mathbb{N}} \bigcup \{U_i : i \in \mathbb{N},\, (x, y) \in W_{ni}\} \\ &= Z \setminus \bigcap_{n\in\mathbb{N}} \bigcup \{U_i : i \in \mathbb{N},\, (x, y) \in C_{ni}\} \\ &= Z \setminus \bigcap_{n\in\mathbb{N}} \bigcup \{U_i : i \in K(x, \psi(x, y), n)\} \end{aligned}$$

is meager and includes $Q[\{(x, y)\}]$. So $Y \setminus A_0$ does not meet V and $Q \subseteq R \cup (A_0 \times Z)$. **Q**

(c) For any $x \in X$, consider the set $E_x = \{(y, z) : (x, y, z) \in R\}$. Then E_x is a Borel subset of $Z \times Z$ and $E_x[\{y\}]$ is meager for every $y \in Z$. By the Kuratowski-Ulam theorem (527E), $\{z : z \in Z,\ E_x^{-1}[\{z\}]$ is not meager$\}$ is meager. But this means that if we write

$$V' = \{(x, z) : \{y : (x, y, z) \in R\} \text{ is not meager}\},$$

then $\{z : (x, z) \in V'\}$ is meager for every $x \in X$, while $V' \in \mathcal{B}\mathfrak{a}(X \times Z)$ (4A3Sa again).

(d) At this point, note that because $R \in \mathcal{B}\mathfrak{a}(X \times Z \times Z)$ there is some countable subset I of Γ such that R belongs to $\Sigma_I \widehat{\otimes} \mathcal{B}(Z) \widehat{\otimes} \mathcal{B}(Z)$, where Σ_I is the σ-algebra generated by the sets $\{x : x(\gamma) = 1\}$ as γ runs over I. Let $\xi < \kappa$ be such that $I \subseteq I_\xi$. Now V' is of the form

$$\{(x, z) : (\tilde{x}, z) \in \tilde{V}'\},$$

where for $x \in X$ I define $\tilde{x} \in Z$ by setting $\tilde{x} = x\theta_\xi$; $\tilde{V}'$ is a Borel set in $Z \times Z$. For $w \in Z \times Z$, $\{z : (w, z) \in \tilde{V}'\} = \{z : (x, z) \in V'\}$ whenever $w = \tilde{x}$, so is meager; thus $\tilde{V}' \in \mathcal{N}(\nu_{\mathbb{N}}) \ltimes \mathcal{M}(Z)$, where $\nu_{\mathbb{N}}$ is the usual measure on Z.

Next, recall that $\{g_\eta : \eta < \kappa\}$ is not meager in $C(Z; Z)$. So by 546N there is an $\eta < \kappa$ such that $\{w : (w, g_\eta(w)) \in \tilde{V}'\}$ is $\nu_{\mathbb{N}}$-negligible. Accordingly

$$\{x : (x, h_{\xi\eta}(x)) \in V'\} = \{x : (x, g_\eta(\tilde{x})) \in V'\} = \{x : (\tilde{x}, g_\eta(\tilde{x})) \in \tilde{V}'\}$$

is ν_Γ-negligible. Set

$$A_1 = \{(x, y) : (x, y, h_{\xi\eta}(x)) \in R\}.$$

Because $h_{\xi\eta}$ is continuous and $R \in \mathcal{B}\mathfrak{a}(X \times Z \times Z)$, $A_1 \in \mathcal{B}\mathfrak{a}(X \times Z)$. Also

$$\{x : \{y : (x, y) \in A_1\} \text{ is not meager}\} = \{x : (x, h_{\xi\eta}(x)) \in V'\} \in \mathcal{N}(\nu_\Gamma),$$

so $A_1 \in \mathcal{N}(\nu_\Gamma) \ltimes \mathcal{M}(Z)$ and $A_1 \in \mathcal{I}$. There is therefore some $(x, y) \in Y \setminus (A_0 \cup A_1 \cup \psi^{-1}[\alpha])$, where $\alpha = \max(\xi, \eta) + 1$. In this case, however, $\psi(x, y) \geq \alpha$, so $(x, y, h_{\xi\eta}(x)) \in Q$; as $(x, y) \notin A_0$, $(x, y, h_{\xi\eta}(x)) \in R$ and $(x, y) \in A_1$, which is absurd. **X**

546P Theorem Let $\mathfrak{A}$ be a measurable algebra, not $\{0\}$, and $\mathfrak{G}_{\mathbb{N}}$ the category algebra of $Z = \{0, 1\}^{\mathbb{N}}$. Suppose that the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}_{\mathbb{N}}$ is a power set σ-quotient algebra. Then there is a quasi-measurable cardinal less than $\tau(\mathfrak{A})$.

proof (a) To begin with (down to the end of (c) below), suppose that $\mathfrak{A}$ is homogeneous with infinite Maharam type, so that there is a set Γ of cardinal $\tau(\mathfrak{A})$ such that $\mathfrak{A}$ is isomorphic to the measure algebra of the usual measure ν_Γ on $X = \{0, 1\}^\Gamma$. Note that ν_Γ is completion regular (416U), so $\mathfrak{A}$ can be regarded as the measure algebra of $\nu_\Gamma \restriction \mathcal{B}\mathfrak{a}(X)$. By 527O, the Dedekind completion of the free product $\mathfrak{A} \otimes \mathfrak{G}_{\mathbb{N}}$ is isomorphic to $\mathfrak{C} = (\mathcal{B}\mathfrak{a}(X) \widehat{\otimes} \mathcal{B}(Z))/\mathcal{L}$, where $\mathcal{L} = (\mathcal{B}\mathfrak{a}(X) \widehat{\otimes} \mathcal{B}(Z)) \cap (\mathcal{N}(\nu_\Gamma) \ltimes \mathcal{M}(Y))$; also $\mathfrak{C}$ is ccc.

Observe that $\mathcal{B}\mathfrak{a}(X) \widehat{\otimes} \mathcal{B}(Z) = \mathcal{B}\mathfrak{a}(X \times Z)$. So we can apply 546C to see that we have a σ-ideal $\mathcal{I}$ of subsets of $X \times Z$ such that

(α) $\mathcal{B}\mathfrak{a}(X \times Z) \cap \mathcal{I} = \mathcal{L}$,

(β) for every $A \subseteq X \times Z$ there is an $E \in \mathcal{B}\mathfrak{a}(X \times Z)$ such that $A \triangle E \in \mathcal{I}$,

(γ) setting $\kappa = \operatorname{add} \mathcal{I}$, there is a normal ideal $\mathcal{J}$ on κ such that $\mathcal{P}\kappa/\mathcal{J}$ is isomorphic to an atomless order-closed subalgebra of a principal ideal of $\mathfrak{C}$.

In particular, κ is quasi-measurable. But $\mathfrak{A} \otimes \mathfrak{G}_{\mathbb{N}}$ and $\mathfrak{C}$ are homogeneous, by 546Ha and 316P, so $\mathcal{P}\kappa/\mathcal{J}$ is isomorphic to an atomless order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{C}$ itself.

Now $\kappa \leq \operatorname{non} \mathcal{M}(Z)$. **P** X can be covered by $\operatorname{cov} \mathcal{N}(\nu_\Gamma)$ negligible sets, and $\operatorname{cov} \mathcal{N}(\nu_\Gamma) \leq \operatorname{cov} \mathcal{N}(\nu_{\mathbb{N}})$ (523B), so $X \times Z$ can be covered by $\operatorname{cov} \mathcal{N}(\nu_{\mathbb{N}})$ members of $\mathcal{L}$ and

$$\kappa \le \operatorname{cov}\mathcal{I} \le \operatorname{cov}\mathcal{L} \le \operatorname{cov}\mathcal{N}(\nu_{\mathbb{N}}) \le \operatorname{non}\mathcal{M}(Z)$$

by 522E (using 522V to move between Lebesgue measure on $\mathbb{R}$ and $\nu_{\mathbb{N}}$ on Z). **Q**

(b) Suppose that $\mathfrak{D}$ has a non-zero principal ideal $\mathfrak{D}_d$ which is a measurable algebra with Maharam type at most $\tau(\mathfrak{A})$. Then $\mathfrak{D}_d$ is a power set σ-quotient algebra (546Bb), and there is a quasi-measurable cardinal (in fact, an atomlessly-measurable cardinal) less than $\tau(\mathfrak{D}_d) \le \tau(\mathfrak{A})$, by 546Bd. So in this case we can stop.

(c) Otherwise, 546M tells us that $\mathfrak{D}$ has an atomless order-closed subalgebra $\mathfrak{E}$ of countable π-weight; by 546Ha, $\mathfrak{E} \cong \mathfrak{G}_{\mathbb{N}}$. In this case, $\operatorname{non}\mathcal{M}(Z) \le \kappa$. **P** We have an injective order-continuous Boolean homomorphism from $\mathcal{B}(Z)/\mathcal{B}(Z) \cap \mathcal{M}(Z)$ to $\mathcal{P}\kappa/\mathcal{J}$. By 546C, there is an $h : \kappa \to Z$ such that, for $E \in \mathcal{B}(Z)$, $h^{-1}[E] \in \mathcal{J}$ iff $E \in \mathcal{M}(Z)$. But this means that no meager subset of Z can include $h[\kappa]$, and $h[\kappa]$ witnesses that $\operatorname{non}\mathcal{M}(Z) \le \kappa$. **Q**

Putting this together with (a), we see that κ is precisely $\operatorname{non}\mathcal{M}(Z)$, and that all the conditions of 546O are satisfied. So we must have $\operatorname{cf}[\Gamma]^{\le\omega} > \kappa$. But this implies that $\#(\Gamma) > \kappa$ (542I). So in this case also we have $\tau(\mathfrak{A}) = \#(\Gamma)$ greater than some quasi-measurable cardinal.

(d) All this has been done on the assumption that $\mathfrak{A}$ is homogeneous and atomless. In general, $\mathfrak{A}$ has a non-zero homogeneous principal ideal $\mathfrak{A}_a$. Now the Dedekind completion of $\mathfrak{A}_a \otimes \mathfrak{G}_{\mathbb{N}}$ can be identified with a principal ideal of $\mathfrak{C}$, so is also a power set σ-quotient algebra (546Bb). Since $\mathfrak{G}_{\mathbb{N}}$ itself is *not* a power set σ-quotient algebra (546I), $\mathfrak{A}_a \ne \{0, a\}$ and $\mathfrak{A}_a$ is atomless. So (a)-(c) tell us that there is a quasi-measurable cardinal strictly less than $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$, and the proof is complete.

546Q Corollary If μ is Lebesgue measure on $\mathbb{R}$, then $\mathfrak{C} = \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{N}(\mu) \ltimes \mathcal{M}(\mathbb{R}))$ is not a power set σ-quotient algebra.

proof Let ν be a probability measure with the same domain and the same null sets as μ, and $\mathfrak{A}$ its measure algebra. Then $\mathfrak{C}$ can be identified with the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}$, where $\mathfrak{G}$ is the category algebra of $\mathbb{R}$ (527O). Also $\mathfrak{G}$ is isomorphic to the category algebra of $\{0,1\}^{\mathbb{N}}$ (522Vb), while $\tau(\mathfrak{A}) = \omega$. So 546P tells us that $\mathfrak{C}$ cannot be a power set σ-quotient algebra.

546X Basic exercises (a) Let $\mathfrak{A}$ be any Dedekind complete ccc Boolean algebra. Show that it is isomorphic to an order-closed subalgebra of a power set σ-quotient algebra. (*Hint*: 314M.)

(b) Show that a power set σ-quotient algebra with countable π-weight is purely atomic.

546Y Further exercises (a) For cardinals κ, write $\mathfrak{G}_\kappa$ for the category algebra of $\{0,1\}^\kappa$. Show that any order-closed subalgebra of $\mathfrak{G}_{\omega_1}$ has a principal ideal isomorphic to one of $\{0,1\}$, $\mathfrak{G}_\omega$ or $\mathfrak{G}_{\omega_1}$, and is therefore a power set σ-quotient algebra iff it is purely atomic.

(b) Let $\mathfrak{A}$ be a measurable algebra, $\mathfrak{B}$ a Boolean algebra, $\mathfrak{C}$ the Dedekind completion of $\mathfrak{A}\otimes\mathfrak{B}$, and $\mathfrak{D}$ an order-closed subalgebra of $\mathfrak{C}$. Show that $\tau(\mathfrak{D}) \le \max(\omega, \tau(\mathfrak{A}), \pi(\mathfrak{B}))$.

546Z Problems (a) Can the category algebra of $\{0,1\}^{\omega_2}$ be a power set σ-quotient algebra?

(b) Let $\mathfrak{G}_{\mathbb{N}}$ be the category algebra of $\{0,1\}^{\mathbb{N}}$. Can there be any non-zero measurable algebra $\mathfrak{A}$ such that the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}_{\mathbb{N}}$ is a power set σ-quotient algebra? 546P tells us only that such an algebra $\mathfrak{A}$ must have large Maharam type.

(c) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $\mathbb{R}$, and $\mathfrak{G}_{\omega_1}$ the category algebra of $\{0,1\}^{\omega_1}$. Can the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}_{\omega_1}$ be a power set σ-quotient algebra? Many of the arguments in 546L-546M can be extended to $\mathfrak{G}_{\omega_1}$ and beyond, but the final steps elude me.

(d) Can there be an atomless power set σ-quotient algebra $\mathfrak{A}$ such that $c(\mathfrak{A}) = \omega$ and $\pi(\mathfrak{A}) = \omega_1$? Note that there seems to be no obstacle to an atomless power set σ-quotient algebra having Maharam type ω (555M).

546 Notes and comments The most notable feature of this section is just how hard we have to work to get results for two 'standard' algebras: the category algebra $\mathfrak{G}_{\mathbb{N}}$ of $\{0,1\}^{\mathbb{N}}$ and the Dedekind completion of $\mathfrak{B}_{\mathbb{N}} \otimes \mathfrak{G}_{\mathbb{N}}$, where $\mathfrak{B}_{\mathbb{N}}$ is the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$. As usual, I give the arguments in general forms, but as far as I know most of the steps have to be gone through for the special cases. The same thing happened, of course, in §543; most of the ideas of 543E, other than some of the infinitary combinatorics from §5A1, are needed to show that $\mathfrak{B}_{\mathbb{N}}$ is not a power set σ-quotient algebra. The byways we have to trace are very interesting (see 546D, for instance), and some of them involve non-trivial facts about 'standard' spaces (546J-546K). Part of the extraordinary nature of the topic lies in the way we have to couple ideas based on Borel sets in $\mathbb{R}^2$ with properties of large cardinals – 'large' in the sense of infinitary combinatorics, that is; in this section everything is bounded by $\mathfrak{c}$.

The general message of the work here seems to be that 'standard' algebras cannot be power set σ-quotient algebras. The arguments as I have presented them use restrictions on Maharam type or π-weight; but it may be that we ought rather to look at the descriptive set theory of the ideals (see the examples in §555 below). Note that many of the algebras we are interested in can be expressed as order-closed subalgebras of power set σ-quotient algebras (546Xa).

All the results here depend on a move from a power set σ-quotient algebra to an order-closed subalgebra with the stronger property that it is of the form $\mathcal{P}\kappa/\mathcal{I}$ where $\mathcal{I}$ is κ-additive, as in 546Cd. If we start from a measurable algebra, the subalgebra will again be measurable, with Maharam type no greater than that of the original algebra; this is why we can move from 543E to 543F. For category algebras this doesn't work in the same way, because an order-closed subalgebra of the category algebra of $\{0,1\}^{\omega_2}$, for instance, can be very different in character. (For examples see Koppelberg & Shelah 96 and Balcar Jech & Zapletal 97.) So 546G gives us only 546I and 546Ya. The discussion of completed free products in 546L-546M depends on a careful analysis of their order-closed subalgebras which seems not to have simple generalizations.

547 Disjoint refinements of sequences of sets

I continue my account of results from Gitik & Shelah 01 and Burke n96. Given a family $\langle A_i\rangle_{i\in I}$ of sets in a measure space, when can we find a disjoint family $\langle A'_i\rangle_{i\in I}$ such that $A'_i \subseteq A_i$ has the same outer measure as A_i for every i? A partial result is in Theorem 547F. Allied questions are: when can we find a set D such that $A_i \cap D$ and $A_i \setminus D$ have the same outer measure as A_i for every i? (547G) or are just non-negligible? (547I).

547A Lemma Let (X,Σ,μ) be a totally finite measure space with Maharam type $\tau(\mu)$. Suppose that $A \subseteq X$ is such that $\min(\mu^*D, \mu^*(A\setminus D)) < \mu^*A$ for every $D \subseteq A$.

(a) There is a non-negligible set $B \subseteq A$ such that the completion $\hat\mu_B$ of the subspace measure on B measures every subset of B.

(b) If μ is atomless, there is an atomlessly-measurable cardinal κ such that $\min(\kappa^{(+\omega)}, 2^\kappa) \le \tau(\mu)$.

(c) If μ is purely atomic and singletons in A are negligible, there is a two-valued-measurable cardinal $\kappa \le \#(A)$.

proof (a)(i) Let $\langle(F_i, D_i)\rangle_{i\in I}$ be a maximal family such that

$$F_i \in \Sigma, \quad \mu^*(F_i \cap A \cap D_i) = \mu^*(F_i \cap A \setminus D_i) = \mu F_i > 0$$

for every i, and $\langle F_i\rangle_{i\in I}$ is disjoint. Because $\mu F_i > 0$ for every i, I must be countable. Set $E = \bigcup_{i\in I} F_i$. **?** If $B = A\setminus E$ is negligible, set $D = \bigcup_{i\in I} F_i \cap A \cap D_i$; then

$$\mu^*A = \mu^*(A\cap E) = \sum_{i\in I} \mu^*(A \cap F_i)$$

(put 132Ae and 132Af together)

$$= \sum_{i\in I} \mu F_i = \sum_{i\in I} \mu^*(F_i \cap A \cap D_i) = \sum_{i\in I} \mu^*(F_i \cap D) = \mu^*D,$$

and similarly $\mu^*A = \mu^*(A\setminus D)$, contrary to hypothesis. **X** So $\mu^*B > 0$.

(ii) If $D \subseteq B$ then $\hat\mu_B$ measures D. **P** Let $H_1 \subseteq X\setminus E$, $H_2 \subseteq X \setminus E$ be measurable envelopes of D and $B\setminus D$ respectively, and set $F = H_1 \cap H_2$. Then

$$\begin{aligned}\mu^*(F\cap B\cap D) &= \mu^*(F\cap D) = \mu(F\cap H_1) = \mu F\\ &= \mu(F\cap H_2) = \mu^*(F\cap B\setminus D) = \mu^*(F\cap A\setminus D).\end{aligned}$$

But $F \cap F_i = \emptyset$ for every $i \in I$, so we must have $\mu F = 0$. Now $B\setminus H_2 \subseteq D \subseteq B\cap H_1$ and $\mu_B((B\cap H_1)\setminus(B\setminus H_2)) = 0$, so $\hat\mu_B$ measures D. **Q**

(b) If μ is atomless, the measure algebra of $\hat\mu_B$ can be identified with the measure algebra of μ_B (322Da) and a principal ideal in the measure algebra of μ (322I), so is atomless, and $\tau(\hat\mu_B) \le \tau(\mu)$ (331Hc). Setting $\kappa = \operatorname{add}\hat\mu_B$, κ is an atomlessly-measurable cardinal (543Ba) and

$$\min(\kappa^{(+\omega)}, 2^\kappa) \le \tau(\hat\mu_B) \le \tau(\mu)$$

by the Gitik-Shelah theorem (543F).

(c) If μ is purely atomic, so are μ_B (214Xe, or use the method of (b)) and $\hat\mu_B$ (212Gd). Also $\hat\mu_B\{x\} = 0$ for every $x \in B$. So $\operatorname{add}\hat\mu_B$ is two-valued-measurable (543B(f-i)), and is at most $\#(B) \le \#(A)$.

547B Lemma Let (X,Σ,μ) be an atomless probability space, $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X, and $A\subseteq X$ a non-negligible set. Then

either (α) there is a set $C\subseteq A$ such that $\mu^*C>0$ and $\mu^*(A_n\setminus C)=\mu^*A_n$ for every $n\in\mathbb{N}$

or (β) there is a quasi-measurable cardinal less than the Maharam type of μ.

proof Suppose that (α) is false. If there is an atomlessly-measurable cardinal less than the Maharam type of μ, we can stop; so let us suppose that there is no such cardinal.

(a) To begin with (down to the end of (e)) let us suppose that $\mu^*A=1$. For $\sigma\in S_2^*=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ choose D_σ, E_σ and D'_σ inductively, as follows. Start with $D_\emptyset=A$. Given that $D_\sigma\subseteq A$ and $\mu^*D_\sigma=1$, where $\sigma\in\{0,1\}^n$, let E_σ be a measurable envelope of $D_\sigma\cap A_n$, and set $D'_\sigma=(D_\sigma\cap A_n)\cup(D_\sigma\setminus E_\sigma)$; then $\mu^*D'_\sigma=1$. By 547Ab, we can choose $D_{\sigma^\frown<0>}\subseteq D'_\sigma$ such that

$$\mu^*D_{\sigma^\frown<0>}=\mu^*(D'_\sigma\setminus D_{\sigma^\frown<0>})=1,$$

and set $D_{\sigma^\frown<1>}=D_\sigma\setminus D_{\sigma^\frown<0>}$. Then

$$\mu^*D_{\sigma^\frown<0>}=\mu^*D_{\sigma^\frown<1>}=1.$$

Continue.

(b) Set $Z=\{0,1\}^{\mathbb{N}}$, and for $\sigma\in S_2^*$ set $Z_\sigma=\{z:\sigma\subseteq z\in Z\}$. Define $f:A\to X\times Z$ by setting $f(x)=(x,\alpha)$, where α is that member of Z such that $x\in D_{\alpha\restriction n}$ for every n. Then for any non-negligible $C\subseteq A$ there are a non-negligible measurable set $E\subseteq X$ and a $\sigma\in S_2^*$ such that $f^{-1}[E\times Z_\sigma]\subseteq C$. **P** Because ($\alpha$) is supposed to be false, there is some $n\in\mathbb{N}$ such that $\mu^*(A_n\setminus C)<\mu^*A_n$. Let F be a measurable envelope of $A_n\setminus C$, so that $A_n\setminus F\subseteq C$ and $\mu^*(A_n\setminus F)>0$. As $A_n\setminus F\subseteq A$, there is some $\sigma\in\{0,1\}^n$ such that $\mu^*(D_\sigma\cap A_n\setminus F)>0$. Set $E=E_\sigma\setminus F$. Now $\mu E=\mu^*(D_\sigma\cap A_n\setminus F)>0$ and

$$f^{-1}[E\times Z_{\sigma^\frown<0>}]=D_{\sigma^\frown<0>}\cap E\subseteq D'_\sigma\cap E_\sigma\setminus F\subseteq D_\sigma\cap A_n\setminus F\subseteq C. \textbf{ Q}$$

(c)(i) Write $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\mathcal{B}(Z))\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Z))$, where, as in §546, $\mathcal{B}(Z)$ is the Borel σ-algebra of Z, $\mathcal{N}(\mu)$ is the null ideal of μ and $\mathcal{M}(Z)$ is the σ-ideal of meager subsets of Z. Let $\mathcal{W}$ be the family of subsets of $X\times Z$ expressible as $\bigcup_{\sigma\in S_2^*}F_\sigma\times Z_\sigma$ where $F_\sigma\in\Sigma$ for every $\sigma\in S_2^*$, and $\mathcal{V}$ the set of those $V\in\mathcal{W}$ such that $V[\{x\}]$ is dense for every $x\in X$.

(ii) If $V\in\mathcal{W}$ and $f^{-1}[(X\times Z)\setminus V]\notin\mathcal{N}(\mu)$ then $V\notin\mathcal{V}$. **P** By (b), there are a non-negligible $E\in\Sigma$ and a $\sigma\in\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ such that $f^{-1}[E\times Z_\sigma]\subseteq f^{-1}[(X\times Z)\setminus V]$, that is, $E\cap D_\sigma$ does not meet $f^{-1}[V]$. Express V as $\bigcup_{\tau\in S_2^*}(F_\tau\times Z_\tau)$ where every F_τ is measurable. Setting

$$E'=\bigcup\{E\cap F_\tau:\tau\in S_2^*,\ E\cap F_\tau\text{ is negligible}\},$$

there must be some $x\in D_\sigma\cap E\setminus E'$. If $\tau\in S_2^*$ and $x\in F_\tau$, then $E\cap F_\tau$ is non-negligible and meets D_τ in y say. But this means that $y\in E$ and $f(y)\in F_\tau\times Z_\tau$ belongs to V; so $y\notin D_\sigma$ and $\tau\not\supseteq\sigma$. Thus $V[\{x\}]\cap Z_\sigma$ is empty and $V[\{x\}]$ is not dense and $V\notin\mathcal{V}$. **Q**

(iii) If $W\in\mathcal{L}$ then $f^{-1}[W]\in\mathcal{N}(\mu)$. **P** By 527I, we can find $W_0\in\mathcal{W}$ and a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ such that $(W\triangle W_0)\cap\bigcap_{n\in\mathbb{N}}V_n=\emptyset$. By (ii) just above, $f^{-1}[(X\times Z)\setminus V_n]$ is negligible for every n, so $f^{-1}[W]\triangle f^{-1}[W_0]$ is negligible. Also $(X\times Z)\setminus V_n$ belongs to $\mathcal{L}$ for every n. Accordingly $W\triangle W_0$ and W_0 belong to $\mathcal{L}$, so

$$f^{-1}[W_0]\subseteq\{x:W_0[\{x\}]\ne\emptyset\}=\{x:W_0[\{x\}]\notin\mathcal{M}(Z)\}$$

is negligible, and $f^{-1}[W]$ is negligible. **Q**

(d) It follows that $\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}\cong\mathcal{P}A/\mathcal{P}A\cap\mathcal{N}(\mu)$. **P** Since $f^{-1}[W]\in\mathcal{N}(\mu)$ for every $W\in\mathcal{L}$, we have a Boolean homomorphism $\pi:\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}\to\mathcal{P}A/\mathcal{P}A\cap\mathcal{N}(\mu)$ defined by setting $\pi W^\bullet=(f^{-1}[W])^\bullet$ for every $W\in\Sigma\widehat{\otimes}\mathcal{B}(Z)$. π is sequentially order-continuous, and because $\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}$ is ccc (527O), π is order-continuous. If $W\in(\Sigma\widehat{\otimes}\mathcal{B}(Z))\setminus\mathcal{L}$, there are a non-negligible measurable set $E\subseteq X$ and a $\sigma\in S_2^*$ such that $(E\times Z_\sigma)\setminus W\in\mathcal{L}$. (Take $W'\in\mathcal{W}$ such that $W\triangle W'\in\mathcal{L}$, and $E\times Z_\sigma\subseteq W'$.) Now $\mu^*f^{-1}[E\times Z_\sigma]=\mu^*(E\cap D_\sigma)>0$, while (by (c)) $f^{-1}[(E\times Z_\sigma)\setminus W]$ is negligible, so $f^{-1}[W]$ is not negligible. Thus $\pi W^\bullet\ne 0$ in $\mathcal{P}A/\mathcal{N}(\mu)$; as W is arbitrary, π is injective. Finally, if $C\in\mathcal{P}A\setminus\mathcal{N}(\mu)$, (b) tells us that $\pi(E\times Z_\sigma)^\bullet\subseteq C^\bullet$ for some non-negligible E and some σ; so that the range of π is order-dense. Because $\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}$ is Dedekind complete (527O), π is a surjection (314F(a-i)), and $\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}\cong\mathcal{P}A/\mathcal{P}A\cap\mathcal{N}(\mu)$. **Q**

(e) But this means that $\Sigma\widehat{\otimes}\mathcal{B}(Z)/\mathcal{L}$ is a power set σ-quotient algebra. By 527O and 546P, there is a quasi-measurable cardinal less than the Maharam type of μ. Thus the result is proved in the case in which $\mu^*A=1$.

(f) For the general case, let H be a measurable envelope of A and $\nu=(\mu H)^{-1}\mu_H$ the normalized subspace measure on H. Then $A\cap H$ is of full outer measure in H. If $C\subseteq A\cap H$ is such that $\nu^*C>0$, then

$$\mu^*C=\mu_H^*C=\mu H\cdot\nu^*C>0,$$

and there is an $n \in \mathbb{N}$ such that

$$\begin{aligned}\mu^*(A_n \setminus H) + \mu H \cdot \nu^*(A_n \cap H) &= \mu^*(A_n \setminus H) + \mu^*(A_n \cap H) = \mu^* A_n \\ &> \mu^*(A_n \setminus C) = \mu^*(A_n \setminus H) + \mu^*(A_n \cap H \setminus C) \\ &= \mu^*(A_n \setminus H) + \mu H \cdot \nu^*(A_n \cap H \setminus C),\end{aligned}$$

that is, $\nu^*(A_n \cap H \setminus C) < \nu^*(A_n \cap H)$. By (a)-(e), there is a quasi-measurable cardinal less than the Maharam type of ν. But the measure algebra of ν is isomorphic, up to a scalar multiple of the measure, to a principal ideal of the measure algebra of μ, so there is a quasi-measurable cardinal less than the Maharam type of μ. Thus (β) is true in this case also.

547C Lemma Let (X, Σ, μ) be a purely atomic probability space, $\langle A_n \rangle_{n \in \mathbb{N}}$ a sequence of subsets of X and $A \subseteq X$ a non-negligible set. Then

either (α) there is an $x \in A$ such that $\mu^*\{x\} > 0$

or (β) there is a set $C \subseteq A$ such that $\mu^* C > 0$ and $\mu^*(A_n \setminus C) = \mu^* A_n$ for every $n \in \mathbb{N}$

or (γ) there is a two-valued-measurable cardinal less than or equal to $\#(A)$.

proof Suppose that (α) and (β) are false. Because μ is purely atomic, there is a countable family $\langle E_i \rangle_{i \in I}$ of atoms for μ such that $\bigcup_{i \in I} E_i$ is conegligible in X. Consider the algebra $\mathfrak{A} = \mathcal{P}A/\mathcal{P}A \cap \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ. For each $n \in \mathbb{N}$ and $i \in I$, set $a_n = (A_n \cap A)^\bullet$ and $e_i = (A \cap E_i)^\bullet$ in $\mathfrak{A}$. Then $\{a_n \cap e_i : n \in \mathbb{N}, i \in I\}$ is order-dense in $\mathfrak{A}$. **P** If $c \in \mathfrak{A} \setminus \{0\}$, then $c = C^\bullet$ for some non-negligible $C \subseteq A$; now there are an $n \in \mathbb{N}$ such that $\mu^*(A_n \setminus C) < \mu^* A_n$ and an $E \in \Sigma$ such that $A_n \setminus C \subseteq E$ and $\mu E < \mu^* A_n$. In this case, $A_n \setminus E$ is a non-negligible subset of C; let $i \in I$ be such that $\mu^*(E_i \cap A_n \setminus E) > 0$. As E_i is an atom, $E_i \cap E$ is negligible and $(A_n \cap E_i) \setminus C$ is negligible. But this means that $a_n \cap e_i \subseteq c$. Also

$$\mu^*(A \cap E_i \cap A_n) \ge \mu^*(C \cap E_i \cap A_n) \ge \mu^*(E_i \cap A_n \setminus E) > 0,$$

so $a_n \cap e_i \ne 0$. **Q**

It follows that $\mathfrak{A}$ is purely atomic. **P?** Otherwise, there is some $B \subseteq A$ such that the principal ideal $\mathfrak{A}_b$ of $\mathfrak{A}$ generated by $b = B^\bullet$ is non-zero and atomless. Now $\mathcal{P}B/\mathcal{P}B \cap \mathcal{N}(\mu) \cong \mathfrak{A}_b$ has countable π-weight (514Fd), and in particular is ccc; as it is Dedekind σ-complete (314C), it is Dedekind complete (316Fa); as it is atomless and not $\{0\}$, it is isomorphic to the category algebra of $\{0,1\}^{\mathbb{N}}$ (546H). But it is also a power set σ-quotient algebra, contradicting 546I. **XQ**

Let d be an atom of $\mathfrak{A}$ and $D \subseteq A$ a set such that $D^\bullet = d$. Then for every $C \subseteq D$, one of $C^\bullet$, $(D \setminus C)^\bullet$ must be zero; that is, $\mathcal{P}D \cap \mathcal{N}(\mu)$ is a maximal ideal, and the completion $\hat{\mu}_D$ of the subspace measure μ_D measures every subset of D.

At this point recall that we are supposing that every singleton subset of A is negligible, so that $\operatorname{add} \hat{\mu}_D \le \#(D)$. By 543B(f-i), $\operatorname{add} \hat{\mu}_D$ is two-valued-measurable, and (γ) is true.

547D Lemma Let (X, Σ, μ) be a probability space in which singletons are negligible. Suppose

either that μ is atomless and there is no quasi-measurable cardinal less than the Maharam type of μ

or that μ is purely atomic and there is no two-valued-measurable cardinal less than or equal to $\#(X)$.

Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of X. Then there is a set $A \subseteq A_0$ such that $\mu^* A = \mu^* A_0$ and $\mu^*(A_n \setminus A) = \mu^* A_n$ for every $n \in \mathbb{N}$.

proof Let $\langle (C_i, E_i) \rangle_{i \in I}$ be a maximal family such that

$$C_i \subseteq A_0, \quad \mu^*(A_n \setminus C_i) = \mu^* A_n \text{ for every } n \ge 1, \quad \mu^* C_i > 0,$$

$$E_i \text{ is a measurable envelope of } C_i$$

for every $i \in I$, and $E_i \cap E_j = \emptyset$ for $i \ne j$. Then I must be countable. Set

$$F = X \setminus \bigcup_{i \in I} E_i, \quad A = \bigcup_{i \in I} C_i.$$

If $n \ge 1$, then

$$\begin{aligned}\mu^*(A_n \setminus A) &= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i \setminus A) \\ &= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i \setminus C_i) \\ &= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i) = \mu^* A_n.\end{aligned}$$

? Suppose, if possible, that $\mu^*A < \mu^*A_0$. Then

$$\mu^*A < \mu^*(A_0 \cap F) + \sum_{i\in I}\mu^*(A_0 \cap E_i) \le \mu^*(A_0 \cap F) + \sum_{i\in I}\mu E_i$$
$$= \mu^*(A_0 \cap F) + \sum_{i\in I}\mu^*(A \cap E_i) \le \mu^*(A_0 \cap F) + \mu^*A,$$

so $\mu^*(A_0 \cap F) > 0$. By 547B or 547C, there is a $C \subseteq A_0 \cap F$ such that $\mu^*C > 0$ and $\mu^*(A_n \setminus C) = \mu^*A_n$ for every $n \ge 1$. Let $E \subseteq F$ be a measurable envelope for C. Then we ought to have added (C, E) to $\langle(C_i, E_i)\rangle_{i\in I}$. **X**

Accordingly $\mu^*A = \mu^*A_0$ and we're home.

547E Lemma Let (X, Σ, μ) be a totally finite measure space in which singletons are negligible. Suppose that there is no quasi-measurable cardinal less than or equal to $\#(X)$. Then for any $A \subseteq X$ there is a disjoint family $\langle D_\xi\rangle_{\xi<\omega_1}$ of subsets of A such that $\mu^*D_\xi = \mu^*A$ for every $\xi < \omega_1$.

proof (a) Let $\langle(F_i, \langle D_{i\xi}\rangle_{\xi<\omega_1})\rangle_{i\in I}$ be a maximal family such that

$F_i \in \Sigma$,

$\langle D_{i\xi}\rangle_{\xi<\omega_1}$ is a disjoint family of subsets of $F_i \cap A$,

$\mu^*D_{i\xi} = \mu F_i > 0$ for every $\xi < \omega_1$

for each $i \in I$, and $\langle F_i\rangle_{i\in I}$ is disjoint. Then I is countable. Set $E = \bigcup_{i\in I} F_i$ and $B = A \setminus E$.

(b) B is negligible. **P?** Otherwise, consider the σ-ideal $\mathcal{N}(\mu_B) = \mathcal{P}B \cap \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ. This is a proper σ-ideal containing singletons; because its additivity (being less than or equal to $\#(X)$) is not quasi-measurable, $\mathcal{N}(\mu_B)$ cannot be ω_1-saturated, and there is a disjoint family $\langle C_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{P}B \setminus \mathcal{N}(\mu)$. For $\xi \le \zeta < \omega_1$ let $H_{\xi\zeta} \subseteq X \setminus E$ be a measurable envelope of $\bigcup_{\xi\le\eta<\zeta} C_\eta$ and set $a_{\xi\zeta} = H^\bullet_{\xi\zeta}$ in the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ. If $\xi \le \xi' \le \zeta' \le \zeta < \omega_1$ then $\bigcup_{\xi'\le\eta<\zeta'} C_\eta \subseteq H_{\xi'\zeta'} \cap H_{\xi\zeta}$, so $\mu(H_{\xi'\zeta'} \setminus H_{\xi\zeta}) = 0$ and $a_{\xi'\zeta'} \subseteq a_{\xi\zeta}$. For each $\xi < \omega_1$, set $a_\xi = \sup_{\xi\le\zeta<\omega_1} a_{\xi\zeta}$, and let $h(\xi) < \omega_1$ be such that $a_\xi = a_{\xi,h(\xi)}$. Next, $\langle a_\xi\rangle_{\xi<\omega_1}$ is non-increasing; set $a = \inf_{\xi<\omega_1} a_\xi$. Let $\alpha < \omega_1$ be such that $a = a_\alpha$. Note that

$$a = a_\alpha \supseteq a_{\alpha\alpha} \ne 0$$

because C_α is non-negligible, and that $a \cap E^\bullet = 0$ because no C_ξ meets E.

Let $F \subseteq X \setminus E$ be such that $a = F^\bullet$. Now let $\langle \alpha_\xi\rangle_{\xi<\omega_1}$ be a strictly increasing family in ω_1 such that $\alpha_0 = 0$ and $\alpha_{\xi+1} \ge h(\alpha_\xi)$ for each ξ. Set $D_\xi = F \cap \bigcup_{\alpha_\xi\le\eta<\alpha_{\xi+1}} C_\eta$ for each $\xi < \omega_1$. Then $\langle D_\xi\rangle_{\xi<\omega_1}$ is a disjoint family of subsets of $F \cap A$. Next, for each ξ, $H_{\alpha_\xi,\alpha_{\xi+1}}$ is a measurable envelope of $\bigcup_{\alpha_\xi\le\eta<\alpha_{\xi+1}} C_\eta$, so

$$\mu^*D_\xi = \mu(F \cap H_{\alpha_\xi,\alpha_{\xi+1}}) = \bar\mu(a \cap a_{\alpha_\xi,\alpha_{\xi+1}}) = \bar\mu(a \cap a_{\alpha_\xi}) = \bar\mu a = \mu F.$$

But this means that we ought to have added $(F, \langle D_\xi\rangle_{\xi<\omega_1})$ to the family $\langle(F_i, \langle D_{i\xi}\rangle_{\xi<\omega_1})\rangle_{i\in I}$. **XQ**

(c) So if we set $D_\xi = \bigcup_{i\in I} D_{i\xi}$ for each $\xi < \omega_1$, $\langle D_\xi\rangle_{\xi<\omega_1}$ is a disjoint family of subsets of A and

$$\mu^*A \ge \mu^*D_\xi = \sum_{i\in I}\mu^*(F_i \cap D_\xi) = \sum_{i\in I}\mu^*D_{i\xi} = \sum_{i\in I}\mu F_i = \mu E \ge \mu^*A$$

for every $\xi < \omega_1$, as required.

547F Theorem Let (X, Σ, μ) be a totally finite measure space in which singletons are negligible. Suppose

either that μ is atomless and there is no quasi-measurable cardinal less than the Maharam type of μ

or that μ is purely atomic and there is no two-valued-measurable cardinal less than or equal to $\#(X)$

or that there is no quasi-measurable cardinal less than or equal to $\#(X)$.

Then for any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that $A'_n \subseteq A_n$ and $\mu^*A'_n = \mu^*A_n$ for every $n \in \mathbb{N}$.

proof (a) If $A \subseteq X$ and $\langle B_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X there is a $D \subseteq A$ such that $\mu^*D = \mu^*A$ and $\mu^*(B_n \setminus D) = \mu^*B_n$ for every n.

P (i) If either μ is atomless and there is no quasi-measurable cardinal less than the Maharam type of μ, or μ is purely atomic and there is no two-valued-measurable cardinal less than or equal to $\#(X)$, this is just 547D.

(ii) If there is no quasi-measurable cardinal less than or equal to $\#(X)$, then 547E tells us that there is a disjoint family $\langle D_\xi\rangle_{\xi<\omega_1}$ of subsets of A such that $\mu^*D_\xi = \mu^*A$ for every ξ. Now, for each n, let μ_{B_n} be the subspace measure on B_n and $(\mu_{B_n})_*$ the corresponding inner measure (413D). Then $\sum_{\xi<\omega_1}(\mu_{B_n})_*D_\xi \le \mu_{B_n}B_n$ is finite, so $(\mu_{B_n})_*(B_n \cap D_\xi) = 0$ for all but countably many ξ. There is therefore some $\xi < \omega_1$ such that, taking $D = D_\xi$, we have $(\mu_{B_n})_*(B_n \cap D) = 0$ for every $n \in \mathbb{N}$. Of course $D \subseteq A$ and $\mu^*D = \mu^*A$, while

$$\mu^*(B_n \setminus D) = \mu^*_{B_n}(B_n \setminus D) = \mu_{B_n} B_n - (\mu_{B_n})_*(B_n \cap D) = \mu_{B_n} B_n = \mu^* B_n$$

for every n. **Q**

(b) Now choose inductively, for $n \in \mathbb{N}$, sets A'_n such that, for each n,

$$A'_n \subseteq A_n \setminus \bigcup_{i<n} A'_i,$$

$$\mu^* A'_n = \mu^*(A_n \setminus \bigcup_{i<n} A'_i),$$

$$\mu^*(A_m \setminus \bigcup_{i\le n} A'_i) = \mu^*(A_m \setminus \bigcup_{i<n} A'_i) \text{ for every } m > n.$$

Then an easy induction shows that $\mu^*(A_m \setminus \bigcup_{i\le n} A'_i) = \mu^* A_m$ whenever $n < m$, so that $\mu^* A'_n = \mu^* A_n$ for each n.

547G Corollary Let (X, Σ, μ) be an atomless probability space such that there is no quasi-measurable cardinal less than the Maharam type of μ. Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence of subsets of X. Then there is a set $D \subseteq X$ such that $\mu^*(A_n \cap D) = \mu^*(A_n \setminus D) = \mu^* A_n$ for every $n \in \mathbb{N}$.

proof Take $\langle A'_n \rangle_{n\in\mathbb{N}}$ from 547F. For each n, we can choose $D_n \subseteq A'_n$ such that $\mu^* D_n = \mu^*(A'_n \setminus D_n) = \mu^* A'_n$ (547Ab). Set $D = \bigcup_{n\in\mathbb{N}} D_n$; then

$$\mu^*(A_n \cap D) \ge \mu^* D_n = \mu^* A'_n = \mu^* A_n,$$

$$\mu^*(A_n \setminus D) \ge \mu^*(A'_n \setminus D_n) = \mu^* A'_n = \mu^* A_n$$

for every n, as required.

547H I do not know how far we can hope to extend 547G to uncountable families in place of $\langle A_n \rangle_{n\in\mathbb{N}}$. If in place of

$$\mu^*(A_n \cap D) = \mu^*(A_n \setminus D) = \mu^* A_n$$

we ask rather for

$$\min(\mu^*(A_n \cap D), \mu^*(A_n \setminus D)) > 0$$

we are led to rather different patterns, as follows.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and κ a cardinal. Then the following are equiveridical:

(i) $\kappa < \pi(\mathfrak{A}_d)$ for every $d \in \mathfrak{A}^+ = \mathfrak{A} \setminus \{0\}$, writing $\mathfrak{A}_d$ for the principal ideal generated by d;

(ii) whenever $A \subseteq \mathfrak{A}^+$ and $\#(A) \le \kappa$ there is a $b \in \mathfrak{A}$ such that $a \cap b$ and $a \setminus b$ are both non-zero for every $a \in A$.

proof (a) Suppose that (i) is false; that there are a non-zero $d \in \mathfrak{A}$ and an order-dense set $A \subseteq \mathfrak{A}^+_d$ such that $\#(A) \le \kappa$. If $b \in \mathfrak{A}$ and $b \cap d = 0$ then $a \cap b = 0$ for every $a \in A$; if $b \cap d \ne 0$ then there is an $a \in A$ such that $a \subseteq b \cap d$ and $a \setminus b = 0$. So A witnesses that (ii) is false.

For the rest of the proof, therefore, I suppose that (i) is true and seek to prove (ii).

(b) We need an elementary calculation. Let $\langle c_i \rangle_{i<n}$ be a stochastically independent family in $\mathfrak{A}$ such that $\bar\mu c_i = \gamma$ for every $i < n$, where $0 < \gamma < 1$. Suppose that $a \in \mathfrak{A}$ and that

$$\sup_{i<n} \bar\mu(a \cap c_i) = \beta\gamma\bar\mu a$$

where $\beta < 1$. Then $n \le \frac{1}{(1-\beta)^2\gamma\bar\mu a}$.

P In $L^2(\mathfrak{A}, \bar\mu)$ set

$$e_i = \sqrt{\tfrac{\gamma}{1-\gamma}}\chi(1 \setminus c_i) - \sqrt{\tfrac{1-\gamma}{\gamma}}\chi(1 \setminus c_i)$$

for each $i < n$. An easy calculation shows that $\langle e_i \rangle_{i<n}$ is orthonormal. Next, for each i,

$$\begin{aligned}
(e_i|\chi a) &= \sqrt{\frac{\gamma}{1-\gamma}}\bar\mu(a \setminus c_i) - \sqrt{\frac{1-\gamma}{\gamma}}\bar\mu(a \cap c_i) \\
&\ge \sqrt{\frac{\gamma}{1-\gamma}}(\bar\mu a - \beta\gamma\bar\mu a) - \sqrt{\frac{1-\gamma}{\gamma}}\beta\gamma\bar\mu a \\
&= \sqrt{\frac{1}{\gamma(1-\gamma)}}(\gamma(\bar\mu a - \beta\gamma\bar\mu a) - (1-\gamma)\beta\gamma\bar\mu a) \\
&= \gamma\bar\mu a\sqrt{\frac{1}{\gamma(1-\gamma)}}(1-\beta) \ge (1-\beta)\bar\mu a\sqrt{\gamma}.
\end{aligned}$$

By 4A4Ji[8],

$$\bar{\mu}a = \|\chi a\|_2^2 \ge \sum_{i<n} |(e_i|\chi a)|^2 \ge n\gamma(1-\beta)^2(\bar{\mu}a)^2$$

and

$$n \le \frac{1}{(1-\beta)^2\gamma\bar{\mu}a}$$

as claimed. **Q**

(c) If $\mathfrak{A}$ has an atom, then $\kappa = 0$ and there is nothing to prove. So we may suppose henceforth that $\mathfrak{A}$ is atomless. Set $\lambda = \min\{\tau(\mathfrak{A}_d) : d \in \mathfrak{A}^+\}$. Then the measure algebra $(\mathfrak{B}_\lambda, \bar{\nu}_\lambda)$ can be embedded in $(\mathfrak{A}, \bar{\mu})$ (332P). In particular, we can find, for each n, a stochastically independent family $\langle c_{n\xi}\rangle_{\xi<\lambda}$ of elements of $\mathfrak{A}$ with $\bar{\mu}c_{n\xi} = \frac{1}{n!}$ for every ξ.

Set $q(n) = 4n((2n)!+(2n+1)!)$ for each $n \in \mathbb{N}$, and let $(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$ be the corresponding version of the λ-localization relation as described in 522L. For $a \in \mathfrak{A}^+$ let $n_a \in \mathbb{N}$ be such that $n_a\bar{\mu}a \ge 1$ and set

$$\begin{aligned} S_a = \{(n,\xi) : n \ge n_a \text{ and either } \bar{\mu}(a \cap c_{2n,\xi}) \le \frac{\bar{\mu}a}{2(2n)!} \text{ or } \bar{\mu}(a \cap c_{2n+1,\xi}) \le \frac{\bar{\mu}a}{2(2n+1)!}\} \\ \subseteq \mathbb{N} \times \lambda. \end{aligned}$$

Then $S_a \in \mathcal{S}_\lambda^{(q)}$. **P** If $n \ge n_a$ then (b), with $\beta = \frac{1}{2}$, tells us that

$$\#(\{\xi : \bar{\mu}(a \cap c_{2n,\xi}) \le \frac{\bar{\mu}a}{2(2n)!}\}) \le \frac{4(2n)!}{\bar{\mu}a} \le 4n(2n)!,$$

$$\#(\{\xi : \bar{\mu}(a \cap c_{2n+1,\xi}) \le \frac{\bar{\mu}a}{2(2n+1)!}\}) \le \frac{4(2n+1)!}{\bar{\mu}a} \le 4n(2n+1)!,$$

so $\#(S_a[\{n\}]) \le 4n((2n)! + (2n+1)!) = q(n)$. **Q**

(d) Now observe that

$$\min\{\pi(\mathfrak{A}_d) : d \in \mathfrak{A}^+\} = \pi(\mathfrak{B}_\lambda)$$

(see 524Mc)

$$= \mathrm{ci}(\mathfrak{B}_\lambda^+) = \mathrm{cov}(\mathfrak{B}_\lambda^+, \supseteq, \mathfrak{B}_\lambda^+) = \mathrm{cov}(\mathfrak{B}_\lambda^+, \supseteq', [\mathfrak{B}_\lambda^+]^{\le\omega})$$

(512Gf)

$$= \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda)$$

(where $(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda)$ is the ordinary λ-localization relation, by 524H and 512Da)

$$= \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$$

by 522L.

(e) Let $A \subseteq \mathfrak{A}^+$ be a set of size less than $\min\{\pi(\mathfrak{A}_d) : d \in \mathfrak{A}^+\}$. Then $\#(A) < \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$, so there must be an $f \in \lambda^{\mathbb{N}}$ such that $f \not\subseteq^* S_a$ for any $a \in A$. Set

$$b_{2n} = c_{2n,f(n)}, \quad b_{2n+1} = c_{2n+1,f(n)}, \quad b'_n = b_n \setminus \sup_{i>n} b_i \text{ for } n \in \mathbb{N},$$

$$b = \sup_{n\in\mathbb{N}} b'_{2n}.$$

Then $a \cap b$ and $a \setminus b$ are both non-zero for every $a \in A$. **P** There is an $n \ge n_a$ such that $(n, f(n)) \notin S_a$, so that

$$\bar{\mu}(a \cap b_{2n}) = \bar{\mu}(a \cap c_{2n,f(n)}) > \frac{\bar{\mu}a}{2(2n)!} \ge \frac{1}{2n(2n)!},$$

$$\bar{\mu}(a \cap b_{2n+1}) = \bar{\mu}(a \cap c_{2n+1,f(n)}) > \frac{\bar{\mu}a}{2(2n+1)!} \ge \frac{1}{2n(2n+1)!}.$$

But

[8] Later editions only.

$$\begin{aligned}\bar\mu(b_{2n}\setminus b'_{2n}) &\le \sum_{i=2n+1}^\infty \bar\mu b_i = \sum_{i=2n+1}^\infty \frac{1}{i!}\\ &\le \frac{1}{(2n)!}\sum_{j=1}^\infty \frac{1}{(2n+1)^j} = \frac{1}{2n(2n)!} < \bar\mu(a\cap b_{2n}),\\ \bar\mu(b_{2n+1}\setminus b'_{2n+1}) &\le \frac{1}{(2n+1)(2n+1)!} < \bar\mu(a\cap b_{2n+1}),\end{aligned}$$

so $a\cap b\supseteq a\cap b'_{2n}$ and $a\setminus b\supseteq a\cap b'_{2n+1}$ are both non-zero. **Q**

(f) As A is arbitrary, (ii) is true.

547I Proposition Let (X,Σ,μ) be a strictly localizable measure space, with null ideal $\mathcal{N}(\mu)$, and κ a cardinal such that

(*) whenever $\mathcal{E}\in[\Sigma\setminus\mathcal{N}(\mu)]^{\le\kappa}$ and $F\in\Sigma\setminus\mathcal{N}(\mu)$, there is a non-negligible measurable $G\subseteq F$ such that $E\setminus G$ is non-negligible for every $E\in\mathcal{E}$.

Then whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of non-negligible subsets of X, there is a $G\in\Sigma$ such that $A_\xi\cap G$ and $A_\xi\setminus G$ are non-negligible for every $\xi<\kappa$.

proof (a) Suppose to begin with that $\mu X=1$. Let $\mathfrak{A}$ be the measure algebra of μ. Then (*) says just that $\kappa<\pi(\mathfrak{A}_d)$ for every $d\in\mathfrak{A}^+$. For each $\xi<\kappa$ let E_ξ be a measurable envelope of A_ξ and set $a_\xi=E_\xi^\bullet$ in $\mathfrak{A}$. By 547H, there is a $b\in\mathfrak{A}$ such that $a_\xi\cap b$ and $a_\xi\setminus b$ are non-zero for every $\xi<\kappa$. Let $G\in\Sigma$ be such that $b=G^\bullet$; then $E_\xi\cap G$ and $E_\xi\setminus G$ are non-negligible for every ξ. But this means that $A_\xi\cap G$ and $A_\xi\setminus G$ are non-negligible for every ξ.

(b) If $\mu X=0$ the result is trivial. For other totally finite μ, we get the result from (a) if we replace μ by a suitable scalar multiple.

(c) For the general case, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X and for $i\in I$ set $J_i=\{\xi:\xi<\kappa,\ A_\xi\cap X_i$ is not negligible$\}$. By (b), applied to the subspace measure on X_i, there is a measurable $G_i\subseteq X_i$ such that $A_\xi\cap G_i$ and $A_\xi\cap X_i\setminus G_i$ are non-negligible for every $\xi\in J_i$. Set $G=\bigcup_{i\in I}G_i$; this works.

547J Corollary Let (X,Σ,μ) be an atomless quasi-Radon measure space and $\langle A_\xi\rangle_{\xi<\omega_1}$ a family of non-negligible subsets of X. Then there is a $D\subseteq X$ such that $A_\xi\cap D$ and $A_\xi\setminus D$ are non-negligible for every $\xi<\omega_1$.

proof (a) Suppose to begin with that μ is a Maharam-type-homogeneous probability measure. Let $\mathfrak{A}$ be the measure algebra of μ. If $\pi(\mathfrak{A})>\omega_1$ we can use 547I. Otherwise, $\mathrm{ci}(\Sigma\setminus\mathcal{N}(\mu))=\omega_1$, where $\mathcal{N}(\mu)$ is the null ideal of μ (put 524Tb and 524Mc together); let $\langle E_\xi\rangle_{\xi<\omega_1}$ be a coinitial family in $\Sigma\setminus\mathcal{N}(\mu)$. Then we can choose $x_{\xi\eta}$ and $y_{\xi\eta}$, for ξ, $\eta<\omega_1$, so that

all the $x_{\xi\eta}$, $y_{\xi\eta}$ are different,

if $A_\xi\cap E_\eta\notin\mathcal{N}(\mu)$ then $x_{\xi\eta}$ and $y_{\xi\eta}$ belong to $A_\xi\cap E_\eta$.

Set $D=\{x_{\xi\eta}:\xi,\eta<\omega_1\}$; then $\mu^*(A_\xi\cap D)=\mu^*A_\xi$ for every ξ. **P?** Otherwise, let E be a measurable envelope of A_ξ and F a measurable envelope of $A_\xi\cap D$. We have $\mu(E\setminus F)>0$, so there is an $\eta<\omega_1$ such that $E_\eta\subseteq E\setminus F$, in which case

$$x_{\xi\eta}\in A_\xi\cap E_\eta\cap D\subseteq F.\ \mathbf{XQ}$$

Similarly, $\mu^*(A_\xi\setminus D)=\mu^*A_\xi$ for every ξ, and we have a suitable set D.

(b) In general, X has a decomposition into Maharam-type-homogeneous subspaces (as in the proofs of 524J and 524P), so the full result follows as in (b)-(c) of the proof of 547I.

547X Basic exercises (a) Let (X,Σ,μ) be a semi-finite measure space with the measurable envelope property (definition: 213Xl). Suppose that $A\subseteq X$ is such that $\min(\mu^*D,\mu^*(A\setminus D))<\mu^*A$ for every $D\subseteq A$. Show that there is a non-negligible set $B\subseteq A$, of finite outer measure, such that the completion $\hat\mu_B$ of the subspace measure on B measures every subset of B.

(b) Let (X,Σ,μ) be a measure space. Show that the following are equiveridical: (i) for every $A\subseteq X$ there is an $A'\subseteq A$ such that $\mu^*A'=\mu^*(A\setminus A')=\mu^*A$ (ii) whenever $\langle A_i\rangle_{i\in I}$ is a finite family of subsets of X, there is a disjoint family $\langle A'_i\rangle_{i\in I}$ such that $A'_i\subseteq A_i$ and $\mu^*A'_i=\mu^*A_i$ for every $i\in I$.

(c) Show that the following are equiveridical: (i) there is no quasi-measurable cardinal; (ii) if (X, Σ, μ) is a probability space such that $\mu\{x\} = 0$ for every $x \in X$ then there is a disjoint family $\langle D_\xi \rangle_{\xi<\omega_1}$ of subsets of X such that $\mu^* D_\xi = 1$ for every $\xi < \omega_1$.

(d) Let (X, Σ, μ) be a probability space and suppose that $\operatorname{non}\mathcal{N}(\mu) = \operatorname{ci}(\Sigma \setminus \mathcal{N}(\mu)) = \kappa$, where $\mathcal{N}(\mu)$ is the null ideal of μ. Show that if $\langle A_\xi \rangle_{\xi<\kappa}$ is any family of subsets of X, then there is a disjoint family $\langle A'_\xi \rangle_{\xi<\kappa}$ of sets such that $A'_\xi \subseteq A_\xi$ and $\mu^* A'_\xi = \mu^* A_\xi$ for every $\xi < \kappa$.

(e) Let $\langle E_\xi \rangle_{\xi<\mathfrak{c}}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$. Show that there is a disjoint family $\langle A_\xi \rangle_{\xi<\mathfrak{c}}$ of sets such that $A_\xi \subseteq E_\xi$ and $\mu^* A_\xi = \mu E_\xi$ for every $\xi < \mathfrak{c}$, where μ is Lebesgue measure on $\mathbb{R}$. (*Hint*: 419I.)

(f) (M.R.Burke) Let $\mathcal{N}$ be the null ideal of Lebesgue measure μ on $\mathbb{R}$. Show that if $2^{\operatorname{non}\mathcal{N}} = \mathfrak{c}$ then there is a family $\langle A_\xi \rangle_{\xi<\mathfrak{c}}$ of subsets of $\mathbb{R}$ such that for every $D \subseteq \mathbb{R}$ there is some $\xi < \mathfrak{c}$ such that $\min(\mu^*(A_\xi \cap D), \mu^*(A_\xi \setminus D)) < \mu^* A_\xi$.

(g) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) $\pi(\mathfrak{A}_a) > \omega$ for every $a \in \mathfrak{A}^+$, where $\mathfrak{A}_a$ is the principal ideal generated by a (β) for every sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$ there is a disjoint sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$ such that $b_n \subseteq a_n$ for every n. (ii) Show that if $\mathfrak{A}$ has the σ-interpolation property then we can add (γ) whenever $A \subseteq \mathfrak{A}^+$ is countable, there is a $b \in \mathfrak{A}$ such that $a \cap b$ and $a \setminus b$ are both non-zero for every $a \in A$.

547Z Problems (a) Suppose that $\langle A_\xi \rangle_{\xi<\omega_1}$ is a family of subsets of $[0,1]$. Must there be a set $D \subseteq [0,1]$ such that $\mu^*(A_\xi \cap D) = \mu^*(A_\xi \setminus D) = \mu^* A_\xi$ for every $\xi < \omega_1$, where μ is Lebesgue measure on $[0,1]$?

(b) Suppose that there is no quasi-measurable cardinal. Let (X, Σ, μ) be an atomless probability space and $\langle A_\xi \rangle_{\xi<\omega_1}$ a family of subsets of X. Must there be a disjoint family $\langle A'_\xi \rangle_{\xi<\omega_1}$ such that $A'_\xi \subseteq A_\xi$ and $\mu^* A'_\xi = \mu^* A_\xi$ for every $\xi < \omega_1$?

547 Notes and comments Of course the most important case of Theorem 547F is when $X = [0,1]$ with Lebesgue measure, so that we have a result provable in ZFC, whether or not there are quasi-measurable cardinals. As far as I know there is no real simplification available for this special case if we wish to avoid special axioms. In many models of set theory, of course, there are other approaches, as in 547E and 547Xd; and I note that it makes a difference that we start with not-necessarily-measurable sets A_n (547Xe).

The arguments here leave many obvious questions open. The first group concerns possible extensions of 547F to uncountable families of sets, as in 547Z. The methods of §546, as I have written them out, do not seem adequate in this context; an answer to 546Zc might help. I remark that Shelah 03 describes a model in which there is a set $A \in \mathcal{P}\mathbb{R} \setminus \mathcal{N}$ such that $\mathcal{P}A \cap \mathcal{N}$ is ω_1-saturated in $\mathcal{P}A$, where $\mathcal{N}$ is the null ideal of Lebesgue measure. Elsewhere we can ask, in 547B and 547F, whether the hypotheses involving quasi-measurable cardinals could be rewritten with atomlessly-measurable cardinals. Only in 547E is it clear that non-atomlessly-measurable quasi-measurable cardinals are relevant (547Xc).

The questions tackled in this section can be re-phrased as questions about structures $(\mathcal{P}X/\mathcal{I}, \mathfrak{A})$ where $\mathcal{I}$ is a σ-ideal of subsets of X and $\mathfrak{A}$ is a σ-subalgebra of the power set σ-quotient algebra $\mathcal{P}X/\mathcal{I}$; a requirement of the form '$\mu^*(A \cap D) = \mu^* A$' becomes (in the context of a totally finite measure μ) '$\operatorname{upr}(a \cap d, \mathfrak{A}) = \operatorname{upr}(a, \mathfrak{A})$', where $\operatorname{upr}(a, \mathfrak{A})$ is the upper envelope of a in $\mathfrak{A}$ (313S[9]).

I include 547H-547J to show that if we are less ambitious then there are quite different, and rather easier, arguments available. The condition (*) of 547I is exact if we are looking for a measurable splitting set G. But I am not at all sure that 547J is in the right form.

[9]Formerly 314V.

Chapter 55

Possible worlds

In my original plans for this volume, I hoped to cover the most important consistency proofs relating to undecidable questions in measure theory. Unhappily my ignorance of forcing means that for the majority of results I have nothing useful to offer. I have therefore restricted my account to the very narrow range of ideas in which I feel I have achieved some understanding beyond what I have read in the standard texts.

For a measure theorist, by far the most important forcings are those of 'adding random reals'. I give three sections (§§552-553 and 555) to these. Without great difficulty, we can determine the behaviour of the cardinals in Cichoń's diagram (552B, 552C, 552F-552I), at least if many random reals are added. Going deeper, there are things to be said about outer measure and Sierpiński sets (552D, 552E), and extensions of Radon measures (552N). In the same section I give a version of the fundamental result that simple iteration of random real forcings gives random real forcings (552P). In §553 I collect results which are connected with other topics dealt with above (Rothberger's property, precalibers, ultrafilters, medial limits) and in which the arguments seem to me to develop properties of measure algebras which may be of independent interest. In preparation for this work, and also for §554, I start with a section (§551) devoted to a rather technical general account of forcings with quotients of σ-algebras of sets, aiming to find effective representations of names for points, sets, functions, measure algebras and filters.

Very similar ideas can also take us a long way with Cohen real forcing (§554). Here I give little more than obvious parallels to the first part of §552, with an account of Freese-Nation numbers sufficient to support Carlson's theorem that a Borel lifting of Lebesgue measure can exist when the continuum hypothesis is false (554I).

One of the most remarkable applications of random reals is in Solovay's proof that if it is consistent to suppose that there is a two-valued-measurable cardinal, then it is consistent to suppose that there is an atomlessly-measurable cardinal (555D). By taking a bit of trouble over the lemmas, we can get a good deal more, including the corresponding theorem relating supercompact cardinals to the normal measure axiom (555P); and similar techniques show the possibility of interesting power set σ-quotient algebras (555G, 555M).

I end the chapter with something quite different (§556). A familiar phenomenon in ergodic theory is that once one has proved a theorem for ergodic transformations one can expect, possibly at the cost of substantial effort, but without having to find any really new idea, a corresponding result for general measure-preserving transformations. There is more than one way to look at this, but here I present a method in which the key step, in each application, is an appeal to the main theorem of forcing. A similar approach gives a description of the completion of the asymptotic density algebra. The technical details take up a good deal of space, but are based on the same principles as those in §551, and are essentially straightforward.

551 Forcing with quotient algebras

In preparation for the discussion of random real forcing in the next two sections, I introduce some techniques which can be applied whenever a forcing notion is described in terms of a Loomis-Sikorski representation of its regular open algebra. The first step is just a translation of the correspondence between names for real numbers in the forcing language and members of $L^0(\mathrm{RO}(\mathbb{P}))$, as described in 5A3L, when $L^0(\mathrm{RO}(\mathbb{P}))$ can be identified with a quotient of a space $L^0(\Sigma)$ of measurable functions. More care is needed, but we can find a similar formulation of names for members of $\{0,1\}^I$ for any set I (551C). Going a step farther, it turns out that there are very useful descriptions of Baire subsets of $\{0,1\}^I$ (551D-551F), Baire measurable functions (551N), the usual measure on $\{0,1\}^I$ (551I-551J) and its measure algebra (551P). In some special cases, these methods can be used to represent iterated forcing notions (551Q). I end with a construction for a forcing extension of a filter on a countable set (551R).

551A Definition (a) A **measurable space with negligibles** is a triple $(\Omega,\Sigma,\mathcal{I})$ where Ω is a set, Σ is a σ-algebra of subsets of Ω and $\mathcal{I}$ is a σ-ideal of subsets of Ω generated by $\Sigma\cap\mathcal{I}$. In this case $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$ is a Dedekind σ-complete Boolean algebra (314C).

In this context I will use the phrase '$\mathcal{I}$-almost everywhere' to mean 'except on a set belonging to $\mathcal{I}$'.

(b) I will say that $(\Omega,\Sigma,\mathcal{I})$ is **non-trivial** if $\Omega\notin\mathcal{I}$, so that $\mathfrak{A}\ne\{0\}$. In this case, the forcing notion $\mathbb{P}$ **associated** with $(\Omega,\Sigma,\mathcal{I})$ is $(\mathfrak{A}^+,\subseteq,\Omega^\bullet,\downarrow)$ (5A3Ab). If $\mathfrak{A}$ is Dedekind complete we can identify $\mathfrak{A}$ with the regular open algebra $\mathrm{RO}(\mathbb{P})$ (514Sb, 5A3M).

(c) I will say that $(\Omega,\Sigma,\mathcal{I})$ is **ω_1-saturated** if $\Sigma\cap\mathcal{I}$ is ω_1-saturated in Σ in the sense of 541A, that is, if there is no uncountable disjoint family in $\Sigma\setminus\mathcal{I}$, that is, if $\mathfrak{A}$ and $\mathbb{P}$ are ccc.

(d) I will say that $(\Omega,\Sigma,\mathcal{I})$ is **complete** if $\mathcal{I}\subseteq\Sigma$ (cf. 211A).

Remark For an account of the general theory of measurable spaces with negligibles, see Fremlin 87.

551B Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion. Recall from 364Jb that $L^0(\mathfrak{A})$ can be regarded as a quotient of the space of Σ-measurable functions from Ω to $\mathbb{R}$. If $h:\Omega\to\mathbb{R}$ is Σ-measurable, write $\vec{h}=(h^\bullet)^{\vec{}}$ where $h^\bullet$ is the equivalence class of h in $L^0(\mathfrak{A})$, identified with $L^0(\mathrm{RO}(\mathbb{P}))$, and $(h^\bullet)^{\vec{}}$ is the $\mathbb{P}$-name for a real number as defined in 5A3L. Then

$$\Vdash_{\mathbb{P}} \vec{h} \text{ is a real number,}$$

and for any $\alpha\in\mathbb{Q}$

$$[\![\vec{h}>\check{\alpha}]\!]=[\![(h^\bullet)^{\vec{}}>\check{\alpha}]\!]=[\![h^\bullet>\alpha]\!]=\{\omega:h(\omega)>\alpha\}^\bullet.$$

From 5A3Lc, we see that if h_0, h_1 are Σ-measurable real-valued functions on Ω, then

$$\Vdash_{\mathbb{P}} (h_0+h_1)^{\vec{}}=\vec{h}_0+\vec{h}_1,\ (h_0\times h_1)^{\vec{}}=\vec{h}_0\times\vec{h}_1,$$

and that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions with limit h,

$$\Vdash_{\mathbb{P}} \vec{h}=\lim_{n\to\infty}\vec{h}_n \text{ in } \mathbb{R}.$$

551C Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion.

(a) If $f:\Omega\to\{0,1\}$ is Σ-measurable, let $\vec{f}$ be the $\mathbb{P}$-name

$$\{(i,f^{-1}[\{i\}]^\bullet):i\in\{0,1\},\ f^{-1}[\{i\}]\notin\mathcal{I}\}.$$

Then $\Vdash_{\mathbb{P}} \vec{f}\in\{0,1\}$ and $[\![\vec{f}=\check{\imath}]\!]=f^{-1}[\{i\}]^\bullet$ for both i. (I will try always to make it clear when this definition of $\vec{f}$ is intended to overrule the definition in 551B; but we see from 551Xf that any confusion is unlikely to matter.)

Observe that if a $\mathbb{P}$-name $\dot{x}$ and $p\in\mathfrak{A}^+$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}$, then there is a measurable $f:\Omega\to\{0,1\}$ such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$; take $f=\chi E$ where $E\in\Sigma$ is such that $E^\bullet=[\![\dot{x}=1]\!]$ in $\mathfrak{A}$.

(b) Now let I be any set, and $f:\Sigma\to\{0,1\}^I$ a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function, where $\mathcal{B}\mathfrak{a}_I=\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the Baire σ-algebra of $\{0,1\}^I$, that is, the σ-algebra of subsets of $\{0,1\}^I$ generated by sets of the form $\{x:x\in\{0,1\}^I,\ x(i)=1\}$ for $i\in I$ (4A3N). For each $i\in I$, set $f_i(\omega)=f(\omega)(i)$ for $\omega\in\Omega$; then $f_i:\Omega\to\{0,1\}$ is measurable, so we have a $\mathbb{P}$-name $\vec{f}_i$ as in (a). Let $\vec{f}$ be the $\mathbb{P}$-name $\{(\langle\vec{f}_i\rangle_{i\in\check{I}},\mathbb{1})\}$ (interpreting the subformula $\langle\dots\rangle_{i\in\check{I}}$ in the forcing language, of course, by the convention of 5A3Eb). Then

$$\Vdash_{\mathbb{P}} \vec{f}\in\{0,1\}^{\check{I}},$$

and for every $i\in I$

$$\Vdash_{\mathbb{P}} \vec{f}(\check{\imath})=\vec{f}_i.$$

(c) In the other direction, if a $\mathbb{P}$-name $\dot{x}$ and $p\in\mathfrak{A}^+$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}^{\check{I}}$, then for each $i\in I$ we have a $\mathbb{P}$-name $\dot{x}(\check{\imath})$ and a measurable $f_i:\Omega\to\{0,1\}$ such that $p\Vdash_{\mathbb{P}}\dot{x}(\check{\imath})=\vec{f}_i$; setting $f(\omega)=\langle f_i(\omega)\rangle_{i\in I}$ for $\omega\in\Omega$, f is $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable and $p\Vdash_{\mathbb{P}}\vec{f}=\dot{x}$.

(d) I ought to remark that there is a problem with equality for the $\mathbb{P}$-names $\vec{f}$. If, in the context of (b)-(c) above, we have two $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable functions f and g, and if $p\in\mathfrak{A}^+$, then

$$\begin{aligned}p\Vdash_{\mathbb{P}}\vec{f}=\vec{g}&\iff \text{for every } i\in I,\ p\Vdash_{\mathbb{P}}\vec{f}_i=\vec{g}_i\\&\iff \text{for every } i\in I,\ p\subseteq\{\omega:f_i(\omega)=g_i(\omega)\}^\bullet \text{ in } \mathfrak{A}.\end{aligned}$$

In particular, $\Vdash_{\mathbb{P}}\vec{f}=\vec{g}$ iff $f_i=g_i$ a.e. for every $i\in I$. If I is uncountable we can easily have $\Vdash_{\mathbb{P}}\vec{f}=\vec{g}$ while $f(\omega)\ne g(\omega)$ for every $\omega\in\Omega$. But if I is countable then we shall have

$$p\Vdash_{\mathbb{P}}\vec{f}=\vec{g}\iff p\subseteq\{\omega:f(\omega)=g(\omega)\}^\bullet.$$

For a context in which these considerations are vital, see (a-ii) of the proof of 551E.

(e) Suppose that x is any point of $\{0,1\}^I$. Then we have a corresponding $\mathbb{P}$-name $\check{x}$, and $\Vdash_{\mathbb{P}}\check{x}\in\{0,1\}^{\check{I}}$. For each $i\in I$, $\Vdash_{\mathbb{P}}\check{x}(\check{\imath})=x(i)^\vee\in\{0,1\}$. If we set $e_x(\omega)=x$ for every $\omega\in\Omega$, then $e_x(\omega)(i)=x(i)$ for every $i\in I$ and $\omega\in\Omega$, so $\Vdash_{\mathbb{P}}\vec{e}_x(\check{\imath})=x(i)^\vee$ for every $i\in I$, and $\Vdash_{\mathbb{P}}\vec{e}_x=\check{x}$.

551D Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $W\subseteq\Omega\times\{0,1\}^I$, let $\vec{W}$ be the $\mathbb{P}$-name

$$\{(\vec{f},E^\bullet):E\in\Sigma\setminus\mathcal{I},\ f:\Omega\to\{0,1\}^I\text{ is }(\Sigma,\mathcal{B}\mathfrak{a}_I)\text{-measurable},\\ (\omega,f(\omega))\in W\text{ for every }\omega\in E\},$$

interpreting $\vec{f}$ as in 551C. I give the definition for arbitrary sets W, but it is useful primarily when $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, as in most of the next proposition. Perhaps I can note straight away that

$$\Vdash_{\mathbb{P}}\vec{W}\subseteq\{0,1\}^{\check{I}}$$

and that if $W=\Omega\times\{0,1\}^I$ then

$$\Vdash_{\mathbb{P}}\vec{W}=\{0,1\}^{\check{I}}$$

(using 551Cb-551Cc).

551E Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $f:\Omega\to\{0,1\}^I$ is $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable, then $\{\omega:(\omega,f(\omega))\in W\}$ belongs to Σ, and $[\![\vec{f}\in\vec{W}]\!]=\{\omega:(\omega,f(\omega))\in W\}^\bullet$.

(b) If V, $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}}\vec{V}\cap\vec{W}=(V\cap W)^{\rightarrow}\text{ and }\vec{V}\cup\vec{W}=(V\cup W)^{\rightarrow},$$

$$\Vdash_{\mathbb{P}}\vec{V}\setminus\vec{W}=(V\setminus W)^{\rightarrow}\text{ and }\vec{V}\triangle\vec{W}=(V\triangle W)^{\rightarrow}.$$

(c) If V, $W\subseteq\Omega\times\{0,1\}^I$ and $V\subseteq W$ then

$$\Vdash_{\mathbb{P}}\vec{V}\subseteq\vec{W}.$$

(d) If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ with union W and intersection V, then

$$\Vdash_{\mathbb{P}}\bigcup_{n\in\mathbb{N}}\vec{W}_n=\vec{W}\text{ and }\bigcap_{n\in\mathbb{N}}\vec{W}_n=\vec{V}.$$

(e) Suppose that $J\subseteq I$ is countable, $z\in\{0,1\}^J$, $E\in\Sigma$ and

$$W=\{(\omega,x):\omega\in E,\ x\in\{0,1\}^I,\ x\restriction J=z\}.$$

Then

$$E^\bullet=[\![\vec{W}=\{x:x\in\{0,1\}^{\check{I}},\ \check{z}\subseteq x\}]\!],$$

$$1\setminus E^\bullet=[\![\vec{W}=\emptyset]\!].$$

proof (a)(i) Let $\mathcal{W}$ be the family of subsets of $\Omega\times\{0,1\}^I$ such that $F_W=\{\omega:(\omega,f(\omega))\in W\}\in\Sigma$. Then $\mathcal{W}$ is a Dynkin class of subsets of $\Omega\times\{0,1\}^I$, just because Σ is a σ-algebra. If $H\in\Sigma$, $J\subseteq I$ is finite, $z\in\{0,1\}^J$ and $W=\{(\omega,x):\omega\in H,\ z\subseteq x\in\{0,1\}^I\}$ then $F_W=H\cap\{\omega:f(\omega)(i)=z(i)\text{ for every }i\in J\}$ belongs to Σ because f is $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable, so $W\in\mathcal{W}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by sets of this form, which is just $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$.

(ii) Now suppose that $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. If $F_W\in\mathcal{I}$ then surely $F_W^\bullet=0\subseteq[\![\vec{f}\in\vec{W}]\!]$. If $F_W\notin\mathcal{I}$ then $(\vec{f},F_W^\bullet)\in\vec{W}$, $F_W^\bullet\Vdash_{\mathbb{P}}\vec{f}\in\vec{W}$ and again $F_W^\bullet\subseteq[\![\vec{f}\in\vec{W}]\!]$.

? If $F_W^\bullet\ne[\![\vec{f}\in\vec{W}]\!]$, set $p=[\![\vec{f}\in\vec{W}]\!]\setminus F_W^\bullet$. Since $p\Vdash_{\mathbb{P}}\vec{f}\in\vec{W}$ there must be a $q\in\mathfrak{A}^+$ and a $\mathbb{P}$-name $\dot{x}$ and an r stronger than both p and q such that

$$r\Vdash_{\mathbb{P}}\dot{x}=\vec{f}\text{ and }(\dot{x},q)\in\vec{W}.$$

Now there must be a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function g and an $E\in\Sigma\setminus\mathcal{I}$ such that $\dot{x}=\vec{g}$, $q=E^\bullet$ and $(\omega,g(\omega))\in W$ for every $\omega\in E$. In this case, $r=G^\bullet$ for some $G\subseteq E\setminus F_W$, and $r\Vdash_{\mathbb{P}}\vec{f}=\vec{g}$.

Because $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, there is a countable set $J\subseteq I$ such that W factors through $\Omega\times\{0,1\}^J$. For each $i\in J$, we have $r\Vdash_{\mathbb{P}}\vec{f}(\check{\imath})=\vec{g}(\check{\imath})$, that is,

$$r\subseteq[\![\vec{f}(\check{\imath})=\vec{g}(\check{\imath})]\!]=\{\omega:f(\omega)(i)=g(\omega)(i)\}^\bullet.$$

So $f(\omega)(i)=g(\omega)(i)$ for $\mathcal{I}$-almost every $\omega\in G$. This is true for every $i\in J$, so $f(\omega)\restriction J=g(\omega)\restriction J$ for $\mathcal{I}$-almost every $\omega\in G$. But this means that, for $\mathcal{I}$-almost every $\omega\in G$, $(\omega,f(\omega))\in W$ iff $(\omega,g(\omega))\in W$. However, $G\subseteq E\setminus F_W$, so $(\omega,g(\omega))\in W$ and $(\omega,f(\omega))\notin W$ for every $\omega\in G$. **X**

So we must have $F_W^\bullet = [\![\vec{f} \in \vec{W}]\!]$, as claimed.

(b) These are now elementary. The point is that if a $\mathbb{P}$-name $\dot{x}$ and $p \in \mathfrak{A}^+$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{V} \cap \vec{W}$, then $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, so there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, and $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{V} \cap \vec{W}$. Now (a) shows that

$$\begin{aligned} [\![\vec{f} \in (V \cap W)^{\vec{}}]\!] &= \{\omega : (\omega, f(\omega)) \in V \cap W\}^\bullet \\ &= \{\omega : (\omega, f(\omega)) \in V\}^\bullet \cap \{\omega : (\omega, f(\omega)) \in W\}^\bullet \\ &= [\![\vec{f} \in \vec{V}]\!] \cap [\![\vec{f} \in W]\!] \supseteq p \end{aligned}$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in (V \cap W)^{\vec{}}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{V} \cap \vec{W} \subseteq (V \cap W)^{\vec{}}.$$

The other seven inequalities are equally straightforward.

(c) This is immediate from the definition in 551D, since we actually have $\vec{V} \subseteq \vec{W}$.

(d) We can repeat the method of (b). If a $\mathbb{P}$-name $\dot{x}$ and $p \in \mathfrak{A}^+$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \bigcap_{n \in \mathbb{N}} \vec{W}_n$, then there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, and $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}_n$ for every n. Now

$$\begin{aligned} [\![\vec{f} \in \vec{V}]\!] &= \{\omega : (\omega, f(\omega)) \in \bigcap_{n \in \mathbb{N}} W_n\}^\bullet \\ &= (\bigcap_{n \in \mathbb{N}} \{\omega : (\omega, f(\omega)) \in W_n\})^\bullet = \inf_{n \in \mathbb{N}} \{\omega : (\omega, f(\omega)) \in W_n\}^\bullet \end{aligned}$$

(because $\Sigma \cap \mathcal{I}$ is a σ-ideal of Σ, so $E \mapsto E^\bullet$ is sequentially order-continuous, by 313Qb)

$$\supseteq p$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \vec{V}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \textstyle\bigcap_{n \in \mathbb{N}} \vec{W}_n \subseteq \vec{V}.$$

On the other hand, (c) tells us that

$$\Vdash_{\mathbb{P}} \vec{V} \subseteq \textstyle\bigcap_{n \in \mathbb{N}} \vec{W}_n, \text{ so we have equality.}$$

Putting this together with (b) (and recalling that $\Vdash_{\mathbb{P}} (\Omega \times \{0,1\}^I)^{\vec{}} = \{0,1\}^{\check{I}}$), we get

$$\Vdash_{\mathbb{P}} \textstyle\bigcup_{n \in \mathbb{N}} \vec{W}_n = \vec{W}.$$

(e)(i) Suppose that $p \in \mathfrak{A}^+$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. Let $f : \Omega \to \{0,1\}^I$ be a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Then

$$p \subseteq [\![\vec{f} \in \vec{W}]\!] = \{\omega : \omega \in E,\ z \subseteq f(\omega)\}^\bullet$$

by (a) above; that is, $p \subseteq E^\bullet$ and

$$p \Vdash_{\mathbb{P}} \vec{f}(\check{\imath}) = z(i)^{\check{}} = \check{z}(\check{\imath})$$

for every $i \in J$, so

$$p \Vdash_{\mathbb{P}} \check{z} \subseteq \vec{f}.$$

As p and $\dot{x}$ are arbitrary,

$$[\![\vec{W} \neq \emptyset]\!] \subseteq E^\bullet$$

and

$$\Vdash_{\mathbb{P}} \vec{W} \subseteq \{x : \check{z} \subseteq x \in \{0,1\}^{\check{I}}\}.$$

(ii) If $E \in \mathcal{I}$ then $\Vdash_{\mathbb{P}} \vec{W} = \emptyset$ and we can stop. Otherwise, suppose that $p \in \mathfrak{A}^+$ is stronger than $E^\bullet$ and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \check{z} \subseteq \dot{x} \in \{0,1\}^{\check{I}}.$$

Let f be a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Then $p \Vdash_{\mathbb{P}} \vec{f_i} = z(i)\check{}$ for each $i \in J$, where $f_i(\omega) = f(\omega)(i)$ for every ω, so $p \subseteq \{\omega : z \subseteq f(\omega)\}^\bullet$. But also $p \subseteq E^\bullet$, so

$$p \subseteq \{\omega : \omega \in E,\, z \subseteq f(\omega)\}^\bullet = [\![\vec{f} \in \vec{W}]\!],$$

and $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. As p and $\dot{x}$ are arbitrary,

$$[\![\{x : \check{z} \subseteq x\} \subseteq \vec{W}]\!] \supseteq E^\bullet$$

and we have

$$[\![\{x : \check{z} \subseteq x\} = \vec{W}]\!] = E^\bullet, \quad [\![\emptyset = \vec{W}]\!] = 1 \setminus E^\bullet.$$

551F Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

(b) Suppose that $(\Omega, \Sigma, \mathcal{I})$ is ω_1-saturated, $p \in \mathfrak{A}^+$, and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

Then there is a $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ such that

$$p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}.$$

proof (a) Let $\mathcal{W}$ be the family of those $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ such that $\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}$. 551Eb and 551Ed tell us that $\mathcal{W}$ is a σ-subalgebra of $\Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$, and 551Ee tells us that $E \times H \in \mathcal{W}$ whenever $E \in \Sigma$ and H is a basic cylinder set in $\{0,1\}^I$. So $\mathcal{W}$ must be the whole of $\Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$.

(b)(i) Suppose that $p \in \mathfrak{A}^+$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \text{ is a basic cylinder set in } \{0,1\}^{\check{I}}.$$

Then there is a $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** We know that

$$p \Vdash_{\mathbb{P}} \text{ there is a } z \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}) \text{ such that } \dot{W} = \{x : z \subseteq x \in \{0,1\}^{\check{I}}\}.$$

So there is a $\mathbb{P}$-name $\dot{z}$ such that

$$p \Vdash_{\mathbb{P}} \dot{z} \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}) \text{ and } \dot{W} = \{x : \dot{z} \subseteq x\};$$

adjusting $\dot{z}$ if necessary, we can suppose that

$$\Vdash_{\mathbb{P}} \dot{z} \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}).$$

But this means that there is a maximal antichain (that is, a partition of unity) $C \subseteq \mathfrak{A}^+$ and a family $\langle z_c \rangle_{c \in C}$ in $\mathrm{Fn}_{<\omega}(I; \{0,1\})$ such that

$$c \Vdash_{\mathbb{P}} \dot{z} = \check{z}_c$$

for every $c \in C$. Because $\mathcal{I}$ is ω_1-saturated, $\mathfrak{A}$ is ccc and C is countable. We can therefore find a partition $\langle E_c \rangle_{c \in C}$ of Ω into members of Σ such that $E_c^\bullet = c$ for every $c \in C$. Consider

$$W_c = E_c \times \{x : z_c \subseteq x \in \{0,1\}^I\} \text{ for } c \in C, \quad W = \bigcup_{c \in C} W_c.$$

Of course $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$. By (a-i),

$$c \Vdash_{\mathbb{P}} \vec{W}_c = \{x : \check{z}_c \subseteq x\} = \{x : \dot{z} \subseteq x\}, \quad c \Vdash_{\mathbb{P}} \vec{W}_d = \emptyset$$

whenever c, $d \in C$ are distinct. Because C is countable, 551Ed tells us that

$$c \Vdash_{\mathbb{P}} \vec{W} = \{x : \dot{z} \subseteq x\}$$

for every $c \in C$; because C is a maximal antichain,

$$\Vdash_{\mathbb{P}} \vec{W} = \{x : \dot{z} \subseteq x\}$$

and

$$p \Vdash_{\mathbb{P}} \vec{W} = \dot{W}. \ \mathbf{Q}$$

(ii) Suppose that $p \in \mathfrak{A}^+$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \text{ is a cozero set in } \{0,1\}^{\check{I}}.$$

Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** Set $p' = [\![\dot{W} = \emptyset]\!]$. If $p \subseteq p'$ we can take $W = \emptyset$ and stop. Otherwise, let $E \in \Sigma$ be such that $E^\bullet = 1 \setminus p'$. We have

$$p \setminus p' \Vdash_{\mathbb{P}} \dot{W} \text{ is the union of a sequence of basic cylinder sets,}$$

so there is a sequence $\langle \dot{W}_n \rangle_{n\in\mathbb{N}}$ of $\mathbb{P}$-names such that

$$p \setminus p' \Vdash_{\mathbb{P}} \dot{W}_n \text{ is a basic cylinder set for every } n \text{ and } \dot{W} = \bigcup_{n\in\mathbb{N}} \dot{W}_n.$$

By (i), we have for each $n \in \mathbb{N}$ a $W_n \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \setminus p' \Vdash_{\mathbb{P}} \dot{W}_n = \vec{W}_n$; now $V = \bigcup_{n\in\mathbb{N}} W_n$ belongs to $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $\Vdash_{\mathbb{P}} \vec{V} = \bigcup_{n\in\mathbb{N}} \vec{W}_n$, so $p \setminus p' \Vdash_{\mathbb{P}} \vec{V} = \dot{W}$. Finally, setting $W = (E \times \{0,1\}^I) \cap V$, $p \Vdash_{\mathbb{P}} \vec{W} = \dot{W}$. **Q**

(iii) Suppose that $p \in \mathfrak{A}^+$, $\alpha < \omega_1$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}),$$

defining $\mathcal{B}\mathfrak{a}_\alpha$ as in 5A4G. Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** Induce on α. The case $\alpha = 0$ is (ii) above. For the inductive step to $\alpha > 0$, we have

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}),$$

so

$$p \Vdash_{\mathbb{P}} \text{ there is a sequence } \langle W_n \rangle_{n\in\mathbb{N}} \text{ in } \bigcup_{\beta<\check{\alpha}} \mathcal{B}\mathfrak{a}_\beta(\{0,1\}^{\check{I}}) \text{ such that } \dot{W} = \bigcup_{n\in\mathbb{N}} \{0,1\}^{\check{I}} \setminus W_n;$$

let $\langle \dot{W}_n \rangle_{n\in\mathbb{N}}$ be a sequence of $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{W}_n \in \bigcup_{\beta<\check{\alpha}} \mathcal{B}\mathfrak{a}_\beta(\{0,1\}^{\check{I}}) \text{ for every } n \in \mathbb{N} \text{ and } \dot{W} = \bigcup_{n\in\mathbb{N}} \{0,1\}^{\check{I}} \setminus \dot{W}_n.$$

For $n \in \mathbb{N}$, $\beta < \alpha$ set

$$b_{n\beta} = [\![\dot{W}_n \in \mathcal{B}\mathfrak{a}_{\check{\beta}}(\{0,1\}^{\check{I}}) \setminus \bigcup_{\gamma<\check{\beta}} \mathcal{B}\mathfrak{a}_\gamma(\{0,1\}^{\check{I}})]\!],$$

and choose $E_{n\beta} \in \Sigma$ such that $E_{n\beta}^\bullet = b_{n\beta}$. Writing $A_n = \{\beta : \beta < \alpha,\ b_{n\beta} \neq 0\}$, $p \subseteq \sup_{\beta\in A_n} b_{n\beta}$. If $\beta \in A_n$, then

$$b_{n\beta} \Vdash_{\mathbb{P}} \dot{W}_n \in \mathcal{B}\mathfrak{a}_{\check{\beta}}(\{0,1\}^{\check{I}}),$$

so by the inductive hypothesis there is a $W_{n\beta} \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $b_{n\beta} \Vdash_{\mathbb{P}} \dot{W}_n = \vec{W}_{n\beta}$. For $\beta \in \alpha \setminus A_n$ set $W_{n\beta} = \emptyset$.

Set $W_n = \bigcup_{\beta<\alpha} (E_{n\beta} \times \{0,1\}^I) \cap W_{n\beta}$. Then

$$\Vdash_{\mathbb{P}} \vec{W}_n = \bigcup_{\beta<\alpha} (E_{n\beta} \times \{0,1\}^I)^{\rightarrow} \cap \vec{W}_{n\beta},$$

so if $\beta \in A_n$

$$b_{n\beta} \Vdash_{\mathbb{P}} \vec{W}_n = \vec{W}_{n\beta} = \dot{W}_n$$

because

$$b_{n\beta} \Vdash_{\mathbb{P}} (E_{n\gamma} \times \{0,1\}^I)^{\rightarrow} = \emptyset$$

if $\gamma < \alpha$ and $\gamma \neq \beta$, and

$$b_{n\beta} \Vdash_{\mathbb{P}} (E_{n\beta} \times \{0,1\}^I)^{\rightarrow} = \{0,1\}^{\check{I}}.$$

As $p \subseteq \sup_{\beta\in A_n} b_{n\beta}$,

$$p \Vdash_{\mathbb{P}} \vec{W}_n = \dot{W}_n.$$

This is true for every $n \in \mathbb{N}$. So if we set $W = \bigcup_{n\in\mathbb{N}} (\Omega \times \{0,1\}^I) \setminus W_n$, we shall have $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and

$$p \Vdash_{\mathbb{P}} \vec{W} = \bigcup_{n\in\mathbb{N}} \{0,1\}^{\check{I}} \setminus \vec{W}_n = \bigcup_{n\in\mathbb{N}} \{0,1\}^{\check{I}} \setminus \dot{W}_n = \dot{W}. \ \mathbf{Q}$$

(iv) Finally, suppose that $p \in \mathfrak{A}^+$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** Because $\mathbb{P}$ is ccc,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is the first uncountable ordinal}$$

(5A3Nb), so

$$\Vdash_{\mathbb{P}} \mathcal{B}\mathfrak{a}(\{0,1\}^{\check{I}}) = \bigcup_{\alpha<\check{\omega}_1} \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^{\check{I}}).$$

For $\alpha < \omega_1$ set

$$b_\alpha = [\![\dot{W} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}})]\!].$$

Then $p \subseteq \sup_{\alpha<\omega_1} b_\alpha$. Again because $\mathfrak{A}$ is ccc, there is a $\gamma < \omega_1$ such that $p \subseteq \sup_{\alpha<\gamma} b_\alpha$. If $\alpha < \gamma$ and $c_\alpha = b_\alpha \setminus \sup_{\beta<\alpha} b_\beta$ is non-zero, choose $W_\alpha \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $c_\alpha \Vdash_{\mathbb{P}} \vec{W}_\alpha = \dot{W}$; for other $\alpha < \gamma$ set $W_\alpha = \emptyset$. Choose $F_\alpha \in \Sigma$ such that $F_\alpha^\bullet = c_\alpha$ for each α. Set $W = \bigcup_{\alpha<\gamma}(F_\alpha \times \{0,1\}^I) \cap W_\alpha \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. As in (iii) just above,

$$c_\alpha \Vdash_{\mathbb{P}} \vec{W} = \vec{W}_\alpha = \dot{W}$$

whenever $c_\alpha \neq 0$, so

$$p \Vdash_{\mathbb{P}} \vec{W} = \dot{W}. \;\mathbf{Q}$$

551G I noted above that there are difficulties in computing $[\![\vec{f} = \vec{g}]\!]$ for functions f, $g : \Sigma \to \{0,1\}^I$. For W, $V \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ the corresponding question about $[\![\vec{W} = \vec{V}]\!]$ turns out to be simpler, at least in some important cases.

Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ the associated forcing notion and I a set. Suppose that Σ is closed under Souslin's operation.

(a) If $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then $F = \{\omega : W[\{\omega\}] \neq \emptyset\}$ belongs to Σ and $[\![\vec{W} \neq \emptyset]\!] = F^\bullet$ in $\mathfrak{A}$.

(b) If W, $V \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then $[\![\vec{W} = \vec{V}]\!] = \{\omega : W[\{\omega\}] = V[\{\omega\}]\}^\bullet$.

proof (a)(i) The point is that there is a Σ-measurable function $f : \Omega \to \{0,1\}^I$ such that $(\omega, f(\omega)) \in W$ for every $\omega \in F$.

P $(\boldsymbol{\alpha})$ Suppose first that I is countable. Let $\mathcal{V}$ be the family of subsets of $\Omega \times \{0,1\}^I$ obtainable by Souslin's operation $\mathcal{S}$ from $\{E \times H : E \in \Sigma,\, H \subseteq \{0,1\}^I$ is closed$\}$. The family $\mathcal{W} = \{V : V \in \mathcal{V},\, (\Omega \times \{0,1\}^I) \setminus V \in \mathcal{V}\}$ is a σ-algebra and contains $E \times H$ whenever $E \in \Sigma$ and $H \subseteq \{0,1\}^I$ is open-and-closed, so $\mathcal{W} \supseteq \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $W \in \mathcal{W} \subseteq \mathcal{V}$. By 423M, there is a selector g for W which is measurable for the σ-algebra T of subsets of Ω generated by $\mathcal{S}(\Sigma)$; but we are supposing that this is just Σ. Also $F = \operatorname{dom} g$ belongs to $\mathrm{T} = \Sigma$. If f is any extension of g to a Σ-measurable function from Ω to $\{0,1\}^I$, then f has the required property.

$(\boldsymbol{\beta})$ For the general case, note that $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ factors through $\Omega \times \{0,1\}^J$ for some countable $J \subseteq I$, that is, there is a $W_1 \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_J$ such that

$$W = \{(\omega, x) : \omega \in \Omega,\, x \in \{0,1\}^I,\, (\omega, x{\restriction}J) \in W_1\}.$$

Now (α) tells us that $F_1 = \{\omega : W_1[\{\omega\}] \neq \emptyset\}$ belongs to Σ and that there is a Σ-measurable $f_1 : \Omega \to \{0,1\}^J$ such that $(\omega, f_1(\omega)) \in W_1$ for every $\omega \in F_1$. Of course $F_1 = F$, and if we set

$$\begin{aligned} f(\omega)(i) &= f_1(\omega)(i) \text{ for } \omega \in \Omega,\, i \in J, \\ &= 0 \text{ for } \omega \in \Omega,\, i \in I \setminus J, \end{aligned}$$

then f is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable and $(\omega, f(\omega)) \in W$ for every $\omega \in F$. $\mathbf{Q}$

(ii) Using 551Ea, it follows that

$$[\![\vec{W} \neq \emptyset]\!] \supseteq [\![\vec{f} \in \vec{W}]\!] = \{\omega : (\omega, f(\omega)) \in W\}^\bullet = F^\bullet.$$

On the other hand, if $a = [\![\vec{W} \neq \emptyset]\!]$ is non-zero, then there is a $\mathbb{P}$-name $\dot{x}$ such that $a \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. By 551Cc, there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable g such that $a \Vdash_{\mathbb{P}} \dot{x} = \vec{g}$, in which case

$$a \subseteq [\![\vec{g} \in \vec{W}]\!] = \{\omega : (\omega, g(\omega)) \in W\}^\bullet \subseteq F^\bullet.$$

So $[\![\vec{W} \neq \emptyset]\!] = F^\bullet$ exactly.

(b) Apply (a) to $W \triangle V$ (using 551Eb, as usual).

551H Examples Cases in which a σ-algebra is closed under Souslin's operation, so that the conditions of 551G can be satisfied, include the following.

(a) If (X, Σ, μ) is a complete locally determined measure space, then Σ is closed under Souslin's operation (431A).

(b) If $(\Omega, \Sigma, \mathcal{I})$ is a complete ω_1-saturated measurable space with negligibles, then Σ is closed under Souslin's operation (431G[1]).

[1] Later editions only.

(c) If X is any topological space, then its Baire-property algebra $\widehat{\mathcal{B}}(X)$ is closed under Souslin's operation (431F).

551I Theorem Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ its associated forcing notion, and I a set. Let W be any member of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. Then
(i) $h(\omega)=\nu_I W[\{\omega\}]$ is defined for every $\omega\in\Omega$, where ν_I is the usual measure of $\{0,1\}^I$;
(ii) $h:\Omega\to[0,1]$ is Σ-measurable;
(iii) $\Vdash_{\mathbb{P}}\nu_{\check I}\vec W=\vec h$,
where in this formula $\vec h$ is the $\mathbb{P}$-name for a real number defined as in 551B, and $\nu_{\check I}$ is an abbreviation for 'the usual measure on $\{0,1\}^{\check I}$'.

proof I follow the method of 551Ea and 551Fa.

(a) Suppose that W is of the form $E\times\{x:z\subseteq x\in\{0,1\}^I\}$, where $z\in\{0,1\}^J$ for some finite $J\subseteq I$. Then (using 551Ee)

$$E^\bullet=[\![\vec W=\{x:\check z\subseteq x\in\{0,1\}^{\check I}\}]\!]\subseteq[\![\nu_{\check I}\vec W=2^{-\#(\check J)}]\!],$$

$$1\setminus E^\bullet=[\![\vec W=\emptyset]\!]\subseteq[\![\nu_{\check I}\vec W=0]\!];$$

while also $h=2^{-\#(J)}\chi E$ so

$$E^\bullet=[\![\vec h=2^{-\#(\check J)}]\!],\quad 1\setminus E^\bullet=[\![\vec h=0]\!].$$

So in this case

$$\Vdash_{\mathbb{P}}\nu_{\check I}\vec W=\vec h.$$

(b) Now 551E shows that the set of those $W\in\mathcal{B}\mathfrak{a}_I$ for which (i)-(iii) are true is a Dynkin class, so by the Monotone Class Theorem once more we have the result.

551J Corollary Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathbb{P}$ its associated forcing notion, P the partially ordered set underlying $\mathbb{P}$, and I a set. If $p\in P$ and $\dot W$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\text{ is }\nu_{\check I}\text{-negligible},$$

then there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $\nu_I W[\{\omega\}]=0$ for every $\omega\in\Omega$ and

$$p\Vdash_{\mathbb{P}}\dot W\subseteq\vec W.$$

proof Because

$$\Vdash_{\mathbb{P}}\text{the usual measure on }\{0,1\}^{\check I}\text{ is a completion regular Radon measure},$$

we know that

$$p\Vdash_{\mathbb{P}}\text{there is a }\nu_{\check I}\text{-negligible member of }\mathcal{B}\mathfrak{a}_{\check I}\text{ including }\dot W.$$

Let $\dot V$ be a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\subseteq\dot V\in\mathcal{B}\mathfrak{a}_{\check I}\text{ and }\nu_{\check I}\dot V=0.$$

By 551Fb, there is a $V\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p\Vdash_{\mathbb{P}}\dot V=\vec V$. Set $h(\omega)=\nu_I V[\{\omega\}]$ for $\omega\in\Omega$; then

$$p\Vdash_{\mathbb{P}}\vec h=\nu_{\check I}\vec V=0$$

(551I), so $p\subseteq E^\bullet$, where $E=h^{-1}[\{0\}]$. Set $W=(E\times\{0,1\}^I)\cap V$. Then $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, $\nu_I W[\{\omega\}]=0$ for every ω, and

$$p\Vdash_{\mathbb{P}}\vec W=\vec V=\dot V\supseteq\dot W,$$

as required.

551K We have been looking here at general sets $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. A special case of obvious importance is when W is of the form $\Omega\times H$ where $H\in\mathcal{B}\mathfrak{a}_I$. For these it is worth refining the results slightly.

Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathbb{P}$ the associated forcing notion, and I a set. For $H\subseteq\{0,1\}^I$ set $\tilde H=(\Omega\times H)^{\rightharpoondown}$ as defined in 551D.
(a) If $H=\{x:z\subseteq x\in\{0,1\}^I\}$, where $z\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$, then

$$\Vdash_{\mathbb{P}}\tilde H=\{x:\check z\subseteq x\in\{0,1\}^{\check I}\}.$$

(b)(i) If G, $H \in \mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \tilde{G} \cup \tilde{H} = (G \cup H)^{\sim},\ \tilde{G} \cap \tilde{H} = (G \cap H)^{\sim},$$
$$\tilde{G} \setminus \tilde{H} = (G \setminus H)^{\sim},\ \tilde{G} \triangle \tilde{H} = (G \triangle H)^{\sim}.$$

(ii) If $\langle H_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \tilde{H}_n = (\bigcup_{n\in\mathbb{N}} H_n)^{\sim},\ \bigcap_{n\in\mathbb{N}} \tilde{H}_n = (\bigcap_{n\in\mathbb{N}} H_n)^{\sim}.$$

(c) If $\alpha < \omega_1$ and $H \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$, once again defining $\mathcal{B}\mathfrak{a}_\alpha$ as in 5A4G, then

$$\Vdash_{\mathbb{P}} \tilde{H} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}).$$

(d) If H is measured by the usual measure ν_I of $\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}} \tilde{H} \text{ has measure } (\nu_I H)^{\check{}} \text{ for the usual measure of } \{0,1\}^{\check{I}}.$$

proof (a) This is 551Ee.

(b) This is a special case of parts (b) and (d) of 551E.

(c) A subset of $\{0,1\}^I$ is a cozero set iff it is empty or expressible as the union of a sequence of basic cylinder sets, so if H is a cozero set then (a) and (b-ii) tell us that

$$\Vdash_{\mathbb{P}} \tilde{H} \text{ is a cozero set in } \{0,1\}^{\check{I}}.$$

Now an induction on α shows that if $H \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$ then

$$\Vdash_{\mathbb{P}} \tilde{H} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}).$$

(d) We have H_0, $H_1 \in \mathcal{B}\mathfrak{a}_I$ such that $H_0 \subseteq H \subseteq H_1$ and $\nu_I H_0 = \nu_I H = \nu_I H_1$. Applying 551I(iii) to $\Omega \times H_0$ and $\Omega \times H_1$,

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H}_0 = \nu_{\check{I}} \tilde{H}_1 = (\nu_I H)^{\check{}},$$

while of course $\Vdash_{\mathbb{P}} \tilde{H}_0 \subseteq \tilde{H} \subseteq \tilde{H}_1$ (551Ec), so

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H} = (\nu_I H)^{\check{}}.$$

551L Remark If I ask you to think of your favourite Baire set in $\{0,1\}^I$, it is likely to come with a definition; for instance, the set H of those $x \in \{0,1\}^{\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=1}^n x(i) = \frac{1}{2}$. The point of 551K is that we shall automatically get

$$\Vdash_{\mathbb{P}} \tilde{H} = \{x : x \in \{0,1\}^{\mathbb{N}},\ \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=1}^n x(i) = \frac{1}{2}\}.$$

P

$$H = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{k\ge m} \bigcup_{z\in L_{nk}} \{x : z \subseteq x \in \{0,1\}^{\mathbb{N}}\},$$

where

$$L_{nk} = \{z : z \in \{0,1\}^{k+1},\ |\frac{1}{k+1}\sum_{i=0}^k z_i - \frac{1}{2}| \le \frac{1}{n+1}\}.$$

So 551K tells us that

$$\Vdash_{\mathbb{P}} \tilde{H} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{k\ge m} \bigcup_{z\in \check{L}_{nk}} \{x : z \subseteq x \in \{0,1\}^{\mathbb{N}}\},$$

and of course

$$\Vdash_{\mathbb{P}} \check{L}_{nk} = \{z : z \in \{0,1\}^{\check{k}+1},\ |\frac{1}{\check{k}+1}\sum_{i=0}^{\check{k}} z_i - \frac{1}{2}| \le \frac{1}{\check{n}+1}\}. \ \mathbf{Q}$$

What I am trying to say here is that the process $H \mapsto (\Omega \times H)^{\rightarrow} = \tilde{H}$ builds a $\mathbb{P}$-name for the 'right' subset of $\{0,1\}^I$, in the sense that any adequately concrete definition of H will also, when interpreted in $V^{\mathbb{P}}$, be a definition of $\tilde{H}$.

551M We can go still farther.

Definition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $\psi : \Omega \times \{0,1\}^I \to \mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, let $\vec{\psi}$ be the $\mathbb{P}$-name

$$\{((\vec{f},\vec{h}),\mathbb{1}) : f \text{ is a } (\Sigma,\mathcal{B}\mathfrak{a}_I)\text{-measurable function from } \Omega \text{ to } \{0,1\}^I,$$
$$h:\Omega\to\mathbb{R} \text{ is } \Sigma\text{-measurable, } h(\omega)=\psi(\omega,f(\omega)) \text{ for every } \omega\in\Omega\},$$

where in this formula $\vec{f}$ is to be interpreted as a $\mathbb{P}$-name for a member of $\{0,1\}^{\check{I}}$, as in 551C, and $\vec{h}$ as a $\mathbb{P}$-name for a real number, as in 551B.

551N Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set. Suppose that $\psi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, and define $\vec{\psi}$ as in 551M.

(a) $\Vdash_{\mathbb{P}} \vec{\psi}$ is a real-valued function on $\{0,1\}^{\check{I}}$.

(b) If $\phi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is another $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable function, and $\alpha\in\mathbb{R}$, then

$$\Vdash_{\mathbb{P}} (\phi+\psi)^{\vec{}}=\vec{\phi}+\vec{\psi},\ (\alpha\phi)^{\vec{}}=\check{\alpha}\vec{\phi}.$$

(c) If $\langle\psi_n\rangle_{n\in\mathbb{N}}$ is a sequence of $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable real-valued functions on $\Omega\times\{0,1\}^I$ and we set $\psi(\omega,x)=\lim_{n\to\infty}\psi_n(\omega,x)$ for every $\omega\in\Omega$ and $x\in\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}} \psi(x)=\lim_{n\to\infty}\vec{\psi}_n(x) \text{ for every } x\in\{0,1\}^{\check{I}}.$$

(d) If $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, then

$$\Vdash_{\mathbb{P}} (\chi W)^{\vec{}}=\chi\vec{W}.$$

(e) $\Vdash_{\mathbb{P}} \vec{\psi}$ is $\mathcal{B}\mathfrak{a}_{\check{I}}$-measurable.

(f) If $h(\omega)=\int\psi(\omega,x)\nu_I(dx)$ is defined for every $\omega\in\Omega$, then

$$\Vdash_{\mathbb{P}} \int\vec{\psi}\,d\nu_{\check{I}} \text{ is defined and equal to } \vec{h}.$$

proof (a)(i) Suppose that we have two members $((\vec{f}_0,\vec{h}_0),\mathbb{1})$ and $((\vec{f}_1,\vec{h}_1),\mathbb{1})$ of $\vec{\psi}$, and that $E\in\Sigma\setminus\mathcal{I}$ is such that $E^\bullet\Vdash_{\mathbb{P}}\vec{f}_0=\vec{f}_1$. Then $E^\bullet\Vdash_{\mathbb{P}}\vec{h}_0=\vec{h}_1$. **P** Let $J\subseteq I$ be a countable set such that ψ factors through $\Omega\times\{0,1\}^J$, in the sense that $\psi(\omega,x)=\psi(\omega,y)$ whenever $\omega\in\Omega$ and x, $y\in\{0,1\}^I$ are such that $x\restriction J=y\restriction J$. For each $i\in J$,

$$E^\bullet\Vdash_{\mathbb{P}}\vec{f}_0(\check{\imath})=\vec{f}_1(\check{\imath}),$$

so that $f_0(\omega)(i)=f_1(\omega)(i)$ for $\mathcal{I}$-almost every $\omega\in E$. Consequently $f_0(\omega)\restriction J=f_1(\omega)\restriction J$ and

$$h_0(\omega)=\psi(\omega,f_0(\omega))=\psi(\omega,f_1(\omega))=h_1(\omega)$$

for $\mathcal{I}$-almost every $\omega\in E$; that is, $E^\bullet\Vdash_{\mathbb{P}}\vec{h}_0=\vec{h}_1$. **Q**

It follows that

$$\Vdash_{\mathbb{P}}\vec{\psi} \text{ is a function}$$

(5A3H).

(ii) By the constructions in 551Cb and 551B,

$$\Vdash_{\mathbb{P}}\vec{\psi}\subseteq\{0,1\}^{\check{I}}\times\mathbb{R}.$$

(iii) If $\dot{x}$ is a $\mathbb{P}$-name and $p\in\mathfrak{A}^+$ is such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}^{\check{I}}$, then there is a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable $f:\Omega\to\{0,1\}^I$ such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$. Setting $h(\omega)=\psi(\omega,f(\omega))$ for $\omega\in\Omega$, $((\vec{f},\vec{h}),\mathbb{1})\in\vec{\psi}$, so

$$p\Vdash_{\mathbb{P}}\dot{x}=\vec{f} \text{ and } (\vec{f},\vec{h})\in\vec{\psi}, \text{ so } \dot{x}\in\operatorname{dom}(\vec{\psi}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\operatorname{dom}(\vec{\psi})=\{0,1\}^{\check{I}}.$$

(b) This is easy. If $p\in\mathfrak{A}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}^{\check{I}}$, take a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable $f:\Omega\to\{0,1\}^I$ such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$; set

$$h_0(\omega)=\phi(\omega,f(\omega)),\quad h_1(\omega)=\psi(\omega,f(\omega));$$

for $\omega\in\Omega$; then

$$\begin{aligned}p\Vdash_{\mathbb{P}}(\phi+\psi)^{\vec{}}(\dot{x})&=(\phi+\psi)^{\vec{}}(\vec{f})=(h_0+h_1)^{\vec{}}\\&=\vec{h}_0+\vec{h}_1=\vec{\phi}(\vec{f})+\vec{\psi}(\vec{f})=\vec{\phi}(\dot{x})+\vec{\psi}(\dot{x}),\\(\alpha\phi)^{\vec{}}(\dot{x})&=(\alpha h_0)^{\vec{}}=\check{\alpha}\vec{h}_0=\check{\alpha}\vec{\phi}(\dot{x})\end{aligned}$$

by 5A3Lc. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} (\phi+\psi)^{\rightarrow} = \vec{\phi}+\vec{\psi},\ (\alpha\phi)^{\rightarrow} = \check{\alpha}\vec{\phi}.$$

(c) In the same way, if $p \in \mathfrak{A}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, take a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Set

$$h_n(\omega) = \psi_n(\omega, f(\omega)), \quad h(\omega) = \psi(\omega, f(\omega))$$

for $\omega \in \Omega$ and $n \in \mathbb{N}$; then $h = \lim_{n\to\infty} h_n$, so

$$p \Vdash_{\mathbb{P}} \vec{\psi}(\dot{x}) = \vec{h} = \lim_{n\to\infty} \vec{h}_n = \lim_{n\to\infty} \vec{\psi}_n(\dot{x}).$$

(d) Take $p \in \mathfrak{A}^+$ and a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$. Let $f : \Omega \to \{0,1\}^I$ be a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; set $h(\omega) = \chi W(\omega, f(\omega))$ for $\omega \in \Omega$, so that

$$\Vdash_{\mathbb{P}} \vec{h} = (\chi W)^{\rightarrow}(\vec{f}), \quad p \Vdash_{\mathbb{P}} \vec{h} = (\chi W)^{\rightarrow}(\dot{x}).$$

If $p = E^\bullet$ where $E \in \Sigma \setminus \mathcal{I}$,

$$\begin{aligned} p \Vdash_{\mathbb{P}} (\chi W)^{\rightarrow}(\dot{x}) = 1 &\\ \iff & p \Vdash_{\mathbb{P}} \vec{h} = 1 \\ \iff & h(\omega) = 1 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ \iff & (\omega, f(\omega)) \in W \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ \iff & p \subseteq [\![\vec{f} \in \vec{W}]\!] \end{aligned}$$

(551Ea)

$$\iff p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W};$$

similarly,

$$\begin{aligned} p \Vdash_{\mathbb{P}} (\chi W)^{\rightarrow}(\dot{x}) = 0 &\\ \iff & h(\omega) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ \iff & (\omega, f(\omega)) \notin W \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ \iff & p \Vdash_{\mathbb{P}} \dot{x} \notin \vec{W}. \end{aligned}$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} (\chi W)^{\rightarrow} = \chi\vec{W}.$$

(e) Assembling (a)-(d), we see that the result is true when ψ is a linear multiple of the characteristic function of a set in $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, whenever ψ is a sum of such functions, and whenever ψ is the limit of a sequence of such sums; that is, whenever ψ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable.

(f) Similarly, (d) and 551I tell us that the result is true for the characteristic function of a member of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. Once again, we can move to a linear combination of such functions, using (b), and thence to a non-negative $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable function, using (c); finally, with (b) again, we get the general case.

551O Measure algebras With a little more effort we can get a representation of the standard measure algebras in the same style. Let I be a set, ν_I the usual measure on $\{0,1\}^I$ and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. It will be important to appreciate that these are abbreviations for formulae in set theory with a single parameter I; so that if we have a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-name τ, we shall have $\mathbb{P}$-names $\mathfrak{B}_\tau$ and $\bar{\nu}_\tau$, uniquely defined as soon as we have settled on the exact formulations we wish to apply when interpreting the basic constructions $\{\dots\}$, $\mathcal{P}$ in the forcing language. Similarly, if we write $\mathbb{P}_I = (\mathfrak{B}_I^+, \subseteq, 1, \downarrow)$ for the forcing notion based on the Boolean algebra $\mathfrak{B}_I$, this also is a formula which can be interpreted in forcing languages.

551P Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, and such that Σ is closed under Souslin's operation. Let $\mathbb{P}$ be its associated forcing notion, and I a set. Set

$$\Lambda = \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I, \quad \mathcal{J} = \{W : W \in \Lambda,\ \nu_I W[\{\omega\}] = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in \Omega\};$$

then $\mathcal{J}$ is a σ-ideal of Λ (cf. 527B); let $\mathfrak{C}$ be the quotient algebra $\Lambda/\mathcal{J}$. For $W \in \Lambda$ and $\omega \in \Omega$ set $h_W(\omega) = \nu_I W[\{\omega\}]$. For $a \in \mathfrak{C}$ let $\vec{a}$ be the $\mathbb{P}$-name

$$\{(\vec{W}, \mathbb{1}) : W \in \Lambda,\, W^\bullet = a\}$$

where the $\mathbb{P}$-names $\vec{W}$ are defined as in 551D. Consider the $\mathbb{P}$-names

$$\dot{\mathfrak{D}} = \{(\vec{a}, \mathbb{1}) : a \in \mathfrak{C}\}, \quad \dot{\pi} = \{(((W^\bullet)^\rightarrow, (\vec{W})^\bullet), \mathbb{1}) : W \in \Lambda\}.$$

(a) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a bijection between $\dot{\mathfrak{D}}$ and $\mathfrak{B}_{\check{I}}$.

(b) If a, $b \in \mathfrak{C}$, $V \in \Lambda$ and $V^\bullet = a$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \,\triangle\, b)^\rightarrow = \dot{\pi}\vec{a} \,\triangle\, \dot{\pi}\vec{b}, \quad \dot{\pi}(a \cap b)^\rightarrow = \dot{\pi}\vec{a} \cap \dot{\pi}\vec{b}, \quad \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a}) = \vec{h}_V,$$

defining $\vec{h}_V$ as in 551B and 551I.

(c) Let $\varepsilon : \Sigma/\Sigma \cap \mathcal{I} \to \mathfrak{C}$ be the canonical map defined by the formula

$$\varepsilon(E^\bullet) = (E \times \{0,1\}^I)^\bullet \text{ for } E \in \Sigma.$$

If $p \in (\Sigma/\Sigma \cap \mathcal{I})^+$ and a, $b \in \mathfrak{C}$, then

$$p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{b}$$

iff $a \cap \varepsilon(p) = b \cap \varepsilon(p)$.

Remarks Note that in the formula

$$\{(((W^\bullet)^\rightarrow, (\vec{W})^\bullet), \mathbb{1}) : W \in \Lambda\}$$

the first $^\bullet$ is interpreted in the ordinary universe as the canonical map from Λ to $\mathfrak{C}$, and the second is interpreted in the forcing language as the canonical map from $\mathcal{B}\mathfrak{a}_{\check{I}}$ to $\mathfrak{B}_{\check{I}}$; while among the brackets $(\ldots)$, some are just separators, some are to be interpreted as an ordered-pair construction in the ordinary universe, and some are to be interpreted as and ordered-pair construction in the forcing language. Similarly, in the formula

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \,\triangle\, b)^\rightarrow = \dot{\pi}\vec{a} \,\triangle\, \dot{\pi}\vec{b}$$

the first $\triangle$ is to be interpreted in the ordinary universe as symmetric difference in the algebra $\mathfrak{C}$, while the second is to be interpreted in the forcing language as symmetric difference in $\mathfrak{B}_{\check{I}}$.

proof (a)(i) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a function with domain $\dot{\mathfrak{D}}$.

P? Suppose, if possible, that V, $W \in \Lambda$ and $E \in \Sigma \setminus \mathcal{I}$ are such that

$$E^\bullet \Vdash_{\mathbb{P}} (V^\bullet)^\rightarrow = (W^\bullet)^\rightarrow,\, \vec{V}^\bullet \ne \vec{W}^\bullet.$$

By 551I(iii) and 551Eb,

$$E^\bullet \Vdash_{\mathbb{P}} \vec{h}_{V \triangle W} = \nu_{\check{I}}(V \triangle W)^\rightarrow \ne 0.$$

On the other hand,

$$E^\bullet \Vdash_{\mathbb{P}} \vec{V} \in (V^\bullet)^\rightarrow = (W^\bullet)^\rightarrow,$$

so there must be a $W_1 \in \Lambda$ and an $F \in \Sigma \setminus \mathcal{I}$ such that $F^\bullet$ is stronger than $E^\bullet$, $W_1 \triangle W \in \mathcal{J}$ and $F^\bullet \Vdash_{\mathbb{P}} \vec{V} = \vec{W}_1$. Now, calculating in $\Sigma/\Sigma \cap \mathcal{I}$,

$$F^\bullet \subseteq \{\omega : V[\{\omega\}] = W_1[\{\omega\}]\}^\bullet$$

(551Gb)

$$\subseteq \{\omega : \nu_I(V[\{\omega\}] \triangle W_1[\{\omega\}]) = 0\}^\bullet = \{\omega : \nu_I(V[\{\omega\}] \triangle W[\{\omega\}]) = 0\}^\bullet$$

(because $W \triangle W_1 \in \mathcal{J}$, so $\nu_I(W[\{\omega\}] \triangle W_1[\{\omega\}]) = 0$ for $\mathcal{I}$-almost every ω)

$$= [\![\vec{h}_{V \triangle W} = 0]\!]$$

(551B); which is impossible. **X** So 5A3H tells us that

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a function with domain } \dot{\mathfrak{D}}. \quad \mathbf{Q}$$

(ii) Let $\dot{B}$ be the $\mathbb{P}$-name

$$\dot{B} = \{(\vec{W}^\bullet, \mathbb{1}) : W \in \Lambda\},$$

Then

$$\Vdash_{\mathbb{P}} \dot{\pi}^{-1} \text{ is a function with domain } \dot{B}.$$

P I aim to use 5A3H with the coordinates reversed. **?** Suppose, if possible, that V, $W \in \Lambda$ and $E \in \Sigma \setminus \mathcal{I}$ are such that

$$E^\bullet \Vdash_{\mathbb{P}} \vec{V}^\bullet = \vec{W}^\bullet,\ (V^\bullet)^\rightharpoondown \neq (W^\bullet)^\rightharpoondown.$$

Setting

$$V_1 = (V \cap (E \times \{0,1\}^I)) \cup (W \cap ((\Omega \setminus E) \times \{0,1\}^I)),$$

we have $E^\bullet \subseteq [\![\vec{V}_1 = \vec{V}]\!]$ (551Gb again), so

$$E^\bullet \Vdash_{\mathbb{P}} \vec{V}_1 = \vec{V} \in (V^\bullet)^\rightharpoondown \neq (W^\bullet)^\rightharpoondown, \text{ so } \vec{V}_1 \notin (W^\bullet)^\rightharpoondown.$$

It follows that $(\vec{V}_1, \mathbb{1}) \notin (W^\bullet)^\rightharpoondown$ and $V_1 \triangle W \notin \mathcal{J}$. Accordingly,

$$\begin{aligned}\{\omega : \omega \in E,\ h_{V\triangle W}(\omega) > 0\} &= \{\omega : \omega \in E,\ \nu_I(W[\{\omega\}] \triangle V[\{\omega\}]) > 0\} \\ &= \{\omega : \nu_I(W[\{\omega\}] \triangle V_1[\{\omega\}]) > 0\} \notin \mathcal{I},\end{aligned}$$

while at the same time

$$E^\bullet \Vdash_{\mathbb{P}} \vec{h}_{V\triangle W} = \nu_{\check{I}}(V \triangle W)^\rightharpoondown = 0,$$

and $h_{V\triangle W}(\omega) = 0$ for $\mathcal{I}$-almost every $\omega \in E$. **X**

So 5A3H, reversed, shows that

$$\Vdash_{\mathbb{P}} \dot{\pi}^{-1} \text{ is a function with domain } \dot{B}. \ \mathbf{Q}$$

(iii) We need to check that

$$\Vdash_{\mathbb{P}} \dot{B} = \mathfrak{B}_{\check{I}}.$$

P$(\boldsymbol{\alpha})$ Suppose that $E \in \Sigma \setminus \mathcal{I}$ and a $\mathbb{P}$-name $\dot{x}$ are such that $E^\bullet \Vdash_{\mathbb{P}} \dot{x} \in \dot{B}$. Then there must be an $F \in \Sigma \setminus \mathcal{I}$ and a $W \in \Lambda$ such that $F^\bullet$ is stronger than $E^\bullet$ and $F^\bullet \Vdash_{\mathbb{P}} \dot{x} = \vec{W}^\bullet \in \mathfrak{B}_{\check{I}}$; as $E^\bullet$ and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{B} \subseteq \mathfrak{B}_{\check{I}}.$$

$(\boldsymbol{\beta})$ Suppose that $E \in \Sigma \setminus \mathcal{I}$ and a $\mathbb{P}$-name $\dot{x}$ are such that $E^\bullet \Vdash_{\mathbb{P}} \dot{x} \in \mathfrak{B}_{\check{I}}$. Then there must be a $\mathbb{P}$-name $\dot{W}$ such that

$$E^\bullet \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{I}} \text{ and } \dot{x} = \dot{W}^\bullet.$$

By 551Fb, there is a $W \in \Lambda$ such that

$$E^\bullet \Vdash_{\mathbb{P}} \vec{W} = \dot{W}, \text{ so } \dot{x} = \vec{W}^\bullet \in \dot{B};$$

as E and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \mathfrak{B}_{\check{I}} \subseteq \dot{B} \text{ and } \dot{B} = \mathfrak{B}_{\check{I}}. \ \mathbf{Q}$$

(iv) Putting these together,

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a bijection between } \dot{\mathfrak{D}} \text{ and } \mathfrak{B}_{\check{I}}.$$

(b) This is now easy. If V, $W \in \Lambda$, $a = V^\bullet$ and $b = W^\bullet$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \triangle b)^\rightharpoondown = \dot{\pi}((V \triangle W)^\bullet)^\rightharpoondown = ((V \triangle W)^\rightharpoondown)^\bullet = (\vec{V} \triangle \vec{W})^\bullet$$

(551Eb)

$$= \vec{V}^\bullet \triangle \vec{W}^\bullet = \dot{\pi}\vec{a} \triangle \dot{\pi}\vec{b},$$

and similarly

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \cap b)^\rightharpoondown = \dot{\pi}\vec{a} \cap \dot{\pi}\vec{b}.$$

Finally,

$$\Vdash_{\mathbb{P}} \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a}) = \bar{\nu}_{\check{I}}\vec{V}^\bullet = \nu_{\check{I}}\vec{V} = \vec{h}_V$$

by 551I(iii) again.

(c) Let $E \in \Sigma \setminus \mathcal{I}$ and V, $W \in \Lambda$ be such that $E^\bullet = p$, $V^\bullet = a$ and $W^\bullet = b$. Then (b) tells us that

$$\Vdash_{\mathbb{P}} \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a} \mathbin{\triangle} \dot{\pi}\vec{b}) = \bar{\nu}_{\check{I}}\dot{\pi}(a \mathbin{\triangle} b)^{\to} = \bar{\nu}_{\check{I}}\dot{\pi}((V \mathbin{\triangle} W)^\bullet)^{\to} = \vec{h}_{V \triangle W}.$$

So

$$\begin{aligned}
p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{b} &\iff p \Vdash_{\mathbb{P}} \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a} \mathbin{\triangle} \dot{\pi}\vec{b}) = 0 \\
&\iff p \Vdash_{\mathbb{P}} \vec{h}_{V \triangle W} = 0 \\
&\iff h_{V \triangle W}(\omega) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\
\text{(551B)} \qquad & \\
&\iff \nu_I(V[\{\omega\}] \triangle W[\{\omega\}]) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\
&\iff (E \times \{0,1\}^I) \cap (V \triangle W) \in \mathcal{J} \\
&\iff \varepsilon(p) \cap (a \mathbin{\triangle} b) = 0 \iff a \cap \varepsilon(p) = b \cap \varepsilon(p).
\end{aligned}$$

551Q Iterated forcing The machinery just developed can be used to establish one of the most important properties of random real forcing.

Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ccc measurable space with negligibles such that Σ is closed under Souslin's operation, $\mathbb{P}$ its associated forcing notion, and I a set. As in 551P, set $\Lambda = \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$,

$$\mathcal{J} = \{W : W \in \Lambda,\ \nu_I W[\{\omega\}] = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in \Omega\}$$

and $\mathfrak{C} = \Lambda/\mathcal{J}$. Then

$$\mathfrak{C} \cong \mathrm{RO}(\mathbb{P} * \mathbb{P}_{\check{I}}),$$

where the $\mathbb{P}$-name $\mathbb{P}_{\check{I}}$ is defined as in 551O.

proof (a) Since I wish to follow Kunen 80 as closely as possible, I should perhaps start with a remark on the interpretation of names for forcing notions. There is, strictly speaking, a distinction to be made between a name for a forcing notion, which is a name for a quadruplet of the form $(P, \le, \mathbb{1}, \downarrow)$, and a quadruplet of names, the first for a set, the second for a pre-order on that set, and so on; and the latter is easier to work with (Kunen 80, §VIII.5, and 5A3O). In the present case, we do not need any new manoeuvres, since the construction of the name $\mathbb{P}_{\check{I}}$ is based on $\mathbb{P}$-names for $\mathfrak{B}_{\check{I}}$, $\subseteq_{\mathfrak{B}_{\check{I}}}$ and $1_{\mathfrak{B}_{\check{I}}}$.

As usual in this section, I will write $\mathfrak{A}$ for $\Sigma/\Sigma \cap \mathcal{I}$.

(b) Now $\mathbb{P} * \mathbb{P}_{\check{I}}$ is based on the set P of pairs $(p, \dot{b})$ where $p \in \mathfrak{A}^+$, $\dot{b} \in B$ and $p \Vdash_{\mathbb{P}} \dot{b} \in \mathfrak{B}^+_{\check{I}}$; here B is the domain of the $\mathbb{P}$-name $\mathfrak{B}^+_{\check{I}}$ (5A3Ba). If we say that $(p, \dot{b}) \le (p', \dot{b}')$ if $p \subseteq p'$ and $p \Vdash_{\mathbb{P}} \dot{b} \subseteq \dot{b}'$, then P is pre-ordered by $\le$ and $\mathbb{P} * \mathbb{P}_{\check{I}}$ is active downwards.

We have a unique function $\theta : P \to \mathfrak{C}^+$ such that

$$\theta(p, \dot{b}) \subseteq \varepsilon(p), \qquad p \Vdash_{\mathbb{P}} \dot{\pi}\theta(p, \dot{b})^{\to} = \dot{b},$$

whenever $(p, \dot{b}) \in P$, where ε, $\dot{\pi}$ and $\vec{a}$, for $a \in \mathfrak{C}$, are defined as in 551P. **P** If $(p, \dot{b}) \in P$, so that $p \Vdash_{\mathbb{P}} \dot{b} \in \mathfrak{B}_{\check{I}}$, there is a $\mathbb{P}$-name $\dot{b}_1$ such that

$$\Vdash_{\mathbb{P}} \dot{b}_1 \in \mathfrak{B}_{\check{I}},$$

$$p \Vdash_{\mathbb{P}} \dot{b}_1 = \dot{b}.$$

Next, there is an $a_0 \in \mathfrak{C}$ such that $\Vdash_{\mathbb{P}} \dot{\pi}\vec{a}_0 = \dot{b}_1$ (551Pa). Set $a = a_0 \cap \varepsilon(p)$. Then 551Pc tells us that

$$p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{a}_0 = \dot{b}_1 = \dot{b} \ne 0,$$

and $a \ne 0$. To see that a is unique, observe that if $c \in \mathfrak{C}$ is such that $p \Vdash_{\mathbb{P}} \dot{\pi}\vec{c} = \dot{b}$, then $c \cap \varepsilon(p) = a \cap \varepsilon(p)$, by 551Pc again; so if $c \subseteq \varepsilon(p)$, $c = a$. We therefore can, and must, take a for $\theta(p, \dot{b})$. **Q**

(c)(i) If $(p, \dot{b})$, $(p', \dot{b}') \in P$ and $(p, \dot{b})$ is stronger than $(p', \dot{b}')$, then $p \subseteq p'$ and $p \Vdash_{\mathbb{P}} \dot{b} \subseteq \dot{b}'$. In this case,

$$p \Vdash_{\mathbb{P}} \dot{\pi}\theta(p', \dot{b}')^{\to} = \dot{b}' \text{ and } \dot{\pi}(\theta(p, \dot{b}) \cap \theta(p', \dot{b}'))^{\to} = \dot{\pi}\theta(p, \dot{b})^{\to} \cap \dot{\pi}\theta(p', \dot{b}')^{\to} = \dot{b} \cap \dot{b}' = \dot{b},$$

while $\theta(p, \dot{b}) \cap \theta(p', \dot{b}') \subseteq \varepsilon(p)$, so $\theta(p, \dot{b}) \cap \theta(p', \dot{b}') = \theta(p, \dot{b})$ and $\theta(p, \dot{b}) \subseteq \theta(p', \dot{b}')$.

(ii) If $(p, \dot{b})$ and $(p', \dot{b}') \in P$ are incompatible, then $\theta(p, \dot{b}) \cap \theta(p', \dot{b}') = 0$. **P?** Otherwise, writing a for $\theta(p, \dot{b}) \cap \theta(p', \dot{b}')$,

$$\varepsilon(p\cap p')=\varepsilon(p)\cap\varepsilon(p')\supseteq a\ne 0$$

so $p\cap p'\ne 0$ and

$$p\cap p'\Vdash_{\mathbb{P}}\dot\pi\vec a\subseteq\dot\pi\theta(p,\dot b)^{\rightharpoondown}\cap\dot\pi\theta(p',\dot b')^{\rightharpoondown}=\dot b\cap\dot b'=0;$$

as $a\subseteq\varepsilon(p\cap p')$, a must be 0; which is absurd. **XQ**

(iii) If $a\in\mathfrak{C}^+$, there is a $(p,\dot b)\in P$ such that $\theta(p,\dot b)\subseteq a$. **P** By 551Pc, $\not\Vdash_{\mathbb{P}}\dot\pi\vec a=0$, that is, there is a $p_0\in\mathfrak{C}^+$ such that $p_0\Vdash_{\mathbb{P}}\dot\pi\vec a\ne 0$. Now there must be a $p\in\mathfrak{A}^+$, stronger than p_0, and a $\dot b\in B$ such that $p\Vdash_{\mathbb{P}}\dot b=\dot\pi\vec a$, in which case $(p,\dot b)\in P$ and $p\Vdash_{\mathbb{P}}\dot\pi\theta(p,\dot b)^{\rightharpoondown}=\dot\pi\vec a$. Accordingly

$$\theta(p,\dot b)=\theta(p,\dot b)\cap\varepsilon(p)=a\cap\varepsilon(p)\subseteq a.\ \mathbf{Q}$$

(d) Observe now that $\mathfrak{C}$ is ccc (527L), therefore Dedekind complete, and (c) tells us that $\theta:P\to\mathfrak{C}^+$ satisfies the conditions of 514Sa. So $\mathrm{RO}(\mathbb{P}*\mathbb{P}_{\dot I})=\mathrm{RO}^{\downarrow}(P)$ is isomorphic to $\mathfrak{C}$.

551R Extending filters The following device will be useful in §553.

Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathfrak{A}$ its quotient algebra, $\mathbb{P}$ the associated forcing notion, I a countable set and $\mathcal{F}$ a filter on I.

(a) For $H\in\Sigma\widehat{\otimes}\mathcal{P}I$, write $\vec H$ for the $\mathbb{P}$-name $\{(\check\imath,H^{-1}[\{i\}]^\bullet):i\in I,\ H^{-1}[\{i\}]\notin\mathcal{I}\}$.

(i) $\Vdash_{\mathbb{P}}\vec H\subseteq\check I$.

(ii) If $\dot F$ is a $\mathbb{P}$-name and $p\in\mathfrak{A}^+$ is such that $p\Vdash_{\mathbb{P}}\dot F\subseteq\check I$, then there is an $H\in\Sigma\widehat{\otimes}\mathcal{P}I$ such that $p\Vdash_{\mathbb{P}}\dot F=\vec H$.

(b) Write $\vec{\mathcal{F}}$ for the $\mathbb{P}$-name

$$\{(\vec H,E^\bullet):H\in\Sigma\widehat{\otimes}\mathcal{P}I,\ E\in\Sigma\setminus\mathcal{I},\ H[\{\omega\}]\in\mathcal{F}\text{ for every }\omega\in E\}.$$

Then

$$\Vdash_{\mathbb{P}}\vec{\mathcal{F}}\text{ is a filter on }\check I.$$

proof (a)(i) is elementary, just because

$$\Vdash_{\mathbb{P}}\check\imath\in\check I$$

for every $i\in I$.

(ii) Because $(\Omega,\Sigma,\mathcal{I})$ is ω_1-saturated, $\mathfrak{A}$ is Dedekind complete and can be identified with $\mathrm{RO}(\mathbb{P})$. We therefore have, for each $i\in I$, an $E_i\in\Sigma$ such that $E_i^\bullet$ can be identified with $[\![\check\imath\in\dot F]\!]$. Set $H=\bigcup_{i\in I}E_i\times\{i\}$. Then

$$[\![\check\imath\in\dot F]\!]=H^{-1}[\{i\}]^\bullet=[\![\check\imath\in\vec H]\!]$$

for every $i\in I$, so

$$p\Vdash_{\mathbb{P}}\dot F=\dot F\cap\check I=\vec H\cap\check I=\vec H.$$

(b)(i) By (a-i), $\Vdash_{\mathbb{P}}\vec{\mathcal{F}}\subseteq\check I$.

(ii) Since $((\Omega\times I)^{\rightharpoondown},\mathbb{1})\in\vec{\mathcal{F}}$ and

$$\Vdash_{\mathbb{P}}(\Omega\times I)^{\rightharpoondown}=\check I,$$

we have

$$\Vdash_{\mathbb{P}}\check I\in\vec{\mathcal{F}}.$$

(iii) If $(\vec H,p)\in\vec{\mathcal{F}}$ then $p\Vdash_{\mathbb{P}}\vec H\ne\emptyset$. **P** Express p as $E^\bullet$ where $E\in\Sigma$ and $H[\{\omega\}]\in\mathcal{F}$ for every $\omega\in E$. Then $E\subseteq\bigcup_{i\in I}H^{-1}[\{i\}]$. So if $q\in\mathfrak{A}^+$ is stronger than p, there must be an $i\in I$ such that $r=q\cap H^{-1}[\{i\}]^\bullet$ is non-zero; in which case r is stronger than q and

$$r\Vdash_{\mathbb{P}}\check\imath\in\vec H,\text{ so }\vec H\ne\emptyset.$$

As q is arbitrary, $p\Vdash_{\mathbb{P}}\vec H\ne\emptyset$. **Q**

It follows at once that $\Vdash_{\mathbb{P}}\emptyset\notin\vec{\mathcal{F}}$.

(iv) If $\dot F_0$, $\dot F_1$ are $\mathbb{P}$-names and $p\in\mathfrak{A}^+$ is such that

$$p\Vdash_{\mathbb{P}}\dot F_0,\ \dot F_1\in\vec{\mathcal{F}},$$

then

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 \in \vec{\mathcal{F}}.$$

P If $q \in \mathfrak{A}^+$ is stronger than p there must be $(\vec{H}_0, E_0^\bullet)$, $(\vec{H}_1, E_1^\bullet) \in \vec{\mathcal{F}}$ and an r stronger than q, $E_0^\bullet$ and $E_1^\bullet$ such that

$$r \Vdash_{\mathbb{P}} \dot{F}_0 = \vec{H}_0 \text{ and } \dot{F}_1 = \vec{H}_1.$$

Now $((H_0 \cap H_1)^{\rightarrow}, (E_0 \cap E_1)^\bullet) \in \vec{\mathcal{F}}$ and

$$r \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 = \vec{H}_0 \cap \vec{H}_1 = (H_0 \cap H_1)^{\rightarrow} \in \vec{\mathcal{F}}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 \in \vec{\mathcal{F}}. \text{ **Q**}$$

Accordingly

$$\Vdash_{\mathbb{P}} \vec{\mathcal{F}} \text{ is closed under } \cap.$$

(v) Suppose that $\dot{F}_0$, $\dot{F}_1$ are $\mathbb{P}$-names and $p \in \mathfrak{A}^+$ is such that

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \subseteq \dot{F}_1 \subseteq \check{I},\ \dot{F}_0 \in \vec{\mathcal{F}}.$$

By (a-ii), there is an $H_1 \in \Sigma\widehat{\otimes}\mathcal{P}I$ such that $p \Vdash_{\mathbb{P}} \dot{F}_1 = \vec{H}_1$. If q is stronger than p, there are an $(\vec{H}_0, E_0^\bullet) \in \vec{\mathcal{F}}$ and an r stronger than both q and $E_0^\bullet$ such that

$$r \Vdash_{\mathbb{P}} \vec{H}_0 = \dot{F}_0 \subseteq \dot{F}_1 = \vec{H}_1.$$

Expressing r as $E^\bullet$ where $E \in \Sigma \setminus \mathcal{I}$, we have

$$E \cap H_0^{-1}[\{i\}] \setminus H_1^{-1}[\{i\}] \in \mathcal{I}$$

for every $i \in I$. Set

$$E_1 = E \setminus \bigcup_{i \in I}(H_0^{-1}[\{i\}] \setminus H_1^{-1}[\{i\}]);$$

then $(\vec{H}_1, E_1^\bullet) \in \vec{\mathcal{F}}$, so

$$r = E_1^\bullet \Vdash_{\mathbb{P}} \dot{F}_1 = \vec{H}_1 \in \vec{\mathcal{F}}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{F}_1 \in \vec{\mathcal{F}}.$$

As p, $\dot{F}_0$ and $\dot{F}_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{\mathcal{F}} \text{ is a filter on } \check{I}.$$

551X Basic exercises (a) Let $(\Omega, \Sigma, \mathcal{I})$ be any measurable space with negligibles. Set $\hat{\Sigma} = \{E \triangle F : E \in \Sigma, F \in \mathcal{I}\}$. (i) Show that $(\Omega, \hat{\Sigma}, \mathcal{I})$ is a complete measurable space with negligibles; we may call it the **completion** of $(\Omega, \Sigma, \mathcal{I})$. (ii) Show that the algebras $\Sigma/\Sigma \cap \mathcal{I}$ and $\hat{\Sigma}/\mathcal{I}$ are canonically isomorphic (cf. 322D). (iii) Show that $(\Omega, \hat{\Sigma}, \mathcal{I})$ is ω_1-saturated iff $(\Omega, \Sigma, \mathcal{I})$ is.

(b) Let (Ω, Σ, μ) be a measure space and $\mathcal{N}(\mu)$ the null ideal of μ. (i) Show that $(\Omega, \Sigma, \mathcal{N}(\mu))$ is a measurable space with negligibles. (ii) Show that if the completion of (Ω, Σ, μ) (212C) is $(\Omega, \hat{\Sigma}, \hat{\mu})$, then $(\Omega, \hat{\Sigma}, \mathcal{N}(\hat{\mu}))$ is the completion of $(\Omega, \Sigma, \mathcal{N}(\mu))$.

(c) Let X be a topological space, $\mathcal{B}(X)$ the Borel σ-algebra of X, $\widehat{\mathcal{B}}(X)$ the Baire-property algebra of X and $\mathcal{M}$ the ideal of meager subsets of X. (i) Show that $(X, \mathcal{B}(X), \mathcal{M})$ is a measurable space with negligibles. (ii) Show that its completion is $(X, \widehat{\mathcal{B}}(X), \mathcal{M})$.

(d) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion. (i) Show that the regular open algebra of $\mathbb{P}$ can be identified with the Dedekind completion of $\Sigma/\Sigma \cap \mathcal{I}$. (ii) Show that if $\mathcal{E}$ is any coinitial subset of $\Sigma \setminus \mathcal{I}$ containing Ω, then the forcing notion $\mathcal{E}$, active downwards, has regular open algebra isomorphic to $\mathrm{RO}(\mathbb{P})$.

(e) Let $(\Omega, \Sigma, \mathcal{I})$ and $(\Upsilon, \mathrm{T}, \mathcal{J})$ be measurable spaces with negligibles. Show that $(\Omega \times \Upsilon, \Sigma\widehat{\otimes}\mathrm{T}, \mathcal{I} \ltimes_{\Sigma\widehat{\otimes}\mathrm{T}} \mathcal{J})$, as defined in 527Bc, is a measurable space with negligibles.

(f) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion. Let $f : \Omega \to \{0, 1\}$ be a measurable function, and define $\mathbb{P}$-names $\dot{x}$, $\dot{y}$ by saying that $\dot{x} = \vec{f}$ where $\vec{f}$ is the $\mathbb{P}$-name for a real number as defined in 551B, while $\dot{y}$ is the $\mathbb{P}$-name for a member of $\{0, 1\}$ as defined in 551Ca. Show that

$$\Vdash_{\mathbb{P}} \text{regarding 0 and 1 as real numbers, } \dot{x} = \dot{y}.$$

(g) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathbb{P}$ the associated forcing notion, and I a set. Suppose that $f : \Omega \to \{0,1\}^I$ is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function, and that $\Gamma_f \subseteq \Omega \times \{0,1\}^I$ is its graph (for once, I distinguish between f and Γ_f). Let $\vec{f}$ and $\vec{\Gamma}_f$ be the $\mathbb{P}$-names for a point of $\{0,1\}^{\check{I}}$ and a subset of $\{0,1\}^{\check{I}}$ defined by the formulae in 551C and 551D respectively. Show that $\Vdash_{\mathbb{P}} \vec{\Gamma}_f = \{\vec{f}\}$.

(h) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, and $(\Omega, \hat{\Sigma}, \mathcal{I})$ its completion; write $\mathbb{P}$, $\hat{\mathbb{P}}$ for the associated forcing notions, so that $\mathbb{P}$ and $\hat{\mathbb{P}}$ are canonically isomorphic. Let I be a set, W a member of $\Sigma\widehat{\otimes}\{0,1\}^I \subseteq \hat{\Sigma}\widehat{\otimes}\{0,1\}^I$ and $\vec{W}$ the $\mathbb{P}$-name, $\hat{\vec{W}}$ the $\hat{\mathbb{P}}$-name defined by the formula in 551D. Explain what it ought to mean to say that $\Vdash_{\hat{\mathbb{P}}} \hat{\vec{W}} = \vec{W}$, and why this is true.

(i) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathbb{P}$ the associated forcing notion, and I a set. Suppose that $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is such that $\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}})$, where $\alpha < \omega_1$. Show that $W[\{\omega\}] \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$ for $\mathcal{I}$-almost every $\omega \in \Omega$.

551Y Further exercises (a) Investigate the difficulties which arise if we try to represent names for Borel subsets of $\{0,1\}^{\check{I}}$ as members of $\Sigma\widehat{\otimes}\{0,1\}^I$, when I is uncountable. Show that some of these are resolvable if Ω is actually the Stone space of RO($\mathbb{P}$).

(b) Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathbb{P}$ the associated forcing notion and P the partially ordered set underlying $\mathbb{P}$. (i) Let $p \in P$ and a $\mathbb{P}$-name $\dot{G}$ be such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is a dense open subset of } \{0,1\}^\omega.$$

Show that there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a dense open set and $p \Vdash_{\mathbb{P}} \dot{G} = \vec{W}$. (ii) Let $p \in P$ and a $\mathbb{P}$-name $\dot{A}$ be such that

$$p \Vdash_{\mathbb{P}} \dot{A} \text{ is a meager subset of } \{0,1\}^\omega.$$

Show that there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a meager set and $p \Vdash_{\mathbb{P}} \dot{A} \subseteq \vec{W}$.

551 Notes and comments There are real metamathematical difficulties in forcing, and we need to find new compromises between formal rigour and intuitive accessibility. In the formulae of this section I am taking a path with rather more explicit declarations than is customary. The definitions of $\vec{u}$ in 5A3L, $\vec{f}$ in 551B and 551Ca, and $\vec{W}$ in 551D, are supposed to be $\mathbb{P}$-names in the exact sense used in Kunen 80. This leads to rather odd sentences like

$$(\vec{f}, p) \in \vec{W} \text{ so } p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}$$

((a-ii) of the proof of 551E), in which the symbol $\in$ is being used in different ways in the two halves; but it has the advantage that we can move from W to $\vec{W}$ without further explanation, as in the statements of 551E-551J. But you will observe that elsewhere I allow such terms as $\mathcal{B}\mathfrak{a}$ and $\nu_{\dots}$ to enter sentences in the forcing language, since these correspond to definitions which can be expanded there. Note that I am being less strict than usual in requiring formulae to be unambiguous (see 551Xf and 551Xg).

There is always room for variation in the matter of which terms should be decorated with ˇs when they appear in expressions of the forcing language, and I have tried to be reasonably consistent; but there are particular difficulties with transferring names for families (5A3Eb), which appear here in such formulae as '$\Vdash_{\mathbb{P}} \lim_{n\to\infty} \vec{h}_n = \lim_{n\to\infty} \vec{\psi}_n(\dot{x})$' (part (c) of the proof of 551N).

I hope that it is not too confusing to have the formula $[\![\dots]\!]$ used in two different ways, not infrequently in the same sentence: sometimes as a 'Boolean value' in the forcing sense, and sometimes in the sense of Chapter 36. If you look back at the definitions in §364 you will see that the expression $f^\bullet$ also shifts in interpretation as we move between the formally distinct algebras $\mathfrak{A}$ and RO($\mathbb{P}$).

552 Random reals I

From the point of view of the measure theorist, 'random real forcing' has a particular significance. Because the forcing notions are defined directly from the central structures of measure theory (552A), they can be expected to provide worlds in which measure-theoretic questions are answered. Thus we find ourselves with many Sierpiński sets (552E), information on cardinal functions (552C, 552F-552J), and theorems on extension of measures (552N). But there is a second reason why any measure theorist or probabilist should pay attention to random real forcing. Natural questions in the forcing language, when translated into propositions about the ground model, are likely to hinge on properties of measure algebras, giving us a new source of challenging problems. Perhaps the deepest intuitions are those associated with iterated random real forcing (552P).

552A Notation (a) As usual, if μ is any measure then $\mathcal{N}(\mu)$ will be its null ideal. It will be convenient to have a special notation for certain sets of finite functions: if I is a set, $\mathrm{Fn}_{<\omega}(I;\{0,1\})$ will be $\bigcup_{K\in[I]^{<\omega}}\{0,1\}^K$.

For any set I I will write ν_I for the usual completion regular Radon probability measure on $\{0,1\}^I$, T_I for its domain and $(\mathfrak{B}_I,\bar\nu_I)$ for its measure algebra; $\mathcal{B}\mathfrak{a}_I=\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ will be the Baire σ-algebra of $\{0,1\}^I$. (It will sometimes be convenient, when applying the results of §551, to regard $\mathfrak{B}_I$ as the quotient $\mathcal{B}\mathfrak{a}_I/\mathcal{B}\mathfrak{a}_I\cap\mathcal{N}(\nu_I)$.) In this context, I will write $\langle e_i\rangle_{i\in I}$ for the standard generating family in $\mathfrak{B}_I$ (525A). $\mathbb{P}_I$ will be the forcing notion $\mathfrak{B}_I^+=\mathfrak{B}_I\setminus\{0\}$, active downwards. For a formula ϕ in the corresponding forcing language I will write $[\![\phi]\!]$ for the truth value of ϕ, interpreted as a member of $\mathfrak{B}_I$ (5A3M). Note that as $\mathbb{P}_I$ is ccc (cf. 511Eb), it preserves cardinals (5A3Nb).

As in §551, the formulae ν_I, $\mathfrak{B}_I$ etc. are to be regarded as formulae of set theory with one free variable into which the parameter I has been substituted, so that we have corresponding names $\nu_{\dot I}$, $\mathfrak{B}_{\dot I}$ in any forcing language, and in particular (once the context has established a forcing notion $\mathbb{P}$) we have $\mathbb{P}$-names $\nu_{\check I}$, $\mathfrak{B}_{\check I}$ for any ground-model set I.

(b) A great deal of the work of this chapter will involve interpretations of names for standard objects (in particular, for cardinals) in forcing languages. Reflecting suggestions in 5A3G and 5A3N, I will try to signal intended interpretations by using the superscript $\check{}$. Thus $\mathfrak{c}$ will always be an abbreviation for 'the initial ordinal equipollent with the set of subsets of the natural numbers', whether I am using the ordinary language of set theory or speaking in a forcing language; and $\check{\mathfrak{c}}$, in a forcing language, will refer to the name $\{(\check\xi,\mathbb{1}):\xi<\mathfrak{c}\}$, where it is to be understood that the symbol $\mathfrak{c}$ must now be interpreted in the ordinary universe. As I shall avoid arguments involving more than one forcing notion (and, in particular, iterated forcing), there will I hope be little scope for confusion, even in such sentences as

$$\Vdash_{\mathbb{P}_\kappa}\mathfrak{b}=\check{\mathfrak{b}}$$

(552C). The leading $\Vdash_{\mathbb{P}_\kappa}$ declares that the rest of the sentence is in the language of $\mathbb{P}_\kappa$-forcing; the first $\mathfrak{b}$, and the $\check{}$, are therefore to be interpreted in that language; but the second $\mathfrak{b}$, being subject to the $\check{}$, is to be interpreted in the ground model. (Many authors would write $\mathfrak{b}^V$ at this point.) Similarly, in

$$\Vdash_{\mathbb{P}_\kappa}2^{\check\lambda}=(\kappa^\lambda)\check{}$$

(552B), the subformula κ^λ is to be interpreted in the ordinary universe, but $2^{\check\lambda}=\#(\mathcal{P}\check\lambda)$ is to be interpreted in the forcing language. I hope that the resulting gains in directness and conciseness will not be at the expense of leaving you uncertain of the meaning.

552B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{P}_\kappa}2^{\check\lambda}=(\kappa^\lambda)\check{},$$

where κ^λ is the cardinal power (interpreted in the ordinary universe, of course).

proof (a) Recall that $\#(\mathfrak{B}_\kappa)=\kappa^\omega$ (524Ma), so that

$$\#(\mathfrak{B}_\kappa^\lambda)=\#(\kappa^{\omega\times\lambda}).$$

If $\dot A$ is a $\mathbb{P}_\kappa$-name for a subset of $\check\lambda$, then we have a corresponding family $\langle[\![\check\eta\in\dot A]\!]\rangle_{\eta<\lambda}$ of truth values; and if $\dot A$, $\dot B$ are two such names, and $[\![\check\eta\in\dot A]\!]=[\![\check\eta\in\dot B]\!]$ for every $\eta<\lambda$, then

$$\Vdash_{\mathbb{P}_\kappa}\check\eta\in\dot A\iff\check\eta\in\dot B$$

for every $\eta<\lambda$, so

$$\Vdash_{\mathbb{P}_\kappa}\dot A=\dot B.$$

So

$$\Vdash_{\mathbb{P}_\kappa}2^{\check\lambda}=\#(\mathcal{P}\check\lambda)\le\#((\mathfrak{B}_\kappa^\lambda)\check{})=(\kappa^\lambda)\check{}.$$

(b) In the other direction, consider first the case in which $\lambda\le\kappa$. Let F be the set of all functions from λ to κ, so that $\#(F)=\kappa^\lambda$. Then there is a set $G\subseteq F$ such that $\#(G)=\kappa^\lambda$ and $\{\eta:\eta<\lambda,\ f(\eta)\ne g(\eta)\}$ is infinite whenever f,

$g \in G$ are distinct. **P** If $\kappa = \kappa^\lambda$ we can take G to be the set of constant functions. Otherwise, for f, $g \in F$, say that $f =^* g$ if $\{\eta : f(\eta) \neq g(\eta)\}$ is finite; this is an equivalence relation. Let $G \subseteq F$ be a set meeting each equivalence class in just one element. Then we have $\#(\{g : g =^* f\}) = \kappa < \kappa^\lambda$ for every $f \in F$, so $\#(G) = \kappa^\lambda$, as required. **Q**

Let $\langle e_{\xi\eta}\rangle_{\xi<\kappa,\eta<\lambda}$ be a stochastically independent family in $\mathfrak{B}_\kappa$ of elements of measure $\frac{1}{2}$. For $f \in G$ let $\dot{A}_f$ be a $\mathbb{P}_\kappa$-name for a subset of $\check{\lambda}$ such that

$$[\![\check{\eta} \in \dot{A}_f]\!] = e_{f(\eta),\eta}$$

for every $\eta < \lambda$. If f, $g \in G$ are distinct, set $I = \{\eta : f(\eta) \neq g(\eta)\}$; then

$$[\![\dot{A}_f \neq \dot{A}_g]\!] = \sup_{\eta<\lambda} e_{f(\eta),\eta} \triangle e_{g(\eta),\eta} = \sup_{\eta\in I} e_{f(\eta),\eta} \triangle e_{g(\eta),\eta} = 1$$

because $\langle e_{f(\eta),\eta} \triangle e_{g(\eta),\eta}\rangle_{\eta\in I}$ is an infinite stochastically independent family of elements of measure $\frac{1}{2}$.

Thus in the forcing language we have a name for an injective function from $\check{G}$ to $\mathcal{P}\check{\lambda}$, corresponding to the map $f \mapsto \dot{A}_f$ from G to names of subsets of λ. So

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} \geq \#(\check{G}) = (\kappa^\lambda)\check{}.$$

Putting this together with (a), we have

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

(c) If $\lambda > \kappa$, then (in the ordinary universe) $2^\lambda = \kappa^\lambda$. Now

$$\Vdash_{\mathbb{P}_\kappa} (\mathcal{P}\lambda)\check{} \subseteq \mathcal{P}\check{\lambda},$$

so

$$\Vdash_{\mathbb{P}_\kappa} (\kappa^\lambda)\check{} = \#((\mathcal{P}\lambda)\check{}) \leq \#(\mathcal{P}\check{\lambda}) = 2^{\check{\lambda}},$$

and again we have

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

552C Theorem Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} = \check{\mathfrak{b}} \text{ and } \mathfrak{d} = \check{\mathfrak{d}}.$$

proof (a) The point is that if $\dot{f}$ is any $\mathbb{P}_\kappa$-name for a member of $\mathbb{N}^\mathbb{N}$, then there is an $h \in \mathbb{N}^\mathbb{N}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \leq^* \check{h},$$

where I write $f \leq^* g$ to mean that $\{n : g(n) < f(n)\}$ is finite, as in 522C. **P** For n, $i \in \mathbb{N}$ set $a_{ni} = [\![\dot{f}(\check{n}) = \check{\imath}]\!]$. Then $D_n = \{a_{ni} : i \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{B}_\kappa$ for each $n \in \mathbb{N}$. Because $\mathfrak{B}_\kappa$ is weakly (σ,∞)-distributive (322F), there is a partition of unity D such that $\{i : a_{ni} \cap d \neq 0\}$ is finite for each n and each $d \in D$. Let $\langle d_k\rangle_{k\in\mathbb{N}}$ be a sequence running over D and take $h(n)$ such that $a_{mi} \cap d_n = 0$ whenever $m \leq n$ and $i > h(n)$. Now

$$[\![\check{h}(\check{m})\check{} < \dot{f}(\check{m})]\!] = [\![h(m)\check{} < \dot{f}(\check{m})]\!] = \sup\{a_{mi} : i > h(m)\} \subseteq 1 \setminus d_n$$

whenever $n \leq m$. So

$$\begin{aligned}[\![\check{h}(n) < \dot{f}(n) \text{ for infinitely many } n]\!] &= \inf_{n\in\mathbb{N}} \sup_{m\geq n} [\![h(m)\check{} < \dot{f}(\check{m})]\!] \\ &\subseteq \inf_{n\in\mathbb{N}} 1 \setminus d_n = 0,\end{aligned}$$

that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \leq^* \check{h}. \textbf{ Q}$$

(b)(i) Let $\langle \dot{f}_\xi\rangle_{\xi<\lambda}$ be a family of $\mathbb{P}_\kappa$-names for members of $\mathbb{N}^\mathbb{N}$, where $\lambda < \mathfrak{b}$. Then for each $\xi < \lambda$ we can find an $h_\xi \in \mathbb{N}^\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \leq^* \check{h}_\xi$. As $\lambda < \mathfrak{b}$, there is an $h \in \mathbb{N}^\mathbb{N}$ such that $h_\xi \leq^* h$ for every $\xi < \lambda$. Now $\Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \leq^* \check{h}$ for every ξ, so $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \leq^* \check{h}$ for every ξ. As λ and $\langle \dot{f}_\xi\rangle_{\xi<\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} \geq \check{\mathfrak{b}}.$$

(ii) Let $\langle h_\xi\rangle_{\xi<\mathfrak{b}}$ be a family in $\mathbb{N}^\mathbb{N}$ which has no $\leq^*$-upper bound in $\mathbb{N}^\mathbb{N}$. Then

$$\Vdash_{\mathbb{P}_\kappa} \{\check{h}_\xi : \xi < \check{\mathfrak{b}}\} \text{ has no } \leq^*\text{-upper bound.}$$

P? Otherwise, there are a $\mathbb{P}_\kappa$-name $\dot{f}$ for a member of $\mathbb{N}^\mathbb{N}$ and an $a \in \mathfrak{B}_\kappa^+$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \le^* \dot{f} \text{ for every } \xi < \check{\mathfrak{b}}.$$

Now there is an $h \in \mathbb{N}^{\mathbb{N}}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f} \le^* \check{h}$. There must be a $\xi < \mathfrak{b}$ such that $h_\xi \not\le^* h$. We have $a \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \le^* \dot{f} \le^* \check{h}$, so there are an a', stronger than a, and an $n \in \mathbb{N}$ such that

$$a' \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi(i) \le \check{h}(i) \text{ for every } i \ge \check{n}.$$

However, there is an $i \ge n$ such that $h(i) < h_\xi(i)$, in which case

$$\Vdash_{\mathbb{P}_\kappa} \check{\imath} \ge \check{n} \text{ and } \check{h}(\check{\imath}) < \check{h}_\xi(\check{\imath});$$

which is impossible. **XQ**

So

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} \le \check{\mathfrak{b}}.$$

(c)(i) Let $\langle \dot{f}_\xi \rangle_{\xi<\lambda}$ be a family of $\mathbb{P}_\kappa$-names for members of $\mathbb{N}^{\mathbb{N}}$ where $\lambda < \check{\mathfrak{d}}$. Then for each $\xi < \lambda$ we can find an $h_\xi \in \mathbb{N}^{\mathbb{N}}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \le^* \check{h}_\xi$. As $\lambda < \mathfrak{d}$, there is an $h \in \mathbb{N}^{\mathbb{N}}$ such that $h \not\le^* h_\xi$ for every $\xi < \lambda$. Now

$$\Vdash_{\mathbb{P}_\kappa} \check{h} \not\le^* \check{h}_\xi \text{ for every } \xi < \check{\lambda},$$

so

$$\Vdash_{\mathbb{P}_\kappa} \check{h} \not\le^* \dot{f}_\xi \text{ for every } \xi < \check{\lambda}.$$

As λ and $\langle \dot{f}_\xi \rangle_{\xi<\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{d} \ge \check{\mathfrak{d}}.$$

(ii) Let $\langle h_\xi \rangle_{\xi<\mathfrak{d}}$ be a family in $\mathbb{N}^{\mathbb{N}}$ which is $\le^*$-cofinal with $\mathbb{N}^{\mathbb{N}}$. Then

$$\Vdash_{\mathbb{P}_\kappa} \{\check{h}_\xi : \xi < \check{\mathfrak{d}}\} \text{ is } \le^*\text{-cofinal with } \mathbb{N}^{\mathbb{N}}.$$

P Let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for a member of $\mathbb{N}^{\mathbb{N}}$. There are an $h \in \mathbb{N}^{\mathbb{N}}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f} \le^* \check{h}$, and a $\xi < \mathfrak{d}$ such that $h \le^* h_\xi$. In this case,

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \le^* \check{h} \le^* \check{h}_\xi. \text{ } \mathbf{Q}$$

So

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{d} \le \check{\mathfrak{d}}.$$

552D Lemma Let λ and κ be infinite cardinals, and A any subset of $\{0,1\}^\lambda$. Then

$$\Vdash_{\mathbb{P}_\kappa} \nu^*_{\check{\lambda}}(\check{A}) = (\nu^*_\lambda A)\check{}.$$

proof (a) ? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}_\kappa} (\nu^*_\lambda A)\check{} \le \nu^*_{\check{\lambda}}(\check{A}).$$

Then there are an $a \in \mathfrak{B}^+_\kappa$ and a $q \in \mathbb{Q}$ such that $q < \nu^*_\lambda A$ and

$$a \Vdash_{\mathbb{P}_\kappa} \nu^*_{\check{\lambda}}(\check{A}) < \check{q}.$$

Let $\dot{E}$ be a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \check{A} \subseteq \dot{E},\ \dot{E} \in \mathcal{B}\mathfrak{a}_{\check{\lambda}} \text{ and } \nu_{\check{\lambda}}\dot{E} < \check{q}.$$

Of course we can arrange that $1 \setminus a \Vdash_{\mathbb{P}_\kappa} \dot{E} = \emptyset$, so that $\Vdash_{\mathbb{P}_\kappa} \dot{E} \in \mathcal{B}\mathfrak{a}_{\check{\lambda}}$ and there is a $W \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{E} = \vec{W}$ (551Fb). Setting $h(x) = \nu_\lambda W[\{x\}]$ for $x \in \{0,1\}^\kappa$,

$$\Vdash_{\mathbb{P}_\kappa} \vec{h} = \nu_{\check{\lambda}} \vec{W}$$

(551I(iii)), so

$$a \Vdash_{\mathbb{P}_\kappa} \vec{h} = \nu_{\check{\lambda}} \dot{E} < \check{q}$$

and $a \subseteq \{x : h(x) < q\}^\bullet$. Take $F \in \mathrm{T}_\kappa$ such that $F^\bullet = a$; then $h(x) < q$ for almost every $x \in F$ and $(\nu_\kappa \times \nu_\lambda)(W \cap (F \times \{0,1\}^\lambda)) < q\nu_\kappa F$.

For each $y \in A$, let $e_y : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be the constant function with value y. Then $\Vdash_{\mathbb{P}_\kappa} \vec{e}_y = \check{y}$ (551Ce), so

$$a \Vdash_{\mathbb{P}_\kappa} \vec{e}_y \in \check{A} \subseteq \vec{W}$$

and $(x,y) \in W$ for almost every $x \in F$. But if we set

$$H = \{y : (x,y) \in W \text{ for } \nu_\kappa\text{-almost every } x \in F\},$$

$H \in \mathrm{T}_\lambda$, $A \subseteq H$ and

$$\nu_\kappa F \cdot \nu_\lambda H \le (\nu_\kappa \times \nu_\lambda)(W \cap (F \times \{0,1\}^\lambda)) < q\nu_\kappa F.$$

It follows that

$$\nu_\lambda^* A \le \nu_\lambda H < q,$$

contrary to hypothesis. **X** So

$$\Vdash_{\mathbb{P}_\kappa} (\nu_\lambda^* A)\check{} \le \nu_{\check\lambda}^*(\check A).$$

(b) In the other direction, let $E \in \mathcal{B}\mathfrak{a}_\lambda$ be such that $A \subseteq E$ and $\nu_\lambda E = \nu_\lambda^* A$, and consider $W = \{0,1\}^\kappa \times E$. Then

$$\Vdash_{\mathbb{P}_\kappa} \check A \subseteq \vec W \text{ and } \nu_{\check\lambda} \vec W = (\nu_\lambda E)\check{},$$

so

$$\Vdash_{\mathbb{P}_\kappa} \dot\nu_{\check\lambda}^* \check A \le (\nu_\lambda^* A)\check{}.$$

552E Theorem Let κ and λ be infinite cardinals, with $\kappa \ge \max(\omega_1, \lambda)$. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a strongly Sierpiński set for } \nu_{\check\lambda} \text{ of size } \check\kappa.$$

proof (a) As $\kappa \ge \lambda$, $\mathbb{P}_\kappa$ is isomorphic to $\mathbb{P} = \mathbb{P}_{\kappa\times\lambda}$. For each $\xi < \kappa$, let $f_\xi : \{0,1\}^{\kappa\times\lambda} \to \{0,1\}^\lambda$ be given by setting $f_\xi(x)(\eta) = x(\xi,\eta)$ for every $x \in \{0,1\}^{\kappa\times\lambda}$ and $\eta < \lambda$; then, taking $\vec f_\xi$ to be the $\mathbb{P}$-name defined by the process of 551Cb,

$$\Vdash_{\mathbb{P}} \vec f_\xi \in \{0,1\}^{\check\lambda}.$$

If ξ, $\xi' < \kappa$ are distinct, then for any finite set $I \subseteq \lambda$

$$\begin{aligned}[\![\vec f_\xi(\eta) = \vec f_{\xi'}(\eta) \text{ for every } \eta \in \check I]\!] &= \{x : f_\xi(x)(\eta) = f_{\xi'}(x)(\eta) \text{ for every } \eta \in I\}^\bullet \\ &= \{x : x(\xi,\eta) = x(\xi',\eta) \text{ for every } \eta \in I\}^\bullet\end{aligned}$$

has measure $2^{-\#(I)}$, so, because λ is infinite, $\bar\nu[\![\vec f_\xi = \vec f_{\xi'}]\!] = 0$ and

$$\Vdash_{\mathbb{P}} \vec f_\xi \ne \vec f_{\xi'}.$$

So, taking $\dot A$ to be the $\mathbb{P}$-name $\{(\vec f_\xi, \mathbb{1}) : \xi < \kappa\}$, we have

$$\Vdash_{\mathbb{P}} \dot A \subseteq \{0,1\}^{\check\lambda} \text{ has cardinal } \check\kappa \ge \omega_1, \text{ so is uncountable.}$$

(As remarked in 5A3Nb, we do not need to distinguish between ω_1 and $\check\omega_1$ in the last formula.)

(b) Let $r \ge 1$ be an integer and $\dot W$ a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot W \text{ is a subset of } (\{0,1\}^{\check\lambda})^{\check r} \text{ which is negligible for the usual measure.}$$

Then there is a Baire subset W of $\{0,1\}^{\kappa\times\lambda} \times (\{0,1\}^\lambda)^r$, negligible for the usual measure on this space, such that

$$\Vdash_{\mathbb{P}} \dot W \subseteq \vec W$$

(551J, applied to $\{0,1\}^{\lambda\times r} \cong (\{0,1\}^\lambda)^r$). Let $J \subseteq \kappa$ be a countable set such that W factors through $\{0,1\}^{J\times\lambda} \times (\{0,1\}^\lambda)^r$, that is, there is a negligible Baire set $W_1 \subseteq \{0,1\}^{J\times\lambda} \times (\{0,1\}^\lambda)^r$ such that $W = \{(x,y) : (x\restriction J\times\lambda, y) \in W_1\}$. If $\xi_0, \dots, \xi_{r-1}$ are distinct elements of $\kappa \setminus J$, then

$$\Vdash_{\mathbb{P}} (\vec f_{\xi_0}, \dots, \vec f_{\xi_{r-1}}) \notin \vec W.$$

P Applying 551Ea to the function $x \mapsto (f_{\xi_0}(x), \dots, f_{\xi_{r-1}}(x))$, we have

$$[\![(\vec f_{\xi_0}, \dots, \vec f_{\xi_{r-1}}) \in \vec W]\!] = \{x : (x, (f_{\xi_0}(x), \dots, f_{\xi_{r-1}}(x))) \in W\}^\bullet.$$

Set $K = J \cup \{\xi_0, \dots, \xi_{r-1}\}$ and for $w \in \{0,1\}^{K\times\lambda}$, $i < r$, $\eta < \lambda$ set $g_i(w)(\eta) = w(\xi_i, \eta)$. Then $w \mapsto (w\restriction J \times \lambda, (g_0(w), \dots, g_{r-1}(w))$ is a measure space isomorphism between $\{0,1\}^{K\times\lambda}$ and $\{0,1\}^{J\times\lambda} \times (\{0,1\}^\lambda)^r$, so

$$W_2 = \{w : w \in \{0,1\}^{K\times\lambda}, (w\restriction J \times \lambda, (g_0(w), \dots, g_{r-1}(w)) \in W_1\}$$

is negligible. Consequently

$$\begin{aligned}\{x : (x, (f_{\xi_0}(x), \dots, f_{\xi_{r-1}}(x))) \in W\} &= \{x : (x\restriction J \times \lambda, (f_{\xi_0}(x), \dots, f_{\xi_{r-1}}(x))) \in W_1\} \\ &= \{x : x\restriction K \times \lambda \in W_2\}\end{aligned}$$

is negligible and $[\![(\vec{f}_{\xi_0},\dots,\vec{f}_{\xi_{r-1}})\in\vec{W}]\!]=0$, that is,

$$\Vdash_{\mathbb{P}}(\vec{f}_{\xi_0},\dots,\vec{f}_{\xi_{r-1}})\notin\vec{W}.\ \mathbf{Q}$$

Now if we set $\dot{B}=\{(\vec{f}_\xi,\mathbb{1}):\xi\in J\}$, we have

$\Vdash_{\mathbb{P}}\dot{B}$ is a countable subset of $\dot{A}$ and $(x_0,\dots,x_{r-1})\notin\dot{W}$ whenever $x_0,\dots,x_{r-1}$ are distinct members of $\dot{A}\setminus\dot{B}$.

As $\dot{W}$ is arbitrary,

$$\Vdash_{\mathbb{P}}\dot{A}\text{ is a strongly Sierpiński set of size }\check{\kappa}.$$

As $\mathbb{P}$ is isomorphic to $\mathbb{P}_\kappa$,

$$\Vdash_{\mathbb{P}_\kappa}\text{ there is a strongly Sierpiński set for }\nu_{\check{\lambda}}\text{ of size }\check{\kappa}.$$

552F Theorem Let κ and λ be infinite cardinals.

(a) If either κ or λ is uncountable,

$$\Vdash_{\mathbb{P}_\kappa}\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})=\omega_1.$$

(b) $\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)=(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$.

proof (a)(i) If λ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa}\check{\lambda}\text{ is uncountable, so }\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})=\omega_1$$

(5A3Nb, 521Hb/523E).

(ii) If κ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa}\text{ there is a Sierpiński set for }\nu_\omega,\text{ so }\omega_1\le\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})\le\operatorname{add}\mathcal{N}(\nu_\omega)=\omega_1$$

(552E, 523B, 537Ba).

(b)(i) Let $\langle H_\xi\rangle_{\xi<\operatorname{add}\mathcal{N}(\nu_\omega)}$ be a family of negligible Borel sets in $\{0,1\}^\omega$ such that $A=\bigcup_{\xi<\operatorname{add}\mathcal{N}(\nu_\omega)}H_\xi$ is not negligible. Then 552D tells us that

$\Vdash_{\mathbb{P}_\omega}\check{H}_\xi$ is negligible for every $\xi<(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$, but $\check{A}=\bigcup_{\xi<(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}}\check{H}_\xi$ is not, so $\operatorname{add}\mathcal{N}(\nu_\omega)\le(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$.

(ii) ? If

$$\neg\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)\ge(\operatorname{add}\mathcal{N}(\nu_\omega))\check{},$$

then there are an $a\in\mathfrak{B}_\kappa^+$ and a $\theta<\operatorname{add}\mathcal{N}(\nu_\omega)$ such that

$$a\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)=\check{\theta}.$$

Now there is a family $\langle\dot{W}_\xi\rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a\Vdash_{\mathbb{P}_\omega}\dot{W}_\xi\in\mathcal{N}(\nu_\omega)\ \forall\ \xi<\check{\theta},\ \bigcup_{\xi<\check{\theta}}\dot{W}_\xi\notin\mathcal{N}(\nu_\omega).$$

By 551J, there is for each $\xi<\theta$ a $W_\xi\in\mathrm{T}_\omega\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that

$$a\Vdash_{\mathbb{P}_\omega}\dot{W}_\xi\subseteq\vec{W}_\xi$$

and all the vertical sections of every W_ξ are negligible. But this means that W_ξ is negligible for the product measure $\nu_\omega\times\nu_\omega$. Because

$$\theta<\operatorname{add}\mathcal{N}(\nu_\omega)=\operatorname{add}\mathcal{N}(\nu_\omega\times\nu_\omega),$$

$\bigcup_{\xi<\theta}W_\xi$ also is negligible, and there is a negligible $W\in\mathrm{T}_\omega\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ including every W_ξ. In this case, 551I(iii) tells us that

$$\Vdash_{\mathbb{P}_\omega}\nu_\omega\vec{W}=0,$$

so

$$a\Vdash_{\mathbb{P}_\omega}\bigcup_{\xi<\check{\theta}}\dot{W}_\xi\subseteq\bigcup_{\xi<\check{\theta}}\vec{W}_\xi\subseteq\vec{W}\text{ is negligible. }\mathbf{X}$$

Putting this together with (i),

$$\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)=(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}.$$

552G Theorem Let κ and λ be infinite cardinals.
(a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \ge \max(\kappa, \operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}$.
(b) (Pawlikowski 86) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) \ge \mathfrak{b}$.
(c) (Miller 82) If $\kappa \ge \mathfrak{c}$ then $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) = \mathfrak{c}$.[2]
(d) (Miller 82) Suppose that κ and λ are uncountable. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \le (\sup_{\delta<\kappa}\delta^\omega)\check{},$$

where each δ^ω is the cardinal power.

proof (a)(i) If $\kappa = \omega$ then of course $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \ge \check\kappa$. If κ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\kappa}) \ge \check\kappa$$

by 552E and 537Ba, so by 523F we have

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \ge \check\kappa.$$

(ii) ? If

$$\neg \Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \ge (\operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}$$

then we have an $a \in \mathfrak{B}_\kappa^+$, a cardinal $\theta < \operatorname{cov}\mathcal{N}(\nu_\lambda)$ and a family $\langle \dot W_\xi\rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot W_\xi : \xi < \check\theta\} \text{ is a cover of } \{0,1\}^{\check\lambda} \text{ by negligible sets.}$$

By 551J again, we have for each $\xi < \theta$ a $(\nu_\kappa \times \nu_\lambda)$-negligible set W_ξ such that $a \Vdash_{\mathbb{P}_\kappa} \dot W_\xi \subseteq \vec W_\xi$. Set

$$V_\xi = \{y : y \in \{0,1\}^\lambda,\ W_\xi^{-1}[\{y\}] \text{ is not } \nu_\kappa\text{-negligible}\};$$

then $\nu_\lambda V_\xi = 0$ for every $\xi < \theta$, so there is a $y \in \{0,1\}^\lambda \setminus \bigcup_{\xi<\theta} V_\xi$. In this case, let $e_y : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be the constant function with value y. Then we have a $\mathbb{P}_\kappa$-name $\vec e_y$ for a member of $\{0,1\}^\lambda$, and for each $\xi < \theta$

$$[\![\vec e_y \in \vec W_\xi]\!] = \{x : (x, e_y(x)) \in W_\xi\}^\bullet = W_\xi^{-1}[\{y\}]^\bullet = 0$$

(551Ea). So

$$\Vdash_{\mathbb{P}_\kappa} \vec e_y \notin \vec W_\xi \text{ for every } \xi < \check\theta,$$

and

$$a \Vdash_{\mathbb{P}_\kappa} \vec e_y \in \{0,1\}^{\check\lambda} \setminus \bigcup_{\xi<\check\theta} \dot W_\xi. \quad \mathbf{X}$$

So

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) \ge (\operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}.$$

(b)(i) Set $S_2^* = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and let $\langle(\sigma_n, \tau_n, k_n)\rangle_{n\in\mathbb{N}}$ enumerate $S_2^* \times S_2^* \times \mathbb{N}$ with cofinal repetitions. Let D be the set of those $\alpha \in \mathbb{N}^\mathbb{N}$ such that

$\langle k_{\alpha(n)}\rangle_{n\in\mathbb{N}}$ is strictly increasing,
$\#(\sigma_{\alpha(m)}) \le \alpha(n)$ whenever $n \in \mathbb{N}$ and $m < k_{\alpha(n+1)}$,
$\sum_{i=k_{\alpha(n)}}^{k_{\alpha(n+1)}-1} 2^{-\#(\sigma_{\alpha(i)})-\#(\tau_{\alpha(i)})} \le 4^{-n}$ for every $n \in \mathbb{N}$.

For $\alpha \in D$ set

$$G_\alpha = \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{(u,y) : u,\ y \in \{0,1\}^\omega,\ u \supseteq \sigma_{\alpha(m)},\ y \supseteq \tau_{\alpha(m)}\}.$$

(ii) For every $\alpha \in D$, G_α is negligible for the product measure on $\{0,1\}^\omega \times \{0,1\}^\omega$. **P** For any $n \in \mathbb{N}$, the measure of G_α is at most

$$\sum_{m=k_{\alpha(n)}}^\infty 2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})} \le \sum_{j=n}^\infty \sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1} 2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})}$$

$$\le \sum_{j=n}^\infty 4^{-j} = \frac{4}{3}\cdot 4^{-n}. \quad \mathbf{Q}$$

(iii) If $G \subseteq \{0,1\}^\omega \times \{0,1\}^\omega$ is negligible, there is an $\alpha \in D$ such that $G \subseteq G_\alpha$. **P** For each $i \in \mathbb{N}$, let $H_i \supseteq G$ be an open set such that $(\nu_\omega \times \nu_\omega)(H_i) \le 2^{-i}$; we can suppose that H_i is not open-and-closed. H_i can be expressed as the

[2] Remember that the final $\mathfrak{c}$ here is to be interpreted in the forcing language.

union of a sequence of open-and-closed sets; it can therefore be expressed as the union of a disjoint sequence of open-and-closed sets; each of these is expressible as the union of a disjoint family of sets of the form $\{u:\sigma\subseteq x\}\times\{y:\tau\subseteq y\}$ where σ, $\tau\in S_2^*$; so H_i is expressible as $\bigcup_{j\in\mathbb{N}}\{u:\sigma'_{ij}\subseteq u\}\times\{y:\tau'_{ij}\subseteq y\}$, with

$$\sum_{j\in\mathbb{N}}2^{-\#(\sigma'_{ij})-\#(\tau'_{ij})}\le 2^{-i}.$$

Re-indexing $\langle(\sigma'_{ij},\tau'_{ij})\rangle_{i,j\in\mathbb{N}}$ as $\langle(\sigma''_m,\tau''_m)\rangle_{m\in\mathbb{N}}$, we have

$$G\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{(u,y):\sigma''_m\subseteq u,\,\tau''_m\subseteq y\},$$

and

$$\sum_{m\in\mathbb{N}}2^{-\#(\sigma''_m)-\#(\tau''_m)}\le\sum_{i=0}^\infty 2^{-i}<\infty.$$

Let $\gamma:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that

$$\sum_{m=\gamma(n)}^{\gamma(n+1)-1}2^{-\#(\sigma''_m)-\#(\tau''_m)}\le 4^{-n}$$

for every $n\in\mathbb{N}$. Now choose $\langle\alpha(n)\rangle_{n\in\mathbb{N}}$ so that

$$k_{\alpha(n)}=\gamma(n),\quad \sigma_{\alpha(n)}=\sigma''_n,\quad \tau_{\alpha(n)}=\tau''_n,\quad \alpha(n)\ge\#(\sigma''_m)\text{ whenever }m<\gamma(n+1)$$

for each $n\in\mathbb{N}$. Then $\alpha\in D$ and $G\subseteq G_\alpha$. **Q**

(iv) Define $h:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ by setting $h(\beta)(n)=n+\sum_{i=0}^n\beta(i)$ for $\beta\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$. For $\beta\in\mathbb{N}^{\mathbb{N}}$ define $f_\beta:\{0,1\}^\omega\to\{0,1\}^\omega$ by setting $f_\beta(u)(n)=u(h(\beta)(n))$ for $\beta\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$; note that f_β is continuous. For α, $\beta\in\mathbb{N}^{\mathbb{N}}$ say that $\alpha\le^*\beta$ if $\{n:\beta(n)<\alpha(n)\}$ is finite.

(v) If $\alpha\in D$, $\beta\in\mathbb{N}^{\mathbb{N}}$ and $\alpha\le^*\beta$, then $C=\{u:u\in\{0,1\}^\omega,\,(u,f_\beta(u))\in G_\alpha\}$ is ν_ω-negligible. **P** Let n_0 be such that $\alpha(n)\le\beta(n)$ for $n\ge n_0$. C is just $\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}C_m$ where

$$C_m=\{u:\sigma_{\alpha(m)}\subseteq u,\,\tau_{\alpha(m)}\subseteq f_\beta(u)\}$$

for each m. We know that $\langle k_{\alpha(j)}\rangle_{j\in\mathbb{N}}$ is strictly increasing; if $m\ge k_{\alpha(n_0)}$, let $j\ge n_0$ be such that $k_{\alpha(j)}\le m<k_{\alpha(j+1)}$, and set

$$\begin{aligned}C'_m&=\{u:\sigma_{\alpha(m)}\subseteq u,\,\tau_{\alpha(m)}(i)=f_\beta(u)(i)\text{ for }j\le i<\#(\tau_{\alpha(m)})\}\\&=\{u:u(i)=\sigma_{\alpha(m)}(i)\text{ for }i<\#(\sigma_{\alpha(m)}),\\&\qquad\qquad u(h(\beta)(i))=\tau_{\alpha(m)}(i)\text{ for }j\le i<\#(\tau_{\alpha(m)})\}\\&\supseteq C_m.\end{aligned}$$

We know that

$$\#(\sigma_{\alpha(m)})\le\alpha(j)\le\beta(j)\le h(\beta)(i)$$

whenever $i\ge j$ and that $h(\beta)$ is a strictly increasing function, so

$$\nu_\omega C'_m=2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})+j}.$$

But this means that

$$\begin{aligned}\sum_{m=k_{\alpha(n_0)}}^\infty\nu_\omega C_m&=\sum_{j=n_0}^\infty\sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1}\nu_\omega C_m\\&\le\sum_{j=n_0}^\infty 2^j\sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1}2^{-\#(\sigma_{\alpha(m)})-\#(\sigma_{\tau(m)})}\le\sum_{j=n_0}^\infty 2^j\cdot 4^{-j}\end{aligned}$$

is finite, and C is negligible. **Q**

(vi) Let Φ be the set of all continuous functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$, and $\mathcal{E}$ the set of $(\nu_\kappa\times\nu_\omega)$-negligible sets in $\mathrm{T}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}(\{0,1\}^\omega)$; let R be the relation

$$\{(W,g):W\in\mathcal{E},\,g\in\Phi,\,\{x:(x,g(x))\in W\}\in\mathcal{N}(\nu_\kappa)\}.$$

Then $(\mathcal{E},R,\Phi)\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\le^*,\mathbb{N}^{\mathbb{N}})$. **P** For $W\in\mathcal{E}$ set

$$\begin{aligned}V_W=\{(u,y):\,&u,\,y\in\{0,1\}^\omega,\\&\{v:v\in\{0,1\}^{\kappa\setminus\omega},\,(u\cup v,y)\in W\}\text{ is not }\nu_{\kappa\setminus\omega}\text{-negligible}\}.\end{aligned}$$

Then V_W is $(\nu_\omega \times \nu_\omega)$-negligible; by (iii), we can find $\phi(W) \in D$ such that $V_W \subseteq G_{\phi(W)}$. In the other direction, given $\beta \in \mathbb{N}^{\mathbb{N}}$, define $\psi(\beta) \in \Phi$ by saying that $\psi(\beta)(x) = f_\beta(x\restriction\omega)$ for $x \in \{0,1\}^\kappa$.

If $W \in \mathcal{E}$ and $\beta \in \mathbb{N}^{\mathbb{N}}$ are such that $\phi(W) \le^* \beta$, we have $\nu_\omega C = 0$ where $C = \{u : (u, f_\beta(u)) \in G_{\phi(W)}\}$, by (ϵ). But if $C' = \{x : (x, \psi(\beta)(x)) \in W\}$, and $u \in \{0,1\}^\omega \setminus C$, then

$$\{v : v \in \{0,1\}^{\kappa\setminus\omega},\, u \cup v \in C'\} = \{v : v \in \{0,1\}^{\kappa\setminus\omega},\, (u \cup v, f_\beta(u)) \in W\}$$

must be $\nu_{\kappa\setminus\omega}$-negligible, since $(u, f_\beta(u)) \notin V_W$. So C' is negligible and $(W, \psi(\beta)) \in R$. As W and β are arbitrary, (ϕ, ψ) is a Galois-Tukey connection and $(\mathcal{E}, R, \Phi) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$. **Q**

Consequently $\operatorname{add}(\mathcal{E}, R, \Phi) \ge \mathfrak{b}$ (522C(i), 512Ea, 512Db).

(vii) By 552C, we do not need to distinguish between the interpretations of $\mathfrak{b}$ in the ordinary universe and in the forcing language. Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{A}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \mathcal{N}(\nu_\omega) \text{ and } \#(\dot{A}) < \mathfrak{b}.$$

Then there are a $b \in \mathfrak{B}_\kappa^+$, stronger than a, a cardinal $\theta < \mathfrak{b}$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{A} = \{\dot{W}_\xi : \xi < \check{\theta}\}.$$

For each $\xi < \theta$, we have a $(\nu_\kappa \times \nu_\omega)$-negligible $W_\xi \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}(\{0,1\}^\omega)$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$ (551J, as usual). Each W_ξ belongs to $\mathcal{E}$. Since $\theta < \mathfrak{b} \le \operatorname{add}(\mathcal{E}, R, \Phi)$, there is a $g \in \Phi$ such that $(W_\xi, g) \in R$ for every $\xi < \theta$, that is, $\{x : (x, g(x)) \in W_\xi\}$ is negligible for every $\xi < \theta$. But this means that

$$\Vdash_{\mathbb{P}_\kappa} \vec{g} \in \{0,1\}^\omega \setminus \vec{W}_\xi$$

for every $\xi < \theta$. So

$$b \Vdash_{\mathbb{P}_\kappa} \vec{g} \notin \bigcup \dot{A} \text{ and } \dot{A} \text{ does not cover } \{0,1\}^\omega.$$

As a and $\dot{A}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov} \mathcal{N}(\nu_\omega) \ge \mathfrak{b}.$$

(c) Write θ for the cardinal power κ^ω, so that $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \check{\theta}$ (552B). **?** If

$$\neg \Vdash_{\mathbb{P}_\kappa} \operatorname{cov} \mathcal{N}(\nu_\omega) = \mathfrak{c},$$

then there must be an $a \in \mathfrak{B}_\kappa^+$, a cardinal $\delta < \theta$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\delta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot{W}_\xi : \xi < \check{\delta}\} \text{ is a cover of } \{0,1\}^\omega \text{ by negligible sets.}$$

For each $\xi < \delta$, let $W_\xi \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}(\{0,1\}^\omega)$ be a $(\nu_\kappa \times \nu_\omega)$-negligible set such that $a \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$; expanding it if necessary, we can suppose that W_ξ is a Baire set. Let $I_\xi \subseteq \kappa$ be a countable set such that $(u, y) \in W_\xi$ whenever $(x, y) \in W_\xi$, $u \in \{0,1\}^\kappa$ and $u\restriction I_\xi = x\restriction I_\xi$. Set

$$W'_\xi = \{(v, y) : (u, y) \in W_\xi,\, v \in \{0,1\}^\kappa,\, \{\eta : \eta < \kappa,\, u(\eta) \ne v(\eta)\} \in [I_\xi]^{<\omega}\}.$$

Then W'_ξ is still $(\nu_\kappa \times \nu_\omega)$-negligible.

Because $\kappa \ge \mathfrak{c}$ and $\delta < \kappa^\omega$, there is a countably infinite $K \subseteq \kappa$ such that $K \cap I_\xi$ is finite for every $\xi < \delta$ (5A1Fc). Enumerate K as $\langle \eta_n \rangle_{n\in\mathbb{N}}$ and define $f : \{0,1\}^\kappa \to \{0,1\}^\omega$ by setting $f(u) = \langle u(\eta_n) \rangle_{n\in\mathbb{N}}$ for $u \in \{0,1\}^\kappa$.

For each $\xi < \kappa$, $\{u : (u, f(u)) \in W_\xi\}$ is ν_κ-negligible. **P** Set $J = \kappa \setminus K$, so that $\{0,1\}^\kappa$ can be identified with $\{0,1\}^J \times \{0,1\}^K$. We can express W'_ξ as $\{(u, y) : (u\restriction J, y) \in V\}$ where $V \subseteq \{0,1\}^J \times \{0,1\}^\omega$ is negligible. Now the map $u \mapsto (u\restriction J, f(u)) : \{0,1\}^\kappa \to \{0,1\}^J \times \{0,1\}^\omega$ is just a copy of the map $u \mapsto (u\restriction J, u\restriction K)$, so is a measure space isomorphism between $\{0,1\}^\kappa$ and $\{0,1\}^J \times \{0,1\}^\omega$, and $V' = \{u : u \in \{0,1\}^\kappa,\, (u\restriction J, f(u)) \in V\}$ is negligible. But observe now that

$$\{u : (u, f(u)) \in W_\xi\} \subseteq \{u : (u, f(u)) \in W'_\xi\} = \{u : (u\restriction J, f(u)) \in V\} = V'$$

is negligible. **Q**

Turn now to 551E. In the language there, we have $[\![\vec{f} \in \vec{W}_\xi]\!] = 0$, that is, $\Vdash_{\mathbb{P}_\kappa} \vec{f} \notin \vec{W}_\xi$ and $a \Vdash_{\mathbb{P}_\kappa} \vec{f} \notin \dot{W}_\xi$. So

$$a \Vdash_{\mathbb{P}_\kappa} \textstyle\bigcup_{\xi<\check{\delta}} \dot{W}_\xi \ne \{0,1\}^{\check{\lambda}},$$

which is impossible. **X**

So we have the result claimed.

(d)(i) If $\operatorname{cf}\kappa > \omega$ then $\sup_{\delta<\kappa} \delta^\omega = \kappa^\omega$; but this means that $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = (\sup_{\delta<\kappa} \delta^\omega)\check{}$ and the result is trivial. So henceforth suppose that $\operatorname{cf}\kappa = \omega$. By 523B, we may also assume that $\lambda = \omega_1$.

(ii) Let D be the set of all pairs $(\boldsymbol{\xi},y)$ where $\boldsymbol{\xi}\in\omega_1^{\mathbb{N}}$ is one-to-one and y is a Baire measurable function from $\{0,1\}^\delta$ to $\{0,1\}^\omega$ for some cardinal $\delta<\kappa$. Then $\#(D)=\sup_{\delta<\kappa}\delta^\omega$. For $(\boldsymbol{\xi},y)\in D$, let $W_{\boldsymbol{\xi}y}\subseteq\{0,1\}^\kappa\times\{0,1\}^{\omega_1}$ be the set

$$\{(u,v):\lim_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,v(\xi_i)=y(u\restriction\alpha)(i)\})=\frac{1}{2}\},$$

where $\boldsymbol{\xi}=\langle\xi_i\rangle_{i\in\mathbb{N}}$ and $\operatorname{dom}y=\{0,1\}^\alpha$. Then $W_{\boldsymbol{\xi}y}$ is a Baire set; also the vertical section $W_{\boldsymbol{\xi}y}[\{u\}]$ is ν_{ω_1}-conegligible for almost every $u\in\{0,1\}^\kappa$. **P** The set

$$V=\{x:x\in\{0,1\}^{\mathbb{N}},\,\lim_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,x(i)=y(u\restriction\alpha)(i)\})=\frac{1}{2}\}$$

is conegligible in $\{0,1\}^{\mathbb{N}}$, by the strong law of large numbers (273F). But $W_{\boldsymbol{\xi}y}[\{u\}]$ is the inverse image of V under the inverse-measure-preserving map $v\mapsto\langle v(\xi_i)\rangle_{i\in\mathbb{N}}$, so is ν_{ω_1}-conegligible. **Q**

(iii) Consequently

$$\Vdash_{\mathbb{P}_\kappa}\vec{W}_{\boldsymbol{\xi}y}\text{ is conegligible in }\{0,1\}^{\omega_1}$$

whenever $(\boldsymbol{\xi},y)\in D$ (551I(iii)). Now

$$\Vdash_{\mathbb{P}_\kappa}\bigcap_{(\boldsymbol{\xi},y)\in\check{D}}\vec{W}_{\boldsymbol{\xi}y}\text{ is empty.}$$

P? Otherwise, there are an $a\in\mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot{x}$ such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{x}\in\bigcap_{(\boldsymbol{\xi},y)\in\check{D}}\vec{W}_{\boldsymbol{\xi}y}.$$

Let $f:\{0,1\}^\kappa\to\{0,1\}^{\omega_1}$ be a $(\mathrm{T}_\kappa,\mathcal{B}\mathfrak{a}_{\omega_1})$-measurable function such that $a\Vdash_{\mathbb{P}_\kappa}\vec{f}=\dot{x}$ (551Cc). Set $\epsilon=\frac{1}{4}\bar{\nu}_\kappa a$ and for each $\xi<\omega_1$ let E_ξ be an open-and-closed subset of $\{0,1\}^\kappa$ such that $\bar{\nu}_\kappa(E_\xi\triangle\{u:f(u)(\xi)=1\})\le\epsilon$. Let $\alpha_\xi<\kappa$ be such that E_ξ is determined by coordinates less than α_ξ. Because $\operatorname{cf}\kappa=\omega$, there is a cardinal $\delta<\kappa$ such that $A=\{\xi:\alpha_\xi\le\delta\}$ is infinite; let $\boldsymbol{\xi}=\langle\xi_i\rangle_{i\in\mathbb{N}}$ enumerate a subset of A. For each $i\in\mathbb{N}$ let $F_i\subseteq\{0,1\}^\delta$ be an open-and-closed set such that $E_{\xi_i}=\{u:u\restriction\delta\in F_i\}$. Define $y:\{0,1\}^\delta\to\{0,1\}^{\mathbb{N}}$ by saying that $y(v)(i)=\chi F_i(v)$ for $v\in\{0,1\}^\delta$ and $i\in\mathbb{N}$; then y is Baire measurable, so $W_{\boldsymbol{\xi}y}$ is defined and $a\Vdash_{\mathbb{P}_\kappa}\vec{f}\in\vec{W}_{\boldsymbol{\xi}y}$.

Set $H=\{u:(u,f(u))\in W_{\boldsymbol{\xi}y}\}$. Then

$$a\subseteq[\![\vec{f}\in\vec{W}_{\boldsymbol{\xi}y}]\!]=H^\bullet$$

so $\nu_\kappa H\ge\bar{\nu}_\kappa a=4\epsilon$.

But consider the sets

$$H_i=\{u:f(u)(\xi_i)=y(u\restriction\delta)(i)\}=\{0,1\}^\kappa\setminus(E_{\xi_i}\triangle\{u:f(u)(\xi_i)=1\})$$

for $i\in\mathbb{N}$. These all have measure at least $1-\epsilon$. For $n\ge1$ set

$$\gamma_n=\nu_\kappa\{u:\#(\{i:i<n,\,u\in H_i\})\le\frac{2n}{3}\};$$

then

$$n(1-\epsilon)\le\sum_{i<n}\nu_\kappa H_i=\int\#(\{i:i<n,\,u\in H_i\})\nu_\kappa(du)\le\frac{2n}{3}\gamma_n+n(1-\gamma_n)$$

and $\gamma_n\le3\epsilon$. So

$$\begin{aligned}H&=\{u:\lim_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,u\in H_i\})=\frac{1}{2}\}\\&\subseteq\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\{u:\#(\{i:i<n,\,u\in H_i\})\le\frac{2}{3}\}\end{aligned}$$

has measure at most 3ϵ; which is impossible. **XQ**

(iv) Thus

$$\Vdash_{\mathbb{P}_\kappa}\bigcap_{(\boldsymbol{\xi},y)\in\check{D}}\vec{W}_{\boldsymbol{\xi}y}\text{ is empty and }\{0,1\}^{\omega_1}\text{ can be covered by }(\sup_{\delta<\kappa}\delta^\omega)\check{}\text{ negligible sets,}$$

which is what we needed to know.

552H Theorem Let κ and λ be infinite cardinals.

(a) $\Vdash_{\mathbb{P}_\kappa}\operatorname{non}\mathcal{N}(\nu_{\check{\lambda}})\le(\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}$.

(b) If $\kappa \ge \max(\lambda, \omega_1)$ then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_{\check\lambda}) = \omega_1.$$

(c) (Pawlikowski 86)

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_\omega) \le \mathfrak{d}.$$

proof (a) Let $A \subseteq \{0,1\}^\lambda$ be a non-negligible set of size $\operatorname{non}\mathcal{N}(\nu_\lambda)$. Then 552D tells us that

$$\Vdash_{\mathbb{P}_\kappa} \check A \text{ is a non-negligible set of size } (\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}, \text{ so } \operatorname{non}\mathcal{N}(\nu_{\check\lambda}) \le (\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}.$$

(b) Put 552E and 537Ba together again, as in part (a) of the proof of 552F.

(c) Continue the argument from the end of (b-vi) of the proof of 552G above. We have $(\mathcal{E}, R, \Phi) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$, so $\operatorname{cov}(\mathcal{E}, R, \Phi) \le \mathfrak{d}$ (522C(i), 512Ea, 512Da). So there is a family $\langle g_\xi\rangle_{\xi<\mathfrak{d}}$ in Φ such that for every $W \in \mathcal{E}$ there is a $\xi < \mathfrak{d}$ such that $(W, g_\xi) \in R$, that is, $\{x : (x, g_\xi(x)) \in W\}$ is negligible, that is, $\Vdash_{\mathbb{P}_\kappa} \vec g_\xi \notin \vec W$. Now

$$\Vdash_{\mathbb{P}_\kappa} \{\vec g_\xi : \xi < \mathfrak{d}\} \text{ is not negligible.}$$

P? Otherwise, there are an $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot W$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot W \text{ is a negligible set containing } \vec g_\xi \text{ for every } \xi < \mathfrak{d}.$$

By 551J once again, there is a $W \in \mathcal{E}$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot W \subseteq \vec W$. But now we have a $\xi < \mathfrak{d}$ such that $\Vdash_{\mathbb{P}_\kappa} \vec g_\xi \notin \vec W$, which is impossible. **XQ**

552I Theorem Let κ and λ be infinite cardinals. Set $\theta_0 = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\kappa]^{\le\omega}, \operatorname{cf}[\lambda]^{\le\omega})$. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check\lambda}) = \check\theta_0.$$

proof (a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_\omega) \ge \operatorname{cf}[\check\kappa]^{\le\omega} = (\operatorname{cf}[\kappa]^{\le\omega})\check{}$. **P** If $\kappa = \omega$ this is trivial. Otherwise it follows from 552E, 537B(a-ii) and 5A3Nd. **Q**

(b) Set

$$\theta_1 = \operatorname{cf}\mathcal{N}(\nu_\lambda) = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\lambda]^{\le\omega})$$

(523N). Then $\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check\lambda}) \ge \check\theta_1$. **P?** Otherwise, there are $a \in \mathbb{P}_\kappa$, $\theta < \theta_1$ and a family $\langle \dot W_\xi\rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot W_\xi : \xi < \check\theta\} \text{ is a cofinal family in } \mathcal{N}(\nu_{\check\lambda}).$$

For each ξ choose a $(\nu_\kappa \times \nu_\lambda)$-negligible $W_\xi \in \mathrm{T}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot W_\xi \subseteq \vec W_\xi$. Then

$$V_\xi = \{y : y \in \{0,1\}^\lambda,\ W_\xi^{-1}[\{y\}] \text{ is not } \nu_\kappa\text{-negligible}\}$$

is ν_λ-negligible. Because $\theta < \operatorname{cf}\mathcal{N}(\nu_\lambda)$, there is a $V \in \mathcal{N}(\nu_\lambda)$ such that $V \not\subseteq V_\xi$ for every $\xi < \theta$, and (enlarging V slightly if necessary) we can arrange that $V \in \mathcal{B}\mathfrak{a}_\lambda$.

Set $W = \{0,1\}^\kappa \times V$. Then $W \in \mathrm{T}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$ and every vertical section of W is negligible, so 551G tells us that

$$\Vdash_{\mathbb{P}_\kappa} \vec W \text{ is negligible in } \{0,1\}^{\check\lambda}.$$

Accordingly

$$a \Vdash_{\mathbb{P}_\kappa} \text{there is a } \xi < \check\theta \text{ such that } \vec W \subseteq \dot W_\xi \subseteq \vec W_\xi,$$

and there must be a $b \in \mathbb{P}_\kappa$, stronger than a, and a $\xi < \theta$ such that $b \Vdash_{\mathbb{P}_\kappa} \vec W \subseteq \vec W_\xi$. But now take any point y of $V \setminus V_\xi$ and consider the constant function e_y on $\{0,1\}^\kappa$ with value y. Then $\{x : (x, e_y(x)) \in W \setminus W_\xi\}$ is conegligible, so 551E tells us that

$$\Vdash_{\mathbb{P}_\kappa} \vec e_y \in \vec W \setminus \vec W_\xi, \text{ so } \vec W \not\subseteq \vec W_\xi,$$

contrary to the choice of ξ. **XQ**

(c) Now

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check\lambda}) \ge \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \check\theta_1) \ge \max(\operatorname{cf}[\check\kappa]^{\le\omega}, \check\theta_1) = \check\theta_0.$$

(d) In the other direction, let $\mu = \nu_\kappa \times \nu_\lambda$ be the product measure on $\{0,1\}^\kappa \times \{0,1\}^\lambda$. Again by 523N, $\operatorname{cf}\mathcal{N}(\mu) = \theta_0$; let $\langle W_\xi\rangle_{\xi<\theta_0}$ be a cofinal family in $\mathcal{N}(\mu)$ consisting of sets in $\mathrm{T}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$. By 551J,

$$\Vdash_{\mathbb{P}_\kappa} \{\vec W_\xi : \xi < \check\theta_1\} \text{ is cofinal with } \mathcal{N}(\nu_{\check\lambda}), \text{ so } \operatorname{cf}\mathcal{N}(\nu_{\check\lambda}) \le \check\theta_0.$$

Putting this together with (c),

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check\lambda}) = \check\theta_0,$$

and the proof is complete.

552J Theorem Let κ and λ be infinite cardinals; set $\theta_0 = \operatorname{shr}\mathcal{N}(\nu_\lambda)$ and let θ_1 be the cardinal power λ^ω. Then

$$\Vdash_{\mathbb{P}_\kappa} \check\theta_0 \le \operatorname{shr}\mathcal{N}(\nu_{\check\lambda}) \le \check\theta_1.$$

proof (a) ? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}_\kappa} \check\theta_0 \le \operatorname{shr}\mathcal{N}(\nu_{\check\lambda}).$$

Then there are an $a \in \mathfrak{B}_\kappa^+$ and a cardinal $\theta' < \theta_0$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \operatorname{shr}\mathcal{N}(\nu_{\check\lambda}) = \check\theta'.$$

Of course θ' is infinite. Let $A \subseteq \{0,1\}^\lambda$ be such that $\nu_\lambda^* A > 0$ but $B \in \mathcal{N}(\nu_\lambda)$ for every $B \in [A]^{\le\theta'}$. By 552D,

$$\Vdash_{\mathbb{P}_\kappa} \nu_{\check\lambda}^*(\check A) > 0.$$

There must therefore be a $\mathbb{P}_\kappa$-name $\dot B$ for a (non-empty) subset of $\check A$ of size at most $\check\theta'$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \nu_{\check\lambda}^*(\dot B) > 0.$$

By 5A3Nc there is a $B \subseteq A$ such that $\#(B) \le \max(\omega,\theta') = \theta'$ and

$$a \Vdash_{\mathbb{P}_\kappa} \dot B \subseteq \check B.$$

By 552D, in the other direction,

$$a \Vdash_{\mathbb{P}_\kappa} 0 < \nu_{\check\lambda}^*(\dot B) \le \nu_{\check\lambda}^*(\check B) = (\nu_\lambda^* B)\check{}$$

and $\nu_\lambda^* B > 0$, contrary to the choice of A. **X**

(b) (In this part of the proof it will be convenient to regard $\mathfrak{B}_\kappa$ as the measure algebra of $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$.)

(i) ? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}_\kappa} \operatorname{shr}\mathcal{N}(\nu_{\check\lambda}) \le \check\theta_1.$$

Then there are an $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot A$ such that

$a \Vdash_{\mathbb{P}_\kappa} \dot A$ is a non-negligible subset of $\{0,1\}^{\check\lambda}$ and every subset of $\dot A$ of cardinal at most $\check\theta_1$ is negligible.

(ii) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. Choose $\langle f_\xi\rangle_{\xi<\theta_1^+}$, $\langle J_\xi\rangle_{\xi<\theta_1^+}$, $\langle K_\xi\rangle_{\xi<\theta_1^+}$, $\langle W_\xi\rangle_{\xi<\theta_1^+}$ and $\langle V_\xi\rangle_{\xi<\theta_1^+}$ inductively, as follows. $K_\xi = \bigcup_{\eta<\xi} J_\eta$. Given that $\xi < \theta_1^+$ and that, for each $\eta < \xi$, $f_\eta : \{0,1\}^\kappa \to \{0,1\}^\lambda$ is a $(\mathcal{B}\mathfrak{a}_\kappa, \mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that $a \Vdash_{\mathbb{P}_\kappa} \vec f_\eta \in \dot A$ (where $\vec f_\eta$ is the $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\check\lambda}$ as defined in 551Cb), then

$$a \Vdash_{\mathbb{P}_\kappa} \{\vec f_\eta : \eta < \check\xi\} \text{ is negligible,}$$

so by 551J there is a set $W_\xi \in \mathcal{B}\mathfrak{a}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$, negligible for the product measure on $\{0,1\}^\kappa\times\{0,1\}^\lambda$, such that $a \Vdash_{\mathbb{P}_\kappa} \vec f_\eta \in \vec W_\xi$ for every $\eta < \xi$.

Set

$$V_\xi = \{(x,y) : x \in \{0,1\}^\kappa,\ y \in \{0,1\}^\lambda,$$
$$\{t : t \in \{0,1\}^{\kappa\setminus K_\xi},\ ((x\restriction K_\xi)\cup t, y) \in W_\xi\} \text{ is not } \nu_{\kappa\setminus K_\xi}\text{-negligible}\}.$$

Then $V_\xi \in \mathcal{B}\mathfrak{a}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$ is negligible, so $\Vdash_{\mathbb{P}_\kappa} \vec V_\xi \in \mathcal{N}(\nu_{\check\lambda})$ and $a \Vdash_{\mathbb{P}_\kappa} \dot A \not\subseteq \vec V_\xi$; let $f_\xi : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be a $(\mathcal{B}\mathfrak{a}_\kappa, \mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that $a \Vdash_{\mathbb{P}_\kappa} \vec f_\xi \in \dot A \setminus \vec V_\xi$ (551Cc). Let $J_\xi \subseteq \kappa$ be a set of cardinal at most λ such that $\{x : f_\xi(x)(\zeta) = 1\}$ is determined by coordinates in J_ξ for every $\zeta < \lambda$, and continue.

(iii) If $\eta < \xi < \theta_1^+$ then $a \Vdash_{\mathbb{P}_\kappa} \vec f_\eta \in \vec V_\xi$. **P** As $J_\eta \subseteq K_\xi$, we have a function $g : \{0,1\}^{K_\xi} \to \{0,1\}^\lambda$ such that $f_\eta(x) = g(x\restriction K_\xi)$ for every $x \in \{0,1\}^\lambda$. Now, for any $s \in \{0,1\}^{K_\xi}$ and $y \in \{0,1\}^\lambda$, the set

$$\{t : t \in \{0,1\}^{\kappa\setminus K_\xi},\ (s\cup t, y) \in W_\xi \setminus V_\xi\}$$

is $\nu_{\kappa\setminus K_\xi}$-negligible, so

$$E = \{(s,t) : (s\cup t, g(s)) \in W_\xi \setminus V_\xi\}$$

is $(\nu_{K_\xi}\times\nu_{\kappa\setminus K_\xi})$-negligible. ($E\in\mathcal{B}\mathfrak{a}_{K_\xi}\widehat{\otimes}\mathcal{B}\mathfrak{a}_{\kappa\setminus K_\xi}$ because W_ξ and V_ξ belong to $\mathcal{B}\mathfrak{a}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda$ and g is $(\mathcal{B}\mathfrak{a}_{K_\xi},\mathcal{B}\mathfrak{a}(\lambda))$-measurable.) But if we identify $\{0,1\}^{K_\xi}\times\{0,1\}^{\kappa\setminus K_\xi}$ with $\{0,1\}^\kappa$, then E becomes $\{x:(x,f_\eta(x))\in W_\xi\setminus V_\xi\}$. Now

(551Ea)
$$a\subseteq[\![\vec{f}_\eta\in\vec{W}_\xi]\!]=\{x:(x,f_\eta(x))\in W_\xi\}^\bullet$$
$$\subseteq\{x:(x,f_\eta(x))\in V_\xi\}^\bullet=[\![\vec{f}_\eta\in\vec{V}_\xi]\!],$$

and $a\Vdash_{\mathbb{P}_\kappa}\vec{f}_\eta\in\vec{V}_\xi$. **Q**

(iv) For each $\xi<\theta_1^+$, V_ξ factors through $\{0,1\}^{K_\xi}\times\{0,1\}^\lambda$ and belongs to $\mathcal{B}\mathfrak{a}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda$. There is therefore a countable set $L_\xi\subseteq K_\xi$ such that V_ξ factors through $\{0,1\}^{L_\xi}\times\{0,1\}^\lambda$. Let S be the set $\{\xi:\xi<\theta_1^+,\ \mathrm{cf}\,\xi>\omega\}$. Because $\theta_1\ge\omega_1$, S is stationary in θ_1^+ (5A1Ac). For each $\xi\in S$, let $g(\xi)<\xi$ be such that $L_\xi\subseteq K_{g(\xi)}$. By the Pressing-Down Lemma there is a $\gamma<\theta_1^+$ such that $S'=\{\xi:\xi\in S,\ g(\xi)=\gamma\}$ is stationary.

For $\xi\in S'$, we have a $V'_\xi\in\mathcal{B}\mathfrak{a}_{K_\gamma}\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda$ such that
$$V_\xi=\{(x,y):x\in\{0,1\}^\kappa,\ y\in\{0,1\}^\lambda,\ (x\restriction K_\gamma,y)\in V'_\xi\}.$$
But $\#(K_\gamma)\le\lambda$, so
$$\#(\mathcal{B}\mathfrak{a}_{K_\gamma}\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda)\le\lambda^\omega=\theta_1<\#(S'),$$
and there are ξ, $\eta\in S'$ such that $\eta<\xi$ and $V'_\eta=V'_\xi$ and $V_\eta=V_\xi$. But also
$$a\Vdash_{\mathbb{P}_\kappa}\vec{f}_\eta\in\vec{V}_\xi\setminus\vec{V}_\eta,$$
so this is impossible. **X**

552K Lemma Let I be a set. Let $q:\mathrm{Fn}_{<\omega}(I;\{0,1\})\to[0,\infty[$ be a function such that $q(\emptyset)=1$ and
$$q(z)=q(z\cup\{(i,0)\})+q(z\cup\{(i,1)\})$$
whenever $z\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$ and $i\in I\setminus\mathrm{dom}\,z$. Then there is a unique Radon measure μ on $\{0,1\}^I$ such that
$$\mu\{x:z\subseteq x\in\{0,1\}^I\}=q(z)$$
for every $z\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$.

proof (a) For each $K\in[I]^{<\omega}$, let μ_K be the measure on the finite set $\{0,1\}^K$ defined by saying that $\mu_KA=\sum_{z\in A}q(z)$ for every $A\subseteq\{0,1\}^K$. For $K\subseteq L\in[I]^{<\omega}$ set $f_{KL}(z)=z\restriction K$ for $z\in\{0,1\}^L$; then f_{KL} is inverse-measure-preserving for μ_K and μ_L. **P** It is enough to consider the case $L=K\cup\{i\}$ where $i\in I\setminus K$. In this case, for $A\subseteq\{0,1\}^K$,
$$\mu_Lf^{-1}[A]=\sum_{z\in f^{-1}[A]}q(z)=\sum_{z\in A}q(z\cup\{(i,0)\})+q(z\cup\{(i,1)\})$$
$$=\sum_{z\in A}q(z)=\mu_KA.\ \mathbf{Q}$$

(b) Let $\mathcal{E}$ be the algebra of open-and-closed sets in $\{0,1\}^I$, that is, the family $\{f_K^{-1}[A]:K\in[I]^{<\omega},\ A\subseteq\{0,1\}^K\}$, where $f_K(x)=x\restriction K$ for $x\in\{0,1\}^I$. Then we can define a functional $\nu:\mathcal{E}\to[0,1]$ by setting
$$\nu f_K^{-1}[A]=\mu_KA\text{ whenever }K\in[I]^{<\omega},\ A\subseteq\{0,1\}^K;$$
by (a), this is well-defined. By 416Qa, there is a unique Radon measure μ on $\{0,1\}^I$ extending ν, so that
$$\mu\{x:z\subseteq x\}=\nu\{x:z\subseteq x\}=\mu_K\{z\}=q(z)$$
whenever $K\subseteq I$ is finite and $z\in\{0,1\}^K$.

552L Lemma Let θ be a regular infinite cardinal such that the cardinal power δ^ω is less than θ for every $\delta<\theta$, and $S\subseteq\theta$ a stationary set such that $\mathrm{cf}\,\xi>\omega$ for every $\xi\in S$. Let $\langle M_\xi\rangle_{\xi<\theta}$ be a family of sets of size less than θ, and I a set of size less than θ; suppose that for each $i\in I$ we are given a function f_i with domain S such that $f_i(\xi)\in\bigcup_{\eta<\xi}M_\eta$ for every $\xi\in S$. Then there is an ω_1-complete filter $\mathcal{F}$ on θ, containing every closed cofinal subset of θ, such that for every $i\in I$ there is a $D\in\mathcal{F}$ such that $D\subseteq S$ and f_i is constant on D.

proof (a) Set $M=\bigcup_{\xi<\theta}M_\xi$, so that $\#(M)\le\theta$; let $\langle x_\xi\rangle_{\xi<\theta}$ run over M. Set
$$F^*=\{\xi:\xi<\theta,\ \textstyle\bigcup_{\eta<\xi}M_\eta=\{x_\eta:\eta<\xi\}\};$$

then F^* is a closed cofinal subset of θ, because θ is regular and uncountable. Set $S_1 = S \cap F^*$, so that S_1 is stationary. For $\xi \in S_1$ and $i \in I$ let $h_\xi(i) < \xi$ be such that $f_i(\xi) = x_{h_\xi(i)}$. For $J \in [I]^{\le\omega}$, $\xi \in S_1$ set

$$D_{\xi J} = \{\eta : \eta \in S \cap F^*,\, h_\eta{\restriction}J = h_\xi{\restriction}J\}.$$

(b) There is a $\xi \in S_1$ such that $D_{\xi J} \cap F \ne \emptyset$ for every closed cofinal set $F \subseteq \theta$ and every $J \in [I]^{\le\omega}$. **P?** Otherwise, for each $\xi \in S_1$ choose $J_\xi \in [I]^{\le\omega}$ and a closed cofinal set F_ξ not meeting $D_{\xi J_\xi}$. Let F be the diagonal intersection $\{\xi : \xi < \theta,\ \xi \in F_\eta$ whenever $\eta \in S_1 \cap \xi\}$, so that F is a closed cofinal set (4A1Bc) and $S_2 = S_1 \cap F$ is stationary. For $\xi \in S_2$ let $g(\xi) < \xi$ be such that $h_\xi[J_\xi] \subseteq g(\xi)$. Then there is a $\gamma < \theta$ such that $S_3 = \{\xi : \xi \in S_2,\ g(\xi) = \gamma\}$ is stationary, by the Pressing-Down Lemma (4A1Cc). Now $h_\xi{\restriction}J_\xi \in [I \times \gamma]^{\le\omega}$ for every $\xi \in S_1$, and $\#([I\times\gamma]^{\le\omega}) \le \max(\#(I), \gamma, \omega)^\omega < \theta$, so there are ξ, $\eta \in S_3$ such that $h_\xi{\restriction}J_\xi = h_\eta{\restriction}J_\eta$ and $\eta < \xi$. But in this case we have $\xi \in F_\eta \cap D_{\eta J_\eta}$, which is supposed to be impossible. **XQ**

(c) If now $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence of closed cofinal sets in θ, and $\langle J_n\rangle_{n\in\mathbb{N}}$ is any sequence in $[I]^{\le\omega}$,

$$\bigcap_{n\in\mathbb{N}} D_{\xi J_n} \cap F_n = D_{\xi J} \cap F$$

is non-empty, where $J = \bigcup_{n\in\mathbb{N}} J_n$ and $F = \bigcap_{n\in\mathbb{N}} F_n$. So we have an ω_1-complete filter $\mathcal{F}$ on θ generated by

$$\{D_{\xi J} : J \in [I]^{\le\omega}\} \cup \{F : F \subseteq \theta \text{ is closed and cofinal}\}.$$

If $i \in I$ then f_i is constant on $D_{\xi,\{i\}} \in \mathcal{F}$, so we're done.

552M Proposition Let κ and λ be infinite cardinals. Then the following are equiveridical:

(i) if $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^\kappa)$ and $\#(\mathcal{A}) \le \lambda$ then there is an extension of ν_κ to a measure measuring every member of $\mathcal{A}$;

(ii) for every function $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$, there is a Baire measure μ on $\{0,1\}^{\kappa+\lambda}$ such that $\mu\{y : y \in \{0,1\}^{\kappa+\lambda},\ z \subseteq y\} = 2^{-\#(K)}$ whenever $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^K$, and $\mu^*\{x \cup f(x) : x \in \{0,1\}^\kappa\} = 1$;

(iii) if (X, Σ, μ) is a locally compact (definition: 342Ad) semi-finite measure space with Maharam type at most κ, $\mathcal{A} \subseteq \mathcal{P}X$ and $\#(\mathcal{A}) \le \lambda$, then there is an extension of μ to a measure measuring every member of $\mathcal{A}$.

proof (i)⇒(ii) Assume (i). If $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$ is a function, set $\mathcal{A} = \{\{x : f(x)(\xi) = 1\} : \kappa \le \xi < \kappa+\lambda\}$, so that $\mathcal{A}$ is a family of subsets of $\{0,1\}^\kappa$ and $\#(\mathcal{A}) \le \lambda$. Let ν be a measure on $\{0,1\}^\kappa$, extending ν_κ and measuring every member of $\mathcal{A}$. Then $\{x : (x \cup f(x))(\xi) = 1\} \in \operatorname{dom}\nu$ for every $\xi \in \kappa+\lambda$, so we have a Baire measure μ on $\{0,1\}^{\kappa+\lambda}$ defined by saying that $\mu E = \nu\{x : x \cup f(x) \in E\}$ for Baire sets $E \subseteq \{0,1\}^{\kappa+\lambda}$. If $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^\kappa$, then

$$\mu\{y : z \subseteq y\} = \nu\{x : z \subseteq x \cup f(x)\} = \nu\{x : z \subseteq x\} = \nu_\kappa\{x : z \subseteq x\} = 2^{-\#(K)};$$

while if $E \in \mathcal{B}\mathfrak{a}_{\kappa+\lambda}$ and $x \cup f(x) \in E$ for every $x \in \{0,1\}^\kappa$, then $\mu E = \nu\{0,1\}^\kappa = 1$, so $\mu^*\{x \cup f(x) : x \in \{0,1\}^\kappa\} = 1$. Thus (ii) is true.

(ii)⇒(i) Assume (ii). Let $\mathcal{A}$ be a family of subsets of $\{0,1\}^\kappa$ with $\#(\mathcal{A}) \le \lambda$. Let $\langle A_\eta\rangle_{\eta<\lambda}$ run over $\mathcal{A} \cup \{\emptyset\}$. Define $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$ by saying that $f(x)(\kappa+\eta) = (\chi A_\eta)(x)$ whenever $\eta < \lambda$ and $x \in \{0,1\}^\kappa$. Let μ be a Baire measure on $\{0,1\}^{\kappa+\lambda}$ satisfying the conditions of (ii). Set $g(x) = x \cup f(x)$ for $x \in \{0,1\}^\kappa$. Because $g[\{0,1\}^\kappa]$ has full outer measure for μ, we have a measure ν on $\{0,1\}^\kappa$ such that $\nu g^{-1}[E] = \mu E$ for every Baire set $E \subseteq \{0,1\}^{\kappa+\lambda}$ (234F[3]); let $\hat\nu$ be the completion of ν. Now $A_\eta = g^{-1}[\{y : y(\kappa+\eta) = 1\}]$ is measured by ν and $\hat\nu$. Also

$$\hat\nu\{x : z \subseteq x\} = \nu\{x : z \subseteq x\} = \nu\{x : z \subseteq g(x)\} = \mu\{y : z \subseteq y\} = 2^{-\#(K)}$$

whenever $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^\kappa$, so $\hat\nu$ extends ν_κ (254G) and is an extension of ν_κ measuring every member of $\mathcal{A}$.

(i)⇒(iii) Suppose that (i) is true.

(α) Let (X, Σ, μ) be a compact probability space of Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X of size at most λ. Then there is a function $h : \{0,1\}^\kappa \to X$ which is inverse-measure-preserving for ν_κ and μ. **P** By 332P, the measure algebra of μ can be embedded into $\mathfrak{B}_\kappa$; by 343B, this embedding can be realized by an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **Q** Now $\mathcal{C} = \{h^{-1}[A] : A \in \mathcal{A}\}$ has cardinal at most λ, so there is an extension ν of ν_κ measuring every member of $\mathcal{C}$; and the image measure νh^{-1} extends μ and measures every member of $\mathcal{A}$.

(β) It follows at once that if (X, Σ, μ) is a compact totally finite measure space with Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X of size at most λ, then μ can be extended to every member of $\mathcal{A}$. (If $\mu X = 0$ this is trivial, and otherwise we can apply (α) to a scalar multiple of μ.)

(γ) Now suppose that (X, Σ, μ) is a locally compact semi-finite measure space with Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X of size at most λ. In the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ, let D be a partition of unity consisting of elements of finite measure; for $d \in D$ choose $E_d \in \Sigma$ such that $E_d^\bullet = d$. If $G \in \Sigma$ then

[3] Formerly 132G.

$$\mu G = \bar{\mu}G^\bullet = \sum_{d\in D}\bar{\mu}(d\cap G^\bullet) = \sum_{d\in D}\mu(E_d\cap G).$$

For each $d\in D$, the subspace measure μ_{E_d} on E_d is compact and totally finite and has Maharam type at most κ (put 331Hc and 322Ja together), so by (β) can be extended to a measure μ'_{E_d} measuring $A\cap E_d$ for every $A\in\mathcal{A}$. Set $\mu' F = \sum_{d\in D}\mu'_{E_d}(F\cap E_d)$ whenever $F\subseteq X$ is such that the sum is defined; then μ' is a measure on X, extending μ and measuring every set in $\mathcal{A}$, as required.

(iii)⇒(i) is trivial.

552N Theorem (Carlson 84) Let κ and λ be infinite cardinals such that κ is greater than the cardinal power λ^ω. Then

$\Vdash_{\mathbb{P}_\kappa}$ if $\mathcal{A}\subseteq\mathcal{P}(\{0,1\}^{\check\kappa})$ and $\#(\mathcal{A})\le\check\lambda$, there is an extension of $\nu_{\check\kappa}$ to a measure measuring every member of $\mathcal{A}$.

proof (a) Let $\langle e_{\xi\zeta}\rangle_{\xi,\zeta<\kappa}$ be a re-indexing of the standard generating family in $\mathfrak{B}_\kappa$. For $J\subseteq\kappa\times\kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_{\xi\zeta}:(\xi,\zeta)\in J\}$. Recall that $\#(L^\infty(\mathfrak{C}_J))\le\max(\omega,\#(J)^\omega)$ for every J (524Ma, 515Mb); we shall also need to know that every element of $\mathfrak{B}_\kappa$ belongs to $\mathfrak{C}_J$ for some countable $J\subseteq\kappa\times\kappa$ (254Rb, 531Jb).

Set $I=(\kappa+\lambda)\setminus\kappa$, where $\kappa+\lambda$ is the ordinal sum, so that I is disjoint from κ and $\#(I)=\lambda$. Let $\dot f$ be a $\mathbb{P}_\kappa$-name for a function from $\{0,1\}^{\check\kappa}$ to $\{0,1\}^{\check I}$.

For each $\xi<\kappa$ let $\dot x_\xi$ be a $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\check\kappa}$ such that $[\![\dot x_\xi(\check\zeta)=1]\!]=e_{\xi\zeta}$ for every $\zeta<\kappa$. For $z\in\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$ and $\xi<\kappa$ set

$$a_{\xi z}=[\![\check z\subseteq\dot x_\xi\cup\dot f(\dot x_\xi)]\!],$$

and let $J_{\xi z}\subseteq\kappa\times\kappa$ be a countable set such that $a_{\xi z}\in\mathfrak{C}_{J_{\xi z}}$.

Set $\theta=(\lambda^\omega)^+\le\kappa$. For $\xi\le\theta$ let $L_0(\xi)\subseteq\kappa$ be the smallest set such that $\xi\subseteq L_0(\xi)$ and $J_{\eta w}\subseteq L_0(\xi)\times L_0(\xi)$ for every $\eta<\xi$ and $w\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$; set $L(\xi)=L_0(\xi)\times L_0(\xi)$. Then $\#(L(\xi))\le\max(\omega,\lambda,\#(\xi))<\theta$ for every $\xi<\theta$, and $L(\xi)=\bigcup_{\eta<\xi}L(\eta)$ for limit $\xi\le\theta$. Set

$$D^*=\{\xi:\xi<\theta\text{ is a limit ordinal, }\xi>\sup(\theta\cap L_0(\eta))\text{ for every }\eta<\xi\};$$

then D^* is a closed cofinal subset of θ, and $\xi\notin L_0(\xi)$ for every $\xi\in D^*$. So

$$S=\{\xi:\xi\in D^*,\ \mathrm{cf}(\xi\cap D^*)\ge\omega_1\}$$

is stationary in θ.

(b) For $J\subseteq\kappa\times\kappa$, let $P_J:L^1(\mathfrak{B}_\kappa)\to L^1(\mathfrak{C}_J)\subseteq L^1(\mathfrak{B}_\kappa)$ be the conditional expectation operator defined by saying that $P_Ju\in L^1(\mathfrak{C}_J)$ and $\int_c P_Ju=\int_c u$ whenever $c\in\mathfrak{C}_J$ and $u\in L^1(\mathfrak{B}_\kappa)$ (254R, 365R). We need to know that $P_{J\cap J'}=P_JP_{J'}$ for all J, $J'\subseteq\kappa$ (254Ra), and that $P_J(u\times v)=u\times P_Jv$ whenever $u\in L^1(\mathfrak{C}_J)$, $v\in L^1(\mathfrak{B}_\kappa)$ and $u\times v\in L^1(\mathfrak{B}_\kappa)$ (242L). It follows that if J, J' and $J''\subseteq\kappa\times\kappa$, $u\in L^1(\mathfrak{C}_J)$ and $J\cap J'=J\cap J''$, then

$$P_{J'}(u)=P_{J'}P_J(u)=P_{J\cap J'}(u)=P_{J\cap J''}(u)=P_{J''}(u).$$

If $h:\kappa\times\kappa\to\kappa\times\kappa$ is any bijection, then we have a corresponding measure-preserving automorphism $\pi:\mathfrak{B}_\kappa\to\mathfrak{B}_\kappa$ defined by saying that $\pi e_{\xi\zeta}=e_{\xi'\zeta'}$ if $(\xi',\zeta')=h(\xi,\zeta)$, and a Banach lattice automorphism $T:L^1(\mathfrak{B}_\kappa)\to L^1(\mathfrak{B}_\kappa)$ defined by saying that $T(\chi a)=\chi\pi a$ for $a\in\mathfrak{B}_\kappa$ (see 365O); note that $T\restriction L^\infty(\mathfrak{B}_\kappa)$ is multiplicative (compare the formulae in 365Ob and 364Ra).

If $J\subseteq\kappa\times\kappa$ and $u\in L^1(\mathfrak{C}_J)$, then $Tu\in L^1(\mathfrak{C}_{h[J]})$. **P** By 314H, $\mathfrak{C}_{h[J]}=\pi[\mathfrak{C}_J]$; now use the fact that $[\![Tu>\alpha]\!]=\pi[\![u>\alpha]\!]$ for every $\alpha\in\mathbb{R}$. **Q** If $h\restriction J$ is the identity, then $\pi a=a$ for every $a\in\mathfrak{C}_J$ and $Tv=v$ for every $v\in L^\infty(\mathfrak{C}_J)$.

(c) Fix $\xi\in S$ and $w\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$ for the moment. Because $\xi\cap D^*$ has uncountable cofinality and $J_{\xi w}$ is countable, there is a $g_w(\xi)\in\xi\cap D^*$ such that $J_{\xi w}\cap L(\xi)\subseteq L(g_w(\xi))$ and $\{\eta:\eta\in L_0(\xi),\ (\xi,\eta)\in J_{\xi w}\}\subseteq L_0(g_w(\xi))$. Let $h_{\xi w}:\kappa\times\kappa\to\kappa\times\kappa$ be the involution defined by saying that

$$\begin{aligned}h_{\xi w}(\eta,\zeta)&=(g_w(\xi),\zeta)\text{ if }\eta=\xi,\\&=(\xi,\zeta)\text{ if }\eta=g_w(\xi),\\&=(\eta,\zeta)\text{ otherwise;}\end{aligned}$$

note that

$$h_{\xi w}[J_{\xi w}]\cap L(\xi)\subseteq L(g_w(\xi))\cup(\{g_w(\xi)\}\times L_0(g_w(\xi))\subseteq L(g_w(\xi)+1),$$

while $h_{\xi w}$ is the identity on $L(g_w(\xi))$, since neither ξ nor $g_w(\xi)$ belongs to $L_0(g_w(\xi))$. Let $T_{\xi w}:L^1(\mathfrak{B}_\kappa)\to L^1(\mathfrak{B}_\kappa)$ be the corresponding Banach lattice isomorphism defined as in (b) above. Then (b) tells us that

$$P_{L(g_w(\xi))}T_{\xi w}=P_{L(g_w(\xi))}.$$

Set

$$u_{\xi w}=P_{L(g_w(\xi)+1)}T_{\xi w}(\chi a_{\xi w})\in L^\infty(\mathfrak{C}_{L(g_w(\xi)+1)}).$$

(d) Setting $M(\eta)=\eta\times L^\infty(\mathfrak{C}_{L(\eta)})$ for $\eta<\theta$, we see that

$$\#(M(\eta))\le\#(\eta)^\omega\le\lambda^\omega<\theta$$

whenever $\eta\ge 2$ (see (a) above), while $(g_w(\xi),u_{\xi w})\in\bigcup_{\eta<\xi}M(\eta)$ whenever $\xi\in S$ and $w\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$. Since $\#(\mathrm{Fn}_{<\omega}(I;\{0,1\}))=\lambda<\theta$, 552L tells us that there is an ω_1-complete filter $\mathcal{F}$ on S, containing $S\setminus\zeta$ for every $\zeta<\theta$, such that for every $w\in\mathrm{Fn}_{<\omega}(I;\{0,1\})$ there is a $D\in\mathcal{F}$ such that g_w and $\xi\mapsto u_{\xi w}$ are constant on D.

(e) For $z\in\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$ and $\xi<\theta$, set $v_{\xi z}=P_{L(\xi)}(\chi a_{\xi z})$. Then there is a $D\in\mathcal{F}$ such that $\xi\mapsto v_{\xi z}$ is constant on D. **P** Set $z'=z\restriction L_0(\theta)$, $z''=z\restriction\kappa\setminus L_0(\theta)$ and $w=z\restriction I$, so that $a_{\xi z}=a_{\xi z'}\cap a_{\xi z''}\cap a_{\xi w}$. Set $m=\#(z'')$, so that $a_{\xi z''}=[\![\check z''\subseteq\dot x_\xi]\!]$ has measure 2^{-m} for every ξ. Also $a_{\xi z''}\in\mathfrak{C}_{\kappa\times(\kappa\setminus L_0(\theta))}$ is stochastically independent of $\mathfrak{C}_{L(\theta)}$, so $P_{L(\theta)}(\chi a_{\xi z''})=2^{-m}\chi 1$; while $a_{\xi z'}$ and $a_{\xi w}$ belong to $\mathfrak{C}_{L(\theta)}$, so, using the formulae in (b),

$$\begin{aligned}v_{\xi z}&=P_{L(\xi)}(\chi a_{\xi z})=P_{L(\xi)}P_{L(\theta)}(\chi a_{\xi z})\\&=P_{L(\xi)}P_{L(\theta)}(\chi a_{\xi z'}\times\chi a_{\xi z''}\times\chi a_{\xi w})\\&=P_{L(\xi)}(\chi a_{\xi z'}\times\chi a_{\xi w}\times P_{L(\theta)}(\chi a_{\xi z''}))\\&=2^{-m}P_{L(\xi)}(\chi a_{\xi z'}\times\chi a_{\xi w}).\end{aligned}$$

Let $\xi_0<\theta$ be such that $\operatorname{dom}z'\subseteq L_0(\xi_0)$. By (d), there are $D_0\in\mathcal{F}$, $\zeta<\theta$ and $u\in L^\infty(\mathfrak{B}_\kappa)$ such that $g_w(\xi)=\zeta$ and $u_{\xi w}=u$ for every $\xi\in D_0$. For $\xi\in D_0\setminus\xi_0$ take $h_{\xi w}$ and $T_{\xi w}$ as in (c). Then, writing $\pi_{\xi w}$ for the measure-preserving automorphism defined from $h_{\xi w}$ as in (b),

$$\begin{aligned}\pi_{\xi w}a_{\xi z'}&=\pi_{\xi w}(\inf_{z'(\eta)=1}e_{\xi\eta}\setminus\sup_{z'(\eta)=0}e_{\xi\eta})=\inf_{z'(\eta)=1}\pi_{\xi w}e_{\xi\eta}\setminus\sup_{z'(\eta)=0}\pi_{\xi w}e_{\xi\eta}\\&=\inf_{z'(\eta)=1}e_{\zeta\eta}\setminus\sup_{z'(\eta)=0}e_{\zeta\eta}=a_{\zeta z'};\end{aligned}$$

consequently $T_{\xi w}(\chi a_{\xi z'})=\chi a_{\zeta z'}$. Now

$$P_{L(\xi)}(\chi a_{\xi z'}\times\chi a_{\xi w})=P_{L(\zeta)}(\chi a_{\xi z'}\times\chi a_{\xi w})$$

(because $\chi a_{\xi z'}\times\chi a_{\xi w}\in L^\infty(\mathfrak{C}_{(\{\xi\}\times L_0(\xi))\cup J_{\xi w}})$, and $(\{\xi\}\times L_0(\xi))\cup J_{\xi w})\cap L(\xi)=J_{\xi w}\cap L(\xi)\subseteq L(\zeta)$)

$$=P_{L(\zeta)}T_{\xi w}(\chi a_{\xi z'}\times\chi a_{\xi w})$$

(see (c) above)

$$=P_{L(\zeta)}(T_{\xi w}\chi a_{\xi z'}\times T_{\xi w}\chi a_{\xi w})$$

(because $T_{\xi w}$ is multiplicative on $L^\infty(\mathfrak{B}_\kappa)$)

$$\begin{aligned}&=P_{L(\zeta)}(\chi a_{\zeta z'}\times T_{\xi w}\chi a_{\xi w})\\&=P_{L(\zeta)}P_{L(\xi)}(\chi a_{\zeta z'}\times T_{\xi w}\chi a_{\xi w})\end{aligned}$$

(because $L(\zeta)\subseteq L(\xi)$)

$$=P_{L(\zeta)}(\chi a_{\zeta z'}\times P_{L(\xi)}T_{\xi w}\chi a_{\xi w})$$

(because $a_{\zeta z'}\in\mathfrak{C}_{\{\zeta\}\times L_0(\xi)}\subseteq\mathfrak{C}_{L(\xi)}$)

$$=P_{L(\zeta)}(\chi a_{\zeta z'}\times P_{L(\zeta+1)}T_{\xi w}\chi a_{\xi w})$$

(because $T_{\xi w}\chi a_{\xi w}\in L^\infty(\mathfrak{C}_{h_{\xi w}[J_{\xi w}]})$ and $h_{\xi w}[J_{\xi w}]\cap L(\xi)\subseteq L(\zeta+1)$)

$$=P_{L(\zeta)}(\chi a_{\zeta z'}\times u_{\xi w})=P_{L(\zeta)}(\chi a_{\zeta z'}\times u).$$

Finally, we get

$$v_{\xi z}=2^{-m}P_{L(\xi)}(\chi a_{\xi z'}\times\chi a_{\xi w})=2^{-m}P_{L(\zeta)}(\chi a_{\zeta z'}\times u)$$

for every $\xi\in D=D_0\setminus\xi_0$, so we have the required constant value. **Q**

(f) For each $z\in\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$ set $v_z=\lim_{\xi\to\mathcal{F}}v_{\xi z}$, the limit being defined in the strong sense that $\{\xi:v_{\xi z}=v_z\}$ belongs to $\mathcal{F}$. Observe that $0\le v_z\le\chi 1$ and that $v_\emptyset=\chi 1$, because $a_{\xi\emptyset}=1$ for every $\xi\in S$. If $z\in\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$ and $\eta\in(\kappa+\lambda)\setminus\operatorname{dom}z$, then $a_{\xi,z\cup\{(\eta,0)\}}$ and $a_{\xi,z\cup\{(\eta,1)\}}$ are disjoint and have union $a_{\xi z}$, so

$$v_{\xi z} = P_{L(\xi)}(\chi a_{\xi z}) = P_{L(\xi)}(\chi a_{\xi,z\cup\{(\eta,0)\}} + \chi a_{\xi,z\cup\{(\eta,1)\}}) = v_{\xi,z\cup\{(\eta,0)\}} + v_{\xi,z\cup\{(\eta,1)\}}$$

for every $\xi \in S$, and $v_z = v_{z\cup\{(\eta,0)\}} + v_{z\cup\{(\eta,1)\}}$. Let $\vec{v}_z$ be the $\mathbb{P}_\kappa$-name for a real number corresponding to v_z as defined in 5A3L.

(g) We have a $\mathbb{P}_\kappa$-name $\dot{\mu}$ for a Baire probability measure on $\{0,1\}^{\kappa+\lambda}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x \in \{0,1\}^{\check{\kappa}+\check{\lambda}}\} = \vec{v}_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$. **P** Start by setting

$$\dot{\mu}_0 = \{((\check{z},\vec{v}_z),1_{\mathfrak{B}_\kappa}) : z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})\},$$

so that $\dot{\mu}_0$ is a $\mathbb{P}_\kappa$-name for a function from $\mathrm{Fn}_{<\omega}(\check{\kappa}+\check{\lambda};\{0,1\})$ to $[0,1]$ and

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_0\check{z} = \vec{v}_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$; note that

$$\Vdash_{\mathbb{P}_\kappa} \check{\kappa}+\check{\lambda} = (\kappa+\lambda)^\vee \text{ and } \mathrm{Fn}_{<\omega}(\check{\kappa}+\check{\lambda};\{0,1\}) = (\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\}))^\vee.$$

Now (f) tells us that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_0\emptyset = 1,$$

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_0(z) = \dot{\mu}_0(z\cup\{(\eta,0)\}) + \dot{\mu}_0(z\cup\{(\eta,1)\})$$
$$\text{whenever } z \in \mathrm{Fn}_{<\omega}(\check{\kappa}+\check{\lambda};\{0,1\}) \text{ and } \eta \in (\check{\kappa}+\check{\lambda})\setminus\mathrm{dom}\,z.$$

By 552K, copied into $V^{\mathbb{P}_\kappa}$,

$$\Vdash_{\mathbb{P}_\kappa} \text{ there is a Radon measure } \mu \text{ on } \{0,1\}^{\check{\kappa}+\check{\lambda}} \text{ such that}$$
$$\mu\{x : z \subseteq x\} = \dot{\mu}_0(z) \text{ for every } z \in \mathrm{Fn}_{<\omega}(\check{\kappa}+\check{\lambda};\{0,1\}).$$

In fact we don't really want the Radon measure here, but its restriction to the Baire σ-algebra. Let $\dot{\mu}$ be a $\mathbb{P}_\kappa$-name for a Baire measure on $\{0,1\}^{\check{\kappa}+\check{\lambda}}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x \in \{0,1\}^{\check{\kappa}+\check{\lambda}}\} = \dot{\mu}_0(\check{z}) = \vec{v}_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$; this is what we have been looking for. **Q**

(h) I had better check that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x\} = (2^{-\#(z)})^\vee$$

whenever $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$. **P** If $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ and $\xi \in S$, then $a_{\xi z}$ belongs to $\mathfrak{C}_{\{\xi\}\times\kappa}$, so is stochastically independent of $\mathfrak{C}_{L(\xi)}$, and

$$v_{\xi z} = P_{L(\xi)}(\chi a_{\xi z}) = (\bar{\nu}_\kappa a_{\xi z})\chi 1 = 2^{-\#(z)}\chi 1.$$

Accordingly $v_z = 2^{-\#(z)}\chi 1$ and

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x\} = \vec{v}_z = (2^{-\#(z)})^\vee. \textbf{ Q}$$

(i) Finally, we come to the key fact:

$$\Vdash_{\mathbb{P}_\kappa} \{\dot{x}_\xi \cup \dot{f}(\dot{x}_\xi) : \xi < \check{\theta}\} \text{ has full outer measure for } \dot{\mu}.$$

P? Otherwise, there are a non-zero $b \in \mathfrak{B}_\kappa$, a rational number $q < 1$ and a sequence $\langle \dot{C}_n\rangle_{n\in\mathbb{N}}$ of $\mathbb{P}_\kappa$-names for basic cylinder sets in $\{0,1\}^{\check{\kappa}+\check{\lambda}}$ such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{x}_\xi \cup \dot{f}(\dot{x}_\xi) \in \bigcup_{n\in\mathbb{N}} \dot{C}_n$$

for every $\xi < \theta$, while also

$$b \Vdash_{\mathbb{P}_\kappa} \sum_{n=0}^\infty \dot{\mu}\dot{C}_n \le \check{q}.$$

Because $\mathfrak{B}_\kappa$ is ccc, we can find for each $n \in \mathbb{N}$ a maximal antichain $\langle b_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathbb{P}_\kappa$, and a sequence $\langle z_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$, such that $b_{ni} \Vdash_{\mathbb{P}_\kappa} \dot{C}_n = \{x : \check{z}_{ni} \subseteq x\}$ for each i. Now

$$[\![\dot{x}_\xi \cup \dot{f}(\dot{x}_\xi) \in \dot{C}_n]\!] = \sup_{i\in\mathbb{N}} b_{ni} \cap [\![\check{z}_{ni} \subseteq \dot{x}_{ni} \cup \dot{f}(\dot{x}_\xi]\!] = \sup_{i\in\mathbb{N}} b_{ni} \cap a_{\xi,z_{ni}}$$

so we must have

$$b \subseteq \sup_{i,n\in\mathbb{N}} b_{ni} \cap a_{\xi,z_{ni}}$$

for every ξ. At the same time,

$$b_{ni}\Vdash_{\mathbb{P}_\kappa}\dot{\mu}\dot{C}_n=\vec{v}_{z_{ni}}$$

for each n and i, so $\dot{\mu}\dot{C}_n$ is represented by $\sum_{i\in\mathbb{N}}\chi b_{ni}\times v_{z_{ni}}$ in $L^\infty(\mathfrak{B}_\kappa)$ and $\min(1,\sum_{n=0}^\infty\dot{\mu}\dot{C}_n)$ is represented by $\chi 1\wedge\sum_{n,i\in\mathbb{N}}\chi b_{ni}\times v_{z_{ni}}$. Since $[\![\sum_{n=0}^\infty\dot{\mu}\dot{C}_n\le\check{q}]\!]$ includes b,

$$\textstyle\sum_{n,i\in\mathbb{N}}\chi b\times\chi b_{ni}\times v_{z_{ni}}\le q\chi b.$$

Let $J\subseteq\kappa\times\kappa$ be a countable set such that $b\cap b_{ni}\in\mathfrak{C}_J$ for every n, $i\in\mathbb{N}$. Then, because $\mathcal{F}$ is ω_1-complete and contains $S\setminus\xi$ for every $\xi<\theta$, there is a $\xi\in S$ such that

$$v_{\xi z_{ni}}=v_{z_{ni}}\text{ whenever }n,\,i\in\mathbb{N},$$

$$J\cap L(\theta)=J\cap L(\xi),\quad J\cap(\{\xi\}\times\kappa)=\emptyset.$$

Set $L=L(\theta)\cup(\{\xi\}\times\kappa)$. If n, $i\in\mathbb{N}$, then

$$\begin{aligned}\int\chi b\times\chi b_{ni}\times v_{z_{ni}}&=\int\chi b\times\chi b_{ni}\times P_{L(\xi)}\chi a_{\xi,z_{ni}}\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni}\times P_{L(\xi)}\chi a_{\xi,z_{ni}})\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni})\times P_{L(\xi)}\chi a_{\xi,z_{ni}}\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni})\times\chi a_{\xi,z_{ni}}\\&=\int P_L(\chi b\times\chi b_{ni})\times\chi a_{\xi,z_{ni}}\end{aligned}$$

(because $J\cap L=J\cap L(\xi)$)

$$=\int P_L(\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}})$$

(because $a_{\xi,z_{ni}}\in\mathfrak{C}_{\{\xi\}\times\kappa}\subseteq\mathfrak{C}_L$)

$$=\int\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}}.$$

Summing over n and i, we have

$$\begin{aligned}\bar{\nu}_\kappa b&\le\sum_{n=0}^\infty\sum_{i=0}^\infty\bar{\nu}_\kappa(b\cap b_{ni}\cap a_{\xi,z_{ni}})=\sum_{n=0}^\infty\sum_{i=0}^\infty\int\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}}\\&=\sum_{n=0}^\infty\sum_{i=0}^\infty\int\chi b\times\chi b_{ni}\times v_{z_{ni}}\le\int q\chi b=q\bar{\nu}_\kappa b,\end{aligned}$$

which is impossible. **XQ**

(j) What all this shows is that

$\Vdash_{\mathbb{P}_\kappa}$ for every $f:\{0,1\}^{\check{\kappa}}\to\{0,1\}^{(\check{\kappa}+\check{\lambda})\setminus\check{\kappa}}$ there is a Baire measure μ on $\{0,1\}^{\check{\kappa}+\check{\lambda}}$ such that $\mu\{y:y\in\{0,1\}^{\check{\kappa}+\check{\lambda}},\,z\subseteq y\}=2^{-\#(K)}$ whenever $K\in[\check{\kappa}]^{<\omega}$ and $z\in\{0,1\}^K$, and $\mu^*\{x\cup f(x):x\in\{0,1\}^{\check{\kappa}}\}=1$.

By 552M, copied into $V^{\mathbb{P}_\kappa}$,

$\Vdash_{\mathbb{P}_\kappa}$ if $\mathcal{A}\subseteq\mathcal{P}(\{0,1\}^{\check{\kappa}})$ and $\#(\mathcal{A})\le\check{\lambda}$, there is an extension of $\nu_{\check{\kappa}}$ to a measure measuring every member of $\mathcal{A}$,

as required.

552O Proposition Suppose that (X,Σ,μ) is a probability space such that for every countable family $\mathcal{A}$ of subsets of X there is a measure on X extending μ and measuring every member of $\mathcal{A}$.

(a) If Y is a universally negligible (definition: 439B) second-countable T_0 space, then $\#(Y)<\operatorname{cov}\mathcal{N}(\mu)$.

(b) $\operatorname{cov}\mathcal{N}(\mu)>\operatorname{non}\mathcal{N}(\nu_\omega)$.

proof (a) ? Otherwise, let $\langle E_y\rangle_{y\in Y}$ be a cover of X by μ-negligible sets, and $f : X \to Y$ a function such that $x \in E_{f(x)}$ for every $x \in X$. Let $\mathcal{U}$ be a countable base for the topology of Y and $\mathcal{A} = \{f^{-1}[U] : U \in \mathcal{U}\}$; let $\tilde{\mu}$ be a measure on X extending μ and measuring every member of $\mathcal{A}$. Consider the image measure $\tilde{\mu}f^{-1}$ on Y. This measures every member of $\mathcal{U}$ so measures every Borel set in Y; let ν be its restriction to the Borel σ-algebra of Y. Then ν is a Borel probability measure on Y. Take any $y \in Y$. Because Y has a T_0 topology, $\mathcal{U}$ must separate the points of Y and $\{y\}$ is a Borel set; now

$$\nu\{y\} = \tilde{\mu}f^{-1}[\{y\}] \le \tilde{\mu}^*E_y \le \mu^*E_y = 0.$$

So ν is zero on singletons and witnesses that Y is not universally negligible. **X**

(b) By Grzegorek's theorem (439Fc), there is a universally negligible set $Y \subseteq [0,1]$ of cardinal $\operatorname{non}\mathcal{N}(\nu_\omega)$. (Recall that the Lebesgue null ideal is isomorphic to $\mathcal{N}(\nu_\omega)$, as noted in 522Va.)

552P Theorem Let κ and λ be infinite cardinals. Then the iterated forcing notion $\mathbb{P}_\kappa * \mathbb{P}_{\check{\lambda}}$ has regular open algebra isomorphic to $\mathfrak{B}_{\max(\kappa,\lambda)}$.

Remark Here $\mathbb{P}_{\check{\lambda}}$ represents a standard $\mathbb{P}_\kappa$-name for random real forcing; see 551O.

proof In Theorem 551Q, take $\Omega = \{0,1\}^\kappa$, $\Sigma = \mathrm{T}_\kappa$, $\mathcal{I} = \mathcal{N}(\nu_\kappa)$ and $I = \lambda$. If we identify $\{0,1\}^\kappa \times \{0,1\}^\lambda$ with $\{0,1\}^{\kappa+\lambda}$, where $\kappa+\lambda$ is the ordinal sum, then $\Lambda = \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda$ becomes a σ-algebra intermediate between $\mathcal{B}\mathfrak{a}_{\kappa+\lambda}$ and $\mathrm{T}_{\kappa+\lambda}$, while

$$\mathcal{J} = \{W : W \in \Lambda,\ \nu_\lambda W[\{x\}] = 0 \text{ for } \nu_\lambda\text{-almost every } x \in \{0,1\}^\kappa\}$$

is just $\Lambda \cap \mathcal{N}(\nu_{\kappa+\lambda})$. It follows at once that the algebra $\mathfrak{A} = \Lambda/\mathcal{J}$ is isomorphic to $\mathfrak{B}_{\kappa+\lambda}$; and 551Q tells us that $\mathrm{RO}(\mathbb{P}_\kappa * \mathbb{P}_{\check{\lambda}})$ is isomorphic to $\mathfrak{A}$. Since we are supposing that κ and λ are infinite, $\mathfrak{B}_{\kappa+\lambda} \cong \mathfrak{B}_{\max(\kappa,\lambda)}$ and we're done.

552X Basic exercises (a) Let κ be an infinite cardinal. Show that $\Vdash_{\mathbb{P}_\kappa} \check{\kappa}^\omega = (\kappa^\omega)\check{}$, where these are all cardinal powers.

(b) (Miller 82) Suppose that $\mathfrak{c} < \omega_\omega$. Show that

$$\Vdash_{\mathbb{P}_{\omega_\omega}} \operatorname{cov}\mathcal{N}(\nu_{\omega_1}) = \omega_\omega < \operatorname{cov}\mathcal{N}(\nu_\omega).$$

(c) Suppose that the continuum hypothesis is true. Show that there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of $[0,1]$ such that there is no measure extending Lebesgue measure which measures every A_n. (*Hint*: there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of functions from ω_1 to itself such that $\{f_n(\xi) : n \in \mathbb{N}\} = \{\eta : \eta \le \xi\}$ for every $\xi < \omega_1$.)

552Y Further exercises (a) Let κ and λ be infinite cardinals, and μ a Baire measure on $\{0,1\}^\lambda$. (i) Show that there is a $\mathbb{P}_\kappa$-name $\dot{\mu}$ for a Baire measure on $\{0,1\}^{\check{\lambda}}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x\} = (\mu\{x : z \subseteq x\})\check{}$ for every $z \in \mathrm{Fn}_{<\omega}(\lambda;\{0,1\})$. (ii) Show that if $A \subseteq \{0,1\}^\lambda$, then $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}^*(\check{A}) = (\mu^*A)\check{}$.

(b)(i) Show that for any non-zero cardinals κ, λ there are cardinals $\theta^{\mathrm{cov}}_{\kappa\lambda}$ and $\theta^{\mathrm{non}}_{\kappa\lambda}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) = \check{\theta}^{\mathrm{cov}}_{\kappa\lambda}, \quad \operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) = \check{\theta}^{\mathrm{non}}_{\kappa\lambda}.$$

(*Hint*: if κ is infinite, $\mathfrak{B}_\kappa$ is homogeneous.) (ii) Show that $\theta^{\mathrm{cov}}_{\kappa\lambda}$ increases with κ and decreases with λ, while $\theta^{\mathrm{non}}_{\kappa\lambda}$ decreases with κ and increases with λ. (*Hint*: 552P.)

552 Notes and comments In any forcing model, all the open questions of ZFC re-present themselves for our attention. The first and most important question concerns the continuum hypothesis, and in most cases we can say something useful. So I start with 552B: 'if you add κ random reals, then the continuum rises to κ^ω'. Any mnemonic of this kind has to come with footnotes concerning the interpretation of the terms, because we cannot rely on the formula 'κ^ω' meaning the same thing in the universe we start from and the forcing model we move to. Indeed, in general forcing models, the symbol 'κ' has to be watched, since I normally reserve it for cardinals, and cardinals sometimes collapse; but here, at least, we have a ccc forcing notion, and cardinals are preserved (5A3N). Actually, 'κ^ω' also is safe in the present context (552Xa); but we find this out afterwards.

One of the central properties of random real forcing concerns iteration: if you do it twice, you still have random real forcing. Of course 'iterated forcing', in a vast variety of forms, is an indispensable technique, and two-stage forcing, as in 552P, is the easiest kind. I do not expect to quote this result very often in this book, but that is because (for random reals) I am interested as much in the forcing notions themselves, and the measure algebras which are their regular open algebras, as in the propositions which are true in the forcing models. So when I see a proof which depends on repeated random real forcing my first impulse is to examine the relevant properties of measure algebras, and this generally leads to a direct proof in terms of single-stage forcing. Note the form of Theorem 552P: as in 551Q, it does not claim that the iteration $\mathbb{P}_\kappa * \mathbb{P}_{\check\lambda}$ is isomorphic to $\mathbb{P}_{\max(\kappa,\lambda)}$, but only that they have isomorphic regular open algebras, and therefore lead to the same mathematical worlds (5A3I).

A typical example is in 552J. Random real forcing does not change outer measures (552D). If we think of $\mathbb{P}_\kappa$ as an iteration $\mathbb{P}_{\kappa\setminus J} * \mathbb{P}_{\check J}$, and we have a $\mathbb{P}_\kappa$-name $\dot E$ for a 'new' negligible set, then, thinking in $V^{\mathbb{P}_{\check J}}$, the set of members of $\{0,1\}^{\check\lambda}$ contained in $\dot E$ will have to be negligible. Back in the ordinary universe, we shall have a $\mathbb{P}_J$-name for a negligible set containing every member of $\{0,1\}^{\check\lambda}$ with a $\mathbb{P}_J$-name which belongs to $\dot E$. In 552J, the idea is that if $\dot A$ is a set in $V^{\mathbb{P}_\kappa}$ and every small subset of $\dot A$ is negligible in $V^{\mathbb{P}_\kappa}$, then at every stage the set of members of $\dot A$ which have been named so far must be negligible in $V^{\mathbb{P}_\kappa}$, just because there are not very many names yet available, and therefore is also negligible in the intermediate universe of the forcing notion $\mathbb{P}_{K_\xi}$. This must be witnessed by a countable structure in the intermediate universe, and the Pressing-Down Lemma tells us that there is a stationary set of levels for which the same countable structure will serve; it follows easily that we have a name in $V^{\mathbb{P}_\kappa}$ for a negligible set including $\dot A$. I invite you to seek out the elements of the formal exposition in 552J which correspond to this sketch.

552E can also be approached as a result about iterated random real forcing. Here, $\dot A$ is just the set of 'random reals' $\dot x_\xi$ built directly from the regular open algebra $\mathfrak{B}_\kappa$. To see that this is a Sierpiński set, we need to look at a negligible set. A negligible set in $V^{\mathbb{P}_\kappa}$ is included in one which has a name $\dot C$ in $V^{\mathbb{P}_J}$ for some countable $J \subseteq \kappa$. Thinking in $V^{\mathbb{P}_J}$, all but countably many of the $\dot x_\xi$ are still random, because they are the 'random reals' of $V^{\mathbb{P}_{\kappa\setminus J}}$, and therefore do not belong to $\dot C$. The proof of 552E is no more than a formal elaboration of this idea, with the extra technical device necessary to reach 'strongly Sierpiński'.

In 552C all we need to know is that $\mathbb{P}_\kappa$ is weakly σ-distributive, and the key fact is that for every name $\dot f$ for a member of $\mathbb{N}^\mathbb{N}$ there is an h in the ordinary universe such that $\Vdash_{\mathbb{P}_\kappa} \dot f \le^* \check h$; this is why such partial orders are sometimes called 'ω^ω-bounding'. The rest of the argument is based on the same ideas as part (d) of the proof of 5A3N.

In 552F-552J I list the results known to me about the additivity, covering number, uniformity, cofinality and shrinking number of the ideals $\mathcal{N}(\nu_\lambda)$ after random real forcing. Covering number, uniformity and shrinking number are the difficult ones, and even the most basic case, when $\lambda = \omega$ and we are forcing with $\mathbb{P}_\omega$, seems not to have been completely sorted out. 552Gb and 552Hc show that there is room for surprises. My method throughout is to use the results of §551 to relate $\mathcal{N}(\nu_\lambda)$ in $V^{\mathbb{P}_\kappa}$ to $\mathcal{N}(\nu_\kappa \times \nu_\lambda)$ in the original universe. Given a $\mathbb{P}_\kappa$-name $\dot W$ for a negligible set in $\{0,1\}^{\check\lambda}$, we have a negligible $W \subseteq \{0,1\}^\kappa \times \{0,1\}^\lambda$ such that $\Vdash_{\mathbb{P}_\kappa} \dot W \subseteq \vec W$, and then a negligible $V \subseteq \{0,1\}^\lambda$, corresponding to the non-negligible horizontal sections of W, such that $(\{0,1\}^\lambda \setminus V)\check{}$ is disjoint from $\vec W$ and $\dot W$ in $V^{\mathbb{P}_\kappa}$.

In 552K-552M I give some lemmas which apparently have nothing to do with forcing. The intention is to express as much as possible of the argument of Carlson's theorem 552N as results in ZFC. In this section I am taking forcing arguments particularly laboriously; but even when you have got to the point where they seem elementary to you, I believe that it is still worth while minimising the regions in which one has to deal with more than one model of set theory at a time. In 552M the parts (ii) and (iii) contrast oddly. Part (ii) is there to serve as a combinatorial form of (i) which will be accessible for the purposes of 552N. Part (iii) is there to give a notion of the scope of 552N, and in particular to show that in random real models we have extension theorems for many measures not obviously similar to the basic measures ν_κ. I have already noted a similar result in 543G.

In §439 I described a number of examples of probability spaces (X, Σ, μ) with a countable family $\mathcal{A} \subseteq \mathcal{P}X$ such that μ has no extension to a measure measuring every member of $\mathcal{A}$. In particular, as observed in 439Xk, Grzegorek's theorem 439Fc gives us an example of a subspace of $[0,1]$ for which the subspace measure fails to be extendable to some countably-generated σ-algebra. These are ZFC examples; we really do need something like 'compactness' in 552M(iii).

Note that Carlson 84 gives a rather sharper form of Theorem 552N, carrying information about the covering numbers of the measures constructed in $V^{\mathbb{P}_\kappa}$.

553 Random reals II

In this section I collect some further properties of random real models which seem less directly connected with the main topics of this book than those treated in §552. The first concerns strong measure zero or 'Rothberger's property' (534E) and gives a bound for the sizes of sets with this property. The second relates perfect sets in $V^{\mathbb{P}_\kappa}$ to negligible F_σ sets in the original universe; it shows that a random real model can have properties relevant to a question in §531 (553F). Following these, I discuss properties of ultrafilters and partially ordered sets which are not obviously connected with measure theory, but where the arguments needed to establish the truth of sentences in $V^{\mathbb{P}_\kappa}$ involve interesting properties of measure algebras (553G-553M). I conclude with a note on medial limits (553N).

553A Notation I repeat some formulae from 552A. For any cardinal κ, ν_κ will be the usual measure on $\{0,1\}^\kappa$, T_κ its domain, $\mathcal{N}(\nu_\kappa)$ its null ideal and $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ its measure algebra. $\mathcal{B}\mathfrak{a}_\kappa$ will be the Baire σ-algebra of $\{0,1\}^\kappa$. $\mathbb{P}_\kappa$ will be the forcing notion $\mathfrak{B}_\kappa^+$, active downwards.

553B Lemma If $A \subseteq \{0,1\}^{\mathbb{N}}$ has Rothberger's property, then for any $f : \mathbb{N} \to \mathbb{N}$ there is a sequence $\langle z_n \rangle_{n\in\mathbb{N}}$ such that $z_n \in \{0,1\}^{f(n)}$ for each n and $A \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} \{x : z_m \subseteq x \in \{0,1\}^{\mathbb{N}}\}$.

proof (Compare (b) of the proof of 534J.) By 534Fd, A has strong measure zero with respect to the metric ρ on $\{0,1\}^{\mathbb{N}}$ defined by saying that

$$\rho(x,y) = 2^{-n} \text{ if } x{\restriction}n = y{\restriction}n \text{ and } x(n) \ne y(n).$$

So for each $n \in \mathbb{N}$ we can find a sequence $\langle A_{ni} \rangle_{i\in\mathbb{N}}$ of subsets of $\{0,1\}^{\mathbb{N}}$ such that $A \subseteq \bigcup_{i\in\mathbb{N}} A_{ni}$ and $\operatorname{diam} A_{ni} \le 2^{-f(2^n(2i+1))}$ for each i. Take $x_{ni} \in A_{ni}$ if A_{ni} is non-empty, any member of $\{0,1\}^{\mathbb{N}}$ otherwise, and set $z_m = x_{ni}{\restriction}f(2^n(2i+1))$ if $m = 2^n(2i+1)$; take z_0 to be any member of $\{0,1\}^{f(0)}$. Then $z_m \in \{0,1\}^{f(m)}$ for every m, and if $n \in \mathbb{N}$ then

$$\begin{aligned} A \subseteq \bigcup_{i\in\mathbb{N}} A_{ni} &\subseteq \bigcup_{i\in\mathbb{N}} \{x : \rho(x, x_{ni}) \le 2^{-f(2^n(2i+1))}\} \\ &= \bigcup_{i\in\mathbb{N}} \{x : x \supseteq z_{2^n(2i+1)}\} \subseteq \bigcup_{m\ge n} \{x : x \supseteq z_m\}, \end{aligned}$$

as required.

553C Proposition Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{every subset of } \{0,1\}^{\mathbb{N}} \text{ with Rothberger's property has size at most } \check{\mathfrak{c}}.$$

proof (See Bartoszyński & Judah 95, 8.2.11.)

(a) Let $\dot{A}$ be a $\mathbb{P}_\kappa$-name for a subset of $\{0,1\}^{\mathbb{N}}$ with Rothberger's property. Take any $f \in \mathbb{N}^{\mathbb{N}}$. Applying 553B in the forcing language, we must have

$\Vdash_{\mathbb{P}_\kappa}$ there is a sequence $\langle z_n \rangle_{n\in\mathbb{N}}$ such that $z_n \in \{0,1\}^{\check{f}(n)}$ for every $n \in \mathbb{N}$ and $\dot{A} \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} \{x : x \supseteq z_n\}$.

Let $\langle \dot{z}_f(n) \rangle_{n\in\mathbb{N}}$ be a sequence of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{z}_f(n) \in \{0,1\}^{\check{f}(n)} \text{ for every } n \in \mathbb{N},$$

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \textstyle\bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} \{x : x \supseteq \dot{z}_f(n)\}.$$

(b) Let $\langle e_\xi \rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$ (525A). Let $J \subseteq \kappa$ be a set of size at most $\mathfrak{c}$ such that $[\![\dot{z}_f(n) = \check{z}]\!]$ belongs to the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\xi : \xi \in J\}$ for every $f \in \mathbb{N}^{\mathbb{N}}$ and $z \in \bigcup_{n\in\mathbb{N}} \{0,1\}^{f(n)}$. Let $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{C}_J)$ be the conditional expectation operator (365R).

Observe that $\mathfrak{C}_J$ and $\mathfrak{C}_J^{\mathbb{N}}$ have cardinal at most $\mathfrak{c}^\omega = \mathfrak{c}$. So we have a family $\langle \dot{y}_\eta \rangle_{\eta<\mathfrak{c}}$ of $\mathbb{P}_\kappa$-names for members of $\{0,1\}^{\mathbb{N}}$ such that whenever $\langle d_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{C}_J$ there is an $\eta < \mathfrak{c}$ such that $[\![\dot{y}_\eta(\check{n}) = 1]\!] = d_n$ for every $n \in \mathbb{N}$.

(c) Let $\dot{x}$ be a $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\mathbb{N}}$, and suppose that $a = [\![\dot{x} \in \dot{A}]\!]$ is non-zero. For $z \in \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ set $b_z = [\![\dot{x} \supseteq \check{z}]\!]$. For m, $n \in \mathbb{N}$, set

$$c_{nm} = \sup_{z\in\{0,1\}^m} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \in \mathfrak{C}_J.$$

Note that if $k \le m$ and $z \in \{0,1\}^m$ then $b_z \subseteq b_{z\restriction k}$, so

$$[\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \subseteq [\![P_J(\chi(a \cap b_{z\restriction k})) > 2^{-n}]\!],$$

and $c_{nm} \subseteq c_{nk}$; set $c_n = \inf_{m\in\mathbb{N}} c_{nm}$. **?** Suppose, if possible, that $c_n = 0$ for every n. Let $f \in \mathbb{N}^{\mathbb{N}}$ be such that $\sum_{n=0}^\infty \bar{\nu}_\kappa c_{n,f(n)} < \bar{\nu}_\kappa a$, and set

$$d = \sup_{n\in\mathbb{N}} (\sup_{z\in\{0,1\}^{f(n)}} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!]) \in \mathfrak{C}_J.$$

Then $\bar{\nu}_\kappa d < \bar{\nu}_\kappa a$, so $a \setminus d \ne 0$; while

$$\bar{\nu}_\kappa((a \setminus d) \cap [\![\dot{x} \supseteq \dot{z}_f(n)]\!]) = \sum_{z\in\{0,1\}^{f(n)}} \bar{\nu}_\kappa(a \cap b_z \cap [\![\dot{z}_f(n) = \check{z}]\!] \setminus d)$$

(because $\Vdash_{\mathbb{P}_\kappa} \dot{z}_f(n) \in \{0,1\}^{f(n)\check{}}$)

$$= \sum_{z\in\{0,1\}^{f(n)}} \int_{[\![\dot z_f(n)=\check z]\!]\setminus d} P_J(\chi(a\cap b_z))$$

$$\le 2^{-n} \sum_{z\in\{0,1\}^{f(n)}} \bar\nu_\kappa[\![\dot z_f(n)=\check z]\!]$$

(because d includes $[\![P_J(\chi(a\cap b_z)) > 2^{-n}]\!]$ for every $z\in\{0,1\}^{f(n)}$)

$$= 2^{-n}$$

for every n. Consequently

$$(a\setminus d)\cap \inf_{n\in\mathbb{N}}\sup_{m\ge n}[\![\dot x\supseteq \dot z_f(m)]\!] = 0,$$

that is,

$$a\setminus d\Vdash_{\mathbb{P}_\kappa}\dot x\supseteq\dot z_f(n) \text{ for only finitely many } n.$$

But this implies that

$$a\setminus d\Vdash_{\mathbb{P}_\kappa}\dot x\notin\dot A,$$

contrary to hypothesis. **X**

(d) Continuing from (c), we find that there are an $a'\in\mathfrak{B}_\kappa^+$, stronger than a, and an $\eta<\mathfrak{c}$ such that $a'\Vdash_{\mathbb{P}_\kappa}\dot x=\dot y_\eta$. **P** Let $n\in\mathbb{N}$ be such that c_n, as defined in (c), is non-zero. For $m\in\mathbb{N}$ and $w\in\{0,1\}^m$, set

$$d_w = \inf_{k\ge m}\sup_{z\in\{0,1\}^k, z\supseteq w}[\![P_J(\chi(a\cap b_z)) > 2^{-n}]\!].$$

Then

$$\sup_{w\in\{0,1\}^m} d_w = \inf_{k\ge m}\sup_{z\in\{0,1\}^k}[\![P_J(\chi(a\cap b_z)) > 2^{-n}]\!] = c_n$$

because

$$\langle\sup_{z\in\{0,1\}^k, z\supseteq w}[\![P_J(\chi(a\cap b_z)) > 2^{-n}]\!]\rangle_{k\ge m}$$

is non-increasing for each w. Also $d_w = d_{w\cup\{(m,0)\}}\cup d_{w\cup\{(m,1)\}}$ for every $w\in\{0,1\}^m$. So if we set

$$d'_\emptyset = 1,$$

$$d'_{w\cup\{(m,0)\}} = d'_w\cap d_{w\cup\{(m,0)\}},\quad d'_{w\cup\{(m,1)\}} = d'_w\setminus d_{w\cup\{(m,0)\}}$$

for every $m\in\mathbb{N}$ and $w\in\{0,1\}^m$, every d'_w will belong to $\mathfrak{C}_J$, and there must be an $\eta<\mathfrak{c}$ such that

$$[\![\dot y_\eta(\check n)=1]\!] = \sup_{w\in\{0,1\}^{n+1}, w(n)=1} d'_w$$

for every $n\in\mathbb{N}$, in which case

$$[\![\check w\subseteq\dot y_\eta]\!] = d'_w \text{ for every } w\in\bigcup_{n\in\mathbb{N}}\{0,1\}^n.$$

If $m\in\mathbb{N}$, then

$$\bar\nu_\kappa(a\cap[\![\dot x{\restriction}\check m=\dot y_\eta{\restriction}\check m]\!]) = \sum_{w\in\{0,1\}^m}\bar\nu_\kappa(a\cap b_w\cap d'_w)$$

$$= \sum_{w\in\{0,1\}^m}\int_{d'_w} P_J(\chi(a\cap b_w)) \ge \sum_{w\in\{0,1\}^m} 2^{-n}\bar\nu_\kappa(c_n\cap d'_w)$$

(because $c_n\cap d'_w\subseteq d_w\subseteq[\![P_J(\chi(a\cap b_w)) > 2^{-n}]\!]$ for every w)

$$= 2^{-n}\bar\nu_\kappa c_n.$$

So if we set $a' = a\cap[\![\dot x=\dot y_\eta]\!]$, then

$$\bar\nu_\kappa a' = \bar\nu_\kappa(\inf_{m\in\mathbb{N}} a\cap[\![\dot x{\restriction}\check m=\dot y_\eta{\restriction}\check m]\!])$$

$$= \inf_{m\in\mathbb{N}}\bar\nu_\kappa(a\cap[\![\dot x{\restriction}\check m=\dot y_\eta{\restriction}\check m]\!]) \ge 2^{-n}\bar\nu_\kappa c_n > 0,$$

and $a'\ne 0$, while $a'\subseteq a$ and $a'\Vdash_{\mathbb{P}_\kappa}\dot x=\dot y_\eta$. So we have a suitable pair a', η. **Q**

(e) Putting (c) and (d) together, we see that for any name $\dot{x}$ for a member of $\{0,1\}^{\mathbb{N}}$,

$$[\![\dot{x}\in\dot{A}]\!]\subseteq\sup_{\eta<\mathfrak{c}}[\![\dot{x}=\dot{y}_\eta]\!].$$

But this means that

$$\Vdash_{\mathbb{P}_\kappa}\dot{A}\subseteq\{\dot{y}_\eta:\eta<\check{\mathfrak{c}}\}$$

and

$$\Vdash_{\mathbb{P}_\kappa}\#(\dot{A})\le\check{\mathfrak{c}}.$$

553D Remark If $\kappa>\mathfrak{c}$ then

$\Vdash_{\mathbb{P}_\kappa}$ for any countable family of subsets of $\{0,1\}^\omega$ there is an extension of ν_ω measuring every member of the family

(552N). By 552O and 552Ga we see that in this case

$\Vdash_{\mathbb{P}_\kappa}$ any universally negligible subset of $\{0,1\}^\omega$ has cardinal less than $\check{\kappa}$.

The proposition here tells us that

$\Vdash_{\mathbb{P}_\kappa}$ any strong measure zero subset of $\{0,1\}^\omega$ has cardinal at most $\check{\mathfrak{c}}$

which is different if $\mathfrak{c}\ll\kappa$.

553E Proposition Let κ and λ be infinite cardinals, and $\dot{K}$ a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa}\dot{K}\text{ is a compact subset of }\{0,1\}^{\check{\lambda}}\text{ which is not scattered.}$$

Then there is a negligible F_σ set $G\subseteq\{0,1\}^\lambda$ such that

$$\Vdash_{\mathbb{P}_\kappa}\dot{K}\cap\tilde{G}\ne\emptyset$$

where $\tilde{G}$ is the $\mathbb{P}_\kappa$-name for an F_σ set in $\{0,1\}^{\check{\lambda}}$ corresponding to G as described in 551K.

proof (a) If $a\in\mathfrak{B}_\kappa^+$, $\dot{A}$ is a $\mathbb{P}_\kappa$-name such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{A}\text{ is an infinite subset of }\{0,1\}^{\check{\lambda}}$$

and $\epsilon>0$, then there is an open-and-closed subset H of $\{0,1\}^\lambda$ such that $\nu_\lambda H\le\epsilon$ and $\bar{\nu}_\kappa(a\cap[\![\dot{A}\cap\tilde{H}=\emptyset]\!])\le\epsilon$. **P** We may suppose that $\epsilon=2^{-k}$ for some $k\in\mathbb{N}$. Let $\langle\dot{y}_i\rangle_{i\in\mathbb{N}}$ be a sequence of $\mathbb{P}_\kappa$-names such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{y}_i\in\dot{A}\text{ and }\dot{y}_i\ne\dot{y}_j$$

whenever i, $j\in\mathbb{N}$ are distinct. Let $N\in\mathbb{N}$ be so large that $e^{-\epsilon N}<\frac{1}{2}\epsilon$. Then

$$a\Vdash_{\mathbb{P}_\kappa}\text{ there is a finite }J\subseteq\check{\lambda}\text{ such that }\dot{y}_i\restriction J\ne\dot{y}_j\restriction J\text{ whenever }i<j<\check{N},$$

that is,

$$\sup_{J\in[\lambda]^{<\omega}}[\![\dot{y}_i\restriction\check{J}\ne\dot{y}_j\restriction\check{J}\text{ whenever }i<j<\check{N}]\!]\supseteq a,$$

and there is a finite set $J\subseteq\lambda$ such that

$$\bar{\nu}_\kappa(a\setminus[\![\dot{y}_i\restriction\check{J}\ne\dot{y}_j\restriction\check{J}\text{ whenever }i<j<\check{N}]\!])\le\frac{1}{2}\epsilon;$$

enlarging J if necessary, we can suppose that $m=\#(J)$ is such that $m\ge k$ and $(1-\frac{N}{2^m})^{2^m\epsilon}\le\frac{1}{2}\epsilon$.

For each $i\in\mathbb{N}$, let $f_i:\{0,1\}^\kappa\to\{0,1\}^\lambda$ be a $(\mathrm{T}_\kappa,\mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that (in the language of 551Cc) $a\Vdash_{\mathbb{P}_\kappa}\dot{y}_i=\vec{f}_i$. Set

$$E=\{x:f_i(x)\restriction J\ne f_j(x)\restriction J\text{ whenever }i<j<N\};$$

then

$$E^\bullet=[\![\dot{y}_i\restriction\check{J}\ne\dot{y}_j\restriction\check{J}\text{ whenever }i<j<\check{N}]\!],$$

and $\bar{\nu}_\kappa(a\setminus E^\bullet)\le\frac{1}{2}\epsilon$.

Let L be a subset of $\{0,1\}^J$ obtained by a stochastic process in which we pick $2^{m-k}=2^m\epsilon$ points independently with the uniform distribution, and take L to be the set of these points. For any $x\in E$,

$$\Pr(f_i(x)\restriction J\notin L\ \forall\ i<N)=\Pr((L\cap\{f_i(x)\restriction J:i<N\})=\emptyset)=(1-\frac{N}{2^m})^{2^m\epsilon}\le\frac{1}{2}\epsilon.$$

By Fubini's theorem, there must be an $L \subseteq \{0,1\}^J$ such that $\#(L) \le 2^m\epsilon$ and

$$\nu_\kappa\{x : x \in E,\ f_i(x){\restriction}J \notin L\ \forall\, i < N\} \le \frac{1}{2}\epsilon\nu_\kappa E \le \frac{1}{2}\epsilon.$$

Set $H = \{y : y \in \{0,1\}^\lambda,\ y{\restriction}J \in L\}$ and $b = [\![\dot A \cap \tilde H = \emptyset]\!]$. Then H is open-and-closed, $\nu_\lambda H = 2^{-m}\#(L) \le \epsilon$ and

$$\begin{aligned} a \cap b &\subseteq a \cap [\![\dot y_i \notin \tilde H\ \forall\, i < \check N]\!] = a \cap \{x : f_i(x) \notin H\ \forall\, i < N\}^\bullet \\ &= a \cap \{x : f_i(x){\restriction}J \notin L \text{ for every } i < N\}^\bullet \\ &\subseteq (a \setminus E^\bullet) \cup \{x : x \in E,\ f_i(x){\restriction}J \notin L \text{ for every } i < N\}^\bullet \end{aligned}$$

has measure at most ϵ, as required. **Q**

(b) Because every non-scattered space has a non-empty closed subset with no isolated points, we may suppose that

$$\Vdash_{\mathbb{P}_\kappa} \dot K \text{ has no isolated points.}$$

For any $\epsilon > 0$ there is a compact negligible set $F \subseteq \{0,1\}^\lambda$ such that

$$\bar\nu_\kappa[\![\dot K \cap \tilde F \ne \emptyset]\!] \ge 1 - \epsilon.$$

P Choose $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ and $\langle \dot K_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 1$ and $\dot K_0 = \dot K$. Given that

$$a_n \Vdash_{\mathbb{P}_\kappa} \dot K_n \text{ is a non-empty compact set in } \{0,1\}^{\check\lambda} \text{ without isolated points}$$

and $\bar\nu_\kappa a_n > 1-\epsilon$, let $H_n \subseteq \{0,1\}^\lambda$ be an open-and-closed set of measure at most 2^{-n} such that $a_{n+1} = a_n \cap [\![\dot K_n \cap \tilde H_n \ne \emptyset]\!]$ has measure greater than $1 - \epsilon$. Now let $\dot K_{n+1}$ be a $\mathbb{P}_\kappa$-name such that $\Vdash_{\mathbb{P}_\kappa} \dot K_{n+1} = \dot K_n \cap \tilde H_n$. Because

$a_{n+1} \Vdash_{\mathbb{P}_\kappa} \dot K_n$ is a compact set without isolated points, $\tilde H_n$ is open-and-closed and $\dot K_n \cap \tilde H_n \ne \emptyset$, so $\dot K_{n+1}$ is a non-empty compact set without isolated points,

the induction continues.

At the end of the induction, set $F = \bigcap_{n\in\mathbb{N}} H_n$ and $a = \inf_{n\in\mathbb{N}} a_n$. Then $\bar\nu_\kappa a \ge 1 - \epsilon$ and $\Vdash_{\mathbb{P}_\kappa} \tilde F = \bigcap_{n\in\mathbb{N}} \tilde H_n$, so

$a \Vdash_{\mathbb{P}_\kappa} \tilde F \cap \dot K$ is the intersection of the non-increasing sequence $\langle \dot K_n\rangle_{n\in\mathbb{N}}$ of non-empty compact sets, so is not empty.

(I am passing over the trivial case $\epsilon \ge 1$, $a = 0$.) So

$$\bar\nu_\kappa[\![\tilde F \cap \dot K \ne \emptyset]\!] \ge \bar\nu_\kappa a \ge 1 - \epsilon.$$

Also, of course, $\nu_\kappa F = 0$, as required. **Q**

(c) Finally, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact negligible sets such that

$$\bar\nu_\kappa[\![\dot K \cap \tilde F_n \ne \emptyset]\!] \ge 1 - 2^{-n}$$

for every n, and set $G = \bigcup_{n\in\mathbb{N}} F_n$; this works.

553F Corollary Suppose that $\operatorname{cf}\mathcal{N}(\nu_\omega) = \omega_1$ and that $\kappa \ge \omega_2$ is a cardinal. Then

$\Vdash_{\mathbb{P}_\kappa} \omega_1$ is a precaliber of every measurable algebra but does not have Haydon's property.

proof By 523N,

$$\operatorname{cf}\mathcal{N}(\nu_{\omega_1}) = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\omega_1]^{\le\omega}) = \omega_1;$$

let $\langle H_\xi\rangle_{\xi<\omega_1}$ be a cofinal family in $\mathcal{N}(\nu_{\omega_1})$. Now 552Ga and 525K tell us that

$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\lambda) > \omega_1$ for every infinite cardinal λ, so ω_1 is a precaliber of every measurable algebra.

Next,

$$\Vdash_{\mathbb{P}_\kappa} \tilde H_\xi \in \mathcal{N}(\nu_{\omega_1}) \text{ for every } \xi < \omega_1$$

(551Kd), while

$$\Vdash_{\mathbb{P}_\kappa} \text{if } K \subseteq \{0,1\}^{\omega_1} \text{ is a non-scattered compact set then } K \text{ meets } \textstyle\bigcup_{\xi<\omega} \tilde H_\xi.$$

P Suppose that $a \in \mathfrak{B}_\kappa^+$ and $\dot K$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot K \subseteq \{0,1\}^{\omega_1} \text{ is a non-scattered compact set.}$$

If $a = 1$ set $\dot K' = \dot K$; otherwise let $\dot K'$ be a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot K' = \dot K, \quad 1 \setminus a \Vdash_{\mathbb{P}_\kappa} \dot K' = \{0,1\}^{\omega_1}.$$

By 553E, there is a negligible set $G \subseteq \{0,1\}^{\omega_1}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{K}' \cap \tilde{G} \neq \emptyset$. Now there is a $\xi < \omega_1$ such that $G \subseteq H_\xi$, so that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{K} \cap \tilde{H}_\xi \supseteq \dot{K}' \cap \tilde{G} \neq \emptyset. \textbf{ Q}$$

By 531N,

$$\Vdash_{\mathbb{P}_\kappa} \omega_1 \text{ does not have Haydon's property.}$$

553G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\langle e_n \rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ stochastically independent of each other and of $\mathfrak{B}$. Let $I \subseteq \mathfrak{A}$ be a finite set and $\mathfrak{B}_I$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup I$. Then for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\bar{\mu}(b \cap e_n) - \bar{\mu}b \cdot \bar{\mu}e_n| \le \epsilon\bar{\mu}b$ whenever $b \in \mathfrak{B}_I$ and $n \ge n_0$.

proof (a) The first step is to show that if $u \in L^1(\mathfrak{A}, \bar{\mu})$ then

for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\int_{b\cap e_n} u - \bar{\mu}e_n \cdot \int_b u| \le \epsilon \int |u|$ whenever $b \in \mathfrak{B}$ and $n \ge n_0$.

P Consider the set U of those $u \in L^1(\mathfrak{A}, \bar{\mu})$ for which this true. This is a linear subspace of $L^1(\mathfrak{A}, \bar{\mu})$. Also it is $\| \|_1$-closed, because if $\int |v - u| \le \frac{1}{3} \int |u|$ and $|\int_{b\cap e_n} v - \bar{\mu}e_n \cdot \int_b v| \le \frac{1}{4}\epsilon \int |v|$ then $|\int_{b\cap e_n} u - \bar{\mu}e_n \cdot \int_b u| \le \epsilon \int |u|$. If we take $\mathfrak{D}_m$ to be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{e_n : n \le m\}$, then $\bar{\mu}(a \cap b \cap e_n) = \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n$ whenever $a \in \mathfrak{D}_m$, $b \in \mathfrak{B}$ and $n \ge m$, so $\chi a \in U$ for every $a \in \mathfrak{D}_m$. Consequently $\chi a \in U$ for every $a \in \mathfrak{D}$, where $\mathfrak{D}$ is the metric closure of $\bigcup_{m\in\mathbb{N}} \mathfrak{D}_m$ in $\mathfrak{A}$. Identifying $L^1(\mathfrak{D}, \bar{\mu}\restriction\mathfrak{D})$ with the closed linear subspace of $L^1(\mathfrak{A}, \bar{\mu})$ generated by $\{\chi a : a \in \mathfrak{D}\}$ (365R, 365F), we see that $U \supseteq L^1(\mathfrak{D}, \bar{\mu}\restriction\mathfrak{D})$. Now suppose that u is any member of $L^1(\mathfrak{A}, \bar{\mu})$. Then we have a conditional expectation Pu of u in $L^1(\mathfrak{D}, \bar{\mu}\restriction\mathfrak{D})$ (365R), and

$$|\int_{b\cap e_n} u - \bar{\mu}b \cdot \int_b u| = |\int_{b\cap e_n} Pu - \bar{\mu}e_n \cdot \int_b Pu|$$

for every $b \in \mathfrak{B}$ and $n \in \mathbb{N}$, while $|Pu| \le P|u|$, so $u \in U$ because $Pu \in U$. **Q**

(b) I show now, by induction on $\#(I)$, that if $a \in \mathfrak{A}$ then

for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\bar{\mu}(a \cap b \cap e_n) - \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n| \le \epsilon\mu a$ whenever $b \in \mathfrak{B}_I$ and $n \ge n_0$.

P If I is empty, we can apply (a) with $u = \chi a$. For the inductive step to $\#(I) = k + 1$, express I as $J \cup \{c\}$ where $\#(J) = k$. Take $a \in \mathfrak{A}$. Let $n_0 \in \mathbb{N}$ be such that

$$|\bar{\mu}((a \cap c) \cap b \cap e_n) - \bar{\mu}((a \cap c) \cap b) \cdot \bar{\mu}e_n| \le \epsilon\mu(a \cap c),$$

$$|\bar{\mu}((a \setminus c) \cap b \cap e_n) - \bar{\mu}((a \setminus c) \cap b) \cdot \bar{\mu}e_n| \le \epsilon\mu(a \setminus c)$$

whenever $b \in \mathfrak{B}_J$ and $n \ge n_0$. Now take $b \in \mathfrak{B}_I$ and $n \ge n_0$. There are b', $b'' \in \mathfrak{B}_J$ such that $b = (b' \cap c) \cup (b'' \setminus c)$, so that

$$\begin{aligned} |\bar{\mu}(a \cap b \cap e_n) - \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n| &= |\bar{\mu}((a \cap c) \cap b' \cap e_n) - \bar{\mu}((a \cap c) \cap b') \cdot \bar{\mu}e_n \\ &\qquad + \bar{\mu}((a \setminus c) \cap b'' \cap e_n) - \bar{\mu}((a \setminus c) \cap b'') \cdot \bar{\mu}e_n| \\ &\le |\bar{\mu}((a \cap c) \cap b' \cap e_n) - \bar{\mu}((a \cap c) \cap b') \cdot \bar{\mu}e_n| \\ &\qquad + |\bar{\mu}((a \setminus c) \cap b'' \cap e_n) - \bar{\mu}((a \setminus c) \cap b'') \cdot \bar{\mu}e_n| \\ &\le \epsilon\bar{\mu}(a \cap c) + \epsilon\bar{\mu}(a \setminus c) = \epsilon\bar{\mu}a. \end{aligned}$$

Thus the induction proceeds. **Q**

(c) Now the result as stated is just the case $a = 1$ in (b).

553H Theorem If $\kappa > \mathfrak{c}$, then

$$\Vdash_{\mathbb{P}_\kappa} \text{there are no rapid } p\text{-point ultrafilters, therefore no Ramsey filters on } \mathbb{N}.$$

proof (See Jech 78, §38.)

(a) Let $\langle e_{\xi n} \rangle_{\xi<\kappa, n\in\mathbb{N}}$ be a re-indexing of the standard generating family in $\mathfrak{B}_\kappa$. Let $\dot{\mathcal{F}}$ be a $\mathbb{P}_\kappa$-name for an ultrafilter, and set $\hat{a} = [\![\dot{\mathcal{F}} \text{ is a rapid } p\text{-point ultrafilter}]\!]$. **?** Suppose, if possible, that $\hat{a} \neq 0$. For each $f \in \mathbb{N}^{\mathbb{N}}$,

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \text{there is a } D \in \dot{\mathcal{F}} \text{ such that } \#(D \cap \check{f}(k)) \le k \text{ for every } k$$

(538Ad); let $\dot{D}_f$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{D}_f \in \dot{\mathcal{F}}$$

and

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \#(\dot{D}_f \cap f(k)\check{\ }) \leq \check{k}$$

for every $k \in \mathbb{N}$. Let $J \subseteq \kappa$ be a set of size at most $\mathfrak{c}$ such that $[\![\check{n} \in \dot{D}_f]\!]$ belongs to the closed subalgebra $\mathfrak{C}$ generated by $\{e_{\xi i} : \xi \in J,\ i \in \mathbb{N}\}$ for every $f \in \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$, and $\hat{a}$ also belongs to $\mathfrak{C}$.

(b) Let $\zeta < \kappa$ be such that the ordinal sum $\zeta + k$ does not belong to J for any $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\dot{C}_k$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that $[\![\check{n} \in \dot{C}_k]\!] = e_{\zeta+k,n}$ for every $n \in \mathbb{N}$. Set $c_k = [\![\dot{C}_k \notin \dot{\mathcal{F}}]\!]$ and let $\dot{A}_k$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that

$$c_k \Vdash_{\mathbb{P}_\kappa} \dot{A}_k = \mathbb{N} \setminus \dot{C}_k \in \dot{\mathcal{F}}, \quad 1 \setminus c_k \Vdash_{\mathbb{P}_\kappa} \dot{A}_k = \dot{C}_k \in \dot{\mathcal{F}}.$$

Then $\Vdash_{\mathbb{P}_\kappa} \dot{A}_k \in \dot{\mathcal{F}}$ for every k, and $[\![\check{n} \in \dot{A}_k]\!] = c_k \triangle e_{\zeta+k,n}$ for every $n \in \mathbb{N}$.

(c) For k, $n \in \mathbb{N}$ set

$$b_{kn} = [\![\check{n} \in \bigcap_{i<k} \dot{A}_i]\!] = \inf_{i<k} [\![\check{n} \in \dot{A}_i]\!] = \inf_{i<k} c_i \triangle e_{\zeta+i,n}.$$

Then we have a non-decreasing $f : \mathbb{N} \to \mathbb{N}$ such that $\bar{\nu}_\kappa(c \cap b_{kn}) \leq (2^{-k+1} - 2^{-2k})\bar{\nu}_\kappa c$ whenever $c \in \mathfrak{C}$, $k \in \mathbb{N}$ and $n \geq f(k)$. **P** Define f inductively, as follows. If $k = 0$ then (interpreting $\inf \emptyset$ as 1) we have $b_{kn} = 1$ for every n so we can take $f(0) = 0$. For the inductive step to $k+1$, let $\mathfrak{C}_k$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\mathfrak{C} \cup \{e_{\zeta+i,n} : i < k,\ n \in \mathbb{N}\}$ and $\mathfrak{D}_k$ the subalgebra generated by $\mathfrak{C}_k \cup \{c_i : i \leq k\}$. Then $\mathfrak{C}_k$ and $\langle e_{\zeta+k,n}\rangle_{n\in\mathbb{N}}$ are stochastically independent, so Lemma 553G tells us that there is an $f(k+1) \geq f(k)$ such that

$$|\bar{\nu}_\kappa(d \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa d| \leq \frac{1}{24} \cdot 2^{-k}\bar{\nu}_\kappa d \text{ whenever } d \in \mathfrak{D}_k \text{ and } n \geq f(k+1).$$

Take $n \geq f(k+1)$ and $c \in \mathfrak{C}$. Then

$$\begin{aligned}
\bar{\nu}_\kappa(c \cap b_{k+1,n}) &= \bar{\nu}_\kappa(c \cap b_{kn} \cap (c_k \triangle e_{\zeta+k,n})) \\
&= \bar{\nu}_\kappa(c \cap b_{kn} \cap c_k) - 2\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k \cap e_{\zeta+k,n}) + \bar{\nu}_\kappa(c \cap b_{kn} \cap e_{\zeta+k,n}) \\
&\leq 2|\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k)| \\
&\qquad + |\bar{\nu}_\kappa(c \cap b_{kn} \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn})| + \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn}) \\
&\leq \frac{1}{12} \cdot 2^{-k}\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k) + \frac{1}{24} \cdot 2^{-k}\bar{\nu}_\kappa(c \cap b_{kn}) + \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn})
\end{aligned}$$

(because all the elements $c \cap b_{kn}$ and $c \cap b_{kn} \cap c_k$ belong to $\mathfrak{D}_k$)

$$\begin{aligned}
&\leq (2^{-k-3} + \frac{1}{2})\bar{\nu}_\kappa(c \cap b_{kn}) \\
&\leq (2^{-k-3} + \frac{1}{2})(2^{-k+1} - 2^{-2k})\bar{\nu}_\kappa c
\end{aligned}$$

(because $n \geq f(k)$)

$$\leq (2^{-k} - 2^{-2k-2})\bar{\nu}_\kappa c.$$

So the construction proceeds. **Q**

(d) Because

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{F}} \text{ is a } p\text{-point ultrafilter and } \dot{A}_k \in \dot{\mathcal{F}} \text{ for every } k,$$

there are a $\mathbb{P}_\kappa$-name $\dot{A}$ for a subset of $\mathbb{N}$ and a $\mathbb{P}_\kappa$-name $\dot{g}$ for a function from $\mathbb{N}$ to itself such that

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{A} \in \dot{\mathcal{F}} \text{ and } \dot{A} \setminus \dot{A}_i \subseteq \dot{g}(k) \text{ whenever } i < k \in \mathbb{N}.$$

Let g (in the ordinary universe) be a non-decreasing function such that $f(k) \leq g(k)$ and $\bar{\nu}_\kappa(\hat{a} \cap [\![\dot{g}(k) > g(k)]\!]) \leq 2^{-k-2}\bar{\nu}_\kappa\hat{a}$ for every k. Set $\hat{a}_1 = \hat{a} \cap [\![\dot{g} \leq \check{g}]\!]$; then $\bar{\nu}_\kappa\hat{a}_1 \geq \frac{1}{2}\bar{\nu}_\kappa\hat{a}$.

(e) Take the function g from (d) and the name $\dot{D}_g$ from (a), and set $d_n = \hat{a} \cap [\![\check{n} \in \dot{D}_g]\!] \in \mathfrak{C}$ for every n. Then

$$\sum_{n=g(k)}^{g(k+1)-1} \bar{\nu}_\kappa(d_n \cap b_{kn}) \leq 2^{-k+1}(k+1)$$

for every $k \in \mathbb{N}$. **P** Set $J = g(k+1) \setminus g(k)$. We have

$$\sum_{n=g(k)}^{g(k+1)-1} \bar{\nu}_\kappa(d_n \cap b_{kn}) = \sum_{n=g(k)}^{g(k+1)-1} \sum_{I\subseteq J} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!])$$

$$= \sum_{I\subseteq J}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!])$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!] \subseteq [\![\check{n} \in \dot{D}_g]\!] \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!]$ is zero if $n \notin I$)

$$= \sum_{I\in[J]^{\le k+1}}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!])$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap J]\!] \subseteq \hat{a} \cap [\![\check{I} \subseteq \dot{D}_g \cap g(k+1)\check{}\,]\!]$ is zero if $\#(I) > k+1$)

$$\le 2^{-k+1} \sum_{I\in[J]^{\le k+1}}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!])$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap \check{J}]\!] \in \mathfrak{C}$ for every n and I, and we are looking only at $n \ge g(k) \ge f(k)$)

$$\le 2^{-k+1}(k+1) \sum_{I\in[J]^{\le k+1}} \bar{\nu}_\kappa[\![\check{I} = \dot{D}_g \cap \check{J}]\!]$$

$$\le 2^{-k+1}(k+1). \text{ }\mathbf{Q}$$

(f) As $\hat{a} \ne 0$, $\hat{a}_1 \ne 0$. Let m be such that $\sum_{k=m}^{\infty} 2^{-k+1}(k+1)$ is less than $\bar{\nu}_\kappa \hat{a}_1$; then

$$\hat{a}_2 = \hat{a}_1 \setminus \sup_{k\ge m} \sup_{g(k)\le n<g(k+1)} (d_n \cap b_{kn})$$

is non-zero. Let $\dot{B}$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{B} = \dot{A} \cap \dot{D}_g \setminus g(m)\check{}$. Then $\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{B} \in \dot{\mathcal{F}}$. But $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \dot{B} = \emptyset$. $\mathbf{P}$ Take any $n \in \mathbb{N}$. If $n < g(m)$ then $\Vdash_{\mathbb{P}_\kappa} n \notin \dot{B}$. If $k \ge m$ and $g(k) \le n < g(k+1)$, then

$$\hat{a}_1 \Vdash_{\mathbb{P}_\kappa} \dot{g}(k) \le g(k)\check{}, \quad \hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{A} \setminus \dot{A}_i \subseteq \dot{g}(k) \text{ for every } i < k,$$

so

$$\hat{a}_1 \cap [\![\check{n} \in \dot{B}]\!] \subseteq \hat{a} \cap [\![\check{n} \in \dot{A} \setminus \dot{g}(k)]\!] \subseteq [\![\check{n} \in \textstyle\bigcap_{i<k} \dot{A}_i]\!] = b_{kn}.$$

Also, of course, $\Vdash_{\mathbb{P}_\kappa} \dot{B} \subseteq \dot{D}_g$, so $\hat{a}_1 \cap [\![\check{n} \in \dot{B}]\!] \subseteq d_n \cap b_{kn}$ is disjoint from $\hat{a}_2$. But this means that $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \check{n} \notin \dot{B}$. As n is arbitrary, $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \dot{B} = \emptyset$. $\mathbf{Q}$ Now

$$\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \emptyset \in \dot{\mathcal{F}},$$

which is impossible. $\mathbf{X}$

(g) So $\hat{a} = 0$, that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{F}} \text{ is not a rapid } p\text{-point ultrafilter.}$$

As $\dot{\mathcal{F}}$ is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \text{there are no rapid } p\text{-point ultrafilters.}$$

(h) Finally, by 538Fa,

$$\Vdash_{\mathbb{P}_\kappa} \text{there are no Ramsey ultrafilters.}$$

553I Lemma Suppose that $S \subseteq \omega_1^2$ is a set, and let P be the set

$$\{I : I \in [\omega_1]^{<\omega},\ I \cap \xi \subseteq S[\{\xi\}] \text{ for every } \xi \in I\},$$

ordered by $\subseteq$. Suppose that whenever $n \in \mathbb{N}$ and $\langle I_\xi\rangle_{\xi<\omega_1}$ is a family in $[\omega_1]^n$ such that $I_\xi \cap \xi = \emptyset$ for every $\xi < \omega_1$, then there are $\xi < \omega_1$, $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq S$. Then P is upwards-ccc.

proof Let $\langle J_\xi\rangle_{\xi<\kappa}$ be any family in P. Then there are distinct ξ, $\eta < \omega_1$ such that $J_\xi \cup J_\eta \in P$. $\mathbf{P}$ By the Δ-system Lemma (4A1Db), there is an uncountable set $A_0 \subseteq \omega_1$ such that $\langle J_\xi\rangle_{\xi\in A_0}$ is a Δ-system with root J say; next, there is an $n \in \mathbb{N}$ such that $A_1 = \{\xi : \xi \in A_0,\ \#(J_\xi \setminus J) = n\}$ is uncountable. If $n = 0$ then $J_\xi \cup J_\eta = J$ belongs to P for any ξ, $\eta \in A_1$ and we can stop. Otherwise, there is an uncountable $A_2 \subseteq A_1$ such that whenever ξ, $\eta \in A_2$ and $\eta < \xi$ then $\max J_\eta < \min(J_\xi \setminus J)$. Re-enumerate $\langle J_\xi \setminus J\rangle_{\xi\in A_2}$ in increasing order to get a family $\langle I_\xi\rangle_{\xi<\omega_1}$ in $[\omega_1]^n$ such that $\min I_\xi \ge \xi$ for every ξ. Our hypothesis tells us that there are $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq S$. Let ξ', $\eta' < \omega_1$ be such that $I_\xi = J_{\xi'} \setminus J$ and $I_\eta = J_{\eta'} \setminus J$, and consider $I = J \cup I_\xi \cup I_\eta$. If $\alpha \in I$ and $\beta \in I \cap \alpha$,

— *either* α, β both belong to $J_{\eta'}$ so $(\alpha,\beta)\in S$

— *or* α, β both belong to $J_{\xi'}$ so $(\alpha,\beta)\in S$

— *or* $\alpha\in I_\xi$ and $\beta\in I_\eta$ so $(\alpha,\beta)\in S$.

So $J_{\xi'}\cup J_{\eta'}=I$ belongs to S. **Q**

Thus P has no uncountable up-antichains and is upwards-ccc.

553J Theorem Let κ be an infinite cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}$ there are two upwards-ccc partially ordered sets whose product is not upwards-ccc.

Remark If $\kappa>\omega$ this is immediate from 552E, 537F and 537G. So we have a new result only if $\kappa=\omega$.

proof (a) Let $\langle e_\xi\rangle_{\xi\in\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. For $J\subseteq\kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\xi:\xi\in J\}$. For $\xi<\omega_1$ let $h_\xi:\xi\to\mathbb{N}$ be an injective function.

(b) Let $\dot S_0$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1^2 such that

$$\begin{aligned}[\![(\check\xi,\check\eta)\in\dot S_0]\!]&=e_{h_\xi(\eta)}\text{ if }\eta<\xi,\\&=0\text{ otherwise.}\end{aligned}$$

Then

$\Vdash_{\mathbb{P}_\kappa}$ whenever $n\in\mathbb{N}$ and $\langle I_\xi\rangle_{\xi<\omega_1}$ is a family in $[\omega_1]^n$ such that $I_\xi\cap\xi=\emptyset$ for every $\xi<\omega_1$, there are $\xi<\omega_1$ and $\eta<\xi$ such that $I_\xi\times I_\eta\subseteq\dot S_0$.

P? Suppose, if possible, otherwise. Then we have an $n\in\mathbb{N}$, an $a\in\mathfrak{B}_\kappa^+$ and a family $\langle\dot I_\xi\rangle_{\xi<\omega_1}$ of $\mathbb{P}_\kappa$-names such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot I_\eta\in[\omega_1\setminus\eta]^{\check n}\text{ and }\dot I_\xi\times\dot I_\eta\not\subseteq\dot S_0\text{ whenever }\eta<\xi<\omega_1.$$

For each $\xi<\omega_1$ there are $a_\xi\in\mathfrak{B}_\kappa^+$, stronger than a, and $I_\xi\in[\omega_1\setminus\xi]^n$ such that $a_\xi\Vdash_{\mathbb{P}_\kappa}\dot I_\xi=\check I_\xi$. By 525Uc we can find an uncountable set $A_0\subseteq\omega_1$ and an $\epsilon>0$ such that $\bar\nu_\kappa(a_\xi\cap a_\eta)\ge\epsilon$ whenever ξ, $\eta\in A_0$. Next, there is an uncountable set $A_1\subseteq A_0$ such that $I_\eta\subseteq\xi$ whenever $\xi\in A_1$ and $\eta\in\xi\cap A_1$; consequently $I_\eta\cap I_\xi=\emptyset$ whenever ξ, $\eta\in A_1$ are distinct, and $\bar\nu_\kappa[\![\check I_\xi\times\check I_\eta\subseteq\dot S_0]\!]=2^{-n^2}$.

Let $\delta>0$ be such that $2^{-n^2}(\epsilon-2\delta)-2\delta>0$. For each $\xi\in A_1$ we can find a finite set $J_\xi\subseteq\kappa$ and an $a'_\xi\in\mathfrak{C}_{J_\xi}$ such that $\bar\nu_\kappa(a_\xi\mathbin{\triangle}a'_\xi)\le\delta$. Let $m\in\mathbb{N}$ be such that

$$A_2=\{\xi:\xi\in A_1,\,J_\xi\cap\omega\subseteq m\}$$

is uncountable. Let $\xi\in A_2$ be such that $A_2\cap\xi$ is infinite. In this case, $\langle h_\zeta[I_\eta]\rangle_{\eta\in A_2\cap\xi}$ is disjoint for each $\zeta\in I_\xi$, so we have an $\eta\in A_2\cap\xi$ such that $h_\zeta[I_\eta]\cap m=\emptyset$ for every $\zeta\in I_\xi$. In this case, $a'_\xi\cap a'_\eta\in\mathfrak{C}_{m\cup(\kappa\setminus\omega)}$ while $[\![\check I_\xi\times\check I_\eta\subseteq\dot S_0]\!]\in\mathfrak{C}_{\omega\setminus m}$, so

$$\begin{aligned}\bar\nu_\kappa(a'_\xi\cap a'_\eta\cap[\![\check I_\xi\times\check I_\eta\subseteq\dot S_0]\!])&=\bar\nu_\kappa(a'_\xi\cap a'_\eta)\cdot\bar\nu_\kappa[\![\check I_\xi\times\check I_\eta\subseteq\dot S_0]\!]\\&=2^{-n^2}\bar\nu_\kappa(a'_\xi\cap a'_\eta).\end{aligned}$$

So if we set $b=a_\xi\cap a_\eta\cap[\![\check I_\xi\times\check I_\eta\subseteq\dot S_0]\!]$,

$$\begin{aligned}\bar\nu_\kappa b&\ge2^{-n^2}\bar\nu_\kappa(a'_\xi\cap a'_\eta)-2\delta\ge2^{-n^2}(\bar\nu_\kappa(a_\xi\cap a_\eta)-2\delta)-2\delta\\&\ge2^{-n^2}(\epsilon-2\delta)-2\delta>0.\end{aligned}$$

But now we have $b\subseteq a$ and

$$b\Vdash_{\mathbb{P}_\kappa}\dot I_\xi\times\dot I_\eta=\check I_\xi\times\check I_\eta\subseteq\dot S_0,$$

which is supposed to be impossible. **XQ**

(c) Let $\dot P_0$ be a $\mathbb{P}_\kappa$-name for a partially ordered set defined from $\dot S_0$ by the process of 553I, so that for a finite set $I\subseteq\omega_1$

$$[\![\check I\in\dot P_0]\!]=\inf_{\xi,\eta\in I,\eta<\xi}e_{h_\xi(\eta)}.$$

By 553I and (b) above,

$$\Vdash_{\mathbb{P}_\kappa}\dot P_0\text{ is upwards-ccc.}$$

(d) Similarly, if $\dot{S}_1$ is a $\mathbb{P}_\kappa$-name for a subset of ω_1^2 such that

$$\begin{aligned}[\![(\check{\xi},\check{\eta}) \in \dot{S}_1]\!] &= 1 \setminus e_{h_\xi(\eta)} \text{ if } \eta < \xi, \\ &= 0 \text{ otherwise,}\end{aligned}$$

and $\dot{P}_1$ is a $\mathbb{P}_\kappa$-name for a partially ordered set defined from $\dot{S}_1$ by the process of 553I, then

$$\Vdash_{\mathbb{P}_\kappa} \dot{P}_1 \text{ is upwards-ccc.}$$

(The point is just that $\langle 1 \setminus e_\xi\rangle_{\xi<\kappa}$ also is a stochastically independent family of elements of measure $\frac{1}{2}$ which τ-generates $\mathfrak{B}_\kappa$.) But now observe that if $\eta < \xi < \omega_2$ then

$$[\![\{\check{\xi},\check{\eta}\} \in \dot{P}_0 \cap \dot{P}_1]\!] = [\![(\check{\xi},\check{\eta}) \in \dot{S}_0 \cap \dot{S}_1]\!] = e_{h_\xi(\eta)} \cap (1 \setminus e_{h_\xi(\eta)}) = 0.$$

So

$$\Vdash_{\mathbb{P}_\kappa} \{\{\{\xi\},\{\xi\}\} : \xi < \omega_1\} \text{ is an up-antichain in } \dot{P}_0 \times \dot{P}_1, \text{ and } \dot{P}_0 \times \dot{P}_1 \text{ is not upwards-ccc.}$$

Thus we have the required example.

553K I extract an elementary step from the proof of the next lemma.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a non-negative additive functional. Then

$$\sum_{i=0}^n \nu a_i \le \nu(\sup_{i\le n} a_i) + \sum_{i<j\le n} \nu(a_i \cap a_j)$$

whenever $a_0, \dots , a_n \in \mathfrak{A}$.

proof Let d be any atom of the subalgebra of $\mathfrak{A}$ generated by $a_0, \dots , a_n$. Suppose that $\#(\{i : i \le n,\ d \subseteq a_i\}) = m$. Then

$$\begin{aligned}\nu(d \cap \sup_{i\le n} a_i) + \sum_{i<j\le n} \nu(d \cap a_i \cap a_j) - \sum_{i=0}^n \nu(d \cap a_i)& \\ &= 0 \text{ if } m \le 1, \\ &= 1 + \tfrac{m(m-1)}{2} - m = \tfrac{1}{2}(m-1)(m-2) \ge 0 \text{ otherwise.}\end{aligned}$$

Summing over d, we have the result.

553L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, I an uncountable set, X a non-empty set and $\mathcal{F}$ an ultrafilter on X. Let $\langle a_{ix}\rangle_{i\in I,x\in X}$ be a family in $\mathfrak{A}$ such that $\inf_{i\in I} \lim_{x\to\mathcal{F}} \bar{\mu}a_{ix} > 0$. Then there are an uncountable set $S \subseteq I$ and a family $\langle b_i\rangle_{i\in S}$ in $\mathfrak{A} \setminus \{0\}$ such that

$$b_i \cap b_j \subseteq \sup_{x\in F} a_{ix} \cap a_{jx}$$

for all i, $j \in S$ and $F \in \mathcal{F}$.

proof (a) We can suppose that $I = \omega_1$. For each $\xi < \omega_1$ set $u_\xi = \lim_{x\to\mathcal{F}} \chi a_{\xi x}$, the limit being taken for the weak topology on $L^2(\mathfrak{A},\bar{\mu})$ (§366), so that

$$\int_a u_\xi = \lim_{x\to\mathcal{F}} \bar{\mu}(a \cap a_{\xi x})$$

for every $a \in \mathfrak{A}$. In particular, $\int u_\xi \ge \epsilon$, where $\epsilon = \inf_{\xi<\omega_1} \lim_{x\to\mathcal{F}} \bar{\mu}a_{\xi x} > 0$; set $b'_\xi = [\![u_\xi > \frac{1}{2}\epsilon]\!]$, so that $b'_\xi \ne 0$.

(b) For ξ, $\eta < \omega_1$ set

$$c_{\xi\eta} = \inf_{F\in\mathcal{F}} \sup_{x\in F} a_{\xi x} \cap a_{\eta x}.$$

For $K \subseteq \omega_1$ set

$$d_K = \inf_{\xi\in K} b'_\xi \setminus \sup_{\xi,\eta\in K \text{ are distinct}} c_{\xi\eta}.$$

If $d_K \ne 0$ then $\epsilon\#(K) < 3$. **P** We may suppose that K is finite and not empty; set $n = \#(K)$. We have $\int_{d_K} u_\xi > \frac{1}{2}\epsilon\bar{\mu}d_K$ for every $\xi \in K$, so

$$F_0 = \{x : x \in X,\ \bar{\mu}(d_K \cap a_{\xi x}) \ge \tfrac{1}{2}\epsilon\bar{\mu}d_K \text{ for every } \xi \in K\}$$

belongs to $\mathcal{F}$. Let $F \in \mathcal{F}$ be such that, setting $c'_{\xi\eta} = \sup_{x\in F} a_{\xi x} \cap a_{\eta x}$, $\bar{\mu}(c'_{\xi\eta} \setminus c_{\xi\eta}) \le \frac{\bar{\mu}d_K}{n^2}$ for all ξ, $\eta \in K$. Take any $x \in F \cap F_0$. If ξ, $\eta \in K$ are distinct,

$$\bar\mu(d_K\cap a_{\xi x}\cap a_{\eta x})\le\bar\mu(c'_{\xi\eta}\setminus c_{\xi\eta})\le\frac{\bar\mu d_K}{n^2},$$

so

$$\frac{n\epsilon}{2}\bar\mu d_K\le\sum_{\xi\in K}\bar\mu(d_K\cap a_{\xi x})\le\bar\mu d_K+\sum_{\xi,\eta\in K,\xi<\eta}\bar\mu(d_K\cap a_{\xi x}\cap a_{\eta x})$$

(553K)

$$\le\bar\mu d_K+\frac{n(n-1)}{2}\cdot\frac{\bar\mu d_K}{n^2}<\frac{3}{2}\bar\mu d_K$$

and $n\epsilon<3$. **Q**

(c) For each infinite $\xi<\omega_1$ there is therefore a maximal subset K_ξ of ξ such that $b_\xi=d_{K_\xi\cup\{\xi\}}$ is non-zero. Every K_ξ is finite, so there is a $K\in[\omega_1]^{<\omega}$ such that $S=\{\xi:\omega\le\xi<\omega_1,\ K_\xi=K\}$ is stationary. **P** By the Pressing-Down Lemma (4A1Cc), there is a $\zeta<\omega_1$ such that $\{\xi:\xi<\omega_1,\ \sup K_\xi=\zeta\}$ is stationary. As $[\zeta+1]^{<\omega}$ is countable, there will be a $K\subseteq\zeta+1$ such that $\{\xi:K_\xi=K\}$ is stationary. **Q** Now suppose that η, $\xi\in S$ and $\eta<\xi$. Then

$$b_\eta\cap b_\xi\setminus c_{\eta\xi}=d_{K_\eta\cup\{\eta\}}\cap d_{K_\xi\cup\{\xi\}}\setminus c_{\xi\eta}=d_{K\cup\{\eta,\xi\}}=0$$

because $K\cup\{\eta\}$ is a subset of ξ properly including K_ξ. So we have an appropriate family $\langle b_\xi\rangle_{\xi\in S}$.

553M Proposition (Laver 87) If $\mathfrak{m}>\omega_1$ and κ is any infinite cardinal, then

$$\Vdash_{\mathbb{P}_\kappa}\text{ every Aronszajn tree is special.}$$

proof (a) It is enough to show that

$$\Vdash_{\mathbb{P}_\kappa}\text{ every Aronszajn tree ordering of }\omega_1\text{ included in the usual ordering is special}$$

(5A1D(b-ii)). Let $\dot\preccurlyeq$ be a $\mathbb{P}_\kappa$-name for an Aronszajn tree ordering of ω_1 included in the usual ordering of ω_1. For α, $\beta<\omega_1$ set $a_{\alpha\beta}=[\![\check\alpha\dot\preccurlyeq\check\beta]\!]$; note that $a_{\alpha\alpha}=1$, $a_{\alpha\beta}=0$ if $\beta<\alpha$ and $a_{\alpha\beta}\supseteq a_{\alpha\gamma}\cap a_{\beta\gamma}$ whenever $\alpha\le\beta\le\gamma<\omega_1$.

If $\mathcal{F}$ is an ultrafilter on ω_1 containing $\omega_1\setminus\zeta$ for every $\zeta<\omega_1$, then $\lim_{\xi\to\mathcal{F}}\bar\nu_\kappa a_{\alpha\xi}=0$ for all but countably many $\alpha<\omega_1$. **P?** Otherwise, there is an $\epsilon>0$ such that $I=\{\alpha:\alpha<\omega_1,\ \lim_{\xi\to\mathcal{F}}\bar\mu a_{\alpha\xi}\ge\epsilon\}$ is uncountable. By 553L, there are an uncountable $S\subseteq I$ and a family $\langle b_\alpha\rangle_{\alpha\in S}$ in $\mathfrak{A}\setminus\{0\}$ such that

$$b_\alpha\cap b_\beta\subseteq\ \sup\nolimits_{\xi\ge\beta}a_{\alpha\xi}\cap a_{\beta\xi}\subseteq a_{\alpha\beta}$$

whenever α, $\beta\in S$ and $\alpha<\beta$. Set $c=\inf_{\alpha<\omega_1}\sup_{\beta\in S\setminus\alpha}b_\beta$, so that $c\ne0$. Let $\dot Y$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1 such that $[\![\check\alpha\in\dot Y]\!]=b_\alpha$ for $\alpha\in S$, $[\![\check\alpha\in\dot Y]\!]=0$ for other α. Then

$$\Vdash_{\mathbb{P}_\kappa}\alpha\ \dot\preccurlyeq\ \beta\text{ whenever }\alpha,\ \beta\in\dot Y\text{ and }\alpha<\beta,$$

$$c\Vdash_{\mathbb{P}_\kappa}\dot Y\text{ is uncountable;}$$

so

$$c\Vdash_{\mathbb{P}_\kappa}\dot Y\text{ is an uncountable branch in the Aronszajn tree,}$$

which is impossible. **XQ**

(b) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. Choose inductively a non-decreasing family $\langle J_\alpha\rangle_{\alpha<\omega_1}$ of countably infinite subsets of κ such that $a_{\beta\alpha}$ belongs to the closed subalgebra $\mathfrak{C}_{J_\alpha}$ of $\mathfrak{B}_\kappa$ generated by $\{e_\xi:\xi\in J_\alpha\}$ whenever $\beta\le\alpha<\omega_1$.

Let P be the partially ordered set of functions f such that

dom f is a finite subset of $\omega_1\times\omega$,

for every $(\alpha,n)\in\operatorname{dom}f$, $f(\alpha,n)\in\mathfrak{C}_{J_\alpha}$ and $\bar\nu_\kappa f(\alpha,n)>\frac{1}{2}$,

$f(\alpha,n)\cap f(\beta,n)\cap a_{\beta\alpha}=0$ whenever (α,n), $(\beta,n)\in\operatorname{dom}f$ and $\beta<\alpha$.

Say that $f\le g$ if $\operatorname{dom}f\subseteq\operatorname{dom}g$ and $g(\alpha,n)\subseteq f(\alpha,n)$ for every $(\alpha,n)\in\operatorname{dom}f$. Then $\le$ is a partial order on P.

P is upwards-ccc. **P** Let $\langle f_\xi\rangle_{\xi<\omega_1}$ be a family in P. Let $A_0\subseteq\omega_1$ be an uncountable set such that $\langle\operatorname{dom}f_\xi\rangle_{\xi\in A_0}$ is a Δ-system with root K say; let $\epsilon>0$, $m\in\mathbb{N}$ be such that

$$A_1=\{\xi:\xi\in A_0,\ \#(\operatorname{dom}f_\xi)=m+\#(K),$$
$$\bar\nu_\kappa f_\xi(\alpha,n)\ge\frac{1}{2}+2\epsilon\text{ whenever }(\alpha,n)\in\operatorname{dom}f_\xi\}$$

is uncountable. Let $A_2 \subseteq A_1$ be an uncountable set such that $\bar{\mu}(f_\eta(\alpha,n) \triangle f_\xi(\alpha,n)) \le \epsilon$ whenever ξ, $\eta \in A_2$ and $(\alpha,n) \in K$; such a set exists because $\mathfrak{C}_{J_\alpha}$ is metrically separable for each α. Let $A_3 \subseteq A_2$ be an uncountable set such that $\beta < \alpha$ whenever $\eta \in A_3$, $\xi \in A_3$, $\eta < \xi$, $(\beta,m) \in \operatorname{dom} f_\eta$ and $(\alpha,n) \in (\operatorname{dom} f_\xi) \setminus K$.

For $\xi \in A_3$, enumerate $(\operatorname{dom} f_\xi) \setminus K$ as $\langle(\alpha_{\xi i}, n_{\xi i})\rangle_{i<m}$. Let $\mathcal{F}$ be an ultrafilter on ω_1 containing $A_3 \setminus \zeta$ for every $\zeta < \omega_1$, and for $i < m$ let $\mathcal{F}_i$ be the ultrafilter $\{F : F \subseteq \omega_1,\ \{\xi : \alpha_{\xi i} \in F\} \in \mathcal{F}\}$. By (a), we have an uncountable $A_4 \subseteq A_3$ such that

$$\lim_{\xi\to\mathcal{F}_i} \bar{\nu}_\kappa a_{\alpha_{\eta j},\xi} < \frac{\epsilon}{m+1}$$

for every i, $j < m$ and every $\eta \in A_4$; that is,

$$\lim_{\xi\to\mathcal{F}} \bar{\nu}_\kappa a_{\alpha_{\eta j},\alpha_{\xi i}} < \frac{\epsilon}{m+1}$$

for every i, $j < m$ and $\eta \in A_4$. But this means that we can find $\eta \in A_4$ and $\xi \in A_3$ such that $\eta < \xi$ and $\bar{\nu}_\kappa a_{\alpha_{\eta j},\alpha_{\xi i}} \le \frac{\epsilon}{m+1}$ for all i, $j < m$. Now consider the function g with domain $\operatorname{dom} f_\eta \cup \operatorname{dom} f_\xi$ such that

$$\begin{aligned} g(\alpha,n) &= f_\eta(\alpha,n) \cap f_\xi(\alpha,n) \text{ if } (\alpha,n) \in K,\\ &= f_\eta(\alpha,n) \text{ if } (\alpha,n) \in \operatorname{dom} f_\eta \setminus K,\\ &= f_\xi(\alpha_{\xi i}, n_{\xi i}) \setminus \sup_{j<m} a_{\alpha_{\eta j},\alpha_{\xi i}} \text{ if } i < m \text{ and } (\alpha,n) = (\alpha_{\xi i}, n_{\xi i}). \end{aligned}$$

Then $g(\alpha,n) \in \mathfrak{C}_{J_\alpha}$ and $\bar{\nu}_\kappa g(\alpha,n) \ge \frac{1}{2} + \epsilon$ for every $(\alpha,n) \in \operatorname{dom} g$. If (α,n) and (β,n) belong to $\operatorname{dom} g$ and $\beta < \alpha$, then

— if both (β,n) and (α,n) belong to $\operatorname{dom} f_\eta$, then

$$g(\beta,n) \cap g(\alpha,n) \cap a_{\beta\alpha} \subseteq f_\eta(\beta,n) \cap f_\eta(\alpha,n) \cap a_{\beta\alpha} = 0;$$

— if both (β,n) and (α,n) belong to $\operatorname{dom} f_\xi$, then

$$g(\beta,n) \cap g(\alpha,n) \cap a_{\beta\alpha} \subseteq f_\xi(\beta,n) \cap f_\xi(\alpha,n) \cap a_{\beta\alpha} = 0;$$

— if $(\beta,n) = (\alpha_{\eta j}, n_{\eta j})$ and $(\alpha,n) = (\alpha_{\xi i}, n_{\xi i})$ then $g(\alpha,n)$ is disjoint from $a_{\alpha_{\eta j},\alpha_{\xi i}} = a_{\beta\alpha}$ so $g(\beta,n) \cap g(\alpha,n) \cap a_{\beta\alpha} = 0$.

So $g \in P$ and is an upper bound for f_η and f_ξ. Thus $\langle f_\xi\rangle_{\xi<\omega_1}$ is not an up-antichain in P; as $\langle f_\xi\rangle_{\xi<\omega_1}$ is arbitrary, P is upwards-ccc. **Q**

(c) For each $\alpha < \omega_1$ let C_α be a countable metrically dense subset of $\{c : c \in \mathfrak{C}_{J_\alpha},\ \bar{\nu}_\kappa c \le \frac{1}{2}\}$. For $\alpha < \omega_1$ and $c \in C_\alpha$, set

$$\begin{aligned} Q_{\alpha c} = \{f : f \in P &\text{ and there is some } n \in \mathbb{N} \text{ such that } (\alpha,n) \in \operatorname{dom} f,\\ &c \subseteq f(\alpha,n) \text{ and } \bar{\nu}_\kappa f(\alpha,n) = \frac{1}{2} + \frac{1}{3}\bar{\nu}_\kappa c\}. \end{aligned}$$

Then $Q_{\alpha c}$ is cofinal with P. **P** Because J_α is infinite, $\mathfrak{C}_{J_\alpha}$ is atomless and there is an $a \in \mathfrak{C}_{J_\alpha}$ such that $c \subseteq a$ and $\bar{\nu}_\kappa a = \frac{1}{2} + \frac{1}{3}\bar{\nu}_\kappa c$. Now take n so large that $i < n$ whenever $(\alpha,i) \in \operatorname{dom} f$, and set $g = f \cup \{((\alpha,n),a)\}$; then $f \le g \in Q_{\alpha c}$. **Q**

(d) Because $\mathfrak{m} > \omega_1$, there is an upwards-directed set $R \subseteq P$ meeting $Q_{\alpha c}$ whenever $\alpha < \omega_1$ and $c \in C_\alpha$. Now, for $n \in \mathbb{N}$, let $\dot{A}_n$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1 such that, for every $\alpha < \omega_1$,

$$\begin{aligned} [\![\check{\alpha} \in \dot{A}_n]\!] &= \inf\{f(\alpha,n) : f \in R,\ (\alpha,n) \in \operatorname{dom} f\} \text{ if } (\alpha,n) \in \bigcup_{f\in R} \operatorname{dom} f,\\ &= 0 \text{ otherwise} \end{aligned}$$

Then $\dot{A}_n$ is a name for an up-antichain for the tree order $\dot{\preccurlyeq}$. **P** If $\beta < \alpha < \omega_1$, then either $[\![\check{\beta} \in \dot{A}_n]\!] = 0$ or $[\![\check{\alpha} \in \dot{A}_n]\!] = 0$ or there are f, $g \in R$ such that $(\alpha,n) \in \operatorname{dom} f$ and $(\beta,n) \in \operatorname{dom} g$. In this case, because R is upwards-directed, there is an $h \in R$ such that both (α,n) and (β,n) belong to $\operatorname{dom} h$, so that

$$[\![\check{\alpha} \in \dot{A}_n]\!] \cap [\![\check{\beta} \in \dot{A}_n]\!] \cap [\![\check{\beta} \dot{\preccurlyeq} \check{\alpha}]\!] \subseteq h(\alpha,n) \cap h(\beta,n) \cap a_{\beta\alpha} = 0.$$

Thus

$$\Vdash_{\mathbb{P}_\kappa} \text{ if } \alpha,\ \beta \in \dot{A}_n \text{ then they are } \dot{\preccurlyeq}\text{-incompatible upwards.}$$

As α and β are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \dot{A}_n \text{ is an up-antichain. } \mathbf{Q}$$

(e) Finally,

$$\Vdash_{\mathbb{P}_\kappa} \bigcup_{n\in\mathbb{N}} \dot{A}_n = \omega_1.$$

P? Otherwise, there is an $\alpha < \omega_1$ such that $a = 1 \setminus \sup_{n\in\mathbb{N}} [\![\check{\alpha} \in \dot{A}_n]\!] \ne 0$. Observe at this point that $[\![\check{\alpha} \in \dot{A}_n]\!] \in \mathfrak{C}_{J_\alpha}$ for every n. So $a \in \mathfrak{C}_{J_\alpha}$. Let $a' \in \mathfrak{C}_{J_\alpha}$ be such that $a' \subseteq a$ and $0 < \bar{\nu}_\kappa a' \le \frac{1}{2}$, and let $c \in C_\alpha$ be such that $\bar{\nu}_\kappa(a' \mathbin{\triangle} c) \le \frac{1}{4}\bar{\nu}_\kappa a'$, so that $c \ne 0$ and $\bar{\nu}_\kappa(c \setminus a') \le \frac{1}{3}\bar{\nu}_\kappa c$. Since R meets $Q_{\alpha c}$, there are $n \in \mathbb{N}$, $f \in R$ such that $c \subseteq f(\alpha, n)$ and $\bar{\nu}_\kappa f(\alpha, n) = \frac{1}{2} + \frac{1}{3}\bar{\nu}_\kappa c$.

If $g \in P$ and $f \le g$, then $g(\alpha, n) \subseteq f(\alpha, n)$ and $\bar{\nu}_\kappa g(\alpha, n) > \frac{1}{2}$, so

$$\bar{\nu}_\kappa(c \setminus g(\alpha, n)) \le \bar{\nu}_\kappa f(\alpha, n) - \bar{\nu}_\kappa g(\alpha, n) \le \frac{1}{3}\bar{\nu}_\kappa c.$$

Because R is upwards-directed, $\{g(\alpha, n) : g \in R,\ (\alpha, n) \in \operatorname{dom} g\}$ is downwards-directed, and

$$\begin{aligned}\bar{\nu}_\kappa(c \setminus [\![\check{\alpha} \in \dot{A}_n]\!]) &= \sup\{\bar{\nu}_\kappa(c \setminus g(\alpha, n)) : g \in R,\ (\alpha, n) \in \operatorname{dom} g\}\\ &= \sup\{\bar{\nu}_\kappa(c \setminus g(\alpha, n)) : g \in R,\ f \le g\} \le \frac{1}{3}\bar{\nu}_\kappa c.\end{aligned}$$

Accordingly

$$\bar{\nu}_\kappa(a' \cap [\![\check{\alpha} \in \dot{A}_n]\!]) \ge \bar{\nu}_\kappa(c \cap [\![\check{\alpha} \in \dot{A}_n]\!]) - \bar{\nu}_\kappa(c \setminus a') \ge \frac{2}{3}\bar{\nu}_\kappa c - \frac{1}{3}\bar{\nu}_\kappa c > 0;$$

but $a' \subseteq a$ is supposed to be disjoint from $[\![\check{\alpha} \in \dot{A}_n]\!]$. **XQ**

So $\langle \dot{A}_n \rangle_{n\in\mathbb{N}}$ is a name for a sequence of antichains covering ω_1, and

$$\Vdash_{\mathbb{P}_\kappa} (\omega_1, \dot{\preccurlyeq}) \text{ is special.}$$

553N Proposition Suppose that there is a medial limit (definition: 538Q), and that κ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a medial limit.}$$

proof (a) Let $\theta : \mathcal{P}\mathbb{N} \to [0, 1]$ be a medial limit. Let Q be the rationally convex hull of the usual basis of ℓ^1, that is, the set of functions $v : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ such that $\{n : v(n) \ne 0\}$ is finite and $\sum_{n=0}^\infty v(n) = 1$. Note that Q is absolute in the sense that

$$\Vdash_{\mathbb{P}} \check{Q} \text{ is the rationally convex hull of the usual basis of } \ell^1$$

for every forcing notion $\mathbb{P}$. Let $\mathcal{F}$ be the filter on Q which is the trace of the weak* neighbourhood filter of θ, that is, the filter generated by sets of the form

$$\{v : v \in Q,\ |\textstyle\sum_{n=0}^\infty v(n)u(n) - \int u(n)\theta(dn)| \le \epsilon\}$$

where $u \in \ell^\infty$ and $\epsilon > 0$. (Identifying $Q \subseteq \ell^1$ with its image in $(\ell^\infty)^* \cong (\ell^1)^{**}$, the weak* closure of Q is convex, so is equal to its bipolar (4A4Eg) and is the set of positive linear functionals on ℓ^∞ taking the value 1 on the order unit $\chi\mathbb{N}$.) Let $\vec{\mathcal{F}}$ be the $\mathbb{P}_\kappa$-name derived from $\mathcal{F}$ and $(\{0, 1\}^\kappa, \mathrm{T}_\kappa, \mathcal{N}_\kappa)$ by the method of 551Rb, so that

$$\Vdash_{\mathbb{P}_\kappa} \vec{\mathcal{F}} \text{ is a filter on } \check{Q}.$$

Let $\dot{\nu}$ be a $\mathbb{P}_\kappa$-name such that

$\Vdash_{\mathbb{P}_\kappa} \dot{\nu}$ is a bounded additive functional on $\mathcal{P}\mathbb{N}$, and identifying $\check{Q}$ with a subset of $(\ell^\infty)^*$, itself identified with the space $M(\mathcal{P}\mathbb{N})$ of bounded additive functionals on $\mathcal{P}\mathbb{N}$, $\dot{\nu}$ is a cluster point of $\vec{\mathcal{F}}$ for the weak* topology.

(b) Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{\mathfrak{e}}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathfrak{e}} \text{ is a sequence of Borel subsets of } \{0, 1\}^{\mathbb{N}}.$$

By 551Fb, we have for each $n \in \mathbb{N}$ a set $W_n \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathfrak{e}}(\check{n}) = \vec{W}_n,$$

where $\vec{W}_n$ is defined as in 551D. Let λ be the product measure on $\{0, 1\}^\kappa \times \{0, 1\}^{\mathbb{N}}$. Because θ is a medial limit, $\iint \chi W_n(x, y)\theta(dn)\lambda(d(x, y))$ is defined and equal to $\int \lambda W_n \theta(dn)$; that is, there are a conegligible Baire set $W \subseteq \{0, 1\}^\kappa \times \{0, 1\}^{\mathbb{N}}$ and a Baire measurable function $\psi : \{0, 1\}^\kappa \times \{0, 1\}^{\mathbb{N}} \to [0, 1]$ such that

$$\psi(x,y)=\int\chi W_n(x,y)\theta(dn)=\lim_{v\to\mathcal{F}}\sum_{n=0}^{\infty}v(n)\chi W_n(x,y)$$

whenever $(x,y)\in W$, and

$$\int\psi\,d\lambda=\int\lambda W_n\theta(dn)=\lim_{v\to\mathcal{F}}\sum_{n=0}^{\infty}v(n)\lambda W_n.$$

Let $\vec{W}$ and $\vec{\psi}$ be the corresponding $\mathbb{P}_\kappa$-names, as in 551D and 551M, so that

$$\Vdash_{\mathbb{P}_\kappa}\vec{W}\in\mathcal{B}\mathfrak{a}_{\mathbb{N}}\text{ and }\vec{\psi}:\{0,1\}^{\mathbb{N}}\to\mathbb{R}\text{ is Baire measurable.}$$

Moreover, since ν_κ-almost every vertical section of W must be ν_ω-conegligible,

$$\Vdash_{\mathbb{P}_\kappa}\nu_\omega\vec{W}=1$$

(551I).

(c) Now suppose that $\dot{s}$ is a $\mathbb{P}_\kappa$-name and that $b\in\mathfrak{B}_\kappa^+$ is stronger than a and such that

$$b\Vdash_{\mathbb{P}_\kappa}\dot{s}\in\vec{W}.$$

(i) By 551Cc, there is a T_κ-measurable $f:\{0,1\}^\kappa\to\{0,1\}^{\mathbb{N}}$ such that

$$b\Vdash_{\mathbb{P}_\kappa}\dot{s}=\vec{f}.$$

Expressing b as $E^\bullet$ where $E\in\mathrm{T}_\kappa\setminus\mathcal{N}_\kappa$, $(x,f(x))\in W$ for ν_κ-almost every $x\in E$, by 551Ea.

For each $m\in\mathbb{N}$, consider

$$C_m=\{(x,v):x\in\{0,1\}^\kappa,\,v\in Q,\,(x,f(x))\in W,$$
$$|\psi(x,f(x))-\sum_{n=0}^{\infty}v(n)\chi W_n(x,f(x))|\le 2^{-m}\}.$$

Then $C_m\in\mathrm{T}_\kappa\widehat{\otimes}\mathcal{P}Q$; and if $x\in\{0,1\}^\kappa$ is such that $(x,f(x))\in W$, $C_m[\{x\}]\in\mathcal{F}$. Consequently

$$b\Vdash_{\mathbb{P}_\kappa}\vec{C}_m\in\vec{\mathcal{F}},$$

where in this formula $\vec{C}_m$ is the $\mathbb{P}_\kappa$-name defined by the method of 551Ra. At the same time,

$$b\Vdash_{\mathbb{P}_\kappa}|\vec{\psi}(\dot{s})-\sum_{n=0}^{\infty}v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})|\le 2^{-\check{m}}\text{ for every }v\in\vec{C}_m.$$

P Suppose we have a c stronger than b and a $\mathbb{P}_\kappa$-name $\dot{v}$ such that $c\Vdash_{\mathbb{P}_\kappa}\dot{v}\in\vec{C}_m$. Then there are a $G\in\mathrm{T}_\kappa\setminus\mathcal{N}_\kappa$ and a $v\in Q$ such that $G^\bullet\subseteq c$, $G^\bullet\Vdash_{\mathbb{P}_\kappa}\dot{v}=\check{v}$, and $(x,v)\in C_m$ for every $x\in G$. Setting

$$h(x)=\psi(x,f(x)),\quad h_n(x)=\chi W_n(x,f(x))$$

for $x\in\{0,1\}^\kappa$ and $n\in\mathbb{N}$, and interpreting $\vec{h}$, $\vec{h}_n$ as in 551B,

$$b\Vdash_{\mathbb{P}_\kappa}\vec{h}=\vec{\psi}(\vec{f})=\vec{\psi}(\dot{s}),$$

and

$$b\Vdash_{\mathbb{P}_\kappa}\vec{h}_n=(\chi W_n)^{\rightharpoonup}(\dot{s})=(\chi\vec{W}_n)(\dot{s})$$

(551Nd)

$$=(\chi\dot{\mathfrak{e}}(\check{n}))(\dot{s})$$

for $n\in\mathbb{N}$. For $x\in G$, moreover, $|h(x)-\sum_{n=0}^{\infty}v(n)h_n(x)|\le 2^{-m}$, so

$$G^\bullet\Vdash_{\mathbb{P}_\kappa}|\vec{\psi}(\dot{s})-\sum_{n=0}^{\infty}\dot{v}(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})|=|\vec{h}-\sum_{n=0}^{\infty}\check{v}(n)\vec{h}_n|\le 2^{-\check{m}}.$$

As c and $\dot{v}$ are arbitrary, we have the result. **Q**

(ii) As m is arbitrary,

$$b\Vdash_{\mathbb{P}_\kappa}\{v:v\in\check{Q},\,|\vec{\psi}(\dot{s})-\sum_{n=0}^{\infty}v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})|\le\epsilon\}\in\vec{\mathcal{F}}\text{ for every }\epsilon>0,$$

that is,

$$b\Vdash_{\mathbb{P}_\kappa}\vec{\psi}(\dot{s})=\lim_{v\to\vec{\mathcal{F}}}\sum_{n=0}^{\infty}v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s}).$$

As b and $\dot{s}$ are arbitrary,

$$a\Vdash_{\mathbb{P}_\kappa}\vec{\psi}(y)=\lim_{v\to\vec{\mathcal{F}}}\sum_{n=0}^{\infty}v(n)\chi(\dot{\mathfrak{e}}(n))(y)\text{ for every }y\in\vec{W};$$

since $\Vdash_{\mathbb{P}_\kappa} \vec{W}$ is conegligible,

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\psi} =_{\text{a.e.}} \lim_{v\to\vec{\mathcal{F}}} \sum_{n=0}^{\infty} v(n)\chi(\dot{\mathfrak{e}}(n)).$$

Looking back at the choice of $\dot{\nu}$, we see that

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\psi}(y) = \int \chi(\dot{\mathfrak{e}}(n))(y)\dot{\nu}(dn) \text{ for } \nu_\omega\text{-almost every } y.$$

(d) As for the integral of $\vec{\psi}$, 551Nf tells us that

$$\Vdash_{\mathbb{P}_\kappa} \int \vec{\psi}\, d\nu_\omega = \vec{h},$$

where I now set $h(x) = \int \psi(x,y)\nu_\omega(dy)$ for $x \in \{0,1\}^\kappa$. Similarly, setting $h_n(x) = \nu_\omega W_n[\{x\}]$, we have

$$a \Vdash_{\mathbb{P}_\kappa} \nu_\omega \dot{\mathfrak{e}}(\check{n}) = \vec{h}_n.$$

Set

$$H = \{x : x \in \{0,1\}^\kappa,\ W[\{x\}] \text{ is conegligible in } \{0,1\}^{\mathbb{N}}\};$$

then H is conegligible in $\{0,1\}^\kappa$. Now remember that θ is a medial limit. If $x \in H$ we have $\psi(x,y) = \int \chi W_n(x,y)\theta(dn)$ for every y in the conegligible set $W[\{x\}]$, so

$$\begin{aligned} h(x) &= \int \psi(x,y)\nu_\omega(dy) = \iint \chi W_n(x,y)\theta(dn)\nu_\omega(dy) \\ &= \iint \chi W_n(x,y)\nu_\omega(dy)\theta(dn) = \int \nu_\omega W_n[\{x\}]\theta(dn) = \lim_{v\to\mathcal{F}} \sum_{n=0}^{\infty} v(n)h_n(x). \end{aligned}$$

So if, for $m \in \mathbb{N}$, we set

$$C_m = \{(x,v) : x \in \{0,1\}^\kappa,\ v \in Q,\ |h(x) - \sum_{n=0}^{\infty} v(n)h_n(x)| \le 2^{-m}\},$$

we shall again have $\Vdash_{\mathbb{P}_\kappa} \vec{C}_m \in \vec{\mathcal{F}}$; and if $G \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$ and $v \in A$ are such that $G^\bullet$ is stronger than p and $G^\bullet \Vdash_{\mathbb{P}_\kappa} \check{v} \in \vec{C}_m$, then

$$G^\bullet \Vdash_{\mathbb{P}_\kappa} |\int \vec{\psi}\, d\nu_\omega - \sum_{n=0}^{\infty} \check{v}(n)\nu_\omega \dot{\mathfrak{e}}(n)| \le 2^{-\check{m}}.$$

So

$$a \Vdash_{\mathbb{P}_\kappa} \{v : |\int \vec{\psi}\, d\nu_\omega - \sum_{n=0}^{\infty} \check{v}(n)\nu_\omega \dot{\mathfrak{e}}(n)| \le 2^{-\check{m}}\} \in \vec{\mathcal{F}}$$

for every m, and

$$a \Vdash_{\mathbb{P}_\kappa} \int \vec{\psi}\, d\nu_\omega = \lim_{v\to\vec{\mathcal{F}}} \sum_{n=0}^{\infty} \check{v}(n)\nu_\omega \dot{\mathfrak{e}}(n) = \int \nu_\omega \dot{\mathfrak{e}}(n)\dot{\nu}(dn).$$

(e) As p and $\dot{\mathfrak{e}}$ are arbitrary, we see that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\nu} \text{ satisfies condition (iv) of 538P, so is a medial functional.}$$

It is now easy to check that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\nu} \ge 0,\ \dot{\nu}\mathbb{N} = 1 \text{ and } \dot{\nu}\{n\} = 0 \text{ for every } n \in \mathbb{N}, \text{ so } \dot{\nu} \text{ is a medial limit.}$$

This completes the proof.

553X Basic exercises (a)(i) Suppose that $A \subseteq \{0,1\}^{\mathbb{N}}$ has Rothberger's property, and that κ is a cardinal. Show that

$$\Vdash_{\mathbb{P}_\kappa} \check{A} \text{ has Rothberger's property in } \{0,1\}^{\mathbb{N}}.$$

(ii) Repeat with $\mathbb{R}$ in place of $\{0,1\}^{\mathbb{N}}$.

(b) Let $W \subseteq \{0,1\}^\omega \times \{0,1\}^\omega$ be the set

$$\{(x,y) : x(2n) = y(2n) \text{ for every } n \in \mathbb{N}\}.$$

Show that, for every $y \in \{0,1\}^\omega$,

$$\Vdash_{\mathbb{P}_\omega} \vec{W} \text{ is homeomorphic to } \{0,1\}^\omega \text{ and } \check{y} \notin \vec{W}.$$

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a non-negative additive functional. Show that if $\langle a_i\rangle_{i\in I}$ is a finite family in $\mathfrak{A}$ then

$$\nu(\sup_{i\in I} a_i) \le \sum_{k=1}^{m}(-1)^{k+1}\sum_{J\in[I]^k}\nu(\inf_{i\in J} a_i) \text{ if } m\ge 1 \text{ is odd,}$$

$$\nu(\sup_{i\in I} a_i) \ge \sum_{k=1}^{m}(-1)^{k+1}\sum_{J\in[I]^k}\nu(\inf_{i\in J} a_i) \text{ if } m\ge 1 \text{ is even.}$$

(d) Let $\mathbb{P}$ be a forcing notion which satisfies Knaster's condition. (i) Show that if $(P,\le)$ is an upwards-ccc partially ordered set then

$$\Vdash_{\mathbb{P}} (\check{P},\check{\le}) \text{ is upwards-ccc.}$$

(ii) Show that if $(T,\le)$ is a Souslin tree then

$$\Vdash_{\mathbb{P}} (\check{T},\check{\le}) \text{ is a Souslin tree.}$$

(e)(i) Suppose that $\mathcal{F}$ is a p-point filter on $\mathbb{N}$, and that $\mathbb{P}$ is a ccc forcing notion. Show that

$$\Vdash_{\mathbb{P}} \text{the filter on } \mathbb{N} \text{ generated by } \check{\mathcal{F}} \text{ is a } p\text{-point filter.}$$

(ii) Suppose that $\mathcal{F}$ is a rapid filter on $\mathbb{N}$, and that κ is a cardinal. Show that

$$\Vdash_{\mathbb{P}_\kappa} \text{the filter on } \mathbb{N} \text{ generated by } \check{\mathcal{F}} \text{ is a rapid filter.}$$

553Y Further exercises (a) Let κ be a cardinal, $\dot{G}$ a $\mathbb{P}_\kappa$-name and $a\in\mathfrak{B}_\kappa^+$ such that

$$a\Vdash_{\mathbb{P}_\kappa} \dot{G} \text{ is a dense open subset of } \{0,1\}^\omega.$$

Show that there is a $W\in\mathrm{T}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a dense open set and $a\Vdash_{\mathbb{P}_\kappa}\dot{G}=\vec{W}$.

(b) Let κ be a cardinal and $W\in\mathrm{T}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ a set such that every vertical section of W is a dense open set. Let C be the space of continuous functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$ with the compact-open topology (definition: 441Yh). Show that $\{f:f\in C,\ \{x:(x,f(x))\in W\}$ is conegligible$\}$ is comeager in C.

(c) Write $\mathcal{M}$ for the ideal of meager subsets of $\{0,1\}^\omega$. Show that

$$\Vdash_{\mathbb{P}_\omega} \operatorname{cov}\mathcal{M}\ge(\operatorname{cov}\mathcal{M})\check{}.$$

(*Hint*: work with the ideal of meager sets in the Polish space of continuous functions from $\{0,1\}^\omega$ to itself.)

(d) Let $\mathcal{K}$ be the family of compact well-ordered subsets of $\mathbb{Q}\cap[0,\infty[$ containing 0. For s, $t\in\mathcal{K}$ say that $s\preccurlyeq t$ if $s=t\cap[0,\gamma]$ for some $\gamma\in\mathbb{R}$; for $s\in\mathcal{K}$ and $\gamma\in\mathbb{Q}$, set $A(s,\gamma)=\{t:t\in\mathcal{K},\ \max t=\gamma,\ s\preccurlyeq t\}$. (i) Show that $(\mathcal{K},\preccurlyeq)$ is a tree, and that $\operatorname{otp}(t)=r(t)+1$ for every $t\in\mathcal{K}$. (ii) Choose $\langle\mathcal{K}_\xi\rangle_{\xi<\omega_1}$, $\langle T_\xi\rangle_{\xi<\omega_1}$ inductively so that $\mathcal{K}_0=T_0=\{\{0\}\}$ and for $0<\xi<\omega_1$

$\mathcal{K}_\xi=\{t:t\in\mathcal{K},\ r(t)=\xi,\ s\in\bigcup_{\eta<\xi}T_\eta$ whenever $s\prec t\}$,

$T_\xi\subseteq\mathcal{K}_\xi$ is countable,

if $\eta<\xi$, $s\in T_\eta$ and $\gamma\in\mathbb{Q}$ are such that $\gamma>\max s$ and $A(s,\gamma)$ meets $\mathcal{K}_\xi$, then $A(s,\gamma)$ meets T_ξ.

Show that if $\eta<\xi<\omega_1$, $s\in T_\eta$, $\gamma\in\mathbb{Q}$ and $\gamma>\max s$, there is a $t\in T_\xi$ such that $\max t=\gamma$ and $s\preccurlyeq t$. (iii) Show that $T=\bigcup_{\xi<\omega_1}T_\xi$ is a special Aronszajn tree.

(e) Show that $\Vdash_{\mathbb{P}_\omega}\mathfrak{p}\ge(\mathfrak{m}_{\sigma\text{-linked}})\check{}$.

553Z Problem Suppose that the generalized continuum hypothesis is true. Is it the case that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{there is a Borel lifting of Lebesgue measure?}$$

(Compare 554I.)

553 Notes and comments To my mind, the chief interest of the results of this section is that they force us to explore aspects of the structures considered in new ways. We know, for instance, that if a set has Rothberger's property (in a separable metrizable space) this can be witnessed by a family of $\mathfrak{d}$ sequences. The point of 553C is that (in random real models) any family of $\mathfrak{d}$ sequences is associated with a set $\dot{Y}$ of size at most the cardinal power $\mathfrak{d}^\omega=\mathfrak{c}$ (taken in the ordinary universe V), such that $\dot{Y}$ must include the given set with Rothberger's property. Remember that

$$\Vdash_{\mathbb{P}_\kappa} (\mathbb{N}^\mathbb{N})\check{} \text{ is cofinal with } \mathbb{N}^\mathbb{N}$$

(see the proof of 552C), so there is no point in looking at 'new' members of $\mathbb{N}^\mathbb{N}$ in part (a) of the proof.

In 553E, we need to distinguish between the $\mathbb{P}_\kappa$-names $\tilde{G}$ and $\check{G}$. It is quite possible to have

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \cap (\{0,1\}^\lambda)\check{} = \emptyset;$$

that is, we might have $\Vdash_{\mathbb{P}_\kappa} \dot{K} = \vec{W}$ where $W \subseteq \{0,1\}^\kappa \times \{0,1\}^\lambda$ has negligible horizontal sections (553Xb). The name $\tilde{G}$ refers not to a copy of the set G but to a re-interpretation of one (or any) of its descriptions as an F_σ set.

In 553H and 553M, we have to look quite deeply into the structure of measure algebras. Lemmas 553G and 553L are already not obvious, and the combinatorial measure theory of the proof of 553H is delicate. 553J is easier. The idea here is to 'randomize' a construction from Galvin 80, where the continuum hypothesis was used to build complementary sets S_0, S_1 with the property of 553I.

I give a bit of space to 'Aronszajn trees' because the results here express yet another contrast between random and Cohen forcing. Cohen forcing creates Souslin trees (554Yc). Random forcing preserves old Souslin trees (553Xd) but does not necessarily produce new ones (553M).

554 Cohen reals

Parallel to the theory of random reals as described in §§552-553, we have a corresponding theory based on category algebras rather than measure algebras. I start with the exactly matching result on cardinal arithmetic (554B), and continue with Lusin sets (balancing the Sierpiński sets of §552) and the cardinal functions of the meager ideal of $\mathbb{R}$ (554C-554E, 554F). In the last third of the section I use the theory of Freese-Nation numbers (§518) to prove Carlson's theorem on Borel liftings (554I).

554A Notation For any set I, I will write $\widehat{\mathcal{B}}_I$ for the Baire-property algebra of $\{0,1\}^I$, $\mathcal{B}\mathfrak{a}_I$ for the Baire σ-algebra of $\{0,1\}^I$, $\mathcal{M}_I$ for the meager ideal of $\{0,1\}^I$, $\mathfrak{G}_I = \widehat{\mathcal{B}}_I/\mathcal{M}_I$ for the category algebra of $\{0,1\}^I$, and $\mathbb{Q}_I$ for the forcing notion $\mathfrak{G}_I^+ = \mathfrak{G}_I \setminus \{\emptyset\}$ active downwards. $\mathcal{C}_I$ will be the family of basic cylinder sets $\{x : z \subseteq x \in \{0,1\}^I\}$ for $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$, and C_I the corresponding set $\{C^\bullet : C \in \mathcal{C}_I\} \subseteq \mathfrak{G}_I$; then C_I is order-dense in $\mathfrak{G}_I$ (because $\mathcal{C}_I$ is a π-base for the topology of $\{0,1\}^I$). It follows that $\tau(\mathfrak{G}_I) \le \pi(\mathfrak{G}_I) \le \max(\omega, \#(I))$. (These inequalities are of course equalities if I is infinite.)

554B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

proof (Compare 552B.)

(a) Since $\#(\mathfrak{G}_\kappa)$ is ccc and has an order-dense subset (the algebra of open-and-closed sets) of size κ, $\#(\mathfrak{G}_\kappa)$ is at most the cardinal power κ^ω.

If $\dot{A}$ is a $\mathbb{Q}_\kappa$-name for a subset of $\check{\lambda}$, then we have a corresponding family $\langle [\![\check{\eta} \in \dot{A}]\!]\rangle_{\eta<\lambda}$ of truth values; and if $\dot{A}$, $\dot{B}$ are two such names, and $[\![\check{\eta} \in \dot{A}]\!] = [\![\check{\eta} \in \dot{B}]\!]$ for every $\eta < \lambda$, then

$$\Vdash_{\mathbb{Q}_\kappa} \dot{A} = \dot{B}.$$

So

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = \#(\mathcal{P}\check{\lambda}) \le \#((\mathfrak{G}_\kappa^\lambda)\check{}) = (\kappa^\lambda)\check{}.$$

(b) Consider first the case in which $\lambda \le \kappa$. Let F be the set of all functions from λ to κ, so that $\#(F) = \kappa^\lambda$. As in part (b) of the proof of 552B, there is a set $G \subseteq F$ such that $\#(G) = \kappa^\lambda$ and $\{\eta : \eta < \lambda,\ f(\eta) \ne g(\eta)\}$ is infinite whenever f, $g \in G$ are distinct. Let $\langle \zeta_{\xi\eta}\rangle_{\xi<\kappa,\eta<\lambda}$ be a family of distinct elements of κ and set $E_{\xi\eta} = \{x : x \in \{0,1\}^\kappa$, $x(\zeta_{\xi\eta}) = 1\}$ for $\xi < \kappa$ and $\eta < \lambda$. For $f \in G$ let $\dot{A}_f$ be a $\mathbb{Q}_\kappa$-name for a subset of λ such that

$$[\![\check{\eta} \in \dot{A}_f]\!] = E^\bullet_{f(\eta),\eta}$$

for every $\eta < \lambda$. If f, $g \in G$ are distinct, set $I = \{\eta : f(\eta) \ne g(\eta)\}$; then

$$[\![\dot{A}_f \ne \dot{A}_g]\!] = \sup_{\eta<\lambda} E^\bullet_{f(\eta),\eta} \mathbin{\triangle} E^\bullet_{g(\eta),\eta} = 1$$

because $\bigcup_{\eta\in I} E_{f(\eta),\eta} \triangle E_{g(\eta),\eta}$ is a dense open set in $\{0,1\}^\kappa$.

Thus in the forcing language we have a name for an injective function from $\check{G}$ to $\mathcal{P}\lambda$, corresponding to the map $f \mapsto \dot{A}_f$ from G to names of subsets of λ. So

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} \ge \#(\check{G}) = (\kappa^\lambda)\check{}.$$

Putting this together with (a), we have

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

(c) If $\lambda > \kappa$, then $2^\lambda = \kappa^\lambda$. Now

$$\Vdash_{\mathbb{Q}_\kappa} (\mathcal{P}\lambda)\check{} \subseteq \mathcal{P}\check{\lambda},$$

so

$$\Vdash_{\mathbb{Q}_\kappa} (\kappa^\lambda)\check{} = \#((\mathcal{P}\lambda)\check{}) \le \#(\mathcal{P}\check{\lambda}) = 2^{\check{\lambda}},$$

and again we have

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

554C Definition If X is a topological space, a subset of X is a **Lusin set** if it is uncountable but meets every meager set in a countable set; equivalently, if it is uncountable but meets every nowhere dense set in a countable set.

554D Proposition Let κ be a cardinal such that $\mathbb{R}$ has a Lusin set of size κ.

(a) Writing $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$, $\operatorname{non}\mathcal{M} = \omega_1$ and $\mathfrak{m}_{\text{countable}} = \operatorname{cov}\mathcal{M} \ge \kappa$.

(b) There is a point-countable family $\mathcal{A}$ of Lebesgue-conegligible subsets of $\mathbb{R}$ with $\#(\mathcal{A}) = \kappa$.

(c) If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra which is not purely atomic, (κ, ω_1) is not a precaliber pair of $\mathfrak{A}$.

proof Let $B \subseteq \mathbb{R}$ be a Lusin set of size κ.

(a) By 522Ra, $\mathfrak{m}_{\text{countable}} = \operatorname{cov}\mathcal{M}$. Any uncountable subset of B is non-meager, so $\operatorname{non}\mathcal{M} = \omega_1$. If $\mathcal{E}$ is a cover of $\mathbb{R}$ by meager sets, then each member of $\mathcal{E}$ meets B in a countable set, so

$$\kappa = \#(B) \le \max(\omega, \#(\mathcal{E}))$$

and $\#(\mathcal{E}) \ge \kappa$; thus $\operatorname{cov}\mathcal{M} \ge \kappa$.

(b) Let $E \subseteq \mathbb{R}$ be a conegligible meager set containing 0, and set $\mathcal{A} = \{x + E : x \in B\}$. Then $\mathcal{A}$ is a family of conegligible sets. If $y \in \mathbb{R}$, then $y - E$ is meager so $\{x : x \in B,\ y \in x + E\} = B \cap (y - E)$ is countable; thus $\mathcal{A}$ is point-countable. In particular, each point of B belongs to countably many members of $\mathcal{A}$, and (because $B \subseteq \bigcup\mathcal{A}$)

$$\kappa = \#(B) \le \max(\omega, \#(\mathcal{A})) \le \#(\mathcal{A}) \le \#(B),$$

so $\#(\mathcal{A}) = \kappa$.

(c) Let $K \subseteq E$ be a compact set of non-zero measure. If $\Gamma \subseteq B$ is uncountable, $\bigcap_{x\in\Gamma} x + K = \emptyset$, $\{x + K : x \in \Gamma\}$ does not have the finite intersection property and $\{(x + K)^\bullet : x \in \Gamma\}$ is not centered in the measure algebra $\mathfrak{A}_L$ of Lebesgue measure. Thus $\langle (x + K)^\bullet \rangle_{x\in B}$ witnesses that (κ, ω_1) is not a precaliber pair of $\mathfrak{A}_L$.

Since $(\mathfrak{A}, \bar{\mu})$ is semi-finite and not purely atomic, there is a subalgebra of a principal ideal of $\mathfrak{A}$ which is isomorphic to $\mathfrak{A}_L$, and (κ, ω_1) is not a precaliber pair of $\mathfrak{A}$, by 516S.

554E Theorem Let κ be an uncountable cardinal. Then

$$\Vdash_{\mathbb{Q}_\kappa} \text{there is a Lusin set } A \subseteq \mathbb{R} \text{ of cardinal } \check{\kappa}.$$

proof (a) (Compare 552E.) Write $\mathbb{P}$ for $\mathbb{Q}_{\kappa\times\omega}$. For each $\xi < \kappa$, let $f_\xi : \{0,1\}^{\kappa\times\omega} \to \{0,1\}^\omega$ be given by setting $f_\xi(x)(n) = x(\xi, n)$ for every $x \in \{0,1\}^{\kappa\times\lambda}$ and $n < \omega$; then, taking $\vec{f_\xi}$ to be the $\mathbb{P}$-name defined by the process of 551Cb,

$$\Vdash_{\mathbb{P}} \vec{f_\xi} \in \{0,1\}^\omega.$$

If ξ, $\xi' < \kappa$ are distinct, then, by 551Cd,

$$\begin{aligned}[\![\vec{f_\xi} = \vec{f_{\xi'}}]\!] &= \{x : f_\xi(x) = f_{\xi'}(x)\}^\bullet \\ &= \{x : x(\xi, n) = x(\xi', n) \text{ for every } n\}^\bullet = 0\end{aligned}$$

because $\{x : x(\xi, n) = x(\xi', n) \text{ for every } n\}$ is closed and nowhere dense. So, taking $\dot{A}$ to be the $\Vdash_{\mathbb{P}}$-name $\{(\vec{f_\xi}, \mathbb{1}) : \xi < \kappa\}$, we have

$$\Vdash_{\mathbb{P}} \dot{A} \subseteq \{0,1\}^\omega \text{ has cardinal } \check{\kappa}.$$

(b) Now suppose that $\dot{W}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{W} \text{ is a nowhere dense zero set in } \{0,1\}^\omega.$$

By 551Fb there is a $W \in \widehat{\mathcal{B}}_{\kappa\times\omega} \widehat{\otimes} \mathcal{B}\mathfrak{a}_\omega$ such that, in the language of 551D, $\Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. Now W is meager in $\{0,1\}^{\kappa\times\omega}$. **P** For $z \in \mathrm{Fn}_{<\omega}(\omega; \{0,1\})$ set $V_z = \{(x, y) : x \in \{0,1\}^{\kappa\times\omega},\ z \subseteq y \in \{0,1\}^\omega\}$. By 551Ee,

$$\Vdash_{\mathbb{P}} \vec{V}_z = \{y : \check{z} \subseteq y \in \{0,1\}^\omega\};$$

and as $\widehat{\mathcal{B}}_{\kappa\times\omega}$ is closed under Souslin's operation (431F),

$$[\![\vec{W} \cap \vec{V}_z = \emptyset]\!] = \{x : W[\{x\}] \cap \{y : y \supseteq z\} = \emptyset\}^\bullet$$

(551Ga). Now we have

$$\begin{aligned}1 &= [\![\vec{W} \text{ is nowhere dense}]\!]\\ &= \inf_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \sup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} [\![\vec{W} \cap \{y : \check{z}' \subseteq y\} = \emptyset]\!]\\ &= \inf_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \sup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} \{x : W[\{x\}] \cap \{y : y \supseteq z'\} = \emptyset\}^\bullet\\ &= \Bigl(\bigcap_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \bigcup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} \{x : W[\{x\}] \cap \{y : y \supseteq z'\} = \emptyset\}\Bigr)^\bullet\\ &= \{x : W[\{x\}] \text{ is nowhere dense}\}^\bullet.\end{aligned}$$

So $\{x : W[\{x\}]$ is meager$\}$ is comeager. Because W has the Baire property in $\{0,1\}^{\kappa\times\omega} \times \{0,1\}^\omega$ (5A4E(c-ii)), it must be meager, by the Kuratowski-Ulam theorem (527D). **Q**

(c) Continuing from (b), there is a meager Baire set $W' \supseteq W$ (5A4E(d-ii)). Let $J \subseteq \kappa$ be a countable set such that W' is determined by coordinates in $(J \times \omega) \mathbin{\dot\cup} \omega$, that is, if $(x, y) \in W'$, $x' \in \{0,1\}^{\kappa\times\omega}$ and $x'{\restriction}J \times \omega = x{\restriction}J \times \omega$ then $(x', y) \in W'$. Take any $\xi \in \kappa \setminus J$. Set $L = (\kappa \setminus \{\xi\}) \times \omega$ and

$$V = \{(x{\restriction}L, y) : (x, y) \in W'\};$$

then $V \subseteq \{0,1\}^L \times \{0,1\}^\omega$ is meager (applying 527D to

$$V \times \{0,1\}^{\{\xi\}\times\omega} \subseteq \{0,1\}^L \times \{0,1\}^{\{\xi\}\times\omega} \equiv \{0,1\}^{\kappa\times\omega} \times \{0,1\}^\omega).$$

Now consider the map $\phi : \{0,1\}^{\kappa\times\omega} \to \{0,1\}^L \times \{0,1\}^\omega$ defined by setting $\phi(x) = (x{\restriction}L, f_\xi(x))$ for $x \in \{0,1\}^{\kappa\times\omega}$. Looking back at the definition of f_ξ, we see that this is a homeomorphism. So $\phi^{-1}[V]$ must be meager, and

$$[\![\vec{f}_\xi \in \vec{W}]\!] \subseteq [\![\vec{f}_\xi \in \vec{W}']\!] = \{x : (x, f_\xi(x)) \in W'\}^\bullet$$

(551Ea)

$$= \{x : (x{\restriction}L, f_\xi(x)) \in V\}^\bullet = (\phi^{-1}[V])^\bullet = 0,$$

that is, $\Vdash_{\mathbb{P}} \vec{f}_\xi \notin \vec{W}$.

This is true for every $\xi \in \kappa \setminus J$. So

$$\Vdash_{\mathbb{P}} \dot{A} \cap \dot{W} \subseteq \{\vec{f}_\xi : \xi \in \check{J}\} \text{ is countable.}$$

As $\dot{W}$ is arbitrary,

$$\Vdash_{\mathbb{P}} \dot{A} \text{ has countable intersection with every nowhere dense zero set.}$$

It follows at once that

$$\Vdash_{\mathbb{P}} \dot{A} \text{ has countable intersection with every nowhere dense set, and is a Lusin set.}$$

As $\mathbb{P}$ and $\mathbb{Q}_\kappa$ are isomorphic,

$$\Vdash_{\mathbb{Q}_\kappa} \{0,1\}^\omega \text{ has a Lusin set of cardinal } \check{\kappa}.$$

(c) The statement of the proposition referred to $\mathbb{R}$ rather than to $\{0,1\}^\omega$. But, writing $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$ and $\mathcal{M}_\omega$ for the ideal of meager subsets of $\{0,1\}^\omega$, $(\mathbb{R}, \mathcal{M})$ and $(\{0,1\}^\omega, \mathcal{M}_\omega)$ are isomorphic (522Vb), and one will have Lusin sets iff the other does. So

$$\Vdash_{\mathbb{Q}_\kappa} \mathbb{R} \text{ has a Lusin set of cardinal } \check{\kappa}.$$

554F Corollary Let κ be a cardinal which is equal to the cardinal power κ^ω. Write $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \operatorname{non}\mathcal{M} = \omega_1 \text{ and } \mathfrak{m}_{\text{countable}} = \operatorname{cov}\mathcal{M} = \mathfrak{c}.$$

proof By 554B, $\Vdash_{\mathbb{Q}_\kappa} \mathfrak{c} = \check{\kappa}$; so we have only to put 554E and 554Da together.

554G Theorem Let κ be an infinite cardinal such that $\mathrm{FN}(\mathfrak{G}_\kappa)=\omega_1$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1.$$

proof (a) We need to know that $\mathfrak{G}_\kappa$ is isomorphic to the simple power algebra $\mathfrak{G}_\kappa^{\mathbb{N}}$. **P** The algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^\kappa$ is isomorphic to a free product of two-element algebras, so is homogeneous (316Q[4]); $\mathfrak{G}_\kappa$ is isomorphic to the Dedekind completion of $\mathcal{E}$, so is homogeneous (316P[4]). Now we have a partition of unity $\langle p_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}_\kappa$ consisting of non-zero elements, so that $\mathfrak{G}_\kappa$ is isomorphic to the simple product of the corresponding principal ideals (315F) and to $\mathfrak{G}_\kappa^{\mathbb{N}}$. **Q** There is therefore a Freese-Nation function $\theta:\mathfrak{G}_\kappa^{\mathbb{N}}\to[\mathfrak{G}_\kappa^{\mathbb{N}}]^{\le\omega}$.

For $\xi<\kappa$, set $E_\xi=\{x:x\in\{0,1\}^\kappa,\,x(\xi)=1\}$; for $J\subseteq\kappa$, let $\mathfrak{C}_J$ be the order-closed subalgebra of $\mathfrak{G}_\kappa$ generated by $\{E_\xi^\bullet:\xi\in J\}$, and let C_J be the set of elements of $\mathfrak{C}_J$ of the form $\inf_{\xi\in K}E_\xi^\bullet\setminus\sup_{\xi\in L}E_\xi^\bullet$ where K, L are disjoint finite subsets of J.

For $v\in\mathfrak{G}_\kappa^{\mathbb{N}}$ let $\vec v$ be the $\mathbb{Q}_\kappa$-name $\{(\check n,v(n)):n\in\mathbb{N},\,v(n)\ne0\}$; then $\Vdash_{\mathbb{Q}_\kappa}\vec v\subseteq\mathbb{N}$, and $[\![\check n\in\vec v]\!]=v(n)$ for every $n\in\mathbb{N}$.

For any $\mathbb{Q}_\kappa$-name $\dot u$, let $J(\dot u)$ be a countable subset of κ such that $[\![\check n\in\dot u]\!]\in\mathfrak{C}_{J(\dot u)}$ for every $n\in\mathbb{N}$.

(b) Let $\dot X$ be a discriminating $\mathbb{Q}_\kappa$-name such that $\Vdash_{\mathbb{Q}_\kappa}\dot X=\mathcal{P}\mathbb{N}$ (5A3Ka). For $\sigma=(\dot u,p)\in\dot X$ set

$$\theta_1(\sigma)=\bigcup_{e\in C_{J(\dot u)}}\theta(\langle[\![\check n\in\dot u]\!]\cap e)\rangle_{n\in\mathbb{N}})\cup\bigcup_{e\in C_{J(\dot u)}}\theta(\langle[\![\check n\in\dot u]\!]\cup(1\setminus e)\rangle_{n\in\mathbb{N}})\in[\mathfrak{G}_\kappa^{\mathbb{N}}]^{\le\omega},$$

$$\theta_2(\sigma)=\{(\vec v,p):v\in\theta_1(\sigma)\},$$

so that $\theta_2(\sigma)$ is a $\mathbb{Q}_\kappa$-name and

$$\Vdash_{\mathbb{Q}_\kappa}\theta_2(\sigma)\text{ is a countable subset of }\mathcal{P}\mathbb{N}.$$

(c) Set

$$\dot\theta=\{((\dot u,\theta_2(\dot u,p)),p):(\dot u,p)\in\dot X\}.$$

By 5A3Kb,

$$\Vdash_{\mathbb{Q}_\kappa}\dot\theta\text{ is a function with domain }\dot X=\mathcal{P}\mathbb{N}.$$

Next,

$$\Vdash_{\mathbb{Q}_\kappa}\dot\theta\text{ takes values in }[\mathcal{P}\mathbb{N}]^{\le\omega}.$$

P Suppose that $\dot x$ is a $\mathbb{Q}_\kappa$-name and $p\in\mathfrak{G}_\kappa^+$ is such that

$$p\Vdash_{\mathbb{Q}_\kappa}\dot x\in\dot\theta.$$

Then there are a $(\dot u,q)\in\dot X$ and a p' stronger than both p and q such that

$$p'\Vdash_{\mathbb{Q}_\kappa}\dot x=(\dot u,\theta_2(\dot u,q))\text{ has second member }\theta_2(\dot u,q)\in[\mathcal{P}\mathbb{N}]^{\le\omega}.$$

As p and $\dot x$ are arbitrary,

$\Vdash_{\mathbb{Q}_\kappa}$ every value of $\dot\theta$, being the second member of an element of $\dot\theta$, is a countable subset of $\mathcal{P}\mathbb{N}$. **Q**

(d) In fact,

$$\Vdash_{\mathbb{Q}_\kappa}\dot\theta\text{ is a Freese-Nation function on }\mathcal{P}\mathbb{N}.$$

P Suppose that $\dot A_1$, $\dot A_2$ are $\mathbb{Q}_\kappa$-names and $p\in\mathfrak{G}_\kappa^+$ is such that

$$p\Vdash_{\mathbb{Q}_\kappa}\dot A_1\subseteq\dot A_2\subseteq\mathbb{N}.$$

Because $\Vdash_{\mathbb{Q}_\kappa}\dot X=\mathcal{P}\mathbb{N}$, there must be $(\dot u_1,q_1)$ and $(\dot u_2,q_2)\in\dot X$ and a p_1 stronger than p, q_1 and q_2 such that

$$p_1\Vdash_{\mathbb{Q}_\kappa}\dot u_1=\dot A_1\text{ and }\dot u_2=\dot A_2.$$

In this case, for both i, $((\dot u_i,\theta_2(\dot u_i,q_i)),q_i)\in\dot\theta$, so we have

$$p_1\Vdash_{\mathbb{Q}_\kappa}\dot\theta(\dot A_i)=\dot\theta(\dot u_i)=\theta_2(\dot u_i,q_i).$$

Let $e\subseteq p_1$ be a member of C_κ, that is, a member of $\mathfrak{G}_\kappa$ which is the equivalence class of a basic cylinder set. We have

$$e\Vdash_{\mathbb{Q}_\kappa}\dot u_1=\dot A_1\subseteq\dot A_2=\dot u_2,$$

so $e\cap[\![\check n\in\dot u_1]\!]\subseteq[\![\check n\in\dot u_2]\!]$ for every $n\in\mathbb{N}$. Express e as $e_1\cap e_2\cap e_3$ where $e_1\in C_{J(\dot u_1)}$, $e_2\in C_{J(\dot u_2)}$ and $e_3\in C_{\kappa\setminus K}$, where $K=J(\dot u_1)\cup J(\dot u_2)$. For each $n\in\mathbb{N}$,

[4]Later editions only.

$$e_1 \cap e_2 \cap [\![\check{n} \in \dot{u}_1]\!] \setminus [\![\check{n} \in \dot{u}_2]\!]$$

belongs to $\mathfrak{C}_K$ and is disjoint from $e_3 \in \mathfrak{C}_{\kappa\setminus K} \setminus \{0\}$, so must be zero; we therefore have

$$e_1 \cap [\![\check{n} \in \dot{u}_1]\!] \subseteq [\![\check{n} \in \dot{u}_2]\!] \cup (1 \setminus e_2)$$

for every n, that is,

$$\langle [\![\check{n} \in \dot{u}_1]\!] \cap e_1 \rangle_{n\in\mathbb{N}} \subseteq \langle [\![\check{n} \in \dot{u}_2]\!] \cup (1 \setminus e_2) \rangle_{n\in\mathbb{N}}$$

in $\mathfrak{G}_\kappa^{\mathbb{N}}$. Because θ is a Freese-Nation function, there is a sequence

$$\langle a_n \rangle_{n\in\mathbb{N}} \in \theta(\langle [\![\check{n} \in \dot{u}_1]\!] \cap e_1 \rangle_{n\in\mathbb{N}}) \cap \theta(\langle [\![\check{n} \in \dot{u}_2]\!] \cup (1 \setminus e_2) \rangle_{n\in\mathbb{N}})$$

such that

$$[\![\check{n} \in \dot{u}_1]\!] \cap e_1 \subseteq a_n \subseteq [\![\check{n} \in \dot{u}_2]\!] \cup (1 \setminus e_2)$$

for every n. Now $v = \langle a_n \rangle_{n\in\mathbb{N}}$ belongs to $\theta_1(\dot{u}_1, p_1) \cap \theta_1(\dot{u}_2, p_2)$, so $(\vec{v}, p_i) \in \theta_2(\dot{u}_i, p_i)$ and

$$p_i \Vdash_{\mathbb{Q}_\kappa} \vec{v} \in \theta_2(\dot{u}_i, p_i) = \dot{\theta}(\dot{u}_i)$$

for both i. Returning to e, we have

$$e \cap [\![\check{n} \in \dot{u}_1]\!] \subseteq e \cap a_n \subseteq e \cap [\![\check{n} \in \dot{u}_2]\!]$$

for every n, because $e \subseteq e_1 \cap e_2$. So

$$e \Vdash_{\mathbb{Q}_\kappa} \dot{u}_1 \subseteq \vec{v} \subseteq \dot{u}_2.$$

Also e is stronger than p and

$$e \Vdash_{\mathbb{Q}_\kappa} \vec{v} \in \dot{\theta}(\dot{u}_1) \cap \dot{\theta}(\dot{u}_2) = \dot{\theta}(\dot{A}_1) \cap \dot{\theta}(\dot{A}_2).$$

As p, $\dot{A}_1$ and $\dot{A}_2$ are arbitrary,

> $\Vdash_{\mathbb{Q}_\kappa}$ for any A, $B \subseteq \mathbb{N}$ there is a $C \in \dot{\theta}(A) \cap \dot{\theta}(B)$ such that $A \subseteq C \subseteq B$; that is, $\dot{\theta}$ is a Freese-Nation function. **Q**

(e) Putting (c) and (d) together, we have

$$\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) \le \omega_1;$$

and since the Freese-Nation number of $\mathcal{P}\mathbb{N}$ is surely uncountable (522T), this is enough.

554H Corollary Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and that κ is an infinite cardinal such that

(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \kappa$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \kappa$ of countable cofinality.

Then $\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$.

proof Any countably generated order-closed subalgebra $\mathfrak{C}$ of $\mathfrak{G}_\kappa$ is (in the language of part (a) of the proof of 554G) included in $\mathfrak{C}_J$ for some countable $J \subseteq \kappa$, which has a countable π-base C_J; so $\mathfrak{C}_J$ and $\mathfrak{C}$ are σ-linked, and $\mathrm{FN}(\mathfrak{C}) \le \mathrm{FN}(\mathcal{P}\mathbb{N}) \le \omega_1$, by 518D. By 518I, the conditions (α) and (β), together with the fact that $\tau(\mathfrak{G}_\kappa) \le \kappa$, now ensure that $\mathrm{FN}(\mathfrak{G}_\kappa) \le \omega_1$, so 554G gives the result.

554I Theorem (T.J.Carlson) Suppose that the continuum hypothesis is true. Then

$$\Vdash_{\mathbb{Q}_{\omega_2}} \mathfrak{c} = \omega_2 \text{ and Lebesgue measure has a Borel lifting.}$$

proof Of course the cardinal power ω_2^ω (in the ordinary universe) is equal to $\max(\mathfrak{c}, \mathrm{cf}[\omega_2]^{\le\omega}) = \omega_2$. Write $\mathfrak{A}$ for the Lebesgue measure algebra. Then from 518D(iii), 554H and 554B we see that

$$\Vdash_{\mathbb{Q}_{\omega_2}} \mathrm{FN}(\mathfrak{A}) = \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1 \text{ and } \#(\mathfrak{A}) = \mathfrak{c} = \omega_2.$$

So 535E(b-ii) tells us that

$$\Vdash_{\mathbb{Q}_{\omega_2}} \text{Lebesgue measure has a Borel lifting.}$$

554X Basic exercises (a) Show that $\#(\mathfrak{G}_\kappa) = \kappa^\omega$ for every infinite cardinal κ.

(b) Devise a definition of 'strongly Lusin' set to match 537Ab, and state and prove a result corresponding to 552E. (*Hint*: 527Xe.)

554Y Further exercises (a) For how many of the results of 552F-552J can you find equivalents with respect to Cohen real forcing? (*Hint*: BARTOSZYŃSKI & JUDAH 95.)

(b)(i) Show that there is a family $\langle e_\xi\rangle_{\xi<\omega_1}$ such that (α) for each ξ, e_ξ is an injective function from ξ to $\mathbb{N}$ (β) if $\eta\le\xi<\omega_1$ then $\{\zeta:\zeta<\eta,\ e_\xi(\zeta)\ne e_\eta(\zeta)\}$ is finite. (*Hint*: choose the e_ξ inductively, taking care that $\mathbb{N}\setminus e_\xi[\xi]$ is infinite for every ξ.) (ii) Let T be the set of functions t such that, for some $\xi<\omega_1$, $t:\xi+1\to\mathbb{N}$ is a function differing from $e_{\xi+1}$ at only finitely many points. Show that $T\cup\{\emptyset\}$, ordered by $\subseteq$, is a special Aronszajn tree. (*Hint*: for any $n\in\mathbb{N}$, $\{t:t(\max(\operatorname{dom}t))=n\}$ is an antichain.)

(c) (TODORČEVIĆ 87) Let κ be an infinite cardinal. Take functions $e_\xi:\xi\to\mathbb{N}$, for $\xi<\omega_1$, as in 554Yb. Let $\dot{\preccurlyeq}$ be the $\mathbb{Q}_\kappa$-name

$$\{((\check\eta,\check\xi),p):\eta\le\xi<\omega_1,\ p\in\mathbb{Q}_\kappa\text{ and for every }\zeta<\eta\text{ either }e_\eta(\zeta)=e_\xi(\zeta)$$
$$\text{or }p(e_\eta(\zeta))\text{ and }p(e_\xi(\zeta))\text{ are defined and equal}\}.$$

Show that $\Vdash_{\mathbb{Q}_\kappa}(\omega_1,\dot{\preccurlyeq})$ is a Souslin tree.

554 Notes and comments The original theories of Cohen and random reals were developed in parallel; see KUNEN 84 for an account of the special properties of null and meager ideals which make this possible. Thus the Sierpiński sets of random real models become Lusin sets in Cohen real models, and the horizontal gap which appears in Cichoń's diagram if we add random reals becomes a vertical gap if we add Cohen reals (552F-552I, 554F). I give a very much briefer account of Cohen reals because I am restricting attention to results which have consequences in measure theory, as in 554Dc and 554I, and I make no attempt to look for reflections of the patterns in §553, which are mostly there for the illumination they throw on the structure of measure algebras. But I do not seek out the shortest route in every case. In particular, I spell out some of the theory of Freese-Nation numbers (554G-554H) for its own sake as well as to provide a proof of Carlson's theorem 554I. Let me remind you that ω_2 has a very special place in the arguments here; see 518Rb and 535Zb.

555 Solovay's construction of real-valued-measurable cardinals

While all the mathematical ideas of Chapter 54 were expressed as arguments in ZFC, many would be of little interest if it appeared that there could be no atomlessly-measurable cardinals. In this section I present R.M.Solovay's theorem that if there is a two-valued-measurable cardinal in the original universe, then there is a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}}\text{ there is an atomlessly-measurable cardinal}$$

(555D). Varying $\mathbb{P}$ we find that we can force models with other kinds of quasi-measurable cardinal (555G, 555M); starting from a stronger hypothesis we can reach the normal measure axiom (555P).

555A Notation As in §§552-553, I will write $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\mathbb{P}_\kappa$ for the forcing notion $\mathfrak{B}_\kappa^+=\mathfrak{B}_\kappa\setminus\{0\}$, active downwards. In this context, as in 525A, $\langle e_\eta\rangle_{\eta<\kappa}$ will be the standard generating family in $\mathfrak{B}_\kappa$.

As in §554, I will write $\mathfrak{G}_\kappa$ for the category algebra of $\{0,1\}^\kappa$, and $\mathbb{Q}_\kappa$ for the forcing notion $\mathfrak{G}_\kappa^+$, active downwards. Recall that $\mathfrak{G}_\kappa$ is isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^\kappa)$ (514If).

555B Theorem Suppose that X is a set, and $\mathcal{I}$ a proper σ-ideal of subsets of X containing singletons. Let $\mathbb{P}$ be a ccc forcing notion, and $\dot{\mathcal{I}}$ a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}}\dot{\mathcal{I}}=\{J:\text{there is an }I\in\check{\mathcal{I}}\text{ such that }J\subseteq I\}.$$

Then

(a)

$\Vdash_{\mathbb{P}}\dot{\mathcal{I}}$ is the ideal of subsets of $\check X$ generated by $\check{\mathcal{I}}$; it is a proper σ-ideal containing singletons.

(b) $\Vdash_{\mathbb{P}}\operatorname{add}\dot{\mathcal{I}}=(\operatorname{add}\mathcal{I})\check{}$.

(c) If $\mathcal{I}$ is ω_1-saturated in $\mathcal{P}X$, then

$$\Vdash_{\mathbb{P}}\dot{\mathcal{I}}\text{ is }\omega_1\text{-saturated in }\mathcal{P}\check X,\text{ so }\mathcal{P}\check X/\dot{\mathcal{I}}\text{ is ccc and Dedekind complete.}$$

(d) If $X=\lambda$ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal on λ, then

$$\Vdash_{\mathbb{P}}\dot{\mathcal{I}}\text{ is a normal ideal on }\check\lambda.$$

proof (a) Because

$$\Vdash_{\mathbb{P}} \check{\mathcal{I}} \text{ is a family of subsets of } \check{X} \text{ closed under finite unions,}$$

we have

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is an ideal of subsets of } \check{X}.$$

Because

$$\Vdash_{\mathbb{P}} \check{I} \in \dot{\mathcal{I}}$$

whenever $I \in \mathcal{I}$,

$$\Vdash_{\mathbb{P}} \check{\mathcal{I}} \subseteq \dot{\mathcal{I}} \text{ and } \dot{\mathcal{I}} \text{ is the ideal generated by } \check{\mathcal{I}}.$$

Next, if $\dot{J}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{J} \in \dot{\mathcal{I}}$, then there is an $I \in \mathcal{I}$ such that $p \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$. **P** We have

$$p \Vdash_{\mathbb{P}} \text{there is an } I \in \check{\mathcal{I}} \text{ such that } \dot{J} \subseteq I.$$

Set

$$A = \{q : \text{there is an } I \in \mathcal{I} \text{ such that } q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}\}.$$

If p' is stronger than p, there is a $q \in A$ stronger than p'. Let $A' \subseteq A$ be a maximal antichain. Then A' is countable and for each $q \in A'$ there is an $I_q \in \mathcal{I}$ such that $q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}_q$. Set $I = \bigcup_{q \in A} I_q$; because $\mathcal{I}$ is a σ-ideal, $I \in \mathcal{I}$. Now $q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$ for every $q \in A'$. If p' is stronger than p there is a $q \in A'$ which is compatible with p', so $p \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$, as required. **Q**

Since $X \notin \mathcal{I}$, we have

$$\Vdash_{\mathbb{P}} \check{X} \notin \dot{\mathcal{I}}.$$

Since $\{x\} \in \mathcal{I}$ for every $x \in X$,

$$\Vdash_{\mathbb{P}} \{x\} \in \dot{\mathcal{I}} \text{ for every } x \in \check{X}.$$

Thus

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a proper ideal of } \mathcal{P}\check{X} \text{ containing singletons.}$$

I defer the final step to (b-i) below.

(b) Set $\theta = \operatorname{add} \mathcal{I}$.

(i) Suppose that $p \in \mathbb{P}$ and that $\dot{\mathcal{A}}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{\mathcal{A}} \subseteq \dot{\mathcal{I}} \text{ and } \#(\dot{\mathcal{A}}) < \check{\theta}.$$

Then there are a q stronger than p, a $\delta < \theta$ and a family $\langle \dot{A}_\xi \rangle_{\xi < \delta}$ of $\mathbb{P}$-names such that

$$q \Vdash_{\mathbb{P}} \dot{\mathcal{A}} = \{\dot{A}_\xi : \xi < \check{\delta}\}.$$

For each $\xi < \delta$, $q \Vdash_{\mathbb{P}} \dot{A}_\xi \in \dot{\mathcal{I}}$, so we have an $I_\xi \in \mathcal{I}$ such that $q \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I}_\xi$. Set $I = \bigcup_{\xi < \delta} I_\xi \in \mathcal{I}$. Then

$$q \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I} \text{ for every } \xi < \check{\delta}, \text{ so } \bigcup \dot{\mathcal{A}} \subseteq \check{I} \text{ and } \bigcup \dot{\mathcal{A}} \in \dot{\mathcal{I}}.$$

As p and $\dot{\mathcal{A}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \operatorname{add} \dot{\mathcal{I}} \ge \check{\theta}.$$

In particular, since we certainly have $\theta \ge \omega_1$,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a } \sigma\text{-ideal.}$$

(ii) In the other direction, there is a family $\langle I_\xi \rangle_{\xi < \theta}$ in $\mathcal{I}$ with no upper bound in $\mathcal{I}$. Now $\Vdash_{\mathbb{P}} \check{I}_\xi \in \dot{\mathcal{I}}$ for every $\xi < \theta$. **?** If $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \bigcup_{\xi < \theta} \check{I}_\xi \in \dot{\mathcal{I}},$$

then there is an $I \in \mathcal{I}$ such that

$$p \Vdash_{\mathbb{P}} \bigcup_{\xi < \check{\theta}} \check{I}_\xi \subseteq \check{I}$$

and $\bigcup_{\xi < \theta} I_\xi \subseteq I \in \mathcal{I}$. **X** So

$$\Vdash_{\mathbb{P}} \bigcup_{\xi < \check{\theta}} \check{I}_\xi \notin \dot{\mathcal{I}} \text{ and } \operatorname{add} \dot{\mathcal{I}} \le \check{\theta}.$$

(c) Let $p \in \mathbb{P}$ and a family $\langle \dot{A}_\eta \rangle_{\eta<\omega_1}$ of $\mathbb{P}$-names be such that

$$p \Vdash_{\mathbb{P}} \langle \dot{A}_\eta \rangle_{\eta<\omega_1} \text{ is a disjoint family of subsets of } \check{X}.$$

For each $x \in X$, $\langle \widehat{p} \cap [\![\check{x} \in \dot{A}_\eta]\!] \rangle_{\eta<\omega_1}$ is a disjoint family in $\mathrm{RO}(\mathbb{P})$, where

$$\widehat{p} = \operatorname{int} \overline{\{q : q \text{ is stronger than } p\}}$$

is the regular open set corresponding to p. So there is an $\alpha_x < \omega_1$ such that $\widehat{p} \cap [\![\check{x} \in \dot{A}_\eta]\!] = 0$ for every $\eta \geq \alpha_x$, that is, $p \Vdash_{\mathbb{P}} \check{x} \notin \dot{A}_\eta$ for every $\eta \geq \alpha_x$. Because $\mathcal{I}$ is ω_1-saturated, therefore ω_2-additive (542B-542C), there is an $\alpha < \omega_1$ such that $I = \{x : x \in X, \alpha_x \geq \alpha\}$ belongs to $\mathcal{I}$. Now $p \Vdash_{\mathbb{P}} \check{x} \notin \dot{A}_\alpha$ for every $x \in X \setminus I$, that is,

$$p \Vdash_{\mathbb{P}} \dot{A}_\alpha \subseteq \check{I} \text{ and } \dot{A}_\alpha \in \dot{\mathcal{I}}.$$

As p and $\langle \dot{A}_\eta \rangle_{\eta<\omega_1}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is ω_1-saturated.

Thus $\Vdash_{\mathbb{P}} \mathcal{P}\check{X}/\dot{\mathcal{I}}$ is ccc. But since we know from (b) that $\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is a σ-ideal, and of course $\Vdash_{\mathbb{P}} \mathcal{P}\check{X}$ is Dedekind complete, we have

$$\Vdash_{\mathbb{P}} \mathcal{P}\check{X}/\dot{\mathcal{I}} \text{ is Dedekind } \sigma\text{-complete, therefore Dedekind complete.}$$

(d) Suppose that $p \in \mathbb{P}$ and that $\langle \dot{A}_\xi \rangle_{\xi<\lambda}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A}_\xi \in \dot{\mathcal{I}} \text{ for every } \xi < \check{\lambda}.$$

For each $\xi < \lambda$ we have an $I_\xi \in \mathcal{I}$ such that $p \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I}_\xi$; let I be the diagonal union

$$\{\xi : \xi < \lambda, \xi \in \bigcup_{\eta<\xi} I_\eta\}.$$

Because $\mathcal{I}$ is a normal ideal on λ, $I \in \mathcal{I}$. Now suppose that q is stronger than p and that $\xi < \lambda$ is such that

$$q \Vdash_{\mathbb{P}} \check{\xi} \in \bigcup_{\eta<\check{\xi}} \dot{A}_\eta.$$

Then

$$q \Vdash_{\mathbb{P}} \check{\xi} \in \bigcup_{\eta<\check{\xi}} \check{I}_\eta,$$

so $\xi \in \bigcup_{\eta<\xi} I_\eta$ and $\xi \in I$ and $q \Vdash_{\mathbb{P}} \check{\xi} \in \check{I}$. As q and ξ are arbitrary,

$$p \Vdash_{\mathbb{P}} \text{ the diagonal union of } \langle \dot{A}_\xi \rangle_{\xi<\check{\lambda}} \text{ is included in } \check{I} \text{ and belongs to } \dot{\mathcal{I}}.$$

As p and $\langle \dot{A}_\xi \rangle_{\xi<\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is normal.}$$

555C Theorem Let $(X, \mathcal{P}X, \mu)$ be a probability space such that $\mu\{x\} = 0$ for every $x \in X$, and $\mathcal{N}$ the null ideal of μ. Let $\kappa > 0$ be a cardinal. Then we can find a $\mathbb{P}_\kappa$-name $\dot{\mu}$ such that

(i) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a probability measure with domain $\mathcal{P}\check{X}$, zero on singletons;

(ii) if $\dot{\mathcal{N}}$ is a $\mathbb{P}_\kappa$-name for the ideal of subsets of $\check{X}$ generated by $\check{\mathcal{N}}$, as in 555B, then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{N}} \text{ is the null ideal of } \dot{\mu}.$$

proof (a) For each function $\sigma : X \to \mathfrak{B}_\kappa$, write $\vec{\sigma}$ for the $\mathbb{P}_\kappa$-name

$$\{(\check{x}, \sigma(x)) : x \in X, \sigma(x) \neq 0\}.$$

Then

$$\Vdash_{\mathbb{P}_\kappa} \vec{\sigma} \subseteq \check{X}$$

and $[\![\check{x} \in \vec{\sigma}]\!] = \sigma(x)$ for every $x \in X$. Moreover, if $\dot{A}$ is any $\mathbb{P}_\kappa$-name such that $\Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{X}$, then $\Vdash_{\mathbb{P}_\kappa} \dot{A} = \vec{\sigma}$, where $\sigma(x) = [\![\check{x} \in \dot{A}]\!]$ for $x \in X$.

(b) For $\sigma \in \mathfrak{B}_\kappa^X$, the functional

$$a \mapsto \int \bar{\nu}_\kappa(\sigma(x) \cap a) \, \mu(dx)$$

is additive and dominated by $\bar{\nu}_\kappa$, so there is a unique $u_\sigma \in L^\infty(\mathfrak{B}_\kappa)$ such that

$$\int_a u_\sigma \, d\bar{\nu}_\kappa = \int \bar{\nu}_\kappa(\sigma(x) \cap a) \mu(dx)$$

for every $a \in \mathfrak{B}_\kappa$ (365E, 365D(d-ii)), and $0 \leq u_\sigma \leq \chi 1$. Observe that if σ, $\tau \in \mathfrak{B}_\kappa^X$, $a \in \mathfrak{B}_\kappa^+$ and $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \vec{\tau}$, then $a \cap \sigma(x) = a \cap \tau(x)$ for every $x \in X$, so that $u_\sigma \times \chi a = u_\tau \times \chi a$.

(c) For $u \in L^\infty(\mathfrak{B}_\kappa)$ let $\vec{u}$ be the corresponding $\mathbb{P}_\kappa$-name for a real number (5A3L, identifying $\mathfrak{B}_\kappa$ with $\mathrm{RO}(\mathbb{P}_\kappa)$ as usual). Consider the $\mathbb{P}_\kappa$-name

$$\dot{\mu} = \{((\vec{\sigma}, \vec{u}_\sigma), \mathbb{1}) : \sigma \in \mathfrak{B}_\kappa^X\}.$$

Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is a function from } \mathcal{P}\check{X} \text{ to } [0,1].$$

P Because $0 \le u_\sigma \le \chi 1$, $\Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma \in [0,1]$ for each σ, and $\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \subseteq \mathcal{P}\check{X} \times [0,1]$. If σ, $\tau \in \mathfrak{B}_\kappa^X$ and $a \in \mathfrak{B}_\kappa^+$ are such that $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \vec{\tau}$, then $u_\sigma \times \chi a = u_\tau \times \chi a$, by (b); but this means that $a \Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma = \vec{u}_\tau$ (5A3M). So $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a function. If $a \in \mathbb{P}$ and $\dot{A}$ are such that $a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{X}$, then there is a $\sigma \in \mathfrak{B}_\kappa^X$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot{A} = \vec{\sigma}$, so that $\Vdash_{\mathbb{P}_\kappa} \operatorname{dom} \dot{\mu} = \mathcal{P}\check{X}$. **Q**

(d) Now we have to check the properties of $\dot{\mu}$.

(i) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is countably additive. **P** Suppose that $\langle \dot{A}_n \rangle_{n \in \mathbb{N}}$ is a sequence of $\mathbb{P}_\kappa$-names and $a \in \mathfrak{B}_\kappa^+$ is such that

$$a \Vdash_{\mathbb{P}_\kappa} \langle \dot{A}_n \rangle_{n \in \mathbb{N}} \text{ is a disjoint sequence of subsets of } \check{X}.$$

For each $n \in \mathbb{N}$, let $\sigma_n \in \mathfrak{B}_\kappa^X$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot{A}_n = \vec{\sigma}_n$; then $\langle a \cap \sigma_n(x) \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}_\kappa$ for each $x \in X$. Set $\sigma(x) = \sup_{n \in \mathbb{N}} \sigma_n(x)$ for each x. Then $[\![\check{x} \in \vec{\sigma}]\!] = \sup_{n \in \mathbb{N}} [\![\check{x} \in \vec{\sigma}_n]\!]$ for each x, so $\Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \bigcup_{n \in \mathbb{N}} \vec{\sigma}_n$. Now for any $b \subseteq a$,

$$\begin{aligned}\int_b u_\sigma d\bar{\nu}_\kappa &= \int \bar{\nu}_\kappa(b \cap \sigma(x))\mu(dx) = \int \sum_{n=0}^\infty \bar{\nu}_\kappa(b \cap \sigma_n(x))\mu(dx) \\ &= \sum_{n=0}^\infty \int \bar{\nu}_\kappa(b \cap \sigma_n(x))\mu(dx) = \sum_{n=0}^\infty \int_b u_{\sigma_n} d\bar{\nu}_\kappa.\end{aligned}$$

So

$$\chi a \times u_\sigma = \sup_{n \in \mathbb{N}} \chi a \times \textstyle\sum_{i=0}^n u_{\sigma_i}$$

in $L^0(\mathfrak{B}_\kappa)$, and

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\textstyle\bigcup_{n \in \mathbb{N}} \dot{A}_n) = \dot{\mu}(\bigcup_{n \in \mathbb{N}} \vec{\sigma}_n) = \dot{\mu}(\vec{\sigma}) = \vec{u}_\sigma = \sum_{n=0}^\infty \vec{u}_{\sigma_n} = \sum_{n=0}^\infty \dot{\mu}(\dot{A}_n).$$

As a and $\langle \dot{A}_n \rangle_{n \in \mathbb{N}}$ are arbitrary, $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is countably additive. **Q**

(ii) Suppose that $\dot{y}$ is a $\mathbb{P}_\kappa$-name and $a \in \mathfrak{B}_\kappa^+$ is such that $a \Vdash_{\mathbb{P}_\kappa} \dot{y} \in \check{X}$. Take any b stronger than a and $y \in X$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot{y} = \check{y}$. Set $\sigma(y) = 1$ and $\sigma(x) = 0$ for $x \in X \setminus \{y\}$. Then

$$\textstyle\int u_\sigma d\bar{\nu}_\kappa = \mu\{y\} = 0,$$

so

$$b \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\{\dot{y}\}) = \dot{\mu}(\{\check{y}\}) = \dot{\mu}(\vec{\sigma}) = \vec{u}_\sigma = 0.$$

Thus

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is zero on singletons.}$$

(iii) If $\sigma(x) = 1$ for every $x \in X$, then $u_\sigma = \chi 1$ so

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\check{X}) = \vec{\chi 1} = 1.$$

(e) $\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{N}} = \{A : \dot{\mu}A = 0\}$. **P** Let $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot{A}$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{X}$. Let $\sigma \in \mathfrak{B}_\kappa^X$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot{A} = \vec{\sigma}$; then $a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\dot{A}) = \vec{u}_\sigma$.

(i) If $a \Vdash_{\mathbb{P}_\kappa} \dot{A} \in \dot{\mathcal{N}}$, then there is an $I \in \mathcal{N}$ such that $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma} \subseteq \check{I}$, that is, $a \cap \sigma(x) = 0$ for every $x \in X \setminus I$. But this means that

$$\textstyle\int_a u_\sigma d\bar{\nu}_\kappa = \int \bar{\nu}_\kappa(\sigma(x) \cap a)\mu(dx) \le \mu I = 0,$$

and $\chi a \times u_\sigma = 0$. So

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\dot{A}) = \vec{u}_\sigma = 0.$$

(ii) If $a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\dot{A}) = 0$, then $\chi a \times u_\sigma = 0$, so

$$\textstyle\int \bar{\nu}_\kappa(\sigma(x) \cap a)\mu(dx) = \int_a u_\sigma d\bar{\nu}_\kappa = 0.$$

Set $I = \{x : a \cap \sigma(x) \ne 0\}$; then $\mu I = 0$ and

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{I} \text{ so } \dot{A} \in \dot{\mathcal{N}}.$$

Putting these together we have what we need. **Q**

555D Corollary (Solovay 71) Suppose that λ is a two-valued-measurable cardinal and that $\kappa \geq \lambda$ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \check{\lambda} \text{ is atomlessly-measurable.}$$

proof Putting 555C and 555B together,

$\Vdash_{\mathbb{P}_\kappa}$ there is a probability measure μ with domain $\mathcal{P}\check{\lambda}$, zero on singletons, such that the null ideal of μ is $\check{\lambda}$-additive.

By 552B and 543Bc,

$$\Vdash_{\mathbb{P}_\kappa} \check{\lambda} \leq \mathfrak{c} \text{ is a real-valued-measurable cardinal, so is atomlessly-measurable.}$$

555E Theorem Let λ be a two-valued-measurable cardinal, and $\mathcal{I}$ a proper κ-additive maximal ideal of $\mathcal{P}\lambda$ containing singletons; let μ be the $\{0,1\}$-valued probability measure on λ with null ideal $\mathcal{I}$. Let $\kappa \geq \lambda$ be a cardinal, and define $\dot{\mu}$ from μ as in Theorem 555C. Set $\theta = \operatorname{Tr}_{\mathcal{I}}(\lambda;\kappa)$ (definition: 5A1La). Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is Maharam-type-homogeneous with Maharam type } \check{\theta}.$$

proof (a) Let $\langle g_\alpha\rangle_{\alpha<\theta}$ be a family in κ^λ such that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\} \in \mathcal{I}$ whenever $\alpha < \beta < \theta$ (541F). Because $\lambda \leq \kappa$, we can suppose that all the g_α are injective. (Just arrange that $g_\alpha(\xi)$ always belongs to some $J_\xi \in [\kappa]^\kappa$ where $\langle J_\xi\rangle_{\xi<\lambda}$ is disjoint.) For $\alpha < \theta$ and $\xi < \lambda$ set $\sigma_\alpha(\xi) = e_{g_\alpha(\xi)}$. For $\sigma \in \mathfrak{B}_\kappa^\lambda$ let $\vec{\sigma}$ be the corresponding $\mathbb{P}_\kappa$-name for a subset of λ as in the prood of 555C. Then for any non-empty finite $K \subseteq \theta$ we have

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha) = (2^{-\#(K)})\check{}.$$

P Set $\sigma(\xi) = \inf_{\alpha\in K} \sigma_\alpha(\xi)$ for each ξ, so that

$$\Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha.$$

Set

$$I = \textstyle\bigcup_{\alpha,\beta\in K \text{ are different}}\{\xi : \xi < \lambda,\ g_\alpha(\xi) = g_\beta(\xi)\};$$

then $I \in \mathcal{I}$. If $a \in \mathfrak{B}_\kappa$, let $J \in [\kappa]^{\leq\omega}$ be such that a belongs to the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\eta : \eta \in J\}$. Then

$$\{\xi : \xi < \lambda,\ \bar{\nu}_\kappa(\sigma(\xi) \cap a) \neq 2^{-\#(K)}\bar{\nu}_\kappa(a)\} \subseteq I \cup \textstyle\bigcup_{\alpha\in K} g_\alpha^{-1}[J] \in \mathcal{I},$$

so

$$\int \bar{\nu}_\kappa(\sigma(\xi) \cap a)\mu(d\xi) = 2^{-\#(K)}\bar{\nu}_\kappa(a).$$

This means that u_σ, as defined in the proof of 555C, is just $2^{-\#(K)}\chi 1$ and $\Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma = (2^{-\#(K)})\check{}$, that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha) = (2^{-\#(K)})\check{}. \text{ } \mathbf{Q}$$

Thus

$\Vdash_{\mathbb{P}_\kappa} \langle\vec{\sigma}_\alpha\rangle_{\alpha<\check{\theta}}$ is a stochastically independent family in $\mathcal{P}\lambda$ of elements of measure $\frac{1}{2}$, and every principal ideal of the measure algebra of $\dot{\mu}$ has Maharam type at least $\check{\theta}$

(331J).

(b) In the other direction, suppose that $a \in \mathfrak{B}_\kappa^+$, δ is a cardinal, $t > 0$ is a rational number and $\langle\dot{A}_\alpha\rangle_{\alpha<\delta}$ is a family of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A}_\alpha \subseteq \check{\lambda},\ \dot{\mu}(\dot{A}_\alpha \triangle \dot{A}_\beta) \geq 3\check{t} \text{ whenever } \alpha < \beta < \check{\delta}.$$

For each $\alpha < \delta$ let $\sigma_\alpha \in \mathfrak{B}_\kappa^\lambda$ be such that $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma}_\alpha = \dot{A}_\alpha$; then

$$\int \bar{\nu}_\kappa(a \cap \sigma_\alpha(\xi) \cap \sigma_\beta(\xi))\mu(d\xi) \geq 3t\bar{\nu}_\kappa a$$

whenever $\alpha < \beta < \delta$. Let $D \subseteq \mathfrak{B}_\kappa$ be a set of size κ which is dense for the measure-algebra topology (521O(a-ii)), and for $\alpha < \delta$, $\xi < \kappa$ take $d_\alpha(\xi) \in D$ such that $\bar{\nu}_\kappa(d_\alpha(\xi) \triangle \sigma_\alpha(\xi)) \leq t\bar{\nu}_\kappa(a)$. Then

$$\int \bar{\nu}_\kappa(d_\alpha(\xi) \triangle d_\beta(\xi))\mu(d\xi) > 0$$

and $\{\xi : d_\alpha(\xi) \ne d_\beta(\xi)\} \notin \mathcal{I}$ whenever $\alpha < \beta < \delta$; as $\mathcal{I}$ is a maximal ideal, $\{\xi : d_\alpha(\xi) = d_\beta(\xi)\} \in \mathcal{I}$ whenever $\alpha < \beta < \delta$, and $\langle d_\alpha\rangle_{\alpha<\delta}$ witnesses that

$$\delta \le \mathrm{Tr}_{\mathcal{I}}(\lambda; D) = \mathrm{Tr}_{\mathcal{I}}(\lambda; \kappa) = \theta.$$

As a, t and $\langle \dot{A}_\alpha\rangle_{\alpha<\delta}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \text{the Maharam type of } \dot{\mu} \text{ is at most } \check{\theta};$$

with (a), this means that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is Maharam-type-homogeneous with Maharam type } \check{\theta}.$$

555F Proposition Let λ be a two-valued-measurable cardinal and $\kappa > 0$. Let μ be a normal witnessing probability on λ and $\dot{\mu}$ the corresponding $\mathbb{P}_\kappa$-name for a measure on $\check{\lambda}$, as in 555C. Then

$\Vdash_{\mathbb{P}_\kappa}$ the covering number of the null ideal of the product measure $\dot{\mu}^{\mathbb{N}}$ on $\check{\lambda}^{\mathbb{N}}$ is $\check{\lambda}$.

proof (a) It may save a moment's thought later on if I remark now that if (Y, T, ν) is any measure space and $W \subseteq Y^{\mathbb{N}}$ is negligible for the product measure $\nu^{\mathbb{N}}$, then there is a family $\langle F_{ij}\rangle_{j\le i\in\mathbb{N}}$ in T such that

$$W \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} \{y : y \in Y^{\mathbb{N}},\ y(j) \in F_{ij} \text{ for every } j \le i\}, \quad \sum_{i=0}^{\infty} \prod_{j=0}^{i} \nu F_{ij} \le 1.$$

P For each $k \in \mathbb{N}$, let $\langle C_{ki}\rangle_{i\in\mathbb{N}}$ be a sequence of measurable cylinders such that $W \subseteq \bigcup_{i\in\mathbb{N}} C_{ki}$ and $\sum_{i=0}^{\infty} \lambda^{\mathbb{N}} C_{ki} \le 2^{-k-1}$; let $\langle C_i\rangle_{i\in\mathbb{N}}$ be a re-listing of the double family $\langle C_{ki}\rangle_{k,i\in\mathbb{N}}$ with enough copies of the empty set interleaved to ensure that C_i is determined by coordinates less than or equal to i for each i; express each C_i as $\{y : y(j) \in F_{ij} \text{ for } j \le i\}$. **Q**

(b) It will also help to be able to do some calculations with sequences of $\mathbb{P}_\kappa$-names for subsets of λ. Let $\mathcal{F}$ be the normal ultrafilter $\{F : \mu F = 1\}$.

(i) Let $\langle \dot{A}_{ij}\rangle_{j\le i\in\mathbb{N}}$ be a family of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A}_{ij} \subseteq \check{\lambda}$$

whenever $j \le i \in \mathbb{N}$. For $j \le i \in \mathbb{N}$ and $\xi < \lambda$ set $\sigma_{ij}(\xi) = [\![\check{\xi} \in \dot{A}_{ij}]\!]$. Then there is a countable set $I_\xi \subseteq \kappa$ such that $\sigma_{ij}(\xi)$ belongs to the closed subalgebra generated by $\{e_\eta : \eta \in I_\xi\}$ whenever $j \le i \in \mathbb{N}$. By 541Rb, there are an $F_0 \in \mathcal{F}$ and a countable set $I \subseteq \kappa$ such that $I_\xi \cap I_{\xi'} \subseteq I$ for all distinct ξ, $\xi' \in F_0$.

(ii) For $J \subseteq \kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\eta : \eta \in J\}$, and $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{C}_J)$ the corresponding conditional expectation operator; see 242J, 254R and 365R for the basic manipulations of these operators. Set $u_{ij\xi} = P_I(\chi\sigma_{ij}(\xi))$ for $\xi < \lambda$. Because $\#(L^\infty(\mathfrak{C}_I)) \le \mathfrak{c} < \lambda$ and $\mathcal{F}$ is a λ-complete ultrafilter, there are $u_{ij} \in \mathfrak{C}_I$ such that $\{\xi : u_{ij\xi} = u_{ij}\}$ belongs to $\mathcal{F}$ whenever $j \le i \in \mathbb{N}$. Set

$$F = F_0 \cap \{\xi : u_{ij\xi} = u_{ij} \text{ whenever } j \le i \in \mathbb{N}\},$$

so that $F \in \mathcal{F}$.

(iii) We have

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\dot{A}_{ij} = \vec{u}_{ij}$$

whenever $j \le i \in \mathbb{N}$. **P** If $a \in \mathfrak{B}_\kappa$, there is a countable $J \subseteq \kappa$ such that a belongs to the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\eta : \eta \in J\}$; we may suppose that $I \subseteq J$. Now $\langle I_\xi \setminus I\rangle_{\xi\in F}$ is disjoint, so $\{\xi : \xi \in F,\ I_\xi \cap J \not\subseteq I\}$ is countable, and $F' = \{\xi : \xi \in F,\ I_\xi \cap J \subseteq I\}$ belongs to $\mathcal{F}$. For $\xi \in F'$ we have

$$P_J(\chi\sigma_{ij}(\xi)) = P_J P_{I_\xi}(\chi\sigma_{ij}(\xi)) = P_{J\cap I_\xi}(\chi\sigma_{ij}(\xi)) \in \mathfrak{C}_I$$

so that

$$P_J(\chi\sigma_{ij}(\xi)) = P_I P_J(\chi\sigma_{ij}(\xi)) = P_I(\chi\sigma_{ij}(\xi)) = u_{ij};$$

consequently

$$\int_a u_{ij} d\bar{\nu}_\kappa = \int_a \chi\sigma_{ij}(\xi) d\bar{\nu}_\kappa = \bar{\nu}_\kappa(a \cap \sigma_{ij}(\xi)).$$

Because F' is μ-conegligible,

$$\int_a u_{ij} d\bar{\nu}_\kappa = \int \bar{\nu}_\kappa(a \cap \sigma_{ij}(\xi))\mu(d\xi).$$

As this is true for every $a \in \mathfrak{B}_\kappa$, the construction in 555C gives $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\dot{A}_{ij} = \vec{u}_{ij}$. **Q**

(iv) Note also that if $i \in \mathbb{N}$ and $\xi_0, \dots, \xi_j \in F$ are distinct, then

$$\int \prod_{j=0}^{i} u_{ij} d\bar{\nu}_\kappa = \bar{\nu}_\kappa(\inf_{j\le i} \sigma_{ij}(\xi_j)).$$

P The algebras $\mathfrak{C}_{I\cup\{\eta\}}$, for $\eta\in\kappa\setminus I$, are relatively stochastically independent over $\mathfrak{C}_I$ in the sense of 458L; by 458H/458Le, the algebras $\mathfrak{C}_{I\cup I_\xi}$, for $\xi\in F$, are relatively stochastically independent over $\mathfrak{C}_I$; but, disentangling the definitions, this is exactly what we need to know. **Q**

(c) We are now ready for the central idea of the proof. Suppose that $\dot W$ is a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot W\subseteq\check\lambda^{\mathbb{N}} \text{ and } \dot\mu^{\mathbb{N}}\dot W=0.$$

Then there is an $F\in\mathcal{F}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot W \text{ is disjoint from } (F^{\mathbb{N}}\setminus\Delta)\check{}$$

where $\Delta=\bigcup_{j<k\in\mathbb{N}}\{x:x\in\lambda^{\mathbb{N}},\,x(j)=x(k)\}$. **P** By (a), we have a family $\langle\dot A_{ij}\rangle_{j\le i\in\mathbb{N}}$ of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot A_{ij}\subseteq\check\lambda \text{ whenever } j\le i\in\mathbb{N},$$

$$\Vdash_{\mathbb{P}_\kappa} \dot W\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n}\{x:x\in\check\lambda^{\mathbb{N}},\,x(j)\in\dot A_{ij}\text{ for every } j\le i\},$$

$$\Vdash_{\mathbb{P}_\kappa} \sum_{i=0}^\infty\prod_{j=0}^i\dot\mu\dot A_{ij}\le 1.$$

Take $\sigma_{ij}(\xi)$, $I\in[\kappa]^{\le\omega}$, $u_{ij}\in\mathfrak{C}_I$ and $F\in\mathcal{F}$ as in (b). Suppose that $x\in F^{\mathbb{N}}$ and $x(j)\ne x(k)$ for distinct j, $k\in\mathbb{N}$. Then

$$[\![\check x\in\dot W]\!]\subseteq\inf_{n\in\mathbb{N}}\sup_{i\ge n}\inf_{j\le i}[\![x(j)\check{}\in\dot A_{ij}]\!]=\inf_{n\in\mathbb{N}}\sup_{i\ge n}\inf_{j\le i}\sigma_{ij}(x(j)).$$

So

$$\bar\nu_\kappa[\![\check x\in\dot W]\!]\le\inf_{n\in\mathbb{N}}\sum_{i=n}^\infty\bar\nu_\kappa(\inf_{j\le i}\sigma_{ij}(x(j)))=\inf_{n\in\mathbb{N}}\sum_{i=n}^\infty\int\prod_{j\le i}u_{ij}d\bar\nu_\kappa$$

by (b-iv). On the other hand, setting $v_i=\prod_{j\le i}u_{ij}$ for $i\in\mathbb{N}$ and $w=\sup_{m\in\mathbb{N}}\sum_{i=0}^m v_i$, we have

$$\Vdash_{\mathbb{P}_\kappa}\vec w=\sum_{i=0}^\infty\vec v_i=\sum_{i=0}^\infty\prod_{j=0}^i\dot\mu\dot A_{ij}\le 1.$$

So $w\le\chi 1$ and

$$\sum_{i=0}^\infty\int\prod_{j\le i}u_{ij}d\bar\nu_\kappa=\int w\,d\bar\nu_\kappa\le 1.$$

Putting these together, we see that $\bar\nu_\kappa[\![\check x\in\dot W]\!]=0$ and $\Vdash_{\mathbb{P}_\kappa}\check x\notin\dot W$. As x is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa}\dot W\text{ is disjoint from }(F^{\mathbb{N}}\setminus\Delta)\check{}.\ \mathbf{Q}$$

(d) We are nearly home. Suppose that $a\in\mathfrak{B}_\kappa^+$ and that $\dot{\mathcal{W}}$ is a $\mathbb{P}_\kappa$-name such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{\mathcal{W}}\text{ is a family of negligible sets in }\check\lambda^{\mathbb{N}}\text{ and }\#(\dot{\mathcal{W}})<\check\lambda.$$

Take any b stronger than a, $\theta<\lambda$ and family $\langle\dot W_\zeta\rangle_{\zeta<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$b\Vdash_{\mathbb{P}_\kappa}\dot{\mathcal{W}}=\{\dot W_\zeta:\zeta<\check\theta\}.$$

For each $\zeta<\theta$ let $\dot W'_\zeta$ be a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa}\dot W'_\zeta\subseteq\check\lambda^{\mathbb{N}}\text{ is negligible},\quad b\Vdash_{\mathbb{P}_\kappa}\dot W'_\zeta=\dot W_\zeta.$$

By (c), we have an $F_\zeta\in\mathcal{F}$ such that

$$\Vdash_{\mathbb{P}_\kappa}\dot W'_\zeta\cap(F_\zeta^{\mathbb{N}}\setminus\Delta)\check{}=\emptyset.$$

Because $\theta<\lambda$, $\bigcap_{\zeta<\theta}F_\zeta$ belongs to $\mathcal{F}$ and is infinite, and there is an $x\in\bigcap_{\zeta<\theta}F_\zeta^{\mathbb{N}}\setminus\Delta$. But now

$$\Vdash_{\mathbb{P}_\kappa}\check x\notin\bigcup_{\zeta<\check\theta}\dot W'_\zeta$$

and

$$b\Vdash_{\mathbb{P}_\kappa}\check x\notin\bigcup\dot{\mathcal{W}},\text{ so }\dot{\mathcal{W}}\text{ does not cover }\check\lambda^{\mathbb{N}}.$$

As b is arbitrary,

$$a\Vdash_{\mathbb{P}_\kappa}\dot{\mathcal{W}}\text{ does not cover }\check\lambda^{\mathbb{N}};$$

as a and $\dot{\mathcal{W}}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa}\text{ the covering number of the null ideal of the product measure on }\check\lambda^{\mathbb{N}}\text{ is at least }\check\lambda.$$

The reverse inequality is trivial, since

$$\Vdash_{\mathbb{P}_\kappa}\dot\mu\{\xi\}=0$$

for every $\xi < \check{\lambda}$; so the proposition is proved.

555G Cohen forcing If we allow ourselves to start from a measurable cardinal, we can find forcing constructions for a variety of power set σ-quotient algebras besides the probability algebras provided by Theorem 555C. In view of §554, an obvious construction is the following.

Theorem Let λ be a two-valued-measurable cardinal and $\kappa \ge \lambda$ a cardinal. Let $\mathcal{I}$ be a λ-additive maximal ideal of subsets of λ, and $\dot{\mathcal{I}}$ a $\mathbb{Q}_\kappa$-name for the ideal of subsets of $\check{\lambda}$ generated by $\check{\mathcal{I}}$, as in 555B. Set $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\lambda/\dot{\mathcal{I}} \cong \mathfrak{G}_{\check{\theta}}.$$

proof (a) For $\eta < \kappa$ let $e_\eta \in \mathfrak{G}_\kappa$ be the equivalence class of $\{x : x \in \{0,1\}^\kappa,\ x(\eta) = 1\}$; for $L \subseteq \kappa$ let $\mathfrak{C}_L$ be the closed subalgebra of $\mathfrak{G}_\kappa$ generated by $\{e_\eta : \eta \in L\}$. For $\sigma \in \mathfrak{G}_\kappa^\lambda$ let $\vec{\sigma}$ be the $\mathbb{P}_\kappa$-name $\{(\check{\xi}, \sigma(\xi)) : \xi < \lambda,\ \sigma(\xi) \ne 0\}$, so that $\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \subseteq \check{\lambda}$, and $[\![\check{\xi} \in \vec{\sigma}]\!] = \sigma(\xi)$ for any $\xi < \lambda$.

Write $\mathcal{F} = \{\lambda \setminus I : I \in \mathcal{I}\} = \mathcal{P}\lambda \setminus \mathcal{I}$, so that $\mathcal{F}$ is a λ-complete ultrafilter on λ.

(b) For $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ set

$$v_z = \{x : z \subseteq x \in \{0,1\}^\kappa\}^\bullet \in \mathfrak{G}_\kappa.$$

Then $\{v_z : z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})\}$ is order-dense in $\mathfrak{G}_\kappa$. For $A \subseteq \lambda$ and $\tau \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$, set $\sigma_\tau(\xi) = v_{\tau(\xi)}$ if $\xi \in A$, 0 if $\xi \in \lambda \setminus A$. Note that

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \subseteq (\mathrm{dom}\,\tau)^{\check{}}.$$

Now if $a \in \mathfrak{G}_\kappa^+$ and $\dot{C}$ is a $\mathbb{Q}_\kappa$-name such that $a \Vdash_{\mathbb{Q}_\kappa} \dot{C} \subseteq \check{\lambda}$, there is a $T \subseteq \bigcup_{A\subseteq\lambda} \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$ such that $a \Vdash_{\mathbb{Q}_\kappa} \dot{C} = \bigcup_{\tau\in\check{T}} \vec{\sigma}_\tau$ and T is countable. **P** Set $D = \{\xi : \xi < \lambda,\ \Vdash_{\mathbb{Q}_\kappa} \check{\xi} \notin \dot{C}\}$. For each $\xi \in \lambda \setminus D$, we have a sequence $\langle\tau_n(\xi)\rangle_{n\in\mathbb{N}}$ in $\mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ such that $[\![\check{\xi} \in \dot{C}]\!] = \sup_{n\in\mathbb{N}} v_{\tau_n(\xi)}$. If $\xi \in D$, then $\xi \notin \mathrm{dom}\,\tau_n$ so $\sigma_{\tau_n}(\xi) = 0$ for every n; accordingly

$$[\![\check{\xi} \in \dot{C}]\!] = 0 = \sup_{n\in\mathbb{N}} [\![\check{\xi} \in \vec{\sigma}_{\tau_n}]\!].$$

While if $\xi \in \lambda \setminus D$,

$$[\![\check{\xi} \in \dot{C}]\!] = \sup_{n\in\mathbb{N}} v_{\tau_n(\xi)} = \sup_{n\in\mathbb{N}} \sigma_{\tau_n}(\xi) = \sup_{n\in\mathbb{N}} [\![\check{\xi} \in \vec{\sigma}_{\tau_n}]\!].$$

So

$$a \Vdash_{\mathbb{Q}_\kappa} \dot{C} = \bigcup_{n\in\mathbb{N}} \vec{\sigma}_{\tau_n}$$

and we can set $T = \{\tau_n : n \in \mathbb{N}\}$. **Q**

(c) There is a family $G_0 \subseteq \kappa^\lambda$, of cardinal θ, such that $\{\xi : g(\xi) = g'(\xi)\} \in \mathcal{I}$ whenever g, $g' \in G$ are distinct (541F), and we can suppose that every member of G_0 is injective (see part (a) of the proof of 555F). Let $G \supseteq G_0$ be a maximal family such that $\{\xi : g(\xi) = g'(\xi)\}$ belongs to $\mathcal{I}$ whenever g, $g' \in G$ are distinct. Then $\#(G) \le \theta$, by the definition of $\mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$, so in fact $\#(G) = \theta$. Enumerate G as $\langle g_\alpha\rangle_{\alpha<\theta}$, and for $\alpha < \theta$, $\xi < \lambda$ set $\rho_\alpha(\xi) = e_{g_\alpha(\xi)} \in \mathfrak{G}_\kappa$.

(d) Suppose that $\alpha < \theta$ and $a \in \mathfrak{G}_\kappa^+$ are such that

$$a \Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^{\,\bullet} \text{ is neither 0 nor 1 in } \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}.$$

Then $g_\alpha^{-1}[\{\eta\}] \notin \mathcal{F}$ for every $\eta < \kappa$. **P?** Otherwise, take b to be one of $a \cap e_\eta$, $a \setminus e_\eta$ and non-zero, and any $\xi \in g_\alpha^{-1}[\{\eta\}]$. If $b \subseteq e_\eta$ then $b \subseteq \rho_\alpha(\xi)$ and $b \Vdash_{\mathbb{Q}_\kappa} \check{\xi} \in \vec{\rho}_\alpha$. If $b \cap e_\eta = 0$ then $\rho_\alpha(\xi) \cap b = 0$ and $b \Vdash_{\mathbb{Q}_\kappa} \check{\xi} \notin \vec{\rho}_\alpha$. So

$$b \Vdash_{\mathbb{Q}_\kappa} g_\alpha^{-1}[\{\eta\}]^{\check{}} \text{ is either included in or disjoint from } \vec{\rho}_\alpha, \text{ and } \vec{\rho}_\alpha^{\,\bullet} \text{ is either 1 or 0 in } \mathcal{P}\lambda/\dot{\mathcal{I}}.$$

But this contradicts the assumption on a. **XQ**

(e) $\Vdash_{\mathbb{Q}_\kappa} \{\vec{\rho}_\alpha^{\,\bullet} : \alpha < \check{\theta}\} \setminus \{0,1\}$ is a Boolean-independent family in $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$.

P? Otherwise, there must be disjoint finite sets J, $K \subseteq \theta$ and $a \in \mathfrak{G}_\kappa^+$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^{\,\bullet} \text{ is neither 0 nor 1 in } \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$$

for every $\alpha \in J \cup K$, but

$$a \Vdash_{\mathbb{Q}_\kappa} \inf_{\alpha\in\check{J}} \vec{\rho}_\alpha^{\,\bullet} \setminus \sup_{\alpha\in\check{K}} \vec{\rho}_\alpha^{\,\bullet} = 0.$$

In this case,

$$I = \{\xi : \xi < \lambda, \text{ there are distinct } \alpha, \beta \in J \cup K \text{ such that } g_\alpha(\xi) = g_\beta(\xi)\}$$

belongs to $\mathcal{I}$. For $\xi < \lambda$ define $\sigma(\xi) \in \mathfrak{G}_\kappa$ by saying that

$$\sigma(\xi) = 1 \text{ if } \xi \in I,$$
$$= \inf_{\alpha\in J} e_{g_\alpha(\xi)} \setminus \sup_{\alpha\in K} e_{g_\alpha(\xi)} \text{ otherwise.}$$

For $\alpha \in J$ and $\xi \in \lambda \setminus I$, $\sigma(\xi) \subseteq \rho_\alpha(\xi)$; so

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \setminus \vec{\rho}_\alpha \text{ is included in } \check{I} \text{ and belongs to } \dot{\mathcal{I}}, \text{ that is, } \vec{\sigma}^\bullet \subseteq \vec{\rho}_\alpha^\bullet.$$

On the other hand, if $\alpha \in K$ and $\xi \in \lambda \setminus I$, $\sigma(\xi) \cap \rho_\alpha(\xi) = 0$; so

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \cap \vec{\rho}_\alpha \text{ is included in } \check{I} \text{ and belongs to } \dot{\mathcal{I}}, \text{ that is, } \vec{\sigma}^\bullet \cap \vec{\rho}_\alpha^\bullet = 0.$$

So we have

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}^\bullet \subseteq \inf_{\alpha\in\check{J}} \vec{\rho}_\alpha^\bullet \setminus \sup_{\alpha\in\check{K}} \vec{\rho}_\alpha^\bullet$$

and

$$a \Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \in \dot{\mathcal{I}}.$$

Let $I' \in \mathcal{I}$ be such that $a \Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \subseteq \check{I}'$ (see part (a) of the proof of 555B), and $L \in [\kappa]^{\le\omega}$ such that $a \in \mathfrak{C}_L$. By (d), $g_\alpha^{-1}[\{\eta\}] \in \mathcal{I}$ for every $\alpha \in J \cup K$ and $\eta < \kappa$, so there must be a $\xi \in \lambda \setminus (I' \cup I)$ such that $g_\alpha(\xi) \notin L$ for every $\alpha \in J \cup K$. In this case, $\sigma(\xi) \in \mathfrak{C}_{\kappa\setminus L}$ so $a \cap \sigma(\xi)$ is non-zero. But $\sigma(\xi) = [\![\check{\xi} \in \vec{\sigma}]\!]$ and $a \Vdash_{\mathbb{Q}_\kappa} \check{\xi} \notin \vec{\sigma}$, so this is impossible. **XQ**

(f) $\Vdash_{\mathbb{Q}_\kappa}$ the subalgebra of $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ generated by $\{\vec{\rho}_\alpha^\bullet : \alpha < \check{\theta}\}$ is order-dense in $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$.
P If $a \in \mathfrak{G}_\kappa^+$ and $\dot{c}$ is a $\mathbb{Q}_\kappa$-name such that

$$a \Vdash_{\mathbb{Q}_\kappa} \dot{c} \in (\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}) \setminus \{0\},$$

there is a $\mathbb{Q}_\kappa$-name $\dot{C}$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \dot{C} \subseteq \check{\lambda},\ \dot{C} \notin \dot{\mathcal{I}} \text{ and } \dot{c} = \dot{C}^\bullet.$$

Take a countable set $T \subseteq \bigcup_{A\subseteq\lambda} \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$ such that $a \Vdash_{\mathbb{Q}_\kappa} \dot{C} = \bigcup_{\tau\in\check{T}} \vec{\sigma}_\tau$, as described in (b). Since $\Vdash_{\mathbb{Q}_\kappa} \dot{\mathcal{I}}$ is a σ-ideal, there are a b stronger than a and a $\tau \in T$ such that $b \Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \notin \dot{\mathcal{I}}$. Set $F_0 = \operatorname{dom}\tau$; since $\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \subseteq \check{F}_0$, $F_0 \notin \mathcal{I}$.. Since $\mathcal{I}$ is a σ-ideal, there is an $n \in \mathbb{N}$ such that $F_1 = \{\xi : \xi \in F_0,\ \#(\tau(\xi)) = n\} \notin \mathcal{I}$, and $F_1 \in \mathcal{F}$.

Let $\langle h_i\rangle_{i<n}$ be a finite sequence of functions from λ to κ such that $\operatorname{dom}\tau(\xi) = \{h_i(\xi) : i < n\}$ for every $\xi \in F_1$. As G was maximal, there is for each $i < n$ an $\alpha_i < \theta$ such that $\{\xi : g_{\alpha_i}(\xi) = h_i(\xi)\}$ belongs to $\mathcal{F}$; set $F_2 = \{\xi : \xi \in F_1,\ g_{\alpha_i}(\xi) = h_i(\xi) \text{ for every } i < n\}$. Note that if $i < j < n$ then $g_{\alpha_i}(\xi) \neq g_{\alpha_j}(\xi)$ for any $\xi \in F_1$, so $\alpha_i \neq \alpha_j$. Next, there must be an $L \subseteq n$ such that

$$F_3 = \{\xi : \xi \in F_2,\ \tau(\xi)(g_{\alpha_i}(\xi)) = 1 \text{ for every } i \in L,$$
$$\tau(\xi)(g_{\alpha_i}(\xi)) = 0 \text{ for every } i \in n \setminus L\}$$

belongs to $\mathcal{F}$. Set $J = \{\alpha_i : i \in L\}$ and $K = \{\alpha_i : i \in n \setminus L\}$; of course $J \cap K = \emptyset$, because all the α_i are different. Then, for $\xi \in F_3$, $\operatorname{dom}\tau_\xi = \{g_{\alpha_i}(\xi) : i < n\}$ and

$$[\![\check{\xi} \in \vec{\sigma}_\tau]\!] = \sigma_\tau(\xi) = v_{\tau(\xi)} = \inf_{i\in L} e_{g_{\alpha_i}(\xi)} \setminus \sup_{i\in n\setminus L} e_{g_{\alpha_i}(\xi)}$$
$$= \inf_{\alpha\in J} e_{g_\alpha(\xi)} \setminus \sup_{\alpha\in K} e_{g_\alpha(\xi)} = \inf_{\alpha\in J} [\![\check{\xi} \in \vec{\rho}_\alpha]\!] \setminus \sup_{\alpha\in K} [\![\check{\xi} \in \vec{\rho}_\alpha]\!].$$

Accordingly

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \triangle \Big(\bigcap_{\alpha\in\check{J}} \vec{\rho}_\alpha \setminus \bigcup_{\alpha\in\check{K}} \vec{\rho}_\alpha\Big) \text{ is disjoint from } \check{F}_3 \text{ and belongs to } \dot{\mathcal{I}},$$

so

$$b \Vdash_{\mathbb{Q}_\kappa} \inf_{\alpha\in\check{J}} \vec{\rho}_\alpha^\bullet \setminus \sup_{\alpha\in\check{K}} \vec{\rho}_\alpha^\bullet = \vec{\sigma}_\tau^\bullet \subseteq \dot{c}.$$

As a and $\dot{c}$ are arbitrary, this proves the result. **Q**

(g) If $\alpha < \theta$ is such that $g_\alpha \in G_0$, then

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^\bullet \neq 0.$$

P? Otherwise, there is a non-zero $a \in \mathfrak{G}_\kappa$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^\bullet = 0,$$

that is,

$$a \Vdash_{\mathbb{Q}_\kappa} \{\xi : \vec{\rho}_\alpha(\xi) \ne 0\} \in \dot{\mathcal{I}},$$

and there is an $I \in \mathcal{I}$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \{\xi : \vec{\rho}_\alpha(\xi) \ne 0\} \subseteq \check{I}.$$

In this case, for $\xi \in \lambda \setminus I$,

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\rho}_\alpha(\xi) = 0,$$

that is,

$$0 = a \cap \rho_\alpha(\xi) = a \cap e_{g_\alpha(\xi)}.$$

But $\lambda \setminus I$ is infinite and g_α is injective, so $\{g_\alpha(\xi) : \xi \in \lambda \setminus I\}$ is infinite and $a = 0$. **XQ**

Similarly,

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\rho}^{\bullet}_\alpha \ne 1.$$

As this is true whenever $g_\alpha \in G_0$, and $\#(G_0) = \theta$, we see that

$$\Vdash_{\mathbb{Q}_\kappa} \#(\{\alpha : \vec{\rho}^{\bullet}_\alpha \notin \{0,1\}\}) = \check{\theta}.$$

(h) Putting (e)-(g) together,

$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ has a Boolean-independent family of size $\check{\theta}$ generating an order-dense subalgebra; being Dedekind complete, it is isomorphic to $\mathrm{RO}(\{0,1\}^{\check{\theta}}) \cong \mathfrak{G}_{\check{\theta}}$

by 515N and 514Ih.

555H Corollary Suppose that λ is a two-valued-measurable cardinal and $\kappa = 2^\lambda$. Then

$\Vdash_{\mathbb{Q}_\kappa}$ there is a non-trivial atomless σ-centered power set σ-quotient algebra.

proof (a) Note first that $\{0,1\}^{\mathfrak{c}}$ is separable (4A2B(e-ii)), so $\mathfrak{G}_{\mathfrak{c}} \cong \mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ is σ-centered (514H(b-iii)).

(b) Taking $\mathcal{I}$ and $\dot{\mathcal{I}}$ as in 555G, we have

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \cong \mathfrak{G}_{\check{\theta}}$$

where $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$. But since θ lies between κ and the cardinal power $\kappa^\lambda = \kappa^\omega = \kappa$, we have

$$\Vdash_{\mathbb{Q}_\kappa} \check{\theta} = \check{\kappa} = \mathfrak{c}$$

(554B), and

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \cong \mathfrak{G}_{\mathfrak{c}} \text{ is } \sigma\text{-centered}$$

by (a). At the same time,

$$\Vdash_{\mathbb{Q}_\kappa} \check{\lambda} \le \mathfrak{c} \text{ so } \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \text{ is atomless}$$

by 541P.

555I The next example relies on some interesting facts which I have not yet had any compelling reason to spell out. I must begin with a definition which has so far been confined to the exercises.

Definition A Boolean algebra $\mathfrak{A}$ has the **Egorov property** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a countable partition B of unity such that $\{a : a \in A_n,\ a \cap b \ne 0\}$ is finite for every $b \in B$ and $n \in \mathbb{N}$.

555J Lemma (a) Suppose that X is a set and $\#(X) < \mathfrak{b}$. Then $\mathcal{P}X$ has the Egorov property.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with the Egorov property and I a σ-ideal of $\mathfrak{A}$. Then $\mathfrak{A}/I$ has the Egorov property.

(c) A ccc Boolean algebra with the Egorov property is weakly (σ,∞)-distributive.

proof (a) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of countable partitions of X; enumerate each A_n as $\langle a_{ni}\rangle_{i<N_n}$ where $N_n \in \mathbb{N}\cup\{\omega\}$ for each n. For each $x \in X$ and $n \in \mathbb{N}$, let $f_x(n) \in N_n$ be such that $x \in a_{n,f_x(n)}$. Because $\#(X) < \mathfrak{b}$, there is an $f : \mathbb{N} \to \mathbb{N}$ such that $\{n : f(n) < f_x(n)\}$ is finite for every $x \in X$ (522C); set $g(x) = \sup\{n : f(n) < f_x(n)\}$. Now set $b_n = \{x : x \in X,\ n = \max_{m\le g(x)} f_m(x)\}$ for each n, and $B = \{b_n : n \in \mathbb{N}\}$; then B is a partition of X, and for any m, $n \in \mathbb{N}$ we have $b_n \cap a_{mi} = \emptyset$ whenever $\max(n, f(m)) < i < N_m$. So $\{a : a \in A_m,\ b_n \cap a \ne \emptyset\}$ is finite for all m, $n \in \mathbb{N}$. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{P}X$ has the Egorov property.

(b) Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of countable partitions of unity in $\mathfrak{A}/I$. For each $n\in\mathbb{N}$, we can choose a countable disjoint family $A_n\subseteq\mathfrak{A}$ such that $C_n=\{a^\bullet:a\in A_n\}$; set $A'_n=A_n\cup\{1\setminus\sup A_n\}$, so that A'_n is a countable partition of unity in $\mathfrak{A}$. Let B be a countable partition of unity in $\mathfrak{A}$ such that $\{a:a\in A'_n,\,a\cap b\ne 0\}$ is finite for every $n\in\mathbb{N}$. Then $D=\{b^\bullet:b\in B\}$ is a countable partition of unity in $\mathfrak{A}/I$ and $\{c:c\in C_n,\,c\cap d\ne 0\}$ is finite for every $d\in D$ and $n\in\mathbb{N}$.

(c) This is elementary, because every partition of unity in $\mathfrak{A}$ is countable.

555K Lemma Suppose that X is a set and $\#(X)<\mathfrak{p}$. Then there is a countable set $\mathcal{A}\subseteq\mathcal{P}X$ such that $\mathcal{P}X$ is the σ-algebra generated by $\mathcal{A}$.

proof (For the background to this result, see Fremlin 84a, §21. Here I give a brisk resumé of the proof there.) Let $\langle I_x\rangle_{x\in X}$ be a family of infinite subsets of $\mathbb{N}$ such that $I_x\cap I_y$ is finite for all distinct $x,\,y\in X$ (5A1Fa). Set $A_n=\{x:n\in I_x\}$ for $n\in\mathbb{N}$.

Take any $A\subseteq X$ and set $P_A=\mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\})\times[X\setminus A]^{<\omega}$, partially ordered by saying that

$(f,J)\le(f',J')$ if f' extends f, $J'\supseteq J$ and whenever $x\in J$ and $i\in I_x\cap\operatorname{dom}f'\setminus\operatorname{dom}f$, then $f'(i)=0$.

Then P_A is σ-centered upwards because $\{(f,J):J\in[X\setminus A]^{<\omega}\}$ is upwards-centered for every $f\in\mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\})$. For $x\in A$ and $m\in\mathbb{N}$ set

$$Q_{xm}=\{(f,J):(f,J)\in P_A,\,f(i)=1\text{ for some }i\in I_x\setminus m\};$$

for $x\in X\setminus A$ set

$$Q'_x=\{(f,J):(f,J)\in P_A,\,x\in J\}.$$

Then every Q_{xm} and every Q'_x is cofinal with P_A. Because $\#(X)<\mathfrak{p}$, there is an upwards-directed $R\subseteq P_A$ meeting every Q_{xm} and every Q'_x. Set $L=\bigcup_{(f,J)\in R}\{i:f(i)=1\}$. Now

$$A=\{x:x\in X,\,I_x\cap L\text{ is infinite}\}=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in L\setminus n}A_m$$

belongs to the σ-algebra generated by $\mathcal{A}=\{A_n:n\in\mathbb{N}\}$, and we have a suitable family.

555L Lemma Suppose that $\#(X)<\max(\mathfrak{s},\operatorname{cov}\mathcal{M})$, where $\mathfrak{s}$ is the splitting number and $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$. Suppose that Σ is a σ-algebra of subsets of X such that (X,Σ) is countably separated, in the sense that there is a sequence in Σ separating the points of X, and that $\mathcal{I}$ is a σ-ideal of Σ containing singletons. Then there is no non-zero Maharam submeasure on $\Sigma/\mathcal{I}$.

proof (a) Let μ be a Maharam submeasure on $\Sigma/\mathcal{I}$. Then we have a Maharam submeasure ν on Σ defined by setting $\nu E=\mu E^\bullet$ for every $E\in\Sigma$, and $\nu\{x\}=0$ for every $x\in X$.

(b) ν is atomless. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X, and $F\in\Sigma$ such that $\nu F>0$. Choose $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively so that $F_0=F$ and, given that $\nu F_n>0$, F_{n+1} is either $F_n\cap E_n$ or $F_n\setminus E_n$ and $\nu F_{n+1}>0$. Then $\bigcap_{n\in\mathbb{N}}F_n$ has at most one member, so $\lim_{n\to\infty}\nu F_n=0$, and there is an n such that $\nu F_n=\nu(F\cap F_n)$ and $\nu(F\setminus F_n)$ are non-zero. **Q**

(c) By 539G,

$$\operatorname{non}\mathcal{N}(\nu)\ge\max(\mathfrak{s},\operatorname{cov}\mathcal{M})>\#(X)$$

and $\nu X=0$, so μ is identically 0.

555M Główczyński's example (Główczyński 91, Balcar Jech & Pazák 05) Let λ be a two-valued-measurable cardinal, and $\mathbb{P}$ a ccc forcing notion such that

$$\Vdash_{\mathbb{P}}\check\lambda<\mathfrak{m}$$

(5A3P). Then, taking $\mathcal{I}$ to be the null ideal of a witnessing measure on λ, and $\dot{\mathcal{I}}$ to be a $\mathbb{P}$-name for the ideal of subsets of $\check\lambda$ generated by $\check{\mathcal{I}}$, as in 555B,

$\Vdash_{\mathbb{P}}\mathcal{P}\check\lambda/\dot{\mathcal{I}}$ is ccc, atomless, Dedekind complete, weakly (σ,∞)-distributive, has Maharam type ω and is not a Maharam algebra.

proof We know from 555B that

$$\Vdash_{\mathbb{P}}\mathcal{P}\check\lambda/\dot{\mathcal{I}}\text{ is ccc and Dedekind complete.}$$

By 541P, because $\mathfrak{m}\le\mathfrak{c}$,

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ is atomless.

By 555J, because $\mathfrak{m} \le \mathfrak{b}$,

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}$ has the Egorov property, so $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ has the Egorov property and is weakly (σ,∞)-distributive.

Moreover, because $\mathfrak{m} \le \mathfrak{p}$, 555K tells us that

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}$ is σ-generated by a countable set, so $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ is σ-generated by a countable set and has countable Maharam type.

Finally, since we certainly have

$\Vdash_{\mathbb{P}} \check{\lambda} < \mathfrak{m} \le \operatorname{cov}\mathcal{M} \le \mathfrak{c}$, so there is a separable metrizable topology on $\check{\lambda}$,

555L shows that

$\Vdash_{\mathbb{P}}$ there is no non-zero Maharam submeasure on $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$.

555N Supercompact cardinals and the normal measure axiom If we allow ourselves to go a good deal farther than 'measurable cardinal' we can use similar techniques to find a forcing language in which NMA is true.

Definition An uncountable cardinal κ is **supercompact** if for every set X there is a κ-additive maximal ideal $\mathcal{I}$ of subsets of $S = [X]^{<\kappa}$ such that

(α) $\{s : s \in S,\, x \notin s\} \in \mathcal{I}$ for every $x \in X$,

(β) if $A \subseteq S$, $A \notin \mathcal{I}$ and $f : A \to X$ is such that $f(s) \in s$ for every $s \in A$, then there is an $x \in X$ such that $\{s : s \in A,\, f(s) = x\} \notin \mathcal{I}$.

(Compare 545D.)

555O Proposition A supercompact cardinal is two-valued-measurable.

proof If κ is supercompact, let $\mathcal{I}$ be a κ-additive maximal ideal of subsets of $S = [\kappa]^{<\kappa}$ satisfying (α) and (β) of 555N. Define $f : S \to \kappa$ by setting $f(s) = \min(\kappa \setminus s)$ for $s \in S$. Then $\mathcal{J} = \{J : J \subseteq \kappa,\, f^{-1}[J] \in \mathcal{I}\}$ is a κ-additive maximal ideal of subsets of κ. If $\xi < \kappa$, then $A = \{s : s \in S,\, s \not\supseteq \xi\} \in \mathcal{I}$, because $\mathcal{I}$ is κ-additive and (α) is true, and $f(s) \ge \xi$ for every $s \in A$; so $\xi \in \mathcal{J}$, and $\mathcal{J}$ contains all singletons. Thus $\mathcal{J}$ witnesses that κ is two-valued-measurable.

555P Theorem (Prikry 75, Fleissner 91) Suppose that κ is a supercompact cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}$ the normal measure axiom and the product measure extension axiom are true.

proof (a) Life will be a little easier if I start by pointing out that we can work with a variation of NMA as stated in 545D. First, for a set X and an uncountable cardinal λ let $\ddagger(X,\lambda)$ be the statement

there is a λ-additive probability measure ν on $S = [X]^{<\lambda}$, with domain $\mathcal{P}S$, such that

(α) $\nu\{s : x \in s \in S\} = 1$ for every $x \in X$,

(β) if $g : S \setminus \{\emptyset\} \to X$ is such that $g(s) \in s$ for every $s \in S \setminus \{\emptyset\}$, then there is a countable set $K \subseteq X$ such that $\nu g^{-1}[K] = 1$.

Now the point is that if $\ddagger(\alpha,\mathfrak{c})$ is true for every ordinal α, then the normal measure axiom is true. **P** Let X be any set. Since, as always, we are working with the axiom of choice, X is equipollent with some ordinal and $\ddagger(X,\mathfrak{c})$ is true; let ν be a measure on $S = [X]^{<\mathfrak{c}}$ as above. Given $A \subseteq S$ and a function $f : A \to X$ which is regressive in the sense of (β) in 545D, then we can extend f to a function $g : S \setminus \{\emptyset\}$ which is regressive in the sense of (β) here. If K is a countable set such that $g^{-1}[K]$ is conegligible, and A is not negligible, then there must be a $\xi \in K$ such that $A \cap g^{-1}[\{\xi\}] = f^{-1}[\{\xi\}]$ is not negligible, as required in 545D. **Q**

(b) For the time being (down to the end of (d) below), fix an ordinal α. Let $\mathcal{I}$ be a κ-additive maximal ideal of subsets of $S = [\alpha]^{<\kappa}$ as in 555N, and ν the corresponding measure on S, setting $\nu A = 0$ and $\nu(S \setminus A) = 1$ if $A \in \mathcal{I}$. By 555C, we have a corresponding $\mathbb{P}_\kappa$-name $\dot{\nu}$ for a measure on $\check{S}$. Now

$$\Vdash_{\mathbb{P}_\kappa} \check{S} \subseteq [\check{\alpha}]^{<\check{\kappa}},$$

so we have a $\mathbb{P}_\kappa$-name $\dot{\mu}$ such that

$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a measure with domain $\mathcal{P}([\check{\alpha}]^{<\check{\kappa}})$ and $\dot{\mu}(D) = \dot{\nu}(D \cap \check{S})$ for every $D \subseteq [\check{\alpha}]^{<\check{\kappa}}$.

By 555C,

$\Vdash_{\mathbb{P}_\kappa} \dot{\nu}$ is a $\check{\kappa}$-additive probability measure, so $\dot{\mu}$ is a $\check{\kappa}$-additive probability measure.

(c) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{s : \xi \in s \in [\check{\alpha}]^{<\check{\kappa}}\} = 1$ for every $\xi < \check{\alpha}$.

P If $a \in \mathfrak{B}_\kappa^+$ and $\dot{\xi}$ are such that $a \Vdash_{\mathbb{P}} \dot{\xi} < \check{\alpha}$, take any b stronger than a and $\xi < \alpha$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot{\xi} = \check{\xi}$. Now $I = \{s : s \in S,\, \xi \notin s\} \in \mathcal{I}$ so

$$b \Vdash_{\mathbb{P}_\kappa} 0 = \dot\nu\check I = \dot\nu\{s : s \in \check S,\, \check\xi \notin s\} = \dot\mu\{s : s \in [\check\alpha]^{<\check\kappa},\, \check\xi \notin s\},$$
$$1 = \dot\mu\{s : s \in [\check\alpha]^{<\check\kappa},\, \check\xi \in s\}.$$

As b and ξ are arbitrary,

$$a \Vdash_{\mathbb{P}_\kappa} \dot\mu\{s : \dot\xi \in s \in [\check\alpha]^{<\check\kappa}\} = 1;$$

as a and $\dot\xi$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\{s : \xi \in s \in [\check\alpha]^{<\check\kappa}\} = 1 \text{ for every } \xi < \check\alpha. \quad \mathbf{Q}$$

(d) Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot f$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot f : [\check\alpha]^{<\check\kappa} \setminus \{\emptyset\} \to \check\alpha \text{ is a function and } \dot f(s) \in s \text{ whenever } \emptyset \ne s \in [\check\alpha]^{<\check\kappa}.$$

For each $s \in S \setminus \{\emptyset\}$, we have

$$a \Vdash_{\mathbb{P}_\kappa} \dot f(\check s) \in \check s;$$

because $\mathbb{P}_\kappa$ is ccc, there is a countable set $J_s \subseteq s$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot f(\check s) \in \check J_s$ (5A3Nc). Let $\langle h_n(s)\rangle_{n\in\mathbb{N}}$ be a sequence running over J_s. For each $n \in \mathbb{N}$, we have a $\beta_n < \alpha$ such that $\{s : s \in S \setminus \{\emptyset\},\, h_n(s) \ne \beta_n\} \in \mathcal{I}$. Set $K = \{\beta_n : n \in \mathbb{N}\}$; since $\mathcal{I}$ is a σ-ideal, $I = \{s : s \in S \setminus \{\emptyset\},\, J_s \not\subseteq K\}$ belongs to $\mathcal{I}$. But in this case, $\Vdash_{\mathbb{P}_\kappa} \dot\mu\check I = \dot\nu\check I = 0$ and

$$a \Vdash_{\mathbb{P}_\kappa} \dot f(\check s) \in \check J_s \subseteq \check K$$

whenever $s \in S \setminus I$, so

$$a \Vdash_{\mathbb{P}_\kappa} \dot\mu(\dot f^{-1}[\check K]) = \dot\nu(\check S \cap \dot f^{-1}[\check K]) \ge \dot\nu(\check S \setminus \check I) = 1,$$

while of course $\Vdash_{\mathbb{P}_\kappa} \check K$ is countable.

(e) What this means is that

$$\Vdash_{\mathbb{P}_\kappa} \ddagger\,(\check\alpha, \check\kappa)$$

for every ordinal α; since forcing adds no new ordinals (5A3Na),

$$\Vdash_{\mathbb{P}_\kappa} \ddagger\,(\alpha, \check\kappa) \text{ for every ordinal } \alpha.$$

But 552B, with 555O, tells us that

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \check\kappa, \text{ so } \ddagger(\alpha, \mathfrak{c}) \text{ for every ordinal } \alpha;$$

with (a) above and 545E, we get

$$\Vdash_{\mathbb{P}_\kappa} \text{NMA and PMEA.}$$

555Q All forcing constructions of quasi-measurable cardinals start from two-valued-measurable cardinals, and there is a reason for this.

Theorem (Solovay 71) If κ is an uncountable cardinal and $\mathcal{I}$ is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, then

$L(\mathcal{I}) \vDash \kappa$ is two-valued-measurable and the generalized continuum hypothesis is true.

Remarks The proof employs techniques not used elsewhere in this treatise, so I omit it entirely, to the point of not explaining what $L(\mathcal{I})$ is or what the symbol $\vDash$ means; I remark only that $L(\mathcal{I})$ is a proper class containing every ordinal and the set $\mathcal{I}$, and that the theorem says that the axioms of ZFC, together with 'κ is two-valued-measurable' and the generalized continuum hypothesis, are true when relativized appropriately to the class $L(\mathcal{I})$. For more, see Jech 78, p. 416, Theorem 82a.

555X Basic exercises (a) Suppose that λ is a real-valued-measurable cardinal with witnessing probability ν, and κ a cardinal. Let $\dot\mu$ be the $\mathbb{P}_\kappa$-name for a measure on $\check\lambda$ as defined in 555C. Show that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\check A = (\nu A)\check{}$$

for any $A \subseteq \lambda$.

(b) Suppose that λ is a two-valued-measurable cardinal. Set $\kappa = 2^\lambda$. Show that

$\Vdash_{\mathbb{P}_\kappa} \check\lambda$ is an atomlessly-measurable cardinal with a witnessing probability with Maharam type $\mathfrak{c} = 2^{\check\lambda}$.

(c) Suppose that λ is a two-valued-measurable cardinal and that the generalized continuum hypothesis is true. Set $\kappa = \lambda^{(+\omega)}$. Show that

$\Vdash_{\mathbb{P}_\kappa} \check{\lambda}$ is an atomlessly-measurable cardinal with a witnessing probability with Maharam type less than $\mathfrak{c}$.

(d) Suppose that λ is a two-valued-measurable cardinal and $\kappa = 2^\lambda$. Show that

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a non-trivial atomless } \sigma\text{-linked power set } \sigma\text{-quotient algebra.}$$

555Y Further exercises (a) Suppose that X is a set, and $\mathcal{I}$ a proper ideal of subsets of X containing singletons. Let $\mathbb{P}$ be a forcing notion such that $\operatorname{sat}\mathbb{P} \le \operatorname{add}\mathcal{I}$, and $\dot{\mathcal{I}}$ a $\mathbb{P}$-name for the ideal of subsets of $\check{X}$ generated by $\check{\mathcal{I}}$, as in 555B. (i) Show that

$$\Vdash_{\mathbb{P}} \operatorname{add}\dot{\mathcal{I}} = (\operatorname{add}\mathcal{I})\check{}.$$

(ii) Suppose that $\operatorname{sat}(\mathcal{P}X/\mathcal{I}) < \operatorname{add}\mathcal{I}$. Set $\theta = \max(\operatorname{sat}\mathbb{P}, \operatorname{sat}(\mathcal{P}X/\mathcal{I}))$. Show that

$$\Vdash_{\mathbb{P}} \check{\theta} \text{ is a cardinal, } \dot{\mathcal{I}} \text{ is } \check{\theta}\text{-saturated in } \mathcal{P}\check{X} \text{ and } \mathcal{P}\check{X}/\dot{\mathcal{I}} \text{ is Dedekind complete.}$$

(iii) Show that if $X = \lambda$ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal on λ, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a normal ideal on } \check{\lambda}.$$

(b) In 555B, show that if $\mathcal{I}$ is θ-saturated in $\mathcal{P}\lambda$, where θ is an uncountable cardinal such that $\operatorname{cf}[\theta]^{\le\omega} < \operatorname{add}\mathcal{I}$, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is } \check{\theta}\text{-saturated in } \mathcal{P}\check{\lambda}.$$

(c) Suppose that λ is a two-valued-measurable cardinal, and that $\mathbb{P}$ is a forcing notion with $\#(\mathbb{P}) < \lambda$. Show that $\Vdash_{\mathbb{P}} \check{\lambda}$ is a two-valued-measurable cardinal.

(d) Suppose that κ is a two-valued-measurable cardinal, and that $\mathfrak{m} = \mathfrak{c}$. Show that

$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c}$ is real-valued-measurable, $\mathfrak{b} = \mathfrak{d} = \check{\mathfrak{c}}$ and the shrinking number of the Lebesgue null ideal is at least $\check{\mathfrak{c}}$.

(e) Let κ be a supercompact cardinal. Show that $\Box_\lambda$ is false for every $\lambda \ge \kappa$.

555Z Problems (a) In 555B, what can we say about the π-weight of $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$?

(b) Suppose that λ is an atomlessly-measurable cardinal with a normal witnessing probability. Let $\langle A_\eta\rangle_{\eta<\omega_1}$ be a family of non-negligible subsets of λ. Must there be a countable set meeting every A_η? (See 555F and 521Xd.)

555 Notes and comments The point of Solovay's theorems 555D and 555Q is that they are relative consistency results. Continuing the discussion in the notes to §541, write '∃2vmc', '∃qmc', '∃amc' for the sentences 'there is a two-valued-measurable cardinal', 'there is a quasi-measurable cardinal' and 'there is an atomlessly-measurable cardinal'. I have already noted that there are fundamental metamathematical reasons why we cannot have a proof, in ZFC, that

if ZFC is consistent then ZFC + ∃qmc is consistent

unless ZFC is actually *inconsistent*. But 555D tells us that

if ZFC + ∃2vmc is consistent, then ZFC + ∃amc is consistent

and 555Q that

if ZFC + ∃qmc is consistent, then ZFC + ∃2vmc is consistent.

Since ∃qmc is actually a consequence of both ∃2vmc and ∃amc, we see that

if one of ZFC + ∃2vmc, ZFC + ∃amc, ZFC + ∃qmc is consistent, so are the others;

that is, ∃2vmc, ∃amc and ∃qmc are equiconsistent in ZFC. Of course they are not in general *equiveridical* (unless all are disprovable); as noted in 555Q, if we start from a universe in which ∃qmc is true, we can move to one in which ∃2vmc and CH are both true, so that ∃amc is false, and there are easier arguments to show that if we start with ∃amc, we can move to

∃amc + 'there are no strongly inaccessible cardinals', so that we have ∃amc true but ∃2vmc false.

The reason for working through these equiconsistency results is to show that assertions like NMA, PMEA and ∃amc, which are of interest in measure theory and general topology, are no more dangerous than appropriate assertions about large cardinals which have been explored in depth (Jech 78, chap. 6; Kanamori 03, §22; Jech 03, §20), and for which we can have corresponding confidence that either they are consistent with ZFC, or that an eventual contradiction would lead to an earthquake, and rescue (if it came) would be from outside measure theory.

In §544 I examined some of the consequences of supposing that there is an atomlessly-measurable cardinal; for instance, that there are many Sierpiński sets (544G). It is not an accident that we get similar properties of random real models (552E). If we want to know if something might be implied by the existence of an atomlessly-measurable cardinal, the first step is to look at what can be determined in the forcing models of 555C. This is often straightforward; for instance, since $\mathfrak{d}$ is not changed by random real forcing, and since $\mathfrak{d}$ must be much lower than any two-valued-measurable cardinal, it must be much lower than any atomlessly-measurable cardinal created by random real forcing. But it is quite unclear that the same can be said about atomlessly-measurable cardinals in general (544Zd). I offer 555F as another example of a phenomenon which appears in Solovay's models but which is not known to be true for all atomlessly-measurable cardinals (555Zb).

In §546 I gave some of the Gitik-Shelah results showing that non-trivial 'simple' algebras cannot be power set σ-quotient algebras. Of course this depends a good deal on what we mean by 'simple'. Looking at the basic cardinal functions, we see that (at least if there can be measurable cardinals) then there can be non-trivial ccc power set σ-quotient algebras which are σ-centered or have countable Maharam type (555H, 555M). But they are still very far away from any algebra which can be specified without (perhaps implicitly) using a two-valued-measurable cardinal at some stage. We cannot have a non-trivial power set σ-quotient algebra with countable π-weight (546Xb), but I do not see how to rule out 'small' π-weight in general (546Zd, 555Za).

556 Forcing with Boolean subalgebras

I propose now to describe a completely different way in which forcing can be used to throw light on problems in measure theory. Rather than finding forcing models of new mathematical universes, we look for models which will express structures of the ordinary universe in new ways. The problems to which this approach seems to be most relevant are those centered on invariant algebras: in ergodic theory, fixed-point algebras; in the theory of relative independence, the core σ-algebras.

Most of the section is taken up with development of basic machinery. The strategic plan is straightforward enough: given a specific Boolean algebra $\mathfrak{C}$ which seems to be central to a question in hand, force with $\mathfrak{C}\setminus\{0\}$, and translate the question into a question in the forcing language. In order to do this, we need an efficient scheme for automatic translation. This is what 556A-556L and 556O are setting up. The translation has to work both ways, since we need to be able to deduce properties of the ground model from properties of the forcing model.

There are four actual theorems for which I offer proofs by these methods. The first three are 556M (a strong law of large numbers), 556N (Dye's theorem on orbit-isomorphic measure-preserving transformations) and 556P (Kawada's theorem on invariant measures). In each of these, the aim is to prove a general form of the theorem from the classical special case in which the algebra $\mathfrak{C}$ is trivial. In the final example 556S (I.Farah's description of the Dedekind completion of the asymptotic density algebra $\mathfrak{Z}$), we have a natural subalgebra $\mathfrak{C}$ of $\mathfrak{Z}$ and a structure in the corresponding forcing universe to which we can apply Maharam's theorem.

556A Forcing with Boolean subalgebras Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+=\mathfrak{C}\setminus\{0\}$, active downwards.

(a) If $a\in\mathfrak{A}$, the **forcing name for** a **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot a=\{(\check b,p):p\in\mathfrak{C}^+,\,b\in\mathfrak{A},\,p\cap b\subseteq a\}.$$

(b) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then the **forcing name for** $\mathfrak{B}$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name $\{(\dot b,1):b\in\mathfrak{B}\}$, where here $1=1_{\mathfrak{A}}=1_{\mathfrak{B}}=1_{\mathfrak{C}}$.

(c) For each of the binary operations $\odot=\cap\,,\ \cup\,,\ \triangle\,,\ \setminus$ on $\mathfrak{A}$, the **forcing name for** $\odot$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\odot=\{(((\dot a,\dot b),(a\odot b)^{\boldsymbol{\cdot}}),1):a,\,b\in\mathfrak{A}\}.$$

(d) The **forcing name for** $\subseteq$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\subseteq\ =\{((\dot a,\dot b),1):a,\,b\in\mathfrak{A},\,a\subseteq b\}.$$

(e) Let $\pi:\mathfrak{A}\to\mathfrak{A}$ be a ring homomorphism such that $\pi c\subseteq c$ for every $c\in\mathfrak{C}$. In this case, I will say that the **forcing name for** π **over** $\mathfrak{C}$ is the $\mathbb{P}$-name $\{((\dot a,(\pi a)^\bullet),1):a\in\mathfrak{A}\}$.

(f) Now suppose that $\mathfrak{A}$ is Dedekind σ-complete. For $u\in L^0(\mathfrak{A})$, the **forcing name for** u **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name $\{((\check\alpha,[\![u>\alpha]\!]^\bullet),1):\alpha\in\mathbb{Q}\}$.

Remark We shall need to agree on what it is that the formula $L^0(\mathfrak{A})$ abbreviates. The primary definition in 364A speaks of functions from $\mathbb{R}$ to $\mathfrak{A}$. Because $\mathbb{R}$ is inadequately absolute this is not convenient here, and I will move to the alternative version in 364Be[5]: a member u of $L^0(\mathfrak{A})$ is a family $\langle[\![u>\alpha]\!]\rangle_{\alpha\in\mathbb{Q}}$ in $\mathfrak{A}$ such that

$$[\![u>\alpha]\!]=\sup_{\beta\in\mathbb{Q},\beta>\alpha}[\![u>\beta]\!]\text{ for every }\alpha\in\mathbb{Q},$$

$$\inf_{n\in\mathbb{N}}[\![u>n]\!]=0,\quad\sup_{n\in\mathbb{N}}[\![u>-n]\!]=1.$$

556B Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $p\in\mathfrak{C}^+$, a, $b\in\mathfrak{A}$ and $\dot a$, $\dot b$ are the forcing names of a, b over $\mathfrak{C}$, then

$$p\Vdash_{\mathbb{P}}\dot a=\dot b$$

iff $\operatorname{upr}(p\cap(a\triangle b),\mathfrak{C})=0$, that is, for every q stronger than p there is an r stronger than q such that $r\cap a=r\cap b$. In particular,

$$p\Vdash_{\mathbb{P}}\dot a=\dot b$$

whenever $p\cap a=p\cap b$.

(b) Writing $\dot\odot$ for the forcing name for $\odot$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}}\dot\odot\text{ is a binary operation on }\dot{\mathfrak{A}}\text{ and }\dot a\,\dot\odot\,\dot b=(a\odot b)^\bullet$$

for each of the binary operations $\odot=\cap,\ \cup,\ \triangle$ and $\setminus$ and all a, $b\in\mathfrak{A}$.

(c) All the standard identities translate. For instance,

$$\Vdash_{\mathbb{P}}x\mathbin{\dot\cap}(y\mathbin{\dot\triangle}z)=(x\mathbin{\dot\cap}y)\mathbin{\dot\triangle}(x\mathbin{\dot\cap}z)\text{ for all }x,\ y,\ z\in\dot{\mathfrak{A}}.$$

(d)

$\Vdash_{\mathbb{P}}\dot{\mathfrak{A}}$, with the operations $\dot\triangle$, $\dot\cap$, $\dot\cup$ and $\dot\setminus$, is a Boolean algebra, with zero $\dot 0$ and identity $\dot 1$.

(e)(i) Writing $\dot\subseteq$ for the forcing name for $\subseteq$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}}\ \dot\subseteq\text{ is the inclusion relation in the Boolean algebra }\dot{\mathfrak{A}}.$$

(ii) For a, $b\in\mathfrak{A}$ and $p\in\mathfrak{C}^+$,

$$p\Vdash_{\mathbb{P}}\dot a\mathrel{\dot\subseteq}\dot b$$

iff $\operatorname{upr}(p\cap a\setminus b,\mathfrak{C})=0$.

(f) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}}\dot{\mathfrak{B}}\text{ is a Boolean subalgebra of }\dot{\mathfrak{A}}.$$

proof (a)(i) Recall that $\operatorname{upr}(a,\mathfrak{C})=\inf\{c:a\subseteq c\in\mathfrak{C}\}$ if the infimum is defined in $\mathfrak{C}$ (313S[6]). So $\operatorname{upr}(a,\mathfrak{C})=0$ iff for every non-zero $c\in\mathfrak{C}$ there is a $c'\in\mathfrak{C}$ such that $a\subseteq c'$ and $c\not\subseteq c'$; that is, for every non-zero $c\in\mathfrak{C}$ there is a non-zero $c'\in\mathfrak{C}$ such that $c'\subseteq c\setminus a$. In the present context, we see that for $p\in\mathfrak{C}^+$ and a, $b\in\mathfrak{A}$, $\operatorname{upr}(p\cap(a\triangle b),\mathfrak{C})=0$ iff for every q stronger than p there is an r stronger than q such that $r\cap(a\triangle b)=0$.

(ii) Suppose that $p\cap a=p\cap b$, that $q\in\mathfrak{C}^+$ is stronger than p, and that $\dot x$ is a $\mathbb{P}$-name such that $q\Vdash_{\mathbb{P}}\dot x\in\dot a$. Then there are an $r\in\mathfrak{C}^+$, a $d\in\mathfrak{A}$ such that $(\check d,r)\in\dot a$, and an s stronger than both r and q such that $s\Vdash_{\mathbb{P}}\dot x=\check d$. In this case

$$s\cap d\subseteq p\cap r\cap d\subseteq p\cap a\subseteq b,$$

so $(\check d,s)\in\dot b$ and

$$s\Vdash_{\mathbb{P}}\dot x=\check d\in\dot b.$$

As q and $\dot x$ are arbitrary,

[5] Later editions only.

[6] Formerly 314V.

$$p \Vdash_{\mathbb{P}} \dot{a} \text{ is a subset of } \dot{b};$$

similarly,

$$p \Vdash_{\mathbb{P}} \dot{b} \text{ is a subset of } \dot{a} \text{ and } \dot{b} = \dot{a}.$$

(iii) If $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$, then for every q stronger than p there is an r stronger than q such that $r \cap a = r \cap b$ and $r \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$, by (ii). As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$.

(iv) Now suppose that $p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$ and that q is stronger than p. Then $(\check{a}, q) \in \dot{a}$, so $q \Vdash_{\mathbb{P}} \check{a} \in \dot{a} = \dot{b}$. There must therefore be a $(\check{d}, r) \in \dot{b}$ and an s stronger than both r and q such that $s \Vdash_{\mathbb{P}} \check{a} = \check{d}$; in this case $d = a$, $s \cap a \subseteq r \cap d \subseteq b$ and $s \cap a \setminus b = 0$.

As q is arbitrary, $\operatorname{upr}(p \cap (a \setminus b), \mathfrak{C}) = 0$. Similarly, $\operatorname{upr}(p \cap (b \setminus a), \mathfrak{C}) = 0$. By 313Sb, $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$.

(b) Of course

$$\Vdash_{\mathbb{P}} \dot{\odot} \subseteq (\dot{\mathfrak{A}} \times \dot{\mathfrak{A}}) \times \dot{\mathfrak{A}},$$

just because

$$\Vdash_{\mathbb{P}} \dot{a} \in \dot{\mathfrak{A}}$$

for every $a \in \mathfrak{A}$. To see that $\dot{\odot}$ is a name for a function with domain $\dot{A} \times \dot{A}$, use 5A3H. If $(((\dot{a}_1, \dot{b}_1), (a_1 \odot b_1)^{\bullet}), 1)$ and $(((\dot{a}_2, \dot{b}_2), (a_2 \odot b_2)^{\bullet}), 1)$ are two members of $\dot{\odot}$, and $p \in \mathfrak{C}^+$ is such that

$$p \Vdash_{\mathbb{P}} (\dot{a}_1, \dot{b}_1) = (\dot{a}_2, \dot{b}_2),$$

then

$$\begin{aligned}\operatorname{upr}(p \cap ((a_1 \odot b_1) \triangle (a_2 \odot b_2))) &\subseteq \operatorname{upr}(p \cap ((a_1 \triangle a_2) \cup (b_1 \triangle b_2))) \\ &= \operatorname{upr}(p \cap (a_1 \triangle a_2)) \cup \operatorname{upr}(p \cap (b_1 \triangle b_2)) = 0\end{aligned}$$

by (a) above and 313Sb. So

$$p \Vdash_{\mathbb{P}} (a_1 \odot b_1)^{\bullet} = (a_2 \odot b_2)^{\bullet}$$

by (a) in the other direction. Thus the condition of 5A3H is satisfied, and

$$\Vdash_{\mathbb{P}} \dot{\odot} \text{ is a function,}$$

while of course

$$\Vdash_{\mathbb{P}} \dot{a} \dot{\odot} \dot{b} = (a \odot b)^{\bullet}$$

for all a, $b \in \mathfrak{A}$. Moreover, setting $\dot{A} = \{((\dot{a}, \dot{b}), 1) : a, b \in \mathfrak{A}\}$, 5A3Hb tells us that

$$\Vdash_{\mathbb{P}} \operatorname{dom} \dot{\odot} = \dot{A} = \dot{\mathfrak{A}} \times \dot{\mathfrak{A}}, \text{ so } \dot{\odot} \text{ is a binary operation on } \dot{\mathfrak{A}}.$$

(c) I work through only the given example. Suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$, $\dot{y}$ and $\dot{z}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y},\ \dot{z} \in \dot{\mathfrak{A}}.$$

If q is stronger than p, there are an r stronger than q and a, b, $c \in \mathfrak{A}$ be such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a},\ \dot{y} = \dot{b} \text{ and } \dot{z} = \dot{c}.$$

Then

$$r \Vdash_{\mathbb{P}} \dot{y} \mathbin{\dot{\triangle}} \dot{z} = \dot{b} \mathbin{\dot{\triangle}} \dot{c} = (b \triangle c)^{\bullet},$$

$$r \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} (\dot{y} \mathbin{\dot{\triangle}} \dot{z}) = (a \cap (b \triangle c))^{\bullet} = ((a \cap b) \triangle (a \cap c))^{\bullet} = (\dot{x} \mathbin{\dot{\cap}} \dot{y}) \mathbin{\dot{\triangle}} (\dot{x} \mathbin{\dot{\cap}} \dot{z}).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} (\dot{y} \mathbin{\dot{\triangle}} \dot{z}) = (\dot{x} \mathbin{\dot{\cap}} \dot{y}) \mathbin{\dot{\triangle}} (\dot{x} \mathbin{\dot{\cap}} \dot{z});$$

as p, $\dot{x}$, $\dot{y}$ and $\dot{z}$ are arbitrary,

$$\Vdash_{\mathbb{P}} x \mathbin{\dot{\cap}} (y \mathbin{\dot{\triangle}} z) = (x \mathbin{\dot{\cap}} y) \mathbin{\dot{\triangle}} (x \mathbin{\dot{\cap}} z) \text{ for all } x,\ y,\ z \in \dot{\mathfrak{A}}.$$

(d) This is now elementary, amounting to repeated use of the technique in (c).

(e)(i) It will be enough to show that

$$\Vdash_{\mathbb{P}} \text{ for all } x,\ y \in \dot{\mathfrak{A}},\ x \mathrel{\dot{\subseteq}} y \iff x \mathbin{\dot{\cap}} y = x.$$

P Suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\, \dot{y} \in \dot{\mathfrak{A}}.$$

(α) Suppose that $p \Vdash_{\mathbb{P}} \dot{x} \mathrel{\dot{\subseteq}} \dot{y}$. If q is stronger than p, there are an r stronger than q and a, $b \in \mathfrak{A}$ such that $a \subseteq b$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} \text{ and } \dot{y} = \dot{b}.$$

Now

$$r \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} \dot{y} = (a \cap b)^{\bullet} = \dot{a} = \dot{x};$$

as q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} \dot{y} = \dot{x}$. ($\beta$) Conversely, suppose that $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} \dot{y} = \dot{x}$. If q is stronger than p there are r stronger than q and a, $b \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a},\ \dot{y} = \dot{b},\ (a \cap b)^{\bullet} = \dot{a};$$

now $(((a \cap b)^{\bullet}, \dot{b}), 1) \in\ \dot{\subseteq}$, so $\Vdash_{\mathbb{P}} (a \cap b)^{\bullet} \mathrel{\dot{\subseteq}} \dot{b}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = (a \cap b)^{\bullet} \mathrel{\dot{\subseteq}} \dot{b} = \dot{y}.$$

As q is arbitrary. $p \Vdash_{\mathbb{P}} \dot{x} \mathrel{\dot{\subseteq}} \dot{y}$.

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ for } x,\, y \in \dot{\mathfrak{A}},\ x \mathrel{\dot{\subseteq}} y \iff x \mathbin{\dot{\cap}} y = x. \quad \mathbf{Q}$$

(ii) Now, for a, $b \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{b} & \\ &\text{iff } p \Vdash_{\mathbb{P}} \dot{a} \mathbin{\dot{\cap}} \dot{b} = \dot{a} \\ &\text{iff } p \Vdash_{\mathbb{P}} (a \cap b)^{\bullet} = \dot{a} \\ &\text{iff } \operatorname{upr}(p \cap (a \,\triangle\, (a \cap b)), \mathfrak{C}) = 0 \\ &\text{iff } \operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0. \end{aligned}$$

(f) This should now be easy. As $\dot{\mathfrak{B}} \subseteq \dot{\mathfrak{A}}$, $\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \subseteq \dot{\mathfrak{A}}$. If $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}} \dot{x}, \dot{y} \in \dot{\mathfrak{B}}$, then for every q stronger than p there are r stronger than q and a, $b \in \mathfrak{B}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$ and $\dot{y} = \dot{b}$. In this case

$$r \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} \dot{y} = (a \cap b)^{\bullet} \in \dot{\mathfrak{B}},\ \dot{x} \mathbin{\dot{\triangle}} \dot{y} = (a \,\triangle\, b)^{\bullet} \in \dot{\mathfrak{B}};$$

as q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot{\cap}} \dot{y},\ \dot{x} \mathbin{\dot{\triangle}} \dot{y} \in \dot{\mathfrak{B}}.$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a subring of } \dot{\mathfrak{A}};$$

as we also have $\Vdash_{\mathbb{P}} \dot{1} \in \dot{\mathfrak{B}}$, we get

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a subalgebra of } \dot{\mathfrak{A}}.$$

556C Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Suppose that $\pi : \mathfrak{A} \to \mathfrak{A}$ is a ring homomorphism such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$. If $\dot{\pi}$ is the forcing name for π over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself.}$$

So if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism fixing every point of $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a Boolean homomorphism from } \dot{\mathfrak{A}} \text{ to itself.}$$

(b) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism fixing every point of $\mathfrak{C}$, then

(i) if π is injective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective;

(ii) if π is surjective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is surjective.

(c) If π, $\phi : \mathfrak{A} \to \mathfrak{A}$ are Boolean homomorphisms fixing every point of $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}\dot{\phi} = (\pi\phi)^{\bullet}.$$

(d) If $\pi \in \operatorname{Aut}\mathfrak{A}$ is a Boolean automorphism of $\mathfrak{A}$ fixing every point of $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot\pi \text{ is a Boolean automorphism and } (\dot\pi)^{-1} = (\pi^{-1})^{\bullet}.$$

(e) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism with fixed-point algebra exactly $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot\pi \text{ is ergodic.}$$

proof (a)(i) It will help to note straight away that $\pi c = c \cap \pi 1$ for every $c \in \mathfrak{C}$. **P** The hypothesis is that $\pi c \subseteq c$; because π is a ring homomorphism, $\pi c \subseteq \pi 1$, so $\pi c \subseteq c \cap \pi 1$. Since also

$$\pi c = \pi 1 \setminus \pi(1 \setminus c) \supseteq \pi 1 \setminus (1 \setminus c) = c \cap \pi 1,$$

we have equality. **Q** Consequently

$$c \cap \pi a = c \cap \pi(1 \cap a) = c \cap \pi 1 \cap \pi a = \pi c \cap \pi a = \pi(c \cap a)$$

whenever $c \in \mathfrak{C}$ and $a \in \mathfrak{A}$.

(ii) $\Vdash_{\mathbb{P}} \dot\pi$ is a function from $\dot{\mathfrak{A}}$ to itself. **P** Of course $\Vdash_{\mathbb{P}} \dot\pi \subseteq \dot{\mathfrak{A}} \times \dot{\mathfrak{A}}$. Suppose that $p \in \mathfrak{C}^+$ and that $((\dot a, (\pi a)^{\bullet}), 1)$, $((\dot b, (\pi b)^{\bullet}), 1)$ are two members of $\dot\pi$ such that $p \Vdash_{\mathbb{P}} \dot a = \dot b$. Then for every q stronger than p there is an r stronger than q such that $r \cap a = r \cap b$ (556Ba), in which case

$$r \cap \pi a = \pi(r \cap a) = \pi(r \cap b) = r \cap \pi b.$$

This shows that $p \Vdash_{\mathbb{P}} (\pi a)^{\bullet} = (\pi b)^{\bullet}$, byt 556Ba in the other direction. As a and b are arbitrary, the condition of 5A3H is satisfied, with $\dot A$ there exactly equal to $\dot{\mathfrak{A}}$ here, and

$$\Vdash_{\mathbb{P}} \dot\pi \text{ is a function with domain } \dot{\mathfrak{A}}. \text{ } \mathbf{Q}$$

(iii) $\Vdash_{\mathbb{P}} \dot\pi$ is a ring homomorphism. **P** We have

$$\Vdash_{\mathbb{P}} \dot\pi(\dot a) = (\pi a)^{\bullet}$$

for every $a \in \mathfrak{A}$. So, writing $\odot$ for either $\cap$ or $\triangle$,

$$\begin{aligned}\Vdash_{\mathbb{P}} \dot\pi(\dot a \,\dot\odot\, \dot b) &= \dot\pi(a \odot b)^{\bullet} = (\pi(a \odot b))^{\bullet} \\ &= (\pi a \odot \pi b)^{\bullet} = (\pi a)^{\bullet} \,\dot\odot\, (\pi b)^{\bullet} = (\dot\pi \dot a) \,\dot\odot\, (\dot\pi \dot b)\end{aligned}$$

for all a, $b \in \mathfrak{A}$, and therefore

$$\Vdash_{\mathbb{P}} \dot\pi(x \,\dot\odot\, y) = (\dot\pi x) \,\dot\odot\, (\dot\pi y) \text{ for all } x,\, y \in \dot{\mathfrak{A}}.$$

As this is true for both $\dot\odot = \cap$ and $\dot\odot = \triangle$,

$$\Vdash_{\mathbb{P}} \dot\pi \text{ is a ring homomorphism. } \mathbf{Q}$$

(iv) If π is a Boolean homomorphism, we also have

$$\Vdash_{\mathbb{P}} \dot\pi(\dot 1) = \dot 1,$$

so

$$\Vdash_{\mathbb{P}} \dot\pi \text{ is a Boolean homomorphism.}$$

(b)(i) Let $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ be such that

$$p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}} \text{ and } \dot\pi \dot x = 0.$$

For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot a = \dot x \text{ and } (\pi a)^{\bullet} = 0.$$

So there is an s stronger than r such that $0 = s \cap \pi a = \pi(s \cap a)$, $s \cap a = 0$ and

$$s \Vdash_{\mathbb{P}} \dot x = \dot a = 0.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot x = 0$; as p and $\dot x$ are arbitrary, $\Vdash_{\mathbb{P}} \dot\pi$ is injective.

(ii) Let $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ be such that $p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}}$. For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot a = \dot x$. Now there is a $b \in \mathfrak{A}$ such that $a = \pi b$, in which case

$$r \Vdash_{\mathbb{P}} \dot x = \dot a = \dot\pi \dot b \in \dot\pi[\dot{\mathfrak{A}}].$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot x \in \dot\pi[\dot{\mathfrak{A}}]$; as p and $\dot x$ are arbitrary, $\Vdash_{\mathbb{P}} \dot\pi$ is surjective.

(c) Suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$. For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$, so that

$$r \Vdash_{\mathbb{P}} \dot{\pi}(\dot{\phi}(\dot{x})) = \dot{\pi}(\dot{\phi}(\dot{a})) = \dot{\pi}((\phi a)^{\bullet}) = (\pi\phi a)^{\bullet} = (\pi\phi)^{\bullet}(\dot{a}) = (\pi\phi)^{\bullet}(\dot{x}).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\pi}(\dot{\phi}(\dot{x})) = (\pi\phi)^{\bullet}(\dot{x});$$

as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}\dot{\phi} = (\pi\phi)^{\bullet}$.

(d) By (c), we have

$$\Vdash_{\mathbb{P}} \dot{\pi}(\pi^{-1})^{\bullet} = (\pi^{-1})^{\bullet}\dot{\pi} = \dot{\iota}$$

where $\iota : \mathfrak{A} \to \mathfrak{A}$ is the identity automorphism. But

$$\Vdash_{\mathbb{P}} \dot{\iota} \text{ is the identity on } \dot{\mathfrak{A}}.$$

P If $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$, then for any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = (\iota a)^{\bullet} = \dot{\iota}\dot{a} = \dot{\iota}\dot{x}.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{\iota}\dot{x} = \dot{x}$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\iota}$ is the identity. **Q** Putting these together,

$$\Vdash_{\mathbb{P}} (\pi^{-1})^{\bullet} \text{ is the inverse of } \dot{\pi}.$$

(e) Suppose that $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ are such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{\pi}\dot{x} = \dot{x}.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{a} = \dot{x} = \dot{\pi}\dot{x} = \dot{\pi}\dot{a} = (\pi a)^{\bullet},$$

and an s stronger than r such that $s \cap a = s \cap \pi a$. Now $s \cap a = \pi(s \cap a)$ and $s \cap a \in \mathfrak{C}$. If $s \cap a = 0$, set $s' = s$; then $s' \Vdash_{\mathbb{P}} \dot{x} = 0$. Otherwise, set $s' = s \cap a$; then $s' \Vdash_{\mathbb{P}} \dot{x} = 1$. Thus in either case we have an s' stronger than q such that $s' \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}$ is ergodic.

556D Regularly embedded subalgebras I am trying to set these results out in maximal generality, as usual. However it seems that we need to move almost at once to the case in which our subalgebra is regularly embedded, and we have more effective versions of 556Ba and 556B(e-ii).

Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and for $a \in \mathfrak{A}$ let $\dot{a}$ be the forcing name for a over $\mathfrak{C}$.

(a) For $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

iff $p \cap a = p \cap b$.

(b) Let $\dot{\subseteq}$ be the forcing name for $\subseteq$ over $\mathfrak{C}$. Then for $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{b}$$

iff $p \cap a \subseteq b$.

proof The point is just that $\operatorname{upr}(a, \mathfrak{C}) = 0$ only when $a = 0$, because infima in $\mathfrak{C}$ are also infima in $\mathfrak{A}$ (313N); so that $\operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0$ iff $p \cap a \subseteq b$, and $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$ iff $p \cap a = p \cap b$.

556E Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$; for $a \in \mathfrak{A}$, write $\dot{a}$ for the forcing name for a over $\mathfrak{C}$.

(a) Let $\dot{A}$ be a $\mathbb{P}$-name, and set

$$B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}\}.$$

Then for $d \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$p \Vdash_{\mathbb{P}} \dot{d} \text{ is an upper bound for } \dot{A} \cap \dot{\mathfrak{A}}$$

iff $p \cap b \subseteq d$ for every $b \in B$, and

$$p \Vdash_{\mathbb{P}} \dot{d} = \sup(\dot{A} \cap \dot{\mathfrak{A}})$$

iff $p\cap d=\sup_{b\in B}p\cap b$.

(b)(i) If $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ with supremum a, then

$$\Vdash_{\mathbb{P}}\dot a=\sup_{i\in\check I}\dot a_i.$$[7]

(ii) If $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ with infimum a, then

$$\Vdash_{\mathbb{P}}\dot a=\inf_{i\in\check I}\dot a_i.$$

(c) $\Vdash_{\mathbb{P}}\operatorname{sat}(\dot{\mathfrak{A}})\le\operatorname{sat}(\mathfrak{A})\check{}$.[8]

(d) $\Vdash_{\mathbb{P}}\tau(\dot{\mathfrak{A}})\le\tau(\mathfrak{A})\check{}$.

proof (a)(i) ? Suppose, if possible, that $b\in B$, $p\cap b\not\subseteq d$ and

$$p\Vdash_{\mathbb{P}}\dot d\text{ is an upper bound for }\dot A\cap\dot{\mathfrak{A}}.$$

Let $q\in\mathfrak{C}^+$, $a\in\mathfrak{A}$ be such that $b=q\cap a$ and $q\Vdash_{\mathbb{P}}\dot a\in\dot A$. Then $p\cap q\ne 0$, so $p\cap q\in\mathfrak{C}^+$ and

$$p\cap q\Vdash_{\mathbb{P}}\dot a\in\dot A\cap\dot{\mathfrak{A}}\text{, therefore }\dot a\subseteq\dot d.$$

It follows that $p\cap q\cap a\subseteq d$ (556Db); but this contradicts the choice of p and b. **X**

Thus $p\cap b\subseteq d$ whenever $b\in B$ and $p\Vdash_{\mathbb{P}}\dot d$ is an upper bound for $\dot A\cap\dot{\mathfrak{A}}$.

(ii) Next, suppose that $p\in\mathfrak{C}^+$ and $d\in\mathfrak{A}$ are such that $p\cap b\subseteq d$ for every $b\in B$. Suppose that q is stronger than p and that $\dot x$ is a $\mathbb{P}$-name such that $q\Vdash_{\mathbb{P}}\dot x\in\dot A\cap\dot{\mathfrak{A}}$. If r is stronger than q, there are an s stronger than r and an $a\in\mathfrak{A}$ such that $s\Vdash_{\mathbb{P}}\dot x=\dot a$. In this case, $s\cap a\in B$ and $s\cap a=p\cap s\cap a\subseteq d$, so

$$s\Vdash_{\mathbb{P}}\dot x=\dot a=(s\cap a)^{\bullet}\mathrel{\dot\subseteq}\dot d.$$

As r is arbitrary, $q\Vdash_{\mathbb{P}}\dot x\mathrel{\dot\subseteq}\dot d$; as q and $\dot x$ are arbitrary,

$$p\Vdash_{\mathbb{P}}\dot d\text{ is an upper bound for }\dot A.$$

(iii) Putting these together, we see that $p\cap b\subseteq d$ for every $b\in B$ iff $p\Vdash_{\mathbb{P}}\dot d$ is an upper bound for $\dot A\cap\dot{\mathfrak{A}}$.

(iv) Now suppose that $p\Vdash_{\mathbb{P}}\dot d=\sup(\dot A\cap\dot{\mathfrak{A}})$. We know that d, and therefore $p\cap d$, is an upper bound of $\{p\cap b:b\in B\}$. If e is any other upper bound of $\{p\cap b:b\in B\}$, then

$$p\Vdash_{\mathbb{P}}\dot e\text{ is an upper bound of }\dot A\text{, so }\dot d\mathrel{\dot\subseteq}\dot e$$

and $p\cap d\subseteq e$, by 556Db; thus $p\cap d=\sup_{b\in B}p\cap b$.

(v) Finally, suppose that $p\cap d=\sup_{b\in B}p\cap b$. Suppose that q is stronger than p and that $\dot x$ is a $\mathbb{P}$-name such that

$$q\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{A}}\text{ is an upper bound of }\dot A\cap\dot{\mathfrak{A}}.$$

If r is stronger than q, there are a s stronger than r and a $c\in\mathfrak{A}$ such that $s\Vdash_{\mathbb{P}}\dot x=\dot c$. In this case, by (i), we must have $s\cap b\subseteq c$ for every $b\in B$; accordingly $s\cap d\subseteq c$ (313Ba), so that

$$s\Vdash_{\mathbb{P}}\dot d\mathrel{\dot\subseteq}\dot c=\dot x.$$

As r is arbitrary,

$$q\Vdash_{\mathbb{P}}\dot d\mathrel{\dot\subseteq}\dot x;$$

as q and $\dot x$ are arbitrary,

$$p\Vdash_{\mathbb{P}}\dot d=\sup(\dot A\cap\dot{\mathfrak{A}}).$$

(b)(i) Of course

$$\Vdash_{\mathbb{P}}\dot a_i\mathrel{\dot\subseteq}\dot a$$

for every $i\in I$, so that

$$\Vdash_{\mathbb{P}}\dot a\text{ is an upper bound for }\{\dot a_i:i\in\check I\}.$$

(Formally speaking: if $p\in\mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\in\{\dot a_i:i\in\check I\}$, then for any q stronger than p there are an r stronger than q and an $i\in I$ such that $r\Vdash_{\mathbb{P}}\dot x=\dot a_i\mathrel{\dot\subseteq}\dot a$; hence $p\Vdash_{\mathbb{P}}\dot x\mathrel{\dot\subseteq}\dot a$.) In the other direction, suppose that $p\in\mathfrak{C}^+$ and that $\dot x$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot a_i\subseteq\dot x\in\dot{\mathfrak{A}}\text{ for every }i\in\check I.$$

[7] See 5A3E for a note on the interpretation of formulae of this kind.

[8] Of course I am not asserting here that '$\Vdash_{\mathbb{P}}\operatorname{sat}(\mathfrak{A})\check{}$ is a cardinal', only that '$\Vdash_{\mathbb{P}}\operatorname{sat}(\dot{\mathfrak{A}})$ is a cardinal and $\operatorname{sat}(\mathfrak{A})\check{}$ is an ordinal'.

For any q stronger than p there are an r stronger than q and a $b \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{b}$. Now, for any $i \in I$,

$$r \Vdash_{\mathbb{P}} \check{\imath} \in \check{I},\ \dot{a}_i \mathrel{\dot{\subseteq}} \dot{x} = \dot{b}$$

and therefore $r \cap a_i \subseteq b$. As i is arbitrary, $r \cap a \subseteq b$ and $r \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{x}$. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{x}$; as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{a} \text{ is the least upper bound of } \{\dot{a}_i : i \in \check{I}\}.$$

(ii) Now

$$\begin{aligned}\Vdash_{\mathbb{P}} \inf_{i \in \check{I}} \dot{a}_i &= 1 \mathrel{\dot{\setminus}} \sup_{i \in \check{I}} (1 \mathrel{\dot{\setminus}} a_i) = 1 \mathrel{\dot{\setminus}} \sup_{i \in \check{I}} (1 \setminus a_i)^{\bullet} \\ &= 1 \mathrel{\dot{\setminus}} (\sup_{i \in I}(1 \setminus a_i))^{\bullet} = (1 \setminus (\sup_{i \in I}(1 \setminus a_i)))^{\bullet} = (\inf_{i \in I} a_i)^{\bullet}.\end{aligned}$$

(c) ? Otherwise, there are a $p \in \mathfrak{C}^+$ and a family $\langle \dot{x}_\xi \rangle_{\xi < \kappa}$, where $\kappa = \operatorname{sat}(\mathfrak{A})$, such that

$$p \Vdash_{\mathbb{P}} \dot{x}_\xi \in \dot{\mathfrak{A}}^+ \text{ for every } \xi < \check{\kappa} \text{ and } \dot{x}_\xi \mathrel{\dot{\cap}} \dot{x}_\eta = 0 \text{ whenever } \xi < \eta < \check{\kappa}.$$

For each $\xi < \kappa$ choose q_ξ stronger than p and $a_\xi \in \mathfrak{A}$ such that $q_\xi \Vdash_{\mathbb{P}} \dot{x}_\xi = \dot{a}_\xi$. Then $q_\xi \Vdash_{\mathbb{P}} \dot{a}_\xi \neq 0$, so $b_\xi = q_\xi \cap a_\xi$ is non-zero. As $\operatorname{sat}(\mathfrak{A}) = \kappa$, there must be $\xi < \eta < \kappa$ such that $b_\xi \cap b_\eta \neq 0$. Set $r = q_\xi \cap q_\eta$; then $r \in \mathfrak{C}^+$ is stronger than p and

$$r \Vdash_{\mathbb{P}} \dot{x}_\xi \mathrel{\dot{\cap}} \dot{x}_\eta = \dot{a}_\xi \mathrel{\dot{\cap}} \dot{a}_\eta = (a_\xi \cap a_\eta)^{\bullet} \neq 0$$

by 556Da, because $r \cap a_\xi \cap a_\eta \neq 0$. **X**

(d) Let $A \subseteq \mathfrak{A}$ be a set of size $\kappa = \tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$. Let $\dot{A}$ be the $\mathbb{P}$-name $\{(\dot{a}, 1) : a \in A\}$; then

$$\Vdash_{\mathbb{P}} \dot{A}\ \tau\text{-generates } \dot{\mathfrak{A}} \text{ and } \#(\dot{A}) \leq \check{\kappa}.$$

P (i) Suppose that $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{\mathfrak{B}}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is an order-closed subalgebra of } \dot{\mathfrak{A}} \text{ including } \dot{A}, \text{ and } \dot{x} \in \dot{\mathfrak{A}}.$$

Consider $\mathfrak{D} = \{a : a \in \mathfrak{A},\ p \Vdash_{\mathbb{P}} \dot{a} \in \dot{\mathfrak{B}}\}$. Then $\mathfrak{D}$ is a subalgebra of $\mathfrak{A}$, by 556Bb, and is order-closed by (b) here; also $A \subseteq \mathfrak{D}$, so $\mathfrak{D} = \mathfrak{A}$. Next, for any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$; since $a \in \mathfrak{D}$, $p \Vdash_{\mathbb{P}} \dot{a} \in \dot{\mathfrak{B}}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{B}}.$$

As p, $\dot{x}$ and $\dot{\mathfrak{B}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{A}\ \tau\text{-generates } \dot{\mathfrak{A}}.$$

(ii) If $\langle a_\xi \rangle_{\xi < \kappa}$ enumerates A, then

$$\Vdash_{\mathbb{P}} \dot{A} = \{\dot{a}_\xi : \xi < \check{\kappa}\} \text{ and } \#(\dot{A}) \leq \#(\check{\kappa}) \leq \check{\kappa}. \text{ } \mathbf{Q}$$

Accordingly

$$\Vdash_{\mathbb{P}} \tau(\dot{\mathfrak{A}}) \leq \check{\kappa}.$$

556F Quotient forcing In 556A-556B I have gone to pains to describe names $\dot{\mathfrak{A}}$, $\dot{\triangle}$, $\dot{\cap}$, $\dot{0}$, $\dot{1}$ constituting a Boolean algebra. Of course we also have much simpler names $\check{\mathfrak{A}}$, $\check{\triangle}$, $\check{\cap}$, $\check{0}$, $\check{1}$ also constituting a Boolean algebra in the forcing language, and these must obviously be related in some way to the construction here. I think the details are worth bringing into the open.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Consider the $\mathbb{P}$-names

$$\dot{\psi} = \{((\check{a}, \dot{a}), 1) : a \in \mathfrak{A}\}, \quad \dot{\mathcal{I}} = \{(\check{a}, p) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ p \cap a = 0\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is a Boolean homomorphism from } \check{\mathfrak{A}} \text{ onto } \dot{\mathfrak{A}}, \text{ and its kernel is } \dot{\mathcal{I}}.$$

(b) Now suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$. Set $\dot{\mathbb{Q}} = (\dot{\mathfrak{A}}^+, \dot{\subseteq}, \dot{1}, \dot{\downarrow})$ and let $\mathbb{P} * \dot{\mathbb{Q}}$ be the iterated forcing notion (5A3O). Then $\operatorname{RO}(\mathbb{P} * \dot{\mathbb{Q}})$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

(c) Suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$ and that $\mathfrak{B}$ is a Boolean algebra such that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \cong \check{\mathfrak{B}}.$$

Then the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ is isomorphic to the Dedekind completion $\mathfrak{C}\widehat{\otimes}\mathfrak{B}$ of the free product $\mathfrak{C}\otimes\mathfrak{B}$ of $\mathfrak{C}$ and $\mathfrak{B}$.

proof (a)(i) It is elementary that

$$\Vdash_{\mathbb{P}} \dot{\psi} : \check{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is a surjective function}$$

just because $\dot{\mathfrak{A}} = \{(\dot{a}, 1) : a \in \mathfrak{A}\}$. By 556Bb,

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is a ring homomorphism; being surjective, it is a Boolean homomorphism.}$$

(ii)($\boldsymbol{\alpha}$) If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathcal{I}}$, then there are a $q \in \mathfrak{C}^+$, an $a \in \mathfrak{A}$, and an r stronger than both p and q such that

$$q \cap a = 0, \quad r \Vdash_{\mathbb{P}} \dot{x} = \check{a}.$$

In this case, $r \cap a = 0$ so, by 556Ba,

$$r \Vdash_{\mathbb{P}} 0 = \dot{a} = \dot{\psi}(\check{a}) = \dot{\psi}(\dot{x}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is included in the kernel of } \dot{\psi}.$$

($\boldsymbol{\beta}$) If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \check{\mathfrak{A}} \text{ and } \dot{\psi}(\dot{x}) = 0,$$

then there are an $a \in \mathfrak{A}$ and an q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{a} \text{ and } \dot{a} = \dot{\psi}(\check{a}) = \dot{\psi}(\dot{x}) = 0.$$

Now there is an r stronger than q such that $r \cap a = 0$, so that $(\check{a}, r) \in \dot{\mathcal{I}}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \check{a} \in \dot{\mathcal{I}}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ the kernel of } \dot{\psi} \text{ is included in } \dot{\mathcal{I}}, \text{ so they coincide.}$$

(b)(i) In order to use the description of iterated forcing in 5A3O, we need to set out an exact $\mathbb{P}$-name for $\dot{\mathfrak{A}}^+$. If we say that $\dot{\mathfrak{A}}^+$ abbreviates $\{x : x \in \dot{\mathfrak{A}},\ x \neq 0\}$, and use the formulation of Comprehension in Kunen 80, Theorem 4.2, we get

$$\dot{\mathfrak{A}}^+ = \{(\dot{x}, p) : \dot{x} \in \operatorname{dom}\dot{\mathfrak{A}},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{x} \neq 0\}.$$

Now 556Ab specifies $\operatorname{dom}\dot{\mathfrak{A}}$ to be $\{\dot{a} : a \in \mathfrak{A}\}$, so we get

$$\begin{aligned}\dot{\mathfrak{A}}^+ &= \{(\dot{a}, p) : a \in \mathfrak{A},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \in \dot{\mathfrak{A}} \text{ and } \dot{a} \neq 0\}\\ &= \{(\dot{a}, p) : a \in \mathfrak{A},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \neq 0\},\end{aligned}$$

$$\operatorname{dom}\dot{\mathfrak{A}}^+ = \{\dot{a} : a \in \mathfrak{A},\ \not\Vdash_{\mathbb{P}} \dot{a} = 0\} = \{\dot{a} : a \in \mathfrak{A}^+\}$$

by 556Da.

(ii) 5A3O now tells us that the underlying set of $\mathbb{P} * \dot{\mathbb{Q}}$ is to be

$$P = \{(p, \dot{a}) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \neq 0\}.$$

For $p \in \mathfrak{C}^+$ and $a \in \mathfrak{A}$,

$$\begin{aligned}p \Vdash_{\mathbb{P}} \dot{a} \neq 0 &\iff \text{for every } q \text{ stronger than } p,\ q \not\Vdash_{\mathbb{P}} \dot{a} = 0\\ &\iff \text{for every non-zero } q \subseteq p,\ q \cap a \neq 0\end{aligned}$$

(556Da). So P is just

$$\{(p, \dot{a}) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \cap a \neq 0 \text{ whenever } q \in \mathfrak{C} \text{ and } 0 \neq q \subseteq p\}.$$

Next, for $(p, \dot{a}),\ (q, \dot{b}) \in P$,

$$(p,\dot a)\text{ is stronger than }(q,\dot b)\iff p\subseteq q\text{ and }p\Vdash_{\mathbb P}\dot a\mathrel{\dot\subseteq}\dot b$$
$$\iff p\subseteq q\text{ and }p\cap a\subseteq b$$

(556Db).

(iii) We can define a function $f:P\to\mathfrak A^+$ by setting

$$f(p,\dot a)=p\cap a$$

whenever $(p,\dot a)\in P$. **P** If you look at the definition of $\dot a$ in 556A, you will see that $((b,1),1)=(\check b,1)$ belongs to $\dot a$ iff $b\subseteq a$, so that $\dot a=\dot b$ only when $a=b$; thus f is a function from P to $\mathfrak A$. And the definition of P ensures that $f(p,\dot a)\ne0$ whenever $(p,\dot a)\in P$. **Q**

(iv)(*α*) If $(p,\dot a)$ is stronger than $(q,\dot b)$ in P, then $p\subseteq q$ and $p\cap a\subseteq b$, so $f(p,\dot a)\subseteq f(q,b)$.

(*β*) If $a\in\mathfrak A^+$, then (because $\mathfrak C$ is regularly embedded) $C=\{q:a\subseteq q\in\mathfrak C\}$ does not have infimum 0 in $\mathfrak C$; let $p\in\mathfrak C^+$ be a lower bound for C. Then $(p,\dot a)\in P$, and $f(p,\dot a)\subseteq a$. Thus $f[P]$ is order-dense in $\mathfrak A$.

(*γ*) If $(p,\dot a)$, $(q,\dot b)$ are incompatible in P, then $f(p,\dot a)\cap f(q,\dot b)=0$. **P?** Otherwise, $c=p\cap a\cap q\cap b$ is non-zero. Let $r\in\mathfrak C^+$ be such that $(r,\dot c)\in P$. Since $r\setminus(p\cap q)\Vdash_{\mathbb P}\dot c=0$, $r\subseteq p\cap q$; since $c\subseteq a\cap b$, $(r,\dot c)$ is stronger than both $(p,\dot a)$ and $(q,\dot b)$, which is supposed to be impossible. **XQ**

(v) Thinking of $\mathfrak A^+$ as an order-dense subset of $\widehat{\mathfrak A}$, and of f as a function from P to $\widehat{\mathfrak A}^+$, 514Sa tells us that

$$\mathrm{RO}(\mathbb P*\dot{\mathbb Q})=\mathrm{RO}(P)\cong\widehat{\mathfrak A},$$

as claimed.

(c) For free products of Boolean algebras, see §315; for Dedekind completions, see §314. This part can be regarded as a corollary of (b) (see 556Ya-556Yb), but can also be approached directly, as follows.

(i) Let $\dot\theta$ be a $\mathbb P$-name such that

$$\Vdash_{\mathbb P}\dot\theta:\dot{\mathfrak A}\to\check{\mathfrak B}\text{ is an isomorphism.}$$

Set

$$R=\{(p,b,a):p\in\mathfrak C^+,\ b\in\mathfrak B^+,\ a\in\mathfrak A^+,\ a\subseteq p\text{ and }p\Vdash_{\mathbb P}\dot\theta(\dot a)=\check b\},$$

and give R the ordering induced by the product partial ordering of $\mathfrak C^+\times\mathfrak B^+\times\mathfrak A^+$.

(ii) $\mathrm{RO}^{\downarrow}(R)\cong\mathfrak C\widehat\otimes\mathfrak B$. **P** Define $f:R\to(\mathfrak C\widehat\otimes\mathfrak B)^+$ by setting $f(p,b,a)=p\otimes b$.

(*α*) Of course $f(p,b,a)\subseteq f(p',b',a')$ whenever $(p,b,a)\le(p',b',a')$.

(*β*) If (p_0,b_0,a_0) and (p_1,b_1,a_1) belong to R and $f(p_0,b_0,a_0)\cap f(p_1,b_1,a_1)\ne0$, set $p=p_0\cap p_1$, $b=b_0\cap b_1$ and $a=a_0\cap a_1$. Then $p\in\mathfrak C^+$, $b\in\mathfrak B^+$, $a\subseteq p$ and

$$p\Vdash_{\mathbb P}\dot\theta(\dot a)=\dot\theta(\dot a_0\mathbin{\dot\cap}\dot a_1)=\dot\theta(\dot a_0)\mathbin{\check\cap}\dot\theta(\dot a_1)=\check b_0\mathbin{\check\cap}\check b_1=\check b.$$

As $p\Vdash_{\mathbb P}\dot\theta(\dot a)\ne\check 0$, a cannot be 0, and $(p,b,a)\in R$; so (p_0,b_0,a_0) and (p_1,b_1,a_1) are downwards compatible in R.

(*γ*) If $d\in(\mathfrak C\widehat\otimes\mathfrak B)^+$, there are $p_0\in\mathfrak C^+$, $b\in\mathfrak B^+$ such that $p_0\otimes b_0\subseteq d$. Now there is a $\mathbb P$-name $\dot x$ such that

$$\Vdash_{\mathbb P}\dot x\in\dot{\mathfrak A}\text{ and }\dot\theta(\dot x)=\check b.$$

Let p stronger than p_0 and $a_0\in\mathfrak A$ be such that $p\Vdash_{\mathbb P}\dot a_0=\dot x$, and set $a=p\cap a_0$; then

$$p\Vdash_{\mathbb P}\dot a=\dot a_0\text{ so }\dot\theta(\dot a)=\dot\theta(\dot a_0)=\check b.$$

As in (β), it follows that $a\ne0$, so that $(p,b,a)\in R$; now $f(p,b,a)\subseteq d$. As d is arbitrary, $f[R]$ is order-dense in $\mathfrak C\widehat\otimes\mathfrak B$.

(*δ*) Thus f satisfies the conditions of 514Sa and $\mathrm{RO}^{\downarrow}(R)\cong\mathfrak C\widehat\otimes\mathfrak B$. **Q**

(iii) $\mathrm{RO}^{\downarrow}(R)\cong\widehat{\mathfrak A}$. **P** Define $g:R\to\widehat{\mathfrak A}^+$ by setting $g(p,b,a)=a$ for $(p,b,a)\in R$.

(*α*) Of course $g(p,b,a)\subseteq g(p',b',a')$ whenever $(p,b,a)\le(p',b',a')$ in R.

(*β*) Suppose that (p_0,b_0,a_0), $(p_1,b_1,a_1)\in R$ and that $a=a_0\cap a_1\ne0$. Set $p=p_0\cap p_1$ and $b=b_0\cap b_1$. Then $p\ne0$ and

$$p\Vdash_{\mathbb P}\dot\theta(\dot a)=\check b$$

as in (ii-β). Since $p\cap a\ne 0$, $p\not\Vdash_{\mathbb{P}}\dot a=\dot 0$ (556Da), so $p\not\Vdash_{\mathbb{P}}\check b=\check 0$ and $b\ne 0$. Thus $(p,b,a)\in R$ and (p_0,b_0,a_0), (p_1,b_1,a_1) are compatible downwards in R.

(γ) If $d\in\widehat{\mathfrak{A}}^+$, there is an $a_0\in\mathfrak{A}^+$ such that $a_0\subseteq d$. In this case, $\not\Vdash_{\mathbb{P}}\dot a_0=\dot 0$ so there is a $p_0\in\mathfrak{C}^+$ such that $p_0\Vdash_{\mathbb{P}}\dot a_0\ne\dot 0$. Now $p_0\Vdash_{\mathbb{P}}\dot\theta(\dot a_0)\in\check{\mathfrak{B}}$ so there are a p stronger than p_0 and a $b\in\mathfrak{B}$ such that $p\Vdash_{\mathbb{P}}\dot\theta(\dot a_0)=\check b$. Set $a=p\cap a_0$; then

$$p\Vdash_{\mathbb{P}}\dot a=\dot a_0\ne\dot 0\text{ and }\check b=\dot\theta(\dot a)\ne\check 0.$$

Consequently a and b are both non-zero and $(p,b,a)\in R$, while $f(p,b,a)\subseteq d$.

(δ) Thus g satisfies the conditions of 514Sa and $\mathrm{RO}^{\downarrow}(R)\cong\widehat{\mathfrak{A}}$. **Q**

(iv) Putting these together, $\mathfrak{C}\widehat{\otimes}\mathfrak{B}$ and $\widehat{\mathfrak{A}}$ are isomorphic.

556G Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Whenever $p\in\mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{A}},$$

there is an $a\in\mathfrak{A}$ such that

$$p\Vdash_{\mathbb{P}}\dot x=\dot a,$$

where $\dot a$ is the forcing name for a over $\mathfrak{C}$.

(b) $\Vdash_{\mathbb{P}}\dot{\mathfrak{A}}$ is Dedekind complete.

proof (a) Set

$$B=\{q\cap b:q\in\mathfrak{C}^+\text{ is stronger than }p,\ b\in\mathfrak{A},\ q\Vdash_{\mathbb{P}}\dot b=\dot x\},\qquad a=\sup B.$$

Then 556Ea tells us that

$$p\Vdash_{\mathbb{P}}\dot a=\sup\{\dot x\}=\dot x.$$

(b) Suppose that $p\in\mathfrak{C}^+$ and that $\dot A$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot A\subseteq\dot{\mathfrak{A}}$. Set

$$B=\{q\cap a:a\in\mathfrak{A},\ q\in\mathfrak{C}^+\text{ and }q\Vdash_{\mathbb{P}}\dot a\in\dot A\},\qquad d=\sup B.$$

Then $p\cap d=\sup_{b\in B}p\cap b$, so $p\Vdash_{\mathbb{P}}\dot d=\sup\dot A$, by 556Ea. As p and $\dot A$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot{\mathfrak{A}}\text{ is Dedekind complete.}$$

556H $L^0(\mathfrak{A})$ Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. For $a\in\mathfrak{A}$ let $\dot a$ be the forcing name for a over $\mathfrak{C}$.

(a) For every $u\in L^0(\mathfrak{A})$,

$$\Vdash_{\mathbb{P}}\dot u\in L^0(\dot{\mathfrak{A}})$$

where $\dot u$ is the forcing name for u over $\mathfrak{C}$.

(b) For u, $v\in L^0(\mathfrak{A})$ and $\alpha\in\mathbb{R}$,

$$\begin{aligned}\Vdash_{\mathbb{P}}\dot u+\dot v&=(u+v)^{\bullet},\\ -\dot u&=(-u)^{\bullet},\\ \dot u\vee\dot v&=(u\vee v)^{\bullet},\\ \dot u\times\dot v&=(u\times v)^{\bullet},\\ \check\alpha\dot u&=(\alpha u)^{\bullet}.\end{aligned}$$

If $u\le v$, then $\Vdash_{\mathbb{P}}\dot u\le\dot v$.

(c) If $\langle u_i\rangle_{i\in I}$ is a family in $L^0(\mathfrak{A})$ with supremum $u\in L^0(\mathfrak{A})$, then

$$\Vdash_{\mathbb{P}}\dot u=\sup_{i\in\check I}\dot u_i\text{ in }L^0(\dot{\mathfrak{A}}).$$

(d) If $p\in\mathfrak{C}^+$ and $\dot w$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot w\in L^0(\dot{\mathfrak{A}})$, then there is a $u\in L^0(\mathfrak{A})$ such that

$$p\Vdash_{\mathbb{P}}\dot w=\dot u.$$

(e) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^0(\mathfrak{A})$, then the following are equiveridical:

(i) $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (definition: 367A),

(ii) $\Vdash_{\mathbb{P}} \langle\dot u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

proof (a) Examining the definition in 556Af, we see that we have

$$\Vdash_{\mathbb{P}} \dot u \text{ is a function from } \mathbb{Q} \text{ to } \dot{\mathfrak{A}} \text{ and } \dot u(\check\alpha) = [\![u>\alpha]\!]^\bullet$$

for every $\alpha\in\mathbb{Q}$. Now 556Eb tells us that, for every $\alpha\in\mathbb{Q}$,

$$\begin{aligned}\Vdash_{\mathbb{P}} \dot u(\check\alpha) = [\![u>\alpha]\!]^\bullet = (\sup_{\beta\in\mathbb{Q},\beta>\alpha}[\![u>\beta]\!])^\bullet = \sup_{\beta\in\mathbb{Q},\beta>\check\alpha}[\![u>\beta]\!]^\bullet = \sup_{\beta\in\mathbb{Q},\beta>\check\alpha}\dot u(\check\beta),\\ 0 = (\inf_{n\in\mathbb{N}}[\![u>n]\!])^\bullet = \inf_{n\in\mathbb{N}}[\![u>n]\!]^\bullet = \inf_{n\in\mathbb{N}}\dot u(\check n),\\ 1 = (\sup_{n\in\mathbb{N}}[\![u>-n]\!])^\bullet = \sup_{n\in\mathbb{N}}[\![u>-n]\!]^\bullet = \sup_{n\in\mathbb{N}}\dot u(-\check n),\end{aligned}$$

so

$$\Vdash_{\mathbb{P}} \dot u\in L^0(\dot{\mathfrak{A}}),$$

and I can write $[\![\dot u>\check\alpha]\!]$ for the $\mathbb{P}$-name $\dot u(\check\alpha)$, so that

$$\Vdash_{\mathbb{P}} [\![\dot u>\check\alpha]\!] = [\![u>\alpha]\!]^\bullet$$

for every $\alpha\in\mathbb{Q}$.

(b)(i) Suppose u, $v\in L^0(\mathfrak{A})$. By 364E, we have

$$[\![u+v>\alpha]\!] = \sup\nolimits_{\beta\in\mathbb{Q}}[\![u>\beta]\!]\cap[\![v>\alpha-\beta]\!]$$

for every $\alpha\in\mathbb{Q}$. If α, $\beta\in\mathbb{Q}$,

$$\Vdash_{\mathbb{P}} [\![\dot u>\check\beta]\!]\mathbin{\dot\cap}[\![\dot v>\check\alpha-\check\beta]\!] = [\![u>\beta]\!]^\bullet\mathbin{\dot\cap}[\![v>\alpha-\beta]\!]^\bullet = ([\![u>\beta]\!]\cap[\![v>\alpha-\beta]\!])^\bullet.$$

Taking the supremum over β, as in 556E(b-i),

$$\begin{aligned}\Vdash_{\mathbb{P}} [\![\dot u+\dot v>\check\alpha]\!] &= \sup_{\beta\in\mathbb{Q}}[\![\dot u>\beta]\!]\mathbin{\dot\cap}[\![\dot v>\check\alpha-\beta]\!] = \sup_{\beta\in\check{\mathbb{Q}}}[\![\dot u>\beta]\!]\mathbin{\dot\cap}[\![\dot v>\check\alpha-\beta]\!]\\ &= (\sup_{\beta\in\mathbb{Q}}[\![u>\beta]\!]\cap[\![v>\alpha-\beta]\!])^\bullet = [\![u+v>\alpha]\!]^\bullet = [\![(u+v)^\bullet>\check\alpha]\!]\end{aligned}$$

for every $\alpha\in\mathbb{Q}$, and

$$\Vdash_{\mathbb{P}} \dot u+\dot v = (u+v)^\bullet.$$

(ii) Concerning $u\vee v$, we have

$$\begin{aligned}\Vdash_{\mathbb{P}} [\![(u\vee v)^\bullet>\check\alpha]\!] &= [\![u\vee v>\alpha]\!]^\bullet = [\![u>\alpha]\!]^\bullet\mathbin{\dot\cup}[\![v>\alpha]\!]^\bullet\\ &= [\![\dot u>\check\alpha]\!]\mathbin{\dot\cup}[\![\dot v>\check\alpha]\!] = [\![\dot u\vee\dot v>\check\alpha]\!]\end{aligned}$$

for every u, $v\in L^0(\mathfrak{A})$ and $\alpha\in\mathbb{Q}$, so

$$\Vdash_{\mathbb{P}} \dot u\vee\dot v = (u\vee v)^\bullet;$$

it follows that if $u\ge 0$ then $\Vdash_{\mathbb{P}} \dot u = \dot u\vee 0\ge 0$.

(iii) If u, $v\in L^0(\mathfrak{A})^+$, $\alpha\in\mathbb{Q}$ and $\alpha\ge 0$, then, just as in (i),

$$\begin{aligned}\Vdash_{\mathbb{P}} [\![\dot u\times\dot v>\check\alpha]\!] &= \sup_{\beta\in\mathbb{Q},\beta>0}[\![\dot u>\beta]\!]\mathbin{\dot\cap}[\![\dot v>\frac{\check\alpha}{\beta}]\!] = (\sup_{\beta\in\mathbb{Q},\beta>0}[\![u>\beta]\!]\cap[\![v>\frac{\alpha}{\beta}]\!])^\bullet\\ &= [\![u\times v>\alpha]\!]^\bullet = [\![(u\times v)^\bullet>\check\alpha]\!];\end{aligned}$$

so $\Vdash_{\mathbb{P}} \dot u\times\dot v = (u\times v)^\bullet$. Using the distributive law we see that the same is true for all u, $v\in L^0(\mathfrak{A})$.

(iv) Take $\alpha\in\mathbb{R}$ and set $w=\alpha\chi 1\in L^0(\mathfrak{A})$. If $\beta\in\mathbb{Q}$ and $\beta<\alpha$, then

$$\Vdash_{\mathbb{P}} [\![\check\alpha\chi 1>\check\beta]\!] = 1 = \dot 1 = [\![w>\beta]\!]^\bullet = [\![\dot w>\check\beta]\!];$$

while if $\beta\ge\alpha$,

$$\Vdash_{\mathbb{P}} [\![\check\alpha\chi 1>\check\beta]\!] = 0 = \dot 0 = [\![w>\beta]\!]^\bullet = [\![\dot w>\check\beta]\!].$$

So

$$\Vdash_{\mathbb{P}} [\![\check{\alpha}\chi 1 > \beta]\!] = [\![\dot{w} > \beta]\!] \text{ for every } \beta \in \mathbb{Q}, \text{ and } \check{\alpha}\chi 1 = \dot{w} = (\alpha\chi 1)^{\bullet}.$$

Putting this together with (iii), we have

$$\Vdash_{\mathbb{P}} \check{\alpha}\dot{u} = (\check{\alpha}\chi 1) \times \dot{u} = (\alpha\chi 1)^{\bullet} \times \dot{u} = (\alpha\chi 1 \times u)^{\bullet} = (\alpha u)^{\bullet}$$

for every $u \in L^0(\mathfrak{A})$. In particular, taking $\alpha = -1$, $\Vdash_{\mathbb{P}} -\dot{u} = (-u)^{\bullet}$.

(v) Finally, if $u \le v$ then $u \vee v = v$, so

$$\Vdash_{\mathbb{P}} \dot{u} \vee \dot{v} = \dot{v} \text{ and } \dot{u} \le \dot{v}.$$

(c)(i) It will help to note that the criterion in 364Mc

if $A \subseteq L^0(\mathfrak{A})$ is non-empty, then $v \in L^0(\mathfrak{A})$ is the supremum of A in $L^0(\mathfrak{A})$ iff $[\![v > \alpha]\!] = \sup_{u \in A} [\![u > \alpha]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$

can be replaced by

if $A \subseteq L^0(\mathfrak{A})$ is non-empty, then $v \in L^0(\mathfrak{A})$ is the supremum of A in $L^0(\mathfrak{A})$ iff $[\![v > \alpha]\!] = \sup_{u \in A} [\![u > \alpha]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{Q}$.

P If the weaker condition is satisfied, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}[\![v > \alpha]\!] &= \sup_{\beta \in \mathbb{Q}, \beta \ge \alpha} [\![v > \beta]\!] = \sup_{\beta \in \mathbb{Q}, \beta \ge \alpha} \sup_{u \in A} [\![u > \beta]\!] \\ &= \sup_{u \in A} \sup_{\beta \in \mathbb{Q}, \beta \ge \alpha} [\![u > \beta]\!] = \sup_{u \in A} [\![u > \alpha]\!]. \ \mathbf{Q}\end{aligned}$$

(ii) Now 556E(b-i) tells us that

$$\Vdash_{\mathbb{P}} [\![\dot{u} > \check{\alpha}]\!] = \sup_{i \in \check{I}} [\![\dot{u}_i > \check{\alpha}]\!]$$

for every $\alpha \in \mathbb{Q}$, so

$$\Vdash_{\mathbb{P}} \dot{u} = \sup_{i \in \check{I}} \dot{u}_i.$$

(d) For each $\alpha \in \mathbb{Q}$ we have an $a_\alpha \in \mathfrak{A}$ such that $p \Vdash_{\mathbb{P}} \dot{a}_\alpha = [\![\dot{w} > \check{\alpha}]\!]$ (556Ga); since $p \Vdash_{\mathbb{P}} \dot{a} = (a \cap p)^{\bullet}$ for every $a \in \mathfrak{A}$, we can suppose that $a_\alpha \subseteq p$ for every α. Now we find that if $\alpha \in \mathbb{Q}$ and $b_\alpha = \sup_{\beta \in \mathbb{Q}, \beta > \alpha} a_\alpha$, then

$$p \Vdash_{\mathbb{P}} \dot{b}_\alpha = \sup_{\beta \in \mathbb{Q}, \beta > \check{\alpha}} [\![\dot{w} > \beta]\!] = \dot{a}_\alpha,$$

so $a_\alpha = b_\alpha$. Similarly, if $b = \inf_{n \in \mathbb{N}} a_n$ and $c = \sup_{n \in \mathbb{N}} a_{-n}$,

$$\begin{aligned}p \Vdash_{\mathbb{P}} \dot{b} &= \inf_{n \in \mathbb{N}} [\![\dot{w} > n]\!] = 0, \\ \dot{c} &= \sup_{n \in \mathbb{N}} [\![\dot{w} > -n]\!] = 1\end{aligned}$$

and $b = 0$, $c = p$. It is now easy to check that there is a $u \in L^0(\mathfrak{A})$ such that

$$\begin{aligned}[\![u > \alpha]\!] &= a_\alpha \text{ if } \alpha \in \mathbb{Q} \text{ and } \alpha > 0, \\ &= a_\alpha \cup (1 \setminus p) \text{ for other } \alpha \in \mathbb{Q},\end{aligned}$$

and that $p \Vdash_{\mathbb{P}} \dot{u} = \dot{w}$.

(e) Recall that $\langle u_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 iff $\langle u_n \rangle_{n \in \mathbb{N}}$ is order-bounded and $0 = \inf_{n \in \mathbb{N}} \sup_{m \ge n} |u_n|$ (367G); and we shall have a similar formulation in the forcing language. So if $\langle u_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0, then

$$\Vdash_{\mathbb{P}} \sup_{m \ge \check{n}} |\dot{u}_m| = (\sup_{m \ge n} |u_m|)^{\bullet}$$

for every $n \in \mathbb{N}$, and

$$\Vdash_{\mathbb{P}} \inf_{n \in \mathbb{N}} \sup_{m \ge n} |\dot{u}_m| = (\inf_{n \in \mathbb{N}} \sup_{m \ge n} |u_m|)^{\bullet} = 0, \text{ so } \langle \dot{u}_n \rangle_{n \in \mathbb{N}} \to^* 0.$$

Conversely, if $\Vdash_{\mathbb{P}} \langle \dot{u}_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0, then $\Vdash_{\mathbb{P}} \langle \dot{u}_n \rangle_{n \in \mathbb{N}}$ is order-bounded, and there is a $\mathbb{P}$-name $\dot{w}$ such that

$$\Vdash_{\mathbb{P}} \dot{w} \in L^0(\dot{\mathfrak{A}}), \ |\dot{u}_n| \le \dot{w} \text{ for every } n \in \mathbb{N}.$$

By (d), there is a $v \in L^0(\mathfrak{A})$ such that $\Vdash_{\mathbb{P}} \dot{w} = \dot{v}$, so that

$$\Vdash_{\mathbb{P}} (v \vee |u_n|)^{\bullet} = \dot{w} \vee |\dot{u}_n| = \dot{v} \text{ for every } n \in \mathbb{N}$$

and $|u_n| \le v$ for every n. We can therefore repeat the calculation just above to see that

$$\Vdash_{\mathbb{P}} (\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_m|)^{\bullet} = \inf_{n\in\mathbb{N}} \sup_{m\ge n} |\dot{u}_m| = 0,$$

so that $\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_m| = 0$ and $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

556I Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism fixing every point of $\mathfrak{C}$; let $\dot{\pi}$ be the forcing name for π over $\mathfrak{C}$.

(a) π is injective iff $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(b) If π is order-continuous, then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is order-continuous.}$$

(c) If π has a support $\operatorname{supp}\pi$ (definition: 381Bb), then

$$\Vdash_{\mathbb{P}} (\operatorname{supp}\pi)^{\bullet} \text{ is the support of } \dot{\pi}.$$

proof (a) We saw in 556C(b-i) that if π is injective then $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective. Now suppose that π is not injective; let $a \in \mathfrak{A}^+$ be such that $\pi a = 0$. Then $\Vdash_{\mathbb{P}} \dot{\pi}\dot{a} = 0$. $1 \cap a \ne 0$, so $\not\Vdash_{\mathbb{P}} \dot{a} = 0$, by 556Da, and

$$\not\Vdash_{\mathbb{P}} \dot{\pi} \text{ is injective.}$$

(b) Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{A}$ such that

$$p \Vdash_{\mathbb{P}} \dot{A} \subseteq \dot{\mathfrak{A}} \text{ and } \sup \dot{A} = 1.$$

Set $B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}\}$. Then $\sup_{b\in B} p \cap b = p \cap 1 = p$, by 556Ea. Because π is order-continuous,

$$p \cap 1 = p = \pi p = \sup_{b\in B} \pi(p \cap b) = \sup_{b\in B} p \cap \pi b.$$

Consider

$$C = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot{a} \in \dot{\pi}[\dot{A}]\}.$$

Then $\pi[B] \subseteq C$. **P** If $q \in \mathfrak{C}^+$, $a \in \mathfrak{A}$ and $q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}$, then

$$q \Vdash_{\mathbb{P}} (\pi a)^{\bullet} = \dot{\pi}\dot{a} \in \dot{\pi}[\dot{A}]$$

so

$$\pi(q \cap a) = q \cap \pi a \in C. \textbf{ Q}$$

Accordingly

$$\{p \cap c : c \in C\} \supseteq \{p \cap \pi b : b \in B\}$$

must have supremum p, and $p \Vdash_{\mathbb{P}} \sup \dot{\pi}[\dot{A}] = 1$.

As p and $\dot{A}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \sup \dot{\pi}[\dot{A}] = 1 \text{ whenever } \dot{A} \subseteq \mathfrak{A} \text{ and } \sup \dot{A} = 1, \text{ so } \dot{\pi} \text{ is order-continuous}$$

(313L(b-iii)).

(c)(i) $\Vdash_{\mathbb{P}}$ if $x \in \dot{\mathfrak{A}}$ and $x \mathbin{\dot\cap} (\operatorname{supp}\pi)^{\bullet} = 0$ then $\dot{\pi}x = x$. **P** Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{x} \mathbin{\dot\cap} (\operatorname{supp}\pi)^{\bullet} = 0.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a},\ (a \cap \operatorname{supp}\pi)^{\bullet} = 0;$$

now $r \cap a \cap \operatorname{supp}\pi = 0$ (556Da). In this case,

$$r \Vdash_{\mathbb{P}} \dot{\pi}\dot{x} = \dot{\pi}(r \cap a)^{\bullet} = (\pi(r \cap a))^{\bullet} = (r \cap a)^{\bullet} = \dot{x}.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{\pi}\dot{x} = \dot{x}$; as p and $\dot{x}$ are arbitrary, we have the result. **Q**

Now 381Ei, applied in the forcing language, tells us that

$$\Vdash_{\mathbb{P}} (\operatorname{supp}\pi)^{\bullet} \text{ supports } \dot{\pi}.$$

(ii) $\Vdash_{\mathbb{P}}$ if $x \in \dot{\mathfrak{A}}$ supports $\dot{\pi}$, then $x \mathbin{\dot\supseteq} (\operatorname{supp}\pi)^{\bullet}$. **P** Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ supports } \dot{\pi}.$$

Then for any q stronger than p we have an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$. Set $b = a \cup (1 \setminus r)$; then $r \Vdash_{\mathbb{P}} \dot{b} = \dot{a}$ supports $\dot{\pi}$. **?** If b does not support π, then there is a non-zero $d \subseteq 1 \setminus b$ such that $d \cap \pi d = 0$ (381Ei again). Since $r \cap d = d \neq 0$, there is an s stronger than r such that $s \Vdash_{\mathbb{P}} \dot{d} \neq 0$. Now

$$s \Vdash_{\mathbb{P}} \dot{d} \dot{\cap} \dot{\pi}\dot{d} = (d \cap \pi d)^{\bullet} = 0, \text{ while } \dot{d} \dot{\cap} \dot{a} = 0 \text{ and } \dot{a} \text{ supports } \dot{\pi},$$

which is impossible. **X**

So $b \supseteq \operatorname{supp} \pi$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{b} \dot{\supseteq} (\operatorname{supp} \pi)^{\bullet}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \dot{\supseteq} (\operatorname{supp} \pi)^{\bullet};$$

as p and $\dot{x}$ are arbitrary, we have the result. **Q**

Putting this together with (i),

$$\Vdash_{\mathbb{P}} (\operatorname{supp} \pi)^{\bullet} \text{ is the least element of } \dot{\mathfrak{A}} \text{ supporting } \dot{\pi}, \text{ and is the support of } \dot{\pi}.$$

556J Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $\dot{\theta}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself},$$

then there is a unique ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\pi},$$

where $\dot{\pi}$ is the forcing name for π over $\mathfrak{C}$.

(b)(i) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean homomorphism},$$

then π is a Boolean homomorphism, and $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean automorphism},$$

that π is a Boolean automorphism.

proof (a)(i) For each $a \in \mathfrak{A}$, 556Ga tells us that there is a $b \in \mathfrak{A}$ such that

$$\Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = \dot{b};$$

by 556Da, this defines b uniquely, so we have a unique function $\pi : \mathfrak{A} \to \mathfrak{A}$ defined by the rule

$$\text{for every } a \in \mathfrak{A}, \ \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = (\pi a)^{\bullet}.$$

(ii) Now, for $\odot = \triangle$ or $\odot = \cap$, and $a, b \in \mathfrak{A}$,

$$\begin{aligned}\Vdash_{\mathbb{P}} (\pi(a \odot b))^{\bullet} &= \dot{\theta}((a \odot b)^{\bullet}) = \dot{\theta}(\dot{a} \dot{\odot} \dot{b}) \\ &= \dot{\theta}\dot{a} \dot{\odot} \dot{\theta}\dot{b} = (\pi a)^{\bullet} \dot{\odot} (\pi b)^{\bullet} = (\pi a \odot \pi b)^{\bullet}\end{aligned}$$

and $\pi(a \odot b) = \pi a \odot \pi b$. So π is a ring homomorphism.

(iii) If $c \in \mathfrak{C}$ then $\pi c \subseteq c$. **P** If $c = 1$ this is trivial. Otherwise,

$$1 \setminus c \Vdash_{\mathbb{P}} \dot{c} = 0, \ (\pi c)^{\bullet} = \dot{\theta} 0 = 0,$$

so $(1 \setminus c) \cap \pi c = 0$ and $\pi c \subseteq c$. **Q**

(iv) We can therefore speak of the forcing name $\dot{\pi}$. If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$, let $a \in \mathfrak{A}$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$; then

$$p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{x}) = \dot{\theta}(\dot{a}) = (\pi a)^{\bullet} = \dot{\pi}(\dot{a}) = \dot{\pi}(\dot{x}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\pi}.$$

(b)(i) If $\Vdash_{\mathbb{P}} \dot\theta$ is a Boolean homomorphism, then

$$\Vdash_{\mathbb{P}} (\pi 1)^{\bullet} = \dot\theta\dot 1 = \dot 1$$

and $\pi 1 = 1$. Now

$$\pi c \subseteq c = 1 \setminus (1 \setminus c) \subseteq 1 \setminus \pi(1 \setminus c) = \pi 1 \setminus (\pi 1 \setminus \pi c) = \pi c$$

so $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) If $\Vdash_{\mathbb{P}} \dot\theta$ is a Boolean automorphism, then the same arguments tell us that there is a Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\phi c = c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot\phi = \dot\theta^{-1}$. But in this case

$$\Vdash_{\mathbb{P}} (\pi\phi)^{\bullet} = \dot\pi\dot\phi = \dot\theta\dot\theta^{-1} = \dot\iota$$

where ι is the identity automorphism on $\mathfrak{A}$; by the uniqueness of the representing homomorphisms of $\mathfrak{A}$, $\pi\phi = \iota$. Similarly, $\phi\pi = \iota$ and $\phi = \pi^{-1}$, so that π is an automorphism.

556K Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. We can identify $\mathfrak{C}$ with the regular open algebra $\mathrm{RO}(\mathbb{P})$ (514Sb). For $u \in L^0(\mathfrak{C})$ write $\vec u$ for the corresponding $\mathbb{P}$-name for a real number as described in 5A3L.

(a)(i) For each $a \in \mathfrak{A}$ there is a $u_a \in L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ defined by saying that $\int_c u_a = \bar\mu(a \cap c)$ for every $c \in \mathfrak{C}$.

(ii) If $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$ are such that

$$p \Vdash_{\mathbb{P}} \dot a = \dot b$$

(where $\dot a$, $\dot b$ are the forcing names for a, b over $\mathfrak{C}$), then

$$p \Vdash_{\mathbb{P}} \vec u_a = \vec u_b.$$

(b) There is a $\mathbb{P}$-name $\dot{\bar\mu}$ such that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar\mu}) \text{ is a probability algebra,}$$

and

$$\Vdash_{\mathbb{P}} \dot{\bar\mu}\dot a = \vec u_a$$

whenever $a \in \mathfrak{A}$ and $\dot a$ is the corresponding forcing name over $\mathfrak{C}$.

(c) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism such that $\pi c = c$ for every $c \in \mathfrak{C}$, and $\dot\pi$ the corresponding forcing name over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot\pi : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is measure-preserving.}$$

(d) If $\dot\phi$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot\phi : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is a measure-preserving Boolean automorphism}$$

then there is a measure-preserving Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot\phi = \dot\pi.$$

(e) If $v \in L^1(\mathfrak{A}, \bar\mu)$ and $u \in L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ is its conditional expectation on $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot v \in L^1(\dot{\mathfrak{A}}, \dot{\bar\mu}) \text{ and } \int \dot v \, d\dot{\bar\mu} = \vec u.$$

proof (a)(i) This is just the Radon-Nikodým theorem (365E).

(ii) If $p \Vdash_{\mathbb{P}} \dot a = \dot b$, then $p \cap a = p \cap b$ (556Da). Consequently

$$\int_c u_a \times \chi p = \int_{c \cap p} u_a = \bar\mu(c \cap p \cap a) = \bar\mu(c \cap p \cap b) = \int_c u_b \times \chi p$$

whenever $c \in \mathfrak{C}$, and $u_a \times \chi p = u_b \times \chi p$; by 5A3M,

$$p \Vdash_{\mathbb{P}} \vec u_a = \vec u_b.$$

(b)(i) Note first the elementary properties of the conditional expectation $a \mapsto u_a : \mathfrak{A} \to L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$: it is additive and positive and order-continuous, and $0 \le u_a \le \chi 1$ for every a. (To extract these facts most efficiently from the presentation in §365, note that $u_a = P(\chi a)$, where $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ is the conditional expectation operator of 365R.) In particular,

$$\Vdash_{\mathbb{P}} \vec u_a \in [0, 1]$$

for every $a \in \mathfrak{A}$. It is also worth observing that if $c \in \mathfrak{C}$ and $a \in \mathfrak{A}$ then $u_{a \cap c} = u_a \times \chi c$ (see 365Pc).

(ii) Now consider the $\mathbb{P}$-name

$$\dot{\bar\mu} = \{((\dot a, \vec u_a), 1) : a \in \mathfrak{A}\}.$$

We have quite a lot to check, of course. First, $\dot{\bar\mu}$ is a name for a function with domain $\dot{\mathfrak{A}}$. **P** If $((\dot a, \vec u_a), 1)$ and $((\dot b, \vec u_b), 1)$ are two members of $\dot{\bar\mu}$, and $p \in \mathfrak{C}^+$ is such that $p \Vdash_{\mathbb{P}} \dot a = \dot b$, then $p \cap a = p \cap b$, so $p \Vdash_{\mathbb{P}} \vec u_a = \vec u_b$, by (a-ii) above. By 5A3H, $\Vdash_{\mathbb{P}} \dot{\bar\mu}$ is a function. Also $\Vdash_{\mathbb{P}} \operatorname{dom} \dot{\bar\mu} = \dot A$, where $\dot A = \{(\dot a, 1) : a \in \mathfrak{A}\} = \dot{\mathfrak{A}}$. **Q**

(iii) We have

$$\Vdash_{\mathbb{P}} \dot{\bar\mu}\dot a = \vec u_a \in [0,1]$$

for every $a \in \mathfrak{A}$, so

$$\Vdash_{\mathbb{P}} \dot{\bar\mu} \text{ is a function from } \dot{\mathfrak{A}} \text{ to } [0,1].$$

Since $u_1 = \chi 1$,

$$\Vdash_{\mathbb{P}} \dot{\bar\mu}1 = \vec u_1 = 1.$$

(iv) Next, $\Vdash_{\mathbb{P}} \dot{\bar\mu}$ is additive. **P** Suppose that $p \in \mathfrak{C}^+$ and $\dot x$, $\dot y$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot x,\ \dot y \in \dot{\mathfrak{A}} \text{ are disjoint.}$$

By 556Ga there are a, $b \in \mathfrak{A}$ such that

$$p \Vdash_{\mathbb{P}} \dot x = \dot a,\ \dot y = \dot b,\ (a \cap b)^{\bullet} = \dot x \mathbin{\dot\cap} \dot y = 0.$$

So $p \cap a \cap b = 0$ and

$$\chi p \times u_{a \cup b} = u_{p \cap (a \cup b)} = u_{p \cap a} + u_{p \cap b} = \chi p \times u_a + \chi p \times u_b = \chi p \times (u_a + u_b);$$

it follows that

$$p \Vdash_{\mathbb{P}} \dot{\bar\mu}(\dot x \mathbin{\dot\cup} \dot y) = \dot{\bar\mu}(\dot a \mathbin{\dot\cup} \dot b) = \dot{\bar\mu}(a \cup b)^{\bullet} = \vec u_{a \cup b} = (u_a + u_b)^{\rightarrow}$$

(5A3M)

$$= \vec u_a + \vec u_b = \dot{\bar\mu}\dot x + \dot{\bar\mu}\dot y.$$

As p, $\dot x$ and $\dot y$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\bar\mu} \text{ is additive. } \mathbf{Q}$$

(v) Suppose that $p \in \mathfrak{C}^+$ and that $\dot A$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot A \subseteq \dot{\mathfrak{A}} \text{ is closed under } \dot\cup \text{ and has supremum } 1.$$

Then for every rational number $\alpha < 1$ there are an $r \in \mathfrak{C}^+$, stronger than p, and a $d \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot d \in \dot A \text{ and } \dot{\bar\mu}\dot d \ge \check\alpha.$$

P Set

$$B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot a \in \dot A\},$$

so that $p \subseteq \sup B$ (556Ea). Because $\bar\mu$ is completely additive, there are $b_0, \dots, b_{n-1} \in B$ such that $\bar\mu(p \cap \sup_{i<n} b_i) > \alpha\bar\mu p$. Express each b_i as $q_i \cap a_i$ where q_i is stronger than p and $q_i \Vdash_{\mathbb{P}} \dot a_i \in \dot A$. For $J \subseteq n$ set $c_J = p \cap \inf_{i \in J} q_i \setminus \sup_{i \in n \setminus J} q_i$ and $d_J = \sup_{i \in J} a_i$; then $\langle c_J \rangle_{J \subseteq n}$ is disjoint and

$$p \cap \sup_{i<n} b_i = \sup_{\emptyset \ne J \subseteq n} c_J \cap d_J.$$

Accordingly

$$\sum_{\emptyset \ne J \subseteq n} \bar\mu(c_J \cap d_J) > \alpha\bar\mu p$$

and there must be a non-empty $J \subseteq n$ such that $c_J \ne 0$ and

$$\alpha\bar\mu c_J < \bar\mu(c_J \cap d_J) = \int_{c_J} u_{d_J}.$$

So $r = c_J \cap [\![u_{d_J} \ge \alpha]\!]$ is non-zero. Set $d = d_J$; then

$$r \Vdash_{\mathbb{P}} \dot a_i \in \dot A \text{ for every } i \in \check J, \text{ therefore } \dot d = \sup_{i \in \check J} \dot a_i \in \dot A,$$

and

$$r \Vdash_{\mathbb{P}} \dot{\bar{\mu}}\dot{d} = \vec{u}_{d_J} > \check{\alpha}. \quad \mathbf{Q}$$

(vi) It follows that

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}} \text{ is completely additive.}$$

P Suppose that $p \in \mathfrak{C}^+$ and that $\dot{A}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A} \subseteq \dot{\mathfrak{A}} \text{ is closed under } \dot{\cup} \text{ and has supremum 1.}$$

Then for every rational $\alpha < 1$ and every q stronger than p there is an r stronger than q such that

$$r \Vdash_{\mathbb{P}} \text{there is an } x \in \dot{A} \text{ such that } \dot{\bar{\mu}}x \ge \check{\alpha};$$

as q is arbitrary,

$$p \Vdash_{\mathbb{P}} \text{there is an } x \in \dot{A} \text{ such that } \dot{\bar{\mu}}x \ge \check{\alpha};$$

as α is arbitrary,

$$p \Vdash_{\mathbb{P}} \sup_{x \in \dot{A}} \dot{\bar{\mu}}x = 1.$$

As p and $\dot{A}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \sup_{x \in A} \dot{\bar{\mu}}x = 1 \text{ whenever } A \subseteq \dot{\mathfrak{A}} \text{ is closed under } \dot{\cup} \text{ and has supremum 1.}$$

We know that

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}} \text{ is additive and } \dot{\bar{\mu}}1 = 1,$$

so we can turn this over to get

$$\Vdash_{\mathbb{P}} \inf_{x \in A} \dot{\bar{\mu}}x = 0 \text{ whenever } A \subseteq \dot{\mathfrak{A}} \text{ is closed under } \dot{\cap} \text{ and has infimum 0, therefore } \dot{\bar{\mu}} \text{ is completely additive.} \quad \mathbf{Q}$$

(vii) Since we already know that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ is Dedekind complete}$$

(556Gb), we have all the elements needed for

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ is a probability algebra.}$$

(c) The point is that if $a \in \mathfrak{A}$ then $u_{\pi a} = u_a$. **P** For any $c \in \mathfrak{C}$,

$$\int_c u_{\pi a} = \bar{\mu}(c \cap \pi a) = \bar{\mu}(\pi(c \cap a)) = \bar{\mu}(c \cap a) = \int_c u_a. \quad \mathbf{Q}$$

Now suppose that $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$. Then there is an $a \in \mathfrak{A}$ such that $p \Vdash_{\mathbb{P}} \dot{a} = \dot{x}$ (556Ga), and

$$p \Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\dot{\pi}\dot{x}) = \dot{\bar{\mu}}(\dot{\pi}\dot{a}) = \dot{\bar{\mu}}(\pi a)^{\bullet} = \vec{u}_{\pi a} = \vec{u}_a = \dot{\bar{\mu}}\dot{x}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is measure-preserving.}$$

(d) By 556J, there is a unique $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot{\phi} = \dot{\pi}$. In this case, for any $a \in \mathfrak{A}$,

$$\Vdash_{\mathbb{P}} \vec{u}_a = \dot{\bar{\mu}}\dot{a} = \dot{\bar{\mu}}(\dot{\phi}\dot{a}) = \dot{\bar{\mu}}(\dot{\pi}\dot{a}) = \dot{\bar{\mu}}(\pi a)^{\bullet} = \vec{u}_{\pi a}.$$

So $u_a = u_{\pi a}$ (5A3M again) and

$$\mu a = \int u_a = \int u_{\pi a} = \mu(\pi a).$$

Thus π is measure-preserving.

(e) Let $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$ be the conditional expectation operator, and let U be the set of those $v \in L^1(\mathfrak{A}, \bar{\mu})$ such that

$$\Vdash_{\mathbb{P}} \dot{v} \in L^1(\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ and } \int \dot{v}\, d\dot{\bar{\mu}} = \vec{Pv}.$$

By (a), $\chi a \in U$ for every $a \in \mathfrak{A}$; by 556Hb, U is closed under addition and rational multiplication; by 556Hc $\sup_{n \in \mathbb{N}} v_n \in U$ for every non-decreasing sequence $\langle v_n \rangle_{n \in \mathbb{N}}$ in U. So $U = L^1(\mathfrak{A}, \bar{\mu})$, as required.

556L Relatively independent subalgebras Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\bar{\mu}}$ be the forcing name for $\bar{\mu}$ described in 556K, so that $\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}},\dot{\bar{\mu}})$ is a probability algebra.

(a) For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including $\mathfrak{C}$, let $\dot{\mathfrak{B}}$ be the forcing name for $\mathfrak{B}$ over $\mathfrak{C}$. If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ including $\mathfrak{C}$, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ (definition: 458La[9]) iff

$$\Vdash_{\mathbb{P}} \langle\dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is stochastically independent in } \dot{\mathfrak{A}}.$$

(b) If $\langle v_i\rangle_{i\in I}$ is a family in $L^0(\mathfrak{A})$ which is relatively independent over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \langle\dot{v}_i\rangle_{i\in\check{I}} \text{ is stochastically independent}$$

(writing $\dot{v}_i$ for the forcing name for v_i over $\mathfrak{C}$).

proof (a)(i) Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. Let $p\in\mathfrak{C}^+$ and $\dot{J}$, $\langle\dot{x}_j\rangle_{j\in\dot{J}}$ be $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}} \dot{J}\in[\check{I}]^{<\omega} \text{ is non-empty, } \dot{x}_j\in\dot{\mathfrak{B}}_j \text{ for every } j\in\dot{J}.$$

Then for every q stronger than p there are an r stronger than q and a family $\langle b_j\rangle_{j\in J}$ such that J is a non-empty finite subset of I, $b_j\in\mathfrak{B}_j$ for every $j\in J$, and

$$r\Vdash_{\mathbb{P}} \dot{J}=\check{J} \text{ and } \dot{x}_j=\dot{b}_j \text{ for every } j\in\check{J}.$$

Set $a=\inf_{j\in J}b_j$. For each $j\in J$ let $u_{b_j}\in L^1(\mathfrak{C},\bar{\mu}{\restriction}\mathfrak{C})$ be the conditional expectation of χb_j on $\mathfrak{C}$, as in 556Ka; then $u_a=\prod_{i\in J}u_{b_j}$, because $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively stochastically independent. But this means that

$$\begin{aligned} r\Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\inf_{j\in\dot{J}}\dot{x}_j) &= \dot{\bar{\mu}}(\inf_{j\in\check{J}}\dot{b}_j)=\dot{\bar{\mu}}\dot{a}=\vec{u}_a\\ &=\prod_{j\in\check{J}}\vec{u}_{b_j}=\prod_{j\in\check{J}}\dot{\bar{\mu}}\dot{b}_j=\prod_{j\in\dot{J}}\dot{\bar{\mu}}\dot{x}_j. \end{aligned}$$

As q is arbitrary,

$$p\Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\inf\nolimits_{j\in\dot{J}}\dot{x}_j)=\prod\nolimits_{j\in\dot{J}}\dot{\bar{\mu}}\dot{x}_j;$$

as p and $\langle\dot{x}_j\rangle_{j\in\dot{J}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \langle\dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

(ii) Now suppose that

$$\Vdash_{\mathbb{P}} \langle\dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

Take a finite set $J\subseteq I$ and $\langle b_j\rangle_{j\in J}\in\prod_{j\in J}\mathfrak{B}_j$. Again set $a=\inf_{j\in J}b_j$ and let u_{b_j} be the conditional expectation of χb_j on $\mathfrak{C}$ for each j. Then

$$\begin{aligned} \Vdash_{\mathbb{P}} (\prod_{j\in J}u_{b_j})^{\vec{}} &= \prod_{j\in\check{J}}\vec{u}_{b_j}=\prod_{j\in\check{J}}\dot{\bar{\mu}}\dot{b}_j\\ &=\dot{\bar{\mu}}(\inf_{j\in\check{J}}\dot{b}_j)=\dot{\bar{\mu}}(\inf_{j\in J}b_j)^{\centerdot}=\vec{u}_a, \end{aligned}$$

so $\prod_{j\in J}u_{b_j}=u_a$. As $\langle b_j\rangle_{j\in J}$ is arbitrary, $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$.

(b) For each $i\in I$, let $\mathfrak{A}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![v_i>\alpha]\!]:\alpha\in\mathbb{Q}\}$, and $\mathfrak{B}_i$ the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_i\cup\mathfrak{C}$. Then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ (458Ld), so $\Vdash_{\mathbb{P}}\langle\dot{\mathfrak{B}}_i\rangle_{i\in\check{I}}$ is independent, by (a) here. Now we have

$$\Vdash_{\mathbb{P}} [\![\dot{v}_i>\alpha]\!]=[\![v_i>\alpha]\!]^{\centerdot}\in\dot{\mathfrak{B}}_i$$

whenever $\alpha\in\mathbb{Q}$ and $i\in I$, so

$$\Vdash_{\mathbb{P}} [\![\dot{v}_i>\alpha]\!]\in\dot{\mathfrak{B}}_i \text{ for every } \alpha\in\mathbb{Q} \text{ and } i\in\check{I}, \text{ and } \langle\dot{v}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

556M Laws of large numbers As an elementary example to show that we can use this machinery to extend a classical result, I give the following. Consider the two statements

(‡) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu=0$ for every $n\in\mathbb{N}$ and every $F\in\mathrm{T}$. Suppose

[9] Formerly 458Ha.

that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Then $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^{n}f_i=0$ a.e.

and

(†) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an independent sequence in $\mathcal{L}^2(\mu)$ such that $\int f_n d\mu=0$ for every $n\in\mathbb{N}$. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Then $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^{n}f_i=0$ a.e.

In 273D I presented (†) as the basic strong law of large numbers from which the other standard forms could be deduced. (‡) may be found in Volume 4 as an exercise (458Yd). What I propose to do is to show how (‡) can be deduced, not exactly from (†), but from (†) in a forcing model; relying on the fundamental theorem of forcing to confirm that if (†) is true in its ordinary sense, then its interpretation in any forcing language will again be true.

proof (a) In order to avoid explanations involving names for real numbers, it seems helpful to re-word (‡). Consider the version

$(‡)_1$ Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu=0$ for every $n\in\mathbb{N}$ and every $F\in$ T. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{Q}\cap\,]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Then $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^{n}f_i=0$ a.e.

Then $(‡)_1$ implies (‡). **P** Given the structure of (‡), with general $\beta_n>0$, let $\delta_n\in\mathbb{Q}\cap\,]0,\beta_n]$ be such that

$$\frac{1}{\delta_n^2}\|f_n\|_2^2\le\frac{1}{\beta_n^2}\|f_n\|_2^2+2^{-n}$$

for every n. Set $\gamma_n=\sup_{m\le n}\delta_m$ for each n; then $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{Q}\cap]0,\infty[$ and $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2}\|f_n\|_2^2$ is finite, so $(‡)_1$ tells us that

$$\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^{n}f_i=\lim_{n\to\infty}\frac{\gamma_n}{\beta_n}\cdot\frac{1}{\gamma_n}\sum_{i=0}^{n}f_i=0\text{ a.e. } \mathbf{Q}$$

(b) Now formulate the assertions $(‡)_1$ and (†) in terms of measure algebras; we get

$(‡)'$ Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and $\langle v_n\rangle_{n\in\mathbb{N}}$ a sequence in $L^2(\mathfrak{A},\bar\mu)$ such that $\langle v_n\rangle_{n\in\mathbb{N}}$ is relatively independent over $\mathfrak{C}$ and $Pv_n=0$ for every $n\in\mathbb{N}$, where $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ is the conditional expectation operator. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{Q}\cap\,]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|v_n\|_2^2<\infty$. Then $\langle\frac{1}{\beta_n}\sum_{i=0}^{n}v_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

and

$(†)'$ Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and $\langle v_n\rangle_{n\in\mathbb{N}}$ an independent sequence in $L^2(\mathfrak{A},\bar\mu)$ such that $\int v_n d\bar\mu=0$ for every n. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|v_n\|_2^2<\infty$. Then $\langle\frac{1}{\beta_n}\sum_{i=0}^{n}v_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

(As usual, the conversions are just a matter of applying the Loomis-Sikorski theorem, with 367F to translate order*-convergence in L^0 into almost-everywhere convergence of functions.)

(c) Assuming $(†)'$, take a structure $(\mathfrak{A},\bar\mu,\mathfrak{C},\langle v_n\rangle_{n\in\mathbb{N}},\langle\beta_n\rangle_{n\in\mathbb{N}})$ as in $(‡)'$, let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and consider the corresponding forcing names $\dot{\mathfrak{A}}$, $\dot{\bar\mu}$ and $\langle\dot v_n\rangle_{n\in\mathbb{N}}$. Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ be the conditional expectation operator. For each $n\in\mathbb{N}$,

$$\Vdash_{\mathbb{P}}\dot v_n\times\dot v_n=(v_n\times v_n)^{\bullet}\in L^1(\dot{\mathfrak{A}},\dot{\bar\mu}),\quad\|\dot v_n\|_2^2=\int\dot v_n^2d\dot{\bar\mu}=(P(v_n^2))^{\rightarrow}$$

by 556Hb and 556Ke. Now

$$\sum_{n=0}^{\infty}\frac{1}{\beta_n}\int P(v_n^2)d(\bar\mu\restriction\mathfrak{C})\le\sum_{n=0}^{\infty}\frac{1}{\beta_n}\|v_n\|_2^2<\infty,$$

so

$$v=\sum_{i=n}^{\infty}\frac{1}{\beta_n}P(v_n^2)$$

is defined in $L^0(\mathfrak{C})$, and

$$\Vdash_{\mathbb{P}}\sum_{n=0}^{\infty}\frac{1}{\check\beta_n}\|\dot v_n^2\|_2^2\le\vec v\text{ is finite.}$$

At the same time,

$$\Vdash_{\mathbb{P}} \int \dot{v}_n d\dot{\bar{\mu}} = P\vec{v}_n = 0 \text{ for every } n \in \mathbb{N}.$$

Applying (†)′ in the forcing language,

$$\Vdash_{\mathbb{P}} \langle (\frac{1}{\check{\beta}_n} \sum_{i=0}^n v_i)^{\bullet} \rangle_{n\in\mathbb{N}} = \langle \frac{1}{\check{\beta}_n} \sum_{i=0}^n \dot{v}_i \rangle_{n\in\mathbb{N}} \text{ order*-converges to 0 in } L^0(\dot{\mathfrak{A}}),$$

so $\langle \frac{1}{\beta_n} \sum_{i=0}^n v_i \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$, by 556He.

Thus (‡)′ is true, and we're home.

556N Dye's theorem Now for something from Volume 3. Let me state two versions of Dye's theorem (388L): the 'full' version

(‡) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra of countable Maharam type, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and π_1, π_2 two measure-preserving automorphisms of $\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that $\phi c = c$ for every $c \in \mathfrak{C}$ and π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

and the 'simple' version

(†) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra of countable Maharam type, and π_1, π_2 two ergodic measure-preserving automorphisms of $\mathfrak{A}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

Here also the machinery of this section provides a proof of (‡) from (†).

proof (a) Assume (†). Take $(\mathfrak{A}, \bar{\mu})$, $\mathfrak{C}$, π_1 and π_2 as in (‡). Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$, $\dot{\pi}_1$ and $\dot{\pi}_2$ be the forcing names for $\mathfrak{A}$, π_1 and π_2 over $\mathfrak{C}$. By 556Cd, 556Ce and 556Kc,

$$\Vdash_{\mathbb{P}} \dot{\pi}_1 \text{ and } \dot{\pi}_2 \text{ are ergodic measure-preserving automorphisms.}$$

By 556Ed,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ has countable Maharam type.}$$

By (†), applied in the forcing universe,

$\Vdash_{\mathbb{P}}$ there is a measure-preserving automorphism θ of $\dot{\mathfrak{A}}$ such that $\dot{\pi}_1$ and $\theta\dot{\pi}_2\theta^{-1}$ generate the same full subgroups of $\operatorname{Aut}\dot{\mathfrak{A}}$.

Let $\dot{\theta}$ be a $\mathbb{P}$-name such that

$\Vdash_{\mathbb{P}} \dot{\theta}$ is a measure-preserving automorphism of $\dot{\mathfrak{A}}$ such that $\dot{\pi}_1$ and $\dot{\theta}\dot{\pi}_2\dot{\theta}^{-1}$ generate the same full subgroups of $\operatorname{Aut}\dot{\mathfrak{A}}$.

By 556J, there is a $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\phi c = c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\phi}$, so that, setting $\pi_3 = \phi\pi_2\phi^{-1}$,

$$\Vdash_{\mathbb{P}} \dot{\pi}_1 \text{ and } \dot{\pi}_3 \text{ generate the same full subgroups of } \operatorname{Aut}\dot{\mathfrak{A}}$$

(using 556Cc and 556Cd).

(b) Since

$$\Vdash_{\mathbb{P}} \dot{\pi}_3 \text{ belongs to the full subgroup of } \operatorname{Aut}\dot{\mathfrak{A}} \text{ generated by } \dot{\pi}_1,$$

we can apply 381I(b-iv) in the forcing language to get

$$\Vdash_{\mathbb{P}} \inf_{n\in\mathbb{Z}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = 0.$$

Now by 556Ic we know that

$$\Vdash_{\mathbb{P}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = (\operatorname{supp}(\pi_1^n\pi_3))^{\bullet}$$

for every $n \in \mathbb{Z}$ (of course we need to check that $\Vdash_{\mathbb{P}} \dot{\pi}_1^n\dot{\pi}_3 = (\pi_1^n\pi_3)^{\bullet}$; but this is easily deduced from 556Cc, an induction on n for $n \geq 0$, and 556Cd). So

$$\Vdash_{\mathbb{P}} (\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi_1^n\pi_3))^{\bullet} = \inf_{n\in\mathbb{Z}} (\operatorname{supp}(\pi_1^n\pi_3)^{\bullet})$$

(556E(b-ii))

$$= \inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi_1^n\pi_3)^{\bullet} = \inf_{n\in\mathbb{Z}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = 0.$$

By 556Da, as usual, $\inf_{n\in\mathbb{Z}}\operatorname{supp}(\pi_1^n\pi_3)=0$; by 381I(b-iv), in the other direction and in the ordinary universe, π_3 belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π_1. Similarly, π_1 belongs to the full subgroup generated by π_3, so π_1 and π_3 generate the same full subgroups, as required by (‡).

556O For the next result, I prepare the ground with a note on 'full local semigroups' as defined in §395[10].

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$, and for a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi c\subseteq c$ for every $c\in\mathfrak{C}$ let $\dot\pi$ be the forcing name for π over $\mathfrak{C}$. Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every point of $\mathfrak{C}$ is fixed by every member of G, and $\dot G$ the $\mathbb{P}$-name $\{(\dot\pi,1):\pi\in G\}$.

(a) $\Vdash_{\mathbb{P}}\dot G$ is a subgroup of $\operatorname{Aut}\dot{\mathfrak{A}}$.

(b) If $\phi:\mathfrak{A}\to\mathfrak{A}$ is a ring homomorphism such that $\phi c\subseteq c$ for every $c\in\mathfrak{C}$, and

$$\Vdash_{\mathbb{P}}\dot\phi\text{ belongs to the full local semigroup generated by }\dot G,$$

then ϕ belongs to the full local semigroup generated by G.

proof (a)(i) If $p\in\mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\in\dot G$, then for every q stronger than p there must be an r stronger than q and a $\pi\in G$ such that

$$r\Vdash_{\mathbb{P}}\dot x=\dot\pi\in\operatorname{Aut}\dot{\mathfrak{A}}$$

(556Ca). As q is arbitrary, $p\Vdash_{\mathbb{P}}\dot x\in\operatorname{Aut}\dot{\mathfrak{A}}$; as p and $\dot x$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot G\subseteq\operatorname{Aut}\dot{\mathfrak{A}}.$$

(ii) Writing ι for the identity automorphism of $\mathfrak{A}$,

$$\Vdash_{\mathbb{P}}\dot\iota\in\dot G\text{ is the identity automorphism of }\dot{\mathfrak{A}}$$

(see part (d) of the proof of 556C). If $p\in\mathfrak{C}^+$ and $\dot x$, $\dot y$ are $\mathbb{P}$-names such that $p\Vdash_{\mathbb{P}}\dot x,\,\dot y\in\dot G$, then for every q stronger than p there are r stronger than q and π_1, $\pi_2\in G$ such that

$$r\Vdash_{\mathbb{P}}\dot x=\dot\pi_1,\ \dot y=\dot\pi_2,\ \dot x\cdot\dot y=\dot\pi_1\dot\pi_2=(\pi_1\pi_2)^\bullet\in\dot G,\ \dot x^{-1}=(\dot\pi_1)^{-1}=(\pi_1^{-1})^\bullet\in\dot G$$

(556Cc, 556Cd), because $\pi_1\pi_2$ and π_1^{-1} belong to G. As q is arbitrary,

$$p\Vdash_{\mathbb{P}}\dot x\cdot\dot y\text{ and }\dot x^{-1}\text{ belong to }\dot G;$$

as p, $\dot x$ and $\dot y$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot G\text{ is a subgroup of }\operatorname{Aut}\dot{\mathfrak{A}}.$$

(b) Take any non-zero $a\in\mathfrak{A}$. Then there is a $p\in\mathfrak{C}^+$ such that $p\Vdash_{\mathbb{P}}\dot a\neq 0$ (556Da). Since

$$p\Vdash_{\mathbb{P}}\dot\phi\text{ belongs to the full local semigroup generated by }\dot G,$$

there must be $\mathbb{P}$-names $\dot x$, $\dot\theta$ such that

$$p\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{A}}\setminus\{0\},\ \dot x\mathrel{\dot\subseteq}\dot a,\ \dot\theta\in\dot G,\ \dot\theta y=\dot\phi y\text{ whenever }y\mathrel{\dot\subseteq}\dot x$$

(395B(a-ii)). Now there are a q stronger than p and $b\in\mathfrak{A}$, $\pi\in G$ such that

$$q\Vdash_{\mathbb{P}}\dot b=\dot x,\ \dot\pi=\dot\theta.$$

Since $q\Vdash_{\mathbb{P}}\dot b\neq 0$, $q\cap b\neq 0$. Suppose that $d\subseteq q\cap b$. Then

$$q\Vdash_{\mathbb{P}}\dot d\mathrel{\dot\subseteq}\dot x,\text{ so }(\pi d)^\bullet=\dot\pi\dot d=\dot\theta\dot d=\dot\phi\dot d=(\phi d)^\bullet$$

and

$$\pi d=q\cap\pi d$$

(see (a-i) of the proof of 556C)

$$=q\cap\phi d$$

(556D(a-i))

$$=\phi(q\cap d)=\phi d.$$

Thus π and ϕ agree on the principal ideal $\mathfrak{A}_{q\cap b}$, while $q\cap b\subseteq a$ is non-zero. As a is arbitrary, ϕ belongs to the full local semigroup generated by G, by 395B(a-ii) in the other direction.

[10]Formerly §394.

556P Kawada's theorem In the same way as in 556M and 556N, we have two versions of 395P[11]:

> (‡) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$, such that $\mathfrak{C}$ is a measurable algebra. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

and

> (†) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ has a subgroup G which is ergodic and fully non-paradoxical. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

Once again, I claim that we can prove (‡) from (†).

proof (a) Take $\mathfrak{A}$, G and $\mathfrak{C}$ as in (‡). If $\mathfrak{A} = \{0\}$, the result is trivial; so let us suppose from now on that $\mathfrak{A} \neq \{0\}$. Let $\bar\lambda$ be a functional such that $(\mathfrak{C}, \bar\lambda)$ is a probability algebra. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$; for $\pi \in \operatorname{Aut}\mathfrak{A}$ let $\dot\pi$ be the forcing name for π over $\mathfrak{C}$. Let $\dot G$ be the $\mathbb{P}$-name $\{(\dot\pi, 1) : \pi \in G\}$.

(b) $\Vdash_{\mathbb{P}} \dot G$ is an ergodic subgroup of $\operatorname{Aut}\dot{\mathfrak{A}}$. **P** I noted in 556Oa that

$$\Vdash_{\mathbb{P}} \dot G \text{ is a subgroup of } \operatorname{Aut}\dot{\mathfrak{A}}.$$

For its ergodicity, copy the argument of 556Ce. Suppose that $p \in \mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}} \text{ and } \theta(\dot x) = \dot x \text{ for every } \theta \in \dot G.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot x = \dot a$. Take any $\pi \in G$. Then

$$r \Vdash_{\mathbb{P}} \dot\pi \in \dot G,\ (\pi a)^{\bullet} = \dot\pi\dot x = \dot x = \dot a,$$

so

$$\pi(r \cap a) = r \cap \pi a = r \cap a$$

(556Da). As π is arbitrary, $r \cap a \in \mathfrak{C}$. If $r \cap a \neq 0$, then $r \cap a \Vdash_{\mathbb{P}} \dot a = 1$; if $r \cap a = 0$, then $r \Vdash_{\mathbb{P}} \dot a = 0$. In either case, we have an s stronger than r such that $s \Vdash_{\mathbb{P}} \dot x \in \{0, 1\}$. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot x \in \{0, 1\}$; as p and $\dot x$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot G \text{ is ergodic. } \mathbf{Q}$$

(c) $\Vdash_{\mathbb{P}} \dot G$ is fully non-paradoxical.

P (i) ? Otherwise,

$$\not\Vdash_{\mathbb{P}} \dot G \text{ satisfies condition (i) of 395D},$$

and there must be a $p \in \mathfrak{C}^+$ and $\mathbb{P}$-names $\dot\theta$, $\dot x$ such that

> $p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}} \setminus \{1\}$, $\dot\theta$ is a Boolean homomorphism from $\dot{\mathfrak{A}}$ to the principal ideal generated by $\dot x$, and $\dot\theta$ belongs to the full local semigroup generated by $\dot G$.

In order to apply 556J and 556O as stated we need a $\mathbb{P}$-name $\dot\theta_1$ such that $\Vdash_{\mathbb{P}} \dot\theta_1$ is a ring homomorphism. If $p = 1$, take $\dot\theta_1 = \dot\theta$; otherwise, take $\dot\theta_1$ such that

$$p \Vdash_{\mathbb{P}} \dot\theta_1 = \dot\theta, \quad 1 \setminus p \Vdash_{\mathbb{P}} \dot\theta_1 \text{ is the identity automorphism.}$$

Then

> $\Vdash_{\mathbb{P}} \dot\theta_1 : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}}$ is a ring homomorphism belonging to the full local semigroup generated by $\dot G$.

(ii) By 556J there is a unique ring homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot\theta_1 = \dot\phi$. By 556Ob, ϕ belongs to the full local semi-group generated by G. Since G is fully non-paradoxical, $\phi 1 = 1$ and $\Vdash_{\mathbb{P}} \dot\theta_1 1 = \dot\phi 1 = 1$. But $p \Vdash_{\mathbb{P}} \dot\theta_1 1 = \dot x \neq 1$. **XQ**

(d) Applying (†) in the forcing language, we see that

> $\Vdash_{\mathbb{P}}$ there is a strictly positive $\dot G$-invariant countably additive functional on $\dot{\mathfrak{A}}$, therefore there is a there is a strictly positive $\dot G$-invariant countably additive functional on $\dot{\mathfrak{A}}$ taking values in $[0, 1]$.

Let $\dot\nu$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot\nu \text{ is a strictly positive } \dot G\text{-invariant } [0,1]\text{-valued countably additive functional on } \dot{\mathfrak{A}}.$$

For each $a \in \mathfrak{A}$, $\Vdash_{\mathbb{P}} \dot\nu\dot a \in [0, 1]$, so there is a unique $u_a \in L^0(\mathfrak{C})^+$ such that

$$\Vdash_{\mathbb{P}} \dot\nu\dot a = \vec u_a$$

[11] Formerly 394P. By an oversight, I originally wrote out a weaker statement of the result; (‡) here is what was actually proved.

(5A3M), and $0 \le u_a \le \chi 1$. Set $\mu a = \int u_a d\bar\lambda$ for $a \in \mathfrak{A}$.

(e) μ is a strictly positive G-invariant countably additive functional on $\mathfrak{A}$.

P (i) If a, $b \in \mathfrak{A}$ are disjoint,

$$\Vdash_{\mathbb{P}} \dot a \mathbin{\dot\cap} \dot b = 0, \text{ so } \vec u_a + \vec u_b = \dot\nu\dot a + \dot\nu\dot b = \dot\nu(\dot a \mathbin{\dot\cup} \dot b) = \dot\nu(a \cup b)^{\bullet} = \vec u_{a\cup b}$$

(using 556Bb); it follows that $u_a + u_b = u_{a\cup b}$ and $\mu a + \mu b = \mu(a \cup b)$. Thus μ is additive.

(ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing family in $\mathfrak{A}$ with supremum a, then, by 556Be and 556Eb,

$\Vdash_{\mathbb{P}}$ $\langle \dot a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\dot{\mathfrak{A}}$ with supremum $\dot a$, so $\langle \vec u_{a_n}\rangle_{n\in\mathbb{N}} = \langle \dot\nu\dot a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $[0,1]$ with supremum $\vec u_a = \dot\nu\dot a$.

Now $\langle u_{a_n}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $L^0(\mathfrak{C})$ with supremum u_a (5A3Ld), so $\mu a = \sup_{n\in\mathbb{N}} \mu a_n$. Thus μ is countably additive.

(iii) Because $u_a \ge 0$, $\mu a \ge 0$ for every $a \in \mathfrak{A}$. If $\mu a = 0$, then $u_a = 0$ so

$$\Vdash_{\mathbb{P}} \dot\nu\dot a = \vec u_a = 0, \text{ therefore } \dot a = 0, \text{ because } \dot\nu \text{ is strictly positive,}$$

and $a = 0$ (556Da). Thus μ is strictly positive.

(iv) Suppose that $\pi \in G$ and $a \in \mathfrak{A}$. Then

$$\Vdash_{\mathbb{P}} \dot\pi \in \dot G \text{ and } \dot\nu \text{ is } \dot G\text{-invariant, so } \vec u_{\pi a} = \dot\nu(\pi a)^{\bullet} = \dot\nu(\dot\pi\dot a) = \dot\nu\dot a = \vec u_a.$$

So $u_{\pi a} = u_a$ and $\mu(\pi a) = \mu a$. Thus μ is G-invariant. **Q**

Accordingly μ is a functional as required by (‡).

556Q For the final application of the methods of this section, I turn to a result of a quite different kind. Here the structure under consideration, the asymptotic density algebra $\mathfrak{Z}$, is off the main line of this treatise, but has some important measure-theoretic properties (see §491); and it turns out that there is a remarkable identification of its Dedekind completion (556S) which can be established by applying Maharam's theorem in a suitable forcing universe of the kind considered here. I start with a couple of easy lemmas, one just a restatement of ideas from Volume 3, and the other a straightforward property of a basic class of forcing notions.

Lemma (a) Let $\mathfrak{A}$ be a Boolean algebra and $\bar\mu : \mathfrak{A} \to [0,1]$ a strictly positive additive functional such that $\bar\mu 1 = 1$. Suppose that whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$, there is an $a \in \mathfrak{A}$ such that $a \subseteq a_n$ for every n and $\bar\mu a = \inf_{n\in\mathbb{N}} \bar\mu a_n$. Then $(\mathfrak{A}, \bar\mu)$ is a probability algebra.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. Suppose that $\kappa \ge \tau(\mathfrak{A})$ is an infinite cardinal and that $\langle e_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\bar\mu(\inf_{\xi\in K} e_\xi) = 2^{-\#(K)}$ for every finite $K \subseteq I$. Then $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

proof (a) Let $A \subseteq \mathfrak{A}$ be a non-empty countable set. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $b_n = \inf_{i\le n} a_i$ for each n. There is a $b \in \mathfrak{A}$, a lower bound for $\{b_n : n \in \mathbb{N}\}$ and therefore for A, such that $\bar\mu b = \inf_{n\in\mathbb{N}} \bar\mu b_n$. If $c \in \mathfrak{A}$ is any lower bound for A, then $b \cup c \subseteq b_n$ for every n, so

$$\bar\mu b + \bar\mu(c \setminus b) = \bar\mu(b \cup c) \le \inf_{n\in\mathbb{N}} \bar\mu b_n = \bar\mu b,$$

and $\bar\mu(c \setminus b) = 0$; as $\bar\mu$ is strictly positive, $c \subseteq b$. Thus $b = \inf A$. As A is arbitrary, $\mathfrak{A}$ is Dedekind σ-complete. But this is the only clause missing from the definition of 'probability algebra'.

(b) By 331Ja[12], $\tau(\mathfrak{A}_d) \ge \kappa$ for every non-zero $d \in \mathfrak{A}$. So $\mathfrak{A}$ is Maharam-type-homogeneous, with Maharam type κ, and $(\mathfrak{A}, \bar\mu) \cong (\mathfrak{B}_\kappa, \bar\mu_\kappa)$ (331I).

556R Proposition Let $\mathbb{P}$ be a countably closed forcing notion. Then, for any set I, writing $(\mathfrak{B}_I, \bar\nu_I)$ for the measure algebra of the usual measure on $\{0,1\}^I$,

$$\Vdash_{\mathbb{P}} (\mathfrak{B}_{\check I}, \bar\nu_{\check I}) \cong (\check{\mathfrak{B}}_I, \check{\bar\nu}_I).$$

proof If I is finite, this is elementary (and does not rely on $\mathbb{P}$ being countably closed), so I shall suppose that I is infinite.

(a)

$\Vdash_{\mathbb{P}}$ if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\check{\mathfrak{B}}_I$, there is an $x \in \check{\mathfrak{B}}_I$ such that $x \mathrel{\check\subseteq} x_n$ for every n and $\check{\bar\nu}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar\nu}_I(x_n)$.

[12] Later editions only.

P Let p be a condition of $\mathbb{P}$ and $\langle \dot{x}_n \rangle_{n\in\mathbb{N}}$ a sequence of $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x}_n \in \check{\mathfrak{B}}_I \text{ and } \dot{x}_{n+1} \check{\subseteq} \dot{x}_n$$

for every n. If q is stronger than p, we can choose $\langle q_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ inductively so that $q_0 = q$ and, for each n, q_{n+1} is stronger than q_n, $b_n \in \mathfrak{B}_I$ and $q_{n+1} \Vdash_{\mathbb{P}} \dot{x}_n = \check{b}_n$. In this case,

$$q_{n+1} \Vdash_{\mathbb{P}} \check{b}_{n+1} = \dot{x}_{n+1} \check{\subseteq} \dot{x}_n = \check{b}_n,$$

so $b_{n+1} \subseteq b_n$ for each n. Setting $b = \inf_{n\in\mathbb{N}} b_n$, $\bar{\nu}_I b = \inf_{n\in\mathbb{N}} \bar{\nu}_I b_n$. Also, because $\mathbb{P}$ is countably closed, there is a condition r stronger than any q_n. So

$$r \Vdash_{\mathbb{P}} \check{b} \check{\subseteq} \check{b}_n = \dot{x}_n \text{ for every } n \in \mathbb{N},\ \check{\bar{\nu}}_I(\check{b}) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(\dot{x}_n).$$

As q is arbitrary,

$p \Vdash_{\mathbb{P}}$ there is an $x \in \check{\mathfrak{B}}_I$ such that $x \check{\subseteq} \dot{x}_n$ for every n and $\check{\bar{\nu}}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(\dot{x}_n)$.

As p and $\langle \dot{x}_n \rangle_{n\in\mathbb{N}}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ if $\langle x_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\check{\mathfrak{B}}_I$, there is an $x \in \check{\mathfrak{B}}_I$ such that $x \check{\subseteq} x_n$ for every n and $\check{\bar{\nu}}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(x_n)$. **Q**

Since we certainly have

$\Vdash_{\mathbb{P}} \check{\mathfrak{B}}_I$ is a Boolean algebra and $\check{\bar{\nu}} : \check{\mathfrak{B}}_I \to [0,1]$ is a strictly positive additive functional such that $\check{\bar{\nu}}_I 1 = 1$,

556Qa, applied in the forcing universe, tells us that

$$\Vdash_{\mathbb{P}} (\check{\mathfrak{B}}, \check{\bar{\nu}}_I) \text{ is a probability algebra.}$$

(b) Let $\langle e_i \rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$. Then $\bar{\nu}_I(\inf_{i\in K} e_i) = 2^{-\#(K)}$ for every finite set $K \subseteq I$, so

$$\Vdash_{\mathbb{P}} \check{\bar{\nu}}_I(\inf_{i\in K} \check{e}_i) = 2^{-\#(K)} \text{ for every finite set } K \subseteq \check{I}.$$

Next, if $\mathfrak{D}$ is the subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in I\}$, then $\mathfrak{D}$ is dense in $\mathfrak{B}_I$ for the measure metric. Now

$\Vdash_{\mathbb{P}} \check{\mathfrak{D}}$ is the subalgebra of $\check{\mathfrak{B}}_I$ generated by $\{\check{e}_i : i \in \check{I}\}$ and $\check{\mathfrak{D}}$ is metrically dense in $\check{\mathfrak{B}}_I$, so $\tau(\check{\mathfrak{B}}_I) \le \#(\check{I})$. By 556Q, $(\check{\mathfrak{B}}_I, \check{\bar{\nu}}_I) \cong (\mathfrak{B}_{\#(\check{I})}, \bar{\nu}_{\#(\check{I})}) \cong (\mathfrak{B}_{\check{I}}, \bar{\nu}_{\check{I}})$,

as required.

556S Theorem (Farah 06) Let $\mathcal{Z}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density 0 and $\mathfrak{Z}$ the asymptotic density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. Then the Dedekind completion of $\mathfrak{Z}$ is isomorphic to the Dedekind completion of the free product $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) \otimes \mathfrak{B}_{\mathfrak{c}}$.

proof (a) For $n \in \mathbb{N}$, set $I_n = \{i : 2^n \le i < 2^{n+1}\}$, so that $\langle I_n \rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{N} \setminus \{0\}$, and $\#(I_n) = 2^n$ for every $n \in \mathbb{N}$. Recall that

$$\mathcal{Z} = \{J : J \subseteq \mathbb{N},\ \lim_{n\to\infty} 2^{-n}\#(J \cap I_n) = 0\}$$

(491Ab). The notation of this proof will be slightly less appalling if I write b_J for $J^\bullet \in \mathfrak{Z}$ when $J \subseteq \mathbb{N}$ and c_K for $(\bigcup_{n\in K} I_n)^\bullet$ when $K \subseteq \mathbb{N}$.

Set

$$\mathfrak{C} = \{c_K : K \subseteq \mathbb{N}\}.$$

Because $K \mapsto c_K : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$ is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\mathfrak{Z}$. Now $\mathfrak{C} \cong \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. **P** If $K \subseteq \mathbb{N}$, then

$$c_K = 0 \iff \bigcup_{n\in K} I_n \in \mathcal{Z} \iff K \text{ is finite.}$$

So the Boolean homomorphism $K \mapsto c_K$ induces a Boolean isomorphism $\pi : \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega} \to \mathfrak{C}$ defined by setting $\pi(K^\bullet) = c_K$ for every $K \subseteq \mathbb{N}$. **Q**

For $p \in \mathfrak{C}^+$, set

$$\mathcal{F}_p = \{K : K \subseteq \mathbb{N},\ p \subseteq c_K\},$$

so that $\mathcal{F}_p$ is a filter on $\mathbb{N}$ containing every cofinite set. Note that if $p \subseteq q$ then $\mathcal{F}_p$ is finer than $\mathcal{F}_q$.

(b) $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$. **P** Suppose that $A \subseteq \mathfrak{C}$ has infimum 0 in $\mathfrak{C}$, and that $b \in \mathfrak{Z}^+$. Let $J_0 \in \mathcal{P}\mathbb{N} \setminus \mathcal{Z}$ be such that $b = b_{J_0}$. Then $\limsup_{n\to\infty} 2^{-n}\#(J_0 \cap I_n) > 0$, so there is an $\epsilon > 0$ such that $K = \{n : \#(J_0 \cap I_n) \ge 2^n\epsilon\}$ is infinite. c_K cannot be a lower bound of A in $\mathfrak{C}$, so there is an $L \subseteq \mathbb{N}$ such that $c_L \in A$ and $c_K \not\subseteq c_L$, that is, $K \setminus L$ is infinite. Set $J = \bigcup_{n\in K\setminus L} J_0 \cap I_n$; then $\#(J \cap I_n) \ge 2^n\epsilon$ for infinitely many n, so $J \notin \mathcal{Z}$ and $0 \ne b_J \subseteq b$. On the other hand, $b_J\cap c_L=0$. So $b\not\subseteq c_L$ and b is not a lower bound of A in $\mathfrak{Z}$. As b is arbitrary, A has infimum 0 in $\mathfrak{Z}$; as A is arbitrary, the embedding $\mathfrak{C}\subsetneqq\mathfrak{Z}$ is order-continuous (313L(b-v)), and $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$. **Q**.

(c) Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Then $\mathbb{P}$ is countably closed. **P** Let $\langle p_m\rangle_{m\in\mathbb{N}}$ be a non-increasing sequence in $\mathfrak{C}^+$. For each $m\in\mathbb{N}$, let $K_m\subseteq\mathbb{N}$ be such that $p_m=c_{K_m}$. Then $K_{m+1}\setminus K_m$ is finite for each m. Let $\langle n_k\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence such that $n_k\in K_m$ whenever $m\le k\in\mathbb{N}$, and set $K=\{n_k:k\in\mathbb{N}\}$; then c_K belongs to $\mathfrak{C}^+$, and $c_K\subseteq p_m$ for every $m\in\mathbb{N}$. **Q**

(d)(i) Let $\dot{\mathfrak{Z}}$ be the forcing name for $\mathfrak{Z}$ over $\mathfrak{C}$, and for $b\in\mathfrak{Z}$ let $\dot b$ be the forcing name for b over $\mathfrak{C}$. Let $\dot\nu$ be the $\mathbb{P}$-name

$$\{((\dot b_J,\check\alpha),p):p\in\mathfrak{C}^+,\ J\subseteq\mathbb{N},\ \lim_{n\to\mathcal{F}_p}2^{-n}\#(J\cap I_n)\text{ is defined and equal to }\alpha\}.$$

(ii) $\Vdash_{\mathbb{P}}\dot\nu$ is a function. **P** Suppose that (J_0,α_0,p_0) and $(J_1,\alpha_1,p_1)\in\mathcal{P}\mathbb{N}\times\mathbb{R}\times\mathfrak{C}^+$ are such that

$$\lim_{n\to\mathcal{F}_{p_0}}2^{-n}\#(J_0\cap I_n)=\alpha_0,\quad \lim_{n\to\mathcal{F}_{p_1}}2^{-n}\#(J_1\cap I_n)=\alpha_1,$$

and that $p\in\mathfrak{C}^+$, $p\subseteq p_0\cap p_1$ and $p\Vdash_{\mathbb{P}}\dot b_{J_0}=\dot b_{J_1}$. Then $p\cap b_{J_0}=p\cap b_{J_1}$ (556Da). Express p as c_K, where $K\subseteq\mathbb{N}$; then $\bigcup_{n\in K}I_n\cap(J_0\triangle J_1)\in\mathcal{Z}$, so $\lim_{n\in K,n\to\infty}2^{-n}\#(I_n\cap(J_0\triangle J_1))=0$, that is, $\lim_{n\to\mathcal{F}_p}2^{-n}\#(I_n\cap(J_0\triangle J_1))=0$. But this means that

$$\alpha_0=\lim_{n\to\mathcal{F}_p}2^{-n}\#(I_n\cap J_0)=\lim_{n\to\mathcal{F}_p}2^{-n}\#(I_n\cap J_1)=\alpha_1,$$

and surely $p\Vdash_{\mathbb{P}}\check\alpha_0=\check\alpha_1$. Thus the condition of 5A3H is satisfied and

$$\Vdash_{\mathbb{P}}\dot\nu\text{ is a function. } \mathbf{Q}$$

(iii) $\Vdash_{\mathbb{P}}\operatorname{dom}\dot\nu=\dot{\mathfrak{Z}}$. **P** Setting

$$\dot A=\{(\dot b_J,p):p\in\mathfrak{C}^+,\ J\subseteq\mathbb{N},\ \lim_{n\to\mathcal{F}_p}2^{-n}\#(J\cap I_n)\text{ is defined}\},$$

5A3H tells us that $\Vdash_{\mathbb{P}}\operatorname{dom}\dot\nu=\dot A$. Of course $\Vdash_{\mathbb{P}}\dot A\subseteq\dot{\mathfrak{Z}}$ just because $\Vdash_{\mathbb{P}}\dot b_J\in\dot{\mathfrak{Z}}$ for every $J\subseteq\mathbb{N}$. In the other direction, if $p\in\mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{Z}}$, there are a q stronger than p and a $b\in\mathfrak{Z}$ such that $q\Vdash_{\mathbb{P}}\dot x=\dot b$. Express q as c_K and b as b_J where $K\subseteq\mathbb{N}$ is infinite and $J\subseteq\mathbb{N}$. Then there is an infinite $L\subseteq K$ such that $\lim_{n\in L,n\to\infty}2^{-n}\#(J\cap I_n)$ is defined, that is, $(\dot b,r)\in\dot A$, where $r=c_L$. So $r\Vdash_{\mathbb{P}}\dot x=\dot b\in\dot A$. As q is arbitrary, $p\Vdash_{\mathbb{P}}\dot x\in\dot A$; as p and $\dot x$ are arbitrary,

$$\Vdash_{\mathbb{P}}\operatorname{dom}\dot\nu=\dot A=\dot{\mathfrak{Z}}.\ \mathbf{Q}$$

(iv) Of course $\lim_{n\to\mathcal{F}_p}2^{-n}\#(J\cap I_n)$, if it is defined, must belong to $[0,1]$. So

$$\Vdash_{\mathbb{P}}\dot\nu\text{ is a function from }\dot{\mathfrak{Z}}\text{ to }[0,1].$$

Next, $((\dot 1,\check 1),1)\in\dot\nu$ (if you can work out how to interpret each 1 in this formula), so $\Vdash_{\mathbb{P}}\dot\nu\dot 1=\check 1=1$.

(v) $\Vdash_{\mathbb{P}}\dot\nu$ is additive. **P** Suppose that $p\in\mathfrak{C}^+$ and that $\dot x_0$, $\dot x_1$ are $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}}\dot x_0,\ \dot x_1\in\dot{\mathfrak{Z}}\text{ are disjoint.}$$

If p_1 is stronger than p there are q_0, q_0', q_1, $r\in\mathbb{P}$, J_0, $J_1\subseteq\mathbb{N}$ and α_0, $\alpha_1\in\mathbb{R}$ such that

$$((\dot b_{J_0},\check\alpha_0),q_0)\in\dot\nu,\quad q_0'\text{ is stronger than both }q_0\text{ and }p_1,\quad q_0'\Vdash_{\mathbb{P}}\dot b_{J_0}=\dot x_0,$$

$$((\dot b_{J_1},\check\alpha_1),q_1)\in\dot\nu,\quad r\text{ is stronger than both }q_1\text{ and }q_0',\quad r\Vdash_{\mathbb{P}}\dot b_{J_1}=\dot x_1.$$

As $r\Vdash_{\mathbb{P}}(b_{J_0}\cap b_{J_1})^{\bullet}=0$, $r\cap b_{J_0}\cap b_{J_1}=0$. Express r as c_K, where $K\in[\mathbb{N}]^\omega$. Then $J_0\cap J_1\cap\bigcup_{n\in K}I_n\in\mathcal{Z}$, so $\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_0\cap J_1\cap I_n)=0$. At the same time,

$$\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_0\cap I_n)=\lim_{n\to\mathcal{F}_{q_0}}2^{-n}\#(J_0\cap I_n)=\alpha_0,$$

$$\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_1\cap I_n)=\lim_{n\to\mathcal{F}_{q_1}}2^{-n}\#(J_1\cap I_n)=\alpha_1,$$

so

$$\begin{aligned}\lim_{n\to\mathcal{F}_r}&2^{-n}\#((J_0\cup J_1)\cap I_n)\\&=\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_0\cap I_n)+\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_1\cap I_n)-\lim_{n\to\mathcal{F}_r}2^{-n}\#(J_0\cap J_1\cap I_n)\\&=\alpha_0+\alpha_1\end{aligned}$$

and $(((\dot b_{J_0\cup J_1},(\alpha_0+\alpha_1)\check{}\,),r)\in\dot\nu$. Accordingly

$$r \Vdash_{\mathbb{P}} \dot\nu(\dot x_0 \mathbin{\dot\cup} \dot x_1) = \dot\nu(\dot b_{J_0\cup J_1}) = (\alpha_0+\alpha_1)\check{} = \check\alpha_0 + \check\alpha_1 = \dot\nu(\dot x_0) + \dot\nu(\dot x_1),$$

while $r \subseteq p_1$. As p_1 is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot\nu(\dot x_0 \mathbin{\dot\cup} \dot x_1) = \dot\nu(\dot x_0) + \dot\nu(\dot x_1).$$

As p, $\dot x_0$ and $\dot x_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\nu \text{ is additive. } \mathbf{Q}$$

(vi) $\Vdash_{\mathbb{P}} \dot\nu$ is strictly positive. $\mathbf{P}$ Let $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ be such that $p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{Z}}$ and $\dot x \ne \dot 0$. If q is stronger than p there are a q' stronger than q and a $J \subseteq \mathbb{N}$ such that $q' \Vdash_{\mathbb{P}} \dot x = \dot b_J$. Express q' as c_K where $K \in [\mathbb{N}]^\omega$. As $q' \Vdash_{\mathbb{P}} \dot b_J \ne \dot 0$, $q' \cap b_J \ne 0$ and $\bigcup_{n\in K} I_n \cap J \notin \mathcal{Z}$. Accordingly $\limsup_{n\in K,n\to\infty} 2^{-n}\#(I_n \cap J) > 0$ and there is an infinite $L \subseteq K$ such that $\alpha = \lim_{n\in L,n\to\infty} 2^{-n}\#(I_n\cap J)$ is defined and greater than 0. Set $r = c_L$; then $((\dot b_J, \check\alpha), r) \in \dot\nu$, so

$$r \Vdash_{\mathbb{P}} \dot\nu(\dot x) = \dot\nu(\dot b_J) = \check\alpha > 0,$$

while r is stronger than q. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot\nu(\dot x) > 0$; as p and $\dot x$ are arbitrary, $\Vdash_{\mathbb{P}} \dot\nu$ is strictly positive. $\mathbf{Q}$

(vii)

$$\Vdash_{\mathbb{P}} \text{ if } \langle x_k\rangle_{k\in\mathbb{N}} \text{ is a non-increasing sequence in } \dot{\mathfrak{Z}}$$
$$\text{there is an } x \in \dot{\mathfrak{Z}} \text{ such that } x \mathrel{\dot\subseteq} x_k \text{ for every } k \text{ and } \dot\nu(x) = \inf_{k\in\mathbb{N}} \dot\nu(x_k).$$

$\mathbf{P}$ Let $p \in \mathfrak{C}^+$ and a sequence $\langle \dot x_k\rangle_{k\in\mathbb{N}}$ of $\mathbb{P}$-names be such that

$$p \Vdash_{\mathbb{P}} \langle \dot x_n\rangle_{n\in\mathbb{N}} \text{ is a non-increasing sequence in } \dot{\mathfrak{Z}}.$$

Let q be stronger than p. Then we can choose $\langle q_k\rangle_{k\in\mathbb{N}}$, $\langle q'_k\rangle_{k\in\mathbb{N}}$, $\langle q''_k\rangle_{k\in\mathbb{N}}$, $\langle J_k\rangle_{k\in\mathbb{N}}$ and $\langle \alpha_k\rangle_{k\in\mathbb{N}}$ inductively so that $q'_0 = q$ and

$$q''_k \text{ is stronger than } q'_k,\ J_k \subseteq \mathbb{N} \text{ and } q''_k \Vdash_{\mathbb{P}} \dot x_k = \dot b_{J_k},$$

$$q_k \text{ is stronger than } q''_k,\ \alpha_k \in [0,1],\ \lim_{n\to\mathcal{F}_{q_k}} 2^{-n}\#(J_k \cap I_n) = \alpha_k$$

(see (iii) above),

$$q'_{k+1} = q_k$$

for every $k \in \mathbb{N}$.

Because $\mathbb{P}$ is countably closed, there is an $r \in \mathfrak{C}^+$ stronger than every q_k. In this case, $((\dot b_{J_k}, \check\alpha_k), r) \in \dot\nu$ for every k, so

$$r \Vdash_{\mathbb{P}} \inf_{k\in\mathbb{N}} \dot\nu(\dot x_k) = \inf_{k\in\mathbb{N}} \check\alpha_k = \check\alpha$$

where $\alpha = \inf_{k\in\mathbb{N}} \alpha_k$. Express r as c_K where $K \subseteq \mathbb{N}$ is infinite. For each $k \in \mathbb{N}$,

$$r \Vdash_{\mathbb{P}} \dot b_{J_{k+1}} = \dot x_{k+1} \mathrel{\dot\subseteq} \dot x_k = \dot b_{J_k},$$

so $r \cap b_{J_{k+1}} \setminus b_{J_k} = 0$ and

$$\lim_{n\in K,n\to\infty} 2^{-n}\#(I_n \cap J_{k+1} \setminus J_k) = 0,$$

while

$$\lim_{n\in K,n\to\infty} 2^{-n}\#(I_n \cap J_k) = \lim_{n\to\mathcal{F}_{q_k}} 2^{-n}\#(I_n\cap J_k) = \alpha_k \ge \alpha.$$

We can therefore find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in K such that

$$2^{-n_k}\#(I_{n_k} \cap J'_k) \ge \alpha - 2^{-k}$$

for every k, where $J'_k = \bigcap_{j\le k} J_j$. Set $r' = (\bigcup_{k\in\mathbb{N}} I_{n_k})^\bullet$ and $J = \bigcup_{k\in\mathbb{N}} I_{n_k} \cap J'_k$. Then $J \setminus J_k$ is finite, so $r' \Vdash_{\mathbb{P}} \dot b_J \mathrel{\dot\subseteq} \dot x_k$ for every k. Also $((\dot b_J, \check\alpha), r') \in \dot\nu$, so

$$r' \Vdash_{\mathbb{P}} \dot\nu(\dot b_J) = \check\alpha = \inf_{k\in\mathbb{N}} \dot\nu(\dot x_k).$$

Thus

$$r' \Vdash_{\mathbb{P}} \text{ there is a lower bound } x \text{ for } \{\dot x_k : k\in\mathbb{N}\} \text{ such that } \dot\nu(x) = \inf_{k\in\mathbb{N}} \dot\nu(\dot x_k).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \text{ there is a lower bound } x \text{ for } \{\dot x_k : k\in\mathbb{N}\} \text{ such that } \dot\nu(x) = \inf_{k\in\mathbb{N}} \dot\nu(\dot x_k).$$

As p and $\langle\dot{x}_k\rangle_{k\in\mathbb{N}}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ if $\langle x_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\dot{\mathfrak{Z}}$ there is an $x \in \dot{\mathfrak{Z}}$ such that $x \mathrel{\dot{\subseteq}} x_k$ for every k and $\dot{\nu}(x) = \inf_{k\in\mathbb{N}} \dot{\nu}(x_k)$. **Q**

(viii)

$\Vdash_{\mathbb{P}}$ there is a family $\langle x_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ in $\dot{\mathfrak{A}}$ such that $\dot{\nu}(\inf_{L\in\mathcal{L}} x_L) = 2^{-\#(\mathcal{L})}$ for every finite set $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$.

P Let $\langle M_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ (5A1Fa). For each $n \in \mathbb{N}$, let $\langle K_{ni}\rangle_{i<n}$ be a family of subsets of I_n such that $\#(\bigcap_{i\in J} K_{ni}) = 2^{n-\#(J)}$ for every non-empty set $J \subseteq n$; such a family exists because $\#(I_n) = 2^n$. For $L \subseteq \mathbb{N}$, set

$$A_L = \bigcup_{n\in\mathbb{N}, n>\min M_L} K_{n,\max(n\cap M_L)}, \quad a_L = b_{A_L} \in \mathfrak{Z}.$$

If $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$ is finite and not empty, let $n_0 \in \mathbb{N}$ be such that $M_L \cap M_{L'} \subseteq n_0$ whenever L, $L' \in \mathcal{L}$ are distinct, and $n_1 \geq n_0$ such that $M_L \cap n_1 \setminus n_0 \neq \emptyset$ for every $L \in \mathcal{L}$. Then $\max(n \cap M_L) \neq \max(n \cap M_{L'})$ whenever L, $L' \in \mathcal{L}$ are distinct and $n \geq n_1$. So

$$\begin{aligned}\lim_{n\to\infty} 2^{-n}\#(I_n \cap \bigcap_{L\in\mathcal{L}} A_L) &= \lim_{n\to\infty} 2^{-n}\#(I_n \cap \bigcap_{L\in\mathcal{L}} K_{n,\max(n\cap M_L)}) \\ &= \lim_{n\to\infty} 2^{-n}2^{n-\#(\mathcal{L})} = 2^{-\#(\mathcal{L})}.\end{aligned}$$

Of course the same result is true for $\mathcal{L} = \emptyset$.

It follows that

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\check{\mathcal{L}}} \dot{a}_L) = 2^{-\#(\check{\mathcal{L}})}$$

for every finite $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$. Accordingly

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\mathcal{L}} \dot{a}_L) = 2^{-\#(\mathcal{L})} \text{ for every finite set } \mathcal{L} \subseteq (\mathcal{P}\mathbb{N})\check{}.$$

But we know also that

$$\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} = (\mathcal{P}\mathbb{N})\check{}$$

(5A3Qb). So the family $\langle\dot{a}_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ of $\mathbb{P}$-names, when interpreted as a $\mathbb{P}$-name $\langle\dot{a}_L\rangle_{L\in(\mathcal{P}\mathbb{N})\check{}}$ as in 5A3Eb, can also be regarded as a $\mathbb{P}$-name for a function defined on the whole power set of the set of natural numbers. If we do this, we get

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\mathcal{L}} \dot{a}_L) = 2^{-\#(\mathcal{L})} \text{ for every finite set } \mathcal{L} \subseteq \mathcal{P}\mathbb{N},$$

witnessing the truth of the result we seek. **Q**

(ix) $\Vdash_{\mathbb{P}} \#(\dot{\mathfrak{A}}) \leq \mathfrak{c}$. **P** Since

$$\dot{\mathfrak{Z}} = \{(\dot{a}, 1) : a \in \mathfrak{Z}\} = \{(\dot{b}_J, 1) : J \in \mathcal{P}\mathbb{N}\}$$

(556Ab), we get

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{Z}} = \{\dot{b}_J : J \in (\mathcal{P}\mathbb{N})\check{}\}, \text{ so } \#(\dot{\mathfrak{Z}}) \leq \#((\mathcal{P}\mathbb{N})\check{}) \leq \#(\mathcal{P}\mathbb{N}) = \mathfrak{c}. \textbf{ Q}$$

(e) Assembling the facts in (d), we see that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{Z}}, \dot{\nu}) \text{ satisfies the conditions of 556Q with } \kappa = \mathfrak{c}, \text{ so } \dot{\mathfrak{Z}} \cong \mathfrak{B}_{\mathfrak{c}}\,.$$

But we also have

$$\Vdash_{\mathbb{P}} \mathfrak{B}_{\mathfrak{c}} \text{ is isomorphic to } \mathfrak{B}_{\mathcal{P}\mathbb{N}} = \mathfrak{B}_{(\mathcal{P}\mathbb{N})\check{}} \cong (\mathfrak{B}_{\mathcal{P}\mathbb{N}})\check{}$$

by 556R. As $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$, we can apply 556Fc to see that $\widehat{\mathfrak{Z}}$ is isomorphic to the Dedekind completed free product $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_{\mathcal{P}\mathbb{N}}$ and therefore to $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})\widehat{\otimes}\mathfrak{B}_{\mathfrak{c}}$, by (a).

This completes the proof.

556X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$ which is not regularly embedded; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. Show that there is an $a \in \mathfrak{A} \setminus \{0\}$ such that $\Vdash_{\mathbb{P}} \dot{a} = 0$, where $\dot{a}$ is the forcing name for a over $\mathfrak{C}$.

(b) Let $\mathbb{P}$ be a countably closed forcing notion. (i) Show that $\Vdash_{\mathbb{P}} \omega_1 = \check{\omega}_1$. (ii) Show that $\Vdash_{\mathbb{P}} [\check{I}]^{\leq\omega} = ([I]^{\leq\omega})\check{}$ for every set I. (iii) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\Vdash_{\mathbb{P}} \check{\mathfrak{A}}$ is Dedekind σ-complete. (iiv) Let (X, ρ) be a complete metric space. Show that $\Vdash_{\mathbb{P}} (\check{X}, \check{\rho})$ is a complete metric space.

(c) Show that the Dedekind completion $\widehat{\mathfrak{Z}}$ of the asymptotic density algebra is a homogeneous Boolean algebra. (*Hint*: 316Q, 316P.)

556Y Further exercises (a) Let $\mathbb{P}$ be a forcing notion, and $\dot{\mathbb{Q}}_1$, $\dot{\mathbb{Q}}_2$ two $\mathbb{P}$-names for forcing notions such that

$$\Vdash_{\mathbb{P}} \mathrm{RO}(\dot{\mathbb{Q}}_1) \cong \mathrm{RO}(\dot{\mathbb{Q}}_2).$$

Show that $\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}}_1) \cong \mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}}_2)$.

(b) Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. Show that $\mathrm{RO}(\mathbb{P} * \check{\mathbb{Q}}) \cong \mathrm{RO}(\mathbb{P}) \widehat{\otimes} \mathrm{RO}(\mathbb{Q})$.

(c) Give an example of a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with an order-closed subalgebra $\mathfrak{C}$ such that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ is not Dedekind } \sigma\text{-complete},$$

where $\mathbb{P}$ is the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ is the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(d) Show that if the Proper Forcing Axiom is true then the asymptotic density algebra $\mathfrak{Z}$ is not homogeneous. (*Hint*: 5A1W.)

556 Notes and comments I did not positively instruct you to do so in the introduction to this section, but I expect that most readers will have passed rather quickly over the nineteen $\dot{}$-infested pages up to 556L, and looked at the target theorems in 556M, 556N and 556P. In each case we have a pair (‡), (†) of propositions, (†) being the special case of (‡) in which an algebra T or $\mathfrak{C}$ is trivial. If, as I hope, you are already acquainted with at least one of the assertions (‡), you will know that it can be proved by essentially the same methods as the corresponding (†), but with some non-trivial technical changes. These technical changes, already incorporated in the proofs of 388L/556N and 395P/556P in Volume 3, and indicated in §458 for 458Yd/556M, certainly do not amount to nineteen pages of mathematics in total; moreover, they explore ideas which are of independent interest. So I cannot on this evidence claim that the approach gives quick proofs of otherwise inaccessible results.

What I do claim is that the general method gives a way of looking at a recurrent phenomenon. Throughout the theory of measure-preserving transformations, ergodic transformations have a special place; and one comes to expect that once one has answered a question for ergodic transformations, the general case will be easy to determine. Similarly, every theorem about independent random variables ought to have a form applying to relatively independent variables. Indeed there are standard techniques for developing such extensions, based on disintegrations, as in §§458-459. What I have tried to do here is to develop a completely different approach, and in the process to indicate a new aspect of the theory of forcing. I note that the method demands preliminary translations into the language of measure algebras, which suits my prejudices as already expressed at length in Volume 3.

The great bulk of the work of this section consists of routine checks that natural formulae are in fact valid. The details demand a certain amount of attention. At the very beginning, in finding a forcing name $\dot{a}$ for an element of a Boolean algebra, we have to take care that we are exactly following our preferred formulation of what a name 'is'. (If my preferred formulation is not yours, you have some work to do, but it should not be difficult, and might be enlightening.) It is not surprising that regularly embedded subalgebras have a special status (556D); it is worth taking a moment to think about why it matters so much (556Xa). In 556H, I do not think it is obvious that $\mathfrak{A}$ must be Dedekind complete, rather than just Dedekind σ-complete, to make the ideas work in the straightforward way that they do (556Yc). When we come to measure algebras (556K), we need to be sure that we have a description of forcing names for real numbers which is compatible with the apparatus here. Again and again, we have sentences with clauses in both the forcing language and in ordinary language, and we must keep the pieces properly segregated in our minds.

The last fifth of the section (556Q-556S) is quite hard work for the result we get, but I think it is particularly instructive, in that it cannot be regarded as a technical extension of a simpler and more important result. It is a good example of a theorem proved by a method unavoidably dependent on the Forcing Theorem (5A3D), and for which it is not at all clear that a proof avoiding forcing can be made manageably simple. Such a proof must exist, but the obvious route to it involves teasing out the requisite parts of the proof of Maharam's theorem, and translating them into properties of the set

$$\{(J, \alpha, K) : K \in [\mathbb{N}]^\omega,\ J \subseteq \mathbb{N},\ \lim_{n \in K, n \to \infty} 2^{-n} \#(J \cap I_n) = \alpha\}$$

as in part (d) of the proof of 556S, but going very much farther. My own experience is that facing up to such challenges is often profitable, but for the moment I am happy to present an adaptation of Farah's original proof.

An easy corollary of Theorem 556S is that $\widehat{\mathfrak{Z}}$ is homogeneous (556Xc). This is striking in view of the fact that $\mathfrak{Z}$ itself may or may not be homogeneous. If the continuum hypothesis is true, then $\mathfrak{Z}$ is indeed homogeneous (Farah 03); but if the Proper Forcing Axiom is true, then $\mathfrak{Z}$ is *not* homogeneous (556Yd), even though its completion is.

Chapter 56

Choice and determinacy

Nearly everyone reading this book will have been taking the axiom of choice for granted nearly all the time. This is the home territory of twentieth-century abstract analysis, and the one in which the great majority of the results have been developed. But I hope that everyone is aware that there are other ways of doing things. In this chapter I want to explore what seem to me to be the most interesting alternatives. In one sense they are minor variations on the standard approach, since I keep strictly to ideas expressible within the framework of Zermelo-Fraenkel set theory; but in other ways they are dramatic enough to rearrange our prejudices. The arguments I will present in this chapter are mostly not especially difficult by the standards of this volume, but they do depend on intuitions for which familiar results which are likely to remain valid under the new rules being considered.

Let me say straight away that the real aim of the chapter is §567, on the axiom of determinacy. The significance of this axiom is that it is (so far) the most striking rival to the axiom of choice, in that it leads us quickly to a large number of propositions directly contradicting familiar theorems; for instance, every subset of the real line is now Lebesgue measurable (567G). But we need also to know which theorems are still true, and the first six sections of the chapter are devoted to a discussion of what can be done in ZF alone (§§561-565) and with countable or dependent choice (§566). Actually §§562-565 are rather off the straight line to §567, because they examine parts of real analysis in which the standard proofs depend only on countable choice or less; but a great deal more can be done than most of us would expect, and the methods are instructive.

Going into details, §561 looks at basic facts from real analysis, functional analysis and general topology which can be proved in ZF. §562 deals with 'codable' Borel sets and functions, using Borel codes to keep track of constructions for objects, so that if we know a sequence of codes we can avoid having to make a sequence of choices. A 'Borel-coded measure' (§563) is now one which behaves well with respect to codable sequences of measurable sets; for such a measure we have an integral with versions of the convergence theorems (§564), and Lebesgue measure fits naturally into the structure (§565). In §566, with ZF + AC(ω), we are back in familiar territory, and most of the results of Volumes 1 and 2 can be proved if we are willing to re-examine some definitions and hypotheses. Finally, in §567, I look at infinite games and half a dozen of the consequences of AD, with a postscript on determinacy in the context of ZF + AC.

561 Analysis without choice

Elementary courses in analysis are often casual about uses of weak forms of choice; a typical argument runs 'for every $\epsilon > 0$ there is an $a \in A$ such that $|a - x| \le \epsilon$, so there is a sequence in A converging to x'. This is a direct call on the countable axiom of choice: setting $A_n = \{a : a \in A,\ |a - x| \le 2^{-n}\}$, we are told that every A_n is non-empty, and conclude that $\prod_{n\in\mathbb{N}} A_n$ is non-empty. In the present section I will abjure such methods and investigate what can still be done with the ideas important in measure theory. We have useful partial versions of Tychonoff's theorem (561D), Baire's theorem (561E), Stone's theorem (561F) and Kakutani's theorem on the representation of L-spaces (561H); moreover, there is a direct construction of Haar measures, regarded as linear functionals (561G).

Unless explicitly stated otherwise, throughout this section (and the next four) I am working entirely without any form of the axiom of choice.

561A Set theory without choice In §§1A1 and 2A1 I tried to lay out some of the basic ideas of set theory with out appealing to the axiom of choice except when this was clearly necessary. The most obvious point is that in the absence of choice

the union of a sequence of countable sets need not be countable

(see the note in 1A1G). In fact Feferman & Levy 63 (see Jech 73, 10.6) have described a model of set theory in which $\mathbb{R}$ is the union of a sequence of countable sets. But not all is lost. The elementary arguments of 1A1E still give

$$\mathbb{N} \simeq \mathbb{Z} \simeq \mathbb{N} \times \mathbb{N} \simeq \mathbb{Q};$$

there is no difficulty in extending them to show such things as

$$\mathbb{N} \simeq [\mathbb{N}]^{<\omega} \simeq \bigcup_{n\ge 1} \mathbb{N}^n \simeq \mathbb{Q}^r \times \mathbb{Q}^r$$

for every integer $r \ge 1$. The Schröder-Bernstein theorem survives (the method in 344D is easily translated back into its original form as a proof of the ordinary Schröder-Bernstein theorem). Consequently we still have enough bijections to establish

$$\mathbb{R} \simeq \mathcal{P}\mathbb{N} \simeq \{0,1\}^{\mathbb{N}} \simeq \mathcal{P}(\mathbb{N} \times \mathbb{N}) \simeq (\mathcal{P}\mathbb{N})^{\mathbb{N}} \simeq \mathbb{R}^{\mathbb{N}} \simeq \mathbb{N}^{\mathbb{N}}.$$

Cantor's theorem that $X \not\simeq \mathcal{P}X$ is unaffected, so we still know that $\mathbb{R}$ is not countable.

We can still use transfinite recursion; see 2A1B. We still have a class On of von Neumann ordinals such that every well-ordered set is isomorphic to exactly one ordinal (2A1Dg) and therefore equipollent with exactly one initial ordinal (2A1Fb). I will say that a set X is **well-orderable** if there is a well-ordering of X. The standard arguments for Zermelo's Well-Ordering Theorem (2A1K) now tell us that for any set X the following are equiveridical:

(i) X is well-orderable;

(ii) X is equipollent with some ordinal;

(iii) there is an injective function from X into a well-orderable set;

(iv) there is a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

(that is, a function f such that $f(A) \in A$ for every non-empty $A \subseteq X$). What this means is that if we are given a family $\langle A_i \rangle_{i\in I}$ of non-empty sets, and $X = \bigcup_{i\in I} A_i$ is well-orderable (e.g., because it is countable), then $\prod_{i\in I} A_i$ is not empty (it contains $\langle f(A_i)\rangle_{i\in I}$ where f is a function as in (iv) above).

Note also that while we still have a first uncountable ordinal ω_1 (the set of countable ordinals), it can have countable cofinality (561Ya). The union of a sequence of finite sets need not be countable (Jech 73, §5.4); but the union of a sequence of finite subsets of a given totally ordered set *is* countable, because we can use the total ordering to simultaneously enumerate each of the finite sets in ascending order. Consequently, if $\gamma : \omega_1 \to \mathbb{R}$ is a monotonic function there is a $\xi < \omega_1$ such that $\gamma(\xi + 1) = \gamma(\xi)$. **P** It is enough to consider the case in which γ is non-decreasing. Set

$$A_n = \{\xi : \gamma(\xi) + 2^{-n} \le \gamma(\xi + 1) \le n\}.$$

Then A_n has at most $2^n \max(0, n - \gamma(0))$ members, so is finite; consequently $\bigcup_{n\in\mathbb{N}} A_n$ is countable, and there is a $\xi \in \omega_1 \setminus \bigcup_{n\in\mathbb{N}} A_n$. Of course we now find that $\gamma(\xi + 1) = \gamma(\xi)$. **Q**

561B Real analysis without choice In fact all the standard theorems of elementary real and complex analysis are essentially unchanged. The kind of tightening required in some proofs, to avoid explicit dependence on the existence of sequences, is similar to the adaptations needed when we move to general topological spaces. For instance, we must define 'compactness' in terms of open covers; compactness and sequential compactness, even for subsets of $\mathbb{R}$, may no longer coincide (561Xc). But we do still have the Heine-Borel theorem in the form 'a subset of $\mathbb{R}^r$ is compact iff it is closed and bounded' (provided, of course, that we understand that 'closed' is not the same thing as 'sequentially closed'); see the proof in 2A2F.

561C Some new difficulties arise when we move away from 'concrete' questions like the Prime Number Theorem and start looking at general metric spaces, or even general subsets of $\mathbb{R}$. For instance, a subset of $\mathbb{R}$, regarded as a topological space, must be second-countable but need not be separable. However we can go a long way if we take care. The following is an elementary example which will be useful below.

Lemma Let F be a non-empty closed subset of $\mathbb{N}^{\mathbb{N}}$. Then there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to F$ such that $f(\alpha) = \alpha$ for every $\alpha \in F$.

proof Set $T = \{\alpha{\restriction}n : \alpha \in F,\ n \in \mathbb{N}\}$. If $\alpha \in \mathbb{N}^{\mathbb{N}} \setminus F$ then, because F is closed, there is some $n \in \mathbb{N}$ such that $\beta{\restriction}n \ne \alpha{\restriction}n$ for any $\beta \in F$, that is, $\alpha{\restriction}n \notin T$. For $\sigma \in T$ define $\beta_\sigma \in \mathbb{N}^{\mathbb{N}}$ inductively by saying that

$$\begin{aligned}\beta_\sigma(n) &= \sigma(n) \text{ if } n < \#(\sigma),\\ &= \inf\{i : \text{ there is some } \alpha \text{ such that } \beta_\sigma{\restriction}n \subseteq \alpha \in F \text{ and } \alpha(n) = i\} \text{ otherwise,}\end{aligned}$$

counting $\inf\emptyset$ as 0 if necessary. We see that in fact $\beta_\sigma{\restriction}n \in T$ for every $n \in \mathbb{N}$, so that $\beta_\sigma \in F$.

We can therefore define $f : \mathbb{N}^{\mathbb{N}} \to F$ by setting

$$\begin{aligned}f(\alpha) &= \alpha \text{ if } \alpha \in F,\\ &= \beta_{\alpha\restriction n} \text{ for the largest } n \text{ such that } \alpha{\restriction}n \in T \text{ otherwise.}\end{aligned}$$

(Because F is not empty, the empty sequence $\alpha{\restriction}0$ belongs to T for every $\alpha \in \mathbb{N}^{\mathbb{N}}$.) This defines a retraction of $\mathbb{N}^{\mathbb{N}}$ onto F. To see that it is continuous, note that in fact if $\alpha \in \mathbb{N}^{\mathbb{N}} \setminus F$, and n is the largest integer such that $\alpha{\restriction}n$ belongs to T, then $f(\beta) = f(\alpha)$ whenever $\beta{\restriction}n + 1 = \alpha{\restriction}n + 1$, so f is continuous at α. While if $\alpha \in F$, $n \in \mathbb{N}$ and $\beta{\restriction}n = \alpha{\restriction}n$, then either $\beta \in F$ so $f(\beta){\restriction}n = f(\alpha){\restriction}n$, or $f(\beta) = \beta_\sigma$ where $\alpha{\restriction}n \subseteq \sigma \subseteq \beta_\sigma$, and again $f(\beta){\restriction}n = \beta_\sigma{\restriction}n = \alpha{\restriction}n$. So we have a suitable function.

561D Tychonoff's theorem It is a classic result (Kelley 50) that Tychonoff's theorem, in a general form, is actually equivalent to the axiom of choice. But nevertheless we have useful partial results which do not depend on the axiom of choice. The following will cover a case which arises in 563I.

Theorem Let $\langle X_i\rangle_{i\in I}$ be a family of compact topological spaces such that I is well-orderable. For each $i \in I$ let $\mathcal{E}_i$ be the family of non-empty closed subsets of X_i, and suppose that there is a choice function for $\bigcup_{i\in I}\mathcal{E}_i$. Then $X = \prod_{i\in I} X_i$ is compact.

proof Since I is well-orderable, we may suppose that $I = \kappa$ for some initial ordinal κ. Fix a choice function ψ for $\bigcup_{i\in I}\mathcal{E}_i$. For $\xi < \kappa$ write $\pi_\xi : X \to X_\xi$ for the coordinate map. If X is empty the result is trivial. Otherwise, let $\mathcal{F}$ be any family of closed subsets of X with the finite intersection property. I seek to define a non-decreasing family $\langle\mathcal{F}_\xi\rangle_{\xi\le\kappa}$ of filters on X such that the image filter $\pi_\xi[[\mathcal{F}_{\xi+1}]]$ (2A1I) is convergent for each $\xi < \kappa$. Start with $\mathcal{F}_0$ the filter generated by $\mathcal{F}$. Given $\mathcal{F}_\xi$, let F_ξ be the set of cluster points of $\pi_\xi[[\mathcal{F}_\xi]]$; because X_ξ is compact, this is a non-empty closed subset of X_ξ, and $x_\xi = \psi(F_\xi)$ is defined. Let $\mathcal{F}_{\xi+1}$ be the filter on X generated by

$$\mathcal{F}_\xi \cup \{\pi_\xi^{-1}[U] : U \text{ is a neighbourhood of } x_\xi \text{ in } X_\xi\}.$$

For limit ordinals $\xi \le \kappa$, let $\mathcal{F}_\xi$ be the filter on X generated by $\bigcup_{\eta<\xi}\mathcal{F}_\eta$.

Now $\mathcal{F}_\kappa$ is a filter including $\mathcal{F}$ converging to $x = \langle x_\xi\rangle_{\xi<\kappa}$, and x must belong to $\bigcap\mathcal{F}$. As $\mathcal{F}$ is arbitrary, X is compact.

Remark The point of the condition 'there is a choice function for $\bigcup_{i\in I}\mathcal{E}_i$' is that it is satisfied if every X_i is the unit interval $[0,1]$, for instance; we could take $\psi(E) = \min E$ for non-empty closed sets $E \subseteq [0,1]$. You will have no difficulty in devising other examples, using the technique of the proof above, or otherwise.

561E Baire's theorem (a) Let (X,ρ) be a complete metric space with a well-orderable dense subset. Then X is a Baire space.

(b) Let X be a compact Hausdorff space with a well-orderable π-base. Then X is a Baire space.

proof (a) Let D be a dense subset of X with a well-ordering $\preccurlyeq$. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X, and G is a non-empty open set, define $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $H_0 = G$. Given H_n, x_n is to be the $\preccurlyeq$-first point of H_n. Given x_n and H_n, ϵ_n is to be the first rational number in $]0,2^{-n}]$ such that $B(x_n,\epsilon_n) \subseteq H_n$. (I leave it to you to decide which rational numbers come first.) Now set $H_{n+1} = \{y : y \in G_n, \rho(y,x_n) < \epsilon_n\}$; continue.

At the end of the induction, $\langle x_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence so has a limit x in X. Since $x_n \in H_n \subseteq H_m$ whenever $m \le n$, $x \in \overline{H}_{n+2} \subseteq H_{n+1} \subseteq G_n$ for every n, and x witnesses that $G \cap \bigcap_{n\in\mathbb{N}} G_n$ is non-empty. As G and $\langle G_n\rangle_{n\in\mathbb{N}}$ are arbitrary, X is a Baire space.

(b) Let $\mathcal{U}$ be a π-base for the topology of X, not containing $\emptyset$, with a well-ordering $\preccurlyeq$. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X, and G is a non-empty open set, define $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ inductively by saying that

U_0 is the $\preccurlyeq$-first member of $\mathcal{U}$ included in G,

U_{n+1} is the $\preccurlyeq$-first member of $\mathcal{U}$ such that $\overline{U}_{n+1} \subseteq U_n \cap G_n$

for each n. Then $\bigcap_{n\in\mathbb{N}}\overline{U}_n$ is non-empty and included in $G \cap \bigcap_{n\in\mathbb{N}} G_n$.

561F Stone's Theorem Let $\mathfrak{A}$ be a well-orderable Boolean algebra. Then there is a compact Hausdorff Baire space Z such that $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed subsets of Z.

proof As in 311E, let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Writing $\mathbb{B}$ for the set of finite subalgebras of $\mathfrak{A}$, $Z = \bigcap_{\mathfrak{B}\in\mathbb{B}} Z_{\mathfrak{B}}$ where

$$Z_{\mathfrak{B}} = \{z : z \in \mathbb{Z}_2^{\mathfrak{A}},\ z{\restriction}\mathfrak{B} \text{ is a Boolean homomorphism}\}.$$

So Z is a closed subset of the compact Hausdorff space $\{0,1\}^{\mathfrak{A}}$, and is compact. Setting $\widehat{a} = \{z : z \in Z,\ z(a) = 1\}$, the map $a \mapsto \widehat{a}$ is a Boolean homomorphism from $\mathfrak{A}$ to the algebra $\mathfrak{E}$ of open-and-closed subsets of Z. If $a \in \mathfrak{A}\setminus\{0\}$ and $\mathfrak{B}$ is a finite subalgebra of $\mathfrak{A}$, then the subalgebra $\mathfrak{C}$ generated by $\{a\}\cup\mathfrak{B}$ is still finite, and there is a Boolean homomorphism $w : \mathfrak{C} \to \mathbb{Z}_2$ such that $w(a) = 1$; extending w arbitrarily to a member of $\{0,1\}^{\mathfrak{A}}$, we obtain a $z \in Z_{\mathfrak{B}}$ such that $z(a) = 1$; as $\mathfrak{B}$ is arbitrary, there is a $z \in Z$ such that $z(a) = 1$. So the map $a \mapsto \widehat{a}$ is injective. If $G \subseteq Z$ is open and $z \in G$, there must be a finite set $A \subseteq \mathfrak{A}$ such that G includes $\{z' : z' \in Z,\ z'{\restriction}A = z{\restriction}A\}$; in this case, setting $c = \inf\{a : a \in A,\ z(a) = 1\} \setminus \sup\{a : a \in A,\ z(a) = 0\}$, $z \in \widehat{c} \subseteq G$. It follows that any member of $\mathfrak{E}$ is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$, so that $a \mapsto \widehat{a}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{E}$.

Because $\mathfrak{A}$ is well-orderable, $\mathbb{Z}_2^{\mathfrak{A}}$ and Z have well-orderable bases, and Z is a Baire space, by 561E.

561G Haar measure Now I come to something which demands a rather less sketchy treatment.

Theorem Let X be a completely regular locally compact Hausdorff topological group.

(i) There is a non-zero left-translation-invariant positive linear functional on $C_k(X)$.

(ii) If ϕ, ϕ' are non-zero left-translation-invariant positive linear functionals on $C_k(X)$ then each is a scalar multiple of the other.

proof (a) Write Φ for $\{g : g \in C_k(X)^+,\ g(e) = \|g\|_\infty = 1\}$ where e is the identity of X. For $f \in C_k(X)^+$ and $g \in \Phi$, set

$$\lceil f : g\rceil = \inf\{\sum_{i=0}^{n} \alpha_i : \alpha_0, \dots, \alpha_n \ge 0$$
$$\text{and there are } a_0, \dots, a_n \in X \text{ such that } f \le \sum_{i=0}^{n} \alpha_i a_i \bullet_l g\},$$

writing $(a\bullet_l g)(x) = g(a^{-1}x)$ as in 441Ac. We have to confirm that this infimum is always defined in $[0,\infty[$. **P** Set $K = \overline{\{x : f(x) > 0\}}$ and $U = \{x : g(x) > \frac{1}{2}\}$, so that K is compact, U is open and $U \ne \emptyset$. Then $K \subseteq \bigcup_{a\in X} aU$, so there are $a_0, \dots, a_n \in X$ such that $K \subseteq \bigcup_{i\le n} a_i U$. In this case

$$f \le \textstyle\sum_{i=0}^{n} 2\|f\|_\infty a_i \bullet_l g$$

and $\lceil f : g\rceil \le 2(n+1)\|f\|_\infty$. **Q**

It is now easy to check that

$$\lceil a\bullet_l f : g\rceil = \lceil f : g\rceil, \quad \lceil f_1 + f_2 : g\rceil \le \lceil f_1 : g\rceil + \lceil f_2 : g\rceil,$$

$$\lceil \alpha f : g\rceil = \alpha\lceil f : g\rceil, \quad \|f\|_\infty \le \lceil f : g\rceil, \quad \lceil f : h\rceil \le \lceil f : g\rceil\lceil g : h\rceil$$

whenever f, f_1, $f_2 \in C_k(X)^+$, g, $h \in \Phi$, $a \in X$ and $\alpha \in [0,\infty[$. (Compare part (c) of the proof of 441C.)

(b) Fix $g_0 \in \Phi$, and for $g \in \Phi$ set

$$\psi_g(f) = \frac{\lceil f : g\rceil}{\lceil g_0 : g\rceil}$$

for $f \in C_k(X)$. Then

$$\psi_g(a\bullet_l f) = \psi_g(f), \quad \psi_g(f_1 + f_2) \le \psi_g(f_1) + \psi_g(f_2),$$

$$\psi_g(\alpha f) = \alpha\psi_g(f), \quad \psi_g(f) \le \lceil f : g_0\rceil,$$

$$\psi_g(f) \le \psi_g(h)\lceil f : h\rceil, \quad 1 \le \psi_g(h)\lceil g_0 : h\rceil$$

whenever f, f_1, $f_2 \in C_k(X)^+$, $h \in \Phi$, $a \in X$ and $\alpha \ge 0$. For a neighbourhood U of the identity e of X, write Φ_U for the set of those $g \in \Phi$ such that $g(x) = 0$ for every $x \in X \setminus U$; because X is locally compact and completely regular, $\Phi_U \ne \emptyset$.

(c)(i) If $f_0, \dots, f_m \in C_k(X)^+$ and $\epsilon > 0$, there is a neighbourhood U of e such that

$$\textstyle\sum_{j=0}^{m} \psi_g(f_j) \le \psi_g(\sum_{j=0}^{m} f_j) + \epsilon$$

whenever $g \in \Phi_U$. **P** Set $f = \sum_{j=0}^{m} f_j$. Let K be the compact set $\overline{\{x : f(x) \ne 0\}}$, and let $\hat f \in C_k(X)$ be such that $\chi K \le \hat f$. Let $\eta > 0$ be such that

$$(1 + (m+1)\eta)(\psi_g(f) + \eta\lceil \hat f : g_0\rceil) \le \psi_g(f) + \epsilon,$$

and set $f^* = f + \eta\hat f$. Then we can express each f_j as $f^* \times h_j$ where $h_j \in C_k(X)^+$ and $\sum_{j=0}^{m} h_j \le \chi X$. Let U be a neighbourhood of e such that $|h_j(x) - h_j(y)| \le \eta$ whenever $x^{-1}y \in U$ and $j \le m$ (compare 4A5Pa).

Take $g \in \Phi_U$. Let $\alpha_0, \dots, \alpha_n \ge 0$ and $a_0, \dots, a_n \in X$ be such that $f^* \le \sum_{i=0}^{n} \alpha_i a_i\bullet_l g$ and $\sum_{i=0}^{n} \alpha_i \le \lceil f^* : g\rceil + \eta$. Then, for any $x \in X$ and $j \le m$,

$$f_j(x) = f^*(x)h_j(x) \le \textstyle\sum_{i=0}^{n} \alpha_i g(a_i^{-1}x)h_j(x) \le \sum_{i=0}^{n} \alpha_i g(a_i^{-1}x)(h_j(a_i) + \eta)$$

because if i is such that $g(a_i^{-1}x) \ne 0$ then $a_i^{-1}x \in U$ and $h_j(x) \le h_j(a_i) + \eta$. So $\lceil f_j : g\rceil \le \sum_{i=0}^{n} \alpha_i(h_j(a_i) + \eta)$. Summing over j,

$$\textstyle\sum_{j=0}^{m}\lceil f_j : g\rceil \le \sum_{i=0}^{n} \alpha_i(1 + (m+1)\eta)$$

because $\sum_{j=0}^{m} h_j(a_i) \le 1$ for every i. As $\alpha_0, \dots, \alpha_n$ and $a_0, \dots, a_n$ are arbitrary,

$$\textstyle\sum_{j=0}^{m}\lceil f_j : g\rceil \le (1 + (m+1)\eta)\lceil f^* : g\rceil \le (1 + (m+1)\eta)(\lceil f : g\rceil + \eta\lceil \hat f : g\rceil),$$

and

$$\sum_{j=0}^{m}\psi_g(f_i)\le(1+(m+1)\eta)(\psi_g(f)+\eta\psi_g(\hat f))$$
$$\le(1+(m+1)\eta)(\psi_g(f)+\eta\lceil\hat f:g_0\rceil)\le\psi_g(f)+\epsilon$$

as required. **Q**

(ii) If $f_0,\dots,f_m\in C_k(X)^+$, $M\ge 0$ and $\epsilon>0$, there is a neighbourhood U of e such that

$$\sum_{j=0}^m\psi_g(\gamma_jf_j)\le\psi_g(\sum_{j=0}^m\gamma_jf_j)+\epsilon$$

whenever $g\in\Phi_U$ and $\gamma_0,\dots,\gamma_m\in[0,M]$. **P** Let $\eta>0$ be such that $\eta(1+\sum_{j=0}^m\lceil f_j:g_0\rceil)\le\epsilon$. By (i), applied finitely often, there is a neighbourhood U of e such that

$$\sum_{j=0}^m\psi_g(\gamma_jf_j)\le\psi_g(\sum_{j=0}^m\gamma_jf_j)+\eta$$

whenever $g\in\Phi_U$ and $\gamma_0,\dots,\gamma_m\in[0,M]$ are multiples of η. Now, given arbitrary $\gamma_0,\dots,\gamma_m\in[0,M]$ and $g\in\Phi_U$, let $\gamma'_0,\dots,\gamma'_m$ be multiples of η such that $\gamma'_j\le\gamma_j<\gamma'_j+\eta$ for each j. Then

$$\sum_{j=0}^m\psi_g(\gamma_jf_j)\le\sum_{j=0}^m\psi_g(\gamma'_jf_j)+\eta\psi_g(f_j)$$
$$\le\psi_g(\sum_{j=0}^m\gamma'_jf_j)+\eta+\eta\sum_{j=0}^m\psi_g(f_j)\le\psi_g(\sum_{j=0}^m\gamma_jf_j)+\epsilon$$

as required. **Q**

(d) Suppose that $f\in C_k(X)^+$, $\epsilon>0$ and that U is a neighbourhood of e such that $|f(x)-f(y)|\le\epsilon$ whenever $x^{-1}y\in U$. Then if $g\in\Phi_U$ and $\gamma>\epsilon$ there are $\alpha_0,\dots,\alpha_n\ge0$ and $a_0,\dots,a_n\in X$ such that $\|f-\sum_{i=0}^n\alpha_ia_i{\bullet_l}g\|_\infty\le\gamma$. **P** For all x, $y\in X$ we have

$$(f(x)-\epsilon)g(x^{-1}y)\le f(y)g(x^{-1}y)\le(f(x)+\epsilon)g(x^{-1}y).$$

Let $\eta>0$ be such that $\eta(1+\lceil f:\tilde g\rceil)\le\gamma-\epsilon$, where $\tilde g(x)=g(x^{-1})$ for $x\in X$. Let V be an open neighbourhood of e such that $|g(x)-g(y)|\le\eta$ whenever $xy^{-1}\in V$. Then we have $a_0,\dots,a_n$ such that $\bigcup_{i\le n}a_iV\supseteq\overline{\{x:f(x)\ne0\}}$, and $h_0,\dots,h_n\in C_k(X)^+$ such that $\sum_{i=0}^nh_i(x)=1$ whenever $f(x)>0$, while $h_i(x)=0$ if $i\le n$ and $x\notin a_iV$. By (c-ii), there is an $h\in\Phi$ such that

$$\sum_{i=0}^n\psi_h(\gamma_if\times h_i)\le\psi_h(\sum_{i=0}^n\gamma_if\times h_i)+\eta$$

whenever $0\le\gamma_i\le\lceil g_0:\tilde g\rceil$ for each i.

Now, for $i\le n$ and x, $y\in X$,

$$h_i(y)f(y)(g(a_i^{-1}x)-\eta)\le h_i(y)f(y)g(y^{-1}x)\le h_i(y)f(y)(g(a_i^{-1}x)+\eta).$$

Accordingly

$$(f(x)-\epsilon)(x{\bullet_l}\tilde g)(y)=(f(x)-\epsilon)g(y^{-1}x)\le f(y)g(y^{-1}x)=\sum_{i=0}^nh_i(y)f(y)g(y^{-1}x)$$
$$\le\sum_{i=0}^nh_i(y)f(y)(g(a_i^{-1}x)+\eta)=\eta f(y)+\sum_{i=0}^nh_i(y)f(y)g(a_i^{-1}x);$$

similarly,

$$(f(x)+\epsilon)(x{\bullet_l}\tilde g)(y)\ge\sum_{i=0}^nh_i(y)f(y)g(a_i^{-1}x)-\eta f(y).$$

Fixing x for the moment, and applying the functional ψ_h to the expressions here (regarded as functions of y), we get

$$(f(x)-\epsilon)\psi_h(\tilde g)\le\eta\psi_h(f)+\psi_h(\sum_{i=0}^ng(a_i^{-1}x)f\times h_i)$$

so

$$f(x)-\gamma\le f(x)-\epsilon-\eta\lceil f:\tilde g\rceil\le f(x)-\epsilon-\eta\frac{\psi_h(f)}{\psi_h(\tilde g)}$$
$$\le\psi_h(\sum_{i=0}^n\frac{g(a_i^{-1}x)}{\psi_h(\tilde g)}f\times h_i)\le\sum_{i=0}^n\frac{g(a_i^{-1}x)}{\psi_h(\tilde g)}\psi_h(f\times h_i)=\sum_{i=0}^n\alpha_ig(a_i^{-1}x)$$

where $\alpha_i = \frac{\psi_h(f\times h_i)}{\psi_h(\tilde{g})}$. On the other side,

$$(f(x)+\epsilon)\psi_h(\tilde{g})+\eta\psi_h(f)\ge\psi_h(\sum_{i=0}^n g(a_i^{-1}x)f\times h_i),$$

so

$$\begin{aligned}f(x)+\gamma&\ge f(x)+\epsilon+\eta\frac{\psi_h(f)}{\psi_h(\tilde{g})}+\eta\\&\ge\psi_h(\sum_{i=0}^n\frac{g(a_i^{-1}x)}{\psi_h(\tilde{g})}f\times h_i)+\eta\ge\sum_{i=0}^n\frac{g(a_i^{-1}x)}{\psi_h(\tilde{g})}\psi_h(f\times h_i)\end{aligned}$$

(because $\frac{g(a_i^{-1}x)}{\psi_h(\tilde{g})}\le\lceil g_0:\tilde{g}\rceil$ for every i)

$$=\sum_{i=0}^n\alpha_i g(a_i^{-1}x).$$

All this is valid for every $x\in X$; so

$$\|f-\sum_{i=0}^n\alpha_i a_i\bullet_l g\|_\infty\le\gamma.\ \mathbf{Q}$$

(e) For any $f\in C_k(X)^+$ and $\epsilon>0$ there are a $\gamma\ge0$ and a neighbourhood U of e such that $|\psi_h(f)-\gamma|\le\epsilon$ for every $h\in\Phi_U$. **P** Let V be a compact neighbourhood of 0 and $K=\overline{\{x:f(x)+g_0(x)>0\}}$; let $f^*\in C_k(X)$ be such that $\chi(KV^{-1}V)\le f^*$. Let δ, $\eta>0$ be such that

$$\delta(1+2(\delta+\lceil f:g_0\rceil))\le\epsilon,\quad\delta\le\tfrac12,\quad\eta(1+\lceil f^*:g_0\rceil)\le\delta.$$

By (d), there are $g\in\Phi_V$, $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m\ge0$ and $a_0,\dots,a_n,b_0,\dots,b_n\in X$ such that

$$\|f-\sum_{i=0}^n\alpha_i a_i\bullet_l g\|_\infty\le\eta,\quad\|g_0-\sum_{j=0}^m\beta_j b_j\bullet_l g\|_\infty\le\eta.$$

We can suppose that all the a_i, b_j belong to KV^{-1}, since $g(a^{-1}x)=0$ if $x\in K$ and $a\notin KV^{-1}$; consequently

$$|f-\sum_{i=0}^n\alpha_i a_i\bullet_l g|\le\eta f^*,\quad|g_0-\sum_{j=0}^m\beta_j b_j\bullet_l g|\le\eta f^*.$$

Set $\alpha=\sum_{i=0}^n\alpha_i$, $\beta=\sum_{j=0}^m\beta_j$ and $\gamma=\frac{\alpha}{\beta}$. ($\beta$ is non-zero because $\|g_0\|_\infty=1$ and $\eta\|f^*\|_\infty\le\frac12$.)

Let $U\subseteq V$ be a neighbourhood of e such that

$$\sum_{i=0}^n\alpha_i\psi_h(a_i\bullet_l g)\le\psi_h(\sum_{i=0}^n\alpha_i a_i\bullet_l g)+\eta,$$

$$\sum_{j=0}^m\beta_j\psi_h(b_j\bullet_l g)\le\psi_h(\sum_{j=0}^m\beta_j b_j\bullet_l g)+\eta$$

for every $h\in\Phi_U$ ((c) above). Take any $h\in\Phi_U$. Then

$$\begin{aligned}|\psi_h(f)-\alpha\psi_h(g)|&=|\psi_h(f)-\sum_{i=0}^n\alpha_i\psi_h(a_i\bullet_l g)|\le|\psi_h(f)-\psi_h(\sum_{i=0}^n\alpha_i a_i\bullet_l g)|+\eta\\&\le\eta\psi_h(f^*)+\eta\le\eta\lceil f^*:g_0\rceil+\eta;\end{aligned}$$

similarly,

$$|1-\beta\psi_h(g)|=|\psi_h(g_0)-\beta\psi_h(g)|\le\eta(\lceil f^*:g_0\rceil+1).$$

But this means that

$$\begin{aligned}|\psi_h(f)-\gamma|&\le\eta(1+\lceil f^*:g_0\rceil)+|\alpha\psi_h(g)-\gamma|\\&\le\delta+\gamma|\beta\psi_h(g)-1|\le\delta(1+\gamma).\end{aligned}$$

Consequently

$$\gamma\le\frac{\psi_h(f)+\delta}{1-\delta}\le2(\delta+\lceil f:g_0\rceil),$$

$$|\psi_h(f)-\gamma|\le\delta(1+2(\delta+\lceil f:g_0\rceil))\le\epsilon,$$

as required. **Q**

(f) We are nearly home. Let $\mathcal{F}$ be the filter on Φ generated by $\{\Phi_U:U$ is a neighbourhood of $e\}$. By (e), $\phi(f)=\lim_{h\to\mathcal{F}}\psi_h(f)$ is defined for every $f\in C_k(X)^+$. From the formulae in (b) we have

$$\phi(a\bullet_l f) = \phi(f), \quad \phi(f_1 + f_2) \le \phi(f_1) + \phi(f_2), \quad \phi(\alpha f) = \alpha\phi(f)$$

whenever f, f_1, $f_2 \in C_k(X)^+$, $a \in X$ and $\alpha \ge 0$. By (c-ii), we have $\phi(f_1) + \phi(f_2) \le \phi(f_1 + f_2)$ for all f_1, $f_2 \in C_k(X)^+$. So ϕ is additive and extends to an invariant positive linear functional on $C_k(X)$ which is non-zero because $\phi(g_0) = 1$.

(g) As for uniqueness, we can repeat the arguments in (e). Suppose that ϕ' is another left-translation-invariant positive linear functional on $C_k(X)$ such that $\phi'(g_0) = 1$, and $f \in C_k(X)^+$. Let K be the closure of $\{x : f(x) + g_0(x) > 0\}$ and V a compact neighbourhood of e; let $f^* \in C_k(X^*)$ be such that $\chi(KV^{-1}V) \le f^*$. Take $\epsilon > 0$. Let δ, $\eta > 0$ be such that

$$\delta \le \frac{1}{2}, \quad \delta(1 + 2(\phi(f) + 1)) \le \epsilon, \quad \delta(1 + 2(\phi'(f) + 1)) \le \epsilon,$$

$$\eta\phi(f^*) \le \delta, \quad \eta\phi'(f^*) \le \delta.$$

Then there is a neighbourhood U of e, included in V, such that $|f(x) - f(y)| \le \eta$ and $|g_0(x) - g_0(y)| \le \eta$ whenever $x^{-1}y \in U$. By (d), there are $g \in \Phi_V$, $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m \ge 0$ and $a_0, \dots, a_n, b_0, \dots, b_m \in X$ such that

$$|f(x) - \sum_{i=0}^n \alpha_i(a_i\bullet_l g)(x)| \le \eta, \quad |g_0(x) - \sum_{j=0}^m \beta_j(b_j\bullet_l g)(x)| \le \eta$$

for every $x \in X$; as in (e), we may suppose that every a_i, b_j belongs to KV^{-1} so that

$$|f - \sum_{i=0}^n \alpha_i a_i\bullet_l g| \le \eta f^*, \quad |g_0 - \sum_{j=0}^m \beta_j b_j\bullet_l g| \le \eta f^*.$$

Consequently, setting $\alpha = \sum_{i=0}^n \alpha_i$, $\beta = \sum_{i=0}^n \beta_i$ and $\gamma = \alpha/\beta$,

$$|\phi(f) - \alpha\phi(g)| = |\phi(f) - \sum_{i=0}^n \alpha_i\phi(a_i\bullet_l g)| \le \eta\phi(f^*) \le \delta,$$

$$|1 - \beta\phi(g)| = |\phi(g_0) - \sum_{j=0}^m \beta_j\phi(b_j\bullet_l g)| \le \eta\phi(f^*) \le \delta.$$

So

$$|\phi(f) - \gamma| \le \eta\phi(f^*) + \gamma|\beta\phi(g) - 1| \le \eta\phi(f^*)(1 + \gamma) \le \delta(1 + \gamma)$$

and

$$\gamma \le \tfrac{\phi(f)+\delta}{1-\delta} \le 2(\phi(f) + 1),$$

$$|\phi(f) - \gamma| \le \delta(1 + 2(\phi(f) + \delta)) \le \epsilon.$$

Similarly, $|\phi'(f) - \gamma| \le \epsilon$ and $|\phi(f) - \phi'(f)| \le 2\epsilon$. As ϵ and f are arbitrary, $\phi = \phi'$.

561H Kakutani's theorem (a) Let U be an Archimedean Riesz space with a weak order unit. Then there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and an order-dense Riesz subspace of $L^0(\mathfrak{A})$, containing $\chi 1$, which is isomorphic to U.

(b) Let U be an L-space with a weak order unit e. Then there is a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ such that U is isomorphic, as normed Riesz space, to $L^1(\mathfrak{A}, \bar{\mu})$, and we can choose the isomorphism to match e with $\chi 1$.

proof All the required ideas are in Volume 3; but we have quite a lot of checking to do.

(a)(i) The first step is to observe that, for any Dedekind σ-complete Boolean algebra $\mathfrak{A}$, the definition of $L^0 = L^0(\mathfrak{A})$ in 364A gives no difficulties, and that the formulae of 364E can be used to define a Riesz space structure on L^0. **P** I recall the formulae in question:

$$[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$\inf_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = 0, \quad \sup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = 1,$$

$$[\![u + v > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u > \alpha]\!] = [\![u > \tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in]0, \infty[$ and $\alpha \in \mathbb{R}$. The distributive laws in 313A-313B are enough to ensure that $u + v$ and γu, so defined, belong to L^0, and also that $u + v = v + u$, $u + (v + w) = (u + v) + w$, $\gamma(u + v) = \gamma u + \gamma v$ for u, v, $w \in L^0$ and $\gamma > 0$. Defining $\mathbf{0} \in L^0$ by saying that

$$[\![\mathbf{0} > \alpha]\!] = 1 \text{ if } \alpha < 0, 0 \text{ if } \alpha \ge 0,$$

we can check that $u + 0 = u$ for every u. Defining $-u \in L^0$ by saying that

$$[\![-u > \alpha]\!] = \sup_{q\in\mathbb{Q},q>\alpha} 1 \setminus [\![u > -q]\!]$$

for $u \in L^0$ and $\alpha \in \mathbb{R}$, we find (again using the distributive laws, of course) that $u + (-u) = \mathbf{0}$; we can now define γu, for $\gamma \le 0$, by saying that $0 \cdot u = \mathbf{0}$ and $\gamma u = (-\gamma)(-u)$ if $\gamma < 0$, and we shall have a linear space. Turning to the ordering, it is nearly trivial to check that the definition

$$u \le v \iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}$$

gives us a partially ordered linear space. It is a Riesz space because the formula

$$[\![u \vee v > \alpha]\!] = [\![u > \alpha]\!] \cup [\![v > \alpha]\!]$$

defines a member of L^0 which must be $\sup\{u, v\}$ in L^0. **Q**

We need to know that if $\mathfrak{A}$ is Dedekind complete, so is L^0; the argument of 364O still applies. Note also that $a \mapsto \chi a : \mathfrak{A} \to L^0$ is order-continuous, by 364M.

(ii) Now suppose that U is an Archimedean Riesz space with an order unit e. Let $\mathfrak{A}$ be the band algebra of U (353B). Then we can argue as in 368E, but with the simplification that the maximal disjoint set C in $U^+ \setminus \{0\}$ is just $\{e\}$, to see that we have an injective Riesz homomorphism $T : U \to L^0(\mathfrak{A})$ defined by taking $[\![Tu > \alpha]\!]$ to be the band generated by $e \wedge (u - \alpha e)^+$ (or, if you prefer, by $(u - \alpha e)^+$, since it comes to the same thing). We shall have $T[U]$ order-dense, as before, with $Te = \chi 1$.

(b)(i) Again, the bit we have to concentrate on is the check that, starting from a totally finite measure algebra $(\mathfrak{A}, \bar\mu)$, we can define $L^1(\mathfrak{A}, \bar\mu)$ as in 365A. We have to be a bit careful, because already in Proposition 321C there is an appeal to AC(ω); but I think we need to know very little about measure algebras to get through the arguments here. Of course another difficulty arises at once in 365A, because I write

$$\|u\|_1 = \int_0^\infty \bar\mu[\![|u| > \alpha]\!]\,d\alpha,$$

and say that the integration is with respect to 'Lebesgue measure', which won't do, at least until I redefine Lebesgue integration as in §565. But we are integrating a monotonic function, so the integral can be thought of as an improper Riemann integral; if you like,

$$\|u\|_1 = \lim_{n\to\infty} 2^{-n}\sum_{i=1}^{4^n} \bar\mu[\![|u| > 2^{-n}i]\!] = \sup_{n\in\mathbb{N}} 2^{-n}\sum_{i=1}^{4^n} \bar\mu[\![|u| > 2^{-n}i]\!].$$

Next, we can't use the Loomis-Sikorski theorem to prove 365C, and have to go back to first principles. To see that $\| \|_1$ is subadditive, and additive on $(L^0)^+$, look first at 'simple' non-negative u, expressed as $u = \sum_{i=0}^n \alpha_i\chi a_i$, and check that $\int u = \|u\|_1 = \sum_{i=0}^n \alpha_i\bar\mu a_i$; now confirm that every element of $(L^0)^+$ is expressible as the supremum of a non-decreasing sequence of such elements, and that $\| \|_1$ is sequentially order-continuous on the left on $(L^0)^+$. (We need 321Be.) This is enough to show that L^1 is a solid linear subspace of L^0 with a Riesz norm and a sequentially order-continuous integral. (I do *not* claim, yet, that L^1 is an L-space, because I do not know, in the absence of countable choice, that every Cauchy filter on L^1 converges.)

(ii) Now let U be an L-space with a weak order unit e. As in (a), let $\mathfrak{A}$ be the band algebra of U and $T : U \to L^0$ an injective Riesz homomorphism onto an order-dense Riesz subspace of L^0 with $Te = \chi 1$. Now U is Dedekind complete (354N, 354Ee). Consequently $T[U]$ must be solid in L^0 (353K).

(iii) For $a \in \mathfrak{A}$, set $\bar\mu a = \|T^{-1}(\chi a)\|$. Because the map $a \mapsto T^{-1}\chi a$ is additive and order-continuous and injective, $(\mathfrak{A}, \bar\mu)$ is a measure algebra; indeed, $\bar\mu$ is actually order-continuous. So we have a space $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Because $\bar\mu$ is order-continuous, 364Mb tells us that $\|w\|_1 = \sup_{v\in B} \|v\|_1$ whenever $B \subseteq L^0$ is a non-empty upwards-directed set in L^0 with supremum w in L^0.

Writing $S \subseteq L^0$ for the linear span of $\{\chi a : a \in \mathfrak{A}\}$, we see that $\|w\|_1 = \|T^{-1}w\|$ for every $w \in S$. Since S is order-dense in L^0 it is order-dense in L^1, and $T^{-1}[S]$ is order-dense in U, therefore norm-dense (354Ef).

(iv) $Tu \in L^1$ for every $u \in U^+$. **P** For $n \in \mathbb{N}$ set $a_n = [\![Tu > 2^n]\!] \setminus [\![Tu > 2^{n+1}]\!]$, $u_n = T^{-1}(\chi a_n)$. Set $w_n = \sum_{i=0}^n 2^i\chi a_i$ for $n \in \mathbb{N}$. Then $w_n \le Tu$ and $\|w_n\|_1 = \|T^{-1}w_n\| \le \|u\|$ for every n. By 364Ma, $w = \sup_{n\in\mathbb{N}} w_n$ is defined in L^0, and $\|w\|_1 = \sup_{n\in\mathbb{N}} \|w_n\|_1$ is finite. But $Tu \le 2w + \chi 1$, so $Tu \in L^1$. **Q**

(v) If $w \in (L^1)^+$ there is a $v \in U^+$ such that $w = Tv$ and $\|v\| = \|w\|_1$. **P** Consider $A = \{u : u \in T^{-1}[S], Tu \le w\}$. This is upwards-directed and norm-bounded, so has a supremum v in U (354N again), and $Tv \ge w'$ whenever $w' \in S$ and $w' \le w$. But S is order-dense in L^0 so $Tv \ge w$. Because T is order-continuous, (iii) tells us that

$$\|Tv\|_1 = \sup_{u\in A} \|Tu\|_1 = \sup_{u\in A} \|u\| = \|v\|,$$

while surely $\|w\|_1 \ge \sup_{u\in A} \|Tu\|_1$. So $Tv = w$. **Q**

(vi) Putting (iv) and (v) together, we see that $T[U] = L^1$ and that T is a normed Riesz space isomorphism, as required.

561I Hilbert spaces: Proposition Let U be a Hilbert space.

(a) If $C \subseteq U$ is a non-empty closed convex set then for any $u \in U$ there is a unique $v \in C$ such that $\|u-v\| = \min\{\|u-w\| : w \in C\}$.

(b) Every closed linear subspace of U is the image of an orthogonal projection, that is, has an orthogonal complement.

(c) Every member of U^* is of the form $u \mapsto (u|v)$ for some $v \in U$.

(d) U is reflexive.

(e) If $C \subseteq U$ is a norm-closed convex set then it is weakly closed.

proof (a) Set $\gamma = \min\{\|u-w\| : w \in C\}$ and let $\mathcal{F}$ be the filter on U generated by sets of the form $F_\epsilon = \{w : w \in C, \|u-w\| \le \gamma+\epsilon\}$ for $\epsilon > 0$. Then $\mathcal{F}$ is Cauchy. **P** Suppose that $\epsilon > 0$ and $w_1, w_2 \in F_\epsilon$. Then

$$\|w_1 - w_2\|^2 = 2\|u-w_1\|^2 + 2\|u-w_2\|^2 - \|2u - w_1 - w_2\|^2 \le 4(\gamma+\epsilon)^2 - 4\gamma^2$$

(because $\frac{1}{2}(w_1+w_2) \in C$)

$$= 8\gamma\epsilon + 4\epsilon^2.$$

So

$$\inf_{F\in\mathcal{F}} \operatorname{diam} F = \inf_{\epsilon>0} \operatorname{diam} F_\epsilon = 0. \quad \mathbf{Q}$$

We therefore have a limit v of $\mathcal{F}$, which is in C because C is closed, and $\|u-v\| = \lim_{w\to\mathcal{F}} \|u-w\| = \gamma$. If now w is any other member of C, $\|u - \frac{1}{2}(v+w)\| \ge \gamma$ so $\|u-w\| > \gamma$.

(b) Let V be a closed linear subspace of U. By (a), we have a function $P : U \to V$ such that Pu is the unique closest element of V to u, that is, $\|u-Pu\| \le \|u-Pu+\alpha v\|$ for every $v \in V$ and $\alpha \in \mathbb{R}$. It follows that $(u-Pu|v) = 0$ for every $v \in V$, that is, that $u-Pu \in V^\perp$. As u is arbitrary, $U = V + V^\perp$; as $V \cap V^\perp = \{0\}$, P must be the projection onto V with kernel $V^\perp$, and is an orthogonal projection.

(c) Take $f \in U^*$. If $f = 0$ then $f(u) = (u|0)$ for every u. Otherwise, set $C = \{w : f(w) = 0\}$. Then C is a proper closed linear subspace of U. Take any $u_0 \in U \setminus C$. Let v_0 be the point of C nearest to u_0, and consider $u_1 = u_0 - v_0$. Then 0 is the point of C nearest to u_1, so that $(u|u_1) = 0$ for every $u \in C$. Set $v = \frac{f(u_1)}{\|u_1\|^2}u_1$; then $(u|v) = 0$ for every $u \in C$, while $f(v) = (v|v)$. So $f(u) = (u|v)$ for every $u \in U$.

(d) From (c) it follows that we can identify U^* with U and therefore U^{**} also becomes identified with U.

(e) If C is empty this is trivial. Otherwise, take any $u \in U \setminus C$. Let v be the point of C nearest to U, and set $f(w) = (w|u-v)$ for $w \in U$. Then $f(w) \le f(v) < f(u)$ for every $w \in C$. So u does not belong to the weak closure of C; as u is arbitrary, C is weakly closed.

561X Basic exercises (a) Let X be any set. (i) Show that $\ell^p(X)$, for $1 \le p \le \infty$, is a Banach space. (ii) Show that $\ell^1(X)^*$ can be identified with $\ell^\infty(X)$. (iii) Show that if $\frac{1}{p} + \frac{1}{q} = 1$ then $\ell^p(X)^*$ can be identified with $\ell^q(X)$.

(b) Let X be any topological space. Show that $C_b(X)$, with $\| \,\|_\infty$, is a Banach space.

(c) Suppose that there is an infinite subset X of $\mathbb{R}$ with no infinite countable subset (Jech 73, §10.1). Show that X is sequentially closed but not closed, second-countable but not separable, sequentially compact but not compact, sequentially complete (that is, every Cauchy sequence converges) but not complete. Show that the topology of $\mathbb{R}$ is not countably tight.

(d)(i) Show that every non-empty closed subset of $\mathbb{N}^\mathbb{N}$ has a lexicographically-first member. (ii) Show that if a T_1 topological space X is a continuous image of $\mathbb{N}^\mathbb{N}$, then there is an injection from X to $\mathcal{P}\mathbb{N}$.

(e) Let X be a topological space. (i) Show that if X is separable, then $X^\mathbb{R}$ is separable. (ii) Show that if X has a countable network, then $X^\mathbb{R}$ has a countable network.

(f)(i) Show that a locally compact Hausdorff space is regular. (ii) Show that a compact Hausdorff space is normal.

(g) Let U be a normed space with a well-orderable subset D such that the linear span of D is dense in X. (i) Show that if V is a linear subspace of U and $f \in V^*$, there is a $g \in U^*$, extending f, with the same norm as f. (ii) Show that the unit ball B of U^* is weak*-compact and has a well-orderable base for its topology. (iii) Show that if $K \subseteq B$ is weak*-closed then K has an extreme point.

(h) Let (X,ρ) be a separable compact metric space, and G the isometry group of X with its topology of pointwise convergence (441G). Show that G is compact. (*Hint*: $X^{\mathbb{N}}$ is compact.)

(i) Let X be a regular topological space and A a subset of X. Show that the following are equiveridical: (i) A is relatively compact in X; (ii) $\overline{A}$ is compact; (iii) every filter on X containing A has a cluster point.

(j) Let X be a well-orderable discrete abelian group. Show that its dual group, as defined in 445A, is a completely regular compact Hausdorff group.

(k) Let U be a Riesz space with a Riesz norm. Let $\Delta : U^+ \to [0,\infty[$ be such that (α) Δ is non-decreasing, (β) $\Delta(\alpha u) = \alpha\Delta(u)$ whenever $u \in U^+$ and $\alpha \ge 0$, (γ) $\Delta(u+v) = \Delta(u)+\Delta(v)$ whenever $u\wedge v = 0$ (δ) $|\Delta(u)-\Delta(v)| \le \|u-v\|$ for all u, $v \in U^+$. Show that Δ has an extension to a member of U^*.

(l) Let U be an L-space. Show that $\|u\| = \sup\{f(u) : f \in U^*, \|f\| \le 1\}$ for every $u \in U$.

(m) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that $L^1(\mathfrak{A},\bar\mu)$ is a Dedekind σ-complete Riesz space and a sequentially complete normed space.

(n) Let $\mathfrak{A}$ be a Boolean algebra. Show that there are a set X, an algebra $\mathcal{E}$ of subsets of X and a surjective Boolean homomorphism from $\mathcal{E}$ onto $\mathfrak{A}$. (*Hint*: 566L.)

(o) Let U be a Hilbert space. (i) Show that a bounded sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U is weakly convergent in U iff $\lim_{n\to\infty}(u_n|u_m)$ is defined for every $m \in \mathbb{N}$. (ii) Show that the unit ball of U is sequentially compact for the weak topology. (iii) Show that if $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is included in the closed linear span of $\{Tv : v \text{ is an eigenvector of } T\}$.

>(p) Let X be a regular second-countable topological space, $\mathcal{C}$ the family of closed subsets of X, and $\mathcal{D}$ the set of disjoint pairs $(F_0,F_1) \in \mathcal{C}\times\mathcal{C}$. (i) Show that X is normal, and that there is a function $\psi : \mathcal{D} \to \mathcal{C}$ such that $F_0 \subseteq \operatorname{int}\psi(F_0,F_1)$ and $F_1 \cap \psi(F_0,F_1) = \emptyset$ whenever $(F_0,F_1) \in \mathcal{D}$. (ii) Show that there is a function $\phi : \mathcal{D} \to C(X)$ such $\phi(F_0,F_1)(x) = 0$, $\phi(F_0,F_1)(y) = 1$ whenever $(F_0,F_1) \in \mathcal{D}$, $x \in F_0$ and $y \in F_1$.

>(q) Let $(X,\mathfrak{T})$ be a regular second-countable topological space, and write $\mathfrak{S}$ for the usual topology on $\mathbb{R}^{\mathbb{N}}$. Show that there are a continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ and a function $\phi : \mathfrak{T} \to \mathfrak{S}$ such that $G = f^{-1}[\phi(G)]$ for every $G \in \mathfrak{T}$.

>(r) (i) Let C be the set of those $R \subseteq \mathbb{N}\times\mathbb{N}$ which are total orderings of subsets of $\mathbb{N}$. Show that C is a closed subset of $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ with its usual topology. (ii) For $\xi < \omega_1$, let C_ξ be the set of those $R \in C$ which are well-orderings of order type ξ of subsets of $\mathbb{N}$. Show that C_ξ is a Borel subset of $\mathcal{P}(\mathbb{N}\times\mathbb{N})$. (*Hint*: induce on ξ.) (iii) Show that there is an injective function from ω_1 to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

561Y Further exercises (a) Suppose that there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of countable sets such that $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$. Show that $\operatorname{cf}\omega_1 = \omega$.

(b)(i) Show that there is a bijection between ω_1 and $\mathbb{N}\times\omega_1$. (ii) Show that ω_2 is not expressible as the union of a sequence of countable sets. (iii) Show that $\mathcal{P}\omega_1$ is not expressible as the union of a sequence of countable sets. (iv) Show that $\mathcal{P}(\mathcal{P}\mathbb{N})$ is not expressible as the union of a sequence of countable sets.

(c) Suppose there is a countable family of doubleton sets with no choice function (Jech 73, §5.5). Show that (i) there is a set I such that $\{0,1\}^I$ has an open-and-closed set which is not determined by coordinates in any countable subset of I (ii) there is a compact metrizable space which is not ccc, therefore not second-countable (iii) there is a complete totally bounded metric space which is neither ccc nor compact.

(d) Show that a complete metric space with a well-orderable π-base is a Baire space.

(e) Let (X,ρ) be a complete metric space. Show that it is separable iff it is second-countable iff it has a countable π-base iff it has a countable network.

(f) Let X be a regular second-countable Hausdorff space. Show that it is metrizable, and that every continuous real-valued function defined on a closed subset of X has a continuous extension to a function on X.

(g)(i) Show that if $\mathfrak{A}$ is a Boolean algebra, there is an essentially unique Dedekind complete Boolean algebra $\widehat{\mathfrak{A}}$ in which $\mathfrak{A}$ can be embedded as an order-dense subalgebra. (ii) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, π has a unique extension to an order-continuous Boolean homomorphism from $\widehat{\mathfrak{A}}$ to $\widehat{\mathfrak{B}}$. (*Hint*: take $\widehat{\mathfrak{A}}$ to be the set of pairs (A,A') of subsets of $\mathfrak{A}$ such that A is the set of lower bounds of A' and A' is the set of upper bounds of A.)

(h) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. (i) Show that there is an essentially unique structure $(\mathfrak{A},\langle\varepsilon_i\rangle_{i\in I})$ such that (α) $\mathfrak{A}$ is a Boolean algebra (β) $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is a Boolean homomorphism for every i (γ) whenever $\mathfrak{B}$ is a Boolean algebra and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ is a Boolean homomorphism for every i, there is a unique Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $\pi\varepsilon_i=\phi_i$ for every i. (ii) Show that if $\nu_i:\mathfrak{A}_i\to\mathbb{R}$ is additive, with $\nu_i 1=1$, for every $i\in I$, there is a unique additive $\nu:\mathfrak{A}\to\mathbb{R}$ such that $\nu(\inf_{i\in J}\varepsilon_i(a_i))=\prod_{i\in J}\nu_i a_i$ whenever $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for $i\in J$.

(i) Suppose that there is an infinite set $I\subseteq\mathbb{R}$ with no countably infinite subset (Jech 73, §10.1). Let $\mathcal{E}$ be the algebra of subsets of $\{0,1\}^I$ determined by coordinates in finite sets. Show that $\mathcal{E}$ is a Dedekind complete measurable algebra.

(j) Show that a locally compact Hausdorff space with a well-orderable base is completely regular.

561 Notes and comments The arguments of this section will I hope give an idea of the kind of discipline which will be imposed for the rest of the chapter. Apart from anything else, we have to fix on the correct definitions. Typically, when defining something like 'compactness' or 'completeness', the definition to use is that which is most useful in the most general context; so that even in metrizable spaces we should prefer filters to sequences (561Ye).

We can distinguish two themes in the methods I have used here. First, in the presence of a well-ordering we can hope to adapt the standard attack on a problem; see 561D, 561E, 561F. Second, if (in the presence of the axiom of choice) there is a *unique* solution to a problem, then we can hope that it is still a unique solution without choice. This is what happens in 561G and also in 561Ia-561Ic. In 561I we just go through the usual arguments with a little more care. In 561G (taken from Naimark 70) we need new ideas. But in the key step, part (d) of the proof, the two variables x and y reflect an adaptation of a repeated-integration argument as in §442. Note that the scope of 561G may be limited if we have fewer locally compact groups than we expect.

A regular second-countable Hausdorff space is metrizable (561Yf). But it may not be separable (561Xc). We do not have Urysohn's Lemma in its usual form, so cannot be sure that a locally compact Hausdorff space is completely regular; a topological group has left, right and bilateral uniformities, but a uniformity need not be defined by pseudometrics and a uniform space need not be completely regular. So in such results as 561G we may need an extra 'completely regular' in the hypotheses.

I give a version of Kakutani's theorem (561H) to show that some of the familiar patterns are distorted in possibly unexpected ways, and that occasionally it is the more abstract parts of the theory which survive best. I suppose I ought to remark explicitly that I define 'measure algebra' exactly as in 321A: a Dedekind σ-complete Boolean algebra with a strictly positive countably additive $[0,\infty]$-valued functional. I do not claim that every σ-finite measure algebra is localizable, nor that every measure algebra can be represented in terms of a measure space. I set up a construction of a normed Riesz space $L^1(\mathfrak{A},\bar\mu)$, but do not claim that this is an L-space. However, if we start from an L-space U with a weak order unit, we can build a measure on its band algebra and proceed to an $L^1(\mathfrak{A},\bar\mu)$ which is isomorphic to U (and is therefore an L-space).

562 Borel codes

The concept of 'Borel set', either in the real line or in general topological spaces, has been fundamental in measure theory since before the modern subject existed. It is at this point that the character of the subject changes if we do not allow ourselves even the countable axiom of choice. I have already mentioned the Feferman-Lévy model in which $\mathbb{R}$ is a countable union of countable sets; immediately, every subset of $\mathbb{R}$ is a countable union of countable sets and is 'Borel' on the definition of 111G. In these circumstances that definition becomes unhelpful.

An alternative which leads to a non-trivial theory, coinciding with the usual theory in the presence of AC, is the algebra of 'codable Borel sets' (562A). This is not necessarily a σ-algebra, but is closed under unions and intersections of 'codable sequences' (562I). When we come to look for measurable functions, the corresponding concept is that of 'codable Borel function' (562J); again, we do not expect the limit of an arbitrary sequence of codable Borel functions to be measurable in any useful sense, but the limit of a codable sequence of codable Borel functions is again a codable Borel function (562Le). The same ideas can be used to give a theory of 'codable Baire sets' in any topological space (562R).

562A Coding with trees (a) Set $S=\bigcup_{n\ge1}\mathbb{N}^n$. For σ, $\tau\in S$ let $\sigma^\frown\tau$ be their concatenation, that is,

$$\begin{aligned}(\sigma^\frown\tau)(n)&=\sigma(n)\text{ if }n<\#(\sigma),\\&=\tau(n-\#(\sigma))\text{ if }\#(\sigma)\le n<\#(\sigma)+\#(\tau).\end{aligned}$$

For $i\in\mathbb{N}$ I will write $<\!i\!>\in\mathbb{N}^1$ for the one-term sequence with value i. For $\sigma\in S$ and $T\subseteq S$, write T_σ for $\{\tau:\tau\in S,\ \sigma^\frown\tau\in T\}$.

(b) Let $\mathcal{T}_0$ be the family of sets $T \subseteq S$ such that $\sigma\restriction n \in T$ whenever $\sigma \in T$ and $n \geq 1$. Recall from 421N[1] that we have a derivation $\partial : \mathcal{T}_0 \to \mathcal{T}_0$ defined by setting

$$\partial T = \{\sigma : \sigma \in S,\, T_\sigma \neq \emptyset\},$$

with iterates ∂^ξ, for $\xi < \omega_1$, defined by setting

$$\partial^0 T = T, \quad \partial^\xi T = \bigcap_{\eta<\xi} \partial(\partial^\eta T) \text{ for } \xi \geq 1.$$

Now for any $T \in \mathcal{T}_0$ there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \leq \eta < \omega_1$. **P** The argument in 421Nd assumed that ω_1 has uncountable cofinality, but we can avoid this assumption, as follows. Let $\langle \epsilon_\sigma\rangle_{\sigma\in S}$ be a summable family of strictly positive real numbers, and set $\gamma_T(\xi) = \sum_{\sigma\in\partial^\xi T} \epsilon_\sigma$; then $\gamma_T : \omega_1 \to [0,\infty[$ is non-increasing, so 561A tells us that there is a $\xi < \omega_1$ such that $\gamma_T(\xi+1) = \gamma_T(\xi)$, that is, $\partial^{\xi+1}T = \partial^\xi T$. Of course we now have $\partial^\eta T = \partial^\xi T$ for every $\eta \geq \xi$. **Q**

(c) We therefore still have a rank function $r : \mathcal{T}_0 \to \omega_1$ defined by saying that $r(T)$ is the least ordinal such that $\partial^{r(T)}T = \partial^{r(T)+1}T$. Now $\partial^{r(T)}T$ is empty iff there is no $\alpha \in \mathbb{N}^\mathbb{N}$ such that $\alpha\restriction n \in T$ for every $n \geq 1$. **P** The argument in 421Nf used the word 'choose'; but we can avoid this by being more specific. If $\sigma \in \partial^{r(T)}T$, then we can define a sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ by saying that $\sigma_0 = \sigma$ and, given $\sigma_n \in \partial^{r(T)}T$, $\sigma_{n+1} = \sigma_n{}^\frown{<}i{>}$ for the least i such that $\sigma_n{}^\frown{<}i{>} \in \partial^{r(T)}T$; $\alpha = \bigcup_{n\in\mathbb{N}} \sigma_n$ will now have $\alpha\restriction n \in T$ for every $n \geq 1$. The argument in the other direction is unchanged. **Q**

Let $\mathcal{T}$ be the set of those $T \in \mathcal{T}_0$ with no infinite branch, that is, such that $\partial^{r(T)}T = \emptyset$.

(d) For $T \in \mathcal{T}$, set $A_T = \{i : {<}i{>} \in T\}$. We need a fact not covered in §421: for any $T \in \mathcal{T}$, $r(T) = \sup\{r(T_{<i>})+1 : i \in A_T\}$. **P** An easy induction on ξ shows that $\partial^\xi(T_\sigma) = (\partial^\xi T)_\sigma$ for any $\xi < \omega_1$, $T \in \mathcal{T}_0$ and $\sigma \in S$. So, for $T \in \mathcal{T}$ and $\xi < \omega_1$,

$$\begin{aligned} r(T) > \xi &\Longrightarrow \partial^\xi T \neq \emptyset \\ &\Longrightarrow \exists\, i,\, {<}i{>} \in \partial^\xi T = \bigcap_{\eta<\xi} \partial^{\eta+1}T \\ &\Longrightarrow \exists\, i \in A_T,\, \partial^\eta(T_{<i>}) = (\partial^\eta T)_{<i>} \neq \emptyset\ \forall\, \eta < \xi \\ &\Longrightarrow \exists\, i \in A_T,\, r(T_{<i>}) > \eta\ \forall\, \eta < \xi \\ &\Longrightarrow \exists\, i \in A_T,\, r(T_{<i>}) \geq \xi; \end{aligned}$$

thus $r(T) \leq \sup\{r(T_{<i>}) + 1 : i \in A_T\}$. In the other direction, if $i \in A_T$ and $\eta < \xi = r(T_{<i>})$, then

$$(\partial^\eta T)_{<i>} = \partial^\eta(T_{<i>}) \neq \emptyset,$$

so ${<}i{>} \in \partial^{\eta+1}T$; as η is arbitrary, ${<}i{>} \in \partial^\xi T$ and $\xi < r(T)$; as i is arbitrary, $r(T) \geq \sup\{r(T_{<i>}) + 1 : i \in A_T\}$. **Q**

(e) Let X be a set and $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X. Define $\phi : \mathcal{T} \to \mathcal{P}X$ inductively by saying that

$$\begin{aligned} \phi(T) &= \bigcup_{i\in A_T} U_i \text{ if } r(T) \leq 1, \\ &= \bigcup_{i\in A_T} X \setminus \phi(T_{<i>}) \text{ if } r(T) > 1. \end{aligned}$$

By (d), this definition is sound. I will call ϕ the **interpretation of Borel codes** defined by X and $\langle U_n\rangle_{n\in\mathbb{N}}$.

For technical reasons, it will be helpful to know that there is a non-empty $T \in \mathcal{T}$ such that $\phi(T) = \emptyset$. **P** Try

$$\begin{aligned} T = \{&{<}0{>}, {<}0{>}^\frown{<}0{>}, {<}0{>}^\frown{<}0{>}^\frown{<}0{>}, {<}0{>}^\frown{<}0{>}^\frown{<}0{>}^\frown{<}0{>}, \\ &{<}0{>}^\frown{<}1{>}, {<}0{>}^\frown{<}1{>}^\frown{<}0{>}\}. \end{aligned} \tag{$*$}$$

Then

$$\phi(T) = X \setminus ((X \setminus (X \setminus E_0)) \cup (X \setminus E_0)) = \emptyset. \ \mathbf{Q}$$

(f) Now suppose that X is a second-countable topological space and that $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ are two sequences running over bases for the topology of X. Let $\phi : \mathcal{T} \to \mathcal{P}X$ and $\phi' : \mathcal{T} \to \mathcal{P}X$ be the interpretations of Borel codes defined by $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively. Then there is a function $\Theta : \mathcal{T} \to \mathcal{T}$ such that $\phi'\Theta = \phi$. **P** Define Θ

[1] Early editions used a slightly different definition of iterated derivations, so that the 'rank' of a tree was not quite the same.

inductively, as follows. The inductive hypothesis will include the assertion that $r(\Theta(T)) \geq 1$ for every $T \in \mathcal{T}$. If $r(T) \leq 1$, then $\phi(T) = \bigcup_{i\in A_T} U_i$ is open. If $\phi(T) \neq \emptyset$, set $\Theta(T) = \{<j> : j \in \mathbb{N},\, V_j \subseteq \phi(T)\}$; then

$$\phi'(\Theta(T)) = \bigcup\{V_j : j \in \mathbb{N},\, V_j \subseteq \phi(T)\} = \phi(T)$$

and $r(\Theta(T)) = 1$. If $\phi(T) = \emptyset$, take $\Theta(T)$ to be any non-empty member of $\mathcal{T}$ such that $\phi'(\Theta(T)) = \emptyset$; e.g., that presented at (*) just above.

For the inductive step to $r(T) > 1$, set

$$\Theta(T) = \{<i> : i \in A_T\} \cup \{<i>^\frown\sigma : i \in A_T,\, \sigma \in \Theta(T_{<i>})\};$$

then $r(\Theta(T)) > 1$ and

$$\begin{aligned}\phi'(\Theta(T)) &= \bigcup_{i\in A_{\Theta(T)}} X \setminus \phi'(\Theta(T)_{<i>}) = \bigcup_{i\in A_T} X \setminus \phi'(\Theta(T_{<i>}))\\ &= \bigcup_{i\in A_T} X \setminus \phi(T_{<i>}) = \phi(T),\end{aligned}$$

so the induction continues. **Q**

(There will be a substantial strengthening of this idea in 562Ka.)

(g) Now say that a **codable Borel set** in X is one expressible as $\phi(T)$ for some $T \in \mathcal{T}$, starting from some enumeration of some base for the topology of X; in view of (f), we can restrict our calculations to a fixed enumeration of a fixed base if we wish. I will write $\mathcal{B}_c(X)$ for the family of codable Borel sets of X.

The definition of 'interpretation of Borel codes' makes it plain that any σ-algebra of subsets of X containing every open set will also contain every codable Borel set; so every codable Borel set is indeed a 'Borel set' on the definition of 111G or 4A3A.

It will sometimes be useful to know that every element of $\mathcal{B}_c(X)$ can be coded by a non-empty member of $\mathcal{T}$; we have only to check the case of the empty set, which is dealt with in the formula (*) in (e).

562B The point of these codings is that we can define explicit functions on $\mathcal{T}$ which will have appropriate reflections in the coded sets.

(a) For instance, there are functions $\Theta_1 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $\Theta_2 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}$ such that

$$\phi(\Theta_1(T_0, T_1)) = \phi(T_0) \setminus \phi(T_1), \quad \phi(\Theta_2(\langle T_n\rangle_{n\in\mathbb{N}})) = \bigcup_{n\in\mathbb{N}} \phi(T_n)$$

for every sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$. **P** If either T_0 or T_1 is empty, set $\Theta_1(T_0, T_1) = T_0$. Otherwise, take

$$\begin{aligned}\Theta_1(T_0, T_1) = \{&<0>, <0>^\frown<0>, <0>^\frown<0>^\frown<0>, <0>^\frown<1>\}\\ &\cup \{<0>^\frown<0>^\frown<0>^\frown\sigma : \sigma \in T_1\} \cup \{<0>^\frown<1>^\frown\sigma : \sigma \in T_0\}.\end{aligned}$$

Next, set $A = \{n : T_n \neq \emptyset\}$, and

$$\begin{aligned}\Theta_2(\langle T_n\rangle_{n\in\mathbb{N}}) = \{&<n> : n \in A\} \cup \{<n>^\frown<0> : n \in A\}\\ &\cup \{<n>^\frown<0>^\frown\sigma : n \in A,\, \sigma \in T_n\}. \textbf{ Q}\end{aligned}$$

Now, setting $\Theta_3(T_0, T_1) = \Theta_1(T_0, \Theta_1(T_0, T_1))$, for instance, we get a similar operator such that

$$\phi(\Theta_3(T_0, T_1)) = \phi(T_0) \cap \phi(T_1)$$

for all T_0, $T_1 \in \mathcal{T}$. I hope that the wide variety of such functions we shall need in the course of the next few sections will give no difficulties. Note that the formulae just given do not need to refer to the space X or the sequence $\langle U_n\rangle_{n\in\mathbb{N}}$.

(b) A more sophisticated version of two such codings will be useful in §564. Let X be a regular second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X containing $\emptyset$, and $\phi : \mathcal{T} \to \mathcal{P}X$ the associated interpretation of Borel codes. Then there are functions Θ, $\Theta' : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ such that

$$\phi(\Theta(T, T')) = \phi(T) \cup \phi(T'), \quad \phi(\Theta'(T, T')) = \phi(T) \cap \phi(T'),$$

$$r(\Theta(T, T')) = r(\Theta'(T, T')) = \max(r(T), r(T'))$$

for all T, $T' \in \mathcal{T}$. **P** The point is just that open sets in a regular second-countable space are F_σ. Because of the slightly awkward form taken by the definition of ϕ, we need to start with an auxiliary function. Define $T \mapsto \tilde{T} : \mathcal{T} \to \mathcal{T}$ by saying that

$$\tilde{T} = \{<n> : \overline{U}_n \subseteq \phi(T)\} \cup \{<n>^\frown<i> : \overline{U}_n \subseteq \phi(T),\, U_i \cap U_n = \emptyset\} \quad \text{if } r(T) \le 1,$$

$$= T \text{ otherwise .}$$

Then $\phi(\tilde{T}) = T$ and $r(\tilde{T}) = \max(2, r(T))$ for every T (because if $r(T) \le 1$ there is some n such that $U_n = \emptyset$ and $<n>^\frown<n> \in \tilde{T}$). We also need to fix a bijection $n \mapsto (i_n, j_n)$ between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

Now define Θ by saying that

— if $\max(r(T), r(T')) \le 1$, $\Theta(T, T') = \Theta(T) \cup \Theta(T')$;

— if $\max(r(T), r(T')) > 1$, then

$$\Theta(T, T') = \{<2n> : <n> \in \tilde{T}\} \cup \{<2n>^\frown\sigma : \sigma \in \tilde{T}_{<n>}\} \cup \{<2n+1> : <n> \in \tilde{T}'\} \cup \{<2n+1>^\frown\sigma : \sigma \in \tilde{T}'_{<n>}\}$$

.

For Θ' induce on $\max(r(T), r(T'))$:

— if $T = T' = \emptyset$, $\Theta'(T, T') = \emptyset$;

— if $\max(r(T), r(T')) = 1$,

$$\Theta'(T, T') = \{<n> : U_n \subseteq \phi(T) \cap \phi(T')\};$$

— if $\max(r(T), r(T')) > 1$,

$$\Theta'(T, T') = \{<n> : i_n \in A_{\tilde{T}},\, j_n \in A_{\tilde{T}'}\} \cup \{<n>^\frown\sigma : n \in \mathbb{N},\, \sigma \in \Theta(\tilde{T}_{<i_n>}, \tilde{T}'_{<j_n>})\}.$$

This works. **Q**

(c) In a different way, we can strengthen the other formulation, as follows. For any countable set K we have a function $\tilde{\Theta}_2 : \bigcup_{J \subseteq K} \mathcal{T}^J \to \mathcal{T}$ such that whenever X is a second-countable space, $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ is an interpretation of Borel codes defined by a sequence running over a base for the topology of X, $J \subseteq K$ and $\tau \in \mathcal{T}^J$ then $\phi(\tilde{\Theta}_2(\tau)) = \bigcup_{j \in J} \phi(\tau(j))$. **P** If K is empty this is trivial. Otherwise, let $\langle k_n \rangle_{n \in \mathbb{N}}$ be a sequence running over K. For $J \subseteq K$ and $\tau \in \mathcal{T}^J$, set $A = \{n : k_n \in J,\, \tau(k_n) \ne \emptyset\}$ and

$$\tilde{\Theta}_2(\tau) = \{<n> : n \in A\} \cup \{<n>^\frown<0> : n \in A\} \cup \{<n>^\frown<0>^\frown\sigma : n \in A,\, \sigma \in \tau(k_n)\}. \textbf{ Q}$$

562C Proposition (a) If X is a second-countable space, then the family of codable Borel subsets of X is an algebra of subsets of X containing every G_δ set and every F_σ set.

(b) [AC(ω)] Every Borel set is a codable Borel set.

proof (a) Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the corresponding surjection. From 562Ba we see that $E_0 \setminus E_1 \in \mathcal{B}_c(X)$ for all E_0, $E_1 \in \mathcal{B}_c(X)$; since $X = \phi(\{<n> : n \in \mathbb{N}\})$ belongs to $\mathcal{B}_c(X)$, $\mathcal{B}_c(X)$ is an algebra of subsets of X.

If $E \subseteq X$ is an F_σ set, there is a sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of closed sets with union E. Set

$$T = \{<n> : n \in \mathbb{N}\} \cup \{<n>^\frown<i> : n,\, i \in \mathbb{N},\, U_i \subseteq X \setminus F_n\}.$$

Then $r(T) = 2$, $\phi(T_{<n>}) = X \setminus F_n$ for every n and $\phi(T) = E$.

Thus every F_σ set belongs to $\mathcal{B}_c(X)$; it follows at once that every G_δ set is also a codable Borel set.

(b) We can repeat the argument in (a), but this time in a more general form. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\mathcal{B}_c(X)$, then for each $n \in \mathbb{N}$ **choose** $T_n \in \mathcal{T} \setminus \{\emptyset\}$ such that $\phi(T_n) = E_n$; set

$$T = \{<n> : n \in \mathbb{N}\} \cup \{<n>^\frown\sigma : n \in \mathbb{N},\, \sigma \in T_n\};$$

then $\bigcup_{n \in \mathbb{N}} X \setminus E_n = \phi(T)$ is a codable Borel set. Because $\mathcal{B}_c(X)$ is an algebra, this is enough to show that it is a σ-algebra and therefore equal to the σ-algebra $\mathcal{B}(X)$.

562D Proposition Let X be a second-countable space and $Y \subseteq X$ a subspace of X. Then $\mathcal{B}_c(Y) = \{Y \cap E : E \in \mathcal{B}_c(X)\}$.

proof Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of X, and set $V_n = Y \cap U_n$ for each n; let $\phi_X : \mathcal{T} \to \mathcal{B}_c(X)$ and $\phi_Y : \mathcal{T} \to \mathcal{B}_c(Y)$ be the interpretations of Borel codes corresponding to $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively. Then an easy induction on the rank of T shows that $\phi_Y(T) = Y \cap \phi_X(T)$ for every $T \in \mathcal{T}$. So

$$\mathcal{B}_c(Y) = \phi_Y[\mathcal{T}] = \{Y \cap \phi_X(T) : T \in \mathcal{T}\} = \{Y \cap E : E \in \mathcal{B}_c(X)\}.$$

***562E** I do not expect to rely on the next result, but it is interesting that one of the basic facts of descriptive set theory still has a version in the new context.

Theorem Let X be a Polish space. Then every codable Borel set in X is analytic.

proof (a) If X is empty, this is trivial; suppose henceforth that X is not empty. Let ρ be a complete metric on X inducing its topology, $\langle x_n\rangle_{n\in\mathbb{N}}$ a sequence running over a dense subset of X, and $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X. Define $\mathcal{T}$, $r : \mathcal{T} \to \omega_1$ and $\phi : \mathcal{T} \to \mathcal{P}X$ as in 562A.

(b) We need to fix on a continuous surjection from a closed subset of $\mathbb{N}^{\mathbb{N}}$ onto X; a convenient one is the following. Set

$$F = \{\alpha : \alpha \in \mathbb{N}^{\mathbb{N}},\ \rho(x_{\alpha(n+1)}, x_{\alpha(n)}) \le 2^{-n} \text{ for every } n \in \mathbb{N}\};$$

then $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed. Define $f : F \to X$ by saying that $f(\alpha) = \lim_{n\to\infty} x_{\alpha(n)}$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. If α, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $\alpha{\restriction}n = \beta{\restriction}n$ where $n \ge 1$, then $\rho(f(\alpha), f(\beta)) \le 2^{-n+2}$, so f is continuous. If $x \in X$, we can define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by saying that $\alpha(n)$ is to be the least i such that $\rho(x, x_i) \le 2^{-n-1}$; then $\rho(x_{\alpha(n)}, x_{\alpha(n+1)}) \le 2^{-n-1} + 2^{-n-2} \le 2^{-n}$ for every n, so $\alpha \in F$, and of course $f(\alpha) = x$. So f is surjective.

(c) There is a family $\langle(F_T, f_T, F'_T, f'_T)\rangle_{T\in\mathcal{T}}$ such that

$$F_T,\ F'_T \text{ are closed subsets of } \mathbb{N}^{\mathbb{N}},$$

$$f_T : F_T \to \phi(T),\ f'_T : F'_T \to X \setminus \phi(T) \text{ are continuous surjections}$$

for each $T \in \mathcal{T}$. **P** Start by fixing a homeomorphism $\alpha \mapsto \langle h_i(\alpha)\rangle_{i\in\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. Define the quadruples (F_T, f_T, F'_T, f'_T) inductively on the rank of T.

If $r(T) \le 1$ then $\phi(T)$ is open. Set

$$F_T = \{\alpha : \alpha \in F,\ x_{\alpha(0)} \in \phi(T),\ \rho(x_{\alpha(n)}, x_{\alpha(0)}) \le \tfrac{1}{2}\rho(x_{\alpha(0)}, X \setminus \phi(T)) \text{ for every } n \ge 1\}$$

(interpreting $\rho(x, \emptyset)$ as ∞ if necessary), and $f_T = f{\restriction}F_T$. Then F_T is a closed subset of $\mathbb{N}^{\mathbb{N}}$ and $\rho(f(\alpha), x_{\alpha(0)}) \le \frac{1}{2}\rho(x_{\alpha(0)}, X \setminus \phi(T))$, so $f(\alpha) \in \phi(T)$, for every $\alpha \in F_T$. If $x \in \phi(T)$ then we can define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by taking

$$\alpha(n) = \min\{i : \rho(x_i, x) \le \min(2^{-n-1}, \tfrac{1}{5}\rho(x, X \setminus \phi(T)))\}$$

for every n, and now we find that $\alpha \in F_T$ and $f_T(\alpha) = x$. As for F'_T and f'_T, just set $F'_T = f^{-1}[X \setminus \phi(T)]$ and $f'_T = f{\restriction}F'_T$.

For the inductive step to $r(T) > 1$, set $A_T = \{i : {<}i{>} \in T\}$, as in 562A. We have $\phi(T) = \bigcup_{i\in A_T} X \setminus \phi(T_{<i>})$ and $X \setminus \phi(T) = \bigcap_{i\in A_T} \phi(T_{<i>})$, while $r(T_{<i>}) < r(T)$ for every $i \in A_T$. Set

$$F_T = \textstyle\bigcup_{i\in A_T}\{{<}i{>}^\frown\alpha : \alpha \in F'_{T_{<i>}}\},$$

$$\begin{aligned} F'_T = \{\alpha : h_i(\alpha) \in F_{T_{<i>}} &\text{ for every } i \in A_T, \\ &f_{T_{<i>}}(h_i(\alpha)) = f_{T_{<j>}}(h_j(\alpha)) \text{ for all } i,\ j \in A_T\}, \end{aligned}$$

$$f_T({<}i{>}^\frown\alpha) = f'_{T_{<i>}}(\alpha) \text{ whenever } i \in A_T \text{ and } \alpha \in F'_{T_{<i>}},$$

$$f'_T(\alpha) = f_{T_{<i>}}(h_i(\alpha)) \text{ whenever } i \in A_T \text{ and } \alpha \in F'_T.$$

It is a straightforward calculation to check that F_T, f_T, F'_T and f'_T have the required properties so that the induction continues. **Q**

(d) In particular, $\phi(T) = f_T[F_T]$ is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$ for every $T \in \mathcal{T}$.

(e) The definition of 'analytic set' in 423A refers to continuous images of $\mathbb{N}^{\mathbb{N}}$, so there is a final step to make. If $E \subseteq X$ is a non-empty codable Borel set, it is a continuous image of a closed subset F_T of $\mathbb{N}^{\mathbb{N}}$; but 561C tells us that F_T is a continuous image of $\mathbb{N}^{\mathbb{N}}$, so E also is, and E is analytic.

562F Resolvable sets The essence of the concept of 'codable Borel set' is that it is not enough to know, in the abstract, that a set is 'Borel'; we need to know its pedigree. For a significant number of elementary sets, however, starting with open sets and closed sets, we can determine codes from the sets themselves.

Definition I will say that a subset E of a topological space X is **resolvable** if there is no non-empty set $F \subseteq X$ such that $F = \overline{F \cap E} = \overline{F \setminus E}$ (see KURATOWSKI 66, §12). It is easy to see that the family of such sets is an algebra of subsets of X containing every open set.

562G Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{E}$ be the algebra of resolvable subsets of X. Then there is a function $\psi : \mathcal{E} \to \mathcal{T}$ such that $\phi(\psi(E)) = E$ for every $E \in \mathcal{E}$.

proof We need to start by settling on functions

$$\Theta_1 : \mathcal{T} \times \mathbb{N} \to \mathcal{T}, \quad \Theta_2 : \mathcal{T} \times \mathcal{T} \times \mathbb{N} \to \mathcal{T}, \quad \Theta_3 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}, \quad \Theta_4 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}$$

such that

$$\phi(\Theta_1(T, n)) = \phi(T) \setminus U_n, \quad \phi(\Theta_2(T, T', n)) = \phi(T) \cup (\phi(T') \cap U_n),$$

$$\phi(\Theta_3(\tau)) = \bigcap_{i\in\mathbb{N}} \phi(\tau(i)), \quad \phi(\Theta_4(\tau)) = \bigcup_{i\in\mathbb{N}} \phi(\tau(i))$$

for $T \in \mathcal{T}$, $n \in \mathbb{N}$ and $\tau \in \mathcal{T}^{\mathbb{N}}$.

Now, given $E \in \mathcal{E}$, define $\langle(F_\xi, \tilde{T}_\xi, T_\xi, n_\xi)\rangle_{\xi<\omega_1}$ inductively, as follows. The inductive hypothesis will be that $F_\xi \subseteq X$ is closed, $F_\xi \subseteq F_\eta$ for every $\eta \le \xi$, $\tilde{T}_\xi$, $T_\xi \in \mathcal{T}$, $\phi(\tilde{T}_\xi) = F_\xi$ and $\phi(T_\xi) = E \setminus F_\xi$. Start with $F_0 = X$, $T_0 = \emptyset$, $\tilde{T}_0 = \{<n> : n \in \mathbb{N}\}$. For the inductive step to $\xi + 1$,

— if $F_\xi = \emptyset$, set $n_\xi = 0$ and $(F_{\xi+1}, \tilde{T}_{\xi+1}, T_{\xi+1}) = (F_\xi, \tilde{T}_\xi, T_\xi)$;

— if there is an n such that $\emptyset \ne F_\xi \cap U_n \subseteq E$, let n_ξ be the least such, and set

$$F_{\xi+1} = F_\xi \setminus U_{n_\xi}, \quad \tilde{T}_{\xi+1} = \Theta_1(\tilde{T}_\xi, n_\xi), \quad T_{\xi+1} = \Theta_2(T_\xi, \tilde{T}_\xi, n_\xi);$$

— otherwise, n_ξ is to be the least n such that $\emptyset \ne F_\xi \cap U_n \subseteq X \setminus E$, and

$$F_{\xi+1} = F_\xi \setminus U_{n_\xi}, \quad \tilde{T}_{\xi+1} = \Theta_1(\tilde{T}_\xi, n_\xi) \quad T_{\xi+1} = T_\xi.$$

(Because E is resolvable, these three cases exhaust the possibilities.) It is easy to check that the inductive hypothesis remains valid at level $\xi + 1$.

For the inductive step to a non-zero limit ordinal ξ, then if there is an $\eta < \xi$ such that $F_\eta = \emptyset$, take the first such η and set $n_\xi = 0$ and $(F_\xi, T_\xi, \tilde{T}_\xi) = (F_\eta, T_\eta, \tilde{T}_\eta)$. Otherwise, we must have $F_\zeta \subseteq F_\eta \setminus U_{n_\eta} \subset F_\eta$ whenever $\eta < \zeta < \xi$, so that $\eta \mapsto n_\eta : \xi \to \mathbb{N}$ is injective. Set

$$\begin{aligned} \tilde{\tau}(i) &= \tilde{T}_\eta \text{ if } \eta < \xi \text{ and } i = n_\eta, \\ &= \tilde{T}_0 \text{ if there is no such } \eta, \\ \tau(i) &= T_\eta \text{ if } \eta < \xi \text{ and } i = n_\eta, \\ &= \emptyset \text{ if there is no such } \eta; \end{aligned}$$

now set

$$F_\xi = \bigcap_{\eta<\xi} F_\eta, \quad \tilde{T}_\xi = \Theta_3(\tilde{\tau}), \quad T_\xi = \Theta_4(\tau).$$

Again, it is easy to check that the induction proceeds.

Now, with the family $\langle(F_\xi, \tilde{T}_\xi, T_\xi, n_\xi)\rangle_{\xi<\omega_1}$ complete, observe that $\langle n_\xi\rangle_{\xi<\omega_1}$ cannot be injective. There is therefore a first $\xi = \xi_E$ for which F_{ξ_E} is empty. Set $\psi(E) = T_{\xi_E}$; then $\phi(\psi(E)) = E \setminus F_{\xi_E} = E$, as required.

562H Codable families of sets Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. Let $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ be sequences running over bases for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$, $\phi' : \mathcal{T} \to \mathcal{B}_c(X)$ the corresponding interpretations of Borel codes. Let us say that a family $\langle E_i\rangle_{i\in I}$ is **ϕ-codable** if there is a family $\langle T_i\rangle_{i\in I}$ in $\mathcal{T}$ such that $\phi(T_i) = E_i$ for every $i \in I$. Then 562Af tells us that $\langle E_i\rangle_{i\in I}$ is ϕ-codable iff it is ϕ'-codable.

We may therefore say that a family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}_c(X)$ is **codable** if it is ϕ-codable for some, therefore any, interpretation of Borel codes defined by the procedure of 562A from a sequence running over a base for the topology of X.

Note that any finite family in $\mathcal{B}_c(X)$ is codable, and that any family of resolvable sets is codable, because we can use 562G to provide codes. If $\langle E_i\rangle_{i\in I}$ and $\langle F_i\rangle_{i\in I}$ are codable families in $\mathcal{B}_c(X)$, then so are $\langle E_i \cup F_i\rangle_{i\in I}$ and $\langle E_i \cap F_i\rangle_{i\in I}$, since we have formulae to transform codes for E, F into codes for $E \cup F$ and $E \cap F$.

562I Proposition Let X be a second-countable space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a codable sequence in $\mathcal{B}_c(X)$. Then
(a) $\bigcup_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{B}_c(X)$;
(b) $\langle\bigcup_{i<n} E_i\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$;
(c) $\langle E_n\setminus\bigcup_{i<n} E_i\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$.

proof Let $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ be an interpretation of Borel codes defined from a sequence running over a base for the topology of X; then we have a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}\setminus\{\emptyset\}$ such that $\phi(T_n)=E_n$ for every n.

(a) Setting

$$T=\{<n>:n\in\mathbb{N}\}\cup\{<n>^\frown<0>:n\in\mathbb{N}\}\cup\{<n>^\frown<0>^\frown\sigma:n\in\mathbb{N},\,\sigma\in T_n\},$$

$$T'=\{<0>\}\cup\{<0>^\frown<n>:n\in\mathbb{N}\}\cup\{<0>^\frown<n>^\frown\sigma:n\in\mathbb{N},\,\sigma\in T_n\},$$

we have $\phi(T)=\bigcup_{n\in\mathbb{N}}E_n$ and $\phi(T')=\bigcap_{n\in\mathbb{N}}E_n$.

(b) Setting

$$T'_n=\{<i>:i<n\}\cup\{<i>^\frown<0>:i<n\}\cup\{<i>^\frown<0>^\frown\sigma:i<n,\,\sigma\in T_i\},$$

$\phi(T'_n)=\bigcup_{i<n}E_i$ for every n.

(c) Setting

$$\begin{aligned}T''=\{<n>:n\in\mathbb{N}\}&\cup\{<n>^\frown<0>:n\in\mathbb{N}\}\cup\{<n>^\frown<0>^\frown\sigma:\sigma\in T_n\}\\&\cup\{<n>^\frown<1>:n\in\mathbb{N}\}\cup\{<n>^\frown<1>^\frown<0>:n\in\mathbb{N}\}\\&\cup\{<n>^\frown<1>^\frown<0>^\frown\sigma:\sigma\in T'_n\},\end{aligned}$$

$\phi(T'')=\bigcup_{n\in\mathbb{N}}(E_n\setminus\bigcup_{i<n}E_i)$.

562J Codable Borel functions Let X and Y be second-countable spaces. A function $f:X\to Y$ is a **codable Borel function** if $\langle f^{-1}[H]\rangle_{H\subseteq Y\text{is open}}$ is a codable family in $\mathcal{B}_c(X)$.

562K Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the corresponding interpretation of Borel codes.

(a) If Y is another second-countable space, $\langle V_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of Y containing $\emptyset$, $\phi_Y:\mathcal{T}\to\mathcal{B}_c(Y)$ the corresponding interpretation of Borel codes, and $f:X\to Y$ is a function, then the following are equiveridical:

(i) f is a codable Borel function;
(ii) $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}_c(X)$;
(iii) there is a function $\Theta:\mathcal{T}\to\mathcal{T}$ such that $\phi(\Theta(T))=f^{-1}[\phi_Y(T)]$ for every $T\in\mathcal{T}$.

(b) If Y and Z are second-countable spaces and $f:X\to Y$, $g:Y\to Z$ are codable Borel functions then $gf:X\to Z$ is a codable Borel function.

(c) If Y and Z are second-countable spaces and $f:X\to Y$, $g:X\to Z$ are codable Borel functions then $x\mapsto(f(x),g(x))$ is a codable Borel function from X to $Y\times Z$.

(d) If Y is a second-countable space then any continuous function from X to Y is a codable Borel function.

proof (a)(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii) This is really a full-strength version of 562Af. Because $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$ is codable, we have a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\phi(T_n)=f^{-1}[V_n]$ for every n. Define $\Theta:\mathcal{T}\to\mathcal{T}$ inductively, as follows. The inductive hypothesis will include the requirement that $r(\Theta(T))\ge r(T)$ for every $T\in\mathcal{T}$. Given T, set $A_T=\{n:<n>\in T\}$. If $r(T)=0$ set $\Theta(T)=T=\emptyset$. If $r(T)=1$ set

$$\begin{aligned}\Theta(T)=\{<n>:n\in A_T\}&\cup\{<n>^\frown<0>:n\in A_T\}\\&\cup\{<n>^\frown<0>^\frown\sigma:n\in A_T,\,\sigma\in T_n\},\end{aligned}$$

so that $r(\Theta(T))\ge 2$ and

$$\phi(\Theta(T))=\bigcup_{n\in A_T}X\setminus(X\setminus\phi(T_n))=\bigcup_{n\in A_T}f^{-1}[V_n]=f^{-1}[\phi_Y(T)].$$

If $r(T)>1$ set

$$\Theta(T)=\{<n>:n\in A_T\}\cup\{<n>^\frown\sigma:n\in A_T,\,\sigma\in\Theta(T_{<n>})\}$$

so that $r(\Theta(T))\ge r(T)$ and

$$\phi(\Theta(T)) = \bigcup_{n\in A_T} X \setminus \phi(\Theta(T_{<n>})) = \bigcup_{n\in A_T} X \setminus f^{-1}[\phi_Y(T_{<n>})]$$
$$= f^{-1}[\bigcup_{n\in A_T} Y \setminus \phi_Y(T_{<n>})] = f^{-1}[\phi_Y(T)],$$

and the induction continues.

(iii)⇒(i) For open $H \subseteq Y$ set $\psi_Y(H) = \{<n> : V_n \subseteq H\}$. Now $\phi(\Theta(\psi_Y(H)) = f^{-1}[\phi_Y(\psi_Y(H))] = f^{-1}[H]$ for every H, so $\langle\phi(\Theta(\psi_Y(H)))\rangle_{H\subseteq Y \text{ is open}}$ is a family of codes for $\langle f^{-1}[H]\rangle_{H\subseteq Y \text{is open}}$.

(b) Take $\langle V_n\rangle_{n\in\mathbb{N}}$, ϕ_Y and $\Theta : \mathcal{T} \to \mathcal{T}$ as in (a). Write $\mathfrak{U}$ for the topology of Z; then we have a function $\theta : \mathfrak{U} \to \mathcal{T}$ such that $\phi_Y(\theta(H)) = g^{-1}[H]$ for every $H \in \mathfrak{U}$. Now $\langle\Theta(\theta(H))\rangle_{H\in\mathfrak{U}}$ is a coding for $\langle(gf)^{-1}[H]\rangle_{H\in\mathfrak{U}}$, so gf is codable.

(c) Let $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ be sequences running over bases for the topologies of Y and Z, and $\langle(i_n, j_n)\rangle_{n\in\mathbb{N}}$ an enumeration of $\mathbb{N}\times\mathbb{N}$. Set $H_n = V_{i_n} \times W_{j_n}$; then $\langle H_n\rangle_{n\in\mathbb{N}}$ is a base for the topology of $Y \times Z$. Let $\Theta : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ be such that $\phi(\Theta(T, T')) = \phi(T) \cap \phi(T')$ for all T, $T' \in \mathcal{T}$ (562Ba). Let $\langle T_n\rangle_{n\in\mathbb{N}}$, $\langle T'_n\rangle_{n\in\mathbb{N}}$ be codings for $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$, $\langle g^{-1}[W_n]\rangle_{n\in\mathbb{N}}$. Then $\langle\Theta(T_{i_n}, T'_{j_n})\rangle_{n\in\mathbb{N}}$ is a coding for $\langle h^{-1}[H_n]\rangle_{n\in\mathbb{N}}$, where $h(x) = (f(x), g(x))$ for $x \in X$. So h is a codable Borel function.

(d) If $f : X \to Y$ is continuous, then $\langle f^{-1}[H]\rangle_{H\subseteq Y \text{ is open}}$ is a family of resolvable sets, therefore codable, as noted in 562H.

Remark Note in part (a)(ii)⇒(iii) of the proof the function Θ is constructed by a definite process from $\langle T_n\rangle_{n\in\mathbb{N}}$; so we shall be able to uniformize the process to define families $\langle\Theta_i\rangle_{i\in I}$ from families $\langle f_i\rangle_{i\in I}$, at least if we can reach a family $\langle T_{in}\rangle_{i\in I,n\in\mathbb{N}}$ such that T_{ni} codes $f_i^{-1}[V_n]$ for all $i \in I$ and $n \in \mathbb{N}$.

562L Proposition Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the associated interpretation of Borel codes.

(a) If $f : X \to \mathbb{R}$ is function, the following are equiveridical:

(i) f is a codable Borel function;

(ii) the family $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ is codable;

(iii) $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable.

(b) Write $\tilde{\mathcal{T}}$ for the set of functions $\tau : \mathbb{R} \to \mathcal{T}$ such that

$$\phi(\tau(\alpha)) = \bigcup_{\beta>\alpha} \phi(\tau(\beta)) \text{ for every } \alpha \in \mathbb{R},$$

$$\bigcap_{n\in\mathbb{N}} \phi(\tau(n)) = \emptyset, \quad \bigcup_{n\in\mathbb{N}} \phi(\tau(-n)) = X.$$

Then

(i) for every $\tau \in \tilde{\mathcal{T}}$ there is a unique codable Borel function $\tilde{\phi}(\tau) : X \to \mathbb{R}$ such that $\phi(\tau(\alpha)) = \{x : \tilde{\phi}(\tau)(x) > \alpha\}$ for every $\alpha \in \mathbb{R}$;

(ii) every codable Borel function from X to $\mathbb{R}$ is expressible as $\tilde{\phi}(\tau)$ for some $\tau \in \tilde{\mathcal{T}}$.

(c) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $f(x) = \sup_{n\in\mathbb{N}} \tilde{\phi}(\tau_n)(x)$ is finite for every $x \in X$, then f is a codable Borel function.

(d) If f, $g : X \to \mathbb{R}$ are codable Borel functions and $\gamma \in \mathbb{R}$, then $f + g$, γf, $|f|$ and $f \times g$ are codable Borel functions.

(e) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$, then there is a codable Borel function f such that $\liminf_{n\to\infty} \tilde{\phi}(\tau_n)(x) = f(x)$ whenever the lim inf is finite.

(f) A subset E of X belongs to $\mathcal{B}_c(X)$ iff $\chi E : X \to \mathbb{R}$ is a codable Borel function.

proof (a)(i)⇒(ii) If $f : X \to \mathbb{R}$ is codable then of course $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$, being a subfamily of $\langle f^{-1}[H]\rangle_{H\subseteq\mathbb{R} \text{ is open}}$, is codable.

(ii)⇒(iii) Similarly, if $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ is codable, its subfamily $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable.

(iii)⇒(i) If $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable, we have a family $\langle T_q\rangle_{q\in\mathbb{Q}}$ in $\mathcal{T}$ coding it (with respect to $\langle U_n\rangle_{n\in\mathbb{N}}$ and ϕ). Let $\langle(q_n, \tilde{q}_n)\rangle_{n\in\mathbb{N}}$ be an enumeration of $\{(q, \tilde{q}) : q, \tilde{q} \in \mathbb{Q}, q < \tilde{q}\}$. As in 562B, we have functions

$$\Theta_1 : \mathcal{T}\times\mathcal{T} \to \mathcal{T}, \quad \Theta_2 : \bigcup_{I\subseteq\mathbb{Q}} \mathcal{T}^I \to \mathcal{T}$$

such that

$$\phi(\Theta_1(T, T')) = \phi(T) \setminus \phi(T'), \quad \phi(\Theta_2(\tau)) = \bigcup_{q\in I} \phi(\tau(q)))$$

for T, $T' \in \mathcal{T}$, $I \subseteq \mathbb{Q}$ and $\tau \in \mathcal{T}^I$. Now for $n \in \mathbb{N}$ consider

$$T'_n = \Theta_2(\langle\Theta_1(T_{q_n}, T_{q'})\rangle_{q'\in\mathbb{Q},q'<\tilde{q}_n}),$$

so that $\phi(T'_n) = f^{-1}[]q_n, \tilde{q}_n[]$ for every n, and $\langle f^{-1}[]q_n, \tilde{q}_n[]\rangle_{n\in\mathbb{N}}$ is codable; by 562Ka, f is a codable Borel function.

(b) This is elementary; given $\tau \in \tilde{\mathcal{T}}$ we can, and must, set $\tilde{\phi}(\tau)(x) = \sup\{\alpha : x \in \phi(\tau(\alpha))\}$ for every $x \in X$; and given f we have a coding τ for $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ which must belong to $\tilde{\mathcal{T}}$ and have $\tilde{\phi}(\tau) = f$.

(c) Given $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as described,

$$\alpha \mapsto \Theta_2(\langle\tau_n(\alpha)\rangle_{n\in\mathbb{N}})$$

will be a Borel code for f.

(d) Use 562K(b)-(d).

(e) A couple more functions will shorten the formulae. Let

$$\Theta_3 : \mathcal{T} \to \mathcal{T}, \quad \Theta_4 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

be such that

$$\phi(\Theta_3(T)) = X \setminus \phi(T), \quad \phi(\Theta_4(T, T')) = \phi(T) \cup \phi(T')$$

for every T, $T' \in \mathcal{T}$. Now, given $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as described, set

$$\tau(\alpha) = \Theta_2(\langle\Theta_2(\langle\Theta_3(\Theta_2(\langle\Theta_3(\tau_m(q))\rangle_{m\ge n}))\rangle_{q\in\mathbb{Q},q>\alpha})\rangle_{n\in\mathbb{N}})$$

for $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\phi(\tau(\alpha)) &= \bigcup_{q>\alpha,n\in\mathbb{N}} X \setminus (\bigcup_{m\ge n}(X \setminus \{x : f_m(x) > q\}))\\ &= \bigcup_{q>\alpha,n\in\mathbb{N}} \bigcap_{m\ge n} \{x : f_m(x) > q\} = \{x : \liminf_{n\to\infty} f_n(x) > \alpha\}\end{aligned}$$

for each α. We don't yet have a code for a real-value function defined everywhere in X. But if we set

$$T = \Theta_2(\langle\Theta_1(\tau(-n), \tau(n))\rangle_{n\in\mathbb{N}}),$$

then

$$\phi(T) = \bigcup_{n\in\mathbb{N}} \phi(\tau(-n)) \setminus \phi(\tau(n)) = \{x : \liminf_{n\to\infty} f_n(x) \text{ is finite}\}.$$

So take

$$\begin{aligned}\tau'(\alpha) &= \Theta_1(\tau(\alpha), \Theta_3(T)) \text{ if } \alpha \ge 0,\\ &= \Theta_4(\tau(\alpha), \Theta_3(T)) \text{ if } \alpha < 0;\end{aligned}$$

this will get $\tau' \in \tilde{\mathcal{T}}$ such that

$$\begin{aligned}\tilde{\phi}(\tau')(x) &= \liminf_{n\to\infty} f_n(x) \text{ if this is finite,}\\ &= 0 \text{ otherwise.}\end{aligned}$$

(f) Elementary.

562M Remarks (a) For some purposes there are advantages in coding real-valued functions by functions from $\mathbb{Q}$ to $\mathcal{T}$ rather than by functions from $\mathbb{R}$ to $\mathcal{T}$; see 364Bd.

(b) As in 562B, it will be useful to observe that the constructions here are largely determinate. For instance, the function Θ of 562K(a-iii) is built by a definite rule from the sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ and the function f there. What this means is that if we have a family $\langle(Y_i, \langle V_{in}\rangle_{n\in\mathbb{N}}, f_i)\rangle_{i\in I}$ such that Y_i is a second-countable space, $\langle V_{in}\rangle_{n\in\mathbb{N}}$ is a sequence running over a base for the topology of Y_i, and $f_i : X \to Y_i$ is a continuous function for each $i \in I$, then there will be a function $\tilde{\Theta} : \mathcal{T} \times I \to \mathcal{T}$ such that $\phi(\tilde{\Theta}(T, i)) = f_i^{-1}[\phi_i(T)]$ for every $i \in I$ and $T \in \mathcal{T}$, where $\phi_i : \mathcal{T} \to \mathcal{B}_c(Y_i)$ is the interpretation of Borel codes corresponding to the sequence $\langle V_{in}\rangle_{n\in\mathbb{N}}$. (Start from

$$T_{in} = \{<j> : U_j \subseteq f_i^{-1}[V_{in}]\}$$

for $i \in I$ and $n \in \mathbb{N}$, and build $\tilde{\Theta}(T, i)$ as 562K.)

(c) Similarly, when we look at 562L(d)-(e), we have something better than just existence proofs for codes for $f + g$ and $\liminf_{n\to\infty} f_n$. For instance, we have a function $\tilde{\Theta}_1 : \tilde{\mathcal{T}} \times \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$ such that $\tilde{\phi}(\tilde{\Theta}_1(\tau, \tau'))$ will always be $\tilde{\phi}(\tau) - \tilde{\phi}(\tau')$ for τ, $\tau' \in \tilde{\mathcal{T}}$. **P** We need to have

$$\phi(\tilde\Theta(\tau,\tau')(\alpha)) = \bigcup_{q\in\mathbb{Q}}\phi(\tau(q))\setminus\phi(\tau'(q-\alpha))$$

for every α, and this is easy to build from a set-difference operator, as in 562Ba, and a general countable-union operator as built in 562Bc. **Q** Equally, we have a function $\tilde\Theta_2:\tilde{\mathcal{T}}^{\mathbb{N}}\to\tilde{\mathcal{T}}^{\mathbb{N}}$ such that

$$\tilde\phi(\tilde\Theta_2(\langle\tau_n\rangle_{n\in\mathbb{N}})(m)) = \inf_{n\ge m}\tilde\phi(\tau_n)$$

for every m whenever $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $\inf_{n\in\mathbb{N}}\tilde\phi(\tau_n)$ is defined as a real-valued function on X. **P** This time we need

$$\phi(\tilde\Theta_2(\langle\tau_n\rangle_{n\in\mathbb{N}})(m)(\alpha)) = \bigcup_{q\in\mathbb{Q},q>\alpha}\bigl(X\setminus\bigcup_{n\ge m}(X\setminus\phi(\tau_n(q)))\bigr)$$

for all m and α, and once again a complementation operator and a general countable-union operator will do the trick. **Q**

562N Codable Borel equivalence (a) If X is a set, we can say that two second-countable topologies $\mathfrak{S}$, $\mathfrak{T}$ on X are **codably Borel equivalent** if the identity functions $(X,\mathfrak{S})\to(X,\mathfrak{T})$ and $(X,\mathfrak{T})\to(X,\mathfrak{S})$ are codable Borel functions. In this case, $\mathfrak{S}$ and $\mathfrak{T}$ give the same algebra $\mathcal{B}_c(X)$ and the same families of codable Borel functions (562Kb).

(b) If $(X,\mathfrak{T})$ is a second-countable space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is any codable sequence in $\mathcal{B}_c(X)$, there is a topology $\mathfrak{S}$ on X, generated by a countable algebra of subsets of X, such that $\mathfrak{S}$ and $\mathfrak{T}$ are codably Borel equivalent and every E_n belongs to $\mathfrak{S}$. **P** Since there is certainly a codable sequence running over a base for the topology of X, we can suppose that such a sequence has been amalgamated with $\langle E_n\rangle_{n\in\mathbb{N}}$, so that $\{E_n:n\in\mathbb{N}\}$ includes a base for $\mathfrak{T}$. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\{E_n:n\in\mathbb{N}\}$ and $\mathfrak{S}$ be the topology generated by $\mathcal{E}$. As $\mathcal{E}$ is an algebra, $\mathfrak{S}$ is zero-dimensional; as $\mathcal{E}$ is countable, $\mathfrak{S}$ is second-countable.

The identity map $(X,\mathfrak{S})\to(X,\mathfrak{T})$ is continuous, therefore a codable Borel function (562Kd). In the reverse direction, we have a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of codes for $\langle E_n\rangle_{n\in\mathbb{N}}$. From these we can build, using our standard operations, codes T_I, for $I\in[\mathbb{N}]^{<\omega}$, T'_{IJ}, for I, $J\in[\mathbb{N}]^{<\omega}$, and $T''_{\mathcal{K}}$, for $\mathcal{K}\in[[\mathbb{N}]^{<\omega}\times[\mathbb{N}]^{<\omega}]^{<\omega}$, such that

T_I codes $\bigcup_{i\in I}E_i$,

T'_{IJ} codes $\bigcup_{i\in I}E_i\setminus\bigcup_{i\in J}E_i$,

$T''_{\mathcal{K}}$ codes $\bigcup_{(I,J)\in\mathcal{K}}(\bigcup_{i\in I}E_i\setminus\bigcup_{i\in J}E_i)$.

But of course $[[\mathbb{N}]^{<\omega}\times[\mathbb{N}]^{<\omega}]^{<\omega}$ is countable and the $T''_{\mathcal{K}}$ can be enumerated as a sequence $\langle T^*_n\rangle_{n\in\mathbb{N}}$ coding a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ running over $\mathcal{E}$. By 562Ka, the identity map $(X,\mathfrak{T})\to(X,\mathfrak{S})$ is a codable Borel function. **Q**

Note that $\mathfrak{S}$ here is necessarily regular; this will be useful at more than one point in the next couple of sections.

562O Resolvable functions Let X be a topological space. I will say that a function $f:X\to[-\infty,\infty]$ is **resolvable** if whenever $\alpha<\beta$ in $\mathbb{R}$ and $F\subseteq X$ is a non-empty set, then at least one of $\{x:x\in F,\ f(x)\le\alpha\}$, $\{x:x\in F,\ f(x)\ge\beta\}$ is not dense in F.

Examples (a) Any semi-continuous function from X to $[-\infty,\infty]$ is resolvable. **P** If $f:X\to[-\infty,\infty]$ is lower semi-continuous, $F\subseteq X$ is non-empty, and $\alpha<\beta$ in $\mathbb{R}$, then $U=\{x:f(x)>\alpha\}$ is open; if $F\cap U\ne\emptyset$ then $\{x:x\in F,\ f(x)\le\alpha\}$ is not dense in F; otherwise $\{x:x\in F,\ f(x)\ge\beta\}$ is not dense in F. **Q**

(b) If $f:X\to\mathbb{R}$ is such that $\{x:f(x)>\alpha\}$ is resolvable for every α, then f is resolvable. **P** Suppose that $F\subseteq X$ is non-empty and $\alpha<\beta$ in $\mathbb{R}$. Set $E=\{x:f(x)>\alpha\}$. If $F\cap E$ is not dense in F, then $\{x:x\in F,\ f(x)\ge\beta\}$ is not dense in F. Otherwise $\{x:x\in F,\ f(x)\le\alpha\}=F\setminus E$ is not dense in F. **Q**

In particular, the characteristic function of a resolvable set is resolvable.

(c) A function $f:\mathbb{R}\to\mathbb{R}$ which has bounded variation on every bounded set is resolvable. **P** If $F\subseteq\mathbb{R}$ is non-empty and $\alpha<\beta$ in $\mathbb{R}$, take $y\in F$. If y is isolated in F, then we have an open set U such that $U\cap F=\{y\}$, so that one of $\{x:x\in F,\ f(x)\le\alpha\}$, $\{x:x\in F,\ f(x)\ge\beta\}$ does not contain y and is not dense in F. Otherwise, y is in the closure of one of $F\cap\,]y,\infty[$, $F\cap\,]-\infty,y[$; suppose the former. For each $n\in\mathbb{N}$ set $I_n=[y+2^{-n-1},y+2^{-n}]$, $\delta_n=\mathrm{Var}_{I_n}(f)$. We have

$$\infty>\mathrm{Var}_{]y,y+1]}(f)=\sum_{n=0}^{\infty}\delta_n,$$

so there is an $n\in\mathbb{N}$ such that $\delta_m\le\frac{1}{4}(\beta-\alpha)$ for $m\ge n$. Take $m>n$ such that $I_m\cap F\ne\emptyset$, and consider $U=\mathrm{int}(I_{m-1}\cup I_m\cup I_{m+1})$. Then $\mathrm{Var}_U(f)\le\frac{3}{4}(\beta-\alpha)$ so U cannot meet both $\{x:x\in F,\ f(x)\ge\beta\}$ and $\{x:x\in F,\ f(x)\ge\alpha\}$, and one of these is not dense in F. **Q**

562P Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{R}$ be the family of resolvable real-valued functions on X. Then there is a function $\tilde\psi:\mathcal{R}\to\mathcal{T}^{\mathbb{R}}$ such that

$$\phi(\tilde{\psi}(f)(\alpha)) = \{x : f(x) > \alpha\}$$

for every $f \in \mathcal{R}$ and $\alpha \in \mathbb{R}$.

proof (a) Start by fixing a bijection

$$k \mapsto (n_k, q_k, q'_k) : \mathbb{N} \to \mathbb{N} \times \{(q, q') : q,\, q' \in \mathbb{Q},\, q < q'\}.$$

Next, fix a function $\Theta_1 : \mathcal{T}^3 \times \mathbb{N} \to \mathcal{T}$ such that

$$\phi(\Theta_1(T, T', T'', n)) = \phi(T) \cup (U_n \setminus (\phi(T') \cup \phi(T'')))$$

for T, T', $T'' \in \mathcal{T}$ and $n \in \mathbb{N}$, and a function $\tilde{\Theta}_2 : \bigcup_{J \subseteq \mathbb{N}} \mathcal{T}^J \to \mathcal{T}$ such that $\phi(\tilde{\Theta}_2(\tau)) = \bigcup_{i \in J} \phi(\tau(i))$ whenever $J \subseteq \mathbb{N}$ and $\tau \in \mathcal{T}^J$. (See 562Ba.)

(b) Given $f \in \mathcal{R}$, define $\zeta < \omega_1$ and a family $\langle(\tau_\xi, \tau'_\xi, k_\xi)\rangle_{\xi \leq \zeta}$ in $\mathcal{T}^{\mathbb{R}} \times \mathcal{T}^{\mathbb{R}} \times \mathbb{N}$ inductively, as follows. The inductive hypothesis will be that $k_\eta \neq k_\xi$ whenever $\eta < \xi < \zeta$. Start with $\tau_0(\alpha) = \tau'_0(\alpha) = \emptyset$ for every $\alpha \in \mathbb{R}$.

Inductive step to a successor ordinal $\xi + 1$ Given τ_ξ and τ'_ξ in $\mathcal{T}^{\mathbb{R}}$, then for $q < q'$ in $\mathbb{Q}$ set $F_\xi(q, q') = X \setminus (\phi(\tau_\xi(q)) \cup \phi(\tau'_\xi(q')))$. Now

—— if there is a $k \in \mathbb{N} \setminus \{k_\eta : \eta < \xi\}$ such that $U_{n_k} \cap F_\xi(q_k, q'_k) \neq \emptyset$ and $f(x) \geq q_k$ for every $x \in U_{n_k} \cap F_\xi(q_k, q'_k)$, take the first such k, and set

$$\begin{aligned}
\tau_{\xi+1}(\alpha) &= \tau_\xi(\alpha) \text{ for every } \alpha \in \mathbb{R},\\
\tau'_{\xi+1}(\alpha) &= \Theta_1(\tau'_\xi(\alpha), \tau_\xi(q_k), \tau'_\xi(q'_k), n_k) \text{ if } \alpha \leq q_k,\\
&= \tau'_\xi(\alpha) \text{ if } \alpha > q_k,\\
k_\xi &= k;
\end{aligned}$$

—— if this is not so, but there is a $k \in \mathbb{N} \setminus \{k_\eta : \eta < \xi\}$ such that $U_{n_k} \cap F_\xi(q_k, q'_k) \neq \emptyset$ and $f(x) \leq q'_k$ for every $x \in U_{n_k} \cap F_\xi(q_k, q'_k)$, take the first such k, and set

$$\begin{aligned}
\tau_{\xi+1}(\alpha) &= \Theta_1(\tau_\xi(\alpha), \tau_\xi(q_k), \tau'_\xi(q'_k), n_k) \text{ if } \alpha \geq q'_k,\\
&= \tau_\xi(\alpha) \text{ if } \alpha < q'_k,\\
\tau'_{\xi+1}(\alpha) &= \tau'_\xi(\alpha) \text{ for every } \alpha \in \mathbb{R},\\
k_\xi &= k;
\end{aligned}$$

—— and if that doesn't happen either, set $\zeta = \xi$ and stop.

Inductive step to a countable limit ordinal ξ Given $\langle(\tau_\eta, \tau'_\eta, k_\eta)\rangle_{\eta < \xi}$, set $I = \{k_\eta : \eta < \xi\}$ and define $g : I \to \xi$ by setting $g(i) = \eta$ whenever $\eta < \xi$ and $k_\eta = i$. Now set

$$\tau_\xi(\alpha) = \tilde{\Theta}_2(\langle\tau_{g(i)}(\alpha)\rangle_{i \in I}), \quad \tau'_\xi(\alpha) = \tilde{\Theta}_2(\langle\tau'_{g(i)}(\alpha)\rangle_{i \in I})$$

for every $\alpha \in \mathbb{R}$.

(c) Now an induction on ξ shows that

$$\phi(\tau_\eta(\alpha)) \subseteq \phi(\tau_\xi(\alpha)), \quad \phi(\tau'_\eta(\alpha)) \subseteq \phi(\tau'_\xi(\alpha)),$$

$$\phi(\tau_\xi(\alpha)) \subseteq \{x : f(x) \leq \alpha\}, \quad \phi(\tau'_\xi(\alpha)) \subseteq \{x : f(x) \geq \alpha\}$$

whenever $\eta \leq \xi$, $\alpha \in \mathbb{R}$ and the codes here are defined. Next, if $k_\eta = k$ is defined, we must have $U_{n_k} \cap F_\eta(q_k, q'_k) \neq \emptyset$ and

—— either $f(x) \geq q_k$ for every $x \in U_{n_k} \cap F_\eta(q_k, q'_k)$ and $\phi(\tau'_{\eta+1}(q_k)) = \phi(\tau'_\eta(q_k)) \cup (U_{n_k} \cap F_\eta(q_k, q'_k))$

—— or $f(x) \leq q'_k$ for every $x \in U_{n_k} \cap F_\eta(q_k, q'_k)$ and $\phi(\tau_{\eta+1}(q_k)) = \phi(\tau_\eta(q_k)) \cup (U_{n_k} \cap F_\eta(q_k, q'_k))$.

In either case, $U_{n_k} \cap F_\eta(q_k, q'_k)$ must be disjoint from $F_\xi(q_k, q'_k)$ for every $\xi > \eta$ for which F_ξ is defined; consequently we cannot have $k_\xi = k$ for any $\xi > \eta$. The induction must therefore stop.

$F_\zeta(q, q') = \emptyset$ whenever q, $q' \in \mathbb{Q}$ and $q < q'$. **P?** Otherwise, because f is resolvable, there is an $n \in \mathbb{N}$ such that $V = U_n \cap F_\zeta(q, q')$ is non-empty and either $f(x) \geq q$ for every $x \in V$ or $f(x) \leq q'$ for every $x \in V$. Let $k \in \mathbb{N}$ be such that $n_k = n$, $q_k = q$ and $q'_k = q'$; then U_{n_k} meets $F_\zeta(q_k, q'_k)$ so $k \neq k_\eta$ for any $\eta < \zeta$. But this means that we ought to have proceeded according to one of the first two alternatives in the single-step inductive stage, and ought not to have stopped at ζ. **XQ**

(d) Now set

$$\tau(\alpha) = \tilde{\Theta}_2(\langle\tau'_\zeta(q_n)\rangle_{n \in \mathbb{N}, q_n > \alpha})$$

for $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\phi(\tau(\alpha)) &= \bigcup_{n\in\mathbb{N},q_n>\alpha} \phi(\tau'_\zeta(q_n))\\ &\subseteq \bigcup_{n\in\mathbb{N},q_n>\alpha} \{x : f(x) \geq q_n\} \subseteq \{x : f(x) > \alpha\}\end{aligned}$$

for every α. **?** If α is such that $\phi(\tau(\alpha)) \subset \{x : f(x) > \alpha\}$, let $x \in X$ and $n \in \mathbb{N}$ be such that $f(x) > q'_n > q_n > \alpha$ and $x \notin \phi(\tau(\alpha))$. Then $x \notin \phi(\tau'_\zeta(q_n))$; but also $f(y) \leq q'_n$ for every $y \in \phi(\tau_\zeta(q'_n))$, so $x \notin \phi(\tau_\zeta(q'_n))$ and $x \in F_\zeta(q_n, q'_n)$, which is supposed to be impossible. **X**

So we can set $\tilde\psi(f) = \tau$.

562Q Codable families of codable functions (a) If X and Y are second-countable spaces, a family $\langle f_i\rangle_{i\in I}$ of functions from X to Y is a **codable family of codable Borel functions** if $\langle f_i^{-1}[H]\rangle_{i\in I,H\subseteq Y \text{ is open}}$ is a codable family in $\mathcal{B}_c(X)$.

(b) Uniformizing the arguments of 562L, it is easy to check that a family $\langle f_i\rangle_{i\in I}$ of real-valued functions on X is a codable family of codable Borel functions iff there is a family $\langle \tau_i\rangle_{i\in I}$ in $\tilde{\mathcal{T}}$ such that, in the language there, $f_i = \tilde\phi(\tau_i)$ for every i.

(c) In this language, 562Le can be rephrased as

if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of real-valued codable Borel functions on X, there is a codable Borel function f such that $f(x) = \liminf_{n\to\infty} f_n(x)$ whenever the lim inf is finite,

and 562P implies that

the family of resolvable real-valued functions on X is a codable family of codable Borel functions.

(d) If X, Y and Z are second-countable spaces, $\langle f_i\rangle_{i\in I}$ is a codable family of codable Borel functions from X to Y, and $\langle g_i\rangle_{i\in I}$ is a codable family of codable Borel functions from Y to Z, then $\langle g_i f_i\rangle_{i\in I}$ is a codable family of codable functions from X to Z; this is because the proof of 562Kb gives a recipe for calculating a code for the composition of codable functions, which can be performed simultaneously on the compositions $g_i f_i$ if we are given codes for the functions g_i and f_i.

562R Codable Baire sets The ideas here can be adapted to give a theory of Baire algebras in general topological spaces. Start by settling on a fixed sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$, with the associated interpretation $\phi : \mathcal{T} \to \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ of Borel codes.

(a) For a topological space X, say that a subset E of X is a **codable Baire set** if it is of the form $f^{-1}[F]$ for some continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and $F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$; write $\mathcal{B}\mathfrak{a}_c(X)$ for the family of such sets. If $E \in \mathcal{B}\mathfrak{a}_c(X)$, then a **code** for E will be a pair (f, T) where $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous, $T \in \mathcal{T}$ and $E = f^{-1}[\phi(T)]$. A family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(X)$ is now a **codable** family if there is a family $\langle(f_i, T_i)\rangle_{i\in I}$ such that (f_i, T_i) codes E_i for every i.

(b)(i) Suppose that $\langle f_i\rangle_{i\in I}$ is a countable family of continuous functions from X to $\mathbb{R}^{\mathbb{N}}$, and $\langle T_i\rangle_{i\in I}$ a family in $\mathcal{T}$. Then there are a continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ and a sequence $\langle T'_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that (f, T'_i) codes the same Baire set as (f_i, T_i) for every $i \in I$. **P** If I is empty, this is trivial. Otherwise, $I \times \mathbb{N}$ is countably infinite, so $(\mathbb{R}^{\mathbb{N}})^I \cong \mathbb{R}^{I\times\mathbb{N}}$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$; let $h : \mathbb{R}^{\mathbb{N}} \to (\mathbb{R}^{\mathbb{N}})^I$ be a homeomorphism, and set $f(x) = h^{-1}(\langle f_i(x)\rangle_{i\in I})$ for each $x \in \mathbb{N}$. Then $f_i = \pi_i h f$ for each i, where $\pi_i(z) = z(i)$ for $z \in (\mathbb{R}^{\mathbb{N}})^I$. Now $\langle(\pi_i h)^{-1}[V_n]\rangle_{i\in I,n\in\mathbb{N}}$ is a family of open sets in $\mathbb{R}^{\mathbb{N}}$, so is codable (562G, or otherwise); let $\langle T_{in}\rangle_{i\in I,n\in\mathbb{N}}$ be a family in $\mathcal{T}$ such that $\phi(T_{in}) = (\pi_i h)^{-1}[V_n]$ whenever $n \in \mathbb{N}$ and $i \in I$. The construction of part (a)(ii)⇒(iii) in the proof of 562K gives us a family $\langle\Theta_i\rangle_{i\in I}$ of functions from $\mathcal{T}$ to $\mathcal{T}$ such that $(\pi_i h)^{-1}[\phi(T)] = \phi(\Theta_i(T))$ whenever $i \in I$ and $T \in \mathcal{T}$. So we can take $T'_i = \Theta_i(T_i)$, and we shall have

$$f_i^{-1}[\phi(T_i)] = f^{-1}[(\pi_i h)^{-1}[\phi(T_i)]] = f^{-1}[\phi(\Theta_i(T_i))] = f^{-1}[\phi(T'_i)]$$

for every i, as required. **Q**

(ii) It follows that if $\langle E_i\rangle_{i\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$ then $\bigcup_{i\in\mathbb{N}} E_i$ and $\bigcap_{i\in\mathbb{N}} E_i$ belong to $\mathcal{B}\mathfrak{a}_c(X)$. **P** By (i), we have a continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and a sequence $\langle T'_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that $E_i = f^{-1}[\phi(T'_i)]$ for every $i \in \mathbb{N}$. Now 562I tells us that $F = \bigcup_{i\in\mathbb{N}} \phi(T'_i)$ and $F' = \bigcap_{i\in\mathbb{N}} \phi(T'_i)$ are codable, so $f^{-1}[F] = \bigcup_{i\in\mathbb{N}} E_i$ and $f^{-1}[F'] = \bigcap_{i\in\mathbb{N}} E_i$ belong to $\mathcal{B}\mathfrak{a}_c(X)$. **Q**

(iii) In particular, $\mathcal{B}\mathfrak{a}_c(X)$ is closed under finite intersections; as it is certainly closed under complementation, it is an algebra of subsets of X. Every zero set belongs to $\mathcal{B}\mathfrak{a}_c(X)$. **P** If $g : X \to \mathbb{R}$ is continuous, set $f(x)(i) = g(x)$ for

$x \in X$, $i \in \mathbb{N}$; then $H = \{z : z \in \mathbb{R}^{\mathbb{N}},\ z(0) = 0\}$ is closed, therefore a codable Borel set, and $g^{-1}[\{0\}] = f^{-1}[H]$ is a codable Baire set. **Q**

(iv) If Y is another topological space and $g : X \to Y$ is continuous, then $\langle g^{-1}[F_i]\rangle_{i\in I}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$ for every codable family $\langle F_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(Y)$. **P** If $\langle (f_i, T_i)\rangle_{i\in I}$ codes $\langle F_i\rangle_{i\in I}$, then $\langle (f_i g, T_i)\rangle_{i\in I}$ codes $\langle g^{-1}[F_i]\rangle_{i\in I}$. **Q**

(c) A function $f : X \to \mathbb{R}$ is a **codable Baire function** if there are a continuous $g : X \to \mathbb{R}^{\mathbb{N}}$ and a codable Borel function $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that $f = hg$. A family $\langle f_i\rangle_{i\in I}$ of codable Baire functions is a **codable** family if there is a family $\langle (g_i, h_i)\rangle_{i\in I}$ such that $g_i : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous function for every $i \in I$ and $\langle h_i\rangle_{i\in I}$ is a codable family of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$.

(i) Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions from X to $\mathbb{R}$. Then there are a continuous function $g : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n = h_n g$ for every $n \in \mathbb{N}$. **P** Let $\langle (g_n, h'_n)\rangle_{n\in\mathbb{N}}$ be such that $g_n : X \to \mathbb{R}^{\mathbb{N}}$ is continuous for every n, $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions from $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, and $f_n = h'_n g_n$ for each n. Now $\langle \{x : h'_n(x) > q\}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ is a codable family in $\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$; let $\langle T_{nq}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ be a family in $\mathcal{T}$ coding it. By (b-i) above, there are a continuous function $g : X \to \mathbb{R}^{\mathbb{N}}$ and a family $\langle T'_{nq}\rangle_{n\in\mathbb{N},q\in\mathbb{N}}$ in $\mathcal{T}$ such that

$$g^{-1}[\phi(T'_{nq})] = g_n^{-1}[\phi(T_{nq})] = \{x : h'_n(x) > q\}$$

for every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$.

To convert $\langle T'_{nq}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ into a code for a sequence of real-valued functions on $\mathbb{R}^{\mathbb{N}}$, I copy ideas from the proof of 562L. Let

$$\Theta_1 : \mathcal{T}\times\mathcal{T} \to \mathcal{T}, \quad \Theta_2 : \bigcup_{I\subseteq\mathbb{Q}} \mathcal{T}^I \to \mathcal{T},$$

$$\Theta_3 : \mathcal{T} \to \mathcal{T}, \quad \Theta_4 : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$$

be such that

$$\phi(\Theta_1(T, T')) = \phi(T) \setminus \phi(T'), \quad \phi(\Theta_2(\tau)) = \bigcup_{q\in I} \phi(\tau(q))),$$

$$\phi(\Theta_3(T)) = X \setminus \phi(T), \quad \phi(\Theta_4(T, T')) = \phi(T) \cup \phi(T')$$

for T, $T' \in \mathcal{T}$, $I \subseteq \mathbb{Q}$ and $\tau \in \mathcal{T}^I$. Now, for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, set

$$\tau'_n(\alpha) = \Theta_2(\langle T'_{nq}\rangle_{q\in\mathbb{Q},q\ge\alpha}),$$

$$T_n = \Theta_2(\langle \Theta_1(\tau'_n(-k), \tau'_n(k))\rangle_{k\in\mathbb{N}}),$$

$$\begin{aligned}\tau_n(\alpha) &= \Theta_1(\tau'_n(\alpha), \Theta_3(T_n)) \text{ if } \alpha \ge 0,\\ &= \Theta_4(\tau'_n(\alpha), \Theta_3(T_n)) \text{ if } \alpha < 0.\end{aligned}$$

We now have a sequence $\langle \tau_n\rangle_{n\in\mathbb{N}}$ in $\tilde{\mathcal{T}}$ (as defined in 562L) coding a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n = h_n g$ for every n (see 562Qb). **Q**

(ii) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions, there is a codable Baire function f such that $f(x) = \liminf_{n\to\infty} f_n(x)$ whenever the lim inf is finite. **P** Take g and $\langle h_n\rangle_{n\in\mathbb{N}}$ as in (i); by 562Le, there is a codable Borel function h such that $h(z) = \liminf_{n\to\infty} h_n(z)$ whenever $z \in \mathbb{R}^{\mathbb{N}}$ is such that the $\lim inf$ is finite, and $f = hg : X \to \mathbb{R}$ will serve. **Q**

(iii) The family of codable Baire functions is a Riesz subspace of $\mathbb{R}^X$ containing all continuous functions and closed under multiplication. (This time, use (i) and 562Ld.)

(iv) The family of continuous real-valued functions on X is a codable family of codable Baire functions. (For $f \in C(X)$, define $g_f \in C(X; \mathbb{R}^{\mathbb{N}})$ by setting $g_f(x)(n) = f(x)$ for every $x \in X$ and $n \in \mathbb{N}$; setting $\pi_0(z) = z(0)$ for $z \in \mathbb{R}^{\mathbb{N}}$, $\langle (g_f, \pi_0)\rangle_{f\in C(X)}$ is a family of codes for $C(X)$.)

562S Proposition Let $(X, \mathfrak{T})$ be a second-countable space. Then there is a second-countable topology $\mathfrak{S}$ on X, codably Borel equivalent to $\mathfrak{T}$, such that $\mathcal{B}_c(X) = \mathcal{B}\mathfrak{a}_c(X, \mathfrak{S})$ and the codable families in $\mathcal{B}_c(X)$ are exactly the codable families in $\mathcal{B}\mathfrak{a}_c(X, \mathfrak{S})$.

proof (a) By 562Nb there is a topology $\mathfrak{S}$ on X, finer than $\mathfrak{T}$, generated by a countable algebra $\mathcal{E}$ of subsets of X, which is codably Borel equivalent to $\mathfrak{T}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$. Define $g_0 : X \to \mathbb{R}^{\mathbb{N}}$ by setting $g_0(x) = \langle \chi U_n(x)\rangle_{n\in\mathbb{N}}$ for each $x \in X$. Then g_0 is continuous. Set $W_n = \{z : z \in \mathbb{R}^{\mathbb{N}},\ z(n) > 0\}$ for each n, so that $W_n \subseteq \mathbb{R}^{\mathbb{N}}$ is open and $U_n = g_0^{-1}[W_n]$; let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$ and such that $V_{2n}=W_n$ for every n. Let $\phi:\mathcal{T}\to\mathcal{B}_c(X)$, $\phi':\mathcal{T}\to\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ be the interpretations of Borel codes corresponding to $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively.

(b) We have a function $\Theta:\mathcal{T}\to\mathcal{T}\setminus\{\emptyset\}$ such that $\phi(T)=g^{-1}[\phi'(\Theta(T))]$ for every $T\in\mathcal{T}$. **P** Induce on $r(T)$. If $r(T)=0$, take $\Theta(T)\in\mathcal{T}\setminus\{\emptyset\}$ such that $\phi'(\Theta(T))=\emptyset$. If $r(T)=1$, set $\Theta(T)=\{{<}2n{>}:{<}n{>}\in T\}$; then

$$\phi'(\Theta(T))=\bigcup\{V_{2n}:{<}n{>}\in T\},\quad g_0^{-1}[\phi'(\Theta(T))]=\bigcup\{U_n:{<}n{>}\in T\}=\phi(T).$$

If $r(T)>1$, set

$$\Theta(T)=\{{<}i{>}:i\in A_T\}\cup\{{<}i{>}^\frown\sigma:i\in A_T,\,\sigma\in\Theta(T_{<i>})\}.\ \mathbf{Q}$$

This means that if we have any codable family in $\mathcal{B}_c(X)$, coded by a family $\langle T_i\rangle_{i\in I}$ in $\mathcal{T}$, $\langle(g_0,\Theta(T_i))\rangle_{i\in I}$ will code the same family in $\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$.

(c) Next, there is a function $\Phi:C((X,\mathfrak{S});\mathbb{R}^{\mathbb{N}})\times\mathcal{T}\to\mathcal{T}\setminus\{\emptyset\}$ such that $g^{-1}[\phi'(T))]=\phi(\Phi(g,T))$ for every $\mathfrak{S}$-continuous $g:X\to\mathbb{R}^{\mathbb{N}}$ and $T\in\mathcal{T}$. **P** If $r(T)\le 1$ and $g^{-1}[\phi'(T)]$ is empty take $\Phi(g,T)\in\mathcal{T}\setminus\{\emptyset\}$ such that $\phi(\Phi(g,T))=\emptyset$. If $r(T)=1$ and $g^{-1}[\phi'(T)]$ is not empty set

$$\Phi(g,T)=\{{<}n{>}:U_n\subseteq g^{-1}[\phi'(T)]\}.$$

If $r(T)>1$ set

$$\Phi(g,T)=\{{<}i{>}:i\in A_T\}\cup\{{<}i{>}^\frown\sigma:i\in A_T,\,\sigma\in\Phi(g,T_{<i>})\}.\ \mathbf{Q}$$

So given any codable family in $\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$, coded by a family $\langle(g_i,T_i)\rangle_{i\in I}$ in $C((X,\mathfrak{S});\mathbb{R}^{\mathbb{N}})\times\mathcal{T}$, $\langle\Phi(g_i,T_i)\rangle_{i\in I}$ will code it in $\mathcal{B}_c(X)$.

562T A different use of Borel codes will appear when we come to re-examine a result in Volume 3. I will defer the application to 566O, but the first part of the argument fits naturally into the ideas of this section.

Theorem (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Then we have an interpretation $\phi:\mathcal{T}\to\mathfrak{A}$ of Borel codes such that

$$\begin{aligned}\phi(T)&=\sup_{i\in A_T}a_i\text{ if }r(T)\le 1,\\&=\sup_{i\in A_T}1\setminus\phi(T_{<i>})\text{ if }r(T)>1.\end{aligned}$$

(b) For $n\in\mathbb{N}$, set $E_n=\{x:x\in\{0,1\}^{\mathbb{N}},\,x(n)=1\}$. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Let $\phi:\mathcal{T}\to\mathfrak{A}$ and $\psi:\mathcal{T}\to\mathcal{P}(\{0,1\}^{\mathbb{N}})$ be the interpretations of Borel codes corresponding to $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle E_n\rangle_{n\in\mathbb{N}}$. If T, $T'\in\mathcal{T}$ are such that $\phi(T)\not\subseteq\phi(T')$, then $\psi(T)\not\subseteq\psi(T')$.

proof (a) Define $\phi(T)$ inductively on the rank of T, as in 562Ae.

(b) Let $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ be a sequence running over $\{T,T'\}\cup\{T_\sigma:\sigma\in S\}\cup\{T'_\sigma:\sigma\in S\}$. Define $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0=\phi(T)\setminus\phi(T')$. Given that $c_n\in\mathfrak{A}\setminus\{0\}$, then

— if $r(T^{(n)})\le 1$ and there is an $i\in A_{T^{(n)}}$ such that $c_n\cap a_i\ne 0$, take the first such i and set $c_{n+1}=c_n\cap a_i$;

— if $r(T^{(n)})>1$ and there is an $i\in A_{T^{(n)}}$ such that $c_n\setminus\phi(T^{(n)}_{<i>})\ne 0$, take the first such i and set $c_{n+1}=c_n\setminus\phi(T^{(n)}_{<i>})$;

— otherwise, set $c_{n+1}=c_n$.

Continue.

At the end of the induction, define $x\in\{0,1\}^{\mathbb{N}}$ by saying that $x(i)=1$ iff there is an $n\in\mathbb{N}$ such that $c_n\subseteq a_i$. Now we find that, for every $m\in\mathbb{N}$,

— if $x\in\psi(T^{(m)})$ there is an $n\in\mathbb{N}$ such that $c_n\subseteq\phi(T^{(m)})$,

— if $x\notin\psi(T^{(m)})$ there is an $n\in\mathbb{N}$ such that $c_n\cap\phi(T^{(m)})=0$.

P Induce on $r(T^{(m)})$. If $r(T^{(m)})\le 1$ then

$$\begin{aligned}x \in \psi(T^{(m)}) &\implies \text{there is an } i \in A_{T^{(m)}} \text{ such that } x \in E_i\\ &\implies \text{there are } i \in A_{T^{(m)}},\ n \in \mathbb{N} \text{ such that } c_n \subseteq a_i\\ &\implies \text{there is an } n \in \mathbb{N} \text{ such that } c_n \subseteq \phi(T^{(m)}),\\ x \notin \psi(T^{(m)}) &\implies x \notin E_i \text{ for every } i \in A_{T^{(m)}}\\ &\implies c_{m+1} \not\subseteq a_i \text{ for every } i \in A_{T^{(m)}}\\ &\implies c_m \cap a_i = 0 \text{ for every } i \in A_{T^{(m)}}\\ &\implies c_m \cap \phi(T^{(m)}) = 0.\end{aligned}$$

If $r(T^{(m)}) > 1$ then

$$\begin{aligned}x \in \psi(T^{(m)}) &\implies \text{there is an } i \in A_{T^{(m)}} \text{ such that } x \notin \psi(T^{(m)}_{<i>})\\ &\implies \text{there are } i \in A_{T^{(m)}},\ n \in \mathbb{N} \text{ such that } c_n \cap \phi(T^{(m)}_{<i>}) = 0\end{aligned}$$

(by the inductive hypothesis, because $T^{(m)}_{<i>}$ is always equal to $T^{(k)}$ for some k, and $r(T^{(m)}_{<i>}) < r(T^{(m)})$)

$$\begin{aligned}&\implies \text{there is an } n \in \mathbb{N} \text{ such that } c_n \subseteq \phi(T^{(m)}),\\ x \notin \psi(T^{(m)}) &\implies x \in \psi(T^{(m)}_{<i>}) \text{ for every } i \in A_{T^{(m)}}\\ &\implies \text{for every } i \in A_{T^{(m)}} \text{ there is an } n \in \mathbb{N} \text{ such that } c_n \subseteq \phi(T^{(m)}_{<i>})\\ &\implies c_{m+1} \not\subseteq 1 \setminus \phi(T^{(m)}_{<i>}) \text{ for every } i \in A_{T^{(m)}}\\ &\implies c_m \setminus \phi(T^{(m)}_{<i>}) = 0 \text{ for every } i \in A_{T^{(m)}}\\ &\implies c_m \cap \phi(T^{(m)}) = 0. \ \mathbf{Q}\end{aligned}$$

In particular, since both T and T' appear in the list $\langle T^{(m)}\rangle_{m\in\mathbb{N}}$, $c_n \cap \phi(T) \neq 0$ and $c_n \cap \phi(T') = 0$ for every n, $x \in \psi(T) \setminus \psi(T')$ and $\psi(T) \not\subseteq \psi(T')$.

562X Basic exercises (a) Let X be a regular second-countable space. Show that a resolvable subset of X is F_σ. (*Hint*: in the proof of 562G, show that $\phi(T_\xi)$ is always F_σ.)

(b) Let X be a second-countable space and $\langle E_{ni}\rangle_{n,i\in\mathbb{N}}$ a family of resolvable subsets of X. Show that $\bigcap_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}} E_{ni}$ is a codable Borel set.

(c) Let X be a second-countable space and $\langle E_i\rangle_{i\in I}$ a codable family in $\mathcal{B}_c(X)$. (i) Show that $\langle E_i\rangle_{i\in J}$ is codable for every $J \subseteq I$. (ii) Show that if I is countable and not empty then $\bigcup_{i\in I} E_i$ and $\bigcap_{i\in I} E_i$ are codable Borel sets. (iii) Show that if $h : J \times \mathbb{N} \to I$ is a function, where J is any other set, then $\langle \bigcup_{n\in\mathbb{N}} E_{h(j,n)}\rangle_{j\in J}$ is a codable family. (iv) Show that if $\langle F_i\rangle_{i\in I}$ is another codable family in $\mathcal{B}_c(X)$ then $\langle E_i \cap F_i\rangle_{i\in I}$ and $\langle E_i \triangle F_i\rangle_{i\in I}$ are codable families.

(d) Let X and Y be second-countable spaces and $f : X \to Y$ a function. Suppose that $\{F : F \subseteq Y,\ f^{-1}[F]$ is resolvable$\}$ includes a countable network for the topology of Y. Show that f is a codable Borel function.

(e) Let X be a second-countable space and $\langle E_i\rangle_{i\in I}$ a family in $\mathcal{B}_c(X)$. (i) Show that $\{J : J \subseteq I,\ \langle E_i\rangle_{i\in J}$ is codable$\}$ is an ideal of subsets of I. (ii) Show that if every E_i is resolvable then $\langle E_i\rangle_{i\in I}$ is codable.

(f) Let X be a second-countable space and $f : X \to \mathbb{R}$ a function. Show that f is a codable Borel function iff $\{(x,\alpha) : x \in X,\ \alpha < f(x)\}$ is a codable Borel subset of $X \times \mathbb{R}$.

(g) Let X be a topological space and $f,\ g : X \to \mathbb{R}$ resolvable functions. (i) Show that $f \vee g$ and αf are resolvable for any $\alpha \in \mathbb{R}$. (ii) Show that if f is bounded then $f + g$ is resolvable. (iii) Show that if f and g are bounded, $f \times g$ is resolvable. (iv) Show that if f and g are non-negative, then $f + g$ is resolvable. (v) Show that if $h : \mathbb{R} \to \mathbb{R}$ is continuous and $h^{-1}[\{\alpha\}]$ is finite for every $\alpha \in \mathbb{R}$, then hf is resolvable.

(h) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{t\downarrow x} f(t)$ is defined in $[-\infty,\infty]$ for every $x \in \mathbb{R}$. Show that f is resolvable.

(i) Let X be a second-countable space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of resolvable real-valued functions on X. Show that there is a codable Borel function g such that $g(x) = \lim_{n\to\infty} f_n(x)$ for any x such that the limit is defined in $\mathbb{R}$.

(j) Let X be a second-countable space and Y a subspace of X. Show that a family $\langle g_i\rangle_{i\in I}$ in $\mathbb{R}^Y$ is a codable family of codable Borel functions iff there is a codable family $\langle f_i\rangle_{i\in I}$ of codable real-valued Borel functions on X such that $g_i = f_i\restriction Y$ for every $i \in I$.

>(k) Let X be a second-countable space. Show that every codable family of codable Baire subsets of X is a codable family of codable Borel subsets of X.

>(l) Let X be a regular second-countable space. Show that every codable family of codable Borel subsets of X is a codable family of codable Baire subsets of X. (*Hint*: 561Xq.)

562Y Further exercises (a) Let X be a Hausdorff second-countable space and A, $B \subseteq X$ disjoint analytic sets. Show that there is a codable Borel set $E \subseteq X$ such that $A \subseteq E \subseteq X \setminus B$.

(b) Show that if X is a Polish space then a subset of X is resolvable iff it is both F_σ and G_δ.

(c) Let X be a Polish space. Show that a function $f : X \to \mathbb{R}$ is resolvable iff $\{x : \alpha < f(x) < \beta\}$ is F_σ for all α, $\beta \in \mathbb{R}$.

(d) Let X be a topological space. Let Φ be the set of functions $f : X \to \mathcal{P}\mathbb{N}$ such that $\{x : n \in f(x)\}$ is open for every $n \in \mathbb{N}$. Write $\mathcal{B}'_c(X)$ for $\{f^{-1}[F] : f \in \Phi,\ F \in \mathcal{B}_c(\mathcal{P}\mathbb{N})\}$; say a family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}'_c(X)$ is codable if there is a family $\langle(f_i, F_i)\rangle_{i\in I}$ in $\Phi \times \mathcal{B}_c(\mathcal{P}\mathbb{N})$ such that $\langle F_i\rangle_{i\in I}$ is codable and $E_i = f_i^{-1}[F_i]$ for every i. (i) Show that if X is second-countable then $\mathcal{B}'_c(X) = \mathcal{B}_c(X)$ and the codable families on the definition here coincide with the codable families of 562H. (ii) Develop a theory of codable Borel sets and functions corresponding to that in 562R.

562 Notes and comments The idea of 'Borel code' is of great importance in mathematical logic, for reasons quite separate from the questions addressed here; see Jech 78, Jech 03 or Kunen 80. (Of course it is not a coincidence that an approach which is effective in the absence of the axiom of choice should also be relevant to absoluteness in the presence of choice.) Every author has his favoured formula corresponding to that in 562Ae. The particular one I have chosen is intended to be economical and direct, but is slightly awkward at the initial stages, and some proofs demand an extra moment's attention to the special case of trees of rank 1. The real motivation for the calculations here will have to wait for §565; Lebesgue measure can be defined in such a way that it is countably additive with respect to *codable* sequences of Borel sets, and there are enough of these to make the theory non-trivial.

Borel codes are wildly non-unique, which is why the concept of codable family is worth defining. But it is also important that certain sets, starting with the open sets, are self-coding in the sense that from the set we can pick out an appropriate code. 'Resolvable' sets and functions (562F, 562O) are common enough to be very useful, and for these we can work with the objects themselves, just as we always have, and leave the coding until we need it.

The Borel codes described here can be used only in second-countable spaces. It is easy enough to find variations of the concept which can be applied in more general contexts (562Yd), though it is not obvious that there are useful theorems to be got in such a way. More relevant to the work of the next few sections is the idea of 'codable Baire set' (562R). Because any codable sequence of codable Baire sets can be factored through a single continuous function to $\mathbb{R}^{\mathbb{N}}$ (562R(b-i)), we have easy paths to the elementary results given here.

563 Borel measures without choice

Having decided that a 'Borel set' is to be one obtainable by a series of operations described by a Borel code, it is a natural step to say that a 'Borel measure' should be one which respects these operations (563A). In regular spaces, such measures have strong inner and outer regularity properties also based on the Borel coding (563D-563F), and we have effective methods of constructing such measures (563H). Analytic sets are universally measurable (563I). We can use similar ideas to give a theory of Baire measures on general topological spaces (563J-563K). In the basic case, of a second-countable space with a codably σ-finite measure, we have a measure algebra with the same basic properties as in the standard theory (563M-563N).

The theory would not be very significant if there were no interesting Borel-coded measures, so you may wish to glance ahead to §565 to confirm that Lebesgue measure can be brought into the framework developed here.

563A Definitions (a) (Foreman & Wehrung 91) Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. I will say that a **Borel-coded measure** on X is a functional $\mu : \mathcal{B}_c(X) \to [0,\infty]$ such that $\mu\emptyset = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint codable family in $\mathcal{B}_c(X)$.

I will try to remember to say 'Borel-coded measure' everywhere in this section, because these are dangerously different from the 'Borel measures' of §434. Their domains are not necessarily σ-algebras and while they are finitely additive they need not be countably additive even in the sense of 326E.

(b) As usual, I will say that a subset of X is **negligible** if it is included in a set of measure 0, which here must be a codable Borel set; the terms 'conegligible', 'almost everywhere', 'null ideal' will take their meanings from this. We can now define the **completion** of μ to be the natural extension of μ to the algebra $\{E\triangle A : E \in \mathcal{B}_c(X),\, A \text{ is } \mu\text{-negligible}\}$.

(c) Some of the other definitions from the ordinary theory can be transferred without difficulty (e.g., 'totally finite', 'probability'), but we may need to make some finer distinctions. For instance, I will say that a Borel-coded measure μ is **semi-finite** if $\sup\{\mu F : F \subseteq E,\, \mu F < \infty\} = \infty$ whenever $\mu E = \infty$; we no longer have the ordinary principle of exhaustion (215A), and the definition in 211F, taken literally, may be too weak. For 'locally finite', however, 411Fa can be taken just as it is, since all open sets are measurable.

(d) For 'σ-finite' we again have to make a choice. The definition in 211C calls only for 'a sequence of measurable sets of finite measure'. Here the following will be more useful: a Borel-coded measure on X is **codably σ-finite** if there is a codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)$ such that $X = \bigcup_{n\in\mathbb{N}} E_n$ and μE_n is finite for every n.

563B Proposition Let $(X,\mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in $\mathcal{B}_c(X)$.

(i) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^\infty \mu E_n$.

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty}\mu E_n$.

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty}\mu E_n$.

(b) If $\mathcal{E}$ is the algebra of resolvable subsets of X, then $\mu\restriction\mathcal{E}$ is countably additive in the sense that $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^\infty \mu E_n$ for any disjoint family $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$.

(c) μ is τ-additive.

proof (a)(i) Set $F_n = E_n \setminus \bigcup_{i<n} E_i$ for $n\in\mathbb{N}$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is codable (562Ic), so

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = \mu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n=0}^\infty \mu F_n \le \sum_{n=0}^\infty \mu E_n.$$

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing then, in the language of (i), $E_n = \bigcup_{i\le n} F_i$ for each n, so

$$\lim_{n\to\infty}\mu E_n = \lim_{n\to\infty}\sum_{i=0}^n \mu F_i = \mu(\bigcup_{n\in\mathbb{N}} E_n).$$

(iii) Apply (ii) to $\langle E_0\setminus E_n\rangle_{n\in\mathbb{N}}$. (It is elementary to check that this is a codable sequence, using ideas in 562B.)

(b) Any sequence of resolvable sets is codable, as observed in 562H.

(c) Suppose that $\mathcal{G}$ is an upwards-directed family of open sets with union H. Set $\gamma = \sup_{G\in\mathcal{G}}\mu G$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of X, and for $n\in\mathbb{N}$ set

$$V_n = \bigcup\{U_i : i\le n,\, U_i\subseteq G \text{ for some } G\in\mathcal{G}\}.$$

Then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union H. As every V_n is resolvable, $\langle V_n\rangle_{n\in\mathbb{N}}$ is codable and

$$\mu H = \lim_{n\to\infty}\mu V_n \le \sup_{G\in\mathcal{G}}\mu G \le \mu H$$

by (a-ii).

563C Corollary Let X be a second-countable space, μ a Borel-coded measure on X and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of resolvable sets in X.

(a)(i) $\bigcup_{n\in\mathbb{N}} E_n$ is measurable;

(ii) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^\infty \mu E_n$;

(iii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^\infty \mu E_n$;

(iv) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty}\mu E_n$.

(b)(i) $\bigcap_{n\in\mathbb{N}} E_n$ is measurable;

(ii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}}\mu E_n$ is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty}\mu E_n$.

proof Use 562G to find a sequence of codes for $\langle E_n\rangle_{n\in\mathbb{N}}$, and apply 563B.

563D Lemma Let $(X,\mathfrak{T})$ be a regular second-countable space and $\mu:\mathfrak{T}\to[0,\infty]$ a functional such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G\subseteq H$,

$\mu G + \mu H = \mu(G\cup H) + \mu(G\cap H)$ for all G, $H\in\mathfrak{T}$,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty}\mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G : G\in\mathfrak{T},\, \mu G < \infty\} = X$.

(a) $\mu(\bigcup_{i\in I} G_i) \le \sum_{i\in I} \mu G_i$ for every countable family $\langle G_i\rangle_{i\in I}$ in $\mathfrak{T}$.

(b) There is a function $\pi^* : \mathfrak{T}\times\mathbb{N}\to\mathfrak{T}$ such that

$$X\setminus G \subseteq \pi^*(G,k), \quad \mu(G\cap\pi^*(G,k)) \le 2^{-k}$$

whenever $G\in\mathfrak{T}$ and $k\in\mathbb{N}$.

(c) Let $\phi : \mathcal{T}\to\mathcal{B}_c(X)$ be an interpretation of Borel codes defined from a sequence running over a base for the topology of X. Then there are functions π, $\pi' : \mathcal{T}\times\mathbb{N}\to\mathfrak{T}$ such that

$$\phi(T)\subseteq\pi(T,n), \quad X\setminus\phi(T)\subseteq\pi'(T,n), \quad \mu(\pi(T,n)\cap\pi'(T,n))\le 2^{-n}$$

for every $T\in\mathcal{T}$ and $n\in\mathbb{N}$.

proof (a) This is elementary. First, $\mu(G\cup H)\le\mu G+\mu H$ for all open sets G and H, because $\mu(G\cap H)\ge 0$. Next, if $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open sets with union G, then

$$\mu G = \lim_{n\to\infty}\mu(\bigcup_{i\le n} G_i) \le \lim_{n\to\infty}\sum_{i=0}^n \mu G_i = \sum_{n=0}^\infty \mu G_n.$$

Now the step to general countable I is immediate.

(b) Set $I=\{n : n\in\mathbb{N},\ \mu U_n<\infty\}$; because μ is locally finite, $\{U_n : n\in I\}$ is a base for $\mathfrak{T}$. Given $G\in\mathfrak{T}$ and $k\in\mathbb{N}$, then for $n\in I$ and $m\in\mathbb{N}$ set

$$W_{nm}=\bigcup\{U_i : i\le m,\ \overline{U}_i\subseteq U_n\cap G\}.$$

Because $\mathfrak{T}$ is regular, $\bigcup_{m\in\mathbb{N}} W_{nm}=U_n\cap G$ and $\mu(U_n\cap G)=\lim_{m\to\infty}\mu W_{nm}$. Let m_n be the least integer such that $\mu W_{nm_n}\ge\mu(U_n\cap G)-2^{-k-n-1}$. Set

$$\pi^*(G,k)=\bigcup_{n\in I} U_n\setminus\overline{W}_{nm_n}.$$

Because $\overline{W}_{nm}\subseteq U_n\cap G$ for all m and n, while $\bigcup_{n\in I} U_n = X$, $\pi^*(G,k)\supseteq X\setminus G$. Now

$$\mu(G\cap\pi^*(G,k))\le\sum_{n\in I}\mu(G\cap U_n\setminus\overline{W}_{nm_n})\le\sum_{n=0}^\infty 2^{-k-n-1}=2^{-k},$$

as required.

(c) Define $\pi(T)$ and $\pi'(T)$ inductively on the rank $r(T)$ of T.

(i) If $r(T)=0$, set $\pi(T,n)=\emptyset$ and $\pi'(T,n)=X$ for every n. If $r(T)=1$ then $G=\phi(T)$ is open; set $\pi(T,n)=G$ and $\pi'(T,n)=\pi^*(G,n)$ for each n.

(ii) For the inductive step to $r(T)>1$, set $A_T=\{i : {<}i{>}\in T\}$, as in 562Ad. Set

$$\pi(T,n)=\bigcup_{i\in A_T}\pi'(T_{<i>},n+i+2),$$

$$\pi'(T,n)=\bigcup_{i\in A_T}(\pi(T_{<i>},n+i+2)\cap\pi'(T_{<i>},n+i+2))\cup\pi^*(\pi(T,n),n+1).$$

Then

$$\phi(T)=\bigcup_{i\in A_T} X\setminus T_{<i>}\subseteq\bigcup_{i\in A_T}\pi'(T_{<i>},n+i+2)=\pi(T,n),$$

$$\begin{aligned}X\setminus\phi(T)&=\bigcap_{i\in A_T}\phi(T_{<i>})\subseteq(\pi(T,n)\cap\bigcap_{i\in A_T}\phi(T_{<i>}))\cup\pi^*(\pi(T,n),n+1)\\&\subseteq\Bigl(\bigcup_{i\in A_T}\pi'(T_{<i>},n+i+2)\cap\bigcap_{i\in A_T}\pi(T_{<i>},n+i+2)\Bigr)\cup\pi^*(\pi(T,n),n+1)\\&\subseteq\bigcup_{i\in A_T}\bigl(\pi'(T_{<i>},n+i+2)\cap\pi(T_{<i>},n+i+2)\bigr)\cup\pi^*(\pi(T,n),n+1)\\&=\pi'(T,n),\end{aligned}$$

$$\begin{aligned}\mu(\pi(T,n)\cap\pi'(T,n))&\le\sum_{i\in A_T}\mu(\pi(T_{<i>},n+i+2)\cap\pi'(T_{<i>},n+i+2))\\&\qquad\qquad+\mu(\pi(T,n)\cap\pi^*(\pi(T,n),n+1))\\&\le\sum_{i\in A_T}2^{-n-i-2}+2^{-n-1}\le 2^{-n}\end{aligned}$$

for every n, so the induction continues.

563E Lemma Let X be a second-countable space and M a non-empty upwards-directed set of Borel-coded measures on X. For each codable Borel set $E \subseteq X$, set $\nu E = \sup_{\mu\in\mathrm{M}} \mu E$. Then ν is a Borel-coded measure on X.

proof Immediate from the definition in 563Aa.

563F Proposition Let $(X,\mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) For any $F \in \mathcal{B}_c(X)$, we have a Borel-coded measure μ_F on X defined by saying that $\mu_F E = \mu(E\cap F)$ for every $E \in \mathcal{B}_c(X)$.

(b) We have a semi-finite Borel-coded measure μ_{sf} defined by saying that

$$\mu_{sf}(E) = \sup\{\mu F : F \in \mathcal{B}_c(X),\, F \subseteq E,\, \mu F < \infty\}$$

for every $E \in \mathcal{B}_c(X)$.

(c)(i) If μ is locally finite it is codably σ-finite.

(ii) If μ is codably σ-finite, it is semi-finite and there is a totally finite Borel-coded measure ν on X with the same null ideal as μ.

(iii) If μ is codably σ-finite, there is a non-decreasing codable sequence of codable Borel sets of finite measure which covers X.

(d) If X is regular then the following are equiveridical:

(i) μ is locally finite;

(ii) μ is semi-finite, outer regular with respect to the open sets and inner regular with respect to the closed sets;

(iii) μ is semi-finite and outer regular with respect to the open sets.

(e) If X is regular and μ is semi-finite, then μ is inner regular with respect to the closed sets of finite measure.

(f) If X is Polish and μ is semi-finite, then μ is inner regular with respect to the compact sets.

(g) If μ is locally finite, and ν is another Borel-coded measure on X agreeing with μ on the open sets, then $\nu = \mu$.

proof Fix a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of X.

(a) The point is just that $\langle E_n \cap F\rangle_{n\in\mathbb{N}}$ is a codable family whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$. (See 562Ba.)

(b) Writing μ_F for the Borel-coded measure corresponding to a set F of finite measure, as in (a), we have an upwards-directed family of measures; by 563E, its supremum μ_{sf} is a Borel-coded measure. If $E \subseteq X$ is a codable Borel set and $\gamma < \mu_{sf}E$, then there is a set F of finite measure such that $\mu(E\cap F) \ge \gamma$; now

$$\gamma \le \mu_{sf}(E\cap F) = \mu(E\cap F) < \infty.$$

(c)(i) Set $I = \{i : i \in \mathbb{N},\, \mu U_i < \infty\}$; then $\langle U_i\rangle_{i\in I}$ is a codable family of sets of finite measure covering X.

(ii) Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a codable sequence of sets of finite measure covering X.

(α) If $E \in \mathcal{B}_c(X)$, then $\langle E \cap \bigcup_{i\le n} H_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union E, so

$$\mu E = \sup_{n\in\mathbb{N}} \mu(E\cap \textstyle\bigcup_{i\le n} H_i) = \sup\{\mu F : F \subseteq E,\, \mu F < \infty\} \le \mu E.$$

As E is arbitrary, μ is semi-finite.

(β) Let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^\infty \epsilon_n \mu H_n$ is finite. Set $\nu E = \sum_{n=0}^\infty \epsilon_n\mu(E\cap H_n)$ for $E \in \mathcal{B}_c(X)$. Of course $\nu\emptyset = 0$ and $\nu X < \infty$. If $\langle E_k\rangle_{k\in\mathbb{N}}$ is a disjoint codable sequence in $\mathcal{B}_c(X)$, then $\langle E_k \cap H_n\rangle_{k\in\mathbb{N}}$ is codable for every n, so

$$\begin{aligned}\nu\Big(\bigcup_{k\in\mathbb{N}} E_k\Big) &= \sum_{n=0}^\infty \epsilon_n\mu\Big(\bigcup_{k\in\mathbb{N}} E_k\cap H_n\Big) = \sum_{n=0}^\infty\sum_{k=0}^\infty \epsilon_n\mu(E_k\cap H_n)\\ &= \sum_{n=0}^\infty\sum_{k=0}^\infty \epsilon_n\mu(E_k\cap H_n) = \sum_{k=0}^\infty\sum_{n=0}^\infty \epsilon_n\mu(E_k\cap H_n) = \sum_{k=0}^\infty \nu E_k.\end{aligned}$$

So ν is a Borel-coded measure.

If $E \in \mathcal{B}_c(X)$ and $\mu E = 0$, then of course $\nu E = \sum_{n=0}^\infty \epsilon_n\mu(E\cap H_n) = 0$. Conversely, if $\nu E = 0$, then $\mu(E\cap H_n) = 0$ for every n; but $\langle E\cap H_n\rangle_{n\in\mathbb{N}}$, like $\langle H_n\rangle_{n\in\mathbb{N}}$, is codable, so $\mu E = \mu(\bigcup_{n\in\mathbb{N}} E\cap H_n) = 0$. Thus μ and ν have the same sets of finite measure; it follows at once that they have the same null ideals.

(iii) All we have to note is that if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of sets of finite measure covering X, then $\langle\bigcup_{i\le n} E_i\rangle_{n\in\mathbb{N}}$ is codable (use 562Bc), so gives the required non-decreasing witness.

(d)(i)$\Rightarrow$(ii)(α) Observe first that $\mu\restriction\mathfrak{T}$ satisfies the conditions of 563D. $\mathbf{P}$ The first three are consequences of the fact that $\mu : \mathcal{B}_c(X) \to [0,\infty]$ is additive. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{T}$, it is a codable sequence of codable Borel sets, by 562G as usual; so $\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ by 563B(a-ii). Finally, we are assuming that μ is locally finite, so the last condition is satisfied. **Q**

Take an interpretation ϕ of Borel codes and functions π, $\pi' : \mathcal{T} \times \mathbb{N} \to \mathfrak{T}$ as in 563Dc.

(β) If $E \in \mathcal{B}_c(X)$ and $\mu E < \gamma$, take n such that $2^{-n} \le \gamma - \mu E$. There is a $T \in \mathcal{T}$ such that $\phi(T) = E$, and now $G = \pi(T, n)$ is open, $E \subseteq G$ and

$$\mu(G \setminus E) \le \mu(G \cap \pi'(T, n)) \le 2^{-n},$$

so $\mu G \le \gamma$.

(γ) If $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$, take $T \in \mathcal{T}$ such that $\phi(T) = E$ and $n \in \mathbb{N}$ such that $2^{-n} < \mu E - \gamma$; now $F = X \setminus \pi'(T, n)$ is closed, $F \subseteq E$ and $\mu(E \setminus F) \le 2^{-n}$, so $\mu F > \gamma$. Next, if we set

$$F_m = F \cap \bigcup\{\overline{U_i} : i \le m,\ \mu\overline{U}_i < \infty\},$$

$\langle F_m\rangle_{m\in\mathbb{N}}$ will be a non-decreasing sequence of closed sets of finite measure with union F. The sets F_m are all resolvable, so $\mu F = \lim_{m\to\infty} \mu F_m$ and there is an m such that $\mu F_m \ge \gamma$, while $F_m \subseteq E$ is a set of finite measure. As E and γ are arbitrary, μ is inner regular with respect to the closed sets and also semi-finite.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i)(α) If $x \in X$ then x belongs to some set of finite measure. **P** Set

$$F = \bigcap\{U_n : n \in \mathbb{N},\ x \in U_n\} \setminus \bigcup\{U_n : n \in \mathbb{N},\ x \notin U_n\}.$$

Then F is a codable Borel set, being the difference of open sets, and the subspace topology on F is indiscrete. If $\mu F = 0$ we can stop. Otherwise, there must be an $F' \subseteq F$ such that $0 < \mu F' < \infty$; but $F' = \mathcal{B}_c(F) = \{\emptyset, F\}$ so $F' = F$ and again F has finite measure. **Q**

(β) Now as μ is outer regular with respect to the open sets, every set of finite measure is included in an open set of finite measure. So μ must be locally finite.

(e) Suppose that μ is semi-finite, $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$. Then there is an $H \in \mathcal{B}_c(X)$ such that $H \subseteq E$ and $\gamma < \mu H < \infty$. Consider the Borel-coded measure μ_H defined from μ and H as in (a). This is totally finite, so (d) tells us that it is outer regular with respect to the open sets and therefore inner regular with respect to the closed sets, and there is a closed set $F \subseteq H$ such that $\mu F = \mu_H F \ge \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the closed sets of finite measure.

(f) Now suppose that X is Polish and μ is semi-finite. Let ρ be a complete metric on X inducing its topology. If $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$, let $F \subseteq E$ be a closed set such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For each $n \in \mathbb{N}$ set $J_n = \{i : \operatorname{diam} U_i \le 2^{-n}\}$. Define $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively by saying that $F_0 = F$ and

$$k_n = \min\{k : \mu(F_n \cap \bigcup_{i\in J_n\cap k} U_i) > \gamma\}, \quad F_{n+1} = F \cap \bigcup_{i\in J_n\cap k_n} \overline{U}_i$$

for each n; set $K = \bigcap_{n\in\mathbb{N}} F_n$. Then $K \subseteq E$ is compact and $\mu K = \lim_{n\to\infty} \mu F_n \ge \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the compact sets.

(g)(i) Consider first the case in which X is regular. In this case both μ and ν must be outer regular with respect to the open sets, by (d); as they agree on the open sets they must be equal.

(ii) Next, suppose that $\mu X = \nu X$ is finite. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\{U_n : n \in \mathbb{N}\}$, and $\mathfrak{S}$ the topology generated by $\mathcal{E}$. As noted in the proof of 562Nb, $\mathfrak{S}$ is codably Borel equivalent to the original topology of X, so μ and ν are still Borel-coded measures with respect to $\mathfrak{S}$, and are still locally finite, because $\mathfrak{S}$ is finer than $\mathfrak{T}$; while $\mathfrak{S}$ is regular. Now any member of $\mathcal{E}$ is expressible in the form $E = \bigcup_{i\le n} G_i \setminus H_i$ where the G_i, H_i are open and $\langle G_i \setminus H_i\rangle_{i\le n}$ is disjoint. So

$$\mu E = \sum_{i=0}^{n} \mu G_i - \mu(G_i \cap H_i) = \nu E.$$

More generally, if $H \in \mathfrak{S}$, there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ with union H; as all the sets in $\mathcal{E}$ are resolvable, $\langle E_n\rangle_{n\in\mathbb{N}}$ is codable and

$$\mu H = \sup_{n\in\mathbb{N}} \mu E_n = \nu H.$$

Thus μ and ν agree on $\mathfrak{S}$; by (i), they are equal.

(iii) Finally, for the general case, set $V_n = \bigcup\{U_i : i \le n,\ \mu U_i < \infty\}$ for each n. Because μ is locally finite, $\bigcup_{n\in\mathbb{N}} V_n = X$. For each $n \in \mathbb{N}$ let μ_{V_n}, ν_{V_n} be the Borel-coded measures defined from V_n as in (a). Then μ_{V_n} and ν_{V_n} are totally finite and agree on the open sets, so are equal. Now $\langle V_n\rangle_{n\in\mathbb{N}}$, being a sequence of open sets, is codable; so if $E \in \mathcal{B}_c(X)$ the sequence $\langle E \cap V_n\rangle_{n\in\mathbb{N}}$ is codable, and

$$\mu E = \lim_{n\to\infty} \mu(E \cap V_n) = \lim_{n\to\infty} \mu_{V_n} E = \nu E.$$

So in this case also we have $\mu = \nu$.

563G Proposition Let X be a set and $\theta : \mathcal{P}X \to [0,\infty]$ a submeasure (definition: 539A).

(a)

$$\Sigma = \{E : E \subseteq X, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\theta{\restriction}\Sigma$ is additive in the sense that $\theta(E \cup F) = \theta E + \theta F$ in $[0,\infty]$ whenever E, $F \in \Sigma$ are disjoint.

(b) If $E \subseteq X$ and for every $\epsilon > 0$ there is an $F \in \Sigma$ such that $E \subseteq F$ and $\theta(F \setminus E) \le \epsilon$, then $E \in \Sigma$.

proof (a) Parts (a)-(c) of the proof of 113C apply unchanged.

(b) Take any $A \subseteq X$ and $\epsilon > 0$. Let $F \in \Sigma$ be such that $E \subseteq F$ and $\theta(F \setminus E) \le \epsilon$. Then

$$\theta A \le \theta(A \cap E) + \theta(A \setminus E) \le \theta(A \cap F) + \theta(A \setminus F) + \theta(F \setminus E) \le \theta A + \epsilon.$$

As ϵ is arbitrary, $\theta A = \theta(A \cap E) + \theta(A \setminus E)$; as A is arbitrary, $E \in \Sigma$.

563H Theorem Let $(X, \mathfrak{T})$ be a regular second-countable space and $\mu : \mathfrak{T} \to [0,\infty]$ a functional such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

$\mu G + \mu H = \mu(G \cup H) + \mu(G \cap H)$ for all G, $H \in \mathfrak{T}$,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G : G \in \mathfrak{T}, \mu G < \infty\} = X$.

Then μ has a unique extension to a Borel-coded measure on X.

proof (a) For $A \subseteq X$ set $\theta A = \inf\{\mu G : A \subseteq G \in \mathfrak{T}\}$. Then θ is a submeasure on $\mathcal{P}X$ (because $\mu(G \cup H) \le \mu G + \mu H$ for all G, $H \in \mathfrak{T}$), extending μ (because $\mu G \le \mu H$ if $G \subseteq H$). Set

$$\Sigma = \{E : E \subseteq X, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}$$

and $\nu = \theta{\restriction}\Sigma$, as in 563G. Let $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ be an interpretation of Borel codes and π, $\pi' : \mathcal{T} \times \mathbb{N} \to \mathfrak{T}$ corresponding functions as in 563Dc. Now $\mathcal{B}_c(X) \subseteq \Sigma$. **P** Given $T \in \mathcal{T}$, $A \subseteq X$ and $n \in \mathbb{N}$, let $G \in \mathfrak{T}$ be such that $A \subseteq G$ and $\mu G \le \theta A + 2^{-n}$. Then

$$\begin{aligned}\theta A &\le \theta(A \cap \phi(T)) + \theta(A \setminus \phi(T)) \le \theta(A \cap \pi(T,n)) + \theta(A \cap \pi'(T,n))\\ &\le \mu(G \cap \pi(T,n)) + \mu(G \cap \pi'(T,n))\\ &= \mu(G \cap (\pi(T,n) \cup \pi'(T,n)) + \mu(G \cap \pi(T,n) \cap \pi'(T,n))\\ &\le \mu G + \mu(\pi(T,n) \cap \pi'(T,n)) \le \theta A + 2^{-n+1}.\end{aligned}$$

As A and n are arbitrary, $\phi(T) \in \Sigma$. **Q**

(b) Let $\langle T_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, where $E_n = \phi(T_n)$ for each n; set $E = \bigcup_{n\in\mathbb{N}} E_n$. Then, for any $k \in \mathbb{N}$, $E \subseteq \bigcup_{n\in\mathbb{N}} \pi(T_n, k+n)$, so

$$\begin{aligned}\sum_{n=0}^{\infty} \nu E_n &= \lim_{n\to\infty} \nu(\bigcup_{i\le n} E_i) \le \nu E \le \nu(\bigcup_{n\in\mathbb{N}} \pi(T_n, k+n))\\ &= \mu(\bigcup_{n\in\mathbb{N}} \pi(T_n, k+n)) \le \sum_{n=0}^{\infty} \mu(\pi(T_n, k+n))\end{aligned}$$

(563Da)

$$\begin{aligned}&= \sum_{n=0}^{\infty} \nu E_n + \nu(\pi(T_n, k+n) \setminus E_n)\\ &\le \sum_{n=0}^{\infty} \nu E_n + \mu(\pi(T_n, k+n) \cap \pi'(T_n, k+n))\\ &\le \sum_{n=0}^{\infty} \nu E_n + 2^{-k-n} = 2^{-k+1} + \sum_{n=0}^{\infty} \nu E_n;\end{aligned}$$

as k is arbitrary, $\sum_{n=0}^{\infty}\nu E_n=\nu E$; as $\langle T_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\nu\restriction\mathcal{B}_c(X)$ is a Borel-coded measure extending μ.

(c) At the same time we see that if λ is any other Borel-coded measure extending μ, we must have $\lambda E\le\theta E=\nu E$ for every $E\in\mathcal{B}_c(X)$. In the other direction,

$$\begin{aligned}\lambda(\phi(T))&\ge\lambda(\pi(T,n))-\lambda(\pi(T,n)\cap\pi'(T,n))\\&=\mu(\pi(T,n))-\mu(\pi(T,n)\cap\pi'(T,n))\ge\nu(\phi(T))-2^{-n}\end{aligned}$$

for every $T\in\mathcal{T}$ and $n\in\mathbb{N}$, so $\lambda E\ge\nu E$ for every $E\in\mathcal{B}_c(X)$. Thus $\nu\restriction\mathcal{B}_c(X)$ is the only Borel-coded measure extending μ.

563I Theorem Let X be a Hausdorff second-countable space, μ a codably σ-finite Borel-coded measure on X, and $A\subseteq X$ an analytic set. Then there are a codable Borel set $E\supseteq A$ and a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of A such that $E\setminus\bigcup_{n\in\mathbb{N}}K_n$ is negligible. Consequently A is measured by the completion of μ.

proof (a) By 563F(c-ii), there is a totally finite Borel-coded measure on X with the same negligible sets as μ; so it will be enough to consider the case in which μ itself is totally finite.

If A is empty, the result is trivial. So we may suppose that there is a continuous surjection $f:\mathbb{N}^{\mathbb{N}}\to A$. For $\sigma\in S^*=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ set $I_\sigma=\{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}$. Fix on a sequence running over a base for the topology of X and the corresponding interpretation $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ of Borel codes.

(b) For $\sigma\in S^*$ and $\xi<\omega_1$ define $E_{\sigma\xi}$ by saying that

$$E_{\sigma 0}=\overline{f[I_\sigma]},$$

$$E_{\sigma,\xi+1}=\bigcup_{i\in\mathbb{N}}E_{\sigma^\frown<i>,\xi},$$

$$E_{\sigma\xi}=\bigcap_{\eta<\xi}E_{\sigma\eta}\text{ if }\xi>0\text{ is a countable limit ordinal.}$$

Then $\langle E_{\sigma\xi}\rangle_{\xi<\omega_1}$ is a non-increasing family of sets including $f[I_\sigma]$.

(c) For every $\xi<\omega_1$, $\langle E_{\sigma\eta}\rangle_{\sigma\in S^*,\eta\le\xi}$ is a codable family of codable Borel sets. **P** It is enough to consider the case $\xi\ge\omega$. Because ξ is countable, we have a function $\tilde\Theta_2:\bigcup_{J\subseteq\xi}\mathcal{T}^J\to\mathcal{T}$ such that $\phi(\tilde\Theta_2(\langle T_\eta\rangle_{\eta\in J}))=\bigcup_{\eta\in J}\phi(T_\eta)$ for every $J\subseteq\xi$ (562Bc). Also, of course, we have a function $\Theta'_1:\mathcal{T}\to\mathcal{T}$ such that $\phi(\Theta'_1(T))=X\setminus\phi(T)$ for every $T\in\mathcal{T}$. Next, all the sets $E_{\sigma0}$ are closed, therefore resolvable. So we have a family $\langle T_{\sigma0}\rangle_{\sigma\in S^*}$ in $\mathcal{T}$ such that $\phi(T_{\sigma0})=E_{\sigma0}$ for every σ. Now we can set

$$T_{\sigma,\eta+1}=\tilde\Theta_2(\langle T_{\sigma^\frown<i>,\eta}\rangle_{i\in\mathbb{N}})$$

if $\eta<\xi$,

$$T_{\sigma\eta}=\Theta'_1(\tilde\Theta_2(\langle\Theta'_1(T_{\sigma\zeta})\rangle_{\zeta<\eta}))$$

if $\eta\le\xi$ is a non-zero limit ordinal, and $\phi(T_{\sigma\eta})$ will be equal to $E_{\sigma\eta}$ as required. **Q**

(d) Let $\langle\epsilon_\sigma\rangle_{\sigma\in S^*}$ be a summable family of strictly positive real numbers, and for $\xi<\omega_1$ set

$$\gamma(\xi)=\sum_{\sigma\in S^*}\epsilon_\sigma\mu(E_{\sigma\xi}).$$

Then $\gamma:\omega_1\to\mathbb{R}$ is non-increasing. There is therefore a $\xi<\omega_1$ such that $\gamma(\xi+1)=\gamma(\xi)$ (561A), that is, $\mu(E_{\sigma,\xi+1})=\mu(E_{\sigma\xi})$ for every $\sigma\in S^*$.

(e)(i) Set $E=E_{\emptyset\xi}$. Of course $A=f[I_\emptyset]\subseteq E$. Now define $\alpha_n\in\mathbb{N}^{\mathbb{N}}$, for $n\in\mathbb{N}$, as follows. Given $\langle\alpha_n(i)\rangle_{i<m}$, set

$$G_{nm}=\bigcup\{E_{\sigma\xi}:\sigma\in\mathbb{N}^m,\,\sigma(i)\le\alpha_n(i)\text{ for every }i<m\},$$

$$G_{nmk}=\bigcup\{E_{\sigma^\frown<j>,\xi}:\sigma\in\mathbb{N}^m,\,j\le k,\,\sigma(i)\le\alpha_n(i)\text{ for every }i<m\},$$

Then $\langle G_{nm}\rangle_{n,m\in\mathbb{N}}$ is codable, and $\lim_{k\to\infty}\mu G_{nmk}=\mu G_{nm}$ for all m, $n\in\mathbb{N}$. **P** By (c), there is a family $\langle T_{\sigma\eta}\rangle_{\sigma\in S^*,\eta\le\xi+1}$ in $\mathcal{T}$ such that $\phi(T_{\sigma\eta})=E_{\sigma\eta}$ whenever $\sigma\in S^*$ and $\eta\le\xi+1$. This time, we need a function $\tilde\Theta_2:\bigcup_{J\subseteq S^*}\mathcal{T}^J\to\mathcal{T}$ such that $\phi(\tilde\Theta_2(\langle T_\sigma\rangle_{\sigma\in J}))=\bigcup_{\sigma\in J}\phi(T_\sigma)$ whenever $J\subseteq S^*$ and $\langle T_\sigma\rangle_{\sigma\in J}$ is a family in $\mathcal{T}$, and a function $\Theta_1:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that $\phi(\Theta_1(T,T'))=\phi(T)\setminus\phi(T')$ for all T, $T'\in\mathcal{T}$. Setting

$$T'_{nm}=\tilde\Theta_2(\langle T_{\sigma\xi}\rangle_{\sigma\in\mathbb{N}^m,\,\sigma(i)\le\alpha_n(i)\forall i<m}),$$

$$T'_{nmk}=\tilde\Theta_2(\langle T_{\sigma^\frown<j>,\xi}\rangle_{\sigma\in\mathbb{N}^m,\,j\le k,\,\sigma(i)\le\alpha_n(i)\forall i<m}),$$

we have $\phi(T'_{nm})=G_{nm}$ and $\phi(T'_{nmk})=G_{nmk}$ for all m, n, $k\in\mathbb{N}$. In particular, all the G_{nm} and G_{nmk} are codable Borel sets, and $\langle G_{nm}\rangle_{n,m\in\mathbb{N}}$ is codable. Moreover, for any particular pair m and n, $\langle G_{nmk}\rangle_{k\in\mathbb{N}}$ is a codable sequence; we therefore have $\lim_{k\to\infty}\mu G_{nmk}=\mu G$, where $G=\bigcup_{k\in\mathbb{N}}G_{nmk}$. Next,

$$G = \bigcup\{E_{\sigma,\xi+1} : \sigma \in \mathbb{N}^m,\, \sigma(i) \le \alpha_n(i) \text{ for every } i < m\},$$

so

$$G \triangle G_{nm} \subseteq \bigcup\{E_{\sigma\xi} \setminus E_{\sigma,\xi+1} : \sigma \in S^*\}.$$

Since $\langle E_{\sigma\xi} \setminus E_{\sigma,\xi+1}\rangle_{\sigma\in S^*}$ is a countable family of negligible sets coded by $\langle \Theta_1(T_{\sigma\xi}, T_{\sigma,\xi+1})\rangle_{\sigma\in S^*}$, $G\triangle G_{nm}$ also is negligible and

$$\mu G_{nm} = \mu G = \lim_{k\to\infty} \mu G_{nmk}. \textbf{ Q}$$

Take the least $\alpha_n(m) \in \mathbb{N}$ such that $\mu G_{n,m,\alpha_n(m)} \ge \mu G_{nm} - 2^{-n-m}$, and continue.

(ii) Set

$$L_n = \{\alpha : \alpha \in \mathbb{N}^{\mathbb{N}},\, \alpha(i) \le \alpha_n(i) \text{ for every } i \in \mathbb{N}\}.$$

Then L_n is compact (561D), and $f[L_n] \subseteq A$. Also $f[L_n] \supseteq \bigcap_{m\in\mathbb{N}} G_{nm}$. **P** If $x \in \bigcap_{m\in\mathbb{N}} G_{nm}$, then for each $m \in \mathbb{N}$ let σ_m be the lexicographically first member of $\{\sigma : \sigma \in \mathbb{N}^m,\, \sigma(i) \le \alpha_n(i) \text{ for every } i < m\}$ such that $x \in E_{\sigma_m,\xi}$, and let $\beta_m \in \mathbb{N}^{\mathbb{N}}$ be such that $\sigma_m \subseteq \beta_m$ and $\beta_m(i) = 0$ for $i \ge m$. Then $\beta_m \in L_n$ for every m, so $\langle\beta_m\rangle_{m\in\mathbb{N}}$ has a cluster point $\alpha \in L_n$. **?** If $f(\alpha) \ne x$, we have an open neighbourhood U of $f(\alpha)$ such that $x \notin \overline{U}$. Let $m \in \mathbb{N}$ be such that $I_{\alpha\restriction m} \subseteq f^{-1}[U]$; then there is a $k \ge m$ such that $\alpha\restriction m = \beta_k\restriction m = \sigma_k\restriction m$. Now

$$x \in E_{\sigma_k,\xi} \subseteq E_{\sigma_k,0} \subseteq \overline{f[I_{\sigma_k}]} \subseteq \overline{f[I_{\alpha\restriction m}]} \subseteq \overline{U}. \textbf{ X}$$

So $x = f(\alpha) \in f[L_n]$. **Q**

But $\langle G_{nm}\rangle_{m\in\mathbb{N}}$ is codable, and $G_{n0} = E$, so we must have

$$\begin{aligned}\mu(E \setminus f[L_n]) &\le \mu(E \setminus G_{n0}) + \sum_{m=0}^{\infty} \mu(G_{nm} \setminus G_{n,m+1}) \\ &= \sum_{m=0}^{\infty} \mu(G_{nm} \setminus G_{n,m,\alpha_n(m)}) \le \sum_{m=0}^{\infty} 2^{-n-m} = 2^{-n+1}.\end{aligned}$$

(Of course $f[L_n]$ is compact, therefore closed, therefore measurable.)

(f) Set $K_n = f[L_n]$ for each n. Then $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence of compact subsets of A; because the K_n are resolvable, $F = \bigcup_{n\in\mathbb{N}} K_n$ is a codable Borel set. For each n,

$$\mu(E \setminus F) \le \mu(H_n \cap E \setminus K_n) \le 2^{-n+1};$$

so $E \setminus F$ is negligible. Thus E and $\langle K_n\rangle_{n\in\mathbb{N}}$ have the required properties.

Of course it now follows that $E \setminus A \subseteq E \setminus F$ is negligible, so that the completion of μ measures A.

563J Baire-coded measures Working from 562R, we can develop a theory of Baire measures on general topological spaces, as follows. If X is a topological space, and $\mathcal{B}\mathfrak{a}_c(X)$ its algebra of codable Baire sets, a **Baire-coded measure** on X will be a function $\mu : \mathcal{B}\mathfrak{a}_c(X) \to [0,\infty]$ such that $\mu\emptyset = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ for every disjoint codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}\mathfrak{a}_c(X)$.

563K Proposition (a) If X and Y are topological spaces, $f : X \to Y$ is a continuous function and μ is a Baire-coded measure on X, then $F \mapsto \mu f^{-1}[F] : \mathcal{B}\mathfrak{a}_c(Y) \to [0,\infty]$ is a Baire-coded measure on Y.

(b) Suppose that μ is a Baire-coded measure on a topological space X, and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$. Then

(i) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$;

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$;

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

(c) Let X be a topological space and M a non-empty upwards-directed family of Baire-coded measures on X. Set $\nu E = \sup_{\mu\in\mathrm{M}} \mu E$ for every codable Baire set $E \subseteq X$. Then ν is a codable Baire measure on X.

proof (a) Use 562R(b-iv).

(b) Recall that, by 562R(b-i), there must be a continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ such that $E_n = f^{-1}[F_n]$ for every n. By (a), $F \mapsto \mu f^{-1}[F] : \mathcal{B}_c(\mathbb{R}^{\mathbb{N}}) \to [0,\infty]$ is a Borel-coded measure on $\mathbb{R}^{\mathbb{N}}$. Applying 563Ba to $\langle F_n\rangle_{n\in\mathbb{N}}$, we get the result here.

(c) As 563E.

563L Proposition Suppose that X is a topological space; write $\mathcal{G}$ for the lattice of cozero subsets of X. Let $\mu : \mathcal{G} \to [0, \infty]$ be such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

$\mu G + \mu H = \mu(G \cup H) + \mu(G \cap H)$ for all G, $H \in \mathcal{G}$,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{G}$ and there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of continuous functions from X to $\mathbb{R}$ such that $G_n = \{x : f_n(x) \ne 0\}$ for every n,[2]

$\mu G = \sup\{\mu H : H \in \mathcal{G},\ H \subseteq G,\ \mu H < \infty\}$ for every $G \in \mathcal{G}$.

Then there is a Baire-coded measure on X extending μ; if μX is finite, then the extension is unique.

proof (a) Suppose to begin with that μX is finite.

(i) For each continuous $f : X \to \mathbb{R}^{\mathbb{N}}$, consider the functional $G \mapsto \mu f^{-1}[G]$ for open $G \subseteq \mathbb{R}^{\mathbb{N}}$. This satisfies the conditions of 563H. **P** Only the fourth requires attention. Fix a metric ρ defining the topology of $\mathbb{R}^{\mathbb{N}}$. If $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets in $\mathbb{R}^{\mathbb{N}}$ with union H, set

$$h_n(z) = \min(1, \rho(z, \mathbb{R}^{\mathbb{N}} \setminus H_n))$$

for $n \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathbb{N}}$, counting $\rho(z, \emptyset)$ as ∞ if necessary. In this case, setting $G_n = f^{-1}[H_n]$, $G_n = \{x : h_n f(x) > 0\}$ for each n; so

$$\mu f^{-1}[H] = \mu(\textstyle\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n = \lim_{n\to\infty} \mu f^{-1}[H_n]. \textbf{ Q}$$

There is therefore a unique Borel-coded measure ν_f on $\mathbb{R}^{\mathbb{N}}$ such that $\nu_f H = \mu f^{-1}[H]$ for every open set $G \subseteq \mathbb{R}^{\mathbb{N}}$.

(ii) If $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous and $F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$, then $\nu_f F = \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}$. **P** By 563Fd, ν_f is outer regular with respect to the open sets, so

$$\begin{aligned}\nu_f F &= \inf\{\nu_f H : H \subseteq \mathbb{R}^{\mathbb{N}} \text{ is open}, F \subseteq H\}\\ &= \inf\{\mu f^{-1}[H] : H \subseteq \mathbb{R}^{\mathbb{N}} \text{ is open}, F \subseteq H\} \ge \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}.\end{aligned}$$

In the other direction, if $G \in \mathcal{G}$ and $f^{-1}[F] \subseteq G$, take any $\epsilon > 0$. There is an open set $H \subseteq \mathbb{R}^{\mathbb{N}}$ such that $\mathbb{R}^{\mathbb{N}} \setminus H \subseteq F$ and $\nu_f(F \cap H) \le \epsilon$. But this means $G \cup f^{-1}[H] = X$ and

$$\mu G \ge \mu X - \mu f^{-1}[H] = \nu_f \mathbb{R}^{\mathbb{N}} - \nu_f H \ge \nu_f F - \epsilon.$$

As ϵ is arbitrary, $\nu_f F \le G$; as G is arbitrary, $\nu_f F \le \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}$. **Q**

(iii) This means that if we set $\nu E = \inf\{\mu G : E \subseteq G \in \mathcal{G}\}$ for $E \in \mathcal{B}\mathfrak{a}_c(X)$, we shall have $\nu f^{-1}[F] = \nu_f F$ whenever $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous and $F \in \mathcal{B}_c(X)$. It follows that ν is a Baire-coded measure on X. **P** Of course $\nu\emptyset = 0$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$, there are a continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of coded Borel sets in $\mathbb{R}^{\mathbb{N}}$ such that $E_n = f^{-1}[F_n]$ for every n, by 562R(b-i). Set $F'_n = F_n \setminus \bigcup_{i<n} F_i$ for $n \in \mathbb{N}$; then $\langle F'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence (562Ic), so

$$\nu(\textstyle\bigcup_{n\in\mathbb{N}} E_n) = \nu_f(\bigcup_{n\in\mathbb{N}} F'_n) = \sum_{n=0}^{\infty} \nu_f F'_n = \sum_{n=0}^{\infty} \nu E_n. \textbf{ Q}$$

Of course ν extends μ.

(iv) As for uniqueness, if ν' is any other Baire-coded measure on X extending μ, and $f : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous function, then $F \mapsto \nu' f^{-1}[F]$ is a Borel-coded measure on $\mathbb{R}^{\mathbb{N}}$ which agrees with ν_f on open sets and is therefore equal to ν_f; it follows at once that $\nu' = \nu$.

(b) For the general case, let $\mathcal{G}^f$ be $\{H : H \in \mathcal{G},\ \mu H < \infty\}$, and for $H \in \mathcal{G}^f$ define $\mu_H : \mathcal{G} \to [0, \infty[$ by setting $\mu_H G = \mu(G \cap H)$ for every $G \in \mathcal{G}$. Then μ_H satisfies all the conditions of the proposition. **P** Everything is elementary; for the hypothesis on non-decreasing sequences in $\mathcal{G}$, note that there is a continuous function $f : X \to \mathbb{R}$ such that $H = \{x : f(x) \ne 0\}$, so that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of real-valued continuous function defining a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$, then $\langle f_n \times f\rangle_{n\in\mathbb{N}}$ defines $\langle G_n \cap H\rangle_{n\in\mathbb{N}}$. **Q**

There is therefore a unique Baire-coded measure ν_H on X extending μ_H. Now if H, $H' \in \mathcal{G}^f$ and $H \subseteq H'$, $\nu_H E = \nu_{H'}(E \cap H)$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$. **P** The functional $E \mapsto \nu_{H'}(E \cap H)$ is a Baire-coded measure on X extending μ_H, so must be equal to ν_H. **Q** In particular, $\nu_H E \le \nu_{H'} E$ for every codable Baire set $E \subseteq X$.

Now set $\nu E = \sup\{\nu_H E : H \in \mathcal{G}^f\}$ for $E \in \mathcal{B}\mathfrak{a}_c(X)$. By 563Kc, ν is a Baire-coded measure on X; and by the final hypothesis of this proposition, ν extends μ.

563M Measure algebras If μ is either a Borel-coded measure or a Baire-coded measure, we can form the quotient Boolean algebra $\mathfrak{A} = \operatorname{dom}\mu/\{E : \mu E = 0\}$ and the functional $\bar\mu : \mathfrak{A} \to [0, \infty]$ defined by setting $\bar\mu E^\bullet = \mu E$ for every

[2]Observe that $\bigcup_{n\in\mathbb{N}} G_n$ is a cozero set, defined by $f : X \to \mathbb{R}$ where $f(x) = \sup_{n\in\mathbb{N}} \min(2^{-n}, |f_n(x)|)$ for each x.

$E \in \operatorname{dom}\mu$; as in 321H, $\bar{\mu}$ becomes a strictly positive additive functional from $\mathfrak{A}$ to $[0,\infty]$. As in §323, we have a topology and uniformity on $\mathfrak{A}$ defined by the pseudometrics $(a,b) \mapsto \bar{\mu}(c \cap (a \triangle b))$ for $c \in \mathfrak{A}$ of finite measure; if μ is semi-finite, the topology is Hausdorff.

563N Theorem Let X be a second-countable space, and μ a codably σ-finite Borel-coded measure on X. Let $\mathfrak{A}$ and $\bar{\mu}$ be as in 563M. Then $\mathfrak{A}$ is complete for its measure-algebra uniformity, therefore Dedekind complete.

proof (a) There is a codable sequence of sets of finite measure covering X. By 562Nb, we can find a codably Borel equivalent second-countable topology $\mathfrak{S}$ on X, generated by a countable algebra $\mathcal{E}$ of subsets of X, for which all these sets are open, so that μ becomes locally finite. Let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$. Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of μ.

(b) $\{H^\bullet : H \in \mathcal{E}\}$ is dense in $\mathfrak{A}$ for the measure-algebra topology. **P** Suppose that a, $c \in \mathfrak{A}$, $\epsilon > 0$ and $\bar{\mu}c < \infty$. Express a as $E^\bullet$ and c as $F^\bullet$ where E, $F \in \mathcal{B}_c(X)$. By 563Fd, there is a $G \in \mathfrak{S}$ such that $E \subseteq G$ and $\mu(G \setminus E) \le \epsilon$. Setting $G_n = \bigcup\{H_i : i \le n,\ H_i \subseteq G\}$, $\langle G_n \cap F\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union $G \cap F$, so there is an $n \in \mathbb{N}$ such that $\mu((G \setminus G_n) \cap F) \le \epsilon$. In this case

$$\bar{\mu}(c \cap (a \triangle G_n^\bullet)) = \mu(F \cap (E \triangle G_n)) \le \mu(F \cap (G \setminus G_n)) + \mu(G \setminus E) \le 2\epsilon,$$

while $G_n \in \mathcal{E}$. As a, c and ϵ are arbitrary, we have the result. **Q**

(c) $\mathfrak{A}$ is complete for the measure-algebra uniformity. **P** Set $\tilde{H}_n = \bigcup\{H_i : i \le n,\ \mu H_i < \infty\}$, $c_n = \tilde{H}_n^\bullet$ for each n. Let $\mathcal{F}$ be a Cauchy filter on $\mathfrak{A}$ for the measure-algebra uniformity. For each $n \in \mathbb{N}$, there is an $A \in \mathcal{F}$ such that $\bar{\mu}(c_n \cap (a \triangle b)) \le 2^{-n}$ for all a, $b \in A$; there is a $b_0 \in A$; and there is an $m \in \mathbb{N}$ such that $\bar{\mu}(c_n \cap (b_0 \triangle H_m^\bullet)) \le 2^{-n}$, so that

$$\{a : \bar{\mu}(c_n \cap (a \triangle H_m^\bullet)) \le 2^{-n+1}\} \in \mathcal{F}. \tag{*}$$

Let m_n be the first m for which (*) is true, and set $d_n = H_{m_n}^\bullet$. Note that

$$\bar{\mu}(c_i \cap (d_{i+1} \triangle d_i)) \le 3 \cdot 2^{-i}$$

for each i, because there must be an $a \in \mathfrak{A}$ such that $\bar{\mu}(c_i \cap (a \triangle d_i)) \le 2^{-i+1}$ and $\bar{\mu}(c_{i+1} \cap (a \triangle d_{i+1})) \le 2^{-i}$.

Set $E = \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} H_{m_i}$; because $\langle \bigcup_{i\ge n} H_{m_i}\rangle_{n\in\mathbb{N}}$ is codable, $E \in \mathcal{B}_c(X)$. Set $d = E^\bullet$. If $n \in \mathbb{N}$, then

$$E \triangle H_{m_n} \subseteq \bigcup_{i\ge n} H_{m_{i+1}} \triangle H_{m_i}$$

and $\langle \tilde{H}_n \cap (H_{m_{i+1}} \triangle H_{m_i})\rangle_{i\in\mathbb{N}}$ is codable, so

$$\begin{aligned}\bar{\mu}(c_n \cap (d \triangle d_n)) = \mu(\tilde{H}_n \cap (E \triangle H_{m_n})) &\le \sum_{i=n}^\infty \mu(\tilde{H}_n \cap (H_{m_{i+1}} \triangle H_{m_i}))\\ &\le \sum_{i=n}^\infty 3 \cdot 2^{-i} = 6 \cdot 2^{-n}.\end{aligned}$$

Take any $c \in \mathfrak{A}$ such that $\bar{\mu}c$ is finite, and $\epsilon > 0$. Express c as $F^\bullet$, where $\mu F < \infty$. Then $\langle F \cap \tilde{H}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union F, so there is an $n \in \mathbb{N}$ such that $\mu(F \setminus \tilde{H}_n) \le \epsilon$ and $2^{-n} \le \epsilon$. Now

$$\begin{aligned}\{a : \bar{\mu}(c \cap (a \triangle d)) \le 9\epsilon\} &\supseteq \{a : \bar{\mu}(c_n \cap (a \triangle d)) \le 8\epsilon\}\\ &\supseteq \{a : \bar{\mu}(c_n \cap (a \triangle d_n)) \le 2\epsilon\} \in \mathcal{F}.\end{aligned}$$

As c and ϵ are arbitrary, $\mathcal{F} \to d$ for the measure-algebra topology; as $\mathcal{F}$ is arbitrary, $\mathfrak{A}$ is complete. **Q**

(d) Now suppose that $A \subseteq \mathfrak{A}$ is a non-empty set, and B the family of its upper bounds, so that B is downwards-directed. As in 323D, the filter $\mathcal{F}(B\downarrow)$ generated by $\{B \cap [0,b] : b \in B\}$ is Cauchy for the measure-algebra uniformity, so has a limit, which is $\inf B = \sup A$. As A is arbitrary, $\mathfrak{A}$ is Dedekind complete.

563X Basic exercises (a) Let X, Y be second-countable spaces, μ a Borel-coded measure on X, and $f : X \to Y$ a codable Borel function. Show that $F \mapsto \mu f^{-1}[F] : \mathcal{B}_c(Y) \to [0,\infty]$ is a Borel-coded measure on Y.

(b) Let X be a regular second-countable space and μ a locally finite Borel-coded measure on X. Show that for every $E \in \mathcal{B}_c(X)$ there are an F_σ set $F \subseteq E$ and a G_δ set $H \supseteq E$ such that $\mu(H \setminus F) = 0$.

(c) Let X be a regular second-countable space. Show that a function μ is a codable Borel measure on X iff it is a codable Baire measure on X. (*Hint*: 562Xk, 562Xl.)

(d) Let X be a topological space. Show that any semi-finite Baire-coded measure on X is inner regular with respect to the closed sets.

563Z Problem Suppose we define 'probability space' in the conventional way, following literally the formulations in 111A, 112A and 211B. Is it relatively consistent with ZF to suppose that every probability space is purely atomic in the sense of 211K?

563 Notes and comments The arguments above are generally drawn from those used earlier in this treatise; the new discipline required is just to systematically respect the self-denying ordinance renouncing the axiom of choice. This does involve us in deeper analyses at a number of points. In 563Dc, for instance, we need functions π, π' defined on $\mathcal{T} \times \mathbb{N}$, not $\mathcal{B}_c(X) \times \mathbb{N}$, because the rank function of $\mathcal{T}$ gives us a foundation for induction. (In 563Db we can use a function π^* defined on $\mathfrak{T} \times \mathbb{N}$, but this is because we have canonical codes for open sets.) In 563I we can no longer assume the existence of measurable envelopes, let alone a whole family of them as used in the standard proof in 431A, and have to find another construction, watching carefully to make sure that we get not only a countable ordinal ξ but a codable family of sets $E_{\sigma\eta}$ leading to the measurable envelopes $E_{\sigma\xi}$; back in 561A, there was a moment when we needed to resist the temptation to suppose that a sequence in ω_1 must have a supremum in ω_1.

Note that we have to distinguish between 'negligible' and 'outer measure zero'. The natural meaning of the latter is 'for every $\epsilon > 0$ there is a measurable set $E \supseteq A$ with $\mu E \leq \epsilon$'. Even for outer regular measures, when a set of outer measure zero must be included in open sets of small measure, we cannot be sure that there is a sequence of such sets from which we can define a set of measure zero including A (565Xb).

In 563K I have kept the proofs short by quoting results from earlier in the section. But you may find it illuminating to look for a list of properties of codable families of codable Baire sets which would support formally independent proofs.

In 563M-563N I am taking care to avoid the phrase 'measure algebra' in the formal exposition. The reason is that the definition in §321 demands a Dedekind σ-complete algebra, and in the generality of 563M there is no reason to suppose that this will be satisfied. In the special context of 563N, of course, there is no difficulty.

There is something I ought to point out here. The problem is not that the principal arguments of §§111-113 and §§121-123 depend on the axiom of choice. If you wish, you can continue to define 'σ-algebra', 'measure', 'outer measure', 'measurable function' and 'integral' with the same forms of words as used in Volume 1, and the basic theorems, up to and including the convergence theorems, will still be true. The problem is that on these definitions the formulae of §§114-115 may not give an outer measure, and we may have nothing corresponding to Lebesgue measure. It does not quite follow that every probability space is purely atomic (there is a question here: see 563Z), but clearly we are not going to get a theory which can respond to any of the basic challenges dealt with in Volume 2 (Fundamental Theorem of Calculus, geometric measure theory, probability distributions, Fourier series), and I think it more useful to develop a new structure which can carry an effective version of the Lebesgue theory (see §565).

564 Integration without choice

I come now to the problem of defining an integral with respect to a Borel- or Baire-coded measure. Since a Borel-coded measure can be regarded as a Baire-coded measure on a second-countable space (562S), I will give the basic results in terms of the wider class. I seek to follow the general plan of Chapter 12, starting from simple functions and taking integrable functions to be almost-everywhere limits of sequences of simple functions (564A); the concept of 'virtually measurable' function has to be re-negotiated (564Ab). The basic convergence theorems from §123 are restricted but recognisable (564F). We also have versions of two of the representation theorems from §436 (564H, 564I).

There is a significant change when we come to the completeness of L^p spaces (564K) and the Radon-Nikodým theorem (564L), where it becomes necessary to choose sequences, and we need a well-orderable dense set of functions to pick from. Subject to this, we have workable notions of conditional expectation operator (564Mc) and product measures (564N, 564O).

564A Definitions (a) Given a topological space X and a Baire-coded measure μ on X (563J), I will write $\mathcal{B}\mathfrak{a}_c(X)^f$ for the ring of codable Baire sets of finite measure; $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ will be the linear subspace of $\mathbb{R}^X$ generated by $\{\chi E : E \in \mathcal{B}\mathfrak{a}_c(X)^f\}$ (see 122Ab, 361D[3]). Then S is a Riesz subspace of $\mathbb{R}^X$, and also an f-algebra in the sense of 352W.

[3] §§361-362 are written on a general assumption of AC. The only essential use of it to begin with, however, is in asserting that an arbitrary Boolean ring can be faithfully represented as a ring of sets; and even that can be dispensed with for a while if we work a little harder, as in 361Ya.

(b) I will write $\mathcal{L}^0$ for the space of real-valued functions f defined almost everywhere in X such that there is a codable Baire function $g : X \to \mathbb{R}$ such that $f =_{\text{a.e.}} g$.

(c) Let $\int : S \to \mathbb{R}$ be the positive linear functional defined by saying that $\int \chi E = \mu E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)^f$. (The arguments of 361E-361G still apply, so there is such a functional.)

(d) $\mathcal{L}^1$ will be the set of those real-valued functions f defined almost everywhere in X for which there is a codable sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in S converging to f almost everywhere and such that $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n| < \infty$; I will call such functions **integrable**.

564B Lemma Let X be a topological space and μ a Baire-coded measure on X.

(a) $\mathcal{L}^1 \subseteq \mathcal{L}^0$.

(b) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-increasing codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ and $\lim_{n\to\infty} h_n(x) = 0$ for almost every x, then $\lim_{n\to\infty} \int h_n = 0$.

(c) If $\langle h_n \rangle_{n\in\mathbb{N}}$ and $\langle h'_n \rangle_{n\in\mathbb{N}}$ are two codable sequences in S such that $\lim_{n\to\infty} h_n$ and $\lim_{n\to\infty} h'_n$ are defined and equal almost everywhere, and $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n|$ and $\sum_{n=0}^{\infty} \int |h'_{n+1} - h'_n|$ are both finite, then $\lim_{n\to\infty} \int h_n$ and $\lim_{n\to\infty} \int h'_n$ are defined and equal.

(d) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in S and $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n|$ is finite, then $\langle h_n \rangle_{n\in\mathbb{N}}$ converges almost everywhere. In particular, if $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in S and $\sup_{n\in\mathbb{N}} \int h_n$ is finite, $\langle h_n \rangle_{n\in\mathbb{N}}$ converges a.e.

(e) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in S^+ and $\liminf_{n\to\infty} \int h_n = 0$, then $\liminf_{n\to\infty} h_n = 0$ a.e.

proof (a) If $f \in \mathcal{L}^1$, there is a codable sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in S converging almost everywhere to f. Now 562R(c-ii) tells us that there is a codable Baire function g equal to $\lim_{n\to\infty} h_n$ wherever this is defined as a real number, so that $f =_{\text{a.e.}} g$ and $f \in \mathcal{L}^0$.

(b) Set $E = \{x : h_0(x) > 0\}$. Take any $\epsilon > 0$. For each $n \in \mathbb{N}$ set $E_n = \{x : h_n(x) > \epsilon\}$. Then $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing codable sequence in $\mathcal{B}\mathfrak{a}_c(X)^f$, and $\bigcap_{n\in\mathbb{N}} E_n \subseteq \{x : \lim_{n\to\infty} h_n(x) \ne 0\}$ is negligible; also E_0 has finite measure. Accordingly $\lim_{n\to\infty} \mu E_n = 0$ (563K(b-iii)). But

$$h_n \le \|h_0\|_\infty \chi E_n + \epsilon \chi E, \quad \int h_n \le \|h_0\|_\infty \mu E_n + \epsilon \mu E$$

for every n, so $\limsup_{n\to\infty} \int h_n \le \epsilon \mu E$. As ϵ is arbitrary, $\lim_{n\to\infty} \int h_n = 0$.

(c) Since

$$\sum_{n=0}^{\infty} \left|\int h_{n+1} - \int h_n\right| \le \sum_{n=0}^{\infty} \int |h_{n+1} - h_n| < \infty,$$

the limit $\lim_{n\to\infty} \int h_n$ is defined in $\mathbb{R}$. Similarly, $\lim_{n\to\infty} \int h'_n$ is defined. To see that the limits are equal, set $g_n = h_n - h'_n$ for each n, so that $\lim_{n\to\infty} g_n = 0$ a.e. and $\sum_{n=0}^{\infty} \int |g_{n+1} - g_n| < \infty$. Then $\int |g_n| \le \sum_{m=n}^{\infty} \int |g_{m+1} - g_m|$ for every n. **P** For $k \ge n$, set $f_k = (|g_n| - \sum_{m=n}^{k} |g_{m+1} - g_m|)^+$. Then $0 \le f_k \le |g_{k+1}|$ for each k, so $\langle f_k \rangle_{k\ge n}$ is a non-increasing codable sequence in S converging to 0 almost everywhere. By (b), $\lim_{n\to\infty} \int f_k = 0$; but $\int f_k \ge \int |g_n| - \sum_{m=n}^{k} \int |g_{m+1} - g_m|$ for every k. **Q**

Consequently

$$|\lim_{n\to\infty} \int h_n - \lim_{n\to\infty} \int h'_n| = \lim_{n\to\infty} \left|\int h_n - \int h'_n\right| \le \lim_{n\to\infty} \int |g_n| = 0$$

as required.

(d) For $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be the least integer such that $\sum_{i=n_k}^{\infty} \int |h_{i+1} - h_i| \le 2^{-k}$, and for $m \ge n_k$ set

$$G_{km} = \{x : \sum_{i=n_k}^{m} |h_{i+1}(x) - h_i(x)| \ge 1\}.$$

Then $\mu G_{km} \le 2^{-k}$, because $\chi G_{km} \le \sum_{i=n_k}^{m} |h_{i+1} - h_i|$, so $\mu G_k \le 2^{-k}$, where $G_k = \bigcup_{m\ge n_k} G_{km}$, by 563K(b-i). (Of course we have to check that all the sequences of sets and functions involved here are codable.) Accordingly, setting $E = \bigcap_{k\in\mathbb{N}} G_k$, $\mu E = 0$. But observe that if $x \in X \setminus E$ there is a $k \in \mathbb{N}$ such that $x \notin G_k$ and $\sum_{i=n_k}^{\infty} |h_{i+1}(x) - h_i(x)| \le 1$; in which case $\lim_{n\to\infty} h_n(x)$ is defined.

(e) For $k \in \mathbb{N}$ let n_k be the least integer such that $n_k > n_i$ for $i < k$ and $\int h_{n_k} \le 4^{-n_k}$. Set $G_k = \{x : h_{n_k}(x) \ge 2^{-k}\}$; then $\mu G_k \le 2^{-k}$ and $\langle G_k \rangle_{k\in\mathbb{N}}$ is codable. So $\mu(\bigcup_{k\ge n} G_k) \le 2^{-n+1}$ for every n and $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} G_k$ is negligible. But $E \supseteq \{x : \liminf_{n\to\infty} h_n(x) > 0\}$.

564C Definition Let X be a topological space and μ a Baire-coded measure on X. For $f \in \mathcal{L}^1$, define its integral $\int f$ by saying that $\int f = \lim_{n\to\infty} \int h_n$ whenever $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ converging to f almost everywhere and $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n|$ is finite. By 564Bb, this definition is sound; and clearly it is consistent with the previous definition of the integral on S.

564D Lemma Let X be a topological space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a codable sequence of codable Baire functions on X. Let $\langle q_i\rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,\infty[$, starting with $q_0=0$. Set

$$f'_n(x)=\max\{q_i : i\le n,\, q_i\le\max(0,f_n(x))\}$$

for $n\in\mathbb{N}$ and $x\in X$. Then $\langle f'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions.

proof Take a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$ and the corresponding interpretation $\phi:\mathcal{T}\to\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ of Borel codes, as in 562A, and $\tilde\phi$ the corresponding interpretation of codes for real-valued codable Borel functions, as in 562L. By 562R(c-i), there are a continuous function $g:X\to\mathbb{R}^{\mathbb{N}}$ and a sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ of codes such that $f_n=\tilde\phi(\tau_n)\circ g$ for every n. We need a sequence $\langle\tau'_n\rangle_{n\in\mathbb{N}}$ of codes such that

$$\begin{aligned}\phi(\tau'_n(\alpha))&=\bigcup_{\substack{i\le n\\ q_i>\alpha}}\bigcap_{\substack{j\in\mathbb{N}\\ q_j<q_i}}\phi(\tau_n(q_j))\text{ if }\alpha\ge 0,\\ &=X\text{ if }\alpha<0;\end{aligned}$$

and this is easy to build using complementation and general union operators as in 562B. Now take $f'_n=\tilde\phi(\tau'_n)\circ g$ for each n.

564E Theorem Let X be a topological space and μ a Baire-coded measure on X.

(a)(i) If f, $g\in\mathcal{L}^0$ and $\alpha\in\mathbb{R}$, then $f+g$, αf, $|f|$ and $f\times g$ belong to $\mathcal{L}^0$.

(ii) If $h:\mathbb{R}\to\mathbb{R}$ is a codable Borel function, $hf\in\mathcal{L}^0$ for every $f\in\mathcal{L}^0$.

(b) If f, $g\in\mathcal{L}^1$ and $\alpha\in\mathbb{R}$, then

(i) $f+g$, αf and $|f|$ belong to $\mathcal{L}^1$;

(ii) $\int f+g=\int f+\int g$, $\int\alpha f=\alpha\int f$;

(iii) if $f\le_{\text{a.e.}}g$ then $\int f\le\int g$.

(c)(i) If $f\in\mathcal{L}^0$, $g\in\mathcal{L}^1$ and $|f|\le_{\text{a.e.}}g$, then $f\in\mathcal{L}^1$.

(ii) If $E\in\mathcal{B}\mathfrak{a}_c(X)$ and $\chi E\in\mathcal{L}^1$ then μE is finite.

proof (a)(i) Use 562R(c-iii).

(ii) We know that f is equal almost everywhere to a product $f'g$ where $g:X\to\mathbb{R}^{\mathbb{N}}$ is continuous and $f':\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ is a codable Borel function. Now hf' is a codable Borel function, by 562Kb, so $hf'g$ is a codable Baire function and $hf=_{\text{a.e.}}hf'g$ belongs to $\mathcal{L}^0$.

(b)(i)-(ii) These proceed by the same arguments as in (a-i). To deal with $|f|$, we need to note that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is any codable sequence in $S=S(\mathcal{B}\mathfrak{a}_c(X)^f)$ then $\sum_{n=0}^\infty\int||h_{n+1}|-|h_n||\le\sum_{n=0}^\infty\int|h_{n+1}-h_n|$.

(iii) If $f\le_{\text{a.e.}}g$, let $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ be codable sequences in S converging almost everywhere to f, g respectively, and such that $\sum_{n=0}^\infty\int|f_{n+1}-f_n|$ and $\sum_{n=0}^\infty\int|g_{n+1}-g_n|$ are finite. Set $h_n=f_n\wedge g_n$ for each n. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is codable, $f=_{\text{a.e.}}\lim_{n\to\infty}h_n$, $\sum_{n=0}^\infty\int|h_{n+1}-h_n|<\infty$ and

$$\int f=\lim_{n\to\infty}\int h_n\le\lim_{n\to\infty}\int g_n=\int g.$$

(c)(i) Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $g=_{\text{a.e.}}\lim_{n\to\infty}h_n$ and $\sum_{n=0}^\infty\int|h_{n+1}-h_n|$ is finite. Set $h'_n=\sup_{i\le n}h_i^+$ for each n; then $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in S and $\sup_{n\in\mathbb{N}}\int h'_n$ is finite, while $|f|\le_{\text{a.e.}}g\le_{\text{a.e.}}\sup_{n\in\mathbb{N}}h'_n$. There is a codable Baire function $\tilde f$ such that $f=_{\text{a.e.}}\tilde f$. Now $\tilde f^+$ is a codable Baire function, so $\langle\tilde f^+\wedge h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of non-negative codable Baire functions.

For each $n\in\mathbb{N}$ consider h''_n where

$$h''_n(x)=\max\{q_i : i\le n,\, q_i\le\max(0,(\tilde f^+\wedge h'_n)(x)\}$$

for $x\in X$. By 564D, $\langle h''_n\rangle_{n\in\mathbb{N}}$ is a codable sequence; it is non-decreasing and converges a.e. to $\tilde f^+=_{\text{a.e.}}f^+$. Because $0\le h''_n\le h'_n$, $h''_n\in S$ for each n, and $\sup_{n\in\mathbb{N}}\int h''_n\le\sup_{n\in\mathbb{N}}\int h'_n$ is finite; so 564Bd tells us that f^+ is integrable.

Similarly, f^- is integrable, so f is integrable.

(ii) Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $\chi E=_{\text{a.e.}}\lim_{n\to\infty}h_n$ and $\sum_{n=0}^\infty\int|h_{n+1}-h_n|$ is finite. Set $h'_n=\sup_{i\le n}h_i$ for each n; then $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S and $\sup_{n\in\mathbb{N}}\int h'_n$ is finite. Set $E_n=\{x:h'_n(x)>\frac{1}{2}\}$; then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in $\mathcal{B}\mathfrak{a}_c(X)^f$. Also $E\setminus\bigcup_{n\in\mathbb{N}}E_n$ is negligible, so

$$\mu E\le\mu(\textstyle\bigcup_{n\in\mathbb{N}}E_n)=\sup_{n\in\mathbb{N}}\mu E_n=\sup_{n\in\mathbb{N}}\int\chi E_n\le 2\sup_{n\in\mathbb{N}}\int h'_n$$

is finite.

564F I come now to versions of the fundamental convergence theorems.

Theorem Let X be a topological space and μ a Baire-coded measure on X. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable codable Baire functions on X.

(a) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\gamma=\sup_{n\in\mathbb{N}}\int f_n$ is finite, then $f=\lim_{n\to\infty}f_n$ is defined a.e. and is integrable, and $\int f=\gamma$.

(b) If every f_n is non-negative and $\liminf_{n\to\infty}\int f_n$ is finite, then $f=\liminf_{n\to\infty}f_n$ is defined a.e. and is integrable, and $\int f\le\liminf_{n\to\infty}\int f_n$.

(c) Suppose that there is a $g\in\mathcal{L}^1$ such that $|f_n|\le_{\text{a.e.}}g$ for every n, and $f=\lim_{n\to\infty}f_n$ is defined a.e. Then $\int f$ and $\lim_{n\to\infty}\int f_n$ are defined and equal.

(d) If every f_n is integrable and $\sum_{n=0}^{\infty}\int|f_{n+1}-f_n|$ is finite, then $f=\lim_{n\to\infty}f_n$ is defined a.e., and $\int f$ and $\lim_{n\to\infty}\int f_n$ are defined and equal.

(e) If every f_n is integrable and $\sum_{n=0}^{\infty}\int|f_n|$ is finite, then $f=\sum_{n=0}^{\infty}f_n$ is defined a.e., and $\int f$ and $\sum_{n=0}^{\infty}\int f_n$ are defined and equal.

proof (a) Replacing f_n by f_n-f_0 for each n, we may suppose that $f_n\ge 0$ for each n. Let $\langle q_i\rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,\infty[$ and set

$$h_n(x)=\max\{q_i:i\le n,\ q_i\le\max(0,f_n(x))\}$$

for each $x\in X$. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions (use 564D again). Moreover, h_n takes only finitely many values, all non-negative, and for $\alpha>0$ the set $E_{n\alpha}=\{x:h_n(x)>\alpha\}$ is a codable Baire set such that $\chi E_{n\alpha}\le_{\text{a.e.}}\frac{1}{\alpha}f_n$; by 564Ec, $E_{n\alpha}$ has finite measure; as α is arbitrary, $h_n\in S$.

Now $\langle h_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and $\int h_n\le\int f_n\le\gamma$ for every n; so by 564Bd $\langle h_n\rangle_{n\in\mathbb{N}}$ converges almost everywhere to an integrable function f_1, with $\int f_1\le\gamma$. Of course $f_1=_{\text{a.e.}}\lim_{n\to\infty}f_n=f$; as $f\ge_{\text{a.e.}}f_n$ for every n, $\int f=\int f_1=\gamma$ exactly.

(b) By 562R(c-i) and 562Mc, $\langle f'_n\rangle_{n\in\mathbb{N}}$ is codable, where $f'_n=\inf_{m\ge n}f_m$ for every n, and of course

$$\int f'_n\le\inf_{m\ge n}\int f_m\le\liminf_{m\to\infty}\int f_m$$

for every n. Now $\langle f'_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so (a) tells us that $\int\liminf_{n\to\infty}f_n=\int\lim_{n\to\infty}f'_n$ is defined and equal to $\lim_{n\to\infty}\int f'_n\le\liminf_{n\to\infty}\int f_n$.

(c) Let g' be a codable Baire function such that $g'=_{\text{a.e.}}g$, and set $f'_n=\operatorname{med}(-g',f_n,g')$ for each n; once again, 562R(c-i) and the ideas of 562Mc show that $\langle g'+f'_n\rangle_{n\in\mathbb{N}}$ is codable. So we can use (b) to see that $\int\liminf_{n\to\infty}g'+f'_n$ is defined and is at most $\liminf_{n\to\infty}\int g'+f'_n=\int g'+\liminf_{n\to\infty}\int f_n$. Subtracting g', we get $\int\liminf_{n\to\infty}f'_n\le\liminf_{n\to\infty}\int f_n$. Similarly, $\int\limsup_{n\to\infty}f'_n\ge\limsup_{n\to\infty}\int f_n$.

Once again, the sequences $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle f'_n\rangle_{n\in\mathbb{N}}$, $\langle|f'_n-f_n|\rangle_{n\in\mathbb{N}}$ and $\langle\{x:f'_n(x)\ne f_n(x)\}\rangle_{n\in\mathbb{N}}$ are all codable. Since all the sets $\{x:f'_n(x)\ne f_n(x)\}$ are negligible, so is their union; but this means that $\lim_{n\to\infty}f'_n=_{\text{a.e.}}\lim_{n\to\infty}f_n$ is defined almost everywhere. So (just as in 123C) the integrals are sandwiched, and $\int\lim_{n\to\infty}f_n=\lim_{n\to\infty}\int f_n$.

(d) Of course $\sum_{n=0}^{\infty}|\int f_{n+1}-\int f_n|$ is finite, so $\gamma=\lim_{n\to\infty}\int f_n$ is defined. Next, (a) tells us that $g=|f_0|+\sum_{n=0}^{\infty}|f_{n+1}-f_n|$ is defined a.e. and is integrable (of course this depends on our having a procedure – induction is allowed – for building a sequence of Baire codes representing $\langle|f_0|+\sum_{i=0}^{n}|f_{i+1}-f_i|\rangle_{n\in\mathbb{N}}$ out of a sequence of codes representing $\langle f_n\rangle_{n\in\mathbb{N}}$). Since $\lim_{n\to\infty}f_n(x)$ is defined whenever $g(x)$ is defined and finite, which is almost everywhere, and $|f_n|\le_{\text{a.e.}}g$ for every n, (c) gives the result we're looking for.

(e) Similarly, $\langle\sum_{i=0}^{n}f_i\rangle_{n\in\mathbb{N}}$ is codable and we can apply (d).

564G Integration over subsets: Proposition Let X be a topological space and μ a Baire-coded measure on X.

(a) If $f\in\mathcal{L}^1$, the functional $E\mapsto\int f\times\chi E:\mathcal{B}\mathfrak{a}_c(X)\to\mathbb{R}$ is additive and truly continuous with respect to μ.[4]

(c) If f, $g\in\mathcal{L}^1$, then $f\le_{\text{a.e.}}g$ iff $\int f\times\chi E\le\int g\times\chi E$ for every $E\in\mathcal{B}\mathfrak{a}_c(X)$. So $f=_{\text{a.e.}}g$ iff $\int f\times\chi E=\int g\times\chi E$ for every $E\in\mathcal{B}\mathfrak{a}_c(X)$.

proof (a) Because $\chi:\mathcal{B}\mathfrak{a}_c(X)\to\mathcal{L}^0$ is additive, $E\mapsto\int f\times\chi E$ is additive. To see that it is truly continuous, take $\epsilon>0$. There is a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $S=S(\mathcal{B}\mathfrak{a}_c(X)^f)$ such that $\sum_{n=0}^{\infty}\int|h_{n+1}-h_n|<\infty$ and $f=_{\text{a.e.}}\lim_{n\to\infty}h_n$. For each n,

$$\int|f-h_n|=\lim_{m\to\infty}\int|h_m-h_n|\le\sum_{m=n}^{\infty}\int|h_{m+1}-h_m|,$$

so there is an n such that $\int|f-h_n|\le\frac{1}{2}\epsilon$. Set $E=\{x:h_n(x)\ne 0\}$ and $\delta=\epsilon/(1+2\|h_n\|_\infty)$. Then E has finite measure. If $F\in\mathcal{B}_c(X)$ and $\mu(E\cap F)\le\delta$, then

[4] The definition of 'truly continuous' in 232Ab assumed that μ was defined on a σ-algebra. I hope it is obvious that the same formulation makes sense when the domain of μ is any Boolean algebra.

$$|\int f\times\chi F|\le\int|f-h_n|+\int|h_n|\times\chi F\le\frac{1}{2}\epsilon+\|h_n\|_\infty\mu(E\cap F)\le\epsilon.$$

As ϵ is arbitrary, the functional is truly continuous.

(b)(i) If $f\le_{\text{a.e.}}g$ and $E\in\mathcal{B}_c(X)$, then $f\times\chi E\le_{\text{a.e.}}g\times\chi E$ so $\int f\times\chi E\le\int g\times\chi E$.

(ii) If $\int f\times\chi E\le\int g\times\chi E$ for every $E\in\mathcal{B}_c(X)$, let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $f-g=_{\text{a.e.}}\lim_{n\to\infty}h_n$ and $\sum_{n=0}^\infty\int|h_{n+1}-h_n|<\infty$. For $k\in\mathbb{N}$ let n_k be the least integer such that $\sum_{m=n_k}^\infty\int|h_{m+1}-h_m|\le 2^{-k}$. For m, $k\in\mathbb{N}$ set $E_{mk}=\{x:h_{n_k}(x)\ge 2^{-m}\}$. Then

$$\begin{aligned}\mu E_{mk}&\le 2^m\int h_{n_k}\times\chi E_{mk}=2^m\int(h_{n_k}-f+g)\times\chi E_{mk}\\&=2^m\lim_{i\to\infty}\int(h_{n_k}-h_i)\times\chi E_{mk}\le 2^m\lim_{i\to\infty}\int|h_{n_k}-h_i|\le 2^{m-k}.\end{aligned}$$

Also $\langle E_{mk}\rangle_{m,k\in\mathbb{N}}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$, so $\mu(\bigcup_{k\ge 2m}E_{mk})\le 2^{-m+1}$ for every m and $\mu E=0$, where

$$E=\bigcup_{l\in\mathbb{N}}\bigcap_{m\ge l}\bigcup_{k\ge 2m}E_{mk}.$$

But for $x\in X\setminus E$, $\limsup_{k\to\infty}h_{n_k}(x)\le 0$. Since $f-g=_{\text{a.e.}}\lim_{k\to\infty}h_{n_k}$, $f\le_{\text{a.e.}}g$.

564H Theorem Let X be a topological space, and $f:C_b(X)\to\mathbb{R}$ a sequentially smooth positive linear functional, where $C_b(X)$ is the space of bounded continuous real-valued functions on X. Then there is a totally finite Baire-coded measure μ on X such that $f(u)=\int u\,d\mu$ for every $u\in C_b(X)$.

proof (a) For cozero sets $G\subseteq X$ set $\mu_0G=\sup\{f(u):u\in C_b(X),\ 0\le u\le\chi G\}$. Then $\mu_0G=\lim_{n\to\infty}f(u_n)$ whenever $G\subseteq X$ is a cozero set and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $C_b(X)^+$ with supremum χG in $\mathbb{R}^X$. **P** Setting $\gamma=\sup_{n\in\mathbb{N}}f(nu\wedge\chi X)$, then of course

$$\mu_0G\ge\gamma=\lim_{n\to\infty}f(u_n).$$

On the other hand, if $v\in C_b(X)$ and $0\le v\le\chi G$, $\langle(v-u_n)^+\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence converging to $\mathbf{0}$ pointwise, so

$$f(v)\le f(u_n)+f(v-u_n)^+\le\gamma+f(v-u_n)^+\to\gamma$$

as $n\to\infty$. As v is arbitrary, $\mu_0G\le\gamma$. **Q**

(b) It follows that μ_0 satisfies the conditions of 563L. **P** Of course $\mu_0\emptyset=0$ and μ_0 is monotonic. If G, $H\subseteq X$ are cozero sets, express them as $\{x:u(x)>0\}$ and $\{x:v(x)>0\}$ where u, $v\in C_b(X)^+$. Set $u_n=nu\wedge\chi X$, $v_n=nv\wedge\chi X$ for each n; then $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ are non-decreasing sequences in $C_b(X)^+$ converging pointwise to χG, χH respectively. Now $\langle u_n\wedge v_n\rangle_{n\in\mathbb{N}}$ and $\langle u_n\vee v_n\rangle_{n\in\mathbb{N}}$ are also non-decreasing sequences in $C_b(X)^+$ converging to $\chi(G\cap H)$, $\chi(G\cup H)$; so (a) tells us that

$$\begin{aligned}\mu_0(G\cup H)+\mu_0(G\cap H)&=\lim_{n\to\infty}f(u_n\wedge v_n)+\lim_{n\to\infty}f(u_n\vee v_n)\\&=\lim_{n\to\infty}f(u_n\wedge v_n+u_n\vee v_n)\\&=\lim_{n\to\infty}f(u_n+v_n)=\mu_0G+\mu_0H.\end{aligned}$$

As for the last condition in 563L, let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of cozero sets such that there is a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $C(X)$ such that $G_n=\{x:v_n(x)\ne 0\}$ for each n. Set $u_n=\chi X\wedge n\sup_{i\le n}|v_i|$ for each n, and $G=\bigcup_{n\in\mathbb{N}}G_n$; then $\langle u_n\rangle_{n\in\mathbb{N}}\uparrow\chi G$, so

$$\mu_0G=\lim_{n\to\infty}f(u_n)\le\lim_{n\to\infty}\mu_0G_n\le\mu_0G,$$

as required. **Q**

(c) We therefore have a Baire-coded measure μ on X extending μ_0. Now take any $u\in C_b(X)$ such that $0\le u\le\chi X$, and $n\ge 1$. For each $i<n$ set $G_i=\{x:u(x)>\frac{i}{n}\}$; then

$$\frac{1}{n}\sum_{i=0}^{n-1}\chi G_i\le u+\frac{1}{n}\chi X,$$

so

$$\frac{1}{n}\sum_{i=0}^{n-1}\mu G_i\le\int u\,d\mu+\frac{1}{n}\mu X.$$

Next, setting

$$v_i = u \wedge \tfrac{i+1}{n}\chi X - u \wedge \tfrac{i}{n}\chi X$$

for $i < n$, $u = \sum_{i=0}^{n-1} v_i$ and $nv_i \le \chi G_i$ for each i, so

$$f(u) = \sum_{i=0}^{n-1} f(v_i) \le \tfrac{1}{n}\sum_{i=0}^{n-1} \mu G_i \le \int u\,d\mu + \tfrac{1}{n}\mu X.$$

As n is arbitrary, $f(u) \le \int u\,d\mu$. On the other hand, $f(\chi X) = \mu X = \int \chi X\,d\mu$ and $f(\chi X - u) \le \int(\chi X - u)d\mu$; so in fact $f(u) = \int u\,d\mu$.

(d) It follows at once that $f(u) = \int u\,d\mu$ for every $u \in C_b(X)^+$ and therefore for every $u \in C_b(X)$, as required.

564I Riesz Representation Theorem Let X be a completely regular locally compact space, and $f : C_k(X) \to \mathbb{R}$ a positive linear functional, where $C_k(X)$ is the space of continuous real-valued functions with compact support. Then there is a Baire-coded measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in C_k(X)$.

proof We can follow the plan of 564H, with minor modifications.

(a) For open sets $G \subseteq X$ write $D_G = \{u : u \in C_k(X),\ 0 \le u \le \chi X,\ \operatorname{supp} u \subseteq G\}$, where $\operatorname{supp} u = \overline{\{x : u(x) \ne 0\}}$. We need to know that if G, $H \subseteq X$ are open and $K \subseteq G\cup H$, $K' \subseteq G\cap H$ are compact, there are $u \in D_G$, $v \in D_H$ such that $\chi K \le u\vee v$ and $\chi K' \le u\wedge v$. **P** Because X is completely regular, the family $\{\operatorname{int}\{x : u(x) = 1\} : u \in D_G\cup D_H\}$ is an open cover of $G\cup H$ and has a finite subfamily covering K; because D_G and D_H are upwards-directed, we can reduce this finite subfamily to two terms, one corresponding to $u_1 \in D_G$ and the other to $v_1 \in D_H$, so that $\chi K \le u_1 \vee v_1$. Next, $\{\operatorname{int}\{x : u(x) = 1\} : u \in D_G\}$ is an open cover of $G \supseteq K'$, so we can find a $u_2 \in D_G$ such that $\chi K' \le u_2$; similarly, there is a $v_2 \in D_H$ such that $\chi K' \le v_2$; set $u = u_1 \vee u_2$ and $v = v_1 \vee v_2$. **Q**

(b) For cozero $G \subseteq X$, set $\mu_0 G = \sup\{f(u) : u \in D_G\}$. If G, $H \subseteq X$ are cozero sets, $u \in D_G$ and $v \in D_H$, then $u \vee v \in D_{G\cup H}$ and $u \wedge v \in D_{G\cap H}$; this is enough to show that $\mu_0 G + \mu_0 H \le \mu_0(G \cup H) + \mu_0(G \cap H)$. If $w \in D_{G\cup H}$ and $w' \in D_{G\cap H}$, (a) tells us that there are $u \in D_G$ and $v \in D_H$ such that $u \vee v \ge \chi(\operatorname{supp} w) \ge w$ and $u \wedge v \ge \chi(\operatorname{supp} w') \ge w'$; this is what we need to show that so that $\mu_0 G + \mu_0 H \ge \mu_0(G \cup H) + \mu_0(G \cap H)$.

If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of cozero sets, defined from a sequence of continuous functions so that $G = \bigcup_{n\in\mathbb{N}} G_n$ is a cozero set, then $D_G = \bigcup_{n\in\mathbb{N}} D_{G_n}$ so that $\mu_0 G = \sup_{n\in\mathbb{N}} \mu_0 G_n$.

If G is a relatively compact cozero set then $\mu_0 G < \infty$. **P** There is a $w \in C_k(X)$ such that $\chi\overline{G} \le w$, so that $\mu G \le f(w)$. **Q** If G is a cozero set and $\gamma < \mu_0 G$, there is a $u \in D_G$ such that $f(u) \ge \gamma$. Now there is a $v \in D_G$ such that $\chi(\operatorname{supp}(u)) \le v$, so that $\mu_0 H \ge \gamma$, where $H = \{x : v(x) > 0\}$; as H is relatively compact, $\mu_0 H$ is finite. Thus $\mu_0 G = \sup\{\mu_0 H : H \subseteq G \text{ is a cozero set}, \mu_0 H < \infty\}$.

The other hypotheses of 563L are elementary, so we have a Baire-coded measure on X extending μ_0.

(c) If $u \in C_k(X)$ and $0 \le u \le \chi X$ and $\epsilon > 0$, let G be a relatively compact cozero set including $\operatorname{supp} u$, and $v \in D_G$ such that $\chi(\operatorname{supp} u) \le v$ and $f(v) \ge \mu G - \epsilon$. The argument of part (c) of the proof of 564H, with v in place of χX, shows that $f(u) \le \int u\,d\mu + \frac{1}{n}\int v\,d\mu$ for every n, so that $f(u) \le \int u\,d\mu$. On the other hand, given $\epsilon > 0$,

$$\begin{aligned} f(u) = f(v) - f(v - u) &\ge \mu G - \epsilon - \int v - u\,d\mu \\ &= \mu G - \int v\,d\mu + \int u\,d\mu - \epsilon \ge \int u\,d\mu - \epsilon. \end{aligned}$$

As ϵ is arbitrary, $f(u) = \int u\,d\mu$. Of course it follows at once that f agrees with $\int\,d\mu$ on the whole of $C_k(X)$.

564J The space L^1 Let X be a topological space and μ a Baire-coded measure on X.

(a) If f, $g \in \mathcal{L}^1$ then $f =_{\text{a.e.}} g$ iff $\int|f - g| = 0$. **P** If $f =_{\text{a.e.}} g$ then $|f - g| = 0$ a.e. and $\int|f - g| = 0$ by the definition in 564Ad. If $\int|f - g| = 0$, let f_1, g_1 be codable Baire functions such that $f =_{\text{a.e.}} f_1$ and $g =_{\text{a.e.}} g_1$ (564Ba); then $|f_1 - g_1|$ is codable. For each $n \in \mathbb{N}$, set $E_n = \{x : |f_1(x) - g_1(x)| \ge 2^{-n}\}$. Then $E_n \in \mathcal{B}\mathfrak{a}_c(X)$ and $|f_1 - g_1| \ge 2^{-n}\chi E_n$ so $\mu E_n = \int \chi E_n = 0$. But $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable sequence so $\bigcup_{n\in\mathbb{N}} E_n = \{x : f_1(x) \ne g_1(x)\}$ is negligible and

$$f =_{\text{a.e.}} f_1 =_{\text{a.e.}} g_1 =_{\text{a.e.}} g. \text{ **Q**}$$

(b) As in §242, we have an equivalence relation $\sim$ on $\mathcal{L}^1$ defined by saying that $f \sim g$ if $f =_{\text{a.e.}} g$. The set L^1 of equivalence classes has a Riesz space structure and a Riesz norm inherited from the addition, scalar multiplication, ordering and integral on $\mathcal{L}^1$.

(c) As in §242, I will define $\int : L^1 \to \mathbb{R}$ by saying that $\int f^\bullet = \int f$ for every $f \in \mathcal{L}^1$. Similarly, we can define $\int_E u$, for $u \in L^1$ and $E \in \mathcal{B}_c(X)$, by saying that $\int_E f^\bullet = \int f \times \chi E$ for $f \in \mathcal{L}^1$.

564K In order to prove that an L^1-space is norm-complete, it seems that we need extra conditions.

Theorem Let X be a second-countable space and μ a codably σ-finite Borel-coded measure on X. Then $L^1(\mu)$ is a separable L-space.

proof (Compare 563N.)

(a) There is a codable sequence of sets of finite measure covering X. By 562Nb, we can find a codably Borel equivalent zero-dimensional second-countable topology on X for which all the these sets are open, so that μ becomes locally finite. Since this procedure does not change $\mathcal{L}^1$ and L^1, we may suppose from the beginning that X is regular and μ is locally finite. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a countable base for the topology of X containing $\emptyset$ and closed under finite unions.

(b) If $E\in\mathcal{B}_c(X)^f$ and $\epsilon>0$, there is an open $G\subseteq X$ such that $E\subseteq G$ and $\mu(G\setminus E)\le\epsilon$, by 563Fd. Next, $G=\bigcup\{U_n:n\in\mathbb{N},\,U_n\subseteq G\}$, so there is a finite set $I\subseteq\mathbb{N}$ such that $G'=\bigcup_{n\in I}U_n\subseteq G$ and $\mu(G\setminus G')\le\epsilon$; now $\mu(G'\triangle E)\le 2\epsilon$ and $G'=U_m$ for some m.

(c) If $f\in\mathcal{L}^1$ and $\epsilon>0$, there is an $h\in S(\mathcal{B}_c(X)^f)$ such that $\int|f-h|\le\epsilon$; now there must be an $n\in\mathbb{N}$ and a family $\langle q_i\rangle_{i\le n}$ in $\mathbb{Q}$ such that $\int|h-\sum_{i=0}^n q_i\chi U_i|\le\epsilon$. The set D of such rational linear combinations of the χU_i is countable; enumerate it as $\langle h_n\rangle_{n\in\mathbb{N}}$. All the h_n are differences of semi-continuous functions, therefore resolvable, so $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence; and for any $u\in L^1$ and $\epsilon>0$ there is an n such that $\|u-h_n^\bullet\|_1\le 2\epsilon$.

(d) This shows that L^1 is separable. To see that it is complete, take a Cauchy filter $\mathcal{F}$ on L^1. For each $k\in\mathbb{N}$ we can take the first $n_k\in\mathbb{N}$ such that $\{u:\|u-h_{n_k}^\bullet\|_1\le 2^{-k}\}$ belongs to $\mathcal{F}$. Now $\int|h_{n_k}-h_{n_{k+1}}|\le 2^{-k}+2^{-k-1}$ for every k, so the codable sequence $\langle h_{n_k}\rangle_{k\in\mathbb{N}}$ converges a.e. to some $f\in\mathcal{L}^1$ (564Fd), and $\int|f-h_{n_k}|\le 6\cdot 2^{-k}$ for every k. So

$$f^\bullet=\lim_{k\to\infty}h_{n_k}^\bullet=\lim\mathcal{F}.$$

(e) Thus L^1 is norm-complete. We know it is a Riesz space with a Riesz norm, so it is a Banach lattice. As for the additivity of the norm on the positive cone, we have only to observe that if f, $g\in\mathcal{L}^1$ and $f^\bullet$, $g^\bullet$ are non-negative, then

$$\begin{aligned}\|f^\bullet+g^\bullet\|_1&=\||f|^\bullet+|g|^\bullet\|_1=\|(|f|+|g|)^\bullet\|_1\\&=\int|f|+|g|=\int|f|+\int|g|=\|f^\bullet\|_1+\|g^\bullet\|_1.\end{aligned}$$

564L Radon-Nikodým theorem Let X be a second-countable space with a codably σ-finite Borel-coded measure μ. Let $\nu:\mathcal{B}_c(X)\to\mathbb{R}$ be a truly continuous additive functional. Then there is an $f\in\mathcal{L}^1(\mu)$ such that $\nu E=\int f\times\chi E$ for every $E\in\mathcal{B}_c(X)$.

proof (a) Let M be the space of bounded additive functionals on $\mathcal{B}_c(X)$; as in 362B, M is an L-space. I will write $\mathcal{L}^1$ for the Riesz space of integrable real-valued codable Borel functions on X. For $f\in\mathcal{L}^1$ and $E\in\mathcal{B}_c(X)$, set $\nu_fE=\int f\times\chi E$; this is defined by 564Ea and 564E(c-i). The map $f\mapsto\nu_f:\mathcal{L}^1\to M$ is a Riesz homomorphism, and norm-preserving in the sense that $\|\nu_f\|=\int|f|$ for every $f\in\mathcal{L}^1$. Accordingly $M_1=\{\nu_f:f\in\mathcal{L}^1\}$ is a Riesz subspace of M isomorphic, as normed Riesz space, to L^1; in particular, it is norm-complete, by 564K, therefore norm-closed.

(b) If $\nu\in M^+$ is truly continuous and $\epsilon>0$, there are an $E\in\mathcal{B}_c(X)^f$ and a $\gamma>0$ such that $\|(\nu-\gamma\nu_{\chi E})^+\|\le\epsilon$. **P** There are $E\in\mathcal{B}_c(X)$ and $\delta>0$ such that $\mu E<\infty$ and $\nu F\le\epsilon$ whenever $\mu(E\cap F)\le\delta$. Set $\gamma=\frac{\|\nu\|}{\delta}$. Then

$$\begin{aligned}(\nu-\gamma\nu_{\chi E})(F)&=\nu F-\tfrac{\|\nu\|}{\delta}\mu(F\cap E)\\&\le 0\text{ if }\mu(F\cap E)\ge\delta,\\&\le\epsilon\text{ otherwise.}\end{aligned}$$

So $\|(\nu-\gamma\nu_{\chi E})^+\|\le\epsilon$. **Q**

(c) Suppose that $\nu\in M$, $E\in\mathcal{B}_c(X)^f$ and $\gamma>0$ are such that $0\le\nu\le\gamma\nu_{\chi E}$. Let $\epsilon>0$. Then there are an $f\in\mathcal{L}^1$ and a $\nu'\in M^+$ such that $\|\nu-\nu_f-\nu'\|\le\epsilon$ and $\nu'\le\frac12\gamma\nu_{\chi E}$. **P** Set $\alpha=\sup_{F\in\mathcal{B}_c(X)}\nu F-\frac12\gamma\mu F$; let $H\in\mathcal{B}_c(X)$ be such that $\nu H-\frac12\gamma\mu H\ge\alpha-\frac13\epsilon$; set $f=\frac12\gamma\chi(H\cap E)$ and $\nu'=(\nu-\nu_f)^+\wedge\frac12\gamma\nu_{\chi E}$.

If $F\in\mathcal{B}_c(X)$ then

$$(\nu_f-\nu)(F)=\frac{1}{2}\gamma\mu(F\cap H\cap E)-\nu F\le\frac{1}{2}\gamma\mu(F\cap H)-\nu(F\cap H)$$
$$=\frac{1}{2}\gamma\mu H-\nu H-\frac{1}{2}\gamma\mu(H\setminus F)+\nu(H\setminus F)$$

(of course μH must be finite, as $\nu H-\frac{1}{2}\gamma\mu H$ is finite)

$$\le-\alpha+\frac{1}{3}\epsilon+\alpha=\frac{1}{3}\epsilon,$$

$$\begin{aligned}(\nu-\nu_f-\tfrac{1}{2}\gamma\nu_{\chi E})(F)&=\nu F-\frac{1}{2}\gamma\mu(F\cap E\cap H)-\frac{1}{2}\gamma\mu(F\cap E)\\&=\nu(F\cap E\setminus H)-\frac{1}{2}\gamma\mu(F\cap E\setminus H)\\&\qquad+\nu(F\setminus E)+\nu(F\cap E\cap H)-\gamma\mu(F\cap E\cap H)\\&\le\nu(F\cap E\setminus H)-\frac{1}{2}\gamma\mu(F\cap E\setminus H)\end{aligned}$$

(because $\nu\le\gamma\nu_{\chi E}$)

$$=\nu((F\cap E)\cup H)-\frac{1}{2}\gamma\mu((F\cap E)\cup H)-\nu H+\frac{1}{2}\gamma\mu H$$
$$\le\alpha-(\alpha-\frac{1}{3}\epsilon)=\frac{1}{3}\epsilon.$$

So $\|(\nu_f-\nu)^+\|\le\frac{1}{3}\epsilon$ and $\|(\nu-\nu_f-\frac{1}{2}\gamma\nu_{\chi E})^+\|\le\frac{1}{3}\epsilon$. But this means that

$$\begin{aligned}\|\nu-\nu_f-\nu'\|&=\|\nu-\nu_f-(\nu-\nu_f)^++((\nu-\nu_f)^+-\frac{1}{2}\gamma\nu_{\chi E})^+\|\\&\le2\|\nu-\nu_f-(\nu-\nu_f)^+\|+\|(\nu-\nu_f-\frac{1}{2}\gamma\nu_{\chi E})^+\|\\&\le2\|(\nu_f-\nu)^+\|+\frac{1}{3}\epsilon\le\epsilon,\end{aligned}$$

as required. **Q**

(d) Again suppose that $\nu\in M$, $E\in\mathcal{B}_c(X)^f$, $\gamma>0$ and $\epsilon>0$ are such that $0\le\nu\le\gamma\nu_{\chi E}$. Then for any $n\in\mathbb{N}$ there are an $f\in\mathcal{L}^1$ and a $\nu'\in M^+$ such that $\|\nu-\nu_f-\nu'\|\le\epsilon$ and $\nu'\le2^{-n}\gamma\nu_{\chi E}$. **P** Induce on n. **Q**

(e) If $\nu\in M^+$ is truly continuous and $\epsilon>0$, there is an $f\in\mathcal{L}^1$ such that $\|\nu-\nu_f\|\le\epsilon$. **P** By (b), there are an $E\in\mathcal{B}_c(X)^f$ and a $\gamma>0$ such that $\|(\nu-\gamma\nu_{\chi E})^+\|\le\frac{1}{3}\epsilon$. Let $n\in\mathbb{N}$ be such that $2^{-n}\gamma\mu E\le\frac{1}{3}\epsilon$. By (d), we have an $f\in\mathcal{L}^1$ and a $\nu'\in M$ such that $\|(\nu\wedge\gamma\nu_{\chi E})-\nu_f-\nu'\|\le\frac{1}{3}\epsilon$ and $0\le\nu'\le2^{-n}\gamma\nu_{\chi E}$. But this means that

$$\begin{aligned}\|\nu-\nu_f\|&\le\|(\nu-\gamma\nu_{\chi E})^+\|+\|(\nu\wedge\gamma\nu_{\chi E})-\nu_f\|\\&\le\frac{1}{3}\epsilon+\frac{1}{3}\epsilon+\|\nu'\|\le\frac{2}{3}\epsilon+2^{-n}\gamma\mu E\le\epsilon.\ \mathbf{Q}\end{aligned}$$

(f) Since any truly continuous $\nu\in M$ has truly continuous positive and negative parts, the space M_{tc} of truly continuous functionals is included in the closure of $M_1=\{\nu_f:f\in\mathcal{L}^1\}$. But I noted in (a) that M_1 is norm-isomorphic to L^1, so is complete, therefore closed, and must include M_{tc}.

564M Inverse-measure-preserving functions (a) Let X and Y be second-countable spaces, with Borel-coded measures μ and ν. Suppose that $\varphi:X\to Y$ is a codable Borel function such that $\mu\varphi^{-1}[F]=\nu F$ for every $F\in\mathcal{B}_c(Y)$. Then $h\varphi\in S_X$ and $\int h\varphi\,d\mu=\int h\,d\nu$ for every $h\in S_Y$, writing $S_X=S(\mathcal{B}_c(X)^f)$, S_Y for the spaces of simple functions. By 562Kb, $f\varphi\in\mathcal{L}^0(\mu)$ for every $f\in\mathcal{L}^0(\nu)$. By 562Qd, $\langle h_n\varphi\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_X whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_Y; consequently $f\varphi\in\mathcal{L}^1(\mu)$ whenever $f\in\mathcal{L}^1(\nu)$, and we have a norm-preserving Riesz homomorphism $T:L^1(\nu)\to L^1(\mu)$ defined by setting $Tf^\bullet=(f\varphi)^\bullet$ for $f\in\mathcal{L}^1(\mu)$.

(b) If ν is codably σ-finite, we have a conditional expectation operator in the reverse direction, as follows. For any $f\in\mathcal{L}^1(\mu)$, consider the functional λ_f defined by setting $\lambda_fF=\int f\times\chi(\varphi^{-1}[F])$ for $F\in\mathcal{B}_c(Y)$. This is additive and truly continuous. **P** Let $\epsilon>0$. By 564Ga, there are an $E_0\in\mathcal{B}_c(X)$ and a $\delta>0$ such that $\mu E_0<\infty$ and $\int|f|\times\chi E\le\epsilon$ whenever $\mu(E\cap E_0)\le2\delta$. Next, there is a non-decreasing codable sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(Y)$ such that $\nu F_n<\infty$ for every n and $Y = \bigcup_{n\in\mathbb{N}} F_n$. In this case, $\langle\varphi^{-1}[F_n]\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in $\mathcal{B}_c(X)$ with union X, so there is an n such that $\mu(E_0 \setminus \varphi^{-1}[F_n]) \le \delta$. Now suppose that $F \in \mathcal{B}_c(Y)$ and $\nu(F \cap F_n) \le \delta$. In this case,

$$\mu(E_0 \cap \varphi^{-1}[F]) \le \mu(E_0 \setminus \varphi^{-1}[F_n]) + \mu(\varphi^{-1}[F_n \cap F]) \le 2\delta,$$

so

$$|\lambda_f F| \le \int |f| \times \chi(\varphi^{-1}[F]) \le \epsilon.$$

As ϵ is arbitrary, λ_f is truly continuous. **Q**

There is therefore a unique $v_f \in L^1(\nu)$ such that $\int_F v_f = \lambda_f F$ for every $F \in \mathcal{B}_c(Y)$. **P** By 564L, there is a $g \in \mathcal{L}^1(\nu)$ such that $\lambda_f F = \int g \times \chi F$ for every $F \in \mathcal{B}_c(Y)$. By 564Gb, any two such functions are equal almost everywhere, so have the same equivalence class in L^1, which we may call v_f. **Q**

We may call v_f the **conditional expectation** of f with respect to the inverse-measure-preserving function φ.

(c) Still supposing that ν is codably σ-finite, we see that $\lambda_f = \lambda_{f'}$ whenever f, $f' \in \mathcal{L}^1(\mu)$ are equal almost everywhere, so that we have an operator $P : L^1(\mu) \to L^1(\nu)$ defined by saying that $Pf^\bullet = v_f$ for every $f \in \mathcal{L}^1(\mu)$; that is, that $\int_F Pu = \int_{\varphi^{-1}[F]} u$ for every $u \in L^1(\mu)$ and $F \in \mathcal{B}_c(Y)$. Because this defines each Pu uniquely, P is linear. It is positive because if $f^\bullet \ge 0$ then $\lambda_f \ge 0$; if now $g \in \mathcal{L}^1(\nu)$ is such that $\int g \times \chi F = \lambda_f F \ge 0$ for every $F \in \mathcal{B}_c(X)$, $g \ge 0$ a.e., by 564Gb, and

$$Pf^\bullet = u_f = g^\bullet \ge 0.$$

It is elementary to check that if T is the operator of (a) above then PT is the identity operator on $L^1(\nu)$.

(d) Now consider the special case in which $Y = X$, the topology of Y is the topology generated by a codable sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)^f$, $\nu = \mu\restriction\mathcal{B}_c(Y)$ and φ is the identity function. (Of course this can be done only when μ is codably σ-finite.) In this case, we can identify $L^1(\nu)$ with its image in $L^1(\mu)$ under T, and P becomes a conditional expectation operator of the kind examined in 242J.

564N Product measures: Theorem Let X and Y be second-countable spaces, and μ, ν semi-finite Borel-coded measures on X, Y respectively.

(a) There is a Borel-coded measure λ on $X \times Y$ such that $\lambda(E \times F) = \mu E \cdot \nu F$ for all $E \in \mathcal{B}_c(X)$ and $F \in \mathcal{B}_c(Y)$.

(b) If ν is codably σ-finite then we can arrange that $\iint f(x, y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$ for every λ-integrable real-valued function f.

(c) If μ and ν are both codably σ-finite then λ is uniquely defined by the formula in (a).

proof (a)(i) Start by fixing sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ running over bases for the topologies of X, Y respectively containing $\emptyset$, and a bijection $n \mapsto (i_n, j_n) : \mathbb{N} \to \mathbb{N}$; then $\langle U_{i_n} \times V_{j_n}\rangle_{n\in\mathbb{N}}$ runs over a base for the topology of $X \times Y$ containing $\emptyset$. Let $\phi : \mathcal{T} \to \mathcal{B}_c(X \times Y)$, $\tilde\phi : \tilde{\mathcal{T}} \to \mathbb{R}^{X\times Y}$, $\phi_Y : \mathcal{T} \to \mathcal{B}_c(Y)$, $\tilde\phi_X : \tilde{\mathcal{T}}_X \to \mathbb{R}^X$ be the interpretations of codes associated with the sequences $\langle U_{i_n} \times V_{j_n}\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ and $\langle U_n\rangle_{n\in\mathbb{N}}$, as described in 562A and 562L. Let $\mathcal{R}_X$ be the space of resolvable real-valued functions on X, and $\tilde\psi_X : \mathcal{R}_X \to \tilde{\mathcal{T}}_X$ a function as in 562P. The argument will depend on the existence of a number of further functions; it may help if I lay them out explicitly.

(α) Let $\Theta_0 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be such that

$$\phi(\Theta_0(T, T')) = \phi(T) \cap \phi(T'), \quad r(\Theta_0(T, T')) = \max(r(T), r(T'))$$

for all T, $T' \in \mathcal{T}$ (562Bb); now define $\Theta_1 : \bigcup_{I\in[\mathbb{N}]^{<\omega}} \mathcal{T}^I \to \mathcal{T}$ by setting

$$\begin{aligned}\Theta_1(\langle T_i\rangle_{i\in I}) &= \{\emptyset\} \cup \{<\!n\!> : n \in \mathbb{N}\} \text{ if } I = \emptyset,\\ &= \Theta_0(\Theta_1(\langle T_i\rangle_{i\in I\cap n}), T_n) \text{ if } n = \max I.\end{aligned}$$

Then

$$\phi(\Theta_1(\langle T_i\rangle_{i\in I})) = (X \times Y) \cap \textstyle\bigcap_{i\in I} \phi(T_i), \quad r(\Theta_1(\langle T_i\rangle_{i\in I})) = \max(1, \sup_{i\in I} r(T_i))$$

whenever $I \subseteq \mathbb{N}$ is finite and $T_i \in \mathcal{T}$ for $i \in I$.

(β) Let $\Theta_2 : \tilde{\mathcal{T}}_X \times \mathcal{T} \to \tilde{\mathcal{T}}_X$ be such that

$$\tilde\phi_X(\Theta_2(\tau, T')) = (\nu\phi_Y(T'))\chi X - \tilde\phi_X(\tau)$$

whenever $\tau \in \tilde{\mathcal{T}}_X$ and $T' \in \mathcal{T}$ is such that $\nu(\phi_Y(T'))$ is finite.

(γ) Let $\Theta_3 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}$ be such that

$$\phi_X(\Theta_3(\langle T_n\rangle_{n\in\mathbb{N}})) = \textstyle\bigcup_{n\in\mathbb{N}} \phi_X(T_n)$$

for every sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$.

(δ) Define $\tilde\Theta_3:\tilde{\mathcal{T}}_X^{\mathbb{N}}\to\mathcal{T}^{\mathbb{R}}$ by saying that

$$\tilde\Theta_3(\langle\tau_n\rangle_{n\in\mathbb{N}})(\alpha)=\Theta_3(\langle\tau_n(\alpha)\rangle_{n\in\mathbb{N}})$$

for every sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\tilde{\mathcal{T}}_X$, so that $\tilde\Theta_3(\langle\tau_n\rangle_{n\in\mathbb{N}})\in\tilde{\mathcal{T}}_X$ and

$$\tilde\phi_X(\Theta_3(\langle\tau_n\rangle_{n\in\mathbb{N}}))=\sup_{n\in\mathbb{N}}\tilde\phi_X(\tau_n)$$

whenever $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}_X$ such that $\sup_{n\in\mathbb{N}}\tilde\phi_X(\tau_n)$ is defined in $\mathbb{R}^X$.

(ϵ) As in 562Mb, we can find a function $\Theta_4:\mathcal{T}\times X\to\mathcal{T}$ such that

$$\phi_Y(\Theta_4(T,x))=\{y:(x,y)\in\phi(T)\}$$

for $T\in\mathcal{T}$ and $x\in X$.

(ii) If $W\subseteq X\times Y$ is open and $F\in\mathcal{B}_c(Y)$, $x\mapsto\nu(F\cap W[\{x\}]):X\to[0,\infty]$ is lower semi-continuous. **P** Take $\gamma\in\mathbb{R}$ and consider $G=\{x:\nu(F\cap W[\{x\}])>\gamma\}$. Given $x\in G$ let K be $\{(m,n):x\in U_m,\ U_m\times V_n\subseteq W\}$; then $W[\{x\}]=\bigcup_{(m,n)\in K}V_n$. Now $\langle V_n\rangle_{(m,n)\in K}$ and $\langle F\cap V_n\rangle_{(m,n)\in K}$ are codable families, so there is a finite set $L\subseteq K$ such that $\nu(\bigcup_{(m,n)\in L}F\cap V_n)>\gamma$ (563Ba). In this case, $H=X\cap\bigcap_{(m,n)\in L}U_m$ is an open neighbourhood of x included in G, and $\nu(F\cap W[\{x'\}])>\gamma$ for every $x'\in H$. As x is arbitrary, G is open; as γ is arbitrary, the function is lower semi-continuous. **Q**

(iii) For T, $T'\in\mathcal{T}$ and $x\in X$, set

$$h_{TT'}(x)=\nu\{y:y\in\phi_Y(T'),\ (x,y)\in\phi(T)\}=\nu(\phi_Y(T')\cap\phi_Y(\Theta_4(T,x))).$$

Then there is a function $\Theta:\mathcal{T}\times\mathcal{T}\to\tilde{\mathcal{T}}_X$ such that

$$\tilde\phi_X(\Theta(T,T'))=h_{TT'}$$

whenever $x\in X$ and T, $T'\in\mathcal{T}$ are such that $\nu\phi_Y(T')$ is finite. **P** Start by fixing on any $\tau_0\in\mathcal{T}_X$. If $\nu\phi_Y(T')=\infty$ set $\Theta(T,T')=\tau_0$. For other T', build Θ by induction on the rank of T, as usual. If $r(T)\le 1$, then $\phi(T)$ is open; by (ii), $h_{TT'}$ is lower semi-continuous, therefore resolvable (562Oa). So we can set $\Theta(T,T')=\tilde\psi_X(h_{TT'})$.

For the inductive step to $r(T)\ge 2$, set $A_T=\{n:{<}n{>}\in T\}$, so that

$$\begin{aligned}\phi(T)&=\bigcup_{n\in A_T}(X\times Y)\setminus\phi(T_{<n>})\\&=\bigcup_{m\in\mathbb{N}}(X\times Y)\setminus((X\times Y)\cap\bigcap_{n\in A_T\cap m}\phi(T_{<n>}))\\&=\bigcup_{m\in\mathbb{N}}(X\times Y)\setminus\phi(\Theta_1(\langle T_{<n>}\rangle_{n\in A_T\cap m}))\end{aligned}$$

and

$$h_{TT'}(x)=\lim_{m\to\infty}\nu\phi_Y(T')-h_{T^{(m)}T'}(x)=\sup_{m\in\mathbb{N}}\nu\phi_Y(T')-h_{T^{(m)}T'}(x)$$

for every x, where

$$T^{(m)}=\Theta_1(\langle T_{<n>}\rangle_{n\in A_T\cap m}),\quad \phi(T^{(m)})=(X\times Y)\cap\bigcap_{n\in A_T\cap m}\phi(T_{<n>})$$

for $m\in\mathbb{N}$. Now $r(T^{(m)})<r(T)$ for every m, so each $\Theta(T^{(m)},T')$ has been defined, and we can speak of $\Theta_2(\Theta(T^{(m)},T'),T')$ for each m; we shall have

$$\begin{aligned}\tilde\phi_X(\Theta_2(\Theta(T^{(m)},T'),T'))(x)&=\nu\phi_Y(T')-\tilde\phi_X(\Theta(T^{(m)},T'))(x)=\nu\phi_Y(T')-h_{T^{(m)}T'}(x)\\&=\nu\{y:y\in\phi_Y(T'),\ (x,y)\in\bigcup_{n\in A_T\cap m}(X\times Y)\setminus\phi(T_{<n>})\}\end{aligned}$$

for $m\in\mathbb{N}$ and $x\in X$. So if we set

$$\Theta(T,T')=\tilde\Theta_3(\langle\Theta_2(\Theta(T^{(m)},T'),T')\rangle_{m\in\mathbb{N}})$$

for every $\alpha\in\mathbb{R}$, we shall have

$$\begin{aligned}\tilde\phi_X(\Theta(T,T')) &= \sup_{m\in\mathbb{N}} \tilde\phi_X(\Theta_2(\Theta(T^{(m)},T'),T'))\\ &= \sup_{m\in\mathbb{N}} (\nu\phi_Y(T'))\chi X - \tilde\phi_X(\Theta(T^{(m)},T'))\\ &= \sup_{m\in\mathbb{N}} (\nu\phi_Y(T'))\chi X - h_{T^{(m)}T'} = h_{TT'},\end{aligned}$$

as required for the induction to proceed. **Q**

(iv) Thus we see that $h_{TT'} \in \mathcal{L}^0(\mu)$ whenever T, $T' \in \mathcal{T}$ and $\nu\phi_Y(T')$ is finite.

(v) Let $\mathcal{B}_c(Y)^f$ be the ring of subsets of Y of finite measure. For $F \in \mathcal{B}_c(Y)^f$ and $W \in \mathcal{B}_c(X \times Y)$ we have T, $T' \in \mathcal{T}$ such that $\phi(T) = W$ and $\phi_Y(T') = F$, and now $\nu(F \cap W[\{x\}]) = h_{TT'}(x)$ for every $x \in X$. So we have a functional $\lambda_F : \mathcal{B}_c(X \times Y) \to [0,\infty]$ defined by saying that

$$\begin{aligned}\lambda_F W &= \int \nu(F \cap W[\{x\}])\mu(dx) \text{ if the integral is defined in } \mathbb{R},\\ &= \infty \text{ otherwise.}\end{aligned}$$

Of course λ_F is additive. If $E \in \mathcal{B}_c(X)$ and $F' \in \mathcal{B}_c(Y)$, then

$$\begin{aligned}\lambda_F(E \times F') &= 0 = \mu E \cdot \nu(F \cap F') \text{ if } \nu(F \cap F') = 0,\\ &= \int \nu(F \cap F')\chi E\, d\mu = \mu E \cdot \nu(F \cap F') \text{ if } \mu E < \infty,\\ &= \infty = \mu E \cdot \nu(F \cap F') \text{ if } \mu E = \infty \text{ and } \nu(F \cap F') > 0.\end{aligned}$$

(To see that $E \times F' \in \mathcal{B}_c(X \times Y)$, use 562Kd.)

(vi) Now suppose that $\langle W_n\rangle_{n\in\mathbb{N}}$ is a codable disjoint sequence in $\mathcal{B}_c(X \times Y)$ with union W, and that $F \in \mathcal{B}_c(Y)^f$. We surely have $\lambda_F W \ge \sum_{n=0}^\infty \lambda_F W_n$. If $\sum_{n=0}^\infty \lambda_F W_n$ is finite, let $\langle T_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\phi(T_n) = W_n$ for each n, and take $T' \in \mathcal{T}$ such that $\phi_Y(T') = F$. Then $\langle h_{T_nT'}\rangle_{n\in\mathbb{N}} = \langle\tilde\phi_X(\Theta(T_n,T'))\rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable Borel functions, so 564Fe tells us that the sum of the integrals is the integral of the sum; but

$$\begin{aligned}\sum_{n=0}^\infty h_{T_nT'}(x) &= \sum_{n=0}^\infty \nu(\phi_Y(T') \cap \phi_Y(\Theta_4(T_n,x))) = \nu(\bigcup_{n\in\mathbb{N}} \phi_Y(T') \cap \phi_Y(\Theta_4(T_n,x)))\\ &= \nu(\bigcup_{n\in\mathbb{N}} F \cap W_n[\{x\}]) = \nu(F \cap W[\{x\}])\end{aligned}$$

for each x, so we have

$$\begin{aligned}\lambda_F W &= \int \nu(F \cap W[\{x\}])\mu(dx) = \int \sum_{n=0}^\infty h_{T_nT'} d\mu\\ &= \sum_{n=0}^\infty \int h_{T_nT'} d\mu = \sum_{n=0}^\infty \lambda_F W_n.\end{aligned}$$

As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, λ_F is a Borel-coded measure.

(vii) If $W \in \mathcal{B}_c(X \times Y)$ and $F \subseteq F'$ in $\mathcal{B}_c(Y)^f$, then

$$\lambda_F W = \int \nu(F \cap W[\{x\}])\mu(dx)$$

(counting $\int h\, d\mu$ as ∞ for a non-negative function $h \in \mathcal{L}^0(\mu) \setminus \mathcal{L}^1(\mu)$)

$$= \int \nu(F' \cap (W \cap (X \times F))[\{x\}])\mu(dx) = \lambda_{F'}(W \cap (X \times F)) \le \lambda_{F'} W.$$

Thus $\langle\lambda_F(W)\rangle_{F\in\mathcal{B}_c(Y)^f}$ is an upwards-directed family for each $W \in \mathcal{B}_c(X \times Y)$; let λW be its supremum. Then λ is a Borel-coded measure on $X \times Y$ (563E). Also

$$\lambda(E \times F) = \lambda_F(E \times F) = \mu E \cdot \nu_F F = \mu E \cdot \nu F$$

whenever $E \in \mathcal{B}_c(X)$ and $F \in \mathcal{B}_c(Y)^f$ have finite measure. For other measurable E and F, if either is negligible then $\lambda(E \times F) = 0$, while if one has infinite measure and the other has non-zero measure then $\lambda(E \times F) = \infty$ because μ and ν are both semi-finite.

Observe that the construction ensures that if $\lambda W < \infty$ and $W \subseteq X \times F$ for some $F \in \mathcal{B}_c(Y)^f$, then $\lambda W = \int \nu W[\{x\}]\mu(dx)$.

(b) Now suppose that ν is codably σ-finite.

(i) Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in $\mathcal{B}_c(Y)^f$ covering Y; since $\langle\bigcup_{i\le n} F_i\rangle_{n\in\mathbb{N}}$ also is codable, we can suppose that $\langle F_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. Let $\langle T'_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\phi_Y(T'_n) = F_n$ for each n. By 562Kd, as usual, $\langle X \times F_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}_c(X \times Y)$, so $\lambda W = \sup_{n\in\mathbb{N}} \lambda(W \cap (X \times F_n))$ whenever λ measures W.

(ii) Let $f : X \times Y \to [0, \infty[$ be an integrable codable Borel function. Then $\iint f(x, y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$.

P$(\boldsymbol{\alpha})$ For n, $k \in \mathbb{N}$ set

$$W_{nk} = \{(x, y) : y \in F_n,\ f(x, y) \ge 2^{-n}k\};$$

then $\langle W_{nk}\rangle_{n,k\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X \times Y)$. Let $\langle T_{nk}\rangle_{n,k\in\mathbb{N}}$ be a family in $\mathcal{T}$ such that $W_{nk} = \phi(T_{nk})$ for n, $k \in \mathbb{N}$. For $n \in \mathbb{N}$, define $v_n : X \times Y \to \mathbb{R}$ by setting

$$v_n = 2^{-n}\textstyle\sum_{k=1}^{4^n} \chi W_{nk};$$

then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on $X \times Y$. Moreover, setting $v_{nx}(y) = v_n(x, y)$, $\langle v_{nx}\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on Y, for each $x \in X$. Now set

$$u_{nk}(x) = \nu W_{nk}[\{x\}], \quad u_n(x) = \int v_n(x, y)\nu(dy)$$

for $x \in X$ and n, $k \in \mathbb{N}$. Then, in the language of part (a) of this proof,

$$u_{nk} = \tilde{\phi}_X(\Theta(T_{nk}, T'_n))$$

for all n and k, so $\langle u_{nk}\rangle_{n,k\in\mathbb{N}}$ is a codable family of codable Borel functions on X. Since

$$u_n = 2^{-n}\textstyle\sum_{k=1}^{4^n} u_{nk}$$

for each n, $\langle u_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on X.

$(\boldsymbol{\beta})$ Next, for each $n \in \mathbb{N}$,

$$\int u_n d\mu = 2^{-n}\sum_{k=1}^{4^n}\int \nu W_{nk}[\{x\}]\mu(dx) = 2^{-n}\sum_{k=1}^{4^n}\lambda W_{nk}$$

(by the final remark in part (a) of the proof)

$$= \int v_n d\lambda.$$

At this point, observe that $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit f. So

$$\lim_{n\to\infty}\int u_n d\mu = \lim_{n\to\infty}\int v_n d\lambda = \int f d\lambda$$

is finite; since $\langle u_n\rangle_{n\in\mathbb{N}}$ also is non-decreasing, $u(x) = \lim_{n\to\infty} u_n(x)$ is finite for μ-almost all x, and

$$\int u\, d\mu = \lim_{n\to\infty}\int u_n d\mu = \int f d\lambda$$

(564Fa). On the other hand, for each $x \in X$, $\langle v_{nx}\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with limit f_x, where $f_x(y) = f(x, y)$ for $y \in Y$; so

$$u(x) = \lim_{n\to\infty}\int v_{nx} d\nu = \int f_x d\nu$$

for almost all x, and

$$\iint f(x, y)\nu(dy)\mu(dx) = \iint f_x d\nu\mu(dx) = \int u\, d\mu = \int f d\lambda. \quad \mathbf{Q}$$

(iii) It follows at once, taking the difference of positive and negative parts, that

$$\iint f(x, y)\nu(dy)\mu(dx) = \int f d\lambda$$

for every λ-integrable codable Borel function f.

(iv) In particular (or more directly), if $W \in \mathcal{B}_c(X \times Y)$ is λ-negligible, then μ-almost every vertical section of W is ν-negligible. So starting from a general λ-integrable function f, we move to a codable Borel function g such that $f =_{\text{a.e.}} g$; now $\int f(x,y)\nu(dy)$ must be defined and equal to $\int g(x,y)\nu(dy)$ for almost every x, and

$$\iint f(x,y)\nu(dy)\mu(dx) = \iint g(x,y)\nu(dy)\mu(dx) = \int g d\lambda = \int f d\lambda.$$

This completes the proof of (b).

(c) Let $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ be codable sequences of sets of finite measure covering X, Y respectively; we may suppose that both sequences are non-decreasing. Then $\langle E_n \times F_n\rangle_{n\in\mathbb{N}} = \langle (E_n \times Y) \cap (X \times F_n)\rangle_{n\in\mathbb{N}}$ is a codable sequence (562Kd). Suppose that λ, λ' are two Borel-coded measures on $X \times Y$ agreeing on measurable rectangles. For each $n \in \mathbb{N}$ let λ_n, λ'_n be the totally finite measures defined by setting

$$\lambda_n W = \lambda(W \cap (E_n \times F_n)), \quad \lambda'_n W = \lambda'(W \cap (E_n \times F_n))$$

for $W \in \mathcal{B}_c(X \times Y)$. Now, given n, set $\mathcal{W}_n = \{W : W \in \mathcal{B}_c(X \times Y),\ \lambda_n W = \lambda'_n W\}$. Then $W \cup W' \in \mathcal{W}_n$ whenever W, $W' \in \mathcal{W}_n$ are disjoint, and $E \times F \in \mathcal{W}_n$ whenever $E \in \mathcal{B}_c(X)$ and $F \in \mathcal{B}_c(Y)$. So $\mathcal{W}_n$ includes the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \mathcal{B}_c(X),\ F \in \mathcal{B}_c(Y)\}$. In particular, $\mathcal{W}_n$ includes any set of the form $\bigcup_{(i,j)\in K} U_i \times V_j$ where $K \subseteq \mathbb{N} \times \mathbb{N}$ is finite. But any open subset of $X \times Y$ is expressible as the union of a non-decreasing codable sequence of such sets, so also belongs to $\mathcal{W}_n$. By 563G, $\lambda_n = \lambda'_n$.

This is true for every $n \in \mathbb{N}$. Since

$$\lambda W = \sup_{n\in\mathbb{N}} \lambda_n W, \quad \lambda' W = \sup_{n\in\mathbb{N}} \lambda'_n W$$

for every $W \in \mathcal{B}_c(X \times Y)$, $\lambda = \lambda'$, as claimed.

564O Theorem Let $\langle (X_k, \rho_k)\rangle_{n\in\mathbb{N}}$ be a sequence of complete metric spaces, and suppose that we have a double sequence $\langle U_{ki}\rangle_{k,i\in\mathbb{N}}$ such that $\{U_{ki} : i \in \mathbb{N}\}$ is a base for the topology of X_k for each k. Let $\langle \mu_k\rangle_{n\in\mathbb{N}}$ be a sequence such that μ_k is a Borel-coded probability measure on X_k for each k. Set $X = \prod_{k\in\mathbb{N}} X_k$. Then X is a Polish space and there is a Borel-coded probability measure λ on X such that $\lambda(\prod_{k\in\mathbb{N}} E_k) = \prod_{k\in\mathbb{N}} \mu_k E_k$ whenever $\langle E_k\rangle_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \mathcal{B}_c(X_k)$ and $\{k : E_k \neq X_k\}$ is finite.

proof (a)(i) Of course X is Polish; we have a complete metric ρ on X defined by saying that $\rho(x,y) = \sup_{k\in\mathbb{N}} \min(2^{-k}, \rho_k(x(k), y(k)))$ for x, $y \in X$, and a countable base generated by sets of the form $\{x : x(k) \in U_{ki}\}$.

(ii) Writing $\mathcal{F}_k$ for the family of closed subsets of X_k for $k \in \mathbb{N}$, we have a choice function ζ on $\bigcup_{k\in\mathbb{N}} \mathcal{F}_k \setminus \{\emptyset\}$. **P** Given a non-empty $F \in \bigcup_{k\in\mathbb{N}} \mathcal{F}_k$, take the first k such that $F \in \mathcal{F}_k$, and define $\langle F_m\rangle_{m\in\mathbb{N}}$, $\langle i_m\rangle_{m\in\mathbb{N}}$ by saying that

$F_0 = F$,

$i_m = \min\{i : i \in \mathbb{N},\ U_{ki} \cap F_m \neq \emptyset,\ \operatorname{diam} U_{ki} \leq 2^{-m}\}$

(taking the diameter as measured by ρ_k, of course),

$F_{m+1} = \overline{F_m \cap U_{ki_m}}$

for each m. Now $\langle F_m\rangle_{m\in\mathbb{N}}$ generates a Cauchy filter in X_k which must have a unique limit; take this limit for $\zeta(F)$. **Q**

(b)(i) Let $\mathrm{T} = \bigotimes_{k\in\mathbb{N}} \mathcal{B}_c(X_k)$ be the algebra of subsets of X generated by $\{\{x : x(k) \in E\} : k \in \mathbb{N},\ E \in \mathcal{B}_c(X_k)\}$. Note that all these sets belong to $\mathcal{B}_c(X)$, by 562K, so $\mathrm{T} \subseteq \mathcal{B}_c(X)$. Set

$$\mathcal{C} = \{\textstyle\prod_{k\in\mathbb{N}} E_k : E_k \in \mathcal{B}_c(X_k) \text{ for every } k \in \mathbb{N},\ \{k : E_k \neq X_k\} \text{ is finite}\},$$

$$\mathcal{C}_o = \{\textstyle\prod_{k\in\mathbb{N}} G_k : G_k \subseteq X_k \text{ is open for every } k \in \mathbb{N},\ \{k : F_k \neq X_k\} \text{ is finite}\},$$

$$\mathcal{C}_c = \{\textstyle\prod_{k\in\mathbb{N}} F_k : F_k \subseteq X_k \text{ is closed for every } k \in \mathbb{N},\ \{k : F_k \neq X_k\} \text{ is finite}\}.$$

Then every member of T can be expressed as the union of a finite disjoint family in $\mathcal{C}$. $\overline{C} \in \mathcal{C}_c$ for every $C \in \mathcal{C}$, so the closure of any member of T can be expressed as the union of finitely many members of $\mathcal{C}_c$ and belongs to T. The complement of a member of $\mathcal{C}_c$ can be expressed as the union of finitely many members of $\mathcal{C}_o$, so any open set belonging to T can be expressed as the union of finitely many members of $\mathcal{C}_o$.

(ii) For $m \in \mathbb{N}$ write $\mathrm{T}_m = \bigotimes_{k\geq m} \mathcal{B}_c(X_k)$ for the algebra of subsets of $\prod_{k\geq m} X_k$ generated by sets of the form $\{x : x(k) \in E_k\}$ for $k \geq m$ and $E_k \in \mathcal{B}_c(X_k)$. Then we have an additive functional $\nu_m : \mathrm{T}_m \to [0,1]$ defined by saying that $\nu_m(\prod_{k\geq m} E_k) = \prod_{k=m}^{\infty} \mu_k E_k$ whenever $E_k \in \mathcal{B}_c(X_k)$ for every $k \geq m$ and $\{k : E_k \neq X_k\}$ is finite (326Q). Now if $m \in \mathbb{N}$ and $W \in \mathrm{T}_m$ then

$$\nu_m W = \int \nu_{m+1}\{v : {<}t{>}^\frown v \in W\}\mu_m(dt),$$

where I write ${<}t{>}^\frown v \in \prod_{k\geq m} X_k$ for $(t, v(m+1), v(m+2), \dots)$. **P** This is elementary for cylinder sets $W = \prod_{k\geq m} E_k$; now any other member of T_m is expressible as a finite disjoint union of such sets. **Q**

(c)(i) For open sets $W \subseteq X$ define

$$\lambda_0 W = \sup\{\nu_0 V : V \in \mathrm{T}, \overline{V} \subseteq W\}.$$

Then if $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union X, $\lim_{n\to\infty} \lambda_0 W_n = 1$. **P** Starting from the double sequence $\langle U_{ki}\rangle_{k,i\in\mathbb{N}}$, it is easy to build a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C}_o$ which runs over a base for the topology of X. Set $W'_n = \bigcup\{U_i : i \le n, \overline{U}_i \subseteq W_n\}$ for each n; then $\langle W'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets belonging to $\mathrm{T} = \mathrm{T}_0$, and $\bigcup_{n\in\mathbb{N}} W'_n = X$. **?** Suppose, if possible, that $\lim_{n\to\infty} \nu_0 W'_n \le 1 - 2^{-l}$ for some $l \in \mathbb{N}$. Then we can define $\langle t_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that $\nu_m V_{mn} \le 1 - 2^{-l-m}$ for every n, where

$$V_{mn} = \{v : v \in \prod_{k\ge m} X_k, \langle t_k\rangle_{k<m} \cup v \in W'_n\}.$$

In this case, define $f_{mn} : X_m \to [0,1]$ by setting

$$f_{mn}(t) = \nu_{m+1}\{w : w \in \prod_{k\ge m+1} X_k, <t>^\frown w \in V_{mn}\}.$$

By (b-ii), $\nu_m V_{mn} = \int f_{mn} d\mu_m$ for every m, while $\langle f_{mn}\rangle_{n\in\mathbb{N}}$ is non-decreasing.

Because every W'_n is a finite union of open cylinder sets, so is V_{mn}, and f_{mn} is lower semi-continuous, therefore resolvable; so

$$\int \sup_{n\in\mathbb{N}} f_{mn} d\mu_m = \sup_{n\in\mathbb{N}} \int f_{mn} d\mu_m = \lim_{n\to\infty} \nu_m V_{mn} \le 1 - 2^{-l-m}.$$

The set $F = \{t : \sup_{n\in\mathbb{N}} f_{mn}(t) \le 1 - 2^{-l-m-1}\}$ must be closed and non-empty, and we can set $t_m = \zeta(F)$, where ζ is the choice function of (a-ii). In this case,

$$V_{m+1,n} = \{w : <t_m>^\frown w \in V_{mn}\}, \quad \nu_{m+1} V_{m+1,n} = f_{mn}(t_m) \le 1 - 2^{-l-m-1}$$

for every n, and the induction continues.

At the end of the induction, however, $x = \langle t_k\rangle_{k\in\mathbb{N}}$ belongs to X, so belongs to W'_n for some n. There must be an m such that W'_n is determined by coordinates less than m, and now $V_{mn} = \prod_{k\ge m} X_k$, so $\nu_m V_{mn} = 1$; which is supposed to be impossible. **X**

We conclude that

$$1 = \lim_{n\to\infty} \nu_0 W'_n = \lim_{n\to\infty} \nu_0 \overline{W'_n} = \lim_{n\to\infty} \lambda_0 W_n$$

because $\overline{W'_n}$ is a closed member of T included in W_n for each n. **Q**

(ii) λ_0 satisfies the conditions of 563H. **P**

(α) Of course $\lambda_0\emptyset = \emptyset$ and $\lambda_0 W \le \lambda_0 W'$ whenever $W \subseteq W'$; also $\lambda_0 X = 1$ is finite.

(β) Suppose that $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets in X with union W, and $\epsilon > 0$. Then there is a closed $V \in \mathrm{T}$ such that $V \subseteq W$ and $\nu_0 V \ge \lambda_0 W - \epsilon$. Set $W'_n = (X \setminus V) \cup W_n$ for each n; then $\langle W'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union X, so by (i) there are an $n \in \mathbb{N}$ such that $\lambda_0 W'_n \ge 1 - \epsilon$, and a closed $V' \in \mathrm{T}$ such that $V' \subseteq W'_n$ and $\nu_0 V' \ge 1 - 2\epsilon$. Now $V \cap V'$ is a closed member of T included in W_n and

$$\lambda W_n \ge \nu_0(V \cap V') \ge \nu_0 V - 2\epsilon \ge \lambda W - 3\epsilon.$$

As ϵ is arbitrary, $\lambda_0 W \le \lim_{n\to\infty} \lambda_0 W_n$; the reverse inequality is trivial, so we have equality.

(γ) Let W, $W' \subseteq X$ be open sets. As in (i), we have non-decreasing sequences $\langle W_n\rangle_{n\in\mathbb{N}}$, $\langle W'_n\rangle_{n\in\mathbb{N}}$ of open members of T such that

$$W = \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{n\in\mathbb{N}} \overline{W}_n, \quad W' = \bigcup_{n\in\mathbb{N}} W'_n = \bigcup_{n\in\mathbb{N}} \overline{W'_n}.$$

In this case

$$W \cap W' = \bigcup_{n\in\mathbb{N}} W_n \cap W'_n = \bigcup_{n\in\mathbb{N}} \overline{W_n \cap W'_n},$$

$$W \cup W' = \bigcup_{n\in\mathbb{N}} W_n \cup W'_n = \bigcup_{n\in\mathbb{N}} \overline{W_n \cup W'_n}.$$

Also

$$\lambda_0 W = \lim_{n\to\infty} \lambda_0 W_n \le \lim_{n\to\infty} \nu_0 W_n \le \lim_{n\to\infty} \nu_0 \overline{W}_n \le \lambda_0 W,$$

so these are all equal; the same applies to the sequences converging to W', $W \cap W'$ and $W \cup W'$, so

$$\begin{aligned}\lambda_0 W + \lambda_0 W' &= \lim_{n\to\infty} \nu_0 W_n + \nu_0 W'_n \\ &= \lim_{n\to\infty} \nu_0(W_n \cap W'_n) + \nu_0(W_n \cup W'_n) \\ &= \lambda_0(W \cap W') + \lambda_0(W \cup W'). \ \mathbf{Q}\end{aligned}$$

(d) By 563H, we have a Borel-coded measure λ on X extending λ_0. Now λ extends ν_0. **P** If $C \in \mathcal{C}$ and $\epsilon > 0$, express C as $\prod_{k\in\mathbb{N}} E_k$ where $E_k \in \mathcal{B}_c(X_k)$ for every k and there is an m such that $E_k = X_k$ for $k > m$. For each $k \le m$, there is an open set $G_k \supseteq E_k$ such that $\mu_k G_k \le \mu_k E_k + \frac{\epsilon}{m+1}$ (563Fd); setting $G_k = X_k$ for $k > m$ and $W = \prod_{k\in\mathbb{N}} G_k$, $C \subseteq W$ and

$$\lambda C \le \lambda W = \lambda_0 W \le \nu_0 W = \prod_{k=0}^{m} \mu_k G_k \le \nu_0 C + \epsilon.$$

As ϵ is arbitrary, $\lambda C \le \nu_0 C$. This is true for every $C \in \mathcal{C}$. As both λ and ν_0 are additive, $\lambda W \le \nu_0 W$ for every $W \in \mathrm{T}$; as $\lambda X = \nu_0 X = 1$, λ agrees with ν_0 on T. **Q**

In particular, λ agrees with ν_0 on $\mathcal{C}$, as required.

564X Basic exercises (a) Let X be a second-countable space and μ a Borel-coded measure on X. Let $E \in \mathcal{B}_c(X)$ and let μ_E be the Borel-coded measure on X defined as in 563Fa. Show that $\int f d\mu_E$ is defined and equal to $\int f \times \chi E\,d\mu$ for every $f \in \mathcal{L}^1(\mu)$.

(b) Let X be a countably compact topological space. (i) Show that $C(X) = C_b(X)$. (ii) Show that every positive linear functional $f : C(X) \to \mathbb{R}$ is sequentially smooth. (iii) Show that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in the normed space $C(X)$ is weakly convergent to 0 iff it is bounded for the weak topology and pointwise convergent to 0. (iv) Prove this without using measure theory. (*Hint*: Fremlin 74, A2F.)

(c) Let X be a topological space and μ a Baire-coded measure on X. (i) Describe constructions for normed Riesz spaces $L^p(\mu)$ for $1 < p \le \infty$. (ii) Show that if X is second-countable, μ is codably σ-finite and $1 < p < \infty$ then $L^p(\mu)$ is a Dedekind complete Banach lattice with an order-continuous norm, while $L^2(\mu)$ is a Hilbert space.

(d) In 564O, show that λ is uniquely defined. Hence show that we have commutative and associative laws for the product measure construction.

564Y Further exercises (a) Let X be a locally compact completely regular topological group. Show that there is a non-zero left-translation-invariant Baire-coded measure on X.

(b) Let I be a set and $X = \{0,1\}^I$. Write Z for $\{0,1\}^{\mathbb{N}}$. For $\theta : \mathbb{N} \to I$ define $g_\theta : X \to Z$ by setting $g_\theta(x) = x\theta$ for $x \in X$. Let $\phi : \mathcal{T} \to \mathcal{B}_c(Z)$ be an interpretation of Borel codes for subsets of Z defined from a sequence running over a base for the topology of Z. Let Σ be the family of subsets of X of the form $\phi'(\theta,T) = g_\theta^{-1}[\phi(T)]$ where $\theta \in I^{\mathbb{N}}$ and $T \in \mathcal{T}$; say that a codable family in Σ is one of the form $\langle\phi'(\theta_i,T_i)\rangle_{i\in I}$. Show that there is a functional $\mu : \Sigma \to [0,1]$ such that $\mu\emptyset = 0$, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint codable sequence in Σ, and $\mu\{x : x{\restriction}J = w\} = 2^{-\#(J)}$ whenever $J \subseteq I$ is finite and $w \in \{0,1\}^J$.

(c) Suppose there is a disjoint sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of doubleton sets such that for every function f with domain $\mathbb{N}$ the set $\{n : f(n) \in I_n\}$ is finite (Jech 73, 4.4). Set $I = \bigcup_{n\in\mathbb{N}} I_n$ and let Σ be the algebra of subsets of $\{0,1\}^I$ determined by coordinates in finite sets. Let $\lambda : \Sigma \to [0,1]$ be the additive functional such that $\lambda\{x : z \subseteq x\} = 2^{-k}$ whenever $J \in [I]^k$ and $z \in \{0,1\}^J$. Show that there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ, covering $\{0,1\}^I$, such that $\sum_{n=0}^{\infty} \lambda E_n < 1$.

564 Notes and comments In the definitions of 564A, I follow the principles of earlier volumes in allowing virtually measurable functions with conegligible domains to be counted as integrable. But you will see that in 564F and elsewhere I work with real-valued Baire measurable functions defined everywhere. The point is that while, if you wish to work through the basic theorems of Fourier analysis under the new rules, you will certainly need to deal with functions which are not defined everywhere, all the main theorems will depend on establising that you have sequences of sets and functions which are codable in appropriate senses. There is no way of coding members of $\mathcal{L}^0$ or $\mathcal{L}^1$ as I have defined them in 564A. What you will need to do is to build parallel structures, so that associated with each almost-everywhere-summable Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$ you have in hand a code τ for a codable Borel function $\tilde{f}$ equal almost everywhere to f, together with a code T for a conegligible codable Borel set E included in $\{x : x \in \operatorname{dom} f,\ f(x) = \tilde{f}(x)\}$. Provided that associated with every relevant sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ you can define appropriate sequences $\langle\tau_n\rangle_{n\in\mathbb{N}}$ and $\langle T_n\rangle_{n\in\mathbb{N}}$, you can hope to deduce the reqired properties of $\langle f_n\rangle_{n\in\mathbb{N}}$ by applying 564F to the sequence coded by $\langle\tau_n\rangle_{n\in\mathbb{N}}$.

Of course there are further significant technical differences between the treatment here and the more orthodox one I have employed elsewhere. In the ordinary theory, using the axiom of choice whenever convenient, a measure μ, thought of as a function defined on a σ-algebra of sets, carries in itself all the information needed to describe the space $\mathcal{L}^0(\mu)$. In the present context, we are dealing with functions μ defined on algebras which do not directly code the topologies on which the definition relies. So it would be safer to write $\mathcal{L}^0(\mathfrak{T},\mu)$. But of course what really matters is the collection of codable families of codable sets, and perhaps we should be thinking of a different level of abstraction. In the proof of 564N I have tried to cast the proof in a language which might be adaptable to other ways of coding sets and functions.

From 564K on, most of the results seem to depend on second-countability; it may be that something can be done with spaces which have well-orderable bases.

I offer 564Yb and 564Yc as positive and negative examples. The point is that in 564Yb there may be few sequences of functions from $\mathbb{N}$ to I, so that we get few codable sequences of sets. Of course, if I is well-orderable then $\{0,1\}^I$ is compact (561D) and we can use 564H. For well-orderable I, any continuous real-valued function on $\{0,1\}^I$ is determined by coordinates in some countable set, so that the methods of 564H and 564Yb will give the same measure.

565 Lebesgue measure without choice

I come now to the actual construction of non-trivial Borel-coded measures. Primary among them is of course Lebesgue measure on $\mathbb{R}^r$; we also have Hausdorff measures (565N-565O). For Lebesgue measure I begin, as in §115, with half-open intervals. The corresponding 'outer measure' may no longer be countably subadditive, so I call it 'Lebesgue submeasure'. Carathéodory's method no longer seems quite appropriate, as it smudges the distinction between 'negligible' and 'outer measure zero', so I use 563H to show that there is a Borel-coded measure agreeing with Lebesgue submeasure on open sets (565C-565D); it is the completion of this Borel-coded measure which I will call Lebesgue measure. We have a version of Vitali's theorem for well-orderable families (in particular, for countable families) of balls (565F). From this we can prove the Fundamental Theorem of Calculus in essentially its standard form (565M).

565A Definitions Throughout this section, except when otherwise stated, $r \geq 1$ will be a fixed integer. As in §115, I will say that a **half-open interval** in $\mathbb{R}^r$ is a set of the form

$$[a,b[= \{x : x \in \mathbb{R}^r,\, a(i) \leq x(i) < b(i) \text{ for } i < r\}.$$

For a half-open interval I, set $\lambda I = 0$ if $I = \emptyset$ and otherwise $\lambda I = \prod_{i=0}^{r-1} b(i) - a(i)$ where $I = [a,b[$. Now for $A \subseteq \mathbb{R}^r$ set

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j\rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals covering } A\}.$$

565B Proposition In the notation of 565A,

(a) the function $\theta : \mathcal{P}\mathbb{R}^r \to [0,\infty]$ is a submeasure,

(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}^r$.

proof (a) As in parts (a-i) to (a-iii) of the proof of 115D, $\theta\emptyset = 0$ and $\theta A \leq \theta B$ whenever $A \subseteq B$. If A, $B \subseteq \mathbb{R}^r$ and $\epsilon > 0$, we have sequences $\langle I_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ of half-open intervals such that

$$A \subseteq \bigcup_{n\in\mathbb{N}} I_n, \quad B \subseteq \bigcup_{n\in\mathbb{N}} J_n,$$

$$\sum_{n=0}^{\infty} \lambda I_n \leq \theta A + \epsilon, \quad \sum_{n=0}^{\infty} \lambda J_n \leq \theta B + \epsilon.$$

Set $K_{2n} = I_n$, $K_{2n+1} = J_n$ for $n \in \mathbb{N}$; then $A \cup B \subseteq \bigcup_{n\in\mathbb{N}} K_n$ so

$$\theta(A\cup B) \leq \sum_{n=0}^{\infty} \lambda K_n = \sum_{n=0}^{\infty} \lambda I_n + \sum_{n=0}^{\infty} \lambda J_n \leq \theta A + \theta B + 2\epsilon.$$

As ϵ is arbitrary, $\theta(A \cup B) \leq \theta A + \theta B$.

(b) The arguments of 114B/115B/115Db nowhere used any form of the axiom of choice, so can be used unchanged.

Definition I will call the submeasure θ **Lebesgue submeasure** on $\mathbb{R}^r$.

565C Lemma Let $\mathcal{I}$ be the family of half-open intervals in $\mathbb{R}^r$; let θ be Lebesgue submeasure, ν the functional defined from θ by the process of 563G, and Σ the domain of ν.

(a) Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{I}$. Then $E = \bigcup_{n\in\mathbb{N}} I_n$ belongs to Σ and $\nu E = \sum_{n=0}^{\infty} \nu I_n$.

(b) Every open set in $\mathbb{R}^r$ belongs to Σ.

(c) If G, $H \subseteq \mathbb{R}^r$ are open, then $\nu G + \nu H = \nu(G \cap H) + \nu(G \cup H)$.

(d) If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets then $\nu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} G_n$.

proof (a)(i) If $i < r$ and $\alpha \in \mathbb{R}$ then $\{x : x \in \mathbb{R}^r,\, x(i) < \alpha\} \in \Sigma$, as in 115F. So every half-open interval belongs to Σ. By 565Cb, $\nu I = \theta I = \lambda I$ for every $I \in \mathcal{I}$.

(ii)($\boldsymbol{\alpha}$) $\theta E = \sum_{n=0}^{\infty} \nu I_n$. $\mathbf{P}$ Because $\langle I_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}$ covering E, θE is at most $\sum_{n=0}^{\infty} \lambda I_n \leq \sum_{n=0}^{\infty} \nu I_n$. In the other direction,

$$\theta E \ge \sup_{n\in\mathbb{N}} \theta(\bigcup_{i\le n} I_i) = \sup_{n\in\mathbb{N}} \nu(\bigcup_{i\le n} I_i) = \sup_{n\in\mathbb{N}} \sum_{i\le n} \nu I_i = \sum_{n=0}^{\infty} \nu I_n. \text{ } \mathbf{Q}$$

(β) $E \in \Sigma$. **P** Let $A \subseteq \mathbb{R}^r$ be such that θA is finite, and $\epsilon > 0$. We have a sequence $\langle J_m \rangle_{m\in\mathbb{N}}$ in $\mathcal{I}$ such that $A \subseteq \bigcup_{m\in\mathbb{N}} J_m$ and $\sum_{m=0}^{\infty} \lambda J_m \le \theta A + \epsilon$ is finite. Let m be such that $\sum_{j=m+1}^{\infty} \lambda J_j \le \epsilon$, and set $K = \bigcup_{j\le m} J_j$; then

$$A \cap E \subseteq (K \cap E) \cup \bigcup_{j>m} J_j,$$

so

$$\theta(A \cap E) \le \theta(K \cap E) + \theta(\bigcup_{j>m} J_j) \le \theta(K \cap E) + \sum_{j=m+1}^{\infty} \lambda J_j \le \theta(K \cap E) + \epsilon.$$

Similarly, $\theta(A \setminus E) \le \theta(K \setminus E) + \epsilon$. Next, by ($\alpha$) applied to $\langle J_j \cap I_i \rangle_{i\in\mathbb{N}}$ or otherwise, $\sum_{i=0}^{\infty} \nu(J_j \cap I_i)$ is finite for every j, so there is an $n \in \mathbb{N}$ such that $\sum_{j=0}^{m} \sum_{i=n+1}^{\infty} \nu(J_j \cap I_i) \le \epsilon$. Set $L = \bigcup_{i\le n} I_i$; then

$$K \cap E \subseteq (K \cap L) \cup \bigcup_{j\le m, i>n} J_j \cap I_i,$$

$$\begin{aligned}\theta(K \cap E) &\le \theta(K \cap L) + \theta(\bigcup_{j\le m, i>n} J_j \cap I_i) \\ &\le \theta(K \cap L) + \sum_{j=0}^{m} \sum_{i=n+1}^{\infty} \nu(J_j \cap I_i) \le \theta(K \cap L) + \epsilon.\end{aligned}$$

Assembling these,

$$\begin{aligned}\theta A &\le \theta(A \cap E) + \theta(A \setminus E) \le \theta(K \cap E) + \theta(K \setminus E) + 2\epsilon \\ &\le \theta(K \cap L) + \theta(K \setminus L) + 3\epsilon = \theta K + 3\epsilon\end{aligned}$$

(because we know that $L \in \Sigma$)

$$\le \theta A + 3\epsilon.$$

As ϵ is arbitrary, $\theta A = \theta(A \cap E) + \theta(A \setminus E)$. This was on the assumption that θA was finite; but of course it is also true if $\theta A = \infty$. As A is arbitrary, $E \in \Sigma$. **Q**

(γ) Accordingly $\nu E = \theta E = \sum_{n=0}^{\infty} \nu I_n$.

(b) Let $\mathcal{I}_0$ be the family of dyadic half-open intervals in $\mathbb{R}^r$ of the form $[2^{-k}z, 2^{-k}(z + \mathbf{1})[$ where $k \in \mathbb{N}$, $z \in \mathbb{Z}^r$ and $\mathbf{1} = (1, \dots, 1)$. Note that $\mathcal{I}_0$ is countable and that if I, $J \in \mathcal{I}_0$ then either $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$. Also any non-empty subset of $\mathcal{I}_0$ has a maximal element.

If $G \subseteq \mathbb{R}^r$ is open, set $\mathcal{J} = \{I : I \in \mathcal{I}_0, I \subseteq G\}$ and let $\mathcal{J}'$ be the set of maximal elements of $\mathcal{J}$. Then $\mathcal{J}'$ is disjoint and countable, so by (a-ii) $G = \bigcup \mathcal{J} = \bigcup \mathcal{J}'$ belongs to Σ.

(c) Because ν is additive on Σ,

$$\nu(G \cup H) + \nu(G \cap H) = \nu G + \nu(H \setminus G) + \nu(H \cap G) = \nu G + \nu H$$

for all open sets G, $H \subseteq \mathbb{R}^r$.

(d) This time, let $\mathcal{J}$ be $\bigcup_{n\in\mathbb{N}}\{I : I \in \mathcal{I}_0, I \subseteq G_n\}$; again, let $\mathcal{J}'$ be the set of maximal elements of $\mathcal{J}$. Then $G = \bigcup \mathcal{J} = \bigcup \mathcal{J}'$, so

$$\nu G = \sum_{J\in\mathcal{J}'} \nu J = \sup_{\mathcal{K}\subseteq\mathcal{J}' \text{ is finite}} \sum_{J\in\mathcal{K}} \nu J \le \sup_{n\in\mathbb{N}} \nu G_n = \lim_{n\to\infty} \nu G_n \le \nu G$$

because $\langle G_n \rangle_{n\in\mathbb{N}}$ is non-decreasing.

565D Definition By 563H there is a unique Borel-coded measure μ on $\mathbb{R}^r$ such that $\mu G = \nu G = \theta G$ for every open set $G \subseteq \mathbb{R}^r$. I will say that **Lebesgue measure** on $\mathbb{R}^r$ is the completion $\hat{\mu}$ of μ; the sets it measures will be **Lebesgue measurable**.

565E Proposition Let $\mathcal{I}$, θ, Σ, ν, μ and $\hat{\mu}$ be as in 565A-565D.

(a) μ is the restriction of θ to the algebra $\mathcal{B}_c(\mathbb{R}^r)$ of codable Borel sets.

(b) For every $A \subseteq \mathbb{R}^r$,

$$\theta A = \inf\{\hat{\mu}E : E \supseteq A \text{ is Lebesgue measurable}\} = \inf\{\mu G : G \supseteq A \text{ is open}\}.$$

(c) $E \in \Sigma$ and $\hat{\mu}E = \nu E = \theta E$ whenever E is Lebesgue measurable.

(d) $\hat{\mu}$ is inner regular with respect to the compact sets and outer regular with respect to the open sets.

proof (a) If $E \in \mathcal{B}_c(\mathbb{R}^r)$, then

$$\mu E = \inf\{\mu G : G \supseteq E \text{ is open}\}$$

(563Fd)

$$= \inf\{\theta G : G \supseteq E \text{ is open}\} \geq \theta E.$$

Next, if $I \subseteq \mathbb{R}^r$ is a half-open interval, it is a codable Borel set and

$$\lambda I = \inf\{\lambda J : J \in \mathcal{I},\, I \subseteq \operatorname{int} J\} \geq \inf\{\theta(\operatorname{int} J) : J \in \mathcal{I},\, I \subseteq \operatorname{int} J\}$$
$$\geq \inf\{\theta G : G \supseteq I \text{ is open}\} = \mu I \geq \theta I = \lambda I.$$

So μ and λ agree on $\mathcal{I}$. If now E is a codable Borel set and $\epsilon > 0$, there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $E \subseteq \bigcup_{n\in\mathbb{N}} I_n$ and $\sum_{n=0}^{\infty} \lambda I_n \leq \theta E + \epsilon$. But every I_n is resolvable (because it belongs to the algebra of sets generated by the open sets), so $\langle I_n\rangle_{n\in\mathbb{N}}$ is a codable sequence (562H) and

$$\mu E \leq \mu(\bigcup_{n\in\mathbb{N}} I_n) \leq \sum_{n=0}^{\infty} \mu I_n = \sum_{n=0}^{\infty} \lambda I_n \leq \theta E + \epsilon.$$

As E and ϵ are arbitrary, $\mu = \theta\restriction\mathcal{B}_c(\mathbb{R}^r)$.

(b) Suppose that $A \subseteq \mathbb{R}^r$. If $E \supseteq A$ is Lebesgue measurable, there are F, $H \in \mathcal{B}_c(\mathbb{R}^r)$ such that $E \triangle F \subseteq H$ and $\mu H = 0$, so that $E \subseteq F \cup H$ and

$$\theta A \leq \theta(F \cup H) = \mu(F \cup H) = \hat{\mu} E.$$

So we have

$$\theta A \leq \inf\{\hat{\mu} E : E \supseteq A \text{ is Lebesgue measurable}\} \leq \inf\{\mu G : G \supseteq A \text{ is open}\}.$$

In the other direction, given $\epsilon > 0$ there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$, covering A, such that $\sum_{n=0}^{\infty} \lambda I_n \leq \theta A + \epsilon$. As in (a) just above, $E = \bigcup_{n\in\mathbb{N}} I_n$ is a codable Borel set and $\mu E \leq \sum_{n=0}^{\infty} \lambda I_n$; now there is an open $G \supseteq E$ such that $\mu G \leq \mu E + \epsilon \leq \theta A + 2\epsilon$. As ϵ is arbitrary,

$$\inf\{\mu G : G \supseteq A \text{ is open}\} \leq \theta A$$

and we have the equalities.

(c) Suppose that E is Lebesgue measurable, $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$. By (b), there is an open set $G \supseteq A$ such that $\mu G \leq \theta A + \epsilon$. Now

$$\theta(A \cap E) + \theta(A \setminus E) \leq \theta(G \cap E) + \theta(G \setminus E) \leq \hat{\mu}(G \cap E) + \hat{\mu}(G \setminus E)$$

(by (b))

$$= \hat{\mu} G = \mu G \leq \theta A + \epsilon.$$

As usual, this is enough to ensure that $E \in \Sigma$. Now (b) again tells us that $\hat{\mu} E = \theta E = \nu E$.

(d) Of course μ is locally finite, while $\mathbb{R}^r$ is a regular topological space. So 563F(d-ii) tells us that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets; it follows that $\hat{\mu}$ also is. Next, every closed set is K_σ, while compact sets are resolvable so all sequences of compact sets are codable, so $\mu F = \sup\{\mu K : K \subseteq F$ is compact$\}$ for every closed set $F \subseteq \mathbb{R}^r$; so $\hat{\mu}$ is inner regular with respect to the compact sets.

565F Vitali's Theorem Let $\mathcal{C}$ be a well-orderable family of non-singleton closed balls in $\mathbb{R}^r$. For $\mathcal{I} \subseteq \mathcal{C}$ set

$$A_{\mathcal{I}} = \bigcap_{\delta>0} \bigcup\{C : C \in \mathcal{I},\, \operatorname{diam} C \leq \delta\}.$$

Let $\mathfrak{T}$ be the family of open subsets of $\mathbb{R}^r$. Then there are functions $\Psi : \mathcal{PC} \to \mathcal{PC}$ and $\Theta : \mathcal{PC} \times \mathbb{N} \to \mathfrak{T}$ such that $\Psi(\mathcal{I}) \subseteq \mathcal{I}$, $\Psi(\mathcal{I})$ is disjoint and countable, $\hat{\mu}(\Theta(\mathcal{I}, k)) \leq 2^{-k}$ and $A_{\mathcal{I}} \subseteq \bigcup \Psi(\mathcal{I}) \cup \Theta(\mathcal{I}, k)$ whenever $\mathcal{I} \subseteq \mathcal{C}$ and $k \in \mathbb{N}$. In particular,

$$A_{\mathcal{I}} \setminus \bigcup \Psi(\mathcal{I}) \subseteq \bigcap_{k\in\mathbb{N}} \Theta(\mathcal{I}, k)$$

is negligible.

proof We use the greedy algorithm of 221A/261B, but watching more carefully. Start by fixing on a well-ordering $\preccurlyeq$ of $\mathcal{C} \cup \{\emptyset\}$. Next, for each $n \in \mathbb{N}$, set $U_n = \{x : x \in \mathbb{R}^r,\, n < \|x\| < n+1\}$, where $\|\,\|$ is the Euclidean norm on $\mathbb{R}^r$. It will be convenient to fix at this point on a family $\langle G_{kn}\rangle_{k,n\in\mathbb{N}}$ of open sets such that $\hat{\mu} G_{kn} \leq 2^{-k-n-2}$ and

$\{x : \|x\| = n\} \subseteq G_{kn}$ for all k and n; for instance, G_{kn} could be an open shell with rational inner and outer radii (except for G_{k0}, which should be an open ball)[5].

Now define $C_{\mathcal{I}nm}$, for $\mathcal{I} \subseteq \mathcal{C}$ and m, $n \in \mathbb{N}$, by saying that

given $\langle C_{\mathcal{I}ni}\rangle_{i<m}$, $C_{\mathcal{I}nm}$ is to be the $\preccurlyeq$-first member of $\{\emptyset\} \cup (\mathcal{I} \cap \mathcal{P}U_n)$ which is disjoint from $\bigcup_{i<m} C_{\mathcal{I}ni}$ and has diameter at least

$$\frac{1}{2}\sup\{\operatorname{diam} C : C \in \{\emptyset\} \cup (\mathcal{I} \cap \mathcal{P}U_n) \text{ is disjoint from } \textstyle\bigcup_{i<m} C_{\mathcal{I}ni}\}.$$

(I take the diameter of the empty set to be 0, as usual.) Set

$$\Psi(\mathcal{I}) = \{C_{\mathcal{I}nm} : n, m \in \mathbb{N}\} \setminus \{\emptyset\}.$$

Because the U_n are disjoint, $\Psi(\mathcal{I})$ is a disjoint subfamily of $\mathcal{I}$, and of course it is countable. Just as in 261B, we find that for each $\mathcal{I} \subseteq \mathcal{C}$ and $n \in \mathbb{N}$ we have

$$A_{\mathcal{I}} \cap U_n \subseteq \textstyle\bigcup_{i<m} C_{\mathcal{I}ni} \cup \bigcup_{i\ge m} C'_{\mathcal{I}ni},$$

where for $C \in \mathcal{C}$ I write C' for the *open* ball with the same centre and *six* times the radius; $\emptyset'$ will be $\emptyset$. Just as in 261B, $\sum_{m=0}^{\infty} \hat\mu C'_{\mathcal{I}nm} \le 6^r \hat\mu B(\mathbf{0}, n+1)$ is finite. So, for each k and n, we can take the first m_{kn} such that $\sum_{i=m_{kn}}^{\infty} \hat\mu C'_{\mathcal{I}ni} \le 2^{-n-k-2}$. Now set

$$\Theta(\mathcal{I}, k) = \textstyle\bigcup_{n\in\mathbb{N}} G_{kn} \cup \bigcup_{n\in\mathbb{N}, i\ge m_{kn}} C'_{\mathcal{I}ni};$$

we shall have $A \setminus \bigcup \Psi(\mathcal{I}) \subseteq \Theta(\mathcal{I}, k)$ and

$$\hat\mu\Theta(\mathcal{I}, k) \le \textstyle\sum_{n=0}^{\infty} \hat\mu G_{kn} + \sum_{n=0}^{\infty}\sum_{i=m_n}^{\infty} \hat\mu C'_{\mathcal{I}ni} \le 2^{-k},$$

as required.

565G Proposition Let $A \subseteq \mathbb{R}^r$ be any set. Then its Lebesgue submeasure is

$$\theta A = \inf\{\textstyle\sum_{n=0}^{\infty} \hat\mu B_n : \langle B_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of closed balls covering } A\}.$$

proof Let $\epsilon > 0$. Then there is a (non-empty) open set $G \supseteq A$ with $\hat\mu G \le \theta A + \epsilon$. Use Vitali's theorem, with $\mathcal{C}$ the family of closed balls with rational centres and non-zero rational radii, to see that there is a disjoint sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of balls included in G such that $\hat\mu(A \setminus \bigcup_{n\in\mathbb{N}} C_n) = 0$ and $\sum_{n\in\mathbb{N}} \hat\mu C_n \le \theta A + \epsilon$. Next, cover $A \setminus \bigcup_{n\in\mathbb{N}} C_n$ by a sequence of half-open intervals with measures summing to not more than ϵ, and expand these to balls with measures summing to not more than $\epsilon r^{r/2}$, as in the proof of 261F. Interleaving this sequence with the C_n, we get a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of balls, covering A, with $\sum_{n=0}^{\infty} \hat\mu B_n \le \theta A + (1 + r^{r/2})\epsilon$. So

$$\theta A \ge \inf\{\textstyle\sum_{n=0}^{\infty} \hat\mu B_n : \langle B_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of closed balls covering } A\}.$$

The reverse inequality is trivial (563C(a-ii)).

565H Corollary Lebesgue measure is invariant under isometries.

proof We can see from its definition that Lebesgue submeasure is translation-invariant, so Lebesgue measure also is. Consequently two balls with the same radii have the same measure. Isometries of $\mathbb{R}^r$ take closed balls to closed balls with the same radii, so 565G gives the result.

565I Lemma (a) Writing $C_k(\mathbb{R}^r)$ for the space of continuous real-valued functions on $\mathbb{R}^r$ with compact support, $C_k(\mathbb{R}^r) \subseteq \mathcal{L}^1$.

(b) There is a countable set $D \subseteq C_k(\mathbb{R}^r)$ such that $\{g^\bullet : g \in D\}$ is norm-dense in L^1.

proof (a) This is elementary; every continuous function is resolvable, therefore a codable Borel function and belongs to $\mathcal{L}^0$; if in addition it has compact support it is dominated by an integrable function and is integrable, by 564E(c-i).

(b)(i) Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}^r$, consisting of bounded sets and closed under finite unions. Let D_0 be the set of functions of the form $x \mapsto \max(0, 1 - 2^k\rho(x, \mathbb{R}^r \setminus U))$ for $U \in \mathcal{U}$ and $k \in \mathbb{N}$, and D the set of rational linear combinations of members of D_0; then D is a countable subset of $C_k(\mathbb{R}^r)$.

(ii) If $E \subseteq \mathbb{R}^r$ is a Borel-codable set of finite measure, and $\epsilon > 0$, then by 565Ed there are a compact set $K \subseteq E$ and an open set $G \supseteq E$ such that $\hat\mu(G \setminus K) \le \epsilon$. Now there are a $U \in \mathcal{U}$ such that $K \subseteq U \in G$ and a $g \in D_0$ such that $\chi K \le g \le \chi U$, so that $\int |g - \chi E| \le \epsilon$.

[5] Of course we can still use the similarity argument from part (g) of the proof of 261B to check that thin shells have small measure.

(iii) It follows that whenever f is a simple codable Borel function, in the sense of 564Aa, and $\epsilon > 0$ there is a $g \in D$ such that $\int |f - g| \le \epsilon$.

(iv) If $f \in \mathcal{L}^1$ and $\epsilon > 0$ there are a simple function g such that $\int |f - g| \le \frac{1}{2}\epsilon$ and an $h \in D$ such that $\int |g - h| \le \frac{1}{2}\epsilon$, so that $\int |f - h| \le \epsilon$.

565J Lemma Suppose that f is an integrable function on $\mathbb{R}^r$, and that $\int_I f \ge 0$ for every half-open interval $I \subseteq \mathbb{R}^r$. Then $f(x) \ge 0$ for almost every $x \in \mathbb{R}^r$.

proof (a) Note first that any finite union E of half-open intervals is expressible as a finite disjoint union of half-open intervals. So $\int_E f \ge 0$.

(b) Suppose that g is a simple codable Borel function such that $\int_E g \le \epsilon$ whenever E is a finite union of half-open intervals. Then $\int g^+ \le \epsilon$. **P** Set $F = \{x : g(x) > 0\}$, and take any $\eta > 0$. Then there are a compact $K \subseteq F$ and an open $G \supseteq F$ such that $\hat\mu(G \setminus K) \le \eta$. There is a set E, a finite union of half-open intervals, such that $K \subseteq E \subseteq G$. In this case,

$$|\int g^+ - \int_E g| \le \int |g \times \chi(E \triangle F)| \le \|g\|_\infty \hat\mu(G \setminus K), \quad \int g^+ \le \epsilon + \eta\|g\|_\infty;$$

as η is arbitrary, we have the result. **Q**

(c) We know that there is a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ of simple Borel functions such that $f =_{\text{a.e.}} \lim_{n\to\infty} g_n$ and the sum $\sum_{n=0}^\infty \int |g_{n+1} - g_n|$ is finite. Set $\epsilon_n = \sum_{i=n}^\infty \int |g_{i+1} - g_i|$ for each n; then $\int |f - g_n| \le \epsilon_n$, because $\int |g_m - g_n| \le \epsilon_n$ for every $m \ge n$. So if E is a finite union of half-open intervals,

$$\int_E g_n = \int g_n \times \chi E \ge \int f \times \chi E - \int |f - g_n| \ge -\epsilon_n;$$

by (a), applied to $-g_n$, $\int g_n^- \le \epsilon_n$. By 564Be,

$$f^- =_{\text{a.e.}} \lim_{n\to\infty} g_n^- =_{\text{a.e.}} \liminf_{n\to\infty} g_n^- = 0$$

almost everywhere, as required.

565K Theorem A monotonic function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

Remark Of course 'almost everywhere' here is with respect to Lebesgue measure on $\mathbb{R}$; in this result and the next two I am taking $r = 1$.

proof We can use the ideas in 222A if we refine them using 565F. First, $\mathcal{C}$ will be the set of closed non-trivial intervals with rational endpoints; take Ψ and Θ as in 565F. It will be enough to deal with the case of non-decreasing f. For $a < b$ in $\mathbb{R}$, set $f^*([a, b]) = [f(a), f(b)]$. I shall repeatedly use the fact that if $\mathcal{I} \subseteq \mathcal{C}$ is disjoint, then

$$\hat\mu(\bigcup_{C\in\mathcal{I}} f^*(C)) = \sum_{C\in\mathcal{I}} \hat\mu f^*(C),$$

because $\mathcal{I}$ is countable and $f^*(C) \cap f^*(C')$ contains at most one point for any distinct C, $C' \in \mathcal{I}$, and we can use 563C(a-iv).

(a) To see that $D^*f < \infty$ a.e., set $E_m = \{x : |x| < m,\ D^*f(x) > 2^m(1 + f(m) - f(-m))\}$ and

$$\begin{aligned}\mathcal{I}_m = \{[\alpha, \beta] : \alpha, \beta \in \mathbb{Q},\ &-m < \alpha < \beta < m,\\ &f(\beta) - f(\alpha) > 2^m(1 + f(m) - f(-m))(\beta - \alpha)\}\\ = \{C : C \in \mathcal{C},\ &C \subseteq\ ]-m, m[\,, \hat\mu f^*(C) > 2^m(1 + f(m) - f(-m))\hat\mu C\}\end{aligned}$$

for each m. Then, in the language of 565F, $E_m \subseteq A_{\mathcal{I}_m}$. (If $x \in E_m$ and $\delta > 0$, then x is an endpoint of a non-trivial closed interval $[\alpha, \beta] \subseteq\ ]-m, m[$, of length less than δ, such that $f(\beta) - f(\alpha) > 2^m(1 + f(m) - f(-m))(\beta - \alpha)$. Now we can expand $[\alpha, \beta]$ slightly to get an interval $[\alpha', \beta'] \in \mathcal{I}_m$ of length at most δ.) So $E_m \subseteq \bigcup \Psi(\mathcal{I}_m) \cup \Theta(\mathcal{I}_m, k)$ for each k. $\Psi(\mathcal{I}_m)$ is a countable family of closed sets, and

$$\begin{aligned}2^m(1 + f(m) - f(-m)) \sum_{C\in\Psi(\mathcal{I}_m)} \hat\mu C &\le \sum_{C\in\Psi(\mathcal{I}_m)} \hat\mu f^*(C)\\ &= \hat\mu\big(\bigcup_{C\in\Psi(\mathcal{I}_m)} f^*(C)\big) \le f(m) - f(-m).\end{aligned}$$

So $\sum_{C\in\Psi(\mathcal{I}_m)} \hat\mu C \le 2^{-m}$. Setting $H_m = \Theta(\mathcal{I}_m, m) \cup \bigcup\{\int C : C \in \Psi(\mathcal{I}_m)\}$, $\mu H_m \le 2^{-m+1}$ and $E_m \setminus \mathbb{Q} \subseteq H_m$.

Set $E = \{x : D^*f(x) = \infty\}$, and take any $n \in \mathbb{N}$. Then

$$E \setminus \mathbb{Q} \subseteq \bigcup_{m\ge n} E_m \setminus \mathbb{Q} \subseteq \bigcup_{m\ge n} H_m.$$

Now 563C(a-ii) tells us that

$$\hat{\mu}(\bigcup_{m\ge n} H_m \le \sum_{m=n}^{\infty}\mu H_m \le 2^{-n+1}$$

for each n, so that $E\setminus\mathbb{Q}$ is included in a negligible G_δ set and $\hat{\mu}E=\hat{\mu}(E\setminus\mathbb{Q})=0$. Thus D^*f is finite a.e.

(b) To see that $D^*f\le_{\text{a.e.}} D_*f$, we use similar ideas, but with an extra layer of complexity, corresponding to the double use of Vitali's theorem. Set $F=\{x:D_*(f)(x)<D^*f(x)\}$. Take any $\epsilon>0$; because $\mathbb{Q}$ is countable, there is a family $\langle\epsilon_{mqq'}\rangle_{m\in\mathbb{N},q,q'\in\mathbb{Q}}$ of strictly positive numbers such that $\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q}}\epsilon_{mqq'}\le\frac{1}{2}\epsilon$. For q, $q'\in\mathbb{Q}$ and m, $k\in\mathbb{N}$ let $\mathcal{I}_{mqk}$, $\mathcal{J}_{mqk}$ be

$$\{C:C\in\mathcal{C},\,C\subseteq\,]-m,m[,\,\hat{\mu}C\le 2^{-k},\,\hat{\mu}f^*(C)\ge q\hat{\mu}C\},$$

$$\{C:C\in\mathcal{C},\,C\subseteq\,]-m,m[,\,\hat{\mu}C\le 2^{-k},\,\hat{\mu}f^*(C)\le q\hat{\mu}C\}$$

respectively. For m, $k\in\mathbb{N}$ and q, $q'\in\mathbb{Q}$ set

$$G_{mqq'k}=\bigcup\{\operatorname{int}C:C\in\mathcal{I}_{mq'k}\}\cap\bigcup\{\operatorname{int}C:C\in\mathcal{J}_{mqk}\};$$

then $\langle G_{mqq'k}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence of open sets of finite measure. So, setting $F_{mqq'}=\bigcap_{k\in\mathbb{N}}G_{mqq'k}$, we can find a family $\langle k(m,q,q')\rangle_{m\in\mathbb{N},q,q'\in\mathbb{Q}}$ in $\mathbb{N}$ such that

$$\hat{\mu}(G_{m,q,q',k(m,q,q')}\setminus F_{mqq'})\le\min(1,\frac{q'-q}{q})\epsilon_{mqq'}$$

whenever $m\in\mathbb{N}$, q, $q'\in\mathbb{Q}$ and $0<q<q'$ (563C(b-ii)). Write $H_{mqq'}$ for $G_{m,q,q',k(m,q,q')}$.

If $m\in\mathbb{N}$ and $0<q<q'$ in $\mathbb{Q}$, set

$$\mathcal{J}'_{mqq'}=\{C:C\in\mathcal{C},\,C\subseteq H_{mqq'},\,\hat{\mu}f^*(C)\le q\hat{\mu}C\}.$$

Then $F_{mqq'}\subseteq H_{mqq'}$, so every point of $F_{mqq'}$ belongs to arbitrarily small intervals belonging to $\mathcal{J}'_{mqq'}$; accordingly $F_{mqq'}\setminus\bigcup\Psi(\mathcal{J}'_{mqq'})$ is negligible.

Now let $\mathcal{I}'_{mqq'}$ be the set

$$\{C:C\in\mathcal{C},\,C\subseteq C'\text{ for some }C'\in\Psi(\mathcal{J}'_{mqq'}),\,\hat{\mu}f^*(C)\ge q'\hat{\mu}C\}.$$

Then every point of $F_{mqq'}\cap\bigcup\Psi(\mathcal{J}'_{mqq'})\setminus\mathbb{Q}$ belongs to arbitrarily small members of $\mathcal{I}'_{mqq'}$, so $F_{mqq'}\setminus\bigcup\Psi(\mathcal{I}'_{mqq'})$ is negligible.

Now we come to the calculation at the heart of the proof. If $m\in\mathbb{N}$ and $0<q<q'$ in $\mathbb{Q}$,

$$\begin{aligned}q'\hat{\mu}F_{mqq'}&\le q'\hat{\mu}(\bigcup\Psi(\mathcal{I}'_{mqq'}))=q'\sum_{C\in\Psi(\mathcal{I}'_{mqq'})}\hat{\mu}C\\&\le\sum_{C\in\Psi(\mathcal{I}'_{mqq'})}\hat{\mu}f^*(C)=\hat{\mu}(\bigcup_{C\in\Psi(\mathcal{I}'_{mqq'})}f^*(C))\le\hat{\mu}(\bigcup_{C\in\Psi(\mathcal{J}'_{mqq'})}f^*(C))\end{aligned}$$

(because every member of $\mathcal{I}'_{mqq'}$ is included in a member of $\Psi(\mathcal{J}'_{mqq'})$)

$$\begin{aligned}&=\sum_{C\in\Psi(\mathcal{J}'_{mqq'})}\hat{\mu}f^*(C)\le q\sum_{C\in\Psi(\mathcal{J}'_{mqq'})}\hat{\mu}C\\&=q\hat{\mu}(\bigcup\Psi(\mathcal{J}'_{mqq'})\le q\hat{\mu}H_{mqq'}\le q\hat{\mu}F_{mqq'}+(q'-q)\epsilon_{mqq'},\end{aligned}$$

and $\hat{\mu}F_{mqq'}\le\epsilon_{mqq'}$, $\hat{\mu}H_{mqq'}\le 2\epsilon_{mqq'}$. But this means that

$$F\setminus\mathbb{Q}\subseteq\bigcup_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'}F_{mqq'}\subseteq\bigcup_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'}H_{mqq'},$$

which has measure at most

$$\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'}\hat{\mu}H_{mqq'}\le 2\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'}\epsilon_{mqq'}\le\epsilon.$$

The process described here gives a recipe, starting from $\epsilon>0$, for finding an open set of measure at most ϵ including $F\setminus\mathbb{Q}$. So we can repeat this for each term of a sequence converging to 0 to define a negligible G_δ set including $F\setminus\mathbb{Q}$, and F must be negligible, as required.

565L Lemma Suppose that $F:\mathbb{R}\to\mathbb{R}$ is a bounded non-decreasing function. Then $\int F'$ is defined and is at most $\lim_{x\to\infty}F(x)-\lim_{x\to-\infty}F(x)$.

proof I copy the ideas of 222C. For each $n \in \mathbb{N}$, define $g_n : \mathbb{R} \to \mathbb{R}$ by setting $a_{nk} = 2^{-n+1}k(n+1) - n$ for $k \le 2^n$,

$$g_n(x) = \frac{2^{n-1}}{n+1}(F(a_{n,k+1}) - F(a_{nk})) \text{ if } k < 2^n \text{ and } a_{nk} \le x < a_{n,k+1},$$
$$= 0 \text{ if } x < -n \text{ or } x \ge n+2.$$

Then g_n is a simple Borel function and $F'(x) = \lim_{n\to\infty} g_n(x)$ whenever $F'(x)$ is defined, which is almost everywhere, by 565K. Also $\int g_n = F(n+2) - F(-n)$. Because the g_n are resolvable, $\langle g_n \rangle_{n\in\mathbb{N}}$ is codable; by Fatou's Lemma (564Fb),

$$\int F' \le \liminf_{n\to\infty} \int g_n = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x).$$

565M Theorem Let $F : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equiveridical:

(i) there is an integrable function f such that $F(x) = \int_{]-\infty,x[} f$ for every $x \in \mathbb{R}$,

(ii) F is of bounded variation, absolutely continuous on every bounded interval, and $\lim_{x\to-\infty} F(x) = 0$,

and in this case $F' =_{\text{a.e.}} f$.

proof (a) If $F(x) = \int_{]-\infty,x[} f$ for every $x \in \mathbb{R}$, take any $\epsilon > 0$. Then there is a $g \in C_k(\mathbb{R})$ such that $\int |f - g| \le \epsilon$ (565Ib). Let x_0 be such that $g(x) = 0$ for $x \le x_0$; then

$$|F(x)| \le \int |f - g| \le \epsilon$$

whenever $x \le x_0$. Set $\delta = \frac{\epsilon}{1+\|g\|_\infty}$. If $a_0 \le b_0 \le \ldots \le a_n \le b_n$ and $\sum_{i=0}^n b_i - a_i \le \delta$, then

$$\sum_{i=0}^n |F(b_i) - F(a_i)| = \sum_{i=0}^n |\int_{[a_i,b_i[} f| \le \sum_{i=0}^n \int_{[a_i,b_i[} |g| + \int_{[a_i,b_i[} |f - g|$$
$$\le \sum_{i=0}^n (b_i - a_i)\|g\|_\infty + \int |f - g| \le \delta\|g\|_\infty + \epsilon \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{x\to-\infty} F(x) = 0$ and F is absolutely continuous on every bounded interval. As for the variation of F, if $a_0 \le a_1 \le \ldots \le a_n$ then

$$\sum_{i=1}^n |F(a_i) - F(a_{i-1})| = \sum_{i=1}^n |\int_{[a_{i-1},a_i[} f| \le \sum_{i=1}^n \int_{[a_{i-1},a_i[} |f|$$
$$= \int_{[a_0,a_n[} |f| \le \int |f|,$$

so $\mathrm{Var}(F) \le \int |f|$ is finite.

Thus (i)$\Rightarrow$(ii).

(b) Moreover, under the conditions of (a), $F' =_{\text{a.e.}} f$. **P** Because f is the difference of two non-negative integrable functions, it is enough to consider the case $f \ge 0$ a.e., so that F is non-decreasing. In this case, applying 565L to the function $x \mapsto \mathrm{med}(F(a), F(x), F(b))$, we see that $\int_{[a,b[} F' \le \int_{[a,b[} f$ whenever $a \le b$ in $\mathbb{R}$; also, applying 565L to F itself, F' is integrable. Applying 565J to $f - F'$, we see that $F' \le_{\text{a.e.}} f$.

Recall that there is a countable subset D of $C_k(\mathbb{R})$ approximating all integrable functions in mean (565Ib). So there is a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ in D such that $\sum_{n=0}^\infty \int |g_n - f|$ is finite. Set $\tilde{g}_n = \sup_{i\le n} g_i^+$ for $n \in \mathbb{N}$; then all the $\tilde{g}_n$ are continuous, therefore resolvable, and $\langle \tilde{g}_n \rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable functions. By 564Fa, $g = \lim_{n\to\infty} \tilde{g}_n$ is defined a.e. and integrable. Let G_n, G be the indefinite integrals of $\tilde{g}_n$, g respectively. Then the arguments just used show that $G' \le_{\text{a.e.}} g$. But note that each G_n, being the indefinite integral of a continuous function, has $G_n' = \tilde{g}_n$ exactly, while $G_n' \le G'$ whenever G' is defined. So

$$g =_{\text{a.e.}} \lim_{n\to\infty} \tilde{g}_n = \lim_{n\to\infty} G_n' \le_{\text{a.e.}} G',$$

and $g =_{\text{a.e.}} G'$.

At this point observe that $\int \liminf_{n\to\infty} |g_n - f| = 0$, by 564Fb, so $f \le_{\text{a.e.}} g$, while $G - F$ is the indefinite integral of the essentially non-negative integrable function $g - f$. So $G' - F' \le_{\text{a.e.}} g - f =_{\text{a.e.}} G' - f$ and $f \le_{\text{a.e.}} F'$. So actually $f =_{\text{a.e.}} F'$, as hoped for. **Q**

(c) Now suppose that $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation, absolutely continuous on every bounded interval and $\lim_{x\to-\infty} F(x) = 0$. By 224D and 565L, F' is integrable; set $G(x) = \int_{]-\infty,x[} F'$ and $H(x) = F(x) - G(x)$ for $x \in \mathbb{R}$. By (b), $H' = F' - G'$ is zero a.e., while H, like F and G, is absolutely continuous on every bounded interval. But this means that H is constant. **P** Suppose that $a<b$ in $\mathbb{R}$ and $\epsilon>0$. Let $\delta\in\,]0,b-a[$ be such that $\sum_{i=0}^n|H(b_i)-H(a_i)|\le\epsilon$ whenever $a\le a_0\le b_0\le a_1\le b_1\le\ldots\le a_n\le b_n\le b$ and $\sum_{i=0}^n b_i-a_i\le\delta$ Set $E=\{x:x\in\,]a,b[,\ H'(x)=0\}$. Let $\mathcal{C}$ be the family of non-trivial closed subintervals $[c,d]$ of $]a,b[$ with rational endpoints such that $|H(d)-H(c)|\le\epsilon(d-c)$; then every point of E belongs to arbitrarily small members of $\mathcal{C}$. By Vitali's theorem (565F) there is a disjoint countable family $\mathcal{I}\subseteq\mathcal{C}$ such that $E\setminus\bigcup\mathcal{I}$ is negligible, so that

$$\sum_{I\in\mathcal{I}}\hat\mu I=\hat\mu(\bigcup\mathcal{I})=b-a.$$

Let $\mathcal{J}\subseteq\mathcal{I}$ be a finite subset such that $\sum_{I\in\mathcal{J}}\hat\mu I\ge b-a-\delta$; express $\mathcal{J}$ as $\langle[b_i,a_{i+1}]\rangle_{i<n}$ where $\langle b_i\rangle_{i<n}$ is strictly increasing. Setting $a_0=a$ and $b_n=b$, we have $a=a_0\le b_0\le\ldots\le a_n\le b_n=b$ and $\sum_{i=0}^n b_i-a_i\le\delta$. So

$$\begin{aligned}|H(b)-H(a)|&\le\sum_{i=0}^n|H(b_i)-H(a_i)|+\sum_{i=0}^{n-1}|H(a_{i+1})-H(b_i)|\\&\le\epsilon+\epsilon\sum_{i=0}^{n-1}(a_{i+1}-b_i)\le\epsilon(1+b-a).\end{aligned}$$

As a, b and ϵ are arbitrary, H is constant. **Q**

So $F-G$ is constant. As both F and G tend to 0 at $-\infty$, they are equal. Thus $F(x)=\int_{]-\infty,x[}F'$ for every x, and F is an indefinite integral.

565N Hausdorff measures Let (X,ρ) be a metric space and $s\in\,]0,\infty[$. As in §471, we can define **Hausdorff s-dimensional submeasure** $\theta_s:\mathcal{P}X\to[0,\infty]$ by writing

$$\begin{aligned}\theta_sA=\sup_{\delta>0}\inf\{\sum_{n=0}^\infty(\operatorname{diam}D_n)^s:\ &\langle D_n\rangle_{n\in\mathbb{N}}\text{ is a sequence of subsets of }X\text{ covering }A,\\&\operatorname{diam}D_n\le\delta\text{ for every }n\in\mathbb{N}\},\end{aligned}$$

counting $\operatorname{diam}\emptyset$ as 0 and $\inf\emptyset$ as ∞. As with Lebesgue submeasure, θ_s is a submeasure.

565O Theorem Let (X,ρ) be a second-countable metric space, and $s>0$. Then there is a Borel-coded measure μ on X such that $\mu K=\theta_sK$ whenever $K\subseteq X$ is compact and θ_sK is finite.

proof (a) To begin with, suppose that X is compact and θ_sX is finite.

(i) Let $\mathcal{U}$ be a countable base for the topology of X closed under finite unions; let $\preccurlyeq$ be a well-ordering of $\mathcal{U}$. Then for any compact $K\subseteq X$, $\delta>0$ and $\epsilon>0$, there are $U_0,\ldots,U_n\in\mathcal{U}$ such that $K\subseteq\bigcup_{i\le n}U_i$, $\operatorname{diam}U_i\le\delta$ for every i and $\sum_{i=0}^n\operatorname{diam}U_i\le\theta_sK+\epsilon$. **P** There is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $K\subseteq\bigcup_{n\in\mathbb{N}}A_n$, $\operatorname{diam}A_n\le\frac12\delta$ for every n and $\sum_{n=0}^\infty\operatorname{diam}A_n\le\theta_sK+\frac12\epsilon$. For each $n\in\mathbb{N}$, set

$$G_n=\{x:\rho(x,A_n)<\min(\tfrac14\delta,2^{-n-3}\epsilon)\}.$$

Then $\overline{A}_n$ is a compact subset of G_n so there is a $\preccurlyeq$-first $U_n\in\mathcal{U}$ such that $\overline{A}_n\subseteq U_n\subseteq G_n$. Now $\operatorname{diam}U_n\le\min(\delta,\operatorname{diam}A_n+2^{-n-2}\epsilon)$ for each n so $\sum_{n=0}^\infty\operatorname{diam}U_n\le\theta_sK+\epsilon$. But as K is compact there is an n such that $K\subseteq\bigcup_{i\le n}U_i$. **Q**

(ii) As in the proof of 471Da, θ_s is a 'metric submeasure', that is, $\theta_s(A\cup B)=\theta_sA+\theta_sB$ whenever A, $B\subseteq X$ and $\rho(A,B)>0$. It follows that $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)=\sum_{n=0}^\infty\theta_sK_n$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X. **P** Recall that $\rho(K,K')>0$ whenever K, K' are disjoint compact subsets of X; this is because $K\times K'$ is compact and $\rho:X\times X\to\mathbb{R}$ is continuous. So

$$\theta_s(\bigcup_{n\in\mathbb{N}}K_n)\ge\theta_s(\bigcup_{i\le n}K_i)=\sum_{i=0}^n\theta_sK_i$$

for every $n\in\mathbb{N}$, and $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)\ge\sum_{n=0}^\infty\theta_sK_n$. In the other direction, let $\epsilon>0$. Let $\preccurlyeq'$ be a well-ordering of $\bigcup_{n\in\mathbb{N}}\mathcal{U}^n$. Then for each $n\in\mathbb{N}$ there is a $\preccurlyeq'$-first finite sequence $U_{n0},\ldots,U_{nm_n}$ in $\mathcal{U}$ such that $K_n\subseteq\bigcup_{i\le m_n}U_{ni}$, $\operatorname{diam}U_{ni}\le\epsilon$ for every i and $\sum_{i=0}^{m_n}\operatorname{diam}U_{ni}\le\theta_sK_n+2^{-n}\epsilon$. Now $\langle U_{ni}\rangle_{n\in\mathbb{N},i\le m_n}$ witnesses that

$$\begin{aligned}\sum_{n=0}^\infty\theta_sK_n+2\epsilon\ge\inf\{\sum_{j=0}^\infty(\operatorname{diam}D_j)^s:\ &\langle D_n\rangle_{n\in\mathbb{N}}\text{ is a sequence of subsets of }X\\&\text{covering }\bigcup_{n\in\mathbb{N}}K_n,\ \operatorname{diam}D_j\le\epsilon\text{ for every }j\in\mathbb{N}\}.\end{aligned}$$

As ϵ is arbitrary, $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)\le\sum_{n=0}^\infty\theta_s(K_n)$. **Q**

(iii) If $G \subseteq X$ is open and $\epsilon > 0$, there is a compact set $K \subseteq G$ such that $\theta_s(G \setminus K) \le \epsilon$. **P** Set $K_0 = \{x : \rho(x, X \setminus G) \ge 1\}$ and for $n \ge 1$ set

$$K_n = \{x : 2^{-n} \le \rho(x, X \setminus G) \le 2^{n+1}\}.$$

Then $\sum_{n=0}^{\infty} \theta_s K_{2n}$ and $\sum_{n=0}^{\infty} \theta_s K_{2n+1}$ are both bounded by $\theta_s X < \infty$, so there is an $n \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \theta_s K_n \le \epsilon$. But this means that $\theta_s(\bigcup_{i \ge n} K_n) \le \epsilon$ (apply (b) to the odd and even terms separately). Set $K = \bigcup_{i \le n} K_i$; this works. **Q**

(iv) Writing $\mathfrak{T}$ for the topology of X, $\theta_s \restriction \mathfrak{T}$ satisfies the conditions of 563H. **P** Of course it is zero at $\emptyset$, monotonic and locally finite. If G, $H \in \mathfrak{T}$ and $\epsilon > 0$, let $K \subseteq G$, $L \subseteq H$ be compact sets such that $\theta_s(G \setminus K) + \theta_s(H \setminus L) \le \epsilon$. Then $K \setminus H$, $K \cap L$ and $L \setminus G$ are disjoint compact sets and

$$(G \cup H) \setminus ((K \setminus H) \cup (K \cap L) \cup (L \setminus G)), \quad (G \cap H) \setminus (K \cap L),$$

$$G \setminus ((K \setminus H) \cup (K \cap L)), \quad H \setminus ((L \setminus G) \cup (K \cap L))$$

are all included in $(G \setminus K) \cup (H \setminus L)$, so all have submeasure at most ϵ. But this means that $\theta_s(G \cup H) + \theta_s(G \cap H)$ and $\theta_s G + \theta_s H$ both differ from $\theta_s(K \setminus H) + 2\theta_s(K \cap L) + \theta_s(L \setminus G)$ by at most 2ϵ (upwards) and differ from each other by at most 2ϵ also. As ϵ is arbitrary, we have the modularity condition.

As for the sequential order-continuity, this is elementary; if $\langle G_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union G, and $\epsilon > 0$, there is a compact $K \subseteq G$ such that $\theta_s(G \setminus K) \le \epsilon$; now $K \subseteq G_n$ for some n, and $\theta_s G \le \theta_s G_n + \epsilon$. **Q**

(v) So 563H tells us that there is a Borel-coded measure μ on X extending $\theta_s \restriction \mathfrak{T}$. Now $\mu K = \theta_s K$ for every compact $K \subseteq X$. **P**

$$\mu K + \mu(X \setminus K) = \mu X = \theta_s X \le \theta_s K + \theta_s(X \setminus K) = \theta_s K + \mu(X \setminus K),$$

so $\mu K \le \theta_s K$. On the other hand, given $\epsilon > 0$, there is a compact $L \subseteq X \setminus K$ such that $\theta_s L \ge \theta_s(X \setminus K) - \epsilon$, and now

$$\theta_s K = \theta_s(K \cup L) - \theta_s L \le \theta_s X - \theta_s(X \setminus K) + \epsilon = \mu K + \epsilon;$$

as ϵ is arbitrary, $\mu K = \theta_s K$. **Q**

(vi) Note that 563H tells us that μ is the only Borel-coded measure extending $\theta_s \restriction \mathfrak{T}$, and must therefore be the only Borel-coded measure agreeing with θ_s on the compact sets.

(b) For the general case, let $\mathcal{K}$ be $\{K : K \subseteq X$ is compact, $\theta_s K < \infty\}$. Then (a) tells us that for every $K \in \mathcal{K}$ there is a unique Borel-coded measure μ_K on K agreeing with θ_s on the compact subsets of K. If K, $L \in \mathcal{K}$ and $K \subseteq L$, $\mu_L \restriction \mathcal{B}_c(K)$ is a Borel-coded measure on K (563E) agreeing with θ_s on the compact subsets of K, so μ_L extends μ_K. We therefore have a Borel-coded measure μ on X defined by setting $\mu E = \sup_{K \in \mathcal{K}} \mu_K(E \cap K)$ for every $E \in \mathcal{B}_c(X)$ (cf. 563E), and μ agrees with θ_s on $\mathcal{K}$, as required.

565X Basic exercises (a) (i) Show that Lebesgue submeasure θ and Lebesgue measure are translation-invariant. (ii) Show that if $A \subseteq \mathbb{R}^r$ and $\alpha \ge 0$ then $\theta(\alpha A) = \alpha^r \theta A$. (iii) Show that if $E \subseteq \mathbb{R}^r$ is measurable and $\alpha \in \mathbb{R}$ then αE is measurable.

(b) Suppose that there is a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of countable subsets of $[0,1]$ with union $[0,1]$. (i) Set $A = \bigcup_{m \le n} A_m + n$. Show that A belongs to the algebra Σ of 565C, that the Lebesgue submeasure of A is ∞, but that $A \cap [0,n]$ is Lebesgue negligible for every n. (ii) Set $B = \{2^{-n}x : n \in \mathbb{N}, x \in A_n\}$. Show that B has Lebesgue submeasure 0, but is not Lebesgue negligible.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. For half-open intervals $I \subseteq \mathbb{R}$ define $\lambda_g I$ by setting

$$\lambda_g \emptyset = 0, \quad \lambda_g\,[a,b[\, = \lim_{x \uparrow b} g(x) - \lim_{x \uparrow a} g(x)$$

if $a < b$. For any set $A \subseteq \mathbb{R}$ set

$$\theta_g A = \inf\{\sum_{j=0}^{\infty} \lambda_g I_j : \langle I_j \rangle_{j \in \mathbb{N}} \text{ is a sequence of half-open intervals covering } A.$$

Show that θ_g is a submeasure on $\mathcal{P}\mathbb{R}$. Show that there is a Borel-coded measure μ_g on $\mathbb{R}$ agreeing with θ_g on open sets.

(d) Apply 564N to relate Lebesgue measure on $\mathbb{R}^2$ to Lebesgue measure on $\mathbb{R}$.

(e) Suppose that there is a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of countable sets with union $[0,1]$. Show that there is a set $A \subseteq [0,1]^2$, with two-dimensional Lebesgue submeasure zero, such that all the vertical sections $A[\{x\}]$, for $x \in [0,1]$, have non-zero one-dimensional Lebesgue measure.

(f) Confirm that the principal results of §281 can be proved without the axiom of choice.

565Y Further exercises (a) Show that if X is a second-countable space and μ is a codably σ-finite Borel-coded measure on X, then there is a non-decreasing function $g : \mathbb{R} \to \mathbb{R}$ such that the Lebesgue-Stieltjes measure μ_g of 565Xc has measure algebra isomorphic to that of μ.

565 Notes and comments In these five sections I have tried to indicate a possible version of Lebesgue's theory which can be used in plain ZF, without succumbing to the temptation to re-write the whole treatise. With the Fundamental Theorem of Calculus (565M), the Radon-Nikodým theorem (564L), Fubini's theorem (564N) and at least some infinite product measures (564O), it is clear that most of the ideas of Volume 2 should be expressible in forms not relying on the axiom of choice. We must expect restrictions of the type already found in the convergence theorems (564F); for versions of the Central Limit Theorem or the strong law of large numbers or Komlós' theorem, for instance, we should certainly start by changing any hypothesis 'let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of random variables' into 'let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a codable sequence of codable Borel functions'. I am not sure how to approach martingales, but the best chance of positive results will be with 'codable martingales' in which we have a full set of Borel codes for countable sets generating each of the σ-algebras involved, along the lines of 564Md. If you glance at the formulae of Chapter 28, you will see that while there are many appeals to the convergence theorems, they are generally applied to sequences of the form $\langle f \times g_n\rangle_{n\in\mathbb{N}}$ where f is integrable and the g_n are continuous; but this means that $\langle g_n\rangle_{n\in\mathbb{N}}$ is necessarily codable (562O) so that $\langle f \times g_n\rangle_{n\in\mathbb{N}}$ will be a codable sequence if f itself is a codable function.

In Volumes 3 and 4 we encounter much more solid obstacles, and I see no way in which Maharam's theorem, or the Lifting Theorem, can be made to work without something approaching the full axiom of choice, or a strong hypothesis declaring the existence of a well-orderable set at a crucial point. I give an example of such a hypothesis in the statement of Vitali's theorem (565F). But in the applications of Vitali's theorem later in this section, we can always work with a countable family of balls, for which well-orderability is not an issue. Separability and second-countability hypotheses can be expected to act in similar ways; so that, for instance, we have 565Ya, which is a kind of primitive case of Maharam's theorem.

566 Countable choice

With AC(ω) measure theory becomes recognisable. The definition of Lebesgue measure used in Volume 1 gives us a true countably additive Radon measure; the most important divergence from the standard theory is the possibility that every subset of $\mathbb{R}$ is Lebesgue measurable (see 567G below). With occasional exceptions (most notably, in the theory of infinite products) we can use the work of Volume 2. In Volume 3, we lose the two best theorems in the abstract theory of measure algebras, Maharam's theorem and the Lifting Theorem; but function spaces and ergodic theory are relatively unaffected. Even in Volume 4, a good proportion of the ideas can be applied in some form.

566A Nearly all mathematicians working on the topics of this treatise spend most of their time thinking in the framework of ZFC. When we move to weaker theories, we have a number of alternative strategies available.

(a) Some of the time, all we have to do is to check that our previous arguments remain valid. In the present context, moving from full ZF + AC to ZF + AC(ω), this is true of most of Volumes 1 and 2 and useful fragments thereafter. In particular, for most of the basic theory of the Lebesgue integral countable choice is adequate. Sometimes, of course, we have to trim our theorems back a bit, as in 566E, 566I, 566M, 566N and 566R.

(b) Some results have to be dropped altogether. For instance, we no longer have a construction of a non-Lebesgue-measurable subset of $\mathbb{R}$, and the Lifting Theorem disappears.

(c) Some results become so much weaker that they change their character entirely. For instance, the Hahn-Banach theorem, Baire's theorem, Stone's theorem and Maharam's theorem survive only in sharply restricted forms (561Xg, 561E, 561F, 566Nb).

(d) Sometimes we find that while proofs rely on the axiom of choice, the results can be proved without it, or with something much weaker. Of course this is often a reason to regard the original proof as inappropriate. Some of the ultrafilters in Volume 4 are there just to save a couple of lines of argument, and renouncing them actually brings ideas into clearer focus. But there are occasions when the less scrupulous approach makes it a good deal easier for us to develop appropriate intuitions. There is an example in the theory of the spaces $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ in Chapter 36. If we think of $S(\mathfrak{A})$ as a quotient of a free linear space (361Ya) and of $L^\infty(\mathfrak{A})$ as the $\| \|_\infty$-completion of $S(\mathfrak{A})$, we can prove all the basic results which come from their identification with spaces of functions on the Stone space of $\mathfrak{A}$; but for most of us such an approach would seriously complicate the process of understanding the nature of the objects being constructed. I used the representation theorems in the theory of free products (§315, §325) for the same reason.

On other occasions, we may need new ideas, as in 566F-566H, 566L and 566P-566Q. A deeper example is in 562T/566O, where I set out alternative routes to the results of 364G and 434T. Here we have quite a lot of extra distance to travel, but at the same time we see some new territory.

(e) More subtly, it may be useful to re-consider some definitions; e.g., the distinction between 'ccc' and 'countable sup property' for Boolean algebras (566Xc). I have made an effort in this book to use definitions which will be appropriate in the absence of the axiom of choice, but in a number of places this would lead to a division of a concept in potentially confusing ways.

The ordinary theory of cardinals depends so essentially on the existence of well-orderings that it is often unclear what we can do without them. However some theorems, which appear to involve the theory of infinite cardinals, can be rescued if we re-interpret the statements. Sometimes the cardinal $\mathfrak{c}$ can be simply replaced by $\mathbb{R}$ or $\mathcal{P}\mathbb{N}$ (343I, 491G). Sometimes a statement '$\#(X) \geq \mathfrak{c}$' can be replaced by 'there is an injection from $\mathcal{P}\mathbb{N}$ into X' or 'there is a surjection from X onto $\mathcal{P}\mathbb{N}$' (344H, 4A2G(j-ii)); similarly, '$\#(X) \leq \mathfrak{c}$' might mean 'there is an injection from X into $\mathcal{P}\mathbb{N}$' or 'there is a surjection from $\mathcal{P}\mathbb{N}$ onto $X \cup \{\emptyset\}$' (4A1O, 4A3Fa). Of course '$\#(X) = \mathfrak{c}$' usually becomes 'there is a bijection between X and $\mathcal{P}\mathbb{N}$' (423K); but it might mean 'there are an injection from $\mathcal{P}\mathbb{N}$ into X and a surjection from $\mathcal{P}\mathbb{N}$ onto X' (4A3Fb), or the other way round, or just two surjections.

When dealing with a property which is invariant under equipollence, it may be right to drop the concept of 'cardinal' altogether, and re-phrase a definition in more primitive terms, as in 566Xk.

Elsewhere, as in 2A1Fd and 4A1E, we have results which refer to initial ordinals and hence to well-orderable sets. But the theory of cardinal functions is so bound up with the idea that cardinal numbers form a well-ordered class that much greater adjustments are necessary. I offer the following idea for consideration. For a metric space (X, ρ) and a dense set $D \subseteq X$, set

$$\theta(X, \rho, D) = \{\{y : y \in X,\ \rho(x, y) < 2^{-n}\} : x \in D,\ n \in \mathbb{N}\},$$

so that $\theta(X, \rho, D)$ is a base for the topology of X. The existence (in ZF) of this function corresponds to the ZFC result that '$w(X)$ is at most the cardinal product $\omega \times d(X)$ for every metrizable space X'.

(f) Another way to preserve the ideas of a theorem in the new environment is to make some small variation in its hypotheses. For instance, Urysohn's Lemma, in its usual form, demands DC. So if we are working with $\mathrm{AC}(\omega)$ alone, we cannot be sure that compact Hausdorff spaces are completely regular; similarly, there may be uniformities not definable from pseudometrics. For a general topologist, this is important. But a measure theorist may be happy to simply add 'completely regular' to the hypotheses of a theorem, as in 561G and 566Xj. In §§412-413 I repeatedly mention families $\mathcal{K}$ which are closed under disjoint finite unions. Results starting from this hypothesis tend to depend on DC; but if we take $\mathcal{K}$ to be closed under $\cup$, $\mathrm{AC}(\omega)$ may well be enough (566D). A more dramatic change, but one which still leads to interesting results, is in 566I.

566B Volume 1 With countable choice, Lebesgue outer measure becomes an outer measure in the usual sense, so we can use Carathéodory's method to define a measure space in the sense of 112A. No further difficulties arise in the work of Chapters 11 and 12, and we can proceed exactly as before to the convergence theorems. Indeed all the theorems of Volume 1 are available, with a single exceptional feature: the construction of non-measurable sets in 134B and 134D, and a non-measurable function in 134Ib. I will return to this point in §567.

566C Volume 2 In Volume 2 also we find that arguments using more than countable choice are the exception rather than the rule. Naturally, they appear oftener in the more abstract topics of Chapter 21. One is in 211L; we can no longer be sure that a strictly localizable space is localizable, though a σ-finite measure space does have to be localizable, since the choice demanded in the proof of 211Ld can then be performed over a countable index set. There is a similar problem in 213J; a strictly localizable space might fail to have locally determined negligible sets, and might have a subset without a measurable envelope. Again, in 214Ia, it is not clear that a subspace of a strictly localizable space must be strictly localizable. In 211P I ask for a non-Borel subset of $\mathbb{R}$, and give an answer involving a non-measurable set; but with $\mathrm{AC}(\omega)$ we have a non-Borel analytic set as in 423L. (See also 566Xb.)

A more important gap arises in the theory of infinite products of probability spaces. The first problem is that if we have an uncountable family $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of probability spaces, there is no assurance that $\prod_{i\in I} X_i$ is non-empty. In concrete cases, this is not usually a serious worry. But there is another one. The proof of 254F makes an appeal to DC. I do not think that there can be a construction of a product measure on even a sequence of arbitrary probability spaces which does not use some form of dependent choice. However a partial version, adequate for many purposes (including the essential needs of Chapter 27), can be done with countable choice alone (566I). We can now continue through §254 with the proviso that every infinite family of probability spaces for which we consider a product measure should be a family of perfect probability spaces with non-empty product. There will be a difficulty in 254L, concerning the product of subspaces of full outer measure, where the modification essentially confines it to non-empty products of conegligible sets. For 254N, it will be helpful to know that (under the conditions of 566I) the product of perfect spaces is again perfect. The proof of this fact (451Jc) is scattered through Volume 4, but (given that we have a product probability measure) needs only countably many choices at each step.

When we come to products of probability spaces in Chapter 27, we shall again have to restrict the applications of the results, but at each point only sufficiently to ensure that we have the product probability measures discussed.

566D Exhaustion The versions of the principle of exhaustion in 215A all seem to require DC rather than AC(ω). For many applications, however, we can make do with a weaker result, as follows. I include some corollaries showing that in many familiar cases we can continue to use the intuitions developed in the main text.

Proposition [AC(ω)] (a) Let P be a partially ordered set such that $p\vee q=\sup\{p,q\}$ is defined for all p, $q\in P$, and $f:P\to\mathbb{R}$ an order-preserving function. Then there is a non-decreasing sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $\lim_{n\to\infty}f(p_n)=\sup_{p\in P}f(p)$.

(b) Let (X,Σ,μ) be a measure space and $\mathcal{E}\subseteq\Sigma$ a non-empty set such that $\sup_{E\in\mathcal{E}}\mu E$ is finite and $E\cup F\in\mathcal{E}$ for every E, $F\in\mathcal{E}$. Then there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, setting $F=\bigcup_{n\in\mathbb{N}}F_n$, $\mu F=\sup_{E\in\mathcal{E}}\mu E$ and $E\setminus F$ is negligible for every $E\in\mathcal{E}$.

(c) Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of sets such that

(α) $K\cup K'\in\mathcal{K}$ for all K, $K'\in\mathcal{K}$,

(β) whenever $E\in\Sigma$ is non-negligible there is a non-negligible $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$.

Then μ is inner regular with respect to $\mathcal{K}$.

(d)(i) Let (X,Σ,μ) be a semi-finite measure space. Then μ is inner regular with respect to the family of sets of finite measure.

(ii) Let (X,Σ,μ) be a perfect measure space. Then whenever $E\in\Sigma$, $f:X\to\mathbb{R}$ is measurable and $\gamma<\mu E$, there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]\ge\gamma$.

proof (a) For each $n\in\mathbb{N}$, set $\gamma_n=\sup_{p\in P}\min(n,f(p)-2^{-n})$. Then there is a sequence $\langle q_n\rangle_{n\in\mathbb{N}}$ in P such that $f(q_n)\ge\gamma_n$ for each n; set $p_n=\sup_{i\le n}q_i$ for each n.

(b) By (a) there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\sup_{n\in\mathbb{N}}\mu F_n=\sup_{E\in\mathcal{E}}\mu E$; set $F=\bigcup_{n\in\mathbb{N}}F_n$.

(c) Because μ is inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K}\cup\{\emptyset\}$, we may suppose that $\emptyset\in\mathcal{K}$. Take $F\in\Sigma$, and consider $\mathcal{E}=\{K:K\in\mathcal{K}\cap\Sigma,\ K\subseteq F\}$. **?** If $\sup_{E\in\mathcal{E}}\mu E<\mu F$, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{E}$ such that $\mu(E\setminus\bigcup_{n\in\mathbb{N}}E_n)=0$ for every $E\in\mathcal{E}$ ((b) above). Set $G=\bigcup_{n\in\mathbb{N}}E_n$; then $\mu G=\sup_{n\in\mathbb{N}}\mu E_n<\mu F$, so $\mu(F\setminus G)>0$. But now there ought to be a non-negligible $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq F\setminus G$, in which case $K\in\mathcal{E}$ and $\mu(K\setminus G)>0$. **X**

(d)(i) Apply (c) with $\mathcal{K}$ the family of sets of finite measure.

(ii) Apply (c) to the subspace measure μ_E and $\mathcal{K}=\{f^{-1}[K]:K\subseteq f[E]\text{ is compact}\}$.

566E The problem recurs in parts of 215B, where I list characterizations of σ-finiteness, and in 215C. It seems equally that a ccc semi-finite measure algebra may fail to be σ-finite, though a σ-finite measure algebra has to be ccc. We have a stripped-down version of 215B, with one of its corollaries used in §235, as follows:

Proposition [AC(ω)] Let (X,Σ,μ) be a semi-finite measure space. Write $\mathcal{N}$ for the σ-ideal of μ-negligible sets.

(a) The following are equiveridical:

(i) μ is σ-finite;

(ii) *either* $\mu X=0$ *or* there is a probability measure ν on X with the same domain and the same negligible sets as μ;

(iii) there is a measurable integrable function $f:X\to\,]0,1]$;

(iv) either $\mu X=0$ or there is a measurable function $f:X\to\,]0,\infty[$ such that $\int fd\mu=1$.

(b) If μ is σ-finite, then

(i) every disjoint family in $\Sigma\setminus\mathcal{N}$ is countable;

(ii) for every $\mathcal{E}\subseteq\Sigma$ there is a countable $\mathcal{E}_0\subseteq\mathcal{E}$ such that $E\setminus\bigcup\mathcal{E}_0$ is negligible for every $E\in\mathcal{E}$.

(c) Suppose that μ is σ-finite, (Y,T,ν) is a semi-finite measure space, and $\phi:X\to Y$ is a (Σ,T)-measurable function such that $\mu\phi^{-1}[F]>0$ whenever $\nu F>0$. Then ν is σ-finite.

proof (a) Use the methods of 215B.

(b) By (a-ii), we may suppose that μ is totally finite.

(i) If $\mathcal{E}\subseteq\Sigma\setminus\mathcal{N}$ is disjoint, then $\mathcal{E}_n=\{E:E\in\mathcal{E},\ \mu E\ge 2^{-n}\}$ is finite for every n, so $\mathcal{E}=\bigcup_{n\in\mathbb{N}}\mathcal{E}_n$ is countable.

(ii) Let $\mathcal{H}$ be the set of finite unions of members of $\mathcal{E}$. By 566Db, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\mu(H\setminus\bigcup_{n\in\mathbb{N}}H_n)=0$ for every $H\in\mathcal{H}$. For each $n\in\mathbb{N}$, choose a finite set $\mathcal{H}_n\subseteq\mathcal{E}$ such that $H_n=\bigcup\mathcal{H}_n$; then $\mathcal{E}_0=\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$ has the required properties.

(c) Again, we may suppose that μ is totally finite. For each $m\in\mathbb{N}$ let $\mathcal{H}_m$ be the set of those $F\in\mathrm{T}$ such that $\nu F<\infty$ and $\mu\phi^{-1}[F]\ge 2^{-m}$. Then any disjoint family in $\mathcal{H}_m$ has at most $\lfloor 2^m\mu X\rfloor$ members, so each $\mathcal{H}_m$ has a maximal disjoint subset; choose a sequence $\langle\mathcal{E}_m\rangle_{m\in\mathbb{N}}$ such that $\mathcal{E}_m$ is a maximal disjoint subset of $\mathcal{H}_m$ for each m. Then $\mathcal{E}=\bigcup_{m\in\mathbb{N}}\mathcal{E}_m$ is a countable family of sets of finite measure in Y. Now $Z=Y\setminus\bigcup\mathcal{E}$ is negligible. **P?** Otherwise, there is a non-negligible set F of finite measure disjoint from $\bigcup\mathcal{E}$; now there is an m such that $F\in\mathcal{H}_m$, so that $\mathcal{E}_m$ was not maximal. **XQ** So $\mathcal{E}\cup\{Z\}$ witnesses that ν is σ-finite.

566F Atomless algebras To make atomless measure spaces and measure algebras recognisable, we need a more penetrating argument than that previously used in 215D.

Lemma [AC(ω)] Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and μ a positive countably additive functional on $\mathfrak{A}$ such that $\mu 1=1$. Suppose that whenever $a\in\mathfrak{A}$ and $\mu a>0$ there is a $b\subseteq a$ such that $0<\mu b<\mu a$. Then there is a function $f:\mathfrak{A}\times[0,1]\to\mathfrak{A}$ such that $f(a,\alpha)\subseteq a$ and $\bar\mu f(a,\alpha)=\min(\alpha,\bar\mu a)$ for $a\in\mathfrak{A}$ and $\alpha\in[0,1]$.

proof (a) Just as in part (a) of the proof of 215D, we see by induction on n that for every $b\in\mathfrak{A}$ such that $\mu b>0$ and every $n\in\mathbb{N}$, there is a $c\subseteq b$ such that $0<\mu c\le 2^{-n}\mu b$.

(b) If $b\in\mathfrak{A}$ and $\mu b>0$, there is a $c\subseteq b$ such that $\frac13\mu b<\mu c\le\frac23\mu b$. **P?** Otherwise, set $\gamma=\sup\{\mu c: c\subseteq b,\ \mu c\le\frac23\mu b\}$ and let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $c_n\subseteq b$ and $\gamma-2^{-n}\le\mu c_n\le\gamma$ for every n. Set $d_n=\sup_{i\le n}c_i$ for each n, and $d=\sup_{n\in\mathbb{N}}d_n$. Inducing on n, we see that $\mu d_n\le\frac23\mu b$ so $\mu d_n\le\frac13\mu b$ for each n, and $\mu d\le\frac13\mu b$. Now by (a) there is an $e\subseteq b\setminus d$ such that $0<\mu e\le\frac13\mu b$. In this case, $\mu(d\cup e)\le\frac23\mu b$, so

$$\gamma\ge\mu(d\cup e)\ge\mu e+\sup_{n\in\mathbb{N}}\mu c_n>\gamma.\ \mathbf{XQ}$$

(c) For each $n\in\mathbb{N}$ and $b\in\mathfrak{A}$ there is a finite partition of b into elements of measure at most $(\frac23)^n\mu b$. **P** Induce on n, using (b) for the inductive step. **Q**

(d) Choose a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of finite partitions of unity such that $\mu c\le 2^{-n}$ for every $n\in\mathbb{N}$ and $c\in C_n$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}C_n$; then $\mathfrak{C}$ is countable. Moreover, whenever $a\in\mathfrak{A}$ and $0\le\beta<\beta'\le\mu a$, there must be a $c\in\mathfrak{C}$ such that $\beta\le\mu(a\cap c)\le\beta'$. **P** Take n such that $2^{-n}\le\beta'-\beta$, and $c=\sup A$ for a minimal $A\subseteq C_n$ such that $\mu(a\cap\sup A)\ge\beta$. **Q**

(e) Let $\preccurlyeq$ be a well-ordering of $\mathfrak{C}$. Define $\langle f_n\rangle_{n\in\mathbb{N}}$ inductively by saying that

$f_0(a,\alpha)=0$ for all $a\in\mathfrak{A}$, $\alpha\in[0,1]$,

$f_{n+1}(a,\alpha)=f_n(a,\alpha)\cup(c\cap a\setminus f_n(a,\alpha))$ where c is the $\preccurlyeq$-first member of C such that

$$\min(\alpha,\mu a)-\mu f_n(a,\alpha)-2^{-n}\le\mu(c\cap a\setminus f_n(a,\alpha))\le\min(\alpha,\mu a)-\mu f_n(a,\alpha).$$

Then for any $a\in\mathfrak{A}$ and $\alpha\in[0,1]$, $\langle f_n(a,\alpha)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of elements included in a and with measures converging to $\min(\alpha,\mu a)$; so if we set $f(a,\alpha)=\sup_{n\in\mathbb{N}}f_n(a,\alpha)$ we shall have an appropriate function.

566G Vitali's theorem The arguments I presented for Vitali's theorem in 221A/261B and 471N-471O, and for the similar result in 472B, involve the inductive construction of a sequence, which ordinarily is a signal that DC is being used. In 565F I suggested a weaker form of Vitali's theorem which is adequate for its most important applications in measure theory. With AC(ω), however, we can get most of the results as previously stated, if we refine our methods slightly.

(a) In 261B, we have a family $\mathcal{I}$ of closed balls in $\mathbb{R}^r$ and we wish to choose inductively a disjoint sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$\operatorname{diam}I_n\ge\frac12\sup\{\operatorname{diam}I: I\in\mathcal{I},\ I\cap\bigcup_{i<n}I_i=\emptyset\}$$

for every n. We have already reduced the problem to the case in which $\sup_{I\in\mathcal{I}}\operatorname{diam}I$ is finite and for any finite disjoint subset of $\mathcal{I}$ there is a member of $\mathcal{I}$ disjoint from all of them. Let $\langle G_m\rangle_{m\in\mathbb{N}}$ run over the family of all open balls with centres in $\mathbb{Q}^r$ and rational radii. For $m\in\mathbb{N}$ set $\mathcal{K}_m=\mathcal{I}\cap\mathcal{P}G_m$, and let $\mathcal{I}'\subseteq\mathcal{I}$ be a countable set such that $\sup_{I\in\mathcal{I}'\cap\mathcal{K}_m}\operatorname{diam}I=\sup_{I\in\mathcal{K}_m}\operatorname{diam}I$ for every $m\in\mathbb{N}$ such that $\mathcal{K}_m$ is non-empty; this can be found with countably many choices.

Now, when we come to choose I_n, we can always pick a member of $\mathcal{I}'$. **P** If $\mathcal{I}_n=\{I: I\in\mathcal{I},\ I\cap\bigcup_{i<n}I_i=\emptyset\}$, $\gamma_n=\sup_{I\in\mathcal{I}_n}\operatorname{diam}I$ and $I\in\mathcal{I}_n$ is such that $\operatorname{diam}I>\frac12\gamma_n$, there is an $m\in\mathbb{N}$ such that $I\subseteq G_m\subseteq\mathbb{R}^r\setminus\bigcup_{i<n}I_i$, in which case there is an $I'\in\mathcal{I}'\cap\mathcal{K}_m$ such that $\operatorname{diam}I'\ge\frac12\gamma_n$, and I' is eligible to be I_n. **Q** Because $\mathcal{I}'$ is well-orderable,

we can set out a rule for making these choices, and the argument can proceed as written, without recourse to the devices of §565.

(b) A similar trick can be used in 472B. Here, given a family $\mathcal{I}$ of closed balls, we wish to choose a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that the centre of B_n does not belong to $\bigcup_{i<n} B_i$ and, subject to this, the diameter of B_n is nearly as large as it could be. This time, take $\mathcal{K}_m$ to be the set of members of $\mathcal{I}$ with centres in G_m, use countably many choices to find a countable set $\mathcal{I}' \subseteq \mathcal{I}$ with adequately large intersections with every $\mathcal{K}_m$, and choose $\langle B_n\rangle_{n\in\mathbb{N}}$ from $\mathcal{I}'$.

At the next step, in 472C, we have to do this repeatedly, but the same method works; in fact, we can work inside a fixed family $\mathcal{I}'$ chosen as above. (See 472Yf.)

(c) The version in 471N-471O is not manageable in quite the same way. If, however, we assume that the metric spaces there are locally compact and separable, we can use the same idea as in (a) above to limit our search to countable subfamilies of the given family $\mathcal{F}$.

566H Bounded additive functionals We come to another obstacle in the proof of 231E. The argument given there relies on DC to show that a countably additive functional is bounded. But we can avoid this, at the cost of an extra manoeuvre, as follows.

Lemma [AC(ω)] Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ an additive functional such that $\{\nu a_n : n \in \mathbb{N}\}$ is bounded for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Then ν is bounded.

proof ? Suppose, if possible, otherwise. Then there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in Σ such that $|\nu b_n| \ge 2^n n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{b_i : i < n\}$; then $\mathfrak{B}_n$ has at most 2^n atoms, so there must be an atom a of $\mathfrak{B}_n$ such that $|\nu(a \cap b_n)| \ge n$. Choose a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ such that c_n is an atom of $\mathfrak{B}_n$ and $|\nu d_n| \ge n$ for every n, where $d_n = c_n \cap b_n$; note that $d_n \in \mathfrak{B}_n$, so that if $n < m$ then either $d_m \subseteq d_n$ or $d_m \cap d_n = 0$. By Ramsey's theorem (4A1G), there is an infinite $I \subseteq \mathbb{N}$ such that

either $\langle d_n\rangle_{n\in I}$ is disjoint

or $d_m \subseteq d_n$ whenever $m, n \in I$ and $n < m$.

Now the first alternative is certainly impossible, because $\{\nu d_n : n \in I\}$ is unbounded. So we have the second. But in this case we can define a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in I such that $n_{k+1} \ge k + |\nu d_{n_k}|$ for each k. Set $a_k = d_{n_k} \setminus d_{n_{k+1}}$ for each k; then $\langle a_k\rangle_{k\in\mathbb{N}}$ is disjoint and $|\nu a_k| \ge k$ for each k, so again we have a contradiction. **X**

566I Infinite products: Theorem [AC(ω)] Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces such that $X = \prod_{i\in I} X_i$ is non-empty. Then there is a complete probability measure λ on X such that

(i) if $E_i \in \Sigma_i$ for every $i \in I$, and $\{i : E_i \ne X_i\}$ is countable, then $\lambda(\prod_{i\in I} E_i)$ is defined and equal to $\prod_{i\in I} \mu_i E_i$;

(ii) λ is inner regular with respect to $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

proof The only point at which the construction in 254A-254F needs re-examination is in the proof that the standard outer measure on X gives it outer measure 1.

(a) I recall the definitions. For a cylinder $C = \prod_{i\in I} C_i$, set $\theta_0 C = \prod_{i\in I} \mu_i C_i$; for $A \subseteq X$, set

$$\theta A = \inf\{\sum_{n=0}^{\infty} \theta_0 C_n : C_n \in \mathcal{C} \text{ for every } n \in \mathbb{N},\ A \subseteq \bigcup_{n\in\mathbb{N}} C_n\};$$

λ will be the measure defined from θ by Carathéodory's method.

? Suppose, if possible, that $\theta X < 1$. Then we have a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of cylinder sets, covering X, with $\sum_{n=0}^{\infty} \theta C_n = 1 - 2\epsilon$ where $\epsilon > 0$. Express each C_n as $\prod_{i\in I} E_{ni}$ where $J_n = \{i : E_{ni} \ne X_i\}$ is finite; let J be the countable set $\bigcup_{n\in\mathbb{N}} J_n$; take $K = \#(J)$ (identifying $\mathbb{N}$ with ω), and a bijection $k \mapsto i_k : K \to J$.

For each $k \in K$ and $n \in \mathbb{N}$, set $L_k = \{i_j : j < k\} \subseteq J$ and $\alpha_{nk} = \prod_{i\in I\setminus L_k} \mu_i E_{ni}$. If J is finite, $L_{\#(J)} = J$ and $\alpha_{n,\#(J)} = 1$ for every n. We have $\alpha_{n0} = \theta_0 C_n$ for each n, so $\sum_{n=0}^{\infty} \alpha_{n0} = 1 - 2\epsilon$. For $n \in \mathbb{N}$, $k \in K$ and $t \in X_{i_k}$ set

$$\begin{aligned} f_{nk}(t) &= \alpha_{n,k+1} \text{ if } t \in E_{n,i_k}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then

$$\int f_{nk} d\mu_{i_k} = \alpha_{n,k+1} \mu_{i_k} E_{ni_k} = \alpha_{nk}.$$

(b) For each $k \in K$, let $h_k : X_{i_k} \to \mathbb{R}$ be the Marczewski functional defined by setting

$$h_k(t) = \sum_{n=0}^{\infty} 3^{-n} \chi E_{ni_k}(t)$$

for $t \in X_k$. Because μ_k is perfect, there is for each $k \in K$ a compact set $Q \subseteq h_k[X_{i_k}]$ such that $\mu_{i_k} h_k^{-1}[Q] \ge 1 - 2^{-k}\epsilon$. Choose $\langle Q_k \rangle_{k\in K}$ such that $Q_k \subseteq h_k[X_{i_k}]$ is compact and $\mu_{i_k} h_k^{-1}[Q_k] \ge 1 - 2^{-k}\epsilon$ for every $k \in K$. Observe that if $k \in K$ and $n \in \mathbb{N}$ then $f_{nk} = \alpha_{nk}\chi E_{ni_k}$ is of the form $\alpha_{nk} g_n h_k$ where $g_n : \mathbb{R} \to [0,1]$ is continuous.

(c) Define non-empty sets $F_k \subseteq H_k \subseteq X_{i_k}$ inductively, for $k \in K$, as follows. The inductive hypothesis will be that $\sum_{n\in M_k} \alpha_{nk} \le 1 - 2^{-k+1}\epsilon$, where $M_k = \{n : n \in \mathbb{N},\ F_j \subseteq E_{ni_j} \text{ whenever } j < k\}$; of course $M_0 = \mathbb{N}$, so the induction starts. Given that $k \in K$ and that

$$1 - 2^{-k+1}\epsilon \ge \sum_{n\in M_k} \alpha_{nk} = \sum_{n\in M_k} \int f_{nk} d\mu_{i_k} = \int (\sum_{n\in M_k} f_{nk}) d\mu_{i_k},$$

the set

$$H_k = \{t : t \in X_{i_k},\ \sum_{n\in M_k} f_{nk}(t) \le 1 - 2^{-k}\epsilon\}$$

must have measure greater than $2^{-k}\epsilon$ and meets $h_k^{-1}[Q_k]$. But observe that $\sum_{n\in M_k} f_{nk} = g'_k h_k$ where $g'_k = \sum_{n\in M_k} \alpha_{nk} g_n$ is lower semi-continuous, so that $H_k = h_k^{-1}[G_k]$ where $G_k = \{\alpha : g'_k(\alpha) \le 1 - 2^{-k}\epsilon\}$ is closed. Since H_k meets $h_k^{-1}[Q_k]$, $Q_k \cap G_k$ is non-empty and has a least member β_k; set $F_k = h_k^{-1}[\{\beta_k\}]$. Because $Q_k \subseteq h_k[X_{i_k}]$, F_k is non-empty.

Examine

$$M_{k+1} = \{n : n \in M_k,\ F_k \subseteq E_{ni_k}\}.$$

There certainly is some $t^* \in F_k$, and because $h_k \restriction F_k$ is constant, $M_{k+1} = \{n : n \in M_k,\ t^* \in E_{ni_k}\}$. In this case

$$\sum_{n\in M_{k+1}} \alpha_{n,k+1} = \sum_{n\in M_k} f_{nk}(t^*) \le 1 - 2^{-k}\epsilon$$

and the induction proceeds.

(d) At the end of the induction, either finite or infinite, choose $t_k \in F_k$ for $k \in K$. We are supposing that X has a member x^*; define $x \in X$ by setting $x(i_k) = t_k$ for $k \in K$ and $x(i) = x^*(i)$ for $i \in I \setminus J$. Then there is supposed to be an $m \in \mathbb{N}$ such that $x \in C_m$, so that $m \in M_k$ for every k. At some stage we shall have $J_m \subseteq L_k$ (allowing $k = \#(K)$ if K is finite) and $\alpha_{mk} = 1$, which is impossible. **X**

566J In particular, 566I applies to all products $\{0,1\}^I$ and $[0,1]^I$ with their usual measures. For these we have Kakutani's theorem that the usual measures are topological measures (415E), which turns out to be valid with countable choice alone.

Theorem [AC(ω)] (a) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be a family of metrizable Radon probability spaces such that every μ_i is strictly positive and $\prod_{i\in I} X_i$ is non-empty. Then the product measure on $X = \prod_{i\in I} X_i$ is a quasi-Radon measure.

(b) If I is well-orderable then the product measure on $\{0,1\}^I$ is a completion regular Radon measure.

proof (a)(i) We had better check immediately that every X_i is separable. The point is that because μ_i is a totally finite measure inner regular with respect to the compact sets, there is a conegligible K_σ set; because μ_i is strictly positive, this is dense; and countable choice is enough to ensure that a compact metrizable space is second-countable, therefore separable. It follows that $\prod_{i\in J} X_i$ is separable, therefore second-countable, for every countable $J \subseteq I$.

(ii) Because every μ_i is a Radon measure it is perfect, so we have a product probability measure on X. Now we can repeat the argument of 416U.

(b) Put (a), 561D and 416G together.

566K Volume 3 Turning to the concerns of Volume 3, the elementary theory of measure algebras is not radically changed. But Lemma 311D is hopelessly lost; we no longer have Stone spaces, and need to re-examine any proof which appears to rely on them. Another result which changes is 313K; order-dense sets, as defined in 313J, need no longer give rise to partitions of unity. So a localizable measure algebra does not need to be isomorphic to a simple product of totally finite measure algebras. Similarly, condition (ii) of 316H is no longer sufficient to prove weak (σ,∞)-distributivity. However some of the constructions which I described in terms of Stone spaces, in particular, the Loomis-Sikorski theorem, the Dedekind completion of a Boolean algebra, the localization of a semi-finite measure algebra, free products and measure-algebra free products, can be done by other methods which remain effective with AC(ω) at most; see 566L, 561Yg, 323Xh and 325Yc.

The theory of ccc algebras is rather different (566M, 566Xc). Maharam's theorem (331I, 332B) is surely unprovable without something like the full axiom of choice; and the Lifting Theorem (341K) is equally inaccessible under the rules of this section. We do however have useful special cases of results in Chapters 33 and 34 (566N).

A good start can be made on the elementary theory of Riesz spaces without any form of the axiom of choice (see 561H), and with AC(ω) we can go a long way, as in 566Q. What is missing is the Hahn-Banach theorem (for non-separable spaces) and many representation theorems. Similarly, the function spaces of Chapter 36 are recognisable, provided that (for general Boolean algebras $\mathfrak{A}$) we think of $S(\mathfrak{A})$ as a quotient space of the free linear space generated by $\mathfrak{A}$, and of $L^\infty(\mathfrak{A})$ as the $\| \|_\infty$-completion of $S(\mathfrak{A})$. Of course we have to take care at every point to avoid the use of Stone spaces. One place at which this involves us in a new argument is in 566O. Most of the arguments of Chapter 24 remain valid, so the basic theory of L^p spaces in §§365-366 survives. What is perhaps surprising is that if we take the trouble we can still reach the most important results on weak compactness (566P, 566Q).

In the ergodic theory of Chapter 38, a good proportion of the classical results survive. There are difficulties with some of the extensions of the classical theory in §§381-382. For instance, the definition of 'full subgroup' of the group of automorphisms of a Boolean algebra in 381Be assumes that order-dense sets include partitions of unity. If not, this definition may fail to be equivalent to the formulation in 381Xh. The latter would seem to be the more natural one to use. However, the definition as given seems to work for the principal needs of Chapter 38 (see 381I).

Frolík's theorem in the generality 382D-382E needs something approaching AC, and with AC(ω) alone there seems no hope of getting results for general Dedekind complete algebras along the lines of the main theorems of §382. For measurable algebras, however, we do have a version of 382Eb (566R).

Many of the later results of Chapter 38 are equally robust, at least in their leading applications to measure algebras. We have to remember that we do not know that measurable algebras have many involutions, and even among those which do there is no assurance that 382Q will be true. So in §§383-384 we find ourselves restricted rather further, to those measurable algebras in which every non-zero element is the support of an involution; but these include the standard examples (566N).

566L The Loomis-Sikorski theorem [AC(ω)] (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there are a set X, a σ-algebra Σ of subsets of X and a σ-ideal $\mathcal{I}$ of Σ such that $\mathfrak{A} \cong \Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Then it is isomorphic to the measure algebra of a measure space.

proof (a)(i) Set $X = \{0,1\}^{\mathfrak{A}}$, and $\Sigma = \widehat{\bigotimes}_{\mathfrak{A}} \mathcal{P}(\{0,1\})$. For $a \in \mathfrak{A}$ set $\widehat{a} = \{x : x \in X,\, x(a) = 1\} \in \Sigma$. Let $\mathcal{I}$ be the σ-ideal of Σ generated by sets of the form

$$\widehat{a \triangle b} \triangle \widehat{a} \triangle \widehat{b}, \quad (\inf_{n\in\mathbb{N}} a_n)\,\widehat{}\,\triangle \bigcap_{n\in\mathbb{N}} \widehat{a_n}$$

for a, $b \in \mathfrak{A}$ and sequences $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, together with the set $\{x : x(1) = 0\}$.

(ii) (The key.) $\widehat{a} \notin \mathcal{I}$ for any $a \in \mathfrak{A} \setminus \{0\}$. **P** If $E \in \mathcal{I}$ then (using AC(ω)) we can find sequences $\langle a_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, together with a double sequence $\langle c_{ni} \rangle_{n,i\in\mathbb{N}}$, such that, setting $c_n = \inf_{i\in\mathbb{N}} c_{ni}$ for each n,

$$F = \{x : x(1) = 0\} \cup \bigcup_{n\in\mathbb{N}} \widehat{a_n \triangle b_n} \triangle \widehat{a_n} \triangle \widehat{b_n} \cup \bigcup_{n\in\mathbb{N}} (\widehat{c_n} \triangle \bigcap_{i\in\mathbb{N}} \widehat{c_{ni}})$$

includes E. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by

$$\{a\} \cup \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\} \cup \{c_n : n \in \mathbb{N}\} \cup \{c_{ni} : n, i \in \mathbb{N}\}.$$

Then $\mathfrak{B}$ is countable, so we can choose inductively a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B} \setminus \{0\}$ such that $d_0 = a$ and, for each $n \in \mathbb{N}$,

$d_{n+1} \subseteq d_n$,
either $d_{n+1} \subseteq a_n$ or $d_{n+1} \cap a_n = 0$,
either $d_{n+1} \subseteq b_n$ or $d_{n+1} \cap b_n = 0$,
either $d_{n+1} \subseteq c_n$ or there is an $i \in \mathbb{N}$ such that $d_{n+1} \cap c_{ni} = 0$.

Define $x \in X$ by saying that

$$x(d) = 1 \text{ if } d \supseteq d_n \text{ for some } n \in \mathbb{N},$$
$$= 0 \text{ otherwise.}$$

Then $x \in \widehat{a} \setminus F$ and $\widehat{a} \not\subseteq E$; as E is arbitrary, $\widehat{a} \notin \mathcal{I}$. **Q**

(iii) Set

$$\Sigma_0 = \{E : E \in \Sigma,\, E \triangle \widehat{a} \in \mathcal{I} \text{ for some } a \in \mathfrak{A}\}.$$

Then Σ_0 is closed under symmetric difference and countable intersections and contains X (because $X \triangle \widehat{1} \in \mathcal{I}$). So Σ_0 is a σ-algebra of sets; as it contains $\widehat{a}$ for every $a \in \mathfrak{A}$, it is equal to Σ.

(iv) From (ii) we see that $\widehat{a \triangle b}$, and therefore $\widehat{a} \triangle \widehat{b}$, do not belong to $\mathcal{I}$ for any distinct a, $b \in \mathfrak{A}$. With (iii), this tells us that we have a function $\pi : \Sigma \to \mathfrak{A}$ defined by setting $\pi E = a$ whenever $E \triangle \widehat{a} \in \mathcal{I}$. Now $\pi X = 1$ and π preserves symmetric difference and countable infima, so is a sequentially order-continuous Boolean homomorphism; its kernel is $\mathcal{I}$, so $\mathfrak{A} \cong \Sigma/\mathcal{I}$, as required.

(b) This is now easy; we can use the familiar argument of 321J.

566M Measure algebras: Proposition [AC(ω)] (a) Let $\mathfrak{A}$ be a Boolean algebra, and $D \subseteq \mathfrak{A}$ an order-dense set. Then $a = \sup\{d : d \in D,\, d \subseteq a\}$ for every $a \in \mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a measurable algebra.

(i) For any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ with the same upper bounds as A.

(ii) $\mathfrak{A}$ is Dedekind complete.

(iii) If $D \subseteq \mathfrak{A}$ is order-dense and $c \in D$ whenever $c \subseteq d \in D$, there is a partition of unity included in D.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Then $a = \sup\{b : b \subseteq a, \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ such that $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ is a semi-finite measure algebra. Then $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ is a σ-finite measure algebra.

proof (a) (Cf. 313K.) **?** Otherwise, there is a non-zero $b \subseteq a$ such that $b \cap d = 0$ whenever $d \in D$ and $d \subseteq a$; but in this case there ought to be a non-zero member of D included in b. **X**

(b) (Cf. 322G, 316E, 322Cc.) Let $\bar{\mu}$ be such that $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra.

(i) Let A^* be the set of suprema of finite subsets of A, and set $\gamma = \sup_{a \in A^*} \bar{\mu}a$. There is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in A^* such that $\sup_{n\in\mathbb{N}} \bar{\nu}a_n = \gamma$; let $B \subseteq A$ be a countable set such that every a_n is the supremum of a finite subset of B. Then any upper bound c of B is an upper bound of A. **P** Take $d \in A$. Then $a \cup a_n \in A^*$, so

$$\bar{\mu}(a \setminus c) \le \bar{\mu}(a \setminus a_n) = \bar{\mu}(a \cup a_n) - \bar{\mu}a_n \le \gamma - \bar{\mu}a_n$$

for every n, and $\bar{\mu}(a \setminus c) = 0$, that is, $a \subseteq c$. **Q**

(ii) follows at once from (i), since $\mathfrak{A}$ is Dedekind σ-complete.

(iii) By (i), there is a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in D with supremum 1; now $\langle d_n \setminus \sup_{i<n} d_i \rangle_{n\in\mathbb{N}}$ is a partition of unity included in D.

(c) (Cf. 322Eb.) Use (a) with $D = \{d : \bar{\mu}d < \infty\}$.

(d) (Cf. 322Nc.) Write $\mathfrak{B}^f$ for the ring $\{b : b \in \mathfrak{B}, \bar{\mu}b < \infty\}$. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}$, with supremum 1, such that $\bar{\mu}a_n < \infty\}$ for every n. For each $n \in \mathbb{N}$, set $\alpha_n = \sup\{\bar{\mu}(b \cap a_n) : b \in \mathfrak{B}^f\}$; choose a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}^f$ such that $\bar{\mu}(b_n \cap a_n) \ge \alpha_n - 2^{-n}$ for every n. **?** If 1 is not the supremum of $\{b_n : n \in \mathbb{N}\}$ in $\mathfrak{B}$, let $b \in \mathfrak{B} \setminus \{0\}$ be such that $b \cap b_n = 0$ for every n. Because $\bar{\mu}\restriction\mathfrak{B}$ is semi-finite, there is a non-zero $b' \in \mathfrak{B}^f$ included in b. But now $0 < \bar{\mu}b' = \sup_{n\in\mathbb{N}} \bar{\mu}(b' \cap a_n)$, so there is an $n \in \mathbb{N}$ such that $\bar{\mu}(b' \cap a_n) > 2^{-n}$; in which case $b' \cup b_n \in \mathfrak{B}^f$ and $\bar{\mu}((b' \cup b_n) \cap a_n) > \alpha_n$, which is impossible. **X**

566N Characterizing the usual measure on $\{0,1\}^{\mathbb{N}}$: Theorem [AC(ω)] (a) Let (X, Σ, μ) be an atomless, perfect, complete, countably separated probability space. Then it is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual measure.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type. Then it is isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$.

(c) An atomless measurable algebra of countable Maharam type is homogeneous.

(d) For any infinite set I, the measure algebra of the usual measure on $\{0,1\}^I$ is homogeneous.

proof (a) (Cf. 344I.) Write ν for the usual measure on $Y = \{0,1\}^{\mathbb{N}}$, and T for its domain.

(i) Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X, and set $F_n = \{y : y \in Y,\, y(n) = 1\}$ for each n. Let $f : \Sigma \times [0,1] \to \Sigma$ and $g : \mathrm{T} \times [0,1] \to \mathrm{T}$ be functions as in 566F. Define $\langle \Sigma_n \rangle_{n\in\mathbb{N}}$, $\langle \mathrm{T}_n \rangle_{n\in\mathbb{N}}$ and $\langle \theta_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that Σ_n and T_n are finite subalgebras of Σ, T respectively, all their atoms being of non-zero measure, and that $\theta_n : \Sigma_n \to \mathrm{T}_n$ is a measure-preserving isomorphism, Start with $\Sigma_0 = \{\emptyset, X\}$, $\mathrm{T}_0 = \{\emptyset, Y\}$ and θ_0 the trivial isomorphism between them. Given Σ_n, T_n and θ_n, let $\mathcal{E}_n$ be the set of atoms of Σ_n. Set

$$E'_n = \bigcup\{E : E \in \mathcal{E}_n,\, \mu(E \setminus E_n) = 0\} \cup \bigcup\{E \cap E_n : E \in \mathcal{E}_n,\, \mu(E \cap E_n) > 0\},$$

so that $\mu(E_n \triangle E'_n) = 0$. Set

$$H_n = \bigcup\{\theta_n E : E \in \mathcal{E}_n,\, \mu(E \setminus E_n) = 0\}$$
$$\cup \bigcup\{g(\theta_n E, \mu(E \cap E_n)) : E \in \mathcal{E}_n,\, \mu(E \cap E_n) > 0\}.$$

Let Σ'_n be the algebra of subsets of X generated by $\Sigma_n \cup \{E'_n\}$ and T'_n the algebra of subsets of Y generated by $\mathrm{T}_n \cup \{H_n\}$; then we have a measure-preserving isomorphism $\theta'_n : \Sigma'_n \to \mathrm{T}'_n$, extending θ_n, with $\theta'_n E'_n = H_n$. Note that the construction ensures that all the atoms of Σ'_n and T'_n have non-zero measure.

Running the same process from the other side, we can define a set $F'_n \subseteq Y$ such that $\nu(F_n \triangle F'_n) = 0$ and a set $G_n \subseteq X$ such that if Σ_{n+1} is the algebra generated by $\Sigma'_n \cup \{G_n\}$ and T_{n+1} is the algebra generated by $\mathrm{T}'_n \cup \{F'_n\}$, then these have non-negligible atoms and there is a measure-preserving isomorphism $\theta_{n+1} : \Sigma_{n+1} \to \mathrm{T}_{n+1}$, extending θ'_n, with $\theta_{n+1} G_n = F'_n$. Continue.

(ii) At the end of the induction, define $\phi_0 : X \to Y$ by setting $\phi_0(x) = \langle \chi G_n(x) \rangle_{n\in\mathbb{N}}$ for each $x \in X$. Then ϕ_0 is measurable. Consider the image measure $\mu\phi_0^{-1}$. This is a topological measure, and because μ is perfect and complete it is a Radon measure. If $I \subseteq \mathbb{N}$ is finite and not empty, then

$$\mu\phi_0^{-1}(\bigcap_{n\in I} F_n) = \mu(\bigcap_{n\in I} G_n) = \nu(\bigcap_{n\in I} F'_n)$$

(because θ_{n+1} is measure-preserving)

$$= \nu(\bigcap_{n\in I} F_n)$$

(because $\nu(F_n \triangle F'_n) = 0$ for every n)

$$= 2^{-\#(I)}.$$

So $\mu\phi_0^{-1} = \nu$.

(iii) For any $m \in \mathbb{N}$ and non-empty $I \subseteq m$,

$$\begin{aligned}\mu(\theta_m^{-1}(\bigcap_{i\in I} F'_i)\triangle\phi_0^{-1}[\bigcap_{i\in I} F'_i]) &\le \sum_{i\in I}\mu(\theta_m^{-1}F'_i\triangle\phi_0^{-1}[F'_i]) = \sum_{i\in I}\mu(G_i\triangle\phi_0^{-1}[F'_i])\\ &= \sum_{i\in I}\mu(\phi_0^{-1}[F_i]\triangle\phi_0^{-1}[F'_i]) = \sum_{i\in I}\nu(F_i\triangle F'_i) = 0,\end{aligned}$$

so $\mu(\theta_m^{-1}H\triangle\phi_0^{-1}[H]) = 0$ for every H in the algebra generated by $\{F'_i : i < m\}$.

(iv) $\mu(\phi_0^{-1}[H_n]\triangle E_n) = 0$ for every $n \in \mathbb{N}$. **P** For any $\epsilon > 0$, there are an $m > n$ and an $H \subseteq Y$, determined by coordinates less than m, such that $\nu(H_n \triangle H) \le \epsilon$. Because $F'_i \in \mathrm{T}_m$ and $F'_i \triangle F_i$ is negligible for every $i < m$, there is an H' in the algebra generated by $\{F'_i : i < m\}$ such that $\nu(H' \triangle H) = 0$. Now

$$\begin{aligned}\mu(E_n\triangle\phi_0^{-1}[H_n]) &\le \mu(E_n\triangle E'_n) + \mu(E'_n\triangle\theta_m^{-1}H')\\ &\qquad + \mu(\theta_m^{-1}H'\triangle\phi_0^{-1}[H']) + \mu(\phi_0^{-1}[H']\triangle\phi_0^{-1}[H_n])\\ &\le 0 + \mu(\theta_m^{-1}H_n\triangle\theta_m^{-1}H') + 0 + \nu(H'\triangle H_n)\end{aligned}$$

(by (iii), and because $\nu = \mu\phi_0^{-1}$)

$$= 2\nu(H'\triangle H_n) = 2\nu(H\triangle H_n) \le 2\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(v) Set

$$X_1 = X \setminus (\bigcup_{n\in\mathbb{N}} E_n\triangle\phi_0^{-1}[H_n]) \cup \bigcup_{n\in\mathbb{N}} \phi_0^{-1}[F_n\triangle F'_n]).$$

Then X_1 is conegligible in X. Because $\langle E_n\rangle_{n\in\mathbb{N}}$ separates the points of X, $\langle\phi_0^{-1}[H_n]\rangle_{n\in\mathbb{N}}$ separates the points of X_1, and $\phi_0{\restriction}X_1$ is injective. Next, $Y_1 = \phi_0[X_1]$ is conegligible in Y. **P** Because μ is perfect, there is a K_σ set $V \subseteq Y_1$ such that $\phi_0^{-1}[V]$ is conegligible in X; but now V must be conegligible in Y because $\nu = \mu\phi_0^{-1}$. **Q**

It follows that if we set $\phi_1 = \phi_0{\restriction}X_1$ then the subspace measure ν_{Y_1} is just the image measure $\mu_{X_1}\phi_1^{-1}$. **P** If $F \subseteq Y_1$ then

$$\mu_{X_1}\phi_1^{-1}[F] = \mu(X_1 \cap \phi_0^{-1}[F]) = \mu\phi_0^{-1}[F] = \nu F = \nu_{Y_1}F$$

if any of these is defined. **Q** But as ϕ_1 is a bijection, this means that it is an isomorphism between (X_1, μ_{X_1}) and (Y_1, ν_{Y_1}).

(vi) There is no reason to suppose that $X \setminus X_1$ and $Y \setminus Y_1$ are equipollent, so ϕ_1 may not be directly extendable to an isomorphism between X and Y. However, there is a negligible subset D of Y_1 which is equipollent with $\mathbb{R}$. **P** Let $K \subseteq Y_1$ be a non-negligible compact set. Set $S = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and define $\langle K_z\rangle_{z\in S}$ inductively, as follows. $K_\emptyset = K$. Given that $z \in \{0,1\}^n$ and that K_z is a non-negligible compact set, take the first $m \ge 2n+2$ such that $J = \{w : w \in \{0,1\}^m, \nu\{y : w \subseteq y \in K_z\} > 0\}$ has more than one member, let w, w' be the lexicographically two first members of J, and set

$$K_{z^\frown 0} = \{y : w \subseteq y \in K_z\}, \quad K_{z^\frown 1} = \{y : w' \subseteq y \in K_z\};$$

continue. This will ensure that $0 < \nu K_z \le 4^{-n}$ for every $z \in \{0,1\}^n$. Set $D = \bigcap_{n\in\mathbb{N}}\bigcup_{z\in\{0,1\}^n} K_z$; then D is negligible and equipollent with $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}$. **Q**

Now set $X_2 = X_1 \setminus \phi_1^{-1}[D]$ and $Y_2 = Y_1 \setminus D$. $\phi_2 = \phi_1 \restriction X_2$ is an isomorphism between the conegligible sets X_2 and Y_2 with their subspace measures. Since $\langle E_n \rangle_{n\in\mathbb{N}}$ separates the points of X, we surely have an injective function from $X \setminus X_2$ to $\mathbb{R}$, while we also have an injective function from $\mathbb{R}$ to $\phi_1^{-1}[D] \subseteq X \setminus X_2$. So $X \setminus X_2$ is equipollent with $\mathbb{R}$. Similarly, $Y \setminus Y_2$ is equipollent with $\mathbb{R}$. So $\phi_2 : X_2 \to Y_2$ can be extended to a bijection $\phi : X \to Y$, which will be the required isomorphism between (X, Σ, μ) and (Y, T, ν).

(b) (Cf. 331I.) Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence running over a τ-generating set $A \subseteq \mathfrak{A}$ with cofinal repetitions. Let $f : \mathfrak{A} \times [0, 1] \to \mathfrak{A}$ be a function as in 566F. Define $g : \mathfrak{A} \times \mathbb{N} \to \mathfrak{A}$ by setting

$$g(a, n) = f(a \cap a_n, \frac{1}{2}\bar{\mu}a) \text{ if } \bar{\mu}(a \cap a_n) \ge \frac{1}{2}\bar{\mu}a,$$

$$= f(a \setminus a_n, \frac{1}{2}\bar{\mu}a) \text{ otherwise }.$$

Define $\langle B_n \rangle_{n\in\mathbb{N}}$ inductively by saying that $B_0 = \{1\}$ and

$$B_{n+1} = \{g(b, n) : b \in B_n\} \cup \{b \setminus g(b, n) : b \in B_n\}$$

for each n. Then each B_n is a partition of unity consisting of 2^n elements of measure 2^{-n}. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} B_n$; then $\mathfrak{B}$ is isomorphic to the measure algebra of the usual measure on $\{0, 1\}^{\mathbb{N}}$.

For $a \in \mathfrak{A}$ and $n \in \mathbb{N}$, set

$$\gamma_n(a) = 2^{-n}\#(\{b : b \in B_n,\ a \cap b \notin \{0, b\}\}).$$

Then $\gamma_{n+1}(a) \le \gamma_n(a)$ for every n, and $\gamma_{n+1}(a_n) \le \frac{1}{2}\gamma_n(a_n)$. Since every member of A appears infinitely often as an a_n, $\lim_{n\to\infty} \gamma_n(a) = 0$ for every $a \in A$. But this means that $A \subseteq \mathfrak{B}$ and $\mathfrak{B} = \mathfrak{A}$. So we have the required isomorphism.

(c) (Cf. 331N.) If $\mathfrak{A}$ is such an algebra, any non-zero principal ideal of $\mathfrak{A}$ is atomless and of countable Maharam type and supports a probability measure, so must be isomorphic to the measure algebra of the usual measure on $\{0, 1\}^{\mathbb{N}}$ and to $\mathfrak{A}$.

(d) For $J \subseteq I$, write ν_J for the usual measure on $\{0, 1\}^J$, T_J for its domain and $(\mathfrak{B}_J, \bar{\nu}_J)$ for its measure algebra. If $a \in \mathfrak{B}_I$ is non-zero, then it is of the form $E^\bullet$ for some $E \in \mathrm{T}_I$ determined by coordinates in a countable subset J of I. Identifying $\{0, 1\}^I$ with $\{0, 1\}^J \times \{0, 1\}^{I\setminus J}$, we have an $F \in \mathrm{T}_J$ such that $E = F \times \{0, 1\}^{I\setminus J}$. Let $b \in \mathfrak{B}_J$ be the equivalence class of F. Now we can think of the probability algebra free product $\mathfrak{B}_J \widehat{\otimes} \mathfrak{B}_{I\setminus J}$ as the metric completion of the algebraic free product $\mathfrak{B}_J \otimes \mathfrak{B}_{I\setminus J}$, and as such isomorphic to $\mathfrak{B}_I$ under an isomorphism which identifies the principal ideal $(\mathfrak{B}_I)_a$ with $(\mathfrak{B}_J)_b \widehat{\otimes} \mathfrak{B}_{I\setminus J}$. By (b), $((\mathfrak{B}_J)_b, \bar{\nu}_J \restriction (\mathfrak{B}_J)_b)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_J, \bar{\nu}_J)$; so we have

$$(\mathfrak{B}_I)_a \cong (\mathfrak{B}_J)_b \widehat{\otimes} \mathfrak{B}_{I\setminus J} \cong \mathfrak{B}_J \widehat{\otimes} \mathfrak{B}_{I\setminus J} \cong \mathfrak{B}_I.$$

As a is arbitrary, $\mathfrak{B}_I$ is homogeneous.

566O Boolean values: Proposition [AC(ω)] (a) Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0, 1\}^{\mathbb{N}}$, and $\mathcal{B}(\{0, 1\}^{\mathbb{N}})$ the Borel σ-algebra. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, π has a unique extension to a sequentially order-continuous Boolean homomorphism from $\mathcal{B}(\{0, 1\}^{\mathbb{N}})$ to $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0 = L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u \in L^0$ corresponds to $\phi \in \Phi$ iff $[\![u > \alpha]\!] = \phi(]\alpha, \infty[)$ for every $\alpha \in \mathbb{R}$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$. Then for any $u \in L^0 = L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ such that

$$[\![u \in E]\!] = \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}$$
$$= \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\} = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\}$$

for every $E \in \Sigma_{\mathrm{um}}$, while

$$[\![u \in]\alpha, \infty[\,]\!] = [\![u > \alpha]\!]$$

for every $\alpha \in \mathbb{R}$.

proof (a) As in §562, let $\mathcal{T}$ be the set of trees without infinite branches in $S = \bigcup_{n\ge1} \mathbb{N}^n$. For $n \in \mathbb{N}$ set $E_n = \{x : x \in \{0, 1\}^{\mathbb{N}},\ x(n) = 1\} \in \mathfrak{B}$ and $a_n = \pi E_n \in \mathfrak{A}$. Let $\phi : \mathcal{T} \to \mathfrak{A}$ and $\psi : \mathcal{T} \to \mathcal{B}(\{0, 1\}^{\mathbb{N}})$ be the corresponding interpretations of Borel codes, as in 562T. Then $\phi(T) = \phi(T')$ whenever $\psi(T) = \psi(T')$ (562T), and (using AC(ω)) it is easy to check that $\psi[\mathcal{T}]=\mathcal{B}(\{0,1\}^{\mathbb{N}})$ (cf. 562Cb), so we have a function $\tilde\pi:\mathcal{B}(\{0,1\}^{\mathbb{N}})\to\mathfrak{A}$ defined by saying that $\tilde\pi(\psi(T))=\phi(T)$ for every $T\in\mathcal{T}$. Now if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence of Borel subsets of $\{0,1\}^{\mathbb{N}}$, we have a $T\in\mathcal{T}$ such that $F_n=\psi(T_{<n>})$ for every n and no $T_{<n>}$ is empty (see 562Ae). In this case

$$\tilde\pi(\bigcup_{n\in\mathbb{N}}\{0,1\}^{\mathbb{N}}\setminus F_n)=\tilde\pi(\psi(T))=\phi(T)$$

$$=\sup_{n\in\mathbb{N}}1\setminus\phi(T_{<n>})=\sup_{n\in\mathbb{N}}1\setminus\tilde\pi F_n.$$

So $\tilde\pi$ is a sequentially order-continuous Boolean homomorphism. Since it agrees with π on $\{E_n:n\in\mathbb{N}\}$ it must agree with π on $\mathfrak{B}$.

Of course the extension is unique because if $\tilde\pi':\mathcal{B}(\{0,1\}^{\mathbb{N}}\to\mathfrak{A}$ is any sequentially order-continuous Boolean homomorphism extending π then $\{E:\tilde\pi'E=\tilde\pi E\}$ is a σ-algebra of sets including $\mathfrak{B}$ and therefore containing every open set.

(b) (Cf. 364G.) Let $\mathcal{E}$ be the algebra of subsets of $\mathbb{R}$ generated by sets of the form $]q,\infty[$ for $q\in\mathbb{Q}$. Then $\mathcal{E}$ is an atomless countable Boolean algebra, so is isomorphic to the algebra $\mathfrak{B}$; let $\theta:\mathfrak{B}\to\mathcal{E}$ be an isomorphism. Define $f:\{0,1\}^{\mathbb{N}}\to[-\infty,\infty]$ by setting $f(x)=\sup\{q:q\in\mathbb{Q},\,x\in\theta^{-1}]q,\infty[\}$. Then f is Borel measurable. Set $\phi E=\tilde\pi(f^{-1}[E])$ for $E\in\mathcal{B}(\mathbb{R})$.

Take any u in L^0. It is easy to check that we have a Boolean homomorphism $\pi:\mathcal{E}\to\mathfrak{A}$ defined by saying that $\pi]q,\infty[=[\![u>q]\!]$ for every $q\in\mathbb{Q}$. By (a), there is a sequentially order-continuous Boolean homomorphism $\tilde\pi:\mathcal{B}(\{0,1\}^{\mathbb{N}})\to\mathfrak{A}$ extending $\pi\theta:\mathfrak{B}\to\mathfrak{A}$.

If $\alpha\in\mathbb{R}$ then

$$\begin{aligned}\tilde\pi\{x:f(x)>\alpha\}&=\tilde\pi(\bigcup\{\theta^{-1}]q,\infty[:q\in\mathbb{Q},\,q>\alpha\})\\&=\sup\{\tilde\pi(\theta^{-1}]q,\infty[):q\in\mathbb{Q},\,q>\alpha\}\\&=\sup\{\pi]q,\infty[:q\in\mathbb{Q},\,q>\alpha\}\\&=\sup\{[\![u>q]\!]:q\in\mathbb{Q},\,q>\alpha\}=[\![u>\alpha]\!].\end{aligned}$$

It follows that

$$\begin{aligned}\phi\mathbb{R}&=\sup_{n\in\mathbb{N}}\tilde\pi(f^{-1}]-n,\infty[)\setminus\inf_{n\in\mathbb{N}}\tilde\pi(f^{-1}]n,\infty[)\\&=\sup_{n\in\mathbb{N}}[\![u>-n]\!]\setminus\inf_{n\in\mathbb{N}}[\![u>n]\!]=1,\end{aligned}$$

and therefore that $\phi\in\Phi$. For the rest of the argument we can follow the method of 364G.

(c) (Cf. 434T.)

(i) To begin with, consider the case in which $\bar\mu$ is totally finite. In this case, we have a non-decreasing function $g:\mathbb{R}\to[0,\infty[$ defined by saying that $g(\alpha)=\bar\mu1-\bar\mu[\![u>\alpha]\!])$ for $\alpha\in\mathbb{R}$. Let ν_g be the corresponding Lebesgue-Stieltjes measure (114Xa), and $(\mathfrak{C},\bar\nu_g)$ its measure algebra. Note that g is continuous on the right, so that $\nu_g]\alpha,\beta]=\bar\mu[\![u>\alpha]\!]-\bar\mu[\![u>\beta]\!]$ whenever $\alpha\le\beta$ in $\mathbb{R}$. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{C}$ generated by $\{]-\infty,\alpha]^\bullet:\alpha\in\mathbb{R}\}$. Then we have a measure-preserving Boolean homomorphism $\pi:\mathfrak{D}\to\mathfrak{A}$ defined uniquely by saying that $\pi(]\alpha,\infty[^\bullet)=[\![u>\alpha]\!]$ for $\alpha\in\mathbb{R}$. Because $\mathfrak{D}$ is dense in $\mathfrak{C}$ for the measure-algebra topology, π has a unique extension to a measure-preserving Boolean homomorphism $\tilde\pi:\mathfrak{C}\to\mathfrak{A}$.

Because $\Sigma_{\mathrm{um}}\subseteq\operatorname{dom}\nu_g$, we can define $[\![u\in E]\!]$ to be $\tilde\pi E^\bullet$ for $E\in\Sigma_{\mathrm{um}}$, and this will give us a sequentially order-continuous Boolean homomorphism from Σ_{um} to $\mathfrak{A}$ such that $[\![u\in]\alpha,\infty[]\!]=[\![u>\alpha]\!]$ for every α. As for the other formulae, they are immediate from the facts that ν_g is inner regular with respect to the compact sets and outer regular with respect to the open sets.

(ii) We need to observe that these properties uniquely define $[\![u\in E]\!]$. **P** Let $\mathcal{E}$ be the algebra of subsets of $\mathbb{R}$ generated by $\{]\alpha,\infty[:\alpha\in\mathbb{R}\}$. The requirement $[\![u\in]\alpha,\infty[]\!]=[\![u>\alpha]\!]$ determines the values of $[\![u\in E]\!]$ for $E\in\mathcal{E}$. Next, if $G\subseteq\mathbb{R}$ is open and $K\subseteq G$ is compact there is an $E\in\mathcal{E}$ such that $K\subseteq E\subseteq G$. Consequently $[\![u\in K]\!]=\inf\{[\![u\in E]\!]:E\in\mathcal{E},\,E\supseteq K\}$ is fixed for every compact $K\subseteq\mathbb{R}$. Finally, the inner regularity condition $[\![u\in E]\!]=\sup\{[\![u\in K]\!]:K\subseteq E$ is compact$\}$ determines $[\![u\in E]\!]$ for other $E\in\Sigma_{\mathrm{um}}$. **Q**

(iii) Now turn to the general case of a localizable measure algebra $(\mathfrak{A},\bar\mu)$ and $u\in L^0(\mathfrak{A})$. Let $\mathfrak{A}^f$ be the ideal of elements of finite measure. Then for each $a\in\mathfrak{A}^f$ we have a corresponding homomorphism $E\mapsto[\![u\in E]\!]_a$ from Σ_{um} to the principal ideal $\mathfrak{A}_a$. If $a\subseteq b\in\mathfrak{A}^f$, we can use the uniqueness described in (ii) to see that $[\![u\in E]\!]_a=a\cap[\![u\in E]\!]_b$ for every E. So if we set $[\![u\in E]\!]=\sup_{a\in\mathfrak{A}^f}[\![u\in E]\!]_a$, we shall have $[\![u\in E]\!]_a=a\cap[\![u\in E]\!]$ whenever $a\in\mathfrak{A}^f$ and $E\in\Sigma_{\mathrm{um}}$. It is now easy to check that $E\mapsto[\![u\in E]\!]$ has the required properties.

566P Weak compactness In the absence of Tychonoff's theorem, the theory of weak compactness in normed spaces becomes uncertain. However AC(ω) is enough to give a couple of the principal results involving classical Banach spaces, starting with Hilbert space.

Theorem [AC(ω)] Let U be a Hilbert space. Then bounded sets in U are relatively weakly compact.

proof If U is finite-dimensional, this is trivial; so let us suppose that U is infinite-dimensional. Let $\mathcal{F}$ be a filter on U containing a bounded set.

(a) For closed subspaces V of U, let $P_V : U \to V$ be the orthogonal projection from U onto V (561Ib), and set $\gamma_V = \liminf_{u\to\mathcal{F}} \|P_V u\|^2$. Because $\mathcal{F}$ contains a bounded set, $\gamma_V \le \gamma_U < \infty$ for every V. If V_0, V_1 are orthogonal subspaces of U, then $\|P_{V_0+V_1}u\|^2 = \|P_{V_0}u\|^2 + \|P_{V_1}u\|^2$ for every $u \in U$, so $\gamma_{V_0+V_1} \ge \gamma_{V_0} + \gamma_{V_1}$.

(b) Set $\gamma = \sup\{\gamma_V : V$ is a finite-dimensional linear subspace of $U\}$, and choose a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of finite-dimensional subspaces of U such that $\gamma = \sup_{n\in\mathbb{N}} \gamma_{V_n}$; because U is infinite-dimensional, we can do this in such a way that $\dim V_n \ge n$ for each n. Let W be the closed linear span of $\bigcup_{n\in\mathbb{N}} V_n$. If V is a finite-dimensional linear subspace of $W^\perp$, then

$$\gamma \ge \gamma_{V+V_n} \ge \gamma_V + \gamma_{V_n}$$

for every n, so $\gamma_V = 0$.

(c) If $F \in \mathcal{F}$, $V \subseteq W^\perp$ is a finite-dimensional linear subspace, and $\epsilon > 0$, then $F \cap \{u : \|P_V u\| \le \epsilon\}$ is non-empty. We can therefore extend $\mathcal{F}$ to the filter $\mathcal{G}$ generated by sets of this type, and $\lim_{u\to\mathcal{G}}(u|w) = 0$ for every $w \in W^\perp$.

(d) Let $\langle e_n\rangle_{n\in\mathbb{N}}$ be an orthonormal basis for W. Define $\langle \mathcal{G}_n\rangle_{n\in\mathbb{N}}$ as follows. $\mathcal{G}_0 = \mathcal{G}$. Given that $\mathcal{G}_n$ is a filter on U containing a bounded set, set $\alpha_n = \liminf_{u\to\mathcal{G}_n}(u|e_n)$, and let $\mathcal{G}_{n+1}$ be the filter generated by $\mathcal{G}_n \cup \{\{u : (u|e_n) < \alpha\} : \alpha > \alpha_n\}$; then $\alpha_n = \lim_{u\to\mathcal{G}_{n+1}}(u|e_n)$. Set $\mathcal{H} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$; then $\alpha_n = \lim_{u\to\mathcal{H}}(u|e_n)$ for each n.

For any $n \in \mathbb{N}$,

$$\textstyle\sum_{i=0}^n \alpha_i^2 = \sum_{i=0}^n \lim_{u\to\mathcal{H}}(u|e_i)^2 = \lim_{u\to\mathcal{H}} \sum_{i=0}^n (u|e_i)^2 \le \liminf_{u\to\mathcal{H}} \|u\|^2 < \infty$$

because $\mathcal{H}$ contains a bounded set. So $\sum_{n=0}^\infty \alpha_n^2$ is finite and $v = \sum_{n=0}^\infty \alpha_n e_n$ is defined in U.

(e) Now

$$(v|e_n) = \alpha_n = \lim_{u\to\mathcal{H}}(u|e_n)$$

for every n; again because $\mathcal{H}$ contains a bounded set, $(v|w) = \lim_{u\to\mathcal{H}}(u|w)$ for every $w \in W$. On the other hand, if $w \in W^\perp$,

$$\lim_{u\to\mathcal{H}}(u|w) = \lim_{u\to\mathcal{G}}(u|w) = 0 = (v|w).$$

Since $W + W^\perp = U$, $\lim_{u\to\mathcal{H}}(u|w) = (v|w)$ for every $w \in U$. By 561Ic, v is the limit of $\mathcal{H}$, and a cluster point of $\mathcal{F}$, for the weak topology on U.

As $\mathcal{F}$ is arbitrary, every bounded set in U is relatively weakly compact (3A3De).

566Q Theorem [AC(ω)] Let U be an L-space. Then a subset of U is weakly relatively compact iff it is uniformly integrable.

proof (a)(i)(α) Recall that U is a Banach lattice with an order-continuous norm (354N), so is Dedekind complete (354Ee) and all its bands are complemented (353I); for a band V in U, let $P_V : U \to V$ be the band projection onto V.

(β) If $u \in U$ there is an $f \in U^*$ such that $\|f\| \le 1$ and $f(u) = \|u\|$. **P** Let V be the band generated by u^+ and $W = V^\perp$ its band complement. Set $f(v) = \int P_V v - \int P_W v$ for $v \in U$. Since $\|v\| = \|P_V v\| + \|P_W v\|$ for every $v \in U$, $\|f\| \le 1$. Also $P_V u = u^+$ and $P_W u = -u^-$ so $f(u) = \int |u| = \|u\|$. **Q**

(γ) If $A \subseteq U$ is weakly bounded it is norm-bounded. **P?** Otherwise, choose for each $n \in \mathbb{N}$ a $u_n \in A$ and $f_n \in U^*$ such that $\|u_n\| \ge n$, $\|f_n\| = 1$ and $f_n(u_n) = \|u_n\| \ge n$. For $f \in U^*$ set $\rho_A(f) = \sup_{u\in A} |f(u)|$. Define $\langle n_k\rangle_{k\in\mathbb{N}}$ by setting $n_k = \lceil 2\cdot 3^k(k + \sum_{i=0}^{k-1} 3^{k-i}\rho_A(f_{n_i}))\rceil$ for each k. Set $f = \sum_{i=0}^\infty 3^{-i} f_{n_i}$. Then for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\rho_A(f) \ge f(u_{n_k}) &= \sum_{i=0}^\infty 3^{-i} f_{n_i}(u_{n_k}) \\
&\ge 3^{-k} f_{n_k}(u_{n_k}) - \sum_{i=0}^{k-1} 3^{-i}\rho_A(f_{n_i}) - \sum_{i=k+1}^\infty 3^{-i}\|u_{n_k}\| \\
&= \tfrac{1}{2\cdot 3^k}\|u_{n_k}\| - \sum_{i=0}^{k-1} 3^{-i}\rho_A(f_{n_i}) \ge k. \text{ **XQ**}
\end{aligned}$$

(ii) Now let $K \subseteq U$ be a weakly relatively countably compact compact set. Let $\mathfrak{A}$ be the band algebra of U. For $V \in \mathfrak{A}$ set $\nu V = \sup_{u\in K} \|P_V u\|$ (counting $\sup\emptyset$ as 0). Then ν is a submeasure on $\mathfrak{A}$. By (i-γ), K is norm-bounded and ν is finite-valued; set $\alpha = \nu U = \sup_{u\in K}\|u\|$.

ν is exhaustive. **P?** Otherwise, let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$ such that $\epsilon = \frac{1}{6}\inf_{n\in\mathbb{N}}\nu V_n$ is greater than 0. For each $n \in \mathbb{N}$ choose $u_n \in K$ and $f_n \in U^*$ such that $\|f_n\| \le 1$ and $f_n(P_n u_n) = \|P_n u_n\| \ge 5\epsilon$, where here I write P_n for P_{V_n}. Let v_0 be a cluster point of $\langle u_n\rangle_{n\in\mathbb{N}}$ in U for the weak topology of U. Note that $\sum_{n=0}^\infty \|P_n u\| \le \|u\|$ for any $u \in U$; let $m \in \mathbb{N}$ be such that $\sum_{n=m}^\infty \|P_n v_0\| \le \epsilon$. For $n \in \mathbb{N}$, set $g_n(u) = f_n(P_n u)$ for $u \in U$.

We can now build a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that

$$n_0 \ge m,$$

$$\textstyle\sum_{i=0}^{k-1} |g_{n_i}(u_{n_k})| \le \epsilon + \sum_{i=0}^{k-1} |g_{n_i}(v_0)|,$$

$$\|P_{n_k} u_{n_i}\| \le 2^{-k}\epsilon \text{ whenever } i < k$$

for every $k \in \mathbb{N}$. Let v_1 be a weak cluster point of $\langle u_{n_k}\rangle_{k\in\mathbb{N}}$, and $l \in \mathbb{N}$ such that $\sum_{k=l}^\infty \|P_{n_k} v_1\| \le \epsilon$. Set $g = \sum_{k=l}^\infty g_{n_k}$; this is defined in U^* because

$$\textstyle\sum_{k=l}^\infty |g_{n_k}(u)| \le \sum_{k=l}^\infty \|P_{n_k} u\| \le \|u\|$$

for every $u \in U$. Of course $|g(v_1)| \le \epsilon$. On the other hand, for any $k \ge l$,

$$\begin{aligned} g(u_{n_k}) &= g_{n_k}(u_{n_k}) + \sum_{i=l}^{k-1} g_{n_i}(u_{n_k}) + \sum_{i=k+1}^{\infty} g_{n_i}(u_{n_k}) \\ &\ge 5\epsilon - \sum_{i=0}^{k-1} |g_{n_i}(u_{n_k})| - \sum_{i=k+1}^{\infty} \|P_{n_i} u_{n_k}\| \\ &\ge 5\epsilon - \sum_{i=0}^{k-1} |g_{n_i}(v_0)| - \epsilon - \sum_{i=k+1}^{\infty} 2^{-i}\epsilon \\ &\ge 4\epsilon - \sum_{n=m}^{\infty} \|P_n v_0\| - 2^{-k}\epsilon \ge 2\epsilon \end{aligned}$$

and v_1 cannot be a weak cluster point of $\langle u_{n_k}\rangle_{k\in\mathbb{N}}$. **XQ**

(iii) In fact ν is uniformly exhaustive. **P?** Otherwise, let $\epsilon > 0$ be such that there are arbitrarily long disjoint strings in $\mathfrak{A}$ of elements of submeasure greater than 2ϵ. Set $q(n) = \lceil \frac{2^n n\alpha}{\epsilon} \rceil$ for each n, and choose a family $\langle V_{ni}\rangle_{n\in\mathbb{N}, i\le q(n)}$ such that $\langle V_{ni}\rangle_{i\le q(n)}$ is a disjoint family in $\mathfrak{A}$ for each n and $\nu V_{ni} > 2\epsilon$ for all n and i; adapting the temporary notation of (ii), I set $P_{ni} = P_{V_{ni}}$ for $i \le q(n)$. Now choose $u_{ni} \in K$ such that $\|P_{ni} u_{ni}\| \ge 2\epsilon$ for all i and n. Because $\sum_{i=0}^{q(n)} \|P_{ni} u\| \le \|u\|$ for every $u \in U$ and $n \in \mathbb{N}$, we can define inductively a sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $i_n \le q(n)$ and $\|P_{ni_n} u_{mi_m}\| \le 2^{-n}\epsilon$ whenever $m < n$.

Now set

$$W_{mn} = V_{mi_m} \cap \textstyle\bigcap_{m<k\le n} V_{ki_k}^{\perp}, \quad Q_{mn} = P_{W_{mn}}$$

for $m \le n$,

$$W_m = \textstyle\bigcap_{n\ge m} W_{mn}, \quad Q_m = P_{W_m}$$

for $m \in \mathbb{N}$. For any $u \in U$ and $m \le n$,

$$|P_{mi_m} u| = P_{mi_m}|u| \le Q_{mn}|u| + \textstyle\sum_{k=m+1}^n P_{ki_k}|u|,$$

$$\|P_{mi_m} u\| \le \|Q_{mn} u\| + \textstyle\sum_{k=m+1}^n \|P_{ki_k} u\|,$$

so

$$\begin{aligned} \|Q_{mn} u_{mi_m}\| &\ge \|P_{mi_m} u_{mi_m}\| - \sum_{k=m+1}^{n} \|P_{ki_k} u_{mi_m}\| \\ &\ge 2\epsilon - \sum_{k=m+1}^{\infty} 2^{-k}\epsilon \ge \epsilon. \end{aligned}$$

Next, if $u \ge 0$, $\langle Q_{mn}u\rangle_{n\ge m}$ is a non-increasing sequence, and its infimum belongs to $\bigcap_{n\ge m} W_{mn}$, so must be equal to $Q_m u$; accordingly $Q_m u$ is the norm-limit of $\langle Q_{mn}u\rangle_{n\ge m}$. The same is therefore true for every $u \in U$, and in particular

$$\|Q_m u_{mi_m}\| = \lim_{n\to\infty} \|Q_{mn}u_{mi_m}\| \ge \epsilon.$$

Consequently $\nu W_m \ge \epsilon$. But $W_m \cap W_n \subseteq V_{ni_n}^{\perp} \cap V_{ni_n} = \{0\}$ whenever $n > m$, so this contradicts (ii). **XQ**

(iv) Now take any $\epsilon > 0$. Then there is a $u^* \in U^+$ such that $\int(|u| - u^*)^+ \le \epsilon$ for every $u \in K$. **P** If $\alpha \le \epsilon$ we can take $u^* = 0$ and stop. Otherwise, there is a largest $n \in \mathbb{N}$ such that there are disjoint $V_0, \dots, V_n \in \mathfrak{A}$ such that $\nu V_i > \epsilon$ for every n. Take $u_0, \dots, u_n \in K$ such that $\|P_{V_i}u_i\| > \epsilon$ for each i. Let $\gamma > 0$ be such that $\|P_{V_i}u_i\| - \frac{\alpha}{\gamma} > \epsilon$ for every $i \le n$, and set $u^* = \gamma\sum_{i=0}^{n} |u_i|$. **?** Suppose that $u \in K$ is such that $\int(|u| - u^*)^+ > \epsilon$. Let W be the band generated by $(|u| - u^*)^+$, so that $\nu W \ge \|P_W u\| > \epsilon$. For each $i \le n$, set $W_i = V_i \cap W^{\perp}$; then

$$|P_W u_i| \le \frac{1}{\gamma}P_V u^* \le \frac{1}{\gamma}|u|, \quad |P_{W_i}u_i| \ge |P_{V_i}u_i| - \frac{1}{\gamma}|u|,$$

$$\nu W_i \ge \|P_{W_i}u_i\| \ge \|P_{V_i}u_i\| - \frac{\alpha}{\gamma} > \epsilon.$$

But now $W_0, \dots, W_n, W$ witnesses that n was not maximal. **X** So $\sup_{u\in K} \int(|u| - u^*)^+ \le \epsilon$, as required. **Q**

As ϵ is arbitrary, K is uniformly integrable. Thus every relatively weakly compact subset of U is uniformly integrable.

(b)(i) In the reverse direction, suppose to begin with that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra, and that $A \subseteq L^1 = L^1(\mathfrak{A}, \bar\mu)$ is uniformly integrable; let $\mathcal{F}$ be a filter on L^1 containing A. Write $\mathcal{V}$ for the set of neighbourhoods of 0 for the weak topology $\mathfrak{T}_s(L^1, (L^1)^*)$[6].

(α) For each $n \in \mathbb{N}$ let $M_n \ge 0$ be such that $\|(|u| - M_n\chi 1)^+\|_1 \le 2^{-n}$ for every $u \in A$, and define sets $K_n \subseteq [-M_n\chi 1, M_n\chi 1]$ and filters $\mathcal{F}_n$ as follows. $\mathcal{F}_0 = \mathcal{F}$. Given that $\mathcal{F}_n$ contains A, define $\phi_n : L^1 \to L^2 = L^2(\mathfrak{A}, \bar\mu)$ by setting $\phi_n(u) = \mathrm{med}(u, -M_n\chi 1, M_n\chi 1)$ for each $u \in L^1$, and consider the filter $\phi_n[[\mathcal{F}_n]]$. This is a filter on the Hilbert space L^2 containing the $\| \|_2$-bounded set $[-M_n\chi 1, M_n\chi 1]$, so the set K_n^* of its $\mathfrak{T}_s(L^2, L^2)$-cluster points is non-empty, by 566P; as K_n^* is $\mathfrak{T}_s(L^2, L^2)$-closed, it is $\mathfrak{T}_s(L^2, L^2)$-compact. As $[-M_n\chi 1, M_n\chi 1]$ is $\| \|_2$-closed and convex, it is $\mathfrak{T}_2(L^2, L^2)$-closed (561Ie) and includes K_n^*. Set $\gamma_n = \inf\{\|u\|_2 : u \in K_n^*\}$. As all the sets $\{u : \|u\|_2 \le \alpha\}$, for $\alpha > \gamma_n$, are $\mathfrak{T}_s(L^2, L^2)$-closed and meet K_n^*, $K_n = \{u : u \in K_n^*, \|u\|_2 \le \gamma_n\}$ is non-empty.

Suppose that $G \in \mathcal{V}$. Because the embedding $L^2 \hookrightarrow L^1$ is norm-continuous, it is weakly continuous, and $G \cap L^2$ is a $\mathfrak{T}_s(L^2, L^2)$-neighbourhood of 0. It follows that $x + G$ meets every member of $\phi_n[[\mathcal{F}_n]]$ for every $x \in K_n^*$; so $K_n + G$ meets every member of $\phi_n[[\mathcal{F}_n]]$. We can therefore extend $\mathcal{F}_n$ to the filter $\mathcal{F}_{n+1}$ generated by

$$\mathcal{F}_n \cup \{\phi_n^{-1}[K_n + G] : G \in \mathcal{V}\}$$

and continue.

(β) Set $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and $B = \{u : u \in L^1, \|u\|_1 \le 1\}$. Then for each $n \in \mathbb{N}$ there is a finite set $J \subseteq L^1$ such that $J + G + 2^{-n+1}B \in \mathcal{G}$ for every $G \in \mathcal{V}$. **P** K_n is a $\mathfrak{T}_s(L^2, L^2)$-closed subset of K_n^*, so is $\mathfrak{T}_s(L^2, L^2)$-compact; also it is included in the sphere $S = \{u : \|u\|_2 = \gamma_n\}$. Because $\| \|_2$ is locally uniformly rotund, it is a Kadec norm (467B) and the norm and weak topologies on S coincide; consequently K_n is $\| \|_2$-compact. Since $\| \|_1$ and $\| \|_2$ give rise to the same topology on any $\| \|_\infty$-bounded set, K_n is $\| \|_1$-compact. There is therefore a finite set $J \subseteq K_n$ such that $K_n \subseteq J + 2^{-n}B$.

Take any $G \in \mathcal{V}$. Then $\|u - \phi_n(u)\| = \|(|u| - M_n\chi 1)^+\| \le 2^{-n}$ for every $u \in A$, so

$$J + G + 2^{-n+1}B \supseteq (K + G) + 2^{-n}B \supseteq A \cap \phi_n^{-1}[K + G] \in \mathcal{F}_{n+1} \subseteq \mathcal{G}. \ \mathbf{Q}$$

(γ) For each $n \in \mathbb{N}$ choose a minimal finite set $J_n \subseteq L^1$ such that $J_n + G + 2^{-n+1}B \in \mathcal{G}$ for every $G \in \mathcal{V}$. Note that $(x + G + 2^{-n+1}B) \cap D$ must be non-empty whenever $n \in \mathbb{N}$, $x \in J_n$, $G \in \mathcal{V}$ and $D \in \mathcal{G}$. **P?** Otherwise,

$$(J_n \setminus \{x\}) + G' + 2^{-n+1}B \supseteq (J_n + (G \cap G') + 2^{-n+1}B) \cap D$$

belongs to $\mathcal{G}$ for every $G' \in \mathcal{V}$, and J_n was not minimal. **XQ**

(δ) For any $n \in \mathbb{N}$ and $u \in J_n$ there is a $v \in J_{n+1}$ such that $\|u - v\|_1 \le 2^{-n+1} + 2^{-n}$. **P?** Otherwise, by (a-i-β) above, we can choose for each $v \in J_{n+1}$ an $f_v \in (L^1)^*$ such that $\|f_v\| = 1$ and $f_v(v - u) = \|v - u\| = 2^{-n+1} + 2^{-n} + \delta_v$ where $\delta_v > 0$; set

$$G = \{w : |f_v(w)| < \tfrac{1}{2}\delta_v \text{ for every } v \in J_{n+1}\} \in \mathcal{V}.$$

Then $u + G + 2^{-n+1}B$ does not meet $J_{n+1} + G + 2^{-n}B$, contradicting (γ) here. **XQ**

[6] Of course $(L^1)^*$ can be identified with $L^\infty(\mathfrak{A})$, but if you don't wish to trace through the arguments for this, and confirm that they can be carried out without appealing to anything more than AC(ω), you can defer the exercise for the time being.

(ϵ) Because $\bigcup_{n\in\mathbb{N}} J_n$ is countable, therefore well-orderable, we can define inductively a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $u_n \in J_n$ and $\|u_n - u_{n+1}\| \le 2^{-n+1} + 2^{-n}$ for every n. Now $\langle u_n\rangle_{n\in\mathbb{N}}$ is Cauchy, so has a limit u in L^1. If $G \in \mathcal{V}$ there is an $n \in \mathbb{N}$ such that $u + G \supseteq u_n + \frac{1}{2}G + 2^{-n}B$, so $u + G$ meets every member of $\mathcal{G}$; thus u is a weak cluster point of $\mathcal{G}$ and of $\mathcal{F}$. As $\mathcal{F}$ is arbitrary, A is relatively weakly compact.

(ii) Now suppose that U is an arbitrary L-space and $A \subseteq U$ is a uniformly integrable set. Then we can choose a sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in U^+ such that $\|(|u| - e_n)^+\| \le 2^{-n}$ for every $n \in \mathbb{N}$ and $u \in A$. Set $e = \sum_{n=0}^{\infty} \frac{1}{1+2^n\|e_n\|} e_n$ in U, and let V be the band in U generated by e. Then $A \subseteq V$, and of course A is uniformly integrable in V. By 561Hb, we have a totally finite measure algebra $(\mathfrak{A}, \bar\mu)$ and a normed Riesz space isomorphism $T : V \to L^1(\mathfrak{A}, \bar\mu)$; now $T[A]$ is uniformly integrable in $L^1(\mathfrak{A}, \bar\mu)$, therefore relatively weakly compact, by (i). But this means that A is relatively weakly compact in V; as the embedding $V \subsetneq U$ is weakly continuous, A is relatively weakly compact in U.

This completes the proof.

566R Automorphisms of measurable algebras: Theorem [AC(ω)] Let $\mathfrak{A}$ be a measurable algebra.

(a) Every automorphism of $\mathfrak{A}$ has a separator.

(b) Every $\pi \in \operatorname{Aut}\mathfrak{A}$ is a product of at most three exchanging involutions belonging to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

proof (a) (Cf. 382Eb.) Take $\pi \in \operatorname{Aut}\mathfrak{A}$. Let $\bar\mu$ be such that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra. For $a \in \mathbb{N}$ set $\psi(a) = \sup_{n\in\mathbb{Z}} \pi^n a$, so that $\pi(\psi(a)) = \psi(a)$. Note that if $a \cap \psi(b) = 0$ then $\psi(a) \cap \psi(b) = 0$. Set $A = \{a : a \in \mathfrak{A}, a \cap \pi a = 0\}$ and choose a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that

$$\sup_{n\in\mathbb{N}} \bar\mu(\psi(a_n)) = \sup_{a\in A} \bar\mu(\psi(a)).$$

Define $\langle b_n\rangle_{n\in\mathbb{N}}$, $\langle c_n\rangle_{n\in\mathbb{N}}$ by saying that

$$c_0 = 0, \quad b_n = a_n \setminus \psi(c_n), \quad c_{n+1} = b_n \cup c_n$$

for each n. Inducing on n we see that b_n and c_n belong to A and that $\psi(c_{n+1}) \supseteq \psi(a_n)$ for every n. Set $c = \sup_{n\in\mathbb{N}} c_n$; then $c \in A$ and $\psi(c) \supseteq \psi(a_n)$ for every n.

Now c is a separator for π. **P?** Otherwise, there is a non-zero $d \subseteq 1 \setminus \psi(c)$ such that $d \cap \pi d = 0$ (381Ei). In this case $d \cup c \in A$ and

$$\bar\mu(\psi(d \cup c)) > \bar\mu(\psi(c)) = \sup_{n\in\mathbb{N}} \bar\mu(\psi(a_n)) = \sup_{a\in A} \bar\mu(\psi(a)),$$

which is impossible. **XQ**

(b) We can now work through the proofs of 382A-382M to confirm that there is no essential use of anything beyond countable choice there, so long as we suppose that we are working with measurable algebras. (There is an inductive construction in the proof of 382J. To do this with AC(ω) rather than DC, we need to check that every element of the construction can be made determinate following an initial countable set of choices; in the case there, we need to check that the existence assertions of 382D and 382I can be represented as functions, as in 566Xg and 566Xi.) Since the proof of 382K speaks of the Stone representation theorem, there seems to be a difficulty here, unless we take the alternative route suggested in 382Yb[7]. But note that while the general Stone theorem has a strength little short of full AC, the representation of a *countable* Boolean algebra $\mathfrak{B}$ as the algebra of open-and-closed subsets of a compact Hausdorff Baire space can be done in ZF alone (561F). In part (f) of the proof of 382K, therefore, take $\mathfrak{B}$ to be a countable subalgebra of $\mathfrak{A}$ such that

e_n, u'_n, u''_n, v'_l, v''_l, d_{lj}, d'_{lj}, $\operatorname{supp}(\pi\phi)^k$, $\operatorname{supp}(\pi\phi_1)^k \in \mathfrak{B}$ whenever $n \in \mathbb{N}$ and j, k, $l \ge 1$,

c_0, c_1, $\operatorname{supp}\phi_2 \in \mathfrak{B}$,

$\mathfrak{B}$ is closed under the functions π, ϕ_1, ϕ_2, ϕ and $\tilde\pi_n$ for $n \in \mathbb{N}$,

and let Z be the Stone space of $\mathfrak{B}$. Now we can perform the arguments of the rest of the proof in Z to show that $c_0 = \inf_{n\ge1} \operatorname{supp}(\pi\phi)^n$ is zero, as required.

566S Volume 4 In Volume 4, naturally, a rather larger proportion of the ideas become inaccessible without strong forms of the axiom of choice. Since we are missing the most useful representation theorems, many results have to be abandoned altogether. More subtly, we seem to lose the result that Radon measures are localizable (416B). Nevertheless, a good deal can still be done, if we follow the principles set out in 566Ae-566Af. Most notably, we have a workable theory of Haar measure on completely regular locally compact topological groups, because the Riesz representation theorems of §436 are still available, and we can use 561G instead of 441C. I should remark, however, that in the absence of Tychonoff's theorem we may have fewer compact groups than we expect. And the theory of dual groups in §445 depends heavily on AC.

[7]Later editions only.

The descriptive set theory of Chapter 42 is hardly touched, and enough of the rest of the volume survives to make it worth checking any point of particular interest. Most of Chapter 46 depends heavily on the Hahn-Banach theorem and therefore becomes limited to cases in which we have a good grasp of dual spaces, as in 561Xg. There are some difficulties in the geometric measure theory of arbitrary metric spaces in §471, but the rest of the chapter seems to stand up. The abstract theory of gauge integrals in §482 is expressed in forms which need DC at least, but I think that the basic facts about the Henstock integral (§483) are unaffected. There are some interesting challenges in Chapter 49, but there the eclectic nature of the arguments means that we cannot expect much of the theory to keep its shape.

566T I give one result which may not be obvious and helps to keep things in order.

Proposition [AC(ω)] Let I be any set, and X a separable metrizable space. Then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X^I)$ of X^I is equal to the σ-algebra $\widehat{\bigotimes}_I\mathcal{B}(X)$ generated by sets of the form $\{x : x(i) \in E\}$ for $i \in I$ and Borel sets $E \subseteq X$.

proof (a) Every open set in X is a cozero set, so $\mathcal{B}(X) = \mathcal{B}\mathfrak{a}(X)$ and $\{x : x(i) \in E\} \in \mathcal{B}\mathfrak{a}(X^I)$ whenever $i \in I$ and $E \in \mathcal{B}(X)$; accordingly $\widehat{\bigotimes}_I\mathcal{B}(X) \subseteq \mathcal{B}\mathfrak{a}(X^I)$.

(b) Fix a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of X. For $\sigma \in S = \bigcup_{J\in[I]^{<\omega}} \mathbb{N}^J$ set

$$C_\sigma = \{x : x \in X,\ x(i) \in U_{\sigma(i)} \text{ for every } i \in \operatorname{dom}\sigma\} \in \widehat{\bigotimes}_I\mathcal{B}(X).$$

Then $\{C_\sigma : \sigma \in S\}$ is a base for the topology of X^I. If $W \subseteq X^I$ is a regular open set, there is a countable set $R \subseteq S$ such that $W = \bigcup_{\sigma\in R} C_\sigma$. **P** Let R^* be the set of those $\sigma \in S$ such that $C_\sigma \subseteq W$, and R the set of minimal members of R^* (ordering S by extension of functions). Then every member of R^* extends some member of R, so

$$\bigcup_{\sigma\in R} C_\sigma = \bigcup_{\sigma\in R^*} C_\sigma = W.$$

For $n \in \mathbb{N}$ set $R_n = \{\sigma : \sigma \in R,\ \#(\sigma) = n,\ \sigma(i) < n \text{ for every } i \in \operatorname{dom}\sigma\}$.

? Suppose, if possible, that $n \in \mathbb{N}$ and R_n is infinite. Then there is a sequence $\langle\sigma_k\rangle_{k\in\mathbb{N}}$ of distinct elements of R_n; set $J_k = \operatorname{dom}\sigma_k$ for each k. Let $M \subseteq \mathbb{N}$ be an infinite set such that $\langle J_k\rangle_{k\in M}$ is a Δ-system with root J say. Then there is a $\sigma \in n^J$ such that $M' = \{k : k \in M,\ \sigma_k\restriction J = \sigma\}$ is infinite.

In this case, however,

$$C_\sigma \subseteq \operatorname{int}\overline{\bigcup_{k\in M'} C_{\sigma_k}} \subseteq \operatorname{int}\overline{W} = W$$

and $\sigma \in R^*$, so that $\sigma_k \notin R$ for $k \in M'$; which is impossible. **X**

Thus every R_n is countable and $R = \bigcup_{n\in\mathbb{N}} R_n$ is countable. **Q**

(c) This shows that every regular open subset of X^I is a countable union of open cylinder sets and belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. Consequently every cozero set belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. **P** If $f : X^I \to \mathbb{R}$ is continuous, then for each rational $q > 0$ set $W_q = \operatorname{int}\{x : |f(x)| \ge q\}$. Then W_q is a regular open set so belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. But now $\{x : f(x) \ne 0\} = \bigcup_{q\in\mathbb{Q},q>0} W_q$ is the union of countably many sets in $\widehat{\bigotimes}_I\mathcal{B}(X)$ and itself belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. **Q**

So $\widehat{\bigotimes}_I\mathcal{B}(X) \supseteq \mathcal{B}\mathfrak{a}(X^I)$ and the two are equal.

566U Dependent choice If we allow ourselves to use the stronger principle DC rather than AC(ω) alone, we get some useful simplifications. The difficulties with the principle of exhaustion in §215 and 566D above disappear, and there is no longer any obstacle to the construction of product measures in 254F, provided only that we know we have a non-empty product space. So a typical theorem on product measures will now begin 'let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces such that $X = \prod_{i\in I} X_i$ is non-empty'. Later, we now have Baire's theorem (both for complete metric spaces and for locally compact Hausdorff spaces) and Urysohn's Lemma (so we can drop the formulation 'completely regular locally compact topological group'). The most substantial gap in Volume 4 which is now filled seems to be in the abstract theory of gauge integrals in §482. But I cannot point to a result which is essential to the structure of this treatise and can be proved in ZF + DC but not in ZF + AC(ω).

566X Basic exercises (a) [AC(ω)] Let (X, ρ) be a metric space. (i) Show that X is compact iff it is sequentially compact iff it is countably compact iff it is complete and totally bounded. (ii) Show that if X is separable then every subspace of X is separable.

(b) [AC(ω)] Show that there is a surjection from $\mathbb{R}$ onto its Borel σ-algebra, so that there must be a non-Borel subset of $\mathbb{R}$.

(c) Let us say that a Boolean algebra $\mathfrak{A}$ has the **countable sup property** if for every $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ with the same upper bounds as A. (i) Show that a Dedekind σ-complete Boolean algebra with the countable sup property is Dedekind complete. (ii) Show that a countably additive functional on a Boolean algebra with the countable sup property is completely additive.

(d) [AC(ω)] Show that if there is a translation-invariant lifting of Lebesgue measure then there is a subset of $\mathbb{R}$ which is not Lebesgue measurable. (*Hint*: 345F.)

(e) [AC(ω)] Show that if $1 < p < \infty$ and $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, the unit ball of $L^p(\mathfrak{A}, \bar{\mu})$ (§366) is weakly compact. (*Hint*: part (b) of the proof of 566Q.)

(f) [AC(ω)] (i) Let $\mathfrak{A}$ be a measurable algebra. Show that the unit ball of $L^\infty = L^\infty(\mathfrak{A})$ is compact for $\mathfrak{T}_s(L^\infty, (L^\infty)^\times)$ (definition: 3A5Ea). (ii) Let U be an L-space with a weak order unit. Show that the unit ball of U^* is weak*-compact.

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that there is a function $f : \operatorname{Aut}\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ such that if $\pi \in \operatorname{Aut}\mathfrak{A}$ and a is a separator for π then $a \cap f(\pi, a) = 0$ and $f(a) \cup \pi f(\pi, a) \cup \pi^2 f(\pi, a)$ is the support of π. (*Hint*: 382D.)

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a well-orderable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let G^* be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that there is a function $f : G^* \times G \to \mathfrak{A}$ such that $\langle f(\pi, \phi)\rangle_{\phi \in G}$ is a partition of unity for each $\pi \in G^*$ and $\pi a = \phi a$ whenever $\pi \in G^*$, $\phi \in G$ and $a \subseteq f(\pi, \phi)$. (*Hint*: 381I.)

(i) [AC(ω)] Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. Let G^* be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that there is a function $g : G^* \to \mathfrak{A}$ such that $g(\pi)$ is a separator for π for every $\pi \in G^*$. (*Hint*: 566Xh, 382Id.)

(j) [AC(ω)] Let $(X, \mathfrak{T})$ be a completely regular locally compact Hausdorff space, and $f : C_k(X) \to \mathbb{R}$ a positive linear functional. Show that there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$.

(k) [AC(ω)] Say that a set X is **measure-free** if whenever μ is a probability measure with domain $\mathcal{P}X$ there is an $x \in X$ such that $\mu\{x\} > 0$. (i) Show that the following are equiveridical: (α) $\mathbb{R}$ is not measure-free; (β) there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic; (γ) there is a measure μ on $[0,1]$ extending Lebesgue measure and measuring every subset of $[0,1]$. (ii) Prove 438B for point-finite families $\langle E_i\rangle_{i\in I}$ such that the index set I is measure-free.

566Y Further exercises (a) [AC(ω)] Show that if U is an L-space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a bounded sequence in U, then there are a subsequence $\langle v_n\rangle_{n\in\mathbb{N}}$ of $\langle u_n\rangle_{n\in\mathbb{N}}$ and a $w \in U$ such that $\langle \frac{1}{n+1}\sum_{i=0}^n w_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to w for every subsequence $\langle w_n\rangle_{n\in\mathbb{N}}$ of $\langle v_n\rangle_{n\in\mathbb{N}}$. (*Hint*: in the proof of 276H, show that we can find a countably-generated filter to replace the ultrafilter $\mathcal{F}$.)

(b) [AC(ω)] (i) Show that a continuous image of a countably compact topological space is countably compact. (ii) Let X be a completely regular compact Hausdorff topological group and μ a left Haar measure on X. Show that if $w \in L^2(\mu)$ then $u \mapsto u * w : L^2(\mu) \to C(X)$ is a compact linear operator. (*Hint*: 444V.)

(c) [AC(ω)] Let $\langle(X_i, \langle U_{in}\rangle_{n\in\mathbb{N}})\rangle_{i\in I}$ be a family such that X_i is a separable metrizable space and $\langle U_{in}\rangle_{n\in\mathbb{N}}$ is a base for the topology of X_i for each $i \in I$. Show that $\mathcal{B}\mathfrak{a}(\prod_{i\in I} X_i) = \widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$.

(d) [DC] Let U be an inner product space and $K \subseteq U$ a convex weakly compact set. Show that K has an extreme point.

566Z Problem Is it relatively consistent with ZF + AC(ω) to suppose that there is a non-zero atomless rigid measurable algebra?

566 Notes and comments In this section I have taken a lightning tour through the material of Volumes 1 to 4, pausing over a rather odd selection of results, mostly chosen to exhibit the alternative arguments which are available. In the first place, I am trying to suggest something of the quality of the world of measure theory, and of analysis in general, under this particular set of rules. Perhaps I should say that my real objective is the next section, with DC rather than AC(ω), because DC is believed to be compatible with the axiom of determinacy, and ZF + DC + AD is not a poor relation of ZFC, as ZF + AC(ω) sometimes seems to be, but a potential rival.

I have a second reason for taking all this trouble, which is a variation on one of the reasons for 'generalization' as found in twentieth-century pure mathematics. When we 'generalize' an argument, moving (for example) from metric spaces to topological spaces, or from Lebesgue measure to abstract measures, we are usually stimulated by some particular question which demands the new framework. But the process frequently has a lasting value which is quite independent of its motivation. It forces us to re-examine the nature of the proofs we are using, discarding or adapting those steps which depend on the original context, and isolating those which belong in some other class of ideas. In the same way, renouncing the use of AC forces us to look more closely at critical points, and decide which of them correspond to some deeper principle.

Something I have not attempted to do is to look for models in which my favourite theorems are actually false. An interesting class of problems is concerned with 'exact engineering', that is, finding combinatorial propositions which will be equivalent, in ZF, to given results which are not provable in ZF. For instance, Baire's theorem for complete metric spaces is actually equivalent to DC (Blair 77), while Baire's theorem for compact Hausdorff spaces may be weaker (Fossy & Morillon 98). I am not presenting any such results here. However, if we take Maharam's theorem as an example of a central result of measure theory with ZFC which is surely unprovable without a strong form of AC, we can ask just how false it can be; and I offer 566Z as a sample target.

567 Determinacy

So far, this chapter has been looking at set theories which are weaker than the standard theory ZFC, and checking which of the principal results of measure theory can still be proved. I now turn to an axiom which directly contradicts the axiom of choice, and leads to a very different world. This is AD, the 'axiom of determinacy', defined in terms of strategies for infinite games (567A-567C). The first step is to confirm that we automatically have a weak version of countable choice which is enough to make Lebesgue measure well-behaved (567D-567E). Next, in separable metrizable spaces all subsets are universally measurable and have the Baire property (567G). Consequently (at least when we can use $\mathrm{AC}(\omega)$) linear operators between Banach spaces are bounded (567H), additive functionals on σ-complete Boolean algebras are countably additive (567J), and many L-spaces are reflexive (567K). In a different direction, we find that ω_1 is two-valued-measurable (567L) and that there are many surjections from $\mathbb{R}$ onto ordinals (567M).

At the end of the section I include two celebrated results in ZFC (567N, 567O) which depend on some of the same ideas.

567A Infinite games I return to an idea introduced in §451.

(a) Let X be a non-empty set and A a subset of $X^{\mathbb{N}}$. In the corresponding infinite game $\mathrm{Game}(X, A)$, players I and II choose members of X alternately, so that I chooses $x(0), x(2), \ldots$ and II chooses $x(1), x(3), \ldots$; a **play** of the game is an element of $X^{\mathbb{N}}$; player I wins the play x if $x \in A$, otherwise II wins. A **strategy** for I is a function $\sigma : \bigcup_{n\in\mathbb{N}} X^n \to X$; a play $x \in X^{\mathbb{N}}$ is **consistent** with σ if $x(2n) = \sigma(\langle x(2i+1)\rangle_{i<n})$ for every n, that is, if I uses the function σ to decide his move from the previous moves by his opponent; σ is a **winning strategy** if every play consistent with σ belongs to A, that is, if I wins whenever he follows the strategy σ. Similarly, a strategy for II is a function $\tau : \bigcup_{n\geq 1} X^n \to X$; a play x is consistent with τ if $x(2n+1) = \tau(\langle x(2i)\rangle_{i\leq n})$ for every n; and τ is a winning strategy for II if $x \notin A$ whenever $x \in X^{\mathbb{N}}$ and x is consistent with τ.

(b) A set $A \subseteq X^{\mathbb{N}}$ is **determined** if either I or II has a winning strategy in $\mathrm{Game}(X, A)$. Note that we need to know the set X as well as the set A to specify the game in question.

(c) It will sometimes be convenient to describe games with 'rules', so that the players are required to choose points in subsets of X (determined by the moves so far) at each move. Such a description can be regarded as specifying A in the form $(A' \cup G) \setminus H$, where G is the set of plays in which II is the first to break a rule, H is the set of plays in which I is the first to break a rule, and A' is the set of plays in which both obey the rules and I wins.

(d) Not infrequently the 'rules' will specify different sets for the moves of the two players, so that I always chooses a point in X_1 and II always chooses a point in X_2; setting $X = X_1 \cup X_2$ we can reduce this to the formalization above.

567B Theorem Let X be a non-empty well-orderable set. Give X its discrete topology and $X^{\mathbb{N}}$ the product topology. If $F \subseteq X^{\mathbb{N}}$ is closed then $\mathrm{Game}(X, F)$ is determined.

proof (a) Fix a well-ordering $\preccurlyeq$ of X. Define $\langle W_\xi\rangle_{\xi\in\mathrm{On}}$ by setting

$$W_0 = \{w : w \in \textstyle\bigcup_{n\in\mathbb{N}} X^{2n+1},\ w \not\subseteq x \text{ for any } x \in F\},$$

$$W_\xi = \{w : w \in \bigcup_{n\in\mathbb{N}} X^{2n+1}, \text{ there is some } t \in X \text{ such that}$$

$$w^\frown{<}t{>}^\frown{<}u{>} \in \bigcup_{\eta<\xi} W_\eta \text{ for every } u \in X\}$$

if $\xi > 0$. (As in 562A, I write $<t> \in X^1$ for the one-term sequence with value t, and $^\frown$ for concatenation of sequences.) If $w \in W_0$ then of course $w^\frown<t>^\frown<u> \in W_0$ for all t, $u \in X$; so $W_0 \subseteq W_1$, and an easy induction now shows that $W_\xi \subseteq W_\eta$ whenever $\xi \le \eta$ in On. There is therefore an ordinal ζ such that $W_{\zeta+1} = W_\zeta$; write W for W_ζ.

For $w \in W$, let $r(w) \le \zeta$ be the least ordinal such that $w \in W_{r(w)}$. If $r(w) > 0$ then there is some $t \in X$ such that such that $w^\frown<t>^\frown<u> \in \bigcup_{\eta<r(w)} W_\eta$, that is, $r(w^\frown<t>^\frown<u>) < r(w)$, for every $u \in X$.

Let V be the set of those $v \in \bigcup_{n\in\mathbb{N}} X^{2n}$ such that there is a $u \in X$ such that $v^\frown<u> \notin W$. Observe that if $w \in \bigcup_{n\in\mathbb{N}} X^{2n+1} \setminus W$ then $w \notin W_{\zeta+1}$ so $w^\frown<t> \in V$ for every $t \in X$.

(b) Suppose that $\emptyset \in V$. Define $\sigma : \bigcup_{n\in\mathbb{N}} X^n \to X$ inductively by saying that

$\sigma(\emptyset)$ is the $\preccurlyeq$-least member t of X such that the one-element sequence $\emptyset^\frown<t>$ does not belong to W,

if $v \in X^{n+1}$ and $w = (\sigma(v\restriction 0), v(0), \sigma(v\restriction 1), v(1), \dots, \sigma(v\restriction n), v(n)) \in V$, take $\sigma(v)$ to be the $\preccurlyeq$-least member t of X such that $w^\frown<t> \notin W$,

for other $v \in X^{n+1}$ take $\sigma(v)$ to be the $\preccurlyeq$-least member of X.

Then σ is a winning strategy for I. **P** If x is a play consistent with σ, then an induction on n shows that $x\restriction 2n \in V$ and $x\restriction 2n+1 \notin W$ for every n. In particular, $x\restriction 2n+1 \notin W_0$, that is, there is a member of F extending $x\restriction 2n+1$, for every n. As F is closed, $x \in F$ and I wins the play x. **Q**

(c) Suppose that $\emptyset \notin V$, that is, $w \in W$ for every $w \in X^1$. Define $\tau : \bigcup_{n\ge 1} X^n \to X$ inductively by saying

if $v \in X^n$ and $w = (v(0), \tau(v\restriction 1), v(2), \tau(v\restriction 2), \dots, v(n-1))$ belongs to $W \setminus W_0$, then $\tau(v)$ is the $\preccurlyeq$-least $t \in X$ such that $r(w^\frown<t>^\frown<u>) < r(w)$ for every $u \in X$,

for other $v \in X^n$, $\tau(v)$ is the $\preccurlyeq$-least member of X.

Then τ is a winning strategy for II. **P** Let x be a play consistent with τ. Then an induction on n tells us that

$$x\restriction 2n+1 \in W, \quad \text{if } x\restriction 2n+1 \notin W_0 \text{ then } r(x\restriction 2n+3) < r(x\restriction 2n+1)$$

for every $n \in \mathbb{N}$. Since $\langle r(x\restriction 2n+1)\rangle_{n\in\mathbb{N}}$ cannot be strictly decreasing, there is some $n \in \mathbb{N}$ such that $x\restriction 2n+1 \in W_0$ and $x \notin F$. Thus II wins the play x. **Q**

(d) Putting (b) and (c) together we see that F is determined.

567C The axiom of determinacy (a) The standard 'axiom of determinacy' is the statement

(AD) Every subset of $\mathbb{N}^\mathbb{N}$ is determined.

Evidently it will follow that every subset of $X^\mathbb{N}$ is determined for any countable set X. (If $X \subseteq \mathbb{N}$, a game on X can be regarded as a game on $\mathbb{N}$ in which there is a rule that the players must always choose points in X. See also 567Xc.)

(b) At the same time, it will be useful to consider a weak form of the axiom of countable choice: for any set X, write $\mathrm{AC}(X;\omega)$ for the statement

$$\prod_{n\in\mathbb{N}} A_n \ne \emptyset \text{ whenever } \langle A_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of non-empty subsets of } X.$$

567D Theorem (Mycielski 64) AD implies $\mathrm{AC}(\mathbb{R};\omega)$.

proof Since we know that $\mathbb{R}$ is equipollent with $\mathbb{N}^\mathbb{N}$, we can look at $\mathrm{AC}(\mathbb{N}^\mathbb{N};\omega)$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of $\mathbb{N}^\mathbb{N}$. Set

$$A = \{x : x \in \mathbb{N}^\mathbb{N},\ \langle x(2n+1)\rangle_{n\in\mathbb{N}} \notin A_{x(0)}\}.$$

Then I has no winning strategy in $\mathrm{Game}(\mathbb{N}, A)$, because if σ is a strategy for I in $\mathrm{Game}(\mathbb{N}, A)$ set $k = \sigma(\emptyset)$; there is a point $y \in A_k$, and II need only play $x(2n+1) = y(n)$ for each n.

So II has a winning strategy τ say. Define $g : \mathbb{N} \to \mathbb{N}^\mathbb{N}$ by saying that $g(n)(i) = \tau(e_{ni})$ for n, $i \in \mathbb{N}$, where $e_{ni} \in \mathbb{N}^{i+1}$, $e_{ni}(0) = n$, $e_{ni}(j) = 0$ for $1 \le j \le i$. If now $n \in \mathbb{N}$, I plays $(n, 0, 0, \dots)$ and II follows the strategy τ, the resulting play $(n, g(n)(0), 0, g(n)(1), 0, \dots)$ must belong to A so $g(n) \in A_n$.

567E Consequences of AC($\mathbb{R};\omega$) Suppose that $\mathrm{AC}(\mathbb{R};\omega)$ is true.

(a) If a set X is the image of a subset Y of $\mathbb{R}$ under a function f, then $\mathrm{AC}(X;\omega)$ is true. **P** If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of X, then there is an $x \in \prod_{n\in\mathbb{N}} f^{-1}[A_n]$, and $\langle f(x(n))\rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$. **Q**

(b) In particular, taking $S = \bigcup_{n\ge 1} \mathbb{N}^n$ as in §562, $\mathrm{AC}(\mathcal{P}S;\omega)$ is true. It follows that (in any second-countable space X) every sequence of codable Borel sets is codable and the family of codable Borel sets is a σ-algebra, coinciding with the Borel σ-algebra $\mathcal{B}(X)$ on its ordinary definition. Moreover, since $\mathcal{B}(X)$ is an image of $\mathcal{P}S$, we have $\mathrm{AC}(\mathcal{B}(X);\omega)$, countable choice for collections of Borel sets. Similarly, the family of codable Borel functions becomes the ordinary family of Borel-measurable functions, and we have countable choice for sets of Borel real-valued functions on X.

(c) Consequently the results of §562-565 give us large parts of the elementary theory of Borel measures on second-countable spaces. At the same time, if X is second-countable, the union of a sequence of meager subsets of X is meager (because we have countable choice for sequences of nowhere dense closed sets), so the Baire-property algebra of X is a σ-algebra.

(d) We also find that the supremum of a sequence of countable ordinals is again countable. **P** Let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a sequence in ω_1. Using AC($\mathbb{R};\omega$)), we can choose for each $n\in\mathbb{N}$ a subset $\preccurlyeq_n$ of $\mathbb{N}\times\mathbb{N}$ which is a well-ordering of $\mathbb{N}$ with order type $\max(\omega,\xi_n)$. Now we have a well-ordering $\preccurlyeq$ of $\mathbb{N}^2$ defined by saying that $(i,j)\preccurlyeq(i',j')$ if $i<i'$ or $i=i'$ and $j\preccurlyeq_i j'$. In this case, the order type ξ of $\preccurlyeq$ will be greater than every ξ_n, so that $\sup_{n\in\mathbb{N}}\xi_n\le\xi$ is countable. **Q**

567F Lemma (see Mycielski & Swierczkowski 64) [AC($\mathbb{R};\omega$)] Suppose that $A\subseteq\{0,1\}^{\mathbb{N}}$ is a continuous image of a subset B of $\{0,1\}^{\mathbb{N}}$ such that $(h^{-1}[B]\cap F)\cup H\subseteq\mathbb{N}^{\mathbb{N}}$ is determined whenever $h:\mathbb{N}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ is continuous, $F\subseteq\mathbb{N}^{\mathbb{N}}$ is closed and $H\subseteq\mathbb{N}^{\mathbb{N}}$ is open.

(a) A is universally measurable.

(b) A has the Baire property in $\{0,1\}^{\mathbb{N}}$.

proof Fix a continuous surjection $f:B\to A$. Let $\mathcal{E}$ be the countable algebra of subsets of $\{0,1\}^{\mathbb{N}}$ determined by coordinates in finite sets.

(a)(i) Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ and $\hat\mu$ its completion. If $Z\subseteq\{0,1\}^{\mathbb{N}}$ is closed and not negligible, then at least one of $Z\cap A$, $Z\setminus A$ has non-zero inner measure.

P Let $\langle E_n\rangle_{n\in\mathbb{N}}$ enumerate $\mathcal{E}$. Set $\epsilon_n=2^{-2n-2}\mu Z$ for $n\in\mathbb{N}$. In $(\{0,1\}\times\mathcal{E})^{\mathbb{N}}$ consider the game in which the players choose $(k_0,K_0),(k_1,K_1),\dots$ such that $K_0=Z$ and for each $n\in\mathbb{N}$

$$k_n\in\{0,1\},\quad K_n\in\mathcal{E},\quad \mu K_{2n+1}\le\epsilon_n.$$

I wins if $y=\langle k_{2n}\rangle_{n\in\mathbb{N}}$ belongs to B and $f(y)\notin\bigcup_{n\in\mathbb{N}}K_{2n+1}$. Observe that when $y\in B$, $f(y)\in\bigcup_{n\in\mathbb{N}}K_{2n+1}$ iff there are m, $n\in\mathbb{N}$ such that $f(w)\in K_{2n+1}$ whenever $w\in B$ and $w\restriction m=y\restriction m$; so I wins iff $y\in B$ and at every stage $((k_0,K_0),\dots,(k_{2m},K_{2m}))$ there is a $w\in B$ such that $w(i)=k_{2i}$ for $i<m$ and $f(w)\notin\bigcup_{i<m}K_{2i+1}$. So the payoff set D of plays $\langle(k_n,K_n)\rangle_{n\in\mathbb{N}}$ won by I is of the form $(h^{-1}[B]\cap F)\cup H$ where $h:(\{0,1\}\times\mathcal{E})^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ is continuous, $F\subseteq(\{0,1\}\times\mathcal{E})^{\mathbb{N}}$ is closed and $H\subseteq(\{0,1\}\times\mathcal{E})^{\mathbb{N}}$ is open. (Here H is the set of plays which are won because II is the first to break a rule.) Consequently D is determined.

case 1 Suppose that I has a winning strategy σ. For each play $\langle(k_n,K_n)\rangle_{n\in\mathbb{N}}$ consistent with σ, $f(\langle k_{2n}\rangle_{n\in\mathbb{N}})$ is defined and belongs to A. Since the set of plays consistent with σ is a closed subset of $(\{0,1\}\times\mathcal{E})^{\mathbb{N}}$, the set C of points obtainable in this way is an analytic subset of Z, therefore measured by $\hat\mu$ (563I). **?** If $\hat\mu C=0$, then there is an open set $H\supseteq C$ such that $\mu H<\epsilon_0$ (563Fd). In this case, II can play in such a way that

$$K_{2n+1}\subseteq H,\quad \mu(H\setminus\bigcup_{i<n}K_{2i+1})<\epsilon_n,$$

$$\text{if } E_n\subseteq H \text{ then } E_n\subseteq\bigcup_{i\le n}K_{2i+1}$$

for every n. But now, taking I's responses under σ, we have a play of Game($\{0,1\}\times\mathcal{E},D$) in which $\bigcup_{n\in\mathbb{N}}K_{2n+1}=H$ includes C, so contains $f(\langle k_n\rangle_{n\in\mathbb{N}})$, and is won by II; which is supposed to be impossible. **X**

So in this case $\mu_*A\ge\mu C>0$.

case 2 Suppose that II has a winning strategy τ. For each $n\in\mathbb{N}$ and $u\in\{0,1\}^n$, let $L_n(u)$ be the second component of $\tau(\langle(u(i),\emptyset)\rangle_{i<n})$; set $G=\bigcup_{n\in\mathbb{N}}\bigcup_{u\in\{0,1\}^n}L_n(u)$, so that $\mu G\le\sum_{n=0}^{\infty}2^n\epsilon_n<\mu Z$. If we take any $y\in B$, then we have a play $\langle(k_n,K_n)\rangle_{n\in\mathbb{N}}$ of Game($\{0,1\}\times\mathcal{E},D$), consistent with τ, in which $k_{2n}=y(n)$ and $K_{2n}=\emptyset$ for each n. Since II wins this play, $f(y)$ must belong to

$$\bigcup_{n\in\mathbb{N}}K_{2n+1}=\bigcup_{n\in\mathbb{N}}L_n(y\restriction n)\subseteq G.$$

As y is arbitrary, $A\subseteq G$ and $\mu_*(Z\setminus A)\ge\mu(Z\setminus G)>0$. **Q**

(ii) Write $\mathcal{K}$ for the family of compact sets $K\subseteq\{0,1\}^{\mathbb{N}}$ such that $A\cap K$ is Borel. If $E\subseteq\{0,1\}^{\mathbb{N}}$ and $\mu E>0$, there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. **P** There is a closed $Z\subseteq E$ such that $\mu Z>0$ (563Fd again). By (i), at least one of $\mu_*(Z\cap A)$, $\mu_*(Z\setminus A)$ is non-zero, and there is a compact set K of non-zero measure which is included in one of $Z\cap A$, $Z\setminus A$. But now $K\in\mathcal{K}$. **Q**

Now (because we have countable choice for subsets of $\mathcal{K}$) there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\sup_{n\in\mathbb{N}}\mu K_n=\sup_{K\in\mathcal{K}}\mu K$; setting $E=\{0,1\}^{\mathbb{N}}\setminus\bigcup_{n\in\mathbb{N}}K_n$, E must be negligible, while $A\setminus E$ is a Borel set; so A is measured by $\hat\mu$. As μ is arbitrary, A is universally measurable.

(b)(i) If $V\in\mathcal{E}\setminus\{\emptyset\}$ then either $V\cap A$ is meager or there is a $V'\in\mathcal{E}\setminus\{\emptyset\}$ such that $V'\subseteq V$ and $V'\setminus A$ is meager. **P** Set $\mathcal{U}=\{E:E\in\mathcal{E}\setminus\{\emptyset\},\ E\subseteq V\}$ and let $\preccurlyeq$ be a well-ordering of $\mathcal{U}$ (in order type ω, if you like). Consider the game on $\{0,1\}\times\mathcal{U}$ in which the players choose $(k_0,U_0),(k_1,U_1),\dots$ such that, for each $n\in\mathbb{N}$,

$$k_n \in \{0,1\}, \quad U_n \in \mathcal{U}, \quad U_{n+1} \subseteq U_n.$$

I wins if $y = \langle k_{2n}\rangle_{n\in\mathbb{N}}$ belongs to B and $f(y) \in \bigcap_{n\in\mathbb{N}} U_n$. Because the U_n are all closed, this game is determined for the same reasons as the game of (a).

case 1 Suppose that I has a winning strategy σ; say that $\sigma_2(w) \in \mathcal{U}$ is the second component of $\sigma(w)$ for each $w \in \bigcup_{n\in\mathbb{N}}(\{0,1\} \times \mathcal{U})^n$. For each $n \in \mathbb{N}$ let $\mathcal{U}_n$ be the set of those $U \in \mathcal{U}$ such that $v{\restriction}n = v'{\restriction}n$ for all v, $v' \in U$. Let $(k', V') = \sigma(\emptyset)$ be I's first move when following σ. Let Q be the set of positions in the game consistent with σ and with II to move, that is, finite sequences

$$q = \langle (k_i, U_i)\rangle_{i\le 2n} \in (\{0,1\} \times \mathcal{U})^{2n+1}$$

such that $(k_{2m}, U_{2m}) = \sigma(\langle (k_{2i+1}, U_{2i+1})\rangle_{i<m})$ for every $m \le n$ and $\langle U_i\rangle_{i\le 2n}$ is non-increasing. For such a q, set $V_q = U_{2n}$ and

$$W_q = \bigcup\{\sigma_2(\langle (k_{2i+1}, U_{2i+1})\rangle_{i<n}{}^\frown{<}(k,U){>}) : k \in \{0,1\},\ U \in \mathcal{U}_n,\ U \subseteq V_q\}.$$

Then W_q is an open subset of V_q; but also it is dense in V_q, because if $W \subseteq V_q$ is open and not empty there is a $U \in \mathcal{U}_n$ included in W and $\sigma_2(\langle (k_{2i+1}, U_{2i+1})\rangle_{i<n}{}^\frown{<}(k,U){>})$ is a non-empty subset of U. Q is countable, so $E = \bigcap_{q\in Q} W_q \cup (\{0,1\}^{\mathbb{N}} \setminus \overline{V}_q)$ is comeager in $\{0,1\}^{\mathbb{N}}$.

? If $V' \setminus A$ is not meager, there is an $x \in E \cap V' \setminus A$. Define $\langle (k_n, U_n)\rangle_{n\in\mathbb{N}}$ inductively, as follows. $(k_0, U_0) = (k', V')$. Given that $q = \langle (k_i, U_i)\rangle_{i\le 2n}$ belongs to Q and $x \in V_q$, then $x \in W_q$ so there are $k \in \{0,1\}$, $U \in \mathcal{U}_n$ such that $x \in \sigma_2(\langle (k_{2i+1}, U_{2i+1})\rangle_{i<n}{}^\frown{<}(k,U){>})$; take the lexicographically first such pair (k, U) for (k_{2n+1}, U_{2n+1}), and set $(k_{2n+2}, U_{2n+2}) = \sigma(\langle (k_{2i+1}, U_{2i+1})\rangle_{i\le n}$. Then $q' = \langle (k_i, U_i)\rangle_{i\le 2n+2}$ belongs to Q and $V_{q'} = U_{2n+2} = \sigma_2(\langle (k_{2i+1}, U_{2i+1})\rangle_{i\le n})$ contains x, so the induction can continue.

At the end of this induction, $\langle (k_n, U_n)\rangle_{n\in\mathbb{N}}$ will be a play of the game consistent with σ in which the only point of $\bigcap_{n\in\mathbb{N}} U_n$ is x and does not belong to A. So either $y = \langle k_{2n}\rangle_{n\in\mathbb{N}}$ does not belong to B or $f(y) \notin \bigcap_{n\in\mathbb{N}} U_n$; in either case, II wins the play; which is supposed to be impossible. **X**

So in this case $V' \setminus A$ is meager.

case 2 Suppose that II has a winning strategy τ; say that $\tau_2(w)$ is the second component of $\tau(w)$ for each $w \in \bigcup_{n\ge 1}(\{0,1\} \times \mathcal{U})^n$. Let Q be the set of objects

$$q = (\langle (k_i, U_i)\rangle_{i<2n}, k)$$

such that $\langle (k_i, U_i)\rangle_{i<2n}$ is a finite sequence in $\{0,1\} \times \mathcal{U}$ consistent with τ (allowing the empty string when $n = 0$) and $k \in \{0,1\}$. For such a q, set $V_q = U_{2n-1}$ (if $n > 0$) or $V_q = V$ (if $n = 0$); set

$$W_q = \bigcup\{\tau_2(\langle (k_{2i}, U_{2i})\rangle_{i<n}{}^\frown{<}(k,U){>}) : U \in \mathcal{U},\ U \subseteq V_q\},$$

so that W_q is a dense subset of V_q. Q is countable, so $E = \bigcap_{q\in Q} W_q \cup (\{0,1\}^{\mathbb{N}} \setminus \overline{V}_q)$ is comeager.

? If there is an x in $A \cap V \cap E$, let $y \in B$ be such that $f(y) = x$, and define $\langle (k_n, U_n)\rangle_{n\in\mathbb{N}}$ as follows. Given that $q = (\langle (k_i, U_i)\rangle_{i<2n}, y(n))$ belongs to Q and $x \in V_q$, then $x \in W_q$ so there is a $U \in \mathcal{U}$ such that $x \in \tau_2(\langle (k_{2i}, U_{2i})\rangle_{i<n}{}^\frown {<}(y(n), U){>})$; take the $\preccurlyeq$-first such U for U_{2n}, set $k_{2n} = y(n)$ and $(k_{2n+1}, U_{2n+1}) = \tau(\langle (k_{2i}, U_{2i})\rangle_{i\le n})$, so that $q' = (\langle (k_i, U_i)\rangle_{i\le 2n+1}, y(n+1)) \in Q$ and $V_{q'} = U_{2n+1} = \tau_2(\langle (k_{2i}, U_{2i})\rangle_{i\le n})$ contains x.

At the end of this induction, $\langle (k_n, U_n)\rangle_{n\in\mathbb{N}}$ will be a play of the game consistent with τ in which $f(\langle k_{2n}\rangle_{n\in\mathbb{N}}) = x \in \bigcap_{n\in\mathbb{N}} U_n$, so that I wins, which is supposed to be impossible. **X**

Thus in this case $A \cap V$ must be meager. **Q**

(ii) Now let G be the union of those $V \in \mathcal{E}$ such that $V \setminus A$ is meager; then $G \setminus A$ is meager. (This is where we need $\mathrm{AC}(\mathbb{R};\omega)$.) If $V \in \mathcal{E}$ and $V \subseteq \{0,1\}^{\mathbb{N}} \setminus G$, then $V' \setminus A$ is non-meager for every whenever $V \in \mathcal{E} \setminus \{\emptyset\}$ and $V' \subseteq V$, so $V \cap A$ is meager; accordingly $G' \cap A$ is meager, where $G' = \{0,1\}^{\mathbb{N}} \setminus \overline{G}$. But this means that $G \triangle A \subseteq (G \setminus A) \cup (G' \cap A) \cup (\overline{G} \setminus G)$ is meager and A has the Baire property.

567G Theorem [AD] In any Hausdorff second-countable space, every subset is universally measurable and has the Baire property.

proof Let X be a Hausdorff second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $A \subseteq X$.

(a) Define $g : X \to \{0,1\}^{\mathbb{N}}$ by setting $g(x) = \langle \chi U_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in X$; then g is injective and Borel measurable. If μ is a Borel probability measure on X, we have a Borel probability measure $\nu = \mu g^{-1}{\restriction}\mathcal{B}(\{0,1\}^{\mathbb{N}})$ on $\{0,1\}^{\mathbb{N}}$. By 567Fa, $g[A]$ is measured by the completion $\hat\nu$ of ν; let F, $H \subseteq \{0,1\}^{\mathbb{N}}$ be Borel sets such that $\nu H = 0$ and $g[A] \triangle F \subseteq H$; then $A \triangle g^{-1}[F] \subseteq g^{-1}[H]$ is μ-negligible, so A is measured by $\hat\mu$. As μ is arbitrary, A is universally measurable.

(b) Set $G=\bigcup\{U_n : n\in\mathbb{N},\, U_n\cap A \text{ has the Baire property}\}$, so that $G\cap A$ has the Baire property. (Remember that as we have a bijection between X and a subset of $\mathbb{R}$, we have countable choice for subsets of X, so that the ideal of meager subsets of X is a σ-ideal and the Baire-property algebra is a σ-algebra.) Set $V=X\setminus(\bigcup_{n\in\mathbb{N}}\partial U_n\cup\overline{G})$; then $G\cup V$ is comeager in X, and $A\setminus V$ has the Baire property. If V is empty, we can stop. Otherwise, let $\mathcal{V}$ be the countable algebra of subsets of V generated by $\{V\cap U_n : n\in\mathbb{N}\}$. Since $A\cap U$ does not have the Baire property (in X) for any non-empty relatively open subset U of V, V has no isolated points and $\mathcal{V}$ is atomless. So there is a Boolean-independent sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ generating $\mathcal{V}$ (see 316M[8]). Define $h:V\to\{0,1\}^{\mathbb{N}}$ by setting $h(x)=\langle\chi V_n(x)\rangle_{n\in\mathbb{N}}$ for $x\in V$. Then $h[V]$ is dense in $\{0,1\}^{\mathbb{N}}$ and $h^{-1}[H]$ is dense in V for every dense open set $H\subseteq\{0,1\}^{\mathbb{N}}$; consequently $h^{-1}[M]$ is meager in V and in X whenever $M\subseteq\{0,1\}^{\mathbb{N}}$ is either nowhere dense or meager. By 567Fb, $h[A]$ has the Baire property in $\{0,1\}^{\mathbb{N}}$; express it as $H\triangle M$ where H is open and M is meager; then $A\cap V=h^{-1}[h[A]]=h^{-1}[H]\triangle h^{-1}[M]$ has the Baire property in X, so A has the Baire property in X, as required.

567H Theorem (a) [AD] Let X be a Polish group and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(b) [AD+AC(ω)] Let X be an abelian topological group which is complete under a metric defining its topology, and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(c) [AD+AC(ω)] Let X be a complete metrizable linear topological space, Y a linear topological space and $T:X\to Y$ a linear operator. Then T is continuous. In particular, every linear operator between Banach spaces is a bounded operator.

proof (a)(i) Let $f:X\to Y$ be a homomorphism, and V a neighbourhood of the identity in Y. Let W be an open neighbourhood of the identity in Y such that $W^{-1}W\subseteq V$. Then there is countable family $\mathcal{H}$ of left translates of W which covers Y. **P** If Y is separable, let D be a countable dense subset of Y, and set $\mathcal{H}=\{yW : y\in D\}$. If Y is Lindelöf, we have only to note that $\{yW : y\in Y\}$ is an open cover of Y, so has a countable subcover. **Q**

(ii) Since X is a Baire space (561Ea), and the ideal of meager subsets of X is a σ-ideal (see part (b) of the proof of 567G), and $\{f^{-1}[H] : H\in\mathcal{H}\}$ is a countable cover of X, there is an $H\in\mathcal{H}$ such that $E=f^{-1}[H]$ is non-meager. Now $E^{-1}E$ is a neighbourhood of the identity in X. **P** By 567Gb, E has the Baire property; let G be a non-empty open set in X such that $G\setminus E$ is meager. Set $U=\{x : Gx\cap G\neq\emptyset\}$; then U is a neighbourhood of the identity in X. If $x\in U$, then

$$Gx\cap G\subseteq(Ex\cap E)\cup(Gx\setminus Ex)\cup(G\setminus E)=(Ex\cap E)\cup(G\setminus E)x\cup(G\setminus E).$$

Since $Gx\cap G$ is non-meager, while $G\setminus E$ and $(G\setminus E)x$ are meager, $Ex\cap E\neq\emptyset$ and $x\in E^{-1}E$. Thus $E^{-1}E\supseteq U$ is a neighbourhood of the identity. **Q**

(iii) Let $y\in Y$ be such that $H=yW$. If $x,\,z\in E$, $y^{-1}f(x)$ and $y^{-1}f(z)$ both belong to W, so

$$f(x^{-1}z)=f(x)^{-1}f(z)\in W^{-1}yy^{-1}W=W^{-1}W\subseteq V.$$

Thus $f^{-1}[V]\supseteq E^{-1}E$ is a neighbourhood of the identity in X. As V is arbitrary, f is continuous at the identity, therefore continuous.

(b) ? Otherwise, there is a neighbourhood V of the identity e_Y of Y such that $f^{-1}[V]$ is not a neighbourhood of the identity e_X of X. Let ρ be a metric on X, defining its topology, under which X is complete. Then for each $n\in\mathbb{N}$ we can choose an $x_n\in X$ such that $\rho(x_n,e_X)\le 2^{-n}$ and $f(x_n)\notin V$. (This is where we need AC(ω).) For finite $J\subseteq\mathbb{N}$ set $u_J=\prod_{n\in J}x_n$, starting from $u_\emptyset=e_X$. We can define an infinite $I\subseteq\mathbb{N}$ inductively by saying that

$$I=\{n : \text{whenever } n\in I \text{ and } J\subseteq I\cap n \text{ then } \rho(u_J,u_Jx_n)\le 2^{-\#(I\cap n)}\}.$$

This will ensure that $v_K=\lim_{n\to\infty}u_{K\cap n}$ is defined for every $K\subseteq I$. Note that $v_{K\cup\{m\}}=v_Kx_m$ whenever $m\in I$ and $K\subseteq I\setminus\{m\}$ (this is where we need to know that X is abelian).

Give $\mathcal{P}I$ its usual topology. Let W be a neighbourhood of e_Y such that $W^{-1}W\subseteq V$. By the argument of (a) above, applied to the map $K\mapsto f(v_K):\mathcal{P}I\to Y$, there is a $y\in Y$ such that $E=\{K : K\subseteq I,\, f(v_K)\in yW\}$ is non-meager in $\mathcal{P}I$. Looking at the topological group $(\mathcal{P}I,\triangle)$, we see that there is a neighbourhood U of $\emptyset$ in $\mathcal{P}I$ included in $\{K\triangle L : K,\,L\in E\}$. Taking any sufficiently large $n\in I$, we have $\{n\}\in U$, so there must be a $K\in E$ such that $n\notin K$ and $K\cup\{n\}\in E$. In this case $f(v_K)\in yW$, $f(v_{K\cup\{n\}})\in yW$ and

$$f(x_n)=f(v_K^{-1}v_{K\cup\{n\}})=f(v_K)^{-1}f(v_{K\cup\{n\}})\in W^{-1}W\subseteq V,$$

which is impossible. **X**

[8] Formerly 393F.

(c) ? Otherwise, there is a neighbourhood V of 0 in Y such that $T^{-1}[V]$ is not a neighbourhood of 0 in X; we can suppose that $\alpha y \in V$ whenever $y \in V$ and $|\alpha| \le 1$. Let ρ be a metric on X, defining its topology, under which X is complete. Let W be a neighbourhood of 0 in Y such that $W - W \in V$. Then for each $n \in \mathbb{N}$ we can choose an $x_n \in X \setminus nT^{-1}[V]$ such that $\rho(x_n, 0) \le 2^{-n}$. Define $I \in [\mathbb{N}]^\omega$ and $\langle v_K \rangle_{K \subseteq I}$ as in (b), but using additive notation rather than multiplicative. This time we are not supposing that Y is separable. However, there must be an $m \in \mathbb{N}$ such that $E = \{K : T(v_K) \in mW\}$ is non-meager. As before, we can find $n \in I \setminus m$ and $K \in E$ such that $n \notin L$ and $K \cup \{n\} \in E$. So the calculation gives

$$Tx_n = Tv_{K\cup\{n\}} - Tv_K \in mW - mW \subseteq mV \subseteq nV,$$

again contrary to the choice of x_n. **X**

567I Proposition [AC($\mathbb{R}$; ω)] Let $\widehat{\mathcal{B}}$ be the Baire-property algebra of $\mathcal{P}\mathbb{N}$. Then every $\widehat{\mathcal{B}}$-measurable real-valued additive functional on $\mathcal{P}\mathbb{N}$ is of the form $a \mapsto \sum_{n\in a} \gamma_n$ for some $\langle \gamma_n \rangle_{n\in\mathbb{N}} \in \ell^1$.

proof As noted in 567Ec, $\widehat{\mathcal{B}}$ is a σ-algebra of subsets of $\mathcal{P}\mathbb{N}$.

(a)(i) If $G \subseteq \mathcal{P}\mathbb{N}$ is a dense open set and $m \in \mathbb{N}$, there are an $m' > m$ and an $L \subseteq m' \setminus m$ such that $\{a : a \subseteq \mathbb{N}, a \cap m' \setminus m = L\} \subseteq G$. **P** The set $H = \{b : b \subseteq \mathbb{N} \setminus m, I \cup b \in G$ for every $I \subseteq m\}$ is a dense open subset of $\mathcal{P}(\mathbb{N} \setminus m)$, so there are an $m' > m$ and an $L \subseteq m' \setminus m$ such that $H \supseteq \{b : b \subseteq \mathbb{N} \setminus m, b \cap m' \setminus m = L\}$; this pair m', L works. **Q**

(ii) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, there are a strictly increasing sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ and sets $L_n \subseteq m_{n+1} \setminus m_n$, for $n \in \mathbb{N}$, such that

$$G \supseteq \{a : a \subseteq \mathbb{N},\ a \cap m_{n+1} \setminus m_n = L_n \text{ for infinitely many } n\}.$$

P Let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of dense open sets such that $G \supseteq \bigcap_{n\in\mathbb{N}} G_n$, and choose $\langle m_n \rangle_{n\in\mathbb{N}}$, $\langle L_n \rangle_{n\in\mathbb{N}}$ inductively such that $m_n < m_{n+1}$, $L_n \subseteq m_{n+1} \setminus m_n$ and $\{a : a \subseteq \mathbb{N}, a \cap m_{n+1} \setminus m_n = L_n\} \subseteq G_n$ for every n. **Q**

(iii) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, and $a \subseteq \mathbb{N}$, then there are b_0, b'_0, b_1, $b'_1 \in G$ such that

$$b_0 \subseteq b'_0, \quad b_1 \subseteq b'_1, \quad (b'_0 \setminus b_0) \cap (b'_1 \setminus b_1) = \emptyset, \quad (b'_0 \setminus b_0) \cup (b'_1 \setminus b_1) = a.$$

P Let $\langle m_n \rangle_{n\in\mathbb{N}}$ and $\langle L_n \rangle_{n\in\mathbb{N}}$ be as in (ii). Set

$$b_0 = \bigcup_{n\in\mathbb{N}} L_{2n}, \quad b'_0 = b_0 \cup (a \cap m_0) \cup \bigcup_{n\in\mathbb{N}} a \cap m_{2n+2} \setminus m_{2n+1},$$

$$b_1 = \bigcup_{n\in\mathbb{N}} L_{2n+1}, \quad b'_1 = b_1 \cup \bigcup_{n\in\mathbb{N}} a \cap m_{2n+1} \setminus m_{2n}. \text{ **Q**}$$

So if $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive, $\sup_{a\subseteq\mathbb{N}} |\nu a| \le 4 \sup_{b\in G} |\nu b|$.

(iv) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in G. **P** Take $\langle m_n \rangle_{n\in\mathbb{N}}$ and $\langle L_n \rangle_{n\in\mathbb{N}}$ as in (ii), and set $a_n = \bigcup_{i\in\mathbb{N}} L_{2^n(2i+1)}$ for each n. **Q**

(b) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable, it is bounded. **P** Let $M \in \mathbb{N}$ be such that $E = \{a : |\nu a| \le M\}$ is non-meager. Then there are an $m \in \mathbb{N}$ and $J \subseteq m$ such that $V_{mJ} \setminus E$ is meager, where $V_{mJ} = \{a : a \cap m = J\}$. For $K \subseteq m$, $a \subseteq \mathbb{N}$ set $\phi_K(a) = a \triangle K$; then ϕ_K is an autohomeomorphism of $\mathcal{P}\mathbb{N}$, so $\phi_K[V_{mJ} \setminus E]$ is meager. Let G be the comeager set $\mathcal{P}\mathbb{N} \setminus \bigcup_{K\subseteq m} \phi_K[V_{mJ} \setminus E]$. Set $\delta = \sum_{i<m} |\nu\{i\}|$; then $|\nu\phi_K(a) - \nu a| \le \delta$ whenever $K \subseteq m$ and $a \subseteq \mathbb{N}$. If $b \in G$, set $K = (b \cap m) \triangle J$; then $\phi_K(b) \in V_{mJ} \setminus (V_{mJ} \setminus E) \subseteq E$, so $|\nu b| \le M + \delta$. So (a-iii) tells us that $|\nu a| \le 4(M + \delta)$ for every $a \subseteq \mathbb{N}$, and ν is bounded. **Q**

(c) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $E = \{a : \nu a \ge \epsilon\}$ is meager for every $\epsilon > 0$. **P?** Otherwise, let $m \in \mathbb{N}$ and $J \subseteq m$ be such that $V_{mJ} \setminus E$ is meager. Let G be the comeager set $\mathcal{P}\mathbb{N} \setminus \bigcup_{K\subseteq m} \phi_K[V_{mJ} \setminus E]$, as in (b). This time, $\nu a = \nu\phi_K(a)$ whenever $K \subseteq m$ and $a \subseteq \mathbb{N}$, so $\nu a \ge \epsilon$ for every $a \in G$. But (a-iv) tells us that there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in G, and now $\sup_{n\in\mathbb{N}} \nu(\bigcup_{i\le n} a_i) = \infty$, contradicting (b). **XQ**

(d) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $\nu = 0$. **P** By (c), applied to ν and $-\nu$, $G = \{a : \nu a = 0\}$ is comeager. By (a-iii), ν must be identically zero. **Q**

(e) Now suppose that ν is any additive $\widehat{\mathcal{B}}$-measurable functional. Set $\gamma_n = \nu\{n\}$ for each n. By (b), $\langle \gamma_n \rangle_{n\in\mathbb{N}} \in \ell^1$. Setting $\nu' a = \nu a - \sum_{n\in a} \gamma_n$ for $a \subseteq \mathbb{N}$, ν' is still additive and $\widehat{\mathcal{B}}$-measurable, and $\nu'\{n\} = 0$ for every n, so (d) tells us that $\nu' = 0$ and $\nu a = \sum_{n\in a} \gamma_n$ for every a, as required.

567J Proposition [AD] A finitely additive functional on a Dedekind σ-complete Boolean algebra is countably additive.

proof Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, ν a finitely additive functional on $\mathfrak{A}$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ a disjoint sequence in $\mathfrak{A}$ with supremum a. Set $\lambda c=\nu(\sup_{n\in c}a_n)$ for $c\subseteq\mathbb{N}$. Then λ is an additive functional on $\mathcal{P}\mathbb{N}$. By 567G, it is $\widehat{\mathcal{B}}(\mathcal{P}\mathbb{N})$-measurable; by 567I,

$$\nu a=\lambda\mathbb{N}=\sum_{n=0}^\infty\lambda\{n\}=\sum_{n=0}^\infty\nu a_n.$$

567K Theorem [AD+AC(ω)] If U is an L-space with a weak order unit, it is reflexive.

proof By 561Hb, U is isomorphic to $L^1(\mathfrak{A},\bar\mu)$ for some totally finite measure algebra $(\mathfrak{A},\bar\mu)$; now U^* can be identified with $L^\infty(\mathfrak{A})$. Next, $L^\infty(\mathfrak{A})^*$ can be identified with the space of bounded finitely additive functionals on $\mathfrak{A}$, as in 363K; by 567J, these are all countably additive. Of course $\mathfrak{A}$ is ccc (566Mb), so they are completely additive and correspond to members of L^1, as in 365Ea. So the canonical embedding of U in U^{**} is surjective.

567L Theorem (R.M.Solovay) [AD] ω_1 is two-valued-measurable.

Remark The definition in 541M speaks of 'regular uncountable cardinals'. In the present context I will use the formulation 'an initial ordinal κ is two-valued-measurable if there is a proper κ-additive 2-saturated ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons', where here 'κ-additive' means that $\bigcup_{\eta<\xi}J_\eta\in\mathcal{I}$ whenever $\xi<\kappa$ and $\langle J_\eta\rangle_{\eta<\xi}$ is a family in $\mathcal{I}$.

proof (a) Let $\mathrm{Str_I}$ be the set of strategies for player I in games of the form $\Gamma(\mathbb{N},.)$, that is, $\mathrm{Str_I}$ is the set of functions from $\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ to $\mathbb{N}$; for $\sigma\in\mathrm{Str_I}$ and $x\in\mathbb{N}^\mathbb{N}$, let $\sigma*x\in\mathbb{N}^\mathbb{N}$ be the play in which I follows the strategy σ and II plays the sequence x, that is,

$$(\sigma*x)(2n)=\sigma(x{\restriction}n),\quad(\sigma*x)(2n+1)=x(n)$$

for $n\in\mathbb{N}$. Similarly, let $\mathrm{Str_{II}}$ be the set of functions from $\bigcup_{n\ge1}\mathbb{N}^n$ to $\mathbb{N}$ and for $\tau\in\mathrm{Str_{II}}$, $x\in\mathbb{N}^\mathbb{N}$, $n\in\mathbb{N}$ set

$$(\tau*x)(2n)=x(n),\quad(\tau*x)(2n+1)=\tau(x{\restriction}(n+1)).$$

We can find bijections $g:\mathbb{N}^\mathbb{N}\to\mathrm{Str_I}\cup\mathrm{Str_{II}}$ and $h:\mathbb{N}^\mathbb{N}\to\mathrm{WO}(\mathbb{N})$, where $\mathrm{WO}(\mathbb{N})\subseteq\mathcal{P}(\mathbb{N}^2)$ is the set of well-orderings of $\mathbb{N}$. **P** Since $S^*=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ and $S=\bigcup_{n\ge1}\mathbb{N}^n$ are countably infinite, $\mathrm{Str_I}=\mathbb{N}^{S^*}$ and $\mathrm{Str_{II}}=\mathbb{N}^S$ are equipollent with $\mathbb{N}^\mathbb{N}$. As $\mathcal{P}(\mathbb{N}^2)\sim\mathcal{P}\mathbb{N}\sim\mathbb{N}^\mathbb{N}$, there is an injection from $\mathrm{WO}(\mathbb{N})$ to $\mathbb{N}^\mathbb{N}$. In the reverse direction, there are an injection from $\mathbb{N}^\mathbb{N}$ to the set F of bijections from $\mathbb{N}$ to itself, and an injection from F to $\mathrm{WO}(\mathbb{N})$; so the Schroeder-Bernstein theorem tells us that $\mathrm{WO}(\mathbb{N})\sim\mathbb{N}^\mathbb{N}$. **Q**

Define $f:\mathrm{WO}(\mathbb{N})\to\omega_1$ by saying that $f(\preccurlyeq)=\mathrm{otp}(\mathbb{N},\preccurlyeq)$ for $\preccurlyeq\in\mathrm{WO}(\mathbb{N})$.

(b) For $x\in\mathbb{N}^\mathbb{N}$ let $L_x\subseteq\mathbb{N}^\mathbb{N}$ be the smallest set such that

$x\in L_x$,

whenever y, $z\in L_x$ then $g(y)*z\in L_x$,

whenever $y\in L_x$ then $\langle y(2n)\rangle_{n\in\mathbb{N}}$ and $\langle y(2^k(2n+1))\rangle_{n\in\mathbb{N}}$ belong to L_x for every $k\in\mathbb{N}$.

Observe that L_x is countable and that $L_y\subseteq L_x$ whenever $y\in L_x$. For $x\in\mathbb{N}^\mathbb{N}$, set $C_x=\{y:y\in\mathbb{N}^\mathbb{N},\,x\in L_y\}$.

(c) For any sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ there is an $x\in\mathbb{N}^\mathbb{N}$ such that $C_x\subseteq\bigcap_{n\in\mathbb{N}}C_{x_n}$. **P** Set $x(0)=0$ and $x(2^k(2n{+}1))=x_k(n)$ for k, $n\in\mathbb{N}$. Then $x_k\in L_x$ for every k. So if $y\in C_x$ and $n\in\mathbb{N}$, we have $x_n\in L_x\subseteq L_y$ and $y\in C_{x_n}$. **Q**

Let $\mathcal{F}$ be the filter on $\mathbb{N}^\mathbb{N}$ generated by $\{C_x:x\in\mathbb{N}^\mathbb{N}\}$; then (because AC($\mathbb{R};\omega$) is true) $\mathcal{F}$ is closed under countable intersections.

(d) Suppose that $A\subseteq\mathbb{N}^\mathbb{N}$ is such that whenever $x\in A$ and $L_y=L_x$ then $y\in A$.

(i) If I has a winning strategy in Game($\mathbb{N},A$) then $A\in\mathcal{F}$. **P** Let $\sigma\in\mathrm{Str_I}$ be a winning strategy for I, and consider $x=g^{-1}(\sigma)\in\mathbb{N}^\mathbb{N}$. Suppose that $y\in\mathbb{N}^\mathbb{N}$ and $x\in L_y$, and consider $z=\sigma*y\in A$. As $z=g(x)*y$ belongs to L_y, $L_z\subseteq L_y$; on the other hand, $y(n)=z(2n+1)$ for every n, so $y\in L_z$ and $L_y\subseteq L_z$. So $L_y=L_z$ and $y\in A$. As y is arbitrary, $C_x\subseteq A$ and $A\in\mathcal{F}$. **Q**

(ii) If II has a winning strategy in Game($\mathbb{N},A$) then $\mathbb{N}^\mathbb{N}\setminus A\in\mathcal{F}$. **P** Let $\tau\in\mathrm{Str_{II}}$ be a winning strategy for II, and consider $x=g^{-1}(\tau)\in\mathbb{N}^\mathbb{N}$. Suppose that $y\in\mathbb{N}^\mathbb{N}$ and $x\in L_y$, and consider $z=\tau*y\in\mathbb{N}^\mathbb{N}\setminus A$. As before, $L_z\subseteq L_y$; this time, $y(n)=z(2n)$ for every n so $y\in L_z$ and $L_y\subseteq L_z$. So $y\notin A$. As y is arbitrary, $C_x\subseteq\mathbb{N}^\mathbb{N}\setminus A$ and $\mathbb{N}^\mathbb{N}\setminus A\in\mathcal{F}$. **Q**

(e) For $x\in\mathbb{N}^\mathbb{N}$ set $\phi(x)=\sup_{y\in L_x}f(h(y))$; because L_x is countable, $\phi(x)<\omega_1$ (567Ed). Let $\mathcal{G}$ be the image filter $\phi[[\mathcal{F}]]$. Because $\mathcal{F}$ is closed under countable intersections, so is $\mathcal{G}$. If $B\subseteq\omega_1$ then $\phi^{-1}[B]$ satisfies the condition of (d), so that one of $\phi^{-1}[B]$, $\mathbb{N}^\mathbb{N}\setminus\phi^{-1}[B]$ belongs to $\mathcal{F}$ and one of B, $\omega_1\setminus B$ belongs to $\mathcal{G}$; as B is arbitrary, $\mathcal{G}$ is an ultrafilter.

(f) Finally, $\mathcal{G}$ does not contain any singletons. **P** If $\xi<\omega_1$, there is an $x\in\mathbb{N}^\mathbb{N}$ such that $f(h(x))=\xi+1$. Now $C_x\in\mathcal{F}$ so $\phi[C_x]\in\mathcal{G}$. If $y\in C_x$ then $x\in L_y$ so $\xi+1\le\phi(y)$; accordingly $\xi\notin\phi[C_x]$ and $\{\xi\}\notin\mathcal{G}$. **Q** So $\mathcal{G}$ (or, if you like, the ideal $\{\omega_1\setminus B:B\in\mathcal{G}\}$) witnesses that ω_1 is two-valued-measurable.

567M Theorem (Moschovakis 70) [AD] Let α be an ordinal such that there is a surjection from $\mathcal{P}\mathbb{N}$ onto α. Then there is a surjection from $\mathcal{P}\mathbb{N}$ onto $\mathcal{P}\alpha$.

proof The formulae will run slightly more smoothly if we work with surjections from $\mathbb{N}^{\mathbb{N}}$ rather than from $\mathcal{P}\mathbb{N}$; of course this makes no difference to the result.

(a) We may suppose that α is uncountable. Let $f:\mathbb{N}^{\mathbb{N}}\to\alpha$ be a surjection. I seek to define inductively a family $\langle g_\xi\rangle_{\xi\le\alpha}$ such that g_ξ is a surjection from $\mathbb{N}^{\mathbb{N}}$ onto $\mathcal{P}\xi$ for every $\xi\le\alpha$. As in the proof of 567L, let $\mathrm{Str_I}$ be the set of functions from $\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ to $\mathbb{N}$, and $\mathrm{Str_{II}}$ the set of functions from $\bigcup_{n\ge 1}\mathbb{N}^n$ to $\mathbb{N}$; fix a surjection $h:\mathbb{N}^{\mathbb{N}}\to\mathrm{Str_I}\cup\mathrm{Str_{II}}$. For $\sigma\in\mathrm{Str_I}$, $\tau\in\mathrm{Str_{II}}$ and $x\in\mathbb{N}^{\mathbb{N}}$ let $\sigma*x$, $\tau*x$ be the plays in games on $\mathbb{N}$ as described in the proof of 567L.

(b) Start by setting $g_n(x)=n\cap x[\mathbb{N}]$ for $x\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$.

(c) For the inductive step to a non-zero limit ordinal $\xi\le\alpha$, given $\langle g_\eta\rangle_{\eta<\xi}$, then for $x\in\mathbb{N}^{\mathbb{N}}$ set

$$\eta_x=f(\langle x(4n)\rangle_{n\in\mathbb{N}}),\qquad \zeta_x=f(\langle x(4n+1)\rangle_{n\in\mathbb{N}}),$$

$$\begin{aligned}E_x&=g_{\eta_x}(\langle x(4n+2)\rangle_{n\in\mathbb{N}})\text{ if }\eta_x<\xi,\\&=\emptyset\text{ otherwise,}\\F_x&=g_{\zeta_x}(\langle x(4n+3)\rangle_{n\in\mathbb{N}})\text{ if }\zeta_x<\xi,\\&=\emptyset\text{ otherwise.}\end{aligned}$$

Next, for $D\subseteq\xi$, set

$$\begin{aligned}A_D=\{x:x\in\mathbb{N}^{\mathbb{N}},\ \eta_x<\xi,\ &E_x=D\cap\eta_x\\&\text{and either }\zeta_x\le\eta_x\text{ or }\zeta_x\ge\xi\text{ or }F_x\ne D\cap\zeta_x\}.\end{aligned}$$

(The idea is that the players are competing to see who can capture the largest initial segment of D with the pair (η_x,E_x) determined by I's moves or the pair (ζ_x,F_x) determined by II's moves; for definiteness, if neither correctly defines an initial segment, then II wins, while if they seize the same segment $(\eta_x,E_x)=(\zeta_x,F_x)$, then I wins.) Finally, define $g:\mathrm{Str_I}\cup\mathrm{Str_{II}}\to\mathcal{P}\xi$ by setting

$$g(\sigma)=\bigcup\{D:\sigma\text{ is a winning strategy for I in Game}(\mathbb{N},A_D)\}\text{ if }\sigma\in\mathrm{Str_I},$$

$$g(\tau)=\bigcup\{D:\tau\text{ is a winning strategy for II in Game}(\mathbb{N},A_D)\}\text{ if }\tau\in\mathrm{Str_{II}}.$$

We find that g is a surjection onto $\mathcal{P}\xi$. **P** Take any $D\subseteq\xi$.

case 1 Suppose that I has a winning strategy σ in $\mathrm{Game}(\mathbb{N},A_D)$. Then $D\subseteq g(\sigma)$. **?** If $D\ne g(\sigma)$, there is a D', distinct from D, such that σ is a winning strategy for I in $\mathrm{Game}(\mathbb{N},D')$. Let $\zeta<\xi$ be such that $D\cap\zeta\ne D'\cap\zeta$. Then there is a $z\in\mathbb{N}^{\mathbb{N}}$ such that $f(\langle z(2n)\rangle_{n\in\mathbb{N}})=\zeta$ and $g_\zeta(\langle z(2n+1)\rangle_{n\in\mathbb{N}})=D\cap\zeta$. In this case, taking $x=\sigma*z$, we have $x(4n+1)=z(2n)$ and $x(4n+3)=z(2n+1)$ for every n, so $\zeta_x=\zeta$ and $F_x=D\cap\zeta$. Since $x\in A_D$, we have $\eta_x<\xi$, $E_x=D\cap\eta_x$ and $\zeta\le\eta_x$. But also $x\in A_{D'}$, so $E_x=D'\cap\eta_x$ and $D\cap\zeta=D'\cap\zeta$, contrary to the choice of ζ. **X** Thus $D=g(\sigma)$.

case 2 Suppose that II has a winning strategy τ in $\mathrm{Game}(\mathbb{N},A_D)$. Then $D\subseteq g(\tau)$. **?** If $D\ne g(\tau)$, there is a D', distinct from D, such that τ is a winning strategy for II in $\mathrm{Game}(\mathbb{N},D')$. Let $\zeta<\xi$ be such that $D\cap\zeta\ne D'\cap\zeta$. Again, there is a $z\in\mathbb{N}^{\mathbb{N}}$ such that $f(\langle z(2n)\rangle_{n\in\mathbb{N}})=\zeta$ and $g_\zeta(\langle z(2n+1)\rangle_{n\in\mathbb{N}})=D\cap\zeta$. This time, taking $x=\tau*z$, we have $x(4n)=z(2n)$ and $x(4n+2)=z(2n+1)$ for every n, so $\eta_x=\zeta$ and $E_x=D\cap\eta_x$. Since $x\notin A_D$, we must have $\eta_x<\zeta_x<\xi$ and $F_x=D\cap\zeta_x$; since also $x\notin A_{D'}$, $F_x=D'\cap\zeta_x$; so that $D\cap\zeta=F_x\cap\zeta=D'\cap\zeta$, which is impossible. **X** Thus $D=g(\tau)$.

Thus in either case $D\in g[\mathrm{Str_I}\cup\mathrm{Str_{II}}]$. As D is arbitrary, $g[\mathrm{Str_I}\cup\mathrm{Str_{II}}]=\mathcal{P}\xi$. **Q**
Setting $g_\xi=gh$, the induction proceeds.

(d) For the inductive step to $\xi+1$ where $\omega\le\xi<\alpha$, set

$$h_\xi(0)=\xi,\ h_\xi(n)=n-1\text{ for }n\in\omega\setminus\{0\},\ h_\xi(\eta)=\eta\text{ if }\omega\le\eta<\xi,$$

$$g_{\xi+1}(x)=h_\xi[g_\xi(x)]\text{ for }x\in\mathbb{N}^{\mathbb{N}}.$$

(e) At the end of the induction, g_α is the required surjection onto $\mathcal{P}\alpha$.

567N Theorem (Martin 70) [AC] Assume that there is a two-valued-measurable cardinal. Then every coanalytic subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

proof Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a coanalytic set.

(a) Set $S = \bigcup_{n\ge1} \mathbb{N}^n$. For υ, $\upsilon' \in S$ say that $\upsilon \preccurlyeq \upsilon'$ if either υ extends υ' or there is an $n < \min(\#(\upsilon), \#(\upsilon'))$ such that $\upsilon{\restriction}n = \upsilon'{\restriction}n$ and $\upsilon(n) < \upsilon'(n)$. Then $\preccurlyeq$ is a total order, and its restriction to $\mathbb{N}^n$ is the lexicographic well-ordering for each $n \ge 1$.

For $w \in \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set $I_w = \{x : w \subseteq x \in \mathbb{N}^{\mathbb{N}}\}$. Fix an enumeration $\langle \upsilon_i\rangle_{i\in\mathbb{N}}$ of S such that $\#(\upsilon_i) \le i+1$ for every $i \in \mathbb{N}$.

(b) $A' = \mathbb{N}^{\mathbb{N}} \setminus A$ is Souslin-F (423Eb); express it as

$$A' = \bigcup_{y\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\ge1} F_{y\restriction n}$$

where F_υ is closed for every $\upsilon \in S$. Replacing F_υ by $\bigcap_{1\le i\le\#(\upsilon)} F_{\upsilon\restriction i}$ if necessary, we may suppose that $F_\upsilon \subseteq F_{\upsilon'}$ whenever $\upsilon \supseteq \upsilon'$.

For $x \in \mathbb{N}^{\mathbb{N}}$, set

$$T_x = \{\upsilon : \upsilon \in S,\ I_{x\restriction\#(\upsilon)} \cap F_\upsilon \ne 0\},$$

and define a relation $\preccurlyeq_x$ on S by saying that

$$\begin{aligned} \upsilon \preccurlyeq_x \upsilon' \iff &\text{ either } \upsilon,\ \upsilon' \in T_x \text{ and } \upsilon \preccurlyeq \upsilon' \\ &\text{ or } \upsilon \in T_x \text{ and } \upsilon' \notin T_x \\ &\text{ or } \upsilon,\ \upsilon' \notin T_x \text{ and } i \le j \text{ where } \upsilon = \upsilon_i,\ \upsilon' = \upsilon_j. \end{aligned}$$

Then $\preccurlyeq_x$ is a total ordering, since it copies the total ordering $\preccurlyeq$ on T_x and the well-ordering induced by the enumeration $\langle \upsilon_i\rangle_{i\in\mathbb{N}}$ on $S \setminus T_x$, and puts one below the other.

Note that if $n \in \mathbb{N}$, x, $y \in \mathbb{N}^{\mathbb{N}}$ are such that $x{\restriction}n = y{\restriction}n$, and $i < n$, then $x{\restriction}\#(\upsilon_i) = y{\restriction}\#(\upsilon_i)$, so $\upsilon_i \in T_x$ iff $\upsilon_i \in T_y$. Consequently, for i, $j < n$, $\upsilon_i \preccurlyeq_x \upsilon_j$ iff $\upsilon_i \preccurlyeq_y \upsilon_j$. It follows that for every $w \in \mathbb{N}^n$ we have a total ordering $\preccurlyeq'_w$ of n defined by saying that $i \preccurlyeq'_w j$ iff $\upsilon_i \preccurlyeq_x \upsilon_j$ whenever $x \in I_w$.

(c) If $x \in \mathbb{N}^{\mathbb{N}}$ and $\preccurlyeq_x$ is not a well-ordering, then $x \notin A$. **P** Let $D \subseteq S$ be a non-empty set with no $\preccurlyeq_x$-least member. Then $D \cap T_x$ is an initial segment of D. Since $S \setminus T_x$ is certainly well-ordered by $\preccurlyeq_x$, $D \cap T_x \ne \emptyset$. Define $\langle D_n\rangle_{n\in\mathbb{N}}$, $\langle y(n)\rangle_{n\in\mathbb{N}}$ as follows. $D_0 = D \cap T_x$. Given that D_n is a non-empty initial segment of D and that $\upsilon \supseteq y{\restriction}n$ for every $\upsilon \in D_n$, then $y{\restriction}n$ cannot be the least member of D, so $D_n \ne \{y{\restriction}n\}$; set $y(n) = \min\{\upsilon(n) : \upsilon \in D_n \setminus \{y{\restriction}n\}\}$,

$$D_{n+1} = \{\upsilon : \upsilon \in D_n \setminus \{y{\restriction}n\},\ \upsilon(n) = y(n)\}.$$

Because $\preccurlyeq_x$ agrees with $\preccurlyeq$ on T_x, D_{n+1} is a non-empty initial segment of D, and the induction continues.

If m, $n \in \mathbb{N}$, then there is an $\upsilon \in T_x$ such that $\upsilon \supseteq y{\restriction}\max(m,n)$, and

$$I_{x\restriction m} \cap F_{y\restriction n} \supseteq I_{x\restriction\#(\upsilon)} \cap F_\upsilon \ne \emptyset.$$

As m is arbitrary and $F_{y\restriction n}$ is closed, $x \in F_{y\restriction n}$; as n is arbitrary, $x \in A'$ and $x \notin A$. **Q**

(d) Let κ be a two-valued-measurable cardinal, and give $\mathbb{N} \times \kappa$ its discrete topology. In $(\mathbb{N} \times \kappa)^{\mathbb{N}}$ consider the set F of sequences $\langle (x(n), \xi(n))\rangle_{n\in\mathbb{N}}$ such that

$$\text{whenever } i,\ j \in \mathbb{N},\ \upsilon_i \subset \upsilon_j \text{ and } x \in F_{\upsilon_j}, \text{ then } \xi(2j) < \xi(2i).$$

Then F is closed for the product topology; by 567B, F is determined.

(e) Suppose I has a winning strategy σ in the game $\mathrm{Game}(\mathbb{N}\times\kappa, F)$. Then I has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$. **P** For $\langle k_i\rangle_{i<n} \in \mathbb{N}^n$ take $\sigma'(\langle k_i\rangle_{i<n})$ to be the first component of $\sigma(\langle (k_i, 0)\rangle_{i<n})$. If x is any play of $\mathrm{Game}(\mathbb{N}, A)$ consistent with σ', then for each n set $\xi(2n+1) = 0$ and let $\xi(2n)$ be the second component of $\sigma(\langle (x(2i+1), 0)\rangle_{i<n})$. Then $\langle (x(n), \xi(n))\rangle_{n\in\mathbb{N}}$ is a play of $\mathrm{Game}(\mathbb{N}\times\kappa, F)$ consistent with σ, so is won by I. **?** If $x \notin A$, let $y \in \mathbb{N}^{\mathbb{N}}$ be such that $x \in F_{y\restriction n}$ for every $n \in \mathbb{N}$. Set $I = \{i : i \in \mathbb{N},\ y \supseteq \upsilon_i\}$; then I is infinite, and there is an infinite $J \subseteq I$ such that $\upsilon_i \subset \upsilon_j$ whenever i, $j \in J$ and $i < j$, while $x \in F_{\upsilon_j}$ for every $j \in J$. But now we see that $\xi(2j) < \xi(2i)$ whenever $i < j$ in J, which is impossible. **X**

Thus $x \in A$; as x was arbitrary, σ' is a winning strategy for I in $\mathrm{Game}(\mathbb{N}, A)$. **Q**

(f) Suppose II has a winning strategy τ in $\mathrm{Game}(\mathbb{N}\times\kappa, F)$. Then II has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$. **P** Fix a normal κ-additive ultrafilter $\mathcal{F}$ on κ (541Ma). For $w = (k_0, \dots, k_{2n}) \in \mathbb{N}^{2n+1}$ consider the function $f_w : [\kappa]^{n+1} \to \mathbb{N}$ defined by saying that $f_w(J)$ is to be the first component of $\tau(\langle (k_{2i}, \xi_i)\rangle_{i\le n})$ where $(\xi_0, \dots, \xi_n)$ is that enumeration of J such that, for i, $j \le n$, $\xi_i \le \xi_j$ iff $i \preccurlyeq'_w j$. Then for each $m \in \mathbb{N}$ there is a $C_{wm} \in \mathcal{F}$ such that either $f_w(J) \le m$ for every $J \in [C_{wm}]^{n+1}$ or $f_w(J) > m$ for every $J \in [C_{wm}]^{n+1}$ (4A1L). Setting $C = \bigcap_{m,n\in\mathbb{N}} \bigcap_{w\in\mathbb{N}^{2n+1}} C_{wm}$, $C \in \mathcal{F}$ and every f_w is constant on $[C]^{n+1}$. Let $\rho(w)$ be the constant value of $f_w{\restriction}[C]^{n+1}$.

Define $\tau' : \bigcup_{n\ge 1}\mathbb{N}^n \to \mathbb{N}$ inductively, saying that $\tau'(k_0,\dots,k_n) = \rho(w)$ whenever $w(2i) = k_i$ for $i \le n$ and $w(2i+1) = \tau'(k_0,\dots,k_i)$ for $i < n$. Suppose that x is a play of Game$(\mathbb{N}, A)$ consistent with τ'. **?** If $x \in A$, then $\preccurlyeq_x$ is a well-ordering, by (c). The order type of $(S, \preccurlyeq_x)$ is countable, so is surely less than $\mathrm{otp}(C) = \kappa$, and we have a function $\theta : S \to C$ such that $\theta(\upsilon) \le \theta(\upsilon')$ iff $\upsilon \preccurlyeq_x \upsilon'$.

Define $\langle\xi(n)\rangle_{n\in\mathbb{N}}$ by saying that

$$\begin{aligned}\xi(n) &= \theta(\upsilon_j) \text{ if } n = 2j \text{ is even },\\ &= \text{ the second component of } \tau((x(0),\xi(0)),(x(2),\xi(2)),\dots,(x(2j),\xi(2j)))\\ &\qquad\qquad \text{if } n = 2j+1 \text{ is odd.}\end{aligned}$$

For i, $j \le n \in \mathbb{N}$, setting $w = x\restriction 2n+1$, we have $\xi(2i)$, $\xi(2j) \in C$ and

$$\xi(2i) \le \xi(2j) \iff \theta(\upsilon_i) \le \theta(\upsilon_j) \iff \upsilon_i \preccurlyeq_x \upsilon_j \iff i \preccurlyeq'_w j.$$

So

$$x(2n+1) = \rho(w) = f_w(\{\xi(2i) : i \le n\})$$

is the first component of $\tau((x(0),\xi(0)),\dots,(x(2n),\xi(2n)))$; thus $\langle(x(n),\xi(n))\rangle_{n\in\mathbb{N}}$ is a play of Game$(\mathbb{N}\times\kappa, F)$ consistent with τ, and is won by II. There must therefore be i, $j \in \mathbb{N}$ such that $\upsilon_i \subset \upsilon_j$, $x \in F_{\upsilon_j}$ and and $\xi(2i) \le \xi(2j)$. Now $\upsilon_j \in T_x$ and $\upsilon_i \preccurlyeq_x \upsilon_j$, so $\upsilon_i \preccurlyeq \upsilon_j$; which is impossible. **X**

So $x \notin A$; as x is arbitrary, τ' is a winning strategy for II in Game$(\mathbb{N}, A)$. **Q**

(g) Putting (d), (e) and (f) together, we see that A is determined.

567O Corollary [AC] If there is a two-valued-measurable cardinal, then every PCA ($= \boldsymbol{\Sigma}^1_2$) subset of $\mathbb{R}$ is Lebesgue measurable.

proof Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be PCA. Then there is a coanalytic subset B of $\mathbb{N}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that A is the projection of B. Of course this means that there is a coanalytic subset B' of $\{0,1\}^{\mathbb{N}}$ such that A is a continuous image of B', since $\mathbb{N}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$ is homeomorphic to a G_δ subset of $\{0,1\}^{\mathbb{N}}$. Now $(h^{-1}[B']\cap F)\cup H$ is coanalytic whenever $h : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is continuous, $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open; by 567N, $(\mathbb{N}, (h^{-1}[B'] \cap F) \cup H)$ is always determined; by 567F, A is measured by the usual measure ν on $\{0,1\}^{\mathbb{N}}$.

Now let $A \subseteq \mathbb{R}$ be PCA. There is a probability measure μ on $\mathbb{R}$ with the same measurable sets and the same negligible sets as Lebesgue measure, so that $(\mathbb{R},\mu)$ and $(\{0,1\}^{\mathbb{N}},\nu)$ are isomorphic; let $\phi : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$ be an isomorphism. Then there are comeager G_δ sets $E \subseteq \mathbb{R}$, $F \subseteq \{0,1\}^{\mathbb{N}}$ such that $\phi\restriction E$ is a homeomorphism between E and F. So $\phi[E\cap A]$ is a PCA set in $\{0,1\}^{\mathbb{N}}$ and is measured by ν; it follows that $E \cap A$ and A are measured by μ and by Lebesgue measure.

567X Basic exercises (a) Let X be a non-empty well-orderable set, with its discrete topology, and $G \subseteq X^{\mathbb{N}}$ an open set. Show that G is determined.

(b) [AC$(\mathbb{R};\omega)$] Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be such that $\{x : {<}n{>}^\frown x \in A\}$ is determined for every $n \in \mathbb{N}$. Show that $\mathbb{N}^{\mathbb{N}} \setminus A$ is determined.

(c) Show that AD is true iff every subset of $\{0,1\}^{\mathbb{N}}$ is determined. (*Hint*: For $x \in \{0,1\}^{\mathbb{N}}$ set $\chi_{\mathrm{I}}(x) = \sup\{n : x(2n) = 1\}$, $\chi_{\mathrm{II}}(x) = \sup\{n : x(2n+1) = 1\}$; set $C_{\mathrm{I}} = \{x : \chi_{\mathrm{I}}(x) > \chi_{\mathrm{II}}(x)\}$, $D = \{x : \chi_{\mathrm{I}}(x) = \chi_{\mathrm{II}}(x) = \infty\}$. Given $x \in D$, define y, $z \in \mathbb{N}^{\mathbb{N}}$ by saying that $y(0) = \min\{k : x(2k) = 1\}$, $z(0) = 2y(0)+1$, $y(n+1) = \min\{k : x(z(n)+2k) = 1\}$, $z(n+1) = z(n)+2y(n+1)+1$; set $f(x) = y$. Show that if $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $f^{-1}[A]\cup C_{\mathrm{I}}$ is determined then A is determined.)

(d) [AC$(\mathbb{R};\omega)$] (i) Show that the intersection of a sequence of closed cofinal subsets of ω_1 is cofinal. (ii) Show that we have a unique topological probability measure on ω_1 which is zero on singletons and inner regular with respect to the closed sets.

(e) Show that there is a game Game$(\mathcal{P}\mathbb{R}, A)$ which is not determined. (*Hint*: 567D, 567G.)

(f) [AD] Show that if $f : [0,1]^2 \to \mathbb{R}$ is a bounded function, then $\iint f(x,y)dxdy$ and $\iint f(x,y)dydx$ are defined and equal, where the integrations are with respect to Lebesgue measure on $[0,1]$.

(g) [AD] Let μ be a Radon measure on a Polish space X, and $\mathcal{E}$ a well-ordered family of subsets of X. Show that $\mu(\bigcup\mathcal{E}) = \sup_{E\in\mathcal{E}}\mu E$. (*Hint*: 567Xf.)

(h) [AD+AC(ω)] Show that there are no interesting Sierpiński sets, in the sense that every uncountable atomless probability space has an uncountable negligible subset.

(i) [AD] Show that every semi-finite measure space is perfect.

(j) [AD] Show that if X is a separable Banach space and Y is a normed space then every linear operator from X to Y is bounded.

(k) [AD] (i) Show that there is no non-principal ultrafilter on $\mathbb{N}$. (ii) Show that $\{0,1\}^{\mathbb{R}}$ is not compact.

(l) [AD] Show that there is no linear lifting of Lebesgue measure on $\mathbb{R}$. (*Hint*: 567J.)

(m) [AD] (i) Show that $\ell^1(\mathbb{R})$ is not reflexive. (ii) Show that $\ell^1(\omega_1)$ is not reflexive.

(n) [AD] (i) Show that there is no injective function from ω_1 to $\mathbb{R}$. (ii) Show that there is no family $\langle f_\xi\rangle_{\xi<\omega_1}$ such that f_ξ is an injective function from ξ to $\mathbb{N}$ for every $\xi<\omega_1$. (iii) Show that there is no function $f:\omega_1\times\mathbb{N}\to\omega_1$ such that $\{f(\xi,n):n\in\mathbb{N}\}$ is a cofinal subset of ξ for every non-zero limit ordinal $\xi<\omega_1$. (*Hint*: 567L.)

(o) [AD] Show that there is a surjective function from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$, but no injective function from $\mathcal{B}(\mathbb{R})$ to $\mathbb{R}$. (*Hint*: 567D, 561Xr.)

(p) [AC] Show that if there is a two-valued-measurable cardinal and $A\subseteq\mathbb{N}^{\mathbb{N}}$ is analytic then A is determined.

(q) [AC] Suppose that there is a two-valued-measurable cardinal. Show that every PCA subset of $\mathbb{R}$ has the Baire property.

567Y Further exercises (a) Let X be a non-empty set and $A\subseteq X^{\mathbb{N}}$. A **quasi-strategy for I** in Game(X,A) is a function $\sigma:\bigcup_{n\in\mathbb{N}}X^n\to\mathcal{P}X\setminus\{\emptyset\}$; it is a winning quasi-strategy if $x\in A$ whenever $x\in X^{\mathbb{N}}$ and $x(2n)\in\sigma(\langle x(2i{+}1)\rangle_{i<n})$ for every n. Similarly, a winning quasi-strategy for II is a function $\tau:\bigcup_{n\ge1}X^n\to\mathcal{P}X\setminus\{\emptyset\}$ such that $x\notin A$ whenever $x\in X^{\mathbb{N}}$ and $x(2n+1)\in\tau(\langle x(2i)\rangle_{i<n})$ for every n. (i) Show that if X is any non-empty discrete space and $F\subseteq X^{\mathbb{N}}$ is closed then at least one player has a winning quasi-strategy in Game(X,F). (ii) Show that DC is true iff there is no game Game(X,A) such that both players have winning quasi-strategies.

(b) [AD] Show that every uncountable subset of $\mathbb{R}$ has a non-empty perfect subset. (*Hint*: Let $A\subseteq\{0,1\}^{\mathbb{N}}$. Enumerate $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ as $\langle v_n\rangle_{n\in\mathbb{N}}$. For $x\in\mathbb{N}^{\mathbb{N}}$ set

$$f(x)=v_{x(0)}{}^\frown{<}\min(1,x(1)){>}^\frown v_{x(2)}{}^\frown{<}\min(1,x(3)){>}^\frown\ldots.$$

Consider Game$(\mathbb{N},f^{-1}[A])$.)

(c) Let Θ be the least ordinal such that there is no surjection from $\mathcal{P}\mathbb{N}$ onto Θ. (i) [AC(ω)] Show that $\operatorname{cf}\Theta>\omega$. (ii) [AD] Show that $\Theta=\omega_\Theta$.

(d) [AC] Suppose that there is a two-valued-measurable cardinal. Show that every uncountable PCA subset of $\mathbb{R}$ has a non-empty perfect subset.

567 Notes and comments The consequences of the axiom of determinacy are so striking that the question of its consistency is particularly pressing. In fact W.H.Woodin has determined its consistency strength, in terms of large cardinals (Kanamori 03, 32.16, or Jech 03, 33.27), and this is less than that of the existence of a supercompact cardinal; so it seems safe enough.

In ZFC, 567B is most naturally thought of as a basic special case of Martin's theorem that every Borel subset of $X^{\mathbb{N}}$, for any discrete space X, is determined (Martin 75, or Kechris 95, 20.5). The idea of the proof is that if II has no winning strategy, then all I has to do is to avoid positions from which II can win. But for a proof in ZF we need more than this. It would not be enough to show that for every first move by I, there is a winning strategy for II from the resulting position; we should need to show that these can be pieced together as a single function $\tau:\bigcup_{n\ge1}X^n\to X$. Turning this round, AD must imply a weak form of the axiom of choice (567D; see also 567Xe). In the particular case of 567B, we have a basic set W_0 of winning positions for II with a trivial family $\langle\tau_w\rangle_{w\in W_0}$ of strategies. (Starting from a position in W_0, II can simply play the $\preccurlyeq$-least point of X to get a position from which I cannot avoid W_0.) From these we can work backwards to find a family $\langle\tau_w\rangle_{w\in W}$ of strategies, where $W=\bigcup_{\xi\in\mathrm{On}}W_\xi$; so that if ${<}u{>}\in W$ for every $u\in X$, we can assemble these into a winning strategy for II in Game(X,F).

The central result of the section is I suppose 567G. From the point of view of a real analyst like myself, as opposed to a logician or set theorist, this is the door into a different world, explored in 567H-567K, 567Xf-567Xm and 567Yb. In 567I we have a result which is already interesting in ZFC. Recall that in ZFC there are non-trivial additive functionals on $\mathcal{P}\mathbb{N}$ which are measurable in the sense of §464; none of them can be Baire-property-measurable.

I have not talked about 'automatic continuity' in this book. If you have seen anything of this subject you will recognise the three parts of 567H as versions of standard results on homomorphisms which are measurable in some sense. I do not know whether the hypothesis 'abelian' is necessary in 567Hb. If you like, 567J can also be thought of as an automatic-continuity result.

You will see that 567H-567J depend on 567Gb rather than on 567Ga; that is, on category rather than on measure. It is not clear how much can be proved if we assume, as an axiom, that every subset of $\mathbb{R}$ is Lebesgue measurable (together with AC(ω) at least, of course), rather than that every subset of $\mathbb{R}$ has the Baire property.

In 567L far more is true, at least with AD+DC; ω_2, as well as ω_1, is two-valued-measurable, and the filter on ω_1 generated by the closed cofinal sets is an ultrafilter (Kanamori 03, §28, or Jech 03, Theorem 33.12). I am not sure what we should think of as a 'real-valued-measurable cardinal' in this context. In the language of 566Xk, AD implies that $\mathbb{R}$ is not measure-free, and Lebesgue measure is κ-additive for every initial ordinal κ (567Xg). For further combinatorial consequences of AD, see Kanamori 03. Note that AD implies CH in the form 'every uncountable subset of $\mathbb{R}$ is equipollent with $\mathbb{R}$' (567Yb). But the relationship of $\mathbb{R}$ with ω_1 is quite different. ZF is enough to build a surjection from $\mathbb{R}$ onto ω_1. AD implies that there is no injection from ω_1 into $\mathbb{R}$ (567Xn) but that there are surjections from $\mathbb{R}$ onto much larger initial ordinals (567M, 567Yc).

In 567N-567O I return to the world of ZFC; they are in this section because they depend on 567B and 567F. Once again, much more is known about determinacy compatible with AC, and may be found in Kanamori 03 or Jech 03.

Appendix to Volume 5

Useful facts

For this volume, the most substantial ideas demanded are, naturally enough, in set theory. Fragments of general set theory are in §5A1, with cardinal arithmetic, infinitary combinatorics and notes on a few standard undecidable propositions. §5A2 contains results from Shelah's pcf theory, restricted to those which are actually used in this book. §5A3 describes the language I will use when I discuss forcing constructions; in essence, I follow Kunen 80, but with some variations which need to be signalled.

As usual, some bits of general topology are needed; I give these in §5A4, starting with a list of cardinal functions to complement the definitions in §511. There is a tiny piece of real analysis in §5A5.

The final section of the appendix, §5A6, consists of statements of results which I mean to include in future editions of Volumes 1-4, and which are relied on at some point in the arguments of this volume. I hope that even in the absence of proofs, these will help readers to make sense of the patterns of ideas here.

5A1 Set theory

As usual, I begin with set theory, continuing from §§2A1 and 4A1. I start with definitions and elementary remarks filling some minor gaps in the deliberately sketchy accounts in the earlier volumes (5A1A). I give a relatively solid paragraph on cardinal arithmetic (5A1E), including an account of cofinalities of ideals $[\kappa]^{\le\lambda}$. 5A1G-5A1J are devoted to infinitary combinatorics, with the Erdős-Rado theorem and Hajnal's Free Set Theorem. 5A1K-5A1O deal with the existence of 'transversals' of various kinds in spaces of functions, that is, large sets of functions which are well separated on some combinatorial criterion.

The last third of the section contains brief accounts of some of the undecidable propositions and special axioms which are used in this volume, with a few of their most basic consequences: the generalized continuum hypothesis, the axiom of constructibility, Jensen's Covering Lemma, square principles, Chang's transfer principle and Todorčević's p-ideal dichotomy.

5A1A Order types (a) If X is a well-ordered set, its **order type** $\operatorname{otp} X$ is the ordinal order-isomorphic to X (2A1Dg).

If S is a set of ordinals, an ordinal-valued function f with domain S is **regressive** if $f(\xi) < \xi$ for every $\xi \in S$ (cf. 4A1Cc).

(b) The non-stationary ideal on a cardinal κ of uncountable cofinality is $(\operatorname{cf}\kappa)$-additive, because the intersection of fewer than $\operatorname{cf}\kappa$ closed cofinal sets is a closed cofinal set (4A1Bd).

(c) If κ is a cardinal, $\lambda < \operatorname{cf}\kappa$ is an infinite regular cardinal and $C \subseteq \kappa$ is a closed cofinal set, then $S = \{\xi : \xi < \kappa, \operatorname{cf}(\xi \cap C) = \lambda\}$ is stationary in κ. **P** If $C' \subseteq \kappa$ is a closed cofinal set, let $\langle\gamma_\xi\rangle_{\xi<\operatorname{otp} C}$ be the increasing enumeration of $C \cap C'$. Then $\operatorname{otp}(C \cap C') \ge \operatorname{cf}\kappa > \lambda$, so γ_λ is defined and belongs to $S \cap C'$. **Q**

(d) If α is an ordinal, there is a closed cofinal set $C \subseteq \alpha$ such that $\operatorname{otp} C = \operatorname{cf}\alpha$. **P** Set $\kappa = \operatorname{cf}\alpha$ and let $\langle\alpha_\xi\rangle_{\xi<\kappa}$ enumerate a cofinal subset of α. Set $\gamma_\xi = \sup_{\eta<\xi} \alpha_\eta$ for $\xi < \kappa$; then $C = \{\gamma_\xi : \xi < \kappa\}$ is a closed cofinal set in α. Let $\langle\gamma'_\xi\rangle_{\xi<\operatorname{otp} C}$ be the increasing enumeration of C; then an induction on ξ shows that $\gamma_\xi \le \gamma'_\xi$ for $\xi < \min(\kappa, \operatorname{otp} C)$, so that $\operatorname{otp} C \le \kappa$. As C is cofinal with α, $\#(C) \ge \kappa$ so $\operatorname{otp} C \ge \kappa$. **Q**

(e) If α is an ordinal and $C \subseteq \alpha$ has closure $\overline{C}$ for the order topology of α, then $\#(\overline{C}) = \#(C)$. **P** If C is finite this is trivial. Otherwise, for any $\beta \in \overline{C} \setminus \{\sup C\}$ set $f(\beta) = \min(C \setminus \beta)$. If β, $\beta' \in \overline{C}$ and $\beta < \beta' < \sup C$ there must be a $\gamma \in C$ such that $\beta \le \gamma < \beta'$ and $f(\beta) \le \gamma < f(\beta')$ (4A2Sa); thus f is injective. So

$$\#(\overline{C}) = \#(\overline{C} \setminus \{\sup C\}) \le \#(C) \le \#(\overline{C}). \text{ Q}$$

5A1B Ordinal arithmetic (a) For ordinals ξ, η their **ordinal sum** $\xi + \eta$ is defined inductively by saying that

$\xi + 0 = \xi$,

$\xi + (\eta + 1) = (\xi + \eta) + 1$,

$\xi + \eta = \sup_{\zeta<\eta} \xi + \zeta$ for non-zero limit ordinals η

(Kunen 80, I.7.18; Jech 78, p. 18; Jech 03, 2.18).

(b) For ordinals ξ, η their **ordinal product** $\xi \cdot \eta$ is defined inductively by saying

$\xi \cdot 0 = 0$,

$\xi \cdot (\eta + 1)$ is the ordinal sum $\xi \cdot \eta + \xi$,

$\xi\cdot\eta=\sup_{\zeta<\eta}\xi\cdot\zeta$ for non-zero limit ordinals η

(Kunen 80, I.7.20; Jech 78, p. 19; Jech 03, 2.19). Note that $0\cdot\eta=0$ and $1\cdot\eta=\eta$ for every η, and that $\sup_{\zeta\in A}\xi\cdot\zeta=\xi\cdot(\sup A)$ for every ξ and every non-empty set A of ordinals. Ordinal multiplication is associative (Kunen 80, I.7.20; Jech 03, 2.21).

(c) For ordinals ξ, η the **ordinal power** ξ^η is defined inductively by saying that

$\xi^0=1$,

$\xi^{\eta+1}$ is the ordinal product $\xi^\eta\cdot\xi$,

$\xi^\eta=\sup_{\zeta<\eta}\xi^\zeta$ for non-zero limit ordinals η

(Kunen 80, I.9.5; Jech 03, 2.20). **Warning!** If ξ and η happen to be cardinals, this is quite different from the 'cardinal power' of 5A1E below.

If ξ, η are ordinals, $\eta\ne 0$ and η is greater than or equal to the ordinal product $\xi\cdot\eta$, then η is at least the ordinal power ξ^ω. **P** Note first that as multiplication is associative, we can induce on n to show that $\xi\cdot\xi^n=\xi^{n+1}$ for every n. Now we are supposing that $\eta\ge 1=\xi^0$. If $n\in\mathbb{N}$ and $\eta\ge\xi^n$, then

$$\eta\ge\xi\cdot\eta\ge\xi\cdot\xi^n=\xi^{n+1}.$$

So $\eta\ge\xi^n$ for every n and $\eta\ge\xi^\omega$. **Q**

5A1C Well-founded sets (a) A partially ordered set P is **well-founded** if every non-empty $A\subseteq P$ has a minimal element, that is, a $p\in A$ such that $q\not<x$ for every $q\in A$.

(b) If P is a well-founded partially ordered set, we have a rank function $r:P\to\mathrm{On}$ defined by saying that

$$r(p)=\sup\{r(q)+1:q<p\}$$

for every $p\in P$ (Kunen 80, III.5.7; Jech 78, p. 21; Jech 03, 2.27).

(c) A partially ordered set P is well-founded iff there is no sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $p_{n+1}<p_n$ for every $n\in\mathbb{N}$. (If $A\subseteq P$ is non-empty and has no minimal element, we can choose inductively a strictly decreasing sequence in A.)

5A1D Trees In §421 I introduced trees of sequences. For this volume a more abstract approach is useful.

(a) A **tree** is a partially ordered set T such that $\{s:s\in T,\ s\le t\}$ is well-ordered for every $t\in T$. In particular, T is well-founded, so has a rank function $r:T\to\mathrm{On}$ defined by saying that

$$r(t)=\mathrm{otp}\{s:s<t\}=\inf\{\xi:r(s)<\xi\text{ whenever }s<t\}$$

for every $t\in T$ (5A1C). (Try to avoid using this terminology in the same sentence as that of 421Ne.)

The **levels** of T are now the sets $\{t:r(t)=\xi\}$ for $\xi\in\mathrm{On}$. The **height** of T is the least ζ such that $r(t)<\zeta$ for every $t\in T$. A **branch** of T is a maximal totally ordered subset. A tree is **well-pruned** if it has at most one minimal element and whenever s, $t\in T$ and $r(s)<r(t)$, there is an $s'\ge s$ such that $r(s')=r(t)$. If T is a tree, a **subtree** of T is a set $T'\subseteq T$ such that $s\in T'$ whenever $s\le t\in T'$; in this case, the rank function of T' is the restriction to T' of the rank function of T.

(b)(i) Let T be a tree in which every level is finite. Then T has a branch meeting every level. **P** If T is empty, this is trivial. Otherwise, let r be the rank function of T, and $\zeta>0$ the height of T; let $\mathcal{F}$ an ultrafilter on T containing $\{t:r(t)\ge\xi\}$ for every $\xi<\zeta$. Set $C=\{t:[t,\infty[\,\in\mathcal{F}\}$. Any two elements of C are upwards-compatible, so C is totally ordered, and C meets every level of T; so C is a branch of the kind we seek. **Q**

(ii) Let $(T,\preccurlyeq')$ be a tree of height ω_1 in which every level is countable. Then there is an ordering $\preccurlyeq$ of ω_1, included in the usual ordering $\le$ of ω_1, such that $(T,\preccurlyeq')$ is isomorphic to $(\omega_1,\preccurlyeq)$. **P** Let $\langle T_\xi\rangle_{\xi<\omega_1}$ be the levels of T. Let $\le'_\xi$ be a well-ordering of T_ξ for each $\xi<\omega_1$, and define $\le'$ on T by saying that $s\le' t$ if either $r(s)<r(t)$ or $r(s)=r(t)=\xi$ and $s\le'_\xi t$; then $\le'$ is a well-ordering of T of order type ω_1. Now the order-isomorphism between $(T,\le')$ and $(\omega_1,\le)$ copies $\preccurlyeq'$ onto a tree ordering of ω_1, isomorphic to $\preccurlyeq'$, and included in the usual ordering. **Q**

(c) An **Aronszajn tree** is a tree T of height ω_1 in which every branch and every level is countable. An Aronszajn tree T is **special** if it is expressible as $\bigcup_{n\in\mathbb{N}}A_n$ where no two elements of any A_n are comparable, that is, every A_n is an up-antichain.

(d)(i) A **Souslin tree** is a tree T of height ω_1 in which every branch and every up-antichain is countable.

(ii) Every Souslin tree is a non-special Aronszajn tree.

(iii) If T is a Souslin tree, it has a subtree which is a well-pruned Souslin tree. (Kunen 80, II.5.11; Jech 78, p. 218; Jech 03, 9.13.)

5A1E Cardinal arithmetic (a)(i) An infinite cardinal which is not regular (4A1Aa) is **singular**. A cardinal κ is a **successor cardinal** if it is of the form λ^+ (2A1Fc); otherwise it is a **limit cardinal**. κ is a **strong limit cardinal** if it is uncountable and $2^\lambda < \kappa$ for every $\lambda < \kappa$. It is is **weakly inaccessible** if it is an uncountable regular limit cardinal; it is **strongly inaccessible** if moreover it is a strong limit cardinal.

(ii) If κ is a cardinal, define $\kappa^{(+\xi)}$, for ordinals ξ, by setting

$$\kappa^{(+0)} = \kappa, \quad \kappa^{(+\xi)} = \sup_{\eta<\xi}(\kappa^{(+\eta)})^+ \text{ if } \xi > 0,$$

that is, $\kappa^{(+\xi)} = \omega_{\zeta+\xi}$ if $\kappa = \omega_\zeta$.

(b)(i) If $\langle\kappa_i\rangle_{i\in I}$ is a family of cardinals, its **cardinal sum** is $\#(\{(\xi, i) : i \in I,\ \xi < \kappa_i\})$, which is at most $\max(\omega, \#(I), \sup_{i\in I} \kappa_i)$.

(ii) For cardinals κ and λ, the **cardinal product** $\kappa \cdot \lambda$ is $\#(\kappa \times \lambda) \le \max(\omega, \kappa, \lambda)$.

(iii) If κ and λ are cardinals there are two natural interpretations of the formula κ^λ: (i) the set of functions from λ to κ (ii) the cardinal of this set. In this volume the latter will be the usual one, but I will try to signal this by using the phrase **cardinal power**. Of course 2^λ is always the cardinal power; the corresponding set of functions will be denoted by $\{0, 1\}^\lambda$.

(c)(i) The cardinal power κ^λ is at most $2^{\max(\omega,\kappa,\lambda)}$ for any cardinals κ and λ. (The set of functions from λ to κ is a subset of $\mathcal{P}(\lambda \times \kappa)$.)

(ii) $\mathfrak{c}^\omega = \#(((\{0, 1\})^{\mathbb{N}})^{\mathbb{N}}) = \#(\{0, 1\}^{\mathbb{N}\times\mathbb{N}}) = \#(\{0, 1\}^{\mathbb{N}}) = \mathfrak{c}$.

(d) $\operatorname{cf} 2^\kappa > \kappa$ for every infinite cardinal κ. (Jech 03, 5.11; Jech 78, p. 46; Erdős Hajnal Máté & Rado 84, 6.9; Kunen 80, 10.41; Just & Weese 96, 11.2.24. Compare (d-v) below.)

(e)(i) If κ and λ are infinite cardinals, then

$$\begin{aligned}\operatorname{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \lambda \ge \kappa,\\ &\ge \kappa \text{ if } \lambda < \kappa.\end{aligned}$$

(ii) Let κ, λ and θ be infinite cardinals such that $\theta \le \lambda \le \kappa$. Then $\operatorname{cf}[\kappa]^\theta \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta})$. **P** Let $\mathcal{A} \subseteq [\kappa]^\lambda$ be a cofinal set of size $\operatorname{cf}[\kappa]^\lambda = \operatorname{cf}[\kappa]^{\le\lambda}$. Then $[\kappa]^{\le\theta} = \bigcup_{A\in\mathcal{A}}[A]^{\le\theta}$, so

$$\operatorname{cf}[\kappa]^{\le\theta} \le \max(\#(\mathcal{A}), \sup_{A\in\mathcal{A}} \operatorname{cf}[A]^{\le\theta}) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta}). \text{ **Q**}$$

(iii) Let κ and λ be infinite cardinals. Then the cardinal power κ^λ is $\max(2^\lambda, \operatorname{cf}[\kappa]^{\le\lambda})$. **P** $\kappa^\lambda \ge 2^\lambda$ because $\kappa \ge 2$; $\kappa^\lambda \ge \#([\kappa]^{\le\lambda}) \ge \operatorname{cf}[\kappa]^{\le\lambda}$ because $f \mapsto f[\lambda]$ is a surjection from the family F of functions from λ to κ onto $[\kappa]^{\le\lambda} \setminus \{\emptyset\}$. In the other direction, if $\kappa \le \lambda$ then $F \subseteq \mathcal{P}(\lambda \times \kappa)$ so $\#(F) \le 2^\lambda = \max(2^\lambda, \operatorname{cf}[\kappa]^{\le\lambda})$. If $\lambda < \kappa$ let $\mathcal{A} \subseteq [\kappa]^{\le\lambda}$ be a cofinal family of cardinal $\operatorname{cf}[\kappa]^{\le\lambda}$; then $F = \bigcup_{A\in\mathcal{A}} A^\lambda$ so

$$\#(F) \le \max(\#(\mathcal{A}), \sup_{A\in\mathcal{A}} \#(A^\lambda)) = \max(\operatorname{cf}[\kappa]^{\le\lambda}, 2^\lambda). \text{ **Q**}$$

(iv) If λ is an infinite cardinal and $\lambda \le \kappa < \lambda^{(+\omega)}$, then $\operatorname{cf}[\kappa]^{\le\omega} = \max(\kappa, \operatorname{cf}[\lambda]^{\le\omega})$. **P** Induce on n to see that

$$\begin{aligned}\operatorname{cf}([\lambda^{(+n+1)}]^{\le\omega}) &= \operatorname{cf}(\bigcup_{\xi<\lambda^{(+n+1)}}[\xi]^{\le\omega}) \le \max(\lambda^{(+n+1)}, \sup_{\xi<\lambda^{(+n+1)}} \operatorname{cf}[\xi]^{\le\omega})\\ &\le \max(\lambda^{(+n+1)}, \lambda^{(+n)}, \operatorname{cf}[\lambda]^{\le\omega}) = \max(\lambda^{(+n+1)}, \operatorname{cf}[\lambda]^{\le\omega}). \text{ **Q**}\end{aligned}$$

Consequently the cardinal power κ^ω is $\max(\mathfrak{c}, \kappa, \operatorname{cf}[\lambda]^{\le\omega}) = \max(\kappa, \lambda^\omega)$.

In particular, if $\omega_1 \le \kappa < \omega_\omega$ then $\operatorname{cf}[\kappa]^{\le\omega} = \kappa$ and $\kappa^\omega = \max(\mathfrak{c}, \kappa)$. Similarly, $(\mathfrak{c}^+)^\omega = \max(\mathfrak{c}^+, \mathfrak{c}) = \mathfrak{c}^+$, $(\mathfrak{c}^{++})^\omega = \mathfrak{c}^{++}$.

(v) If κ is a singular infinite cardinal, then $\operatorname{cf}([\kappa]^{\le\operatorname{cf}\kappa}) > \kappa$. **P** Set $\lambda = \operatorname{cf}\kappa$, and let $\langle\kappa_\xi\rangle_{\xi<\lambda}$ be a strictly increasing family of cardinals with supremum κ. If $\langle A_\eta\rangle_{\eta<\kappa}$ is a family in $[\kappa]^{\le\lambda}$, then for each $\xi < \lambda$ take $\alpha_\xi \in \kappa \setminus \bigcup_{\eta<\kappa_\xi} A_\eta$; set $A = \{\alpha_\xi : \xi < \lambda\} \in [\kappa]^{\le\lambda}$; then $A \not\subseteq A_\eta$ for every $\eta < \kappa$. **Q**

(f) If λ is a regular uncountable cardinal, $\theta \ge 2$ is a cardinal and $\kappa = \sup_{\delta<\lambda} \theta^\delta$, where θ^δ is the cardinal power, then

$$\#([\kappa]^{<\lambda}) = \sup_{\delta<\lambda} \kappa^\delta = \kappa.$$

P Of course $\kappa \le \#([\kappa]^{<\lambda})$ because $\lambda \ge 2$, while

$$\#([\kappa]^{<\lambda}) = \#(\bigcup_{\delta<\lambda}[\kappa]^{\delta}) \le \max(\lambda, \omega, \sup_{\delta<\lambda}\#([\kappa]^{\delta})) \le \sup_{\delta<\lambda}\kappa^{\delta}$$

because $\lambda \le \sup_{\delta<\lambda} 2^{\delta} \le \kappa$. If $\delta < \lambda$ then, because λ is regular,

$$\kappa^{\delta} = \sup_{\zeta<\lambda}(\theta^{\zeta})^{\delta} \le \sup_{\zeta<\lambda}\theta^{\max(\omega,\zeta,\delta)} \le \kappa. \textbf{ Q}$$

(g) Let X, Y and Z be sets, with $\#(X) \le 2^{\#(Z)}$ and $0 < \#(Y) \le \#(Z)$. Then there is a function $f : X \times Z^{\mathbb{N}} \to Y$ such that whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of X and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y there is a $z \in Z^{\mathbb{N}}$ such that $f(x_n, z) = y_n$ for every $n \in \mathbb{N}$. **P** We can suppose that $X \subseteq \mathcal{P}Z$ and $Y \subseteq Z$; moreover, the case of finite X is trivial, so we can suppose that Z is infinite. For each countably infinite set $I \subseteq Z$, let $g_I : I^{\mathbb{N}} \to (\mathcal{P}I)^{\mathbb{N}} \times I^{\mathbb{N}}$ be a surjection. Now let $f : X \times Z^{\mathbb{N}} \to Y$ be such that

whenever $z \in Z^{\mathbb{N}}$ is such that $I = z[\mathbb{N}]$ is infinite, $g_I(z) = (\langle a_n\rangle_{n\in\mathbb{N}}, \langle y_n\rangle_{n\in\mathbb{N}})$ and $x \in X$ is such that there is just one n for which $a_n = x \cap I$, then $f(x, z) = y_n$.

In this case, if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct members of X and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y, let I be a countably infinite subset of Z containing every y_n and such that $x_m \cap I \ne x_n \cap I$ for $m < n$; let $z \in I^{\mathbb{N}}$ be such that $g_I(z) = (\langle x_n \cap I\rangle_{n\in\mathbb{N}}, \langle y_n\rangle_{n\in\mathbb{N}})$; we shall have $f(x_n, z) = y_n$ for every n. **Q**

(h) If κ is an infinite cardinal, then 2^{κ} is at most the cardinal power $(\sup_{\lambda<\kappa} 2^{\lambda})^{\operatorname{cf}\kappa}$. **P** Let $\langle\alpha_{\xi}\rangle_{\xi<\operatorname{cf}\kappa}$ be a family in κ with supremum κ. Set $D = \bigcup_{\alpha<\kappa}\mathcal{P}\alpha$; then

$$\#(D) \le \max(\kappa, \sup_{\alpha<\kappa} 2^{\#(\alpha)}) = \sup_{\lambda<\kappa} 2^{\lambda}.$$

Let F be the set of functions from $\operatorname{cf}\kappa$ to D; we have an injection $A \mapsto \langle A \cap \alpha_{\xi}\rangle_{\xi<\operatorname{cf}\kappa}$ from $\mathcal{P}\kappa$ to F, so $2^{\kappa} \le \#(F)$. **Q**

If $\omega \le \lambda < \kappa$ and $2^{\theta} = 2^{\lambda}$ for $\lambda \le \theta < \kappa$ but $2^{\kappa} > 2^{\lambda}$ then κ is regular. **P** $2^{\kappa} \le (2^{\lambda})^{\operatorname{cf}\kappa} = 2^{\max(\lambda,\operatorname{cf}\kappa)}$. **Q**

5A1F Three fairly simple facts (a) There is a family $\langle a_I\rangle_{I\subseteq\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $a_I \cap a_J$ is finite whenever I, $J \subseteq \mathbb{N}$ are distinct.

(b) Let X be a set, $f : [X]^{<\omega} \to [X]^{\le\omega}$ a function, and $Y \subseteq X$. Then there is a $Z \subseteq X$ such that $Y \subseteq Z$, $f(I) \subseteq Z$ for every $I \in [Z]^{<\omega}$, and $\#(Z) \le \max(\omega, \#(Y))$.

(c) Let $\kappa \ge \mathfrak{c}$ be a cardinal and $\mathcal{A}$ a family of countable subsets of κ such that $\#(\mathcal{A})$ is less than the cardinal power κ^{ω}. Then there is a countably infinite $K \subseteq \kappa$ such that $I \cap K$ is finite for every $I \in \mathcal{A}$.

proof (a) For each $n \in \mathbb{N}$, set $K_n = \{i : 2^n \le i < 2^{n+1}$, and let $f_n : \mathcal{P}n \to K_n$ be a bijection; set $a_I = \{f_n(I \cap n) : n \in \mathbb{N}\}$. (Or apply 5A1Mc below with $\kappa = \omega$.)

(b) Define $\langle Z_n\rangle_{n\in\mathbb{N}}$ inductively by setting $Z_0 = Y$ and $Z_{n+1} = Z_n \cup \bigcup\{f(I) : I \in [Z_n]^{<\omega}\}$ for each n. Then $\#(Z_n) \le \max(\omega, \#(Y))$ for each n, so setting $Z = \bigcup_{n\in\mathbb{N}} Z_n$ we still have $\#(Z) \le \max(\omega, \#(Y))$, while $f(I) \subseteq Z$ for every $I \in [Z]^{<\omega}$.

(c) If $\#(\mathcal{A}) < \kappa$ this is trivial, as we can take $K \subseteq \kappa \setminus \bigcup\mathcal{A}$. Otherwise, let $\lambda \le \kappa$ be the least cardinal such that $\#(\mathcal{A}) < \lambda^{\omega}$. Then $\operatorname{cf}\lambda = \omega$. **P?** Otherwise,

$$\lambda^{\omega} = \max(\lambda, \sup_{\theta<\lambda}\theta^{\omega}) \le \#(\mathcal{A}). \textbf{ XQ}$$

Let $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of cardinals with supremum λ, starting from $\lambda_0 = 0$ and $\lambda_1 = \omega$ (of course $\lambda > \omega$ because $\#(\mathcal{A}) \ge \mathfrak{c}$). For $n \in \mathbb{N}$ let $\phi_n : [n \times \lambda_n]^{<\omega} \to \lambda_{n+1} \setminus \lambda_n$ be an injective function. For $f : \mathbb{N} \to \lambda$ define $C_f \subseteq \lambda$ by setting

$$C_f = \{\phi_n(f \cap (n \times \lambda_n)) : n \in \mathbb{N}\}.$$

If f, $g \in \lambda^{\mathbb{N}}$ are distinct, then there are an $i \in \mathbb{N}$ such that $f(i) \ne g(i)$ and an $m > i$ such that both $f(i)$ and $g(i)$ are less than λ_m, so that $f \cap (n \times \lambda_n) \ne g \cap (n \times \lambda_n)$ for every $n \ge m$ and $C_f \cap C_g$ is finite. It follows that for any $I \in \mathcal{A}$ the set $B_I = \{f : f \in \lambda^{\mathbb{N}}, C_f \cap I$ is infinite$\}$ has cardinal at most $\mathfrak{c}$. Since $\max(\#(\mathcal{A}), \mathfrak{c}) < \lambda^{\omega}$, there must be an $f \in \lambda^{\mathbb{N}}$ such that $C_f \cap I$ is finite for every $I \in \mathcal{A}$, and we can set $K = C_f$.

5A1G Partition calculus (a) The Erdős-Rado theorem Let κ be an infinite cardinal. Set $\kappa_1 = \kappa$, $\kappa_{n+1} = 2^{\kappa_n}$ for $n \ge 1$. If $n \ge 1$, $\#(B) \le \kappa$, $\#(A) > \kappa_n$ and $f : [A]^n \to B$ is a function, there is a $C \in [A]^{\kappa^+}$ such that f is constant on $[C]^n$. (Erdős Hajnal Máté & Rado 84, 16.5; Kanamori 03, 7.3; Just & Weese 97, 15.13.)

(b) Let κ be a cardinal of uncountable cofinality, and $Q \subseteq [\kappa]^2$. Then *either* there is a stationary $A \subseteq \kappa$ such that $[A]^2 \subseteq Q$ *or* there is an infinite closed $B \subseteq \kappa$ such that $[B]^2 \cap Q = \emptyset$. **P** (Cf. Erdős Hajnal Máté & Rado 84, 11.3.) Let $C \subseteq \kappa$ be a closed cofinal set with $\operatorname{otp}(C) = \operatorname{cf}\kappa$ (5A1Ad). Let S_0 be $\{\alpha : \alpha \in C, \operatorname{cf}\alpha = \omega\}$, so that S_0 is stationary (5A1Ac). For each $\alpha \in S_0$ let $\langle f_{\alpha}(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in α with supremum α. Set

$$\mathcal{I}_\alpha = \{I : I \subseteq \alpha\cap C,\ [I\cup\{\alpha\}]^2\cap Q=\emptyset,\ \#(I\cap f_\alpha(n))\le n \text{ for every } n\in\mathbb{N}\}.$$

Let I_α be a maximal member of $\mathcal{I}_\alpha$. If there is any α such that I_α is infinite, we have the second alternative, witnessed by $B=I_\alpha\cup\{\alpha\}$, and we can stop. Otherwise, there is an $n\in\mathbb{N}$ such that $S_1=\{\alpha:\alpha\in S_0,\ I_\alpha\subseteq f_\alpha(n)\}$ is stationary. As $f_\alpha(n)<\alpha$ for every $\alpha\in S_1$, the Pressing-Down Lemma (4A1Cc) tells us that there is a $\gamma<\kappa$ such that $S_2=\{\alpha:\alpha\in S_1,\ f_\alpha(n)=\gamma\}$ is stationary. Because

$$\#([\gamma\cap C]^{<\omega})\le\max(\omega,\#(\gamma\cap C))<\operatorname{cf}\kappa,$$

there is an $I\subseteq\gamma\cap C$ such that $A=\{\alpha:\alpha\in S_2,\ I_\alpha=I\}$ is stationary.

? Suppose, if possible, that $[A]^2\not\subseteq Q$. Take α, $\beta\in A$ such that $\alpha<\beta$ and $\{\alpha,\beta\}\notin Q$. We know that $[I\cup\{\alpha\}]^2$ and $[I\cup\{\beta\}]^2$ are both disjoint from Q. So $[J\cup\{\beta\}]^2$ is disjoint from Q, where $J=I\cup\{\alpha\}$. If $m\le n$,

$$f_\beta(m)\le f_\beta(n)=\gamma=f_\alpha(n)<\alpha,$$

so $\#(J\cap f_\beta(m))=\#(I_\beta\cap f_\beta(m))\le m$; while if $m>n$ then $\#(J\cap f_\beta(m))\le\#(I_\beta\cap f_\beta(n))+1\le m$. So $J\in\mathcal{I}_\beta$; but J properly includes I_β, so this is impossible. **X**

Thus $[A]^2\subseteq Q$ and we have the first alternative. **Q**

5A1H Δ-systems and free sets: Proposition Let κ and λ be infinite cardinals and $\langle I_\xi\rangle_{\xi<\kappa}$ a family of sets of size less than λ.

(a) If $\operatorname{cf}\kappa>\lambda$, there are a $\Gamma\in[\kappa]^\kappa$ and a set J of cardinal less than κ such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$.

(b) If $\kappa>\lambda$ is regular and the cardinal power θ^δ is less than κ for every $\theta<\kappa$ and $\delta<\lambda$, then there is a $\Gamma'\in[\kappa]^\kappa$ such that $\langle I_\xi\rangle_{\xi\in\Gamma'}$ is a Δ-system.

(c) If $\kappa>\lambda$ there is a $\Gamma''\in[\kappa]^\kappa$ such that $\eta\notin I_\xi$ for any distinct ξ, $\eta\in\Gamma''$.

proof (a) ? Otherwise, choose $\langle\Gamma_\alpha\rangle_{\alpha<\lambda}$ and $\langle J_\alpha\rangle_{\alpha<\lambda}$ as follows. $J_\alpha=\bigcup_{\beta<\alpha}\bigcup_{\xi\in\Gamma_\beta}I_\xi$. Given J_α, let $\Gamma_\alpha\subseteq\kappa$ be maximal subject to the requirement that $I_\xi\cap I_\eta\subseteq J_\alpha$ for all distinct ξ, $\eta\in\Gamma_\alpha$. Then we see by induction that $\#(J_\alpha)<\kappa$ so $\#(\Gamma_\alpha)<\kappa$ for every $\alpha<\lambda$; because $\operatorname{cf}\kappa>\lambda$, $\bigcup_{\alpha<\lambda}\Gamma_\alpha$ cannot be the whole of κ.

Take any $\xi\in\kappa\setminus\bigcup_{\alpha<\lambda}\Gamma_\alpha$. As $\#(I_\xi)<\lambda$, there must be an $\alpha<\lambda$ such that $I_\xi\cap J_\alpha=I_\xi\cap J_{\alpha+1}$. As $\xi\notin\Gamma_\alpha$, there is an $\eta\in\Gamma_\alpha$ such that $I_\xi\cap I_\eta\not\subseteq J_\alpha$; but now $I_\xi\cap I_\eta\setminus J_\alpha\subseteq I_\xi\cap J_{\alpha+1}\setminus J_\alpha$. **X**

(b) Let J and Γ be as in (a). Because $\operatorname{cf}\kappa>\lambda$, there must be some cardinal $\delta<\lambda$ such that $\Gamma_1=\{\xi:\xi\in\Gamma,\ \#(I_\xi\cap J)\le\delta\}$ has cardinal κ. Now $\#([J]^{\le\delta})\le\#(J)^\delta<\operatorname{cf}\kappa$, so there must be a $K\subseteq J$ such that $\Gamma'=\{\xi:\xi\in\Gamma_1,\ I_\xi\cap J=K\}$ has cardinal κ; and $\langle I_\xi\rangle_{\xi\in\Gamma'}$ is a Δ-system with root K.

(c) It is enough to consider the case in which $\xi\in I_\xi$ for every $\xi<\kappa$.

(i) If $\operatorname{cf}\kappa\ge\lambda^+$, take Γ and J from (a). Then we can choose $\langle\xi_\delta\rangle_{\delta<\kappa}$ inductively so that

$$\xi_\delta\in\Gamma\setminus(J\cup\bigcup_{\beta<\delta}I_{\xi_\beta})$$

for every $\delta<\kappa$; and $\{\xi_\delta:\delta<\kappa\}$ will serve for Γ''.

(ii) If $\operatorname{cf}\kappa=\theta\le\lambda$, let $\langle\kappa_\alpha\rangle_{\alpha<\theta}$ be a strictly increasing family of regular cardinals with supremum κ, starting from $\kappa_0\ge\lambda^{++}$. For each $\alpha<\theta$, (i) tells us that there is an $A_\alpha\in[\kappa_\alpha]^{\kappa_\alpha}$ such that $\eta\notin I_\xi$ for any distinct ξ, $\eta\in A_\alpha$. Set

$$B_\alpha=A_\alpha\setminus\bigcup_{\beta<\alpha}(B_\beta\cup\bigcup_{\xi\in A_\beta}I_\xi);$$

then $\#(B_\alpha)=\kappa_\alpha$ for each $\alpha<\theta$. Choose $\langle C_{\alpha\gamma}\rangle_{\alpha<\theta,\gamma<\lambda^+}$ and $\langle\zeta_\alpha\rangle_{\alpha<\theta}$ inductively, as follows. Given that $\langle C_{\beta\gamma}\rangle_{\beta<\alpha,\gamma<\lambda^+}$ is disjoint, then for each $\xi\in B_\alpha$ there is a $\zeta<\lambda^+$ such that $I_\xi\cap\bigcup_{\beta<\alpha}C_{\beta\gamma}$ is empty for every $\gamma\ge\zeta$; because $\lambda^+<\operatorname{cf}\kappa_\alpha$, there is a $\zeta_\alpha<\lambda^+$ such that

$$B'_\alpha=\{\xi:\xi\in B_\alpha,\ I_\xi\cap C_{\beta\gamma}=\emptyset \text{ whenever } \beta<\alpha \text{ and } \zeta_\alpha\le\gamma<\lambda^+\}$$

has cardinal κ_α. Let $\langle C_{\alpha\gamma}\rangle_{\gamma<\lambda^+}$ be a partition of B'_α into sets of size κ_α, and continue.

At the end of the induction, $\gamma=\sup_{\alpha<\theta}\zeta_\alpha$ is less than λ^+. Set $\Gamma''=\bigcup_{\alpha<\theta}C_{\alpha\gamma}$. Then $\#(\Gamma'')=\kappa$. If ξ, η are distinct members of Γ'', let α, $\beta<\theta$ be such that $\xi\in C_{\alpha\gamma}$ and $\eta\in C_{\beta\gamma}$. If $\alpha<\beta$ then $\xi\in A_\alpha$ and $\eta\in B_\beta$ so $\eta\notin I_\xi$. If $\alpha=\beta$ then both ξ and η belong to A_α so $\eta\notin I_\xi$. If $\beta<\alpha$ then $\eta\in C_{\beta\gamma}$ while $\gamma\ge\zeta_\alpha$ and $\xi\in B'_\alpha$, so $\eta\notin I_\xi$. So Γ'' will serve.

Remark (c) above is Hajnal's Free Set Theorem.

5A1I (a) I spell out the applications of these results which are used in this volume. Let κ be an infinite cardinal and $\langle I_\xi\rangle_{\xi<\kappa}$ a family of countable sets.

(i) If $\operatorname{cf}\kappa\ge\omega_2$, there are a $\Gamma\in[\kappa]^\kappa$ and a set J of cardinal less than κ such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$.

(ii) If κ is infinite and regular and the cardinal power λ^ω is less than κ for every $\lambda < \kappa$, there is a $\Gamma' \in [\kappa]^\kappa$ such that $\langle I_\xi \rangle_{\xi \in \Gamma'}$ is a Δ-system.

(iii) If $\kappa \ge \omega_2$ there is a $\Gamma'' \in [\kappa]^\kappa$ such that $\eta \notin I_\xi$ for any distinct ξ, $\eta \in \Gamma''$.

(b) If, in 5A1Hc, we are willing to settle for a weaker result, there is an easier proof which generalizes to more complex systems. Let λ be an infinite cardinal. Then there is a κ_0 such that for every cardinal $\kappa \ge \kappa_0$, every $n \in \mathbb{N}$ and every function $f : [\kappa]^n \to [\kappa]^{<\lambda}$ there is an $A \in [\kappa]^{\lambda^+}$ such that $\xi \notin f(I)$ whenever $I \in [A]^n$ and $\xi \in A \setminus I$. **P** By the Erdős-Rado theorem (5A1Ga), there is a κ_0 such that for every $\kappa \ge \kappa_0$, $n \ge 1$ and function $g : [\kappa]^n \to \mathbb{N}$ there is an $A \in [\kappa]^{\lambda^+}$ such that g is constant on $[A]^n$. Now, given $n \in \mathbb{N}$, $\kappa \ge \kappa_0$ and $f : [\kappa]^n \to [\kappa]^{<\lambda}$, define $g : [\kappa]^{n+1} \to \mathbb{N}$ by saying that if $J = \{\xi_0, \dots, \xi_n\}$ with $\xi_0 < \xi_1 < \dots < \xi_n$, then $g(J) = \min(\{n+1\} \cup \{j : j \le n,\, \xi_j \in f(J \setminus \{\xi_j\})\})$. Let $A \in [\kappa]^{\lambda^+}$ be such that g is constant on $[A]^{n+1}$. We can suppose that A has order type λ^+. **?** Suppose that the constant value of g in $[A]^{n+1}$ is $j \le n$. Let B be the set of the first λ members of A, I_0 the set of the first j members of A and I_1 the set of the first $n-j$ members of $A \setminus B$. Then we have $g(I_0 \cup \{\xi\} \cup I_1) = j$ for every $\xi \in B \setminus I_0$, so that $B \setminus I_0 \subseteq f(I_0 \cup I_1)$; but $\#(f(I_0 \cup I_1)) < \lambda$. **X** So the constant value of g on $[A]^{n+1}$ is $n+1$, and A satisfies the required condition. **Q**

(c) In the same complex of ideas, we have an elementary fact about the case $\lambda < \kappa = \omega$. If $n \in \mathbb{N}$ and $\langle K_i \rangle_{i \in \mathbb{N}}$ is a sequence in $[\mathbb{N}]^{\le n}$, there is an infinite $\Gamma \subseteq \mathbb{N}$ such that $\langle K_i \rangle_{i \in \Gamma}$ is a Δ-system. **P** Let $K \subseteq \mathbb{N}$ be a maximal set such that $I = \{i : K \subseteq K_i\}$ is infinite; then $\{i : i \in I,\, K_i \cap L \ne \emptyset\}$ is finite for every finite $L \subseteq \mathbb{N} \setminus K$, so we can choose Γ inductively by saying that $\Gamma = \{i : i \in I,\, K_i \cap K_j = K$ whenever $j \in \Gamma \cap i\}$. **Q**

5A1J Lemma Suppose that θ, λ and κ are cardinals, with $\theta < \lambda < \operatorname{cf}\kappa$, and that S is a stationary subset of κ. Let $\langle I_\xi \rangle_{\xi \in S}$ be a family in $[\lambda]^{\le\theta}$. Then there is a set $M \subseteq \lambda$ such that $\operatorname{cf}(\#(M)) \le \theta$ and $\{\xi : \xi \in S,\, I_\xi \subseteq M\}$ is stationary in κ.

proof For $M \subseteq \lambda$, set $S_M = \{\xi : \xi \in S,\, I_\xi \subseteq M\}$. Let $M \subseteq \lambda$ be a set of minimal cardinality such that S_M is stationary in κ. Set $\delta = \#(M)$. **?** If $\operatorname{cf}\delta > \theta$, enumerate M as $\langle \alpha_\eta \rangle_{\eta < \delta}$. For each $\xi \in S_M$, set $\beta_\xi = \sup\{\eta : \alpha_\eta \in I_\xi\}$; because $\#(I_\xi) \le \theta < \operatorname{cf}\delta$, $\beta_\xi < \delta$. Because $\delta \le \lambda < \operatorname{cf}\kappa$, there is a $\beta < \delta$ such that $S' = \{\xi : \xi \in S_M,\, \beta_\xi = \beta\}$ is stationary in κ (5A1Ab). Consider $M' = \{\alpha_\eta : \eta \le \beta\}$; then $\#(M') < \#(M)$ but $S_{M'} \supseteq S'$ so is stationary in κ, contrary to the choice of M. **X**

Thus M and $S = S_M$ will serve.

5A1K Lemma Let $\langle X_i \rangle_{i \in I}$ be a non-empty family of infinite sets, with product X. Then there is a set $Y \subseteq X$, with $\#(Y) = \#(X)$, such that for every finite $L \subseteq Y$ there is an $i \in I$ such that $x(i) \ne y(i)$ for any distinct x, $y \in L$.

proof Set $\kappa = \#(X)$.

(a) We can well-order I in such a way that $\#(X_i) \le \#(X_j)$ whenever $i \le j$ in I. It will therefore be enough to deal with the case in which $I = \delta$ is an ordinal and $\#(X_\alpha) \le \#(X_\beta)$ whenever $\alpha \le \beta < \delta$. I proceed by induction on δ.

(b) If δ is finite then $\kappa = \max_{\alpha<\delta} \#(X_\alpha)$ and the result is trivial, since we can take the x_ξ to be all different at a single coordinate.

(c) Suppose there is a $\gamma < \delta$ such that $\#(\delta \setminus \gamma) < \#(\delta)$. Then, in particular, the order type of $\delta \setminus \gamma$ is less than the order type of δ. Set $I_0 = \gamma$, $I_1 = \delta \setminus \gamma$ and $Y_j = \prod_{\alpha \in I_j} X_\alpha$ for both j. Then $X \cong Y_0 \times Y_1$, so $\kappa = \max(\#(Y_0), \#(Y_1))$; say $\kappa = \#(Y_j)$. By the inductive hypothesis, there is a family $\langle y_\xi \rangle_{\xi<\kappa}$ in Y_j such that for any $L \in [\kappa]^{<\omega}$ there is an $\alpha \in I_j$ such that $\xi \mapsto y_\xi(\alpha) : L \to X_\alpha$ is injective. Taking x_ξ to be any member of X extending y_ξ, for each $\xi < \kappa$, we have a suitable family $\langle x_\xi \rangle_{\xi<\kappa}$ in X, and the induction proceeds.

(d) Suppose that δ is infinite and that $\#(\delta \setminus \gamma) = \#(\delta) = \lambda$ for every $\gamma < \delta$. Enumerate $[\delta]^{<\omega}$ as $\langle J_\zeta \rangle_{\zeta<\lambda}$, and choose $\langle \alpha_\zeta \rangle_{\zeta<\lambda}$ such that

$$J_\zeta \subseteq \alpha_\zeta \in \delta \setminus \{\alpha_\eta : \eta < \zeta\}$$

for each $\alpha < \lambda$. We have

$$\#(X_{\alpha_\zeta}) \ge \max(\omega, \sup_{\beta \in J_\zeta} \#(X_\beta)) \ge \#(\textstyle\prod_{\beta \in J_\zeta} X_\beta),$$

so there is an injective function $f_\zeta : \prod_{\beta \in J_\zeta} X_\beta \to X_{\alpha_\zeta}$ for each $\zeta < \lambda$. Let $\langle z_\xi \rangle_{\xi<\kappa}$ be any enumeration of X. Because all the α_ζ are distinct, we can find $x_\xi \in X$, for each $\xi < \kappa$, such that $x_\xi(\alpha_\zeta) = f_\zeta(z_\xi \restriction J_\zeta)$ for every ζ. Now if $L \in [\kappa]^{<\omega}$ there must be a $\zeta < \lambda$ such that $z_\xi \restriction J_\zeta \ne z_\eta \restriction J_\zeta$ for any distinct ξ, $\eta \in L$; so that $\xi \mapsto x_\xi(\alpha_\zeta)$ is injective on L. Thus $\langle x_\xi \rangle_{\xi<\kappa}$ is a suitable family in X and the induction proceeds in this case also.

5A1L Definitions (a) Let X and Y be sets and $\mathcal{I}$ an ideal of subsets of X. Write $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ for the **transversal number**

$$\sup\{\#(F) : F \subseteq Y^X,\ \{x : f(x) = g(x)\} \in \mathcal{I} \text{ for all distinct } f,\, g \in F\}.$$

(b) Let κ be a cardinal. Write $\mathrm{Tr}(\kappa)$ for

$$\mathrm{Tr}_{[\kappa]^{<\kappa}}(\kappa;\kappa) = \sup\{\#(F) : F \subseteq \kappa^\kappa,\ \#(f \cap g) < \kappa \text{ for all distinct } f,\, g \in F\}.$$

5A1M Lemma (a) For any infinite cardinal κ,

$$\kappa^+ \le \mathrm{Tr}(\kappa) \le 2^\kappa.$$

(b) For any infinite cardinal κ,

$$\max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) \ge \min(2^\kappa, \kappa^{(+\omega)}).$$

(c) If κ is such that $2^\delta \le \kappa$ for every $\delta < \kappa$, then $\mathrm{Tr}(\kappa) = 2^\kappa$, and in fact there is an $F \subseteq \kappa^\kappa$ such that $\#(F) = 2^\kappa$ and $\#(f \cap g) < \kappa$ for all distinct $f,\, g \in F$.

proof (a) We can build inductively a family $\langle f_\alpha\rangle_{\alpha<\kappa^+}$ in κ^κ, as follows. Given $\langle f_\alpha\rangle_{\alpha<\beta}$, where $\beta < \kappa^+$, let $\theta : \beta \to \kappa$ be any injection. Now choose $f_\beta : \kappa \to \kappa$ so that

$$f_\beta(\xi) \ne f_\alpha(\xi) \text{ whenever } \alpha < \beta \text{ and } \theta(\alpha) \le \xi.$$

This will mean that if $\alpha < \beta$, then

$$\{\xi : f_\alpha(\xi) = f_\beta(\xi)\} \subseteq \theta(\alpha)$$

has cardinal less than κ. So at the end of the induction, $F = \{f_\alpha : \alpha < \kappa^+\}$ will witness that $\mathrm{Tr}(\kappa) \ge \kappa^+$. On the other hand, $\mathrm{Tr}(\kappa) \le \#(\kappa^\kappa) = 2^\kappa$.

(b)? If not, then take $\lambda = \max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) < \min(2^\kappa, \kappa^{(+\omega)})$. For each $\xi < \kappa$ take an injective function $\phi_\xi : \mathcal{P}\xi \to \lambda$. Because $\lambda < 2^\kappa$, we have an injective function $h : \lambda^+ \to \mathcal{P}\kappa$. For $\alpha < \lambda^+$ set $g_\alpha(\xi) = \phi_\xi(h(\alpha) \cap \xi)$ for every $\xi < \kappa$; then $\langle g_\alpha\rangle_{\alpha<\lambda^+}$ is a family in λ^κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha \ne \beta$.

Apply 5A1J with $S = \lambda^+$, $I_\alpha = g_\alpha[\kappa]$ to see that there is a set $M \subseteq \lambda$ with $\mathrm{cf}(\#(M)) \le \kappa$ and $S_1 = \{\alpha : \alpha < \lambda^+,\ g_\alpha[\kappa] \subseteq M\}$ stationary in λ^+. Because $\lambda < \kappa^{(+\omega)}$, we must have $\#(M) \le \kappa$. If $f : M \to \kappa$ is any injection, $\langle fg_\alpha\rangle_{\alpha\in S}$ will witness that $\mathrm{Tr}(\kappa) \ge \#(S_1) = \lambda^+$; which is impossible. **X**

(c) For each $\xi < \kappa$, let $\phi_\xi : \mathcal{P}\xi \to \kappa$ be injective. For $A \subseteq \kappa$, define $f_A \in \kappa^\kappa$ by writing

$$f_A(\xi) = \phi_\xi(A \cap \xi) \text{ for every } \xi < \kappa.$$

Then $F = \{f_A : A \subseteq \kappa\}$ has the required property, and $\mathrm{Tr}(\kappa) \ge 2^\kappa$; by (a), we have equality.

5A1N Almost-square-sequences: Lemma Let κ, λ be regular infinite cardinals, with $\lambda > \max(\omega_1, \kappa)$. Then we can find a stationary set $S \subseteq \lambda^+$ and a family $\langle C_\alpha\rangle_{\alpha\in S}$ of sets such that

(i) for each $\alpha \in S$, C_α is a closed cofinal set in α of order type κ;

(ii) if $\alpha,\, \beta \in S$ and γ is a limit point of both C_α and C_β then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

proof (a) For each $\gamma < \lambda^+$ fix an injection $f_\gamma : \gamma \to \lambda$. Let S_0 be the set of ordinals $\alpha < \lambda^+$ of cofinality κ; then S_0 is stationary in λ^+ (5A1Ac). For each $\alpha \in S_0$ choose an increasing family $\langle N_{\alpha\delta}\rangle_{\delta<\lambda}$ of subsets of λ^+ such that

(α) $N_{\alpha 0}$ is a cofinal subset of α of cardinal κ;

(β) if $\delta < \lambda$ then

$$N_{\alpha,\delta+1} = \bigcup\{f_\gamma[N_{\alpha\delta}] \cup f_\gamma^{-1}[\delta] : \gamma \in N_{\alpha\delta}\} \cup \overline{N}_{\alpha\delta} \cup \delta$$

(taking the closure $\overline{N}_{\alpha\delta}$ in the order topology of λ^+);

(γ) if $\delta < \lambda$ is a non-zero limit ordinal then $N_{\alpha\delta} = \bigcup_{\delta'<\delta} N_{\alpha\delta'}$.

Then $\#(N_{\alpha\delta}) \le \max(\kappa, \#(\delta)) < \lambda$ for each $\delta < \lambda$ (using 5A1Ae). Because λ is regular, $\sup(N_{\alpha\delta} \cap \lambda) < \lambda$ for every δ. It follows that $\{\delta : \delta < \lambda,\ N_{\alpha\delta} \cap \lambda = \delta\}$ is a closed cofinal set in λ, and in particular contains an ordinal of cofinality ω_1, for every $\alpha \in S_0$. Let $\delta < \lambda$ be such that $\mathrm{cf}\,\delta = \omega_1$ and

$$S_1 = \{\alpha : \alpha \in S_0,\ N_{\alpha\delta} \cap \lambda = \delta\}$$

is stationary in λ^+. For $\alpha \in S_1$, set $C^*_\alpha = \alpha \cap \overline{N}_{\alpha\delta}$; then C^*_α is a closed cofinal set in α and $\#(C^*_\alpha) < \lambda$ so $\mathrm{otp}(C^*_\alpha) < \lambda$. Let $\zeta < \lambda$ be such that

$$S = \{\alpha : \alpha \in S_1,\ \mathrm{otp}(C^*_\alpha) = \zeta\}$$

is stationary in λ^+. Observe that as $\operatorname{cf} C^*_\alpha = \operatorname{cf}\alpha = \kappa$ for each $\alpha \in S$, $\operatorname{cf}\zeta = \kappa$.

(b) Take any closed cofinal set $C \subseteq \zeta$ of order type κ and for each $\alpha \in S$ let C_α be the image of C in C^*_α under the order-isomorphism between ζ and C^*_α. Then C_α will be a closed cofinal subset of α of order type κ.

I claim that if α, $\beta \in S$ and γ is a common limit point of C_α, C_β then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

P case 1 Suppose $\kappa = \omega$. In this case the only limit point of C_α will be α itself, and similarly for β, so that in this case we have $\alpha = \beta$ and there is nothing more to do.

case 2 Suppose $\operatorname{cf}\gamma = \omega < \kappa$. Then γ is a limit point of $C_\alpha \subseteq C^*_\alpha \subseteq \overline{N}_{\alpha\delta}$, so there is an increasing sequence in $N_{\alpha\delta}$ with supremum γ; as $N_{\alpha\delta} = \bigcup_{\delta'<\delta} N_{\alpha\delta'}$ and $\operatorname{cf}\delta = \omega_1$, this sequence lies entirely within $N_{\alpha\delta'}$ for some $\delta' < \delta$, and $\gamma \in \overline{N}_{\alpha\delta'} \subseteq N_{\alpha,\delta'+1}$. Now, for $\delta'+1 \le \xi < \delta$, $N_{\alpha,\xi+1} \supseteq f_\gamma^{-1}[\xi] \cup f_\gamma[N_{\alpha\xi}]$; consequently

$$N_{\alpha\delta} \cap \gamma = f_\gamma^{-1}[N_{\alpha\delta} \cap \lambda] = f_\gamma^{-1}[\delta].$$

Similarly, $N_{\beta\delta} \cap \gamma = f_\gamma^{-1}[\delta]$. Now

$$C^*_\alpha \cap \gamma = \overline{N}_{\alpha\delta} \cap \gamma = \overline{f_\gamma^{-1}[\delta]} \cap \gamma = C^*_\beta \cap \gamma.$$

Accordingly the increasing enumerations of C^*_α and C^*_β must agree on $C^*_\alpha \cap \gamma = C^*_\beta \cap \gamma$, and $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

case 3 Suppose that $\operatorname{cf}\gamma > \omega$ and $\kappa > \omega$. Because $\gamma = \sup(C_\alpha \cap \gamma) = \sup(C_\beta \cap \gamma)$,

$$D = \{\gamma' : \gamma' < \gamma \text{ is a limit point of both } C_\alpha \text{ and } C_\beta,\ \operatorname{cf}\gamma' = \omega\}$$

is cofinal with γ, and

$$C_\alpha \cap \gamma = \bigcup_{\gamma' \in D} C_\alpha \cap \gamma' = C_\beta \cap \gamma,$$

using case 2. **Q**

Thus S and $\langle C_\alpha\rangle_{\alpha\in S}$ have the required properties.

5A1O Corollary Let κ, λ be regular infinite cardinals with $\lambda > \max(\omega_1, \kappa)$. Then we can find a stationary subset S of λ^+ and a family $\langle g_\alpha\rangle_{\alpha\in S}$ of functions from κ to λ^+ such that, for all distinct α, $\beta \in S$,

(i) $g_\alpha[\kappa] \subseteq \alpha$,

(ii) $\#(g_\alpha \cap g_\beta) < \kappa$,

(iii) if $\theta < \kappa$ is a limit ordinal and $g_\alpha(\theta) = g_\beta(\theta)$ then $g_\alpha{\restriction}\theta = g_\beta{\restriction}\theta$.

proof Take $\langle C_\alpha\rangle_{\alpha\in S}$ from 5A1N above and let g_α be the increasing enumeration of C_α.

5A1P The generalized continuum hypothesis (a) The generalized continuum hypothesis is the assertion

$$\text{(GCH)}\quad 2^\kappa = \kappa^+ \text{ for every infinite cardinal } \kappa.$$

(b) If GCH is true, then for infinite cardinals κ, λ

$$\begin{aligned}\operatorname{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \kappa \le \lambda,\\ &= \kappa \text{ if } \lambda < \operatorname{cf}\kappa,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

P If $\kappa \le \lambda$, use 5A1E(e-i). If $\lambda < \kappa$ then

$$\operatorname{cf}[\kappa]^{\le\lambda} \le \#([\kappa]^{\le\lambda}) \le \#(\mathcal{P}\kappa) = 2^\kappa = \kappa^+.$$

If $\lambda < \theta = \operatorname{cf}\kappa$, then $[\kappa]^{\le\lambda} = \bigcup_{\xi<\kappa}[\xi]^{\le\lambda}$ so

$$\operatorname{cf}[\kappa]^{\le\lambda} \le \max(\kappa, \sup_{\xi<\kappa} \operatorname{cf}[\xi]^{\le\lambda}) \le \max(\kappa, \sup_{\xi<\kappa} \#(\xi)^+) = \kappa$$

and we have equality (using the other part of 5A1E(e-i)). If $\lambda = \theta$ then 5A1E(e-v) tells us that $\operatorname{cf}[\kappa]^\lambda$ is greater than κ, so must be κ^+. If $\theta < \lambda < \kappa$ then, by 5A1E(e-ii),

$$\kappa < \operatorname{cf}[\kappa]^{\le\theta} \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta}) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \lambda^+) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \kappa),$$

so again $\operatorname{cf}[\kappa]^{\le\lambda} = \kappa^+$. **Q**

(c) If GCH is true, then for infinite cardinals κ and λ, the cardinal power κ^λ is 2^λ if $\kappa \le \lambda$, κ if $\lambda < \operatorname{cf}\kappa$, κ^+ otherwise. (Put (b) and 5A1E(e-iii) together.)

5A1Q L, $0^\sharp$ and Jensen's Covering Lemma (a)(i) Let L be the class of **constructible** sets (Jech 03, §13; Jech 78, §12; Kanamori 03, §3; Kunen 80, chap. VI).

(ii) The **axiom of constructibility** is '$V{=}L$', 'every set is constructible'. $V{=}L$ implies GCH (Jech 03, 13.20; Jech 78, Theorem 34; Kunen 80, §VI.4).

(iii) I will call on the following three properties of L in the remarks below. To make sense of them you will of course need to look at the proper definition. Only the third has any real content. Every ordinal belongs to L; if A, $B \in L$ then $A \cap B \in L$; if κ is a cardinal, then $\#(L \cap \mathcal{P}\kappa) \le \kappa^+$.

(b) $0^\sharp$, if it exists, is a set of sentences in a countable formal language (Jech 03, §18; Kanamori 03, §9). I will not attempt to explain further; I mention $0^\sharp$ only so that you will be able to explore the literature for proofs of the assertions below. I will write '$\exists 0^\sharp$' for the assertion '$0^\sharp$ exists'.

Jensen's Covering Lemma is the assertion

(CL) for every uncountable set A of ordinals, there is a constructible set of the same cardinality including A.

Now Jensen's Covering Theorem is

$$\text{CL iff not-}\exists 0^\sharp$$

(Jech 03, Theorem 18.30.)

(c) The importance to us of $0^\sharp$ is that there are relatively direct proofs that $V{=}L$ implies not-$\exists 0^\sharp$ (Jech 03, §18), and that not-$\exists 0^\sharp$ is true in any set forcing extension of a model of not-$\exists 0^\sharp$; see Jech 03, Exercise 18.2 or Jech 78, Exercise 30.2. So CL implies that $\Vdash_{\mathbb{P}}$ CL for every forcing notion $\mathbb{P}$ of the kind considered in §5A3.

5A1R Theorem Assume that CL is true.

(a) For infinite cardinals κ and λ,

$$\begin{aligned}\operatorname{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \kappa \le \lambda,\\ &= \kappa \text{ if } \lambda < \operatorname{cf}\kappa,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

(b) If κ and λ are infinite cardinals, then the cardinal power κ^λ is 2^λ if $\kappa \le 2^\lambda$, κ if $\lambda < \operatorname{cf}\kappa$ and $2^\lambda \le \kappa$, and κ^+ otherwise.

proof (a)(i) If $\omega_1 \le \lambda < \kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa^+$. **P** By CL, every $A \in [\kappa]^\lambda$ is included in a $B \in [L]^\lambda$; now $\kappa \in L$ so $B \cap \kappa \in L$ and $A \subseteq B \cap \kappa \in [\kappa]^\lambda$. Thus $L \cap [\kappa]^\lambda$ is cofinal with $[\kappa]^\lambda$ and $[\kappa]^{\le\lambda}$. But $\#(L \cap \mathcal{P}\kappa) \le \kappa^+$ (5A1Q(a-iii)), so $\operatorname{cf}[\kappa]^\lambda \le \kappa^+$. **Q**

(ii) It follows that $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa^+$ for all infinite cardinals κ and λ. **P** The case $\lambda \ge \kappa$ is trivial, so only the case $\lambda = \omega < \kappa$ remains. But

$$\operatorname{cf}[\kappa]^{\le\omega} \le \max(\operatorname{cf}[\kappa]^{\omega_1}, \operatorname{cf}[\omega_1]^{\le\omega}) \le \max(\kappa^+, \omega_1) = \kappa^+$$

by 5A1E(e-ii) and (i). **Q**

(iii) If $\lambda < \operatorname{cf}\kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa$. **P** $[\kappa]^{\le\lambda} = \bigcup_{\xi<\kappa}[\xi]^{\le\lambda}$, so

$$\operatorname{cf}[\kappa]^{\le\lambda} \le \max(\kappa, \sup_{\xi<\kappa} \operatorname{cf}[\xi]^{\le\lambda}) \le \max(\kappa, \sup_{\xi<\kappa} \#(\xi)^+) = \kappa. \text{ **Q**}$$

(iv) If $\operatorname{cf}\kappa \le \lambda < \kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} > \kappa$. **P** Set $\theta = \operatorname{cf}\kappa$. Then

$$\begin{aligned}\kappa < \operatorname{cf}[\kappa]^{\le\theta} &\le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta})\\ &\le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \lambda^+) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \kappa)\end{aligned}$$

(5A1E(e-v), 5A1E(e-ii), (ii) above), so $\operatorname{cf}[\kappa]^{\le\lambda} > \kappa$. **Q**

Putting this together with 5A1E(e-i), (ii) and (iii) we have the result.

(b) As 5A1Pc.

5A1S Square principles (a)(i) Let Sing be the class of non-zero limit ordinals which are not regular cardinals. Global Square is the statement

there is a family $\langle C_\xi\rangle_{\xi\in\mathrm{Sing}}$ such that
for every $\xi \in \mathrm{Sing}$, C_ξ is a closed cofinal set in ξ;
$\operatorname{otp} C_\xi < \xi$ for every $\xi \in \mathrm{Sing}$;
if $\xi \in \mathrm{Sing}$ and $\zeta > 0$ is such that $\zeta = \sup(\zeta \cap C_\xi)$, then $\zeta \in \mathrm{Sing}$ and $C_\zeta = \zeta \cap C_\xi$.

(ii) For an infinite cardinal κ, let $\square_\kappa$ be the statement

there is a family $\langle C_\xi\rangle_{\xi<\kappa^+}$ of sets such that

for every $\xi<\kappa^+$, $C_\xi\subseteq\xi$ is a closed cofinal set in ξ;

if $\operatorname{cf}\xi<\kappa$ then $\#(C_\xi)<\kappa$;

whenever $\xi<\kappa^+$ and $\zeta<\xi$ is such that $\zeta=\sup(\zeta\cap C_\xi)$, then $C_\zeta=\zeta\cap C_\xi$.

(b) The axiom of constructibility implies Global Square (Friedman & Koepke 97). Global Square implies that $\square_\kappa$ is true for every infinite cardinal κ (Devlin 84, VI.6.2). Jensen's Covering Lemma implies that $\square_\kappa$ is true for every singular infinite cardinal κ (Devlin 84, V.5.6).

(c) If κ is an uncountable cardinal and $\langle C_\xi\rangle_{\xi<\kappa^+}$ is a family as in (a-ii), then $\operatorname{otp}C_\xi\le\kappa$ for every $\xi<\kappa^+$. **P** If $\operatorname{cf}\xi<\kappa$ this is immediate from the second clause of $\square_\kappa$. Otherwise, κ is regular, and $\operatorname{cf}C_\xi=\kappa$. **?** If $\operatorname{otp}C_\xi>\kappa$ then $\operatorname{otp}C_\xi>\kappa+\omega$ so there is a $\zeta\in C_\xi$ such that $\operatorname{otp}(\zeta\cap C_\xi)=\kappa+\omega$; but now $\operatorname{cf}\zeta=\omega<\kappa$ and $C_\zeta=\zeta\cap C_\xi$ has cardinal κ, which is not allowed. **XQ**

5A1T Lemma Suppose that κ is an uncountable cardinal with countable cofinality such that $\square_\kappa$ is true. Then there is a family $\langle I_\xi\rangle_{\xi<\kappa^+}$ of countably infinite subsets of κ such that

$I_\xi\cap I_\eta$ is finite whenever $\eta<\xi<\kappa^+$,

$\{\xi:\xi<\kappa^+,\,I\cap I_\xi$ is infinite$\}$ is countable for every countable $I\subseteq\kappa$.

proof Let $\langle C_\xi\rangle_{\xi<\kappa^+}$ be a family as in 5A1S(a-ii). Let $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ be a sequence of infinite cardinals less than κ with supremum κ. Define $\langle f_\xi\rangle_{\xi<\kappa^+}$ in $\prod_{n\in\mathbb{N}}\kappa_n^+$ as follows. $f_0(n)=0$ for every n. Given f_ξ, set $f_{\xi+1}(n)=f_\xi(n)+1$ for every n. Given $\langle f_\eta\rangle_{\eta<\xi}$, where $\xi<\kappa^+$ is a non-zero limit ordinal, set

$$f_\xi(n)=\sup\{f_\eta(n):\eta\in C_\xi,\,\#(\eta\cap C_\xi)<\kappa_n\}$$

for each n; because $\{\eta:\eta\in C_\xi,\,\#(\eta\cap C_\xi)<\kappa_n\}$ has cardinal at most κ_n, $f_\xi(n)<\kappa_n^+$. Continue.

We find that if $\eta<\xi<\lambda^+$ then $\{n:f_\xi(n)\le f_\eta(n)\}$ is finite. **P** Induce on ξ. If $\xi=0$ there is nothing to prove. If $\xi=\zeta+1$ then $\{n:f_\xi(n)\le f_\eta(n)\}=\{n:f_\zeta(n)<f_\eta(n)\}$ is finite. If ξ is a non-zero limit ordinal, let $\zeta\in C_\xi$ be such that $\eta<\zeta$. Because $\operatorname{otp}C_\xi\le\kappa$ (5A1Sc), $\#(\zeta\cap C_\xi)<\kappa_m$ for some m. Now $f_\xi(n)\ge f_\zeta(n)$ for every $n\ge m$, so $\{n:f_\xi(n)\le f_\eta(n)\}\subseteq m\cup\{n:f_\zeta(n)\le f_\eta(n)\}$ is finite. **Q**

If $I\subseteq\mathbb{N}\times\kappa$ is countable, then $B=\{\xi:\xi<\kappa^+,\,I\cap f_\xi$ is infinite$\}$ is countable, where in this formula I am identifying f_ξ with its graph, as usual. **P?** Otherwise, let $B'\subseteq B$ be a set with order type ω_1, and set $\xi=\sup B'<\kappa^+$. Set

$$I'=\{(n,\alpha):(n,\alpha)\in I,\,\alpha\le f_\eta(n)\text{ for some }\eta\in C_\xi\}.$$

Because I and I' are countable, while $\operatorname{cf}C_\xi=\operatorname{cf}\xi=\omega_1$, there is a $\zeta\in C_\xi$ such that $\zeta=\sup(\zeta\cap C_\xi)$ and

$$I'=\{(n,\alpha):(n,\alpha)\in I,\,\alpha\le f_\eta(n)\text{ for some }\eta\in\zeta\cap C_\xi\}.$$

Take $\eta\in B'$ such that $\eta>\zeta$, and $\zeta'\in C_\xi$ such that $\zeta'>\eta$. Then there is an $m\in\mathbb{N}$ such that $\#(\zeta\cap C_\xi)<\kappa_m$ and $f_\zeta(n)<f_\eta(n)<f_{\zeta'}(n)$ for every $n\ge m$. As $\eta\in B$, there is an $n\ge m$ such that $(n,f_\eta(n))\in I$; as $f_\eta(n)<f_{\zeta'}(n)$, $(n,f_\eta(n))\in I'$ and there is an $\eta'\in\zeta\cap C_\xi$ such that $f_\eta(n)\le f_{\eta'}(n)$. But now we have $\eta'\in C_\zeta$ and $\#(\eta'\cap C_\zeta)\le\#(C_\zeta)<\kappa_n$ and $f_\zeta(n)<f_{\eta'}(n)$, contrary to the choice of f_ζ. **XQ**

Thus if we set $I_\xi=f_\xi$ for $\xi<\kappa^+$ we have an appropriate family of sets in $\mathbb{N}\times\kappa^+$ which can be transferred to κ^+ by any bijection.

5A1U Chang's transfer principle (a) If λ_0, λ_1, κ_0 and κ_1 are cardinals, then $(\kappa_1,\lambda_1)\twoheadrightarrow(\kappa_0,\lambda_0)$ means

whenever $f:[\kappa_1]^{<\omega}\to\lambda_1$ is a function, there is an $A\in[\kappa_1]^{\kappa_0}$ such that $\#(f[[A]^{<\omega}])\le\lambda_0$.

For the original model-theoretic version of this principle, and the proof that it comes to the same thing, see Kanamori 03, 8.1. For various combinatorial consequences, see 'Chang's conjecture' in Erdős Hajnal Máté & Rado 84.

In this book, I write $\mathrm{CTP}(\kappa,\lambda)$ for the statement

$$(\kappa,\lambda)\twoheadrightarrow(\omega_1,\omega).$$

What is commonly called 'Chang's conjecture' is $\mathrm{CTP}(\omega_2,\omega_1)$. For a model of GCH + $\mathrm{CTP}(\omega_{\omega+1},\omega_\omega)$, see Levinski Magidor & Shelah 90.

(b) Suppose that $\mathrm{CTP}(\kappa,\lambda)$ is true.

(i) If $f:[\kappa]^{<\omega}\to[\lambda]^{\le\omega}$ is a function, then there is an uncountable $A\subseteq\kappa$ such that $\bigcup\{f(I):I\in[A]^{<\omega}\}$ is countable. **P** Enumerate $\mathbb{N}\times\mathbb{N}$ as $\langle(k_n,m_n)\rangle_{n\in\mathbb{N}}$ in such a way that $m_n\le n$ for every $n\in\mathbb{N}$. For $I\in[\kappa]^{<\omega}$ let $\langle f_k(I)\rangle_{k\in\mathbb{N}}$ be a sequence running over $f(I)\cup\{0\}$. (I am passing over the trivial case $\lambda=0$.) Now, for $n\in\mathbb{N}$ and $I\in[\kappa]^n$, enumerate I in ascending order as $\langle\xi_i\rangle_{i<n}$ and set $g(I)=f_{k_n}(\{\xi_i:i<m_n\})$. There is an uncountable $A\subseteq\kappa$

such that $B=\{g(I):I\in[A]^{<\omega}\}$ is countable; we may suppose that A has order type ω_1. If $J\in[A]^{<\omega}$ and $k\in\mathbb{N}$, let $n\in\mathbb{N}$ be such that $k_n=k$ and $m_n=\#(J)$; let $I\in[A]^n$ be such that J consists of the first m_n elements of I; then $f_k(J)=g(I)$ belongs to B. As J and k are arbitrary, $\bigcup\{f(I):I\in[A]^{<\omega}\}\subseteq B$ is countable. **Q**

(ii) If $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of countable subsets of λ, then there is a countable $A\subseteq\lambda$ such that $\{\xi:A_\xi\subseteq A\}$ is uncountable. **P** In (i), take $f(I)=\bigcup_{\xi\in I}A_\xi$ for $I\in[\kappa]^{<\omega}$. **Q**

(c) CL implies that $\mathrm{CTP}(\kappa,\lambda)$ is false except when $\kappa=\omega_1$ and $\lambda=\omega$ (Jech 03, 18.29).

5A1V Todorčević's *p*-ideal dichotomy (a) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then $\mathcal{I}$ is a ***p*-ideal** if for every sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ there is an $I\in\mathcal{I}$ such that $I_n\setminus I$ is finite for every $n\in\mathbb{N}$. (Compare 538Ab.)

(b) Now Todorčević's p-ideal dichotomy is the statement

(TPID) whenever X is a set and $\mathcal{I}\subseteq[X]^{\le\omega}$ is a p-ideal of countable subsets of X, then *either* there is a $B\in[X]^{\omega_1}$ such that $[B]^{\le\omega}\subseteq\mathcal{I}$ *or* X is expressible as $\bigcup_{n\in\mathbb{N}}X_n$ where $\mathcal{I}\cap\mathcal{P}X_n\subseteq[X_n]^{<\omega}$ for every $n\in\mathbb{N}$.

This is a consequence of the Proper Forcing Axiom, and implies that $\square_\kappa$ is false for every $\kappa\ge\omega_1$ (Todorčević 00).

***5A1W Analytic *P*-ideals: Theorem** Suppose that the Proper Forcing Axiom is true. Take a non-empty set $D\subseteq[0,\infty[^{\mathbb{N}}$ and set

$$\mathcal{I}=\{I:I\subseteq\mathbb{N},\ \lim_{n\to\infty}\sup_{z\in D}\textstyle\sum_{i\in I\setminus n}z(i)=0\},$$

so that $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$. Let $\mathfrak{A}$ be the quotient Boolean algebra $\mathcal{P}\mathbb{N}/\mathcal{I}$. Then for every $\pi\in\operatorname{Aut}\mathfrak{A}$ there are sets I, $J\in\mathcal{I}$ and a bijection $h:\mathbb{N}\setminus I\to\mathbb{N}\setminus J$ representing π in the sense that $\pi(A^\bullet)=(h^{-1}[A])^\bullet$ for every $A\subseteq\mathbb{N}$. (Farah 00, 3.4.6.)

5A2 Pcf theory

In §§542-543 I call on some results from Shelah's pcf theory. As I have still not found any satisfactory textbook for this material, I copy out part of the appendix of Fremlin 93, itself drawn largely from Burke & Magidor 90.

5A2A Reduced products We need the following elementary generalization of the construction in 351M. Let $\langle P_i\rangle_{i\in I}$ be a family of partially ordered sets with product P.

(a) Let $\mathcal{F}$ be a filter on I. We have an equivalence relation $\equiv_{\mathcal{F}}$ on P, given by saying that $f\equiv_{\mathcal{F}}g$ if $\{i:f(i)=g(i)\}\in\mathcal{F}$. I write $P|\mathcal{F}$ for the set of equivalence classes under this relation, the **partial order reduced product** of $\langle P_i\rangle_{i\in I}$ modulo $\mathcal{F}$. Now $P|\mathcal{F}$ is again a partially ordered set, writing

$$f^\bullet\le g^\bullet\iff f\le_{\mathcal{F}}g\iff\{i:f(i)\le g(i)\}\in\mathcal{F}.$$

Observe that if every P_i is totally ordered and $\mathcal{F}$ is an ultrafilter, then $P|\mathcal{F}$ is totally ordered.

(b) Suppose that $P_i\ne\emptyset$ for every $i\in I$. For any filter $\mathcal{F}$ on I we have

$$\min_{i\in I}\operatorname{add}P_i=\operatorname{add}P\le\sup_{F\in\mathcal{F}}\operatorname{add}(\textstyle\prod_{i\in F}P_i)=\sup_{F\in\mathcal{F}}\min_{i\in F}\operatorname{add}P_i\le\operatorname{add}(P|\mathcal{F}),$$

$$\operatorname{cf}(P|\mathcal{F})\le\min_{F\in\mathcal{F}}\operatorname{cf}(\textstyle\prod_{i\in F}P_i)\le\operatorname{cf}P.$$

P By 511Hg, $\operatorname{add}(\prod_{i\in F}P_i)=\min_{i\in F}\operatorname{add}P_i$ for any $F\in\mathcal{F}$, and in particular when $F=I$. For $p\in P|\mathcal{F}$ choose $f_p\in P$ such that $f_p^\bullet=p$. If $F\in\mathcal{F}$ then $p\mapsto f_p{\restriction}F$ is a Tukey function from $P|\mathcal{F}$ to $\prod_{i\in F}P_i$, so $P|\mathcal{F}\preccurlyeq_{\mathrm{T}}\prod_{i\in F}P_i$ and 513Ee tells us that $\operatorname{add}(\prod_{i\in F}P_i)\le\operatorname{add}(P|\mathcal{F})$ and $\operatorname{cf}(P|\mathcal{F})\le\operatorname{cf}(\prod_{i\in F}P_i)$. Also $f\mapsto f{\restriction}F$ is a dual Tukey function from P to $\prod_{i\in F}P_i$, so $\prod_{i\in F}P_i\preccurlyeq_{\mathrm{T}}P$ and $\operatorname{cf}(\prod_{i\in I}P_i)\le\operatorname{cf}P$. **Q**

(c) Note that if $\mathcal{F}$, $\mathcal{G}$ are filters on I and $\mathcal{F}\subseteq\mathcal{G}$, then $\operatorname{add}(P|\mathcal{F})\le\operatorname{add}(P|\mathcal{G})$ and $\operatorname{cf}(P|\mathcal{F})\ge\operatorname{cf}(P|\mathcal{G})$. **P** If $f\le_{\mathcal{F}}g$ then $f\le_{\mathcal{G}}g$. So we have a canonical surjective order-preserving map $\psi:P|\mathcal{F}\to P|\mathcal{G}$ given by saying that $\psi(\pi_{\mathcal{F}}(f))=\pi_{\mathcal{G}}(f)$ for every $f\in P$. By 513E(b-iii), ψ is a dual Tukey function, so $P|\mathcal{G}\preccurlyeq_{\mathrm{T}}P|\mathcal{F}$ and we can use 513Ee again. **Q**

5A2B Theorem Let $\lambda>0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P=\prod_{\zeta<\lambda}\theta_\zeta$. For any filter $\mathcal{F}$ on λ, let $P|\mathcal{F}$ be the corresponding reduced product and $\pi_{\mathcal{F}}:P\to P|\mathcal{F}$ the canonical map. For any cardinal δ set

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda, \operatorname{cf}(P|\mathcal{F}) = \delta\},$$

$$\mathfrak{F}^*_\delta = \bigcup_{\delta' \ge \delta} \mathfrak{F}_{\delta'};$$

if $\mathfrak{F}^*_\delta \ne \emptyset$, let $\mathcal{G}_\delta$ be the filter $\bigcap \mathfrak{F}^*_\delta$. Now

(a) if $\mathfrak{F}^*_\delta \ne \emptyset$, then $\operatorname{add}(P|\mathcal{G}_\delta) \ge \delta$;

(b) for every δ there is a set $F \in [P]^{\le\delta}$ such that $\pi_{\mathcal{F}}[F]$ is cofinal with $P|\mathcal{F}$ for every $F \in \mathfrak{F}_\delta$;

(c) $\mathfrak{F}_{\operatorname{cf} P} \ne \emptyset$.

proof The case of finite λ is trivial throughout, as then

$$\operatorname{cf} P = \max_{\zeta<\lambda} \theta_\zeta,$$

$$\mathfrak{F}_\delta = \{\mathcal{F} : \text{there is a } \zeta < \lambda \text{ such that } \{\zeta\} \in \mathcal{F} \text{ and } \theta_\zeta = \delta\},$$

$$\mathfrak{F}^*_\delta = \{\mathcal{F} : \{\zeta : \theta_\zeta \ge \delta\} \in \mathcal{F}\},$$

$$\mathcal{G}_\delta = \{G : \{\zeta : \theta_\zeta \ge \delta\} \subseteq G \subseteq \lambda\}.$$

So henceforth let us take it that λ is infinite.

For any filter $\mathcal{F}$ on λ, write $f \le_{\mathcal{F}} g$ if $f, g \in P$ and $\{\zeta : f(\zeta) \le g(\zeta)\} \in \mathcal{F}$, that is, if $\pi_{\mathcal{F}} f \le \pi_{\mathcal{F}} g$ in $P|\mathcal{F}$. Note that if $\mathcal{F}$ is an ultrafilter, then $P|\mathcal{F}$ is a non-empty totally ordered set with no greatest member, so its additivity and cofinality are the same; thus

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda, \operatorname{add}(P|\mathcal{F}) = \delta\}$$

for every δ, and

$$\min_{\zeta\in F} \theta_\zeta \le \delta \le \operatorname{cf}(\prod_{\zeta\in F} \theta_\zeta)$$

whenever $F \in \mathcal{F} \in \mathfrak{F}_\delta$, by 5A2Ab.

Write $L = \{\zeta : \zeta < \lambda, \theta_\zeta = \lambda^+\}$, $M = \lambda \setminus L$. If $\mathcal{F}$ is an ultrafilter on λ and $L \in \mathcal{F}$, then $\operatorname{cf}(\prod_{\zeta\in L} \theta_\zeta) = \lambda^+$, because the set of constant functions is cofinal with $\prod_{\zeta\in L} \theta_\zeta$; so $\operatorname{cf}(P|\mathcal{F})$ must be λ^+; otherwise, $M \in \mathcal{F}$ and $\operatorname{cf}(P|\mathcal{F}) > \lambda^+$.

(a) Set $\delta' = \operatorname{add}(P|\mathcal{G}_\delta)$.

(i) δ' is a regular infinite cardinal (513Ca) and

$$\delta' \ge \min_{\zeta<\lambda} \theta_\zeta > \lambda$$

by 5A2Ab again. If $\delta = \lambda^+$ then of course $\delta' \ge \delta$; so suppose that $\delta > \lambda^+$. In this case $L \notin \mathcal{F}$ for any $\mathcal{F} \in \mathfrak{F}^*_\delta$, so $M \in \mathcal{G}_\delta$ and $\delta' \ge \min_{\zeta\in M} \theta_\zeta > \lambda^+$.

(ii) ? If $\delta' < \delta$ then (translating 513C(a-i) into a statement about $\le_{\mathcal{G}_\delta}$) there is a family $\langle f_\alpha\rangle_{\alpha<\delta'}$ in P such that $f_\alpha \le_{\mathcal{G}_\delta} f_\beta$ whenever $\alpha \le \beta < \delta'$ but there is no $f \in P$ such that $f_\alpha \le_{\mathcal{G}_\delta} f$ for every $\alpha < \delta'$. Choose $h_\xi \in P$, $\alpha_\xi < \delta'$ inductively, for $\xi < \lambda^+$, as follows. $h_0 = f_0$. Given h_ξ, set

$$B_{\xi\alpha} = \{\zeta : \zeta \in M, h_\xi(\zeta) \ge f_\alpha(\zeta)\}$$

for each $\alpha < \delta'$; let $\alpha_\xi < \delta'$ be such that $f_{\alpha_\xi} \not\le_{\mathcal{G}_\delta} h_\xi$, so that $B_{\xi\alpha} \notin \mathcal{G}_\delta$ when $\alpha_\xi \le \alpha < \delta'$. Choose $\mathcal{F}_\xi \in \mathfrak{F}^*_\delta$ such that $B_{\xi,\alpha_\xi} \notin \mathcal{F}_\xi$. Now, because $\operatorname{cf}(P|\mathcal{F}_\xi) \ge \delta > \delta'$, there is an $h_{\xi+1} \in P$ such that $f_\alpha \le_{\mathcal{F}_\xi} h_{\xi+1}$ for every $\alpha < \delta'$; we may take $h_{\xi+1} \ge h_\xi$.

For non-zero limit ordinals $\xi < \lambda^+$ take $h_\xi(\zeta) = \sup_{\eta<\xi} h_\eta(\zeta)$ for every $\zeta < \lambda$.

Set $\alpha = \sup_{\xi<\lambda^+} \alpha_\xi < \delta'$. Then $\langle B_{\xi\alpha}\rangle_{\xi<\lambda^+}$ is a non-decreasing family in $\mathcal{P}\lambda$. So there must be a $\xi < \lambda^+$ such that $B_{\xi\alpha} = B_{\xi+1,\alpha}$.

By the choice of $h_{\xi+1}$, $B_{\xi+1,\alpha} \in \mathcal{F}_\xi$. So $B_{\xi\alpha} \in \mathcal{F}_\xi$ and $f_\alpha \le_{\mathcal{F}_\xi} h_\xi$. Because $\alpha \ge \alpha_\xi$, $f_{\alpha_\xi} \le_{\mathcal{G}_\delta} f_\alpha$, so

$$f_{\alpha_\xi} \le_{\mathcal{F}_\xi} f_\alpha \le_{\mathcal{F}_\xi} h_\xi$$

and $B_{\xi,\alpha_\xi} \in \mathcal{F}_\xi$; contrary to the choice of $\mathcal{F}_\xi$. **X**

(b)(i) If $\mathfrak{F}_\delta = \emptyset$, we can take $F = \emptyset$; so suppose that $\mathfrak{F}_\delta$ is non-empty. As in (a-i) above, we must have $\delta \ge \min_{\zeta<\lambda} \theta_\zeta > \lambda$, and the case $\delta = \lambda^+$ is again elementary. **P** Take F to be the set of constant functions with values less than λ^+. If $\mathcal{F} \in \mathfrak{F}_\delta$ then $M \notin \mathcal{F}$ and $L \in \mathcal{F}$. So for any $h \in P$ we have $\alpha = \sup_{\zeta\in L} h(\zeta) < \lambda^+$, and there is an $f \in F$ such that $f(\zeta) = \alpha$ for every ζ, in which case $h \le_{\mathcal{F}} f$; thus $\pi_{\mathcal{F}}[F]$ is cofinal with $P|\mathcal{F}$, as required. **Q**

So suppose from now on that $\delta > \lambda^+$, so that $M \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}_\delta$. Of course δ, being the cofinality of a non-empty totally ordered set with no greatest member, is regular (511He-511Hf, 513C(a-i)).

(ii) ? Suppose, if possible, that there is no F of the required type. In this case, we can find families $\langle f_{\xi\alpha}\rangle_{\xi<\lambda^+,\alpha<\delta}$ in P and $\langle \mathcal{F}_\xi\rangle_{\xi<\lambda^+}$ in $\mathfrak{F}_\delta$ such that

(α) $f_{\eta\alpha} \le_{\mathcal{F}_\xi} f_{\xi 0}$ whenever $\alpha < \delta$, $\eta < \xi < \lambda^+$;
(β) $\{\pi_{\mathcal{F}_\xi}(f_{\xi\alpha}) : \alpha < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$ for every $\xi < \lambda^+$;
(γ) $f_{\eta\alpha} \le f_{\xi\alpha}$ whenever $\alpha < \delta$, $\eta \le \xi < \lambda^+$;
(δ) if $\xi < \lambda^+$, $\alpha < \delta$ and $\operatorname{cf}\alpha = \lambda^+$ then

$$f_{\xi\alpha}(\zeta) = \min\{\sup_{\beta\in C} f_{\xi\beta}(\zeta) : C \text{ is a closed cofinal set in } \alpha\}$$

for every $\zeta \in M$;
(ϵ) $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ whenever $\xi < \lambda^+$, $\beta \le \alpha < \delta$.

P Construct $\langle f_{\xi\alpha}\rangle_{\xi<\lambda^+,\alpha<\delta}$ inductively, taking $\lambda^+ \times \delta$ with its lexicographic well-ordering. Given that $\langle f_{\eta\beta}\rangle_{(\eta,\beta)<(\xi,\alpha)}$ satisfies the inductive hypothesis so far, proceed according to the nature of α, as follows.

Zero If $\alpha = 0$, then, because $\#(\xi \times \delta) \le \delta$, the counter-hypothesis tells us that there is an $\mathcal{F}_\xi \in \mathfrak{F}_\delta$ such that $\{\pi_{\mathcal{F}_\xi}(f_{\eta\beta}) : \eta < \xi,\ \beta < \delta\}$ is not cofinal with $P|\mathcal{F}_\xi$. Accordingly we can find $f_{\xi 0} \in P$ such that

$$f_{\eta\alpha} \le_{\mathcal{F}_\xi} f_{\xi 0} \text{ whenever } \eta < \xi \text{ and } \alpha < \delta,$$

and because $\operatorname{add} P \ge \delta > \#(\xi)$, we can also insist that

$$f_{\eta 0} \le f_{\xi 0} \text{ whenever } \eta < \xi.$$

Now take a family $\langle g_{\xi\beta}\rangle_{\beta<\delta}$ in P such that $\{\pi_{\mathcal{F}_\xi}(g_{\xi\beta}) : \beta < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$.

Successor If $\alpha = \beta + 1$ is a successor ordinal, set

$$f_{\xi\alpha}(\zeta) = \max(f_{\xi\beta}(\zeta), g_{\xi\beta}(\zeta), \sup\nolimits_{\eta<\xi} f_{\eta\alpha}(\zeta)) \text{ for every } \zeta < \lambda;$$

this is acceptable because $\operatorname{cf}\theta_\zeta > \lambda$ for every ζ.

Cofinality λ^+ If $\operatorname{cf}\alpha = \lambda^+$, set

$$\begin{aligned} f_{\xi\alpha}(\zeta) &= \sup_{\eta<\xi} f_{\eta\alpha}(\zeta) \text{ if } \zeta \in L, \\ &= \min\{\sup_{\beta\in C} f_{\xi\beta}(\zeta) : C \text{ is a closed cofinal set in } \alpha\} \text{ if } \zeta \in M. \end{aligned}$$

This time, note that if $\zeta \in M$, then $f_{\xi\alpha}(\zeta) < \theta_\zeta$ because there is a closed cofinal set in α of cardinal $\lambda^+ < \theta_\zeta$.

Otherwise If α is a non-zero limit ordinal and $\operatorname{cf}\alpha \ne \lambda^+$, choose $f_{\xi\alpha}$ such that

$$f_{\eta\alpha} \le f_{\xi\alpha} \text{ for every } \eta < \xi,$$

$$f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha} \text{ for every } \beta < \alpha;$$

this is possible because $\operatorname{add} P$ and $\operatorname{add}(P|\mathcal{F}_\xi)$ are both at least $\lambda^+ > \max(\#(\xi), \#(\alpha))$.

Now let us work through the list of conditions to be satisfied.

(α) is written into the case $\alpha = 0$ of the induction.

(β) Because $g_{\xi\alpha} \le f_{\xi,\alpha+1}$ for every α and $\{\pi_{\mathcal{F}_\xi}(g_{\xi\alpha}) : \alpha < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$, $\{\pi_{\mathcal{F}_\xi}(f_{\xi\alpha}) : \alpha < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$.

(γ) The construction ensures that we shall have $f_{\eta\alpha}(\zeta) \le f_{\xi\alpha}(\zeta)$ in all the required cases except possibly when $\operatorname{cf}\alpha = \lambda^+$ and $\zeta \in M$. But in this case, taking $\eta < \xi$ and a closed cofinal set $C \subseteq \alpha$ such that $f_{\xi\alpha} = \sup_{\beta\in C} f_{\xi\beta}(\zeta)$, the inductive hypothesis will assure us that

$$f_{\eta\alpha}(\zeta) \le \sup\nolimits_{\beta\in C} f_{\eta\beta}(\zeta) \le \sup\nolimits_{\beta\in C} f_{\xi\beta}(\zeta) = f_{\xi\alpha}(\zeta),$$

so there is no problem.

(δ) is written into the formula for the inductive step when $\operatorname{cf}\alpha = \lambda^+$.

(ϵ) We certainly have $f_{\xi\alpha} \le f_{\xi,\alpha+1}$, so $f_{\xi\alpha} \le_{\mathcal{F}_\xi} f_{\xi,\alpha+1}$, for every α. If $\operatorname{cf}\alpha = \lambda^+$, then because the intersection of fewer than $\operatorname{cf}\alpha$ closed cofinal subsets of α is again a closed cofinal set in α (4A1Bd), there will be a closed cofinal set $C \subseteq \alpha$ such that $f_{\xi\alpha}(\zeta) = \sup_{\beta\in C} f_{\xi\beta}(\zeta)$ for every $\zeta \in M$. So $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta \in C$; by the inductive hypothesis, $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta < \alpha$. For other limit ordinals α, we have $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta < \alpha$ directly from the choice of $f_{\xi\alpha}$.

So the procedure works. **Q**

(iii) The next step is to find a non-decreasing family $\langle h_\eta\rangle_{\eta<\lambda^+}$ in P and a strictly increasing family $\langle\gamma(\eta)\rangle_{\eta<\lambda^+}$ in δ such that

$f_{\xi,\gamma(\eta)}(\zeta) < h_\eta(\zeta)$ whenever $\xi, \eta < \lambda^+$ and $\zeta \in M$ (choosing h_η);
$h_\eta \le_{\mathcal{F}_\xi} f_{\xi,\gamma(\eta+1)}$ whenever $\xi, \eta < \lambda^+$ (choosing $\gamma(\eta+1)$);
$\gamma(\eta) = \sup_{\eta'<\eta} \gamma(\eta')$ whenever $\eta < \lambda^+$ is a limit ordinal (so $\gamma(0) = 0$).

Set $h(\zeta) = \sup_{\eta<\lambda^+} h_\eta(\zeta)$ for $\zeta \in M$, $h(\zeta) = 0$ for $\zeta \in L$, $\alpha = \sup_{\eta<\lambda^+} \gamma(\eta) < \delta$ (because $\delta = \operatorname{cf}\delta > \lambda^+$); then $\operatorname{cf}\alpha = \lambda^+$ and $\{\gamma(\eta) : \eta < \lambda^+\}$ is a closed cofinal set in α. So

$$f_{\xi\alpha}(\zeta) \le \sup_{\eta<\lambda^+} f_{\xi,\gamma(\eta)}(\zeta) \le h(\zeta)$$

for every $\xi < \lambda^+$ and $\zeta \in M$, by (i-δ). So if we set

$$A_\xi = \{\zeta : \zeta \in M,\ f_{\xi\alpha}(\zeta) = h(\zeta)\}$$

for each $\xi < \lambda^+$, we shall have $A_\eta \subseteq A_\xi$ whenever $\eta \le \xi < \lambda^+$, by (i-α).

(iv) As $\#(M) \le \lambda$, there must be some $\xi < \lambda^+$ such that $A_\xi = A_{\xi+1}$. Let $C \subseteq \alpha$ be a closed cofinal set such that

$$f_{\xi+1,\alpha}(\zeta) = \sup_{\beta\in C} f_{\xi+1,\beta}(\zeta)$$

for every $\zeta \in M$. Set $C' = \gamma^{-1}[C]$. Then C' is a closed cofinal subset of λ^+. **P** It is closed because $\gamma : \lambda^+ \to \alpha$ is order-continuous, therefore continuous (4A2Ro). Next, $\gamma[\lambda^+]$ is closed and cofinal in α, while $\operatorname{cf}\alpha = \lambda^+$ is uncountable, so $C \cap \gamma[\lambda^+]$ is cofinal with α and $\gamma[\lambda^+]$ (4A1Bd), and C' is cofinal with λ^+. **Q**

For each $\eta \in C'$ write η' for the next member of C' greater than η; then

$$h_\eta \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta+1)} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta')},$$

$$f_{\xi\alpha} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,0} = f_{\xi+1,\gamma(0)}$$

so there is a $\zeta_\eta \in M$ such that

$$h_\eta(\zeta_\eta) \le f_{\xi+1,\gamma(\eta')}(\zeta_\eta), \quad f_{\xi\alpha}(\zeta_\eta) \le f_{\xi+1,\gamma(0)}(\zeta_\eta) < h_0(\zeta_\eta) \le h(\zeta_\eta).$$

Let $\zeta \in M$ be such that

$$B = \{\eta : \eta \in C',\ \zeta_\eta = \zeta\}$$

is cofinal with λ^+. Then $f_{\xi\alpha}(\zeta) < h(\zeta)$ so $\zeta \notin A_\xi$. On the other hand,

$$f_{\xi+1,\alpha}(\zeta) = \sup_{\beta\in C} f_{\xi+1,\beta}(\zeta) \ge \sup_{\eta\in B} f_{\xi+1,\gamma(\eta')}(\zeta) \ge \sup_{\eta\in B} h_\eta(\zeta) = h(\zeta)$$

because $\langle h_\eta\rangle_{\eta<\lambda^+}$ is non-decreasing. So $\zeta \in A_{\xi+1}$; which is impossible. **X**

This contradiction completes the proof of (b).

(c)(i) Set $\Delta = \{\delta : \mathfrak{F}_\delta \ne \emptyset\}$, $\mathcal{G} = \bigcup_{\delta\in\Delta} \mathcal{G}_\delta$. Since $\mathcal{G}_\delta \subseteq \mathcal{G}_{\delta'}$ for $\delta \le \delta'$ in Δ, $\mathcal{G}$ is a filter on λ and there is an ultrafilter $\mathcal{H}$ on λ including $\mathcal{G}$. For any $\delta \in \Delta$, $\mathcal{H} \supseteq \mathcal{G}_\delta$, so

$$\operatorname{cf}(P|\mathcal{H}) = \operatorname{add}(P|\mathcal{H}) \ge \operatorname{add}(P|\mathcal{G}_\delta) \ge \delta,$$

using 5A2Ac and (a) above. Consequently $\delta^* = \operatorname{cf}(P|\mathcal{H})$ is the greatest element of Δ.

(ii) For each $\delta \le \delta^*$ choose a set $F_\delta \in [P]^{\le\delta}$ such that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $P|\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}_\delta$ (using (b) above). Set $F = \bigcup_{\delta\in\Delta} F_\delta$ and

$$G = \{\sup I : I \in [F]^{<\omega}\} \subseteq P.$$

Then $\#(G) \le \delta^*$. I claim that G is cofinal with P. **P?** Suppose, if possible, otherwise; take $h \in P$ such that $h \not\le g$ for every $g \in G$. Write

$$A_g = \{\zeta : h(\zeta) > g(\zeta)\}$$

for each $g \in G$. Because G is upwards-directed, $\{A_g : g \in G\}$ is a filter base, and there is an ultrafilter $\mathcal{F}$ on λ containing every A_g. Now there is a $\delta \in \Delta$ such that $\mathcal{F} \in \mathfrak{F}_\delta$, so that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $P|\mathcal{F}$, and there is an $f \in F_\delta$ such that $h \le_{\mathcal{F}} f$. But in this case $A = \{\zeta : h(\zeta) \le f(\zeta)\}$ and $A_f = \lambda \setminus A$ both belong to $\mathcal{F}$. **XQ**

(iii) Accordingly $\operatorname{cf} P \le \#(G) \le \delta^*$. But also of course $\delta^* = \operatorname{cf}(P|\mathcal{H}) \le \operatorname{cf} P$, so $\delta^* = \operatorname{cf} P$. Now we have $\mathcal{H} \in \mathfrak{F}_{\delta^*} = \mathfrak{F}_{\operatorname{cf} P}$.

5A2C Theorem Let $\lambda > 0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P = \prod_{\zeta<\lambda} \theta_\zeta$. Let $\mathcal{F}$ be an ultrafilter on λ and κ a regular infinite cardinal with $\lambda < \kappa \le \operatorname{cf}(P|\mathcal{F})$. Then there is a family $\langle\theta'_\zeta\rangle_{\zeta<\lambda}$ of regular infinite cardinals such that $\lambda < \theta'_\zeta \le \theta_\zeta$ for every $\zeta < \lambda$ and $\operatorname{cf}(P'|\mathcal{F}) = \kappa$, where $P' = \prod_{\zeta<\lambda} \theta'_\zeta$.

proof If $\kappa = \lambda^+$ we may take $\theta'_\zeta = \lambda^+$ for every ζ; if $\kappa = \operatorname{cf}(P|\mathcal{F})$ we may take $\theta'_\zeta = \theta_\zeta$; so let us assume that $\lambda^+ < \kappa < \operatorname{cf}(P|\mathcal{F})$. In this case $M = \{\zeta : \zeta < \lambda,\ \theta_\zeta > \lambda^+\}$ must belong to $\mathcal{F}$.

(a) For each ordinal $\gamma < \kappa$ choose a relatively closed cofinal set $C_\gamma \subseteq \gamma$ with $\operatorname{otp}(C_\gamma) = \operatorname{cf}\gamma$. Choose families $\langle f_\alpha\rangle_{\alpha<\kappa}$, $\langle g_{\alpha\gamma}\rangle_{\alpha,\gamma<\kappa}$ in P inductively, as follows. Given $\langle f_\beta\rangle_{\beta<\alpha}$, where $\alpha < \kappa$, and $\gamma < \kappa$, define $g_{\alpha\gamma} \in P$ by setting

$$\begin{aligned} g_{\alpha\gamma}(\zeta) &= \sup\{f_\beta(\zeta) : \beta \in C_\gamma \cap \alpha\} + 1 \text{ if this is less than } \theta_\zeta,\\ &= 0 \text{ otherwise.}\end{aligned}$$

Now choose $f_\alpha \in P$ such that

$$f_\beta \le_{\mathcal{F}} f_\alpha \ \forall \ \beta < \alpha, \quad g_{\alpha\gamma} \le_{\mathcal{F}} f_\alpha \ \forall \ \gamma < \kappa;$$

this is possible because $\kappa < \mathrm{cf}(P|\mathcal{F})$. Observe that if $\alpha = \beta + 1$ then $C_\alpha = \{\beta\}$ so that $g_{\alpha\alpha} = f_\beta + 1$ and $f_\alpha \not\le_{\mathcal{F}} f_\beta$. Continue.

(b) Suppose that for each $\zeta < \lambda$ we are given a set $S_\zeta \subseteq \theta_\zeta$ with $\#(S_\zeta) \le \lambda$. Then there is an $\alpha < \kappa$ such that

$$\text{for every } h \in \prod_{\zeta<\lambda} S_\zeta, \text{ if } f_\alpha \le_{\mathcal{F}} h \text{ then } f_\beta \le_{\mathcal{F}} h \text{ for every } \beta < \kappa.$$

P? If not, then (because κ is regular) we can find a family $\langle h_\xi \rangle_{\xi<\kappa}$ in $\prod_{\zeta<\lambda} S_\zeta$ and a strictly increasing family $\langle \phi(\xi) \rangle_{\xi<\kappa}$ in κ such that

$$f_{\phi(\xi)} \le_{\mathcal{F}} h_\xi \le_{\mathcal{F}} f_{\phi(\xi+1)} \text{ for all } \xi < \kappa,$$

$$\phi(\xi) = \sup_{\eta<\xi} \phi(\eta) \text{ for limit ordinals } \xi < \kappa.$$

Set

$$C = \{\xi : \xi < \kappa, \, \phi(\xi) = \xi\},$$

so that C is a closed cofinal set in κ. Let $\alpha \in C$ be such that $\alpha = \sup(C \cap \alpha)$ and $\mathrm{cf}\,\alpha = \lambda^+$. Then (because $\lambda^+ \ge \omega_1$) $C_\alpha \cap C$ is cofinal with α.

For $\beta \in C \cap C_\alpha$ and $\zeta < \lambda$ we have

$$\#(C_\alpha \cap \beta) \le \mathrm{otp}(C_\alpha \cap \beta) < \mathrm{otp}(C_\alpha) = \lambda^+ \le \theta_\zeta,$$

so

$$\theta_\zeta > \sup_{\xi \in C_\alpha \cap \beta} f_\xi(\zeta) + 1 = g_{\beta\alpha}(\zeta).$$

Now

$$g_{\beta\alpha} \le_{\mathcal{F}} f_\beta = f_{\phi(\beta)} \le_{\mathcal{F}} h_\beta \le_{\mathcal{F}} f_{\phi(\beta+1)} \le_{\mathcal{F}} f_{\beta'},$$

where β' is the next member of $C \cap C_\alpha$ greater than β. So there is a $\zeta_\beta < \lambda$ such that

$$g_{\beta\alpha}(\zeta_\beta) \le h_\beta(\zeta_\beta) \le f_{\beta'}(\zeta_\beta).$$

Because $\lambda < \mathrm{cf}\,\alpha$ there is a $\zeta < \lambda$ such that

$$B = \{\beta : \beta \in C \cap C_\alpha, \, \zeta_\beta = \zeta\}$$

is cofinal with α. But now observe that if $\beta, \gamma \in B$ and $\beta' < \gamma$ then $\beta' \in C_\alpha \cap \gamma$ so

$$h_\beta(\zeta) \le f_{\beta'}(\zeta) < g_{\gamma\alpha}(\zeta) \le h_\gamma(\zeta).$$

It follows that

$$\lambda^+ = \#(B) = \#(\{h_\beta(\zeta) : \beta \in B\}) \le \#(S_\zeta) \le \lambda,$$

which is absurd. **XQ**

(c) Consequently $E = \{f_\alpha^\bullet : \alpha < \kappa\}$ has a least upper bound in $P|\mathcal{F}$. **P?** If not, choose a family $\langle h_\xi \rangle_{\xi<\lambda^+}$ in P inductively, as follows. Because $\kappa < \mathrm{cf}(P|\mathcal{F})$, there is an $h_0 \in P$ such that $f_\alpha \le_{\mathcal{F}} h_0$ for every $\alpha < \kappa$. Given h_ξ such that $h_\xi^\bullet$ is an upper bound for E, then $h_\xi^\bullet$ cannot be the least upper bound of E, so there is an $h_{\xi+1} \in P$ such that $h_{\xi+1}^\bullet$ is an upper bound of E strictly less than $h_\xi^\bullet$. For non-zero limit ordinals $\xi < \lambda^+$, set

$$S_{\xi\zeta} = \{h_\eta(\zeta) : \eta < \xi\} \subseteq \theta_\zeta$$

for each $\zeta < \lambda$. By (b) above, there is an $\alpha_\xi < \kappa$ such that

$$\text{for every } h \in \prod_{\zeta<\lambda} S_{\xi\zeta} \text{ either } f_{\alpha_\xi} \not\le_{\mathcal{F}} h \text{ or } f_\alpha \le_{\mathcal{F}} h \ \forall \ \alpha < \kappa.$$

Set

$$h_\xi(\zeta) = \min(\{\eta : \eta \in S_{\xi\zeta}, \, f_{\alpha_\xi}(\zeta) \le \eta\} \cup \{h_0(\zeta)\}) \in S_{\xi\zeta}$$

for each $\zeta < \lambda$. Then $f_{\alpha_\xi} \le_{\mathcal{F}} h_\xi$ (because $f_{\alpha_\xi}(\zeta) \le h_\xi(\zeta)$ whenever $f_{\alpha_\xi}(\zeta) \le h_0(\zeta)$) and $h_\xi \in \prod_{\zeta<\lambda} S_{\xi\zeta}$, so $f_\alpha \le_{\mathcal{F}} h_\xi$ for every $\alpha < \kappa$ and $h_\xi^\bullet$ is an upper bound for E. Also, if $\eta < \xi$, then $h_\xi(\zeta) \le h_\eta(\zeta)$ whenever $f_{\alpha_\xi}(\zeta) \le h_\eta(\zeta)$, so $h_\xi \le_{\mathcal{F}} h_\eta$. Continue.

Having got the family $\langle h_\xi \rangle_{\xi<\lambda^+}$, set

$$S_\zeta = \bigcup_{\xi<\lambda^+} S_{\xi\zeta} = \{h_\xi(\zeta) : \xi < \lambda^+\} \subseteq \theta_\zeta$$

for each $\zeta < \lambda$. For each $\alpha < \kappa$, $\zeta < \lambda$ set

$$g_\alpha(\zeta) = \min(\{\eta : f_\alpha(\zeta) \le \eta \in S_\zeta\} \cup \{h_0(\zeta)\}) \in S_\zeta.$$

Then, by the same arguments as above,

$$f_\alpha \le_{\mathcal{F}} g_\alpha \le_{\mathcal{F}} h_\xi \text{ for every } \alpha < \kappa,\ \xi < \lambda^+.$$

For each $\alpha < \kappa$ there is a non-zero limit ordinal $\xi < \lambda^+$ such that $g_\alpha(\zeta) \in S_{\xi\zeta}$ for every $\zeta < \lambda$, because $\langle S_{\xi\zeta}\rangle_{\xi<\lambda^+}$ is non-decreasing for each ζ. Because $\lambda^+ < \kappa$ there is a non-zero limit ordinal $\xi < \lambda^+$ such that

$$A = \{\alpha : g_\alpha(\zeta) \in S_{\xi\zeta}\ \forall\ \zeta < \lambda\}$$

is cofinal with κ. In particular, there is an $\alpha \in A$ such that $\alpha \ge \alpha_\xi$. In this case

$$f_{\alpha_\xi} \le_{\mathcal{F}} f_\alpha \le_{\mathcal{F}} g_\alpha \le_{\mathcal{F}} h_{\xi+1} \le_{\mathcal{F}} h_\xi \not\le_{\mathcal{F}} h_{\xi+1},$$

so there is a $\zeta < \lambda$ such that

$$f_{\alpha_\xi}(\zeta) \le f_\alpha(\zeta) \le g_\alpha(\zeta) \le h_{\xi+1}(\zeta) < h_\xi(\zeta).$$

But now observe that

$$f_{\alpha_\xi}(\zeta) \le g_\alpha(\zeta) \in S_{\xi\zeta}$$

so $h_\xi(\zeta) \le g_\alpha(\zeta) < h_\xi(\zeta)$, which is absurd. **XQ**

(d) Let $g \in P$ be such that $g^\bullet = \sup E$ in $P|\mathcal{F}$ and $g(\zeta) > 0$ for every $\zeta < \lambda$. For each $\zeta < \lambda$ set $\hat\theta_\zeta = \operatorname{cf} g(\zeta) < \theta_\zeta$ and choose a cofinal set $D_\zeta \subseteq g(\zeta)$ of order type $\hat\theta_\zeta$. For $\alpha < \kappa$, $\zeta < \lambda$ set

$$\hat g_\alpha(\zeta) = \min\{\eta : f_\alpha(\zeta) \le \eta \in D_\zeta\}$$

if $f_\alpha(\zeta) < g(\zeta)$, $\min D_\zeta$ otherwise. Then $\hat g_\alpha \le_{\mathcal{F}} \hat g_\beta$ whenever $\alpha \le \beta < \kappa$. Also if $h \in Y = \prod_{\zeta<\lambda} D_\zeta$ then $h^\bullet < g^\bullet$ so there is an $\alpha < \kappa$ such that $h^\bullet \le f_\alpha^\bullet \le \hat g_\alpha^\bullet$. Thus $\{\hat g_\alpha^\bullet : \alpha < \kappa\}$ is cofinal with $\{h^\bullet : h \in Y\}$.

(e) Because each D_ζ is order-isomorphic to $\hat\theta_\zeta$, we can identify Y with $\hat P = \prod_{\zeta<\lambda}\hat\theta_\zeta$, and see that $\operatorname{cf}(\hat P|\mathcal{F}) = \operatorname{cf}\{h^\bullet : h \in Y\}$ is either 1 or κ. But of course the former is absurd, because it could be so only if $\{\zeta : g(\zeta)$ is a successor ordinal$\}$ belonged to $\mathcal{F}$, and in this case there would have to be an $\alpha < \kappa$ such that $g \le_{\mathcal{F}} f_\alpha$; but we saw in (a) above that $f_{\alpha+1} \not\le_{\mathcal{F}} f_\alpha$.

Accordingly $\operatorname{cf}(\hat P|\mathcal{F}) = \kappa$.

(f) It may be that some of the $\hat\theta_\zeta$ are less than or equal to λ. But taking $I = \{\zeta : \hat\theta_\zeta \le \lambda\}$, we have $I \notin \mathcal{F}$. **P?** If $I \in \mathcal{F}$, then for each $\zeta \in I$ set $S_\zeta = D_\zeta$ and for $\zeta \in \lambda \setminus I$ set $S_\zeta = \{0\}$. By (b), there is an $\alpha < \kappa$ such that

$$\text{for every } h \in \textstyle\prod_{\zeta<\lambda} S_\zeta, \text{ if } f_\alpha \le_{\mathcal{F}} h \text{ then } f_\beta \le_{\mathcal{F}} h\ \forall\ \beta < \kappa.$$

But as $f_{\alpha+1} \le_{\mathcal{F}} g$, and $I \in \mathcal{F}$, there must be an $h \in \prod_{\zeta<\lambda} S_\zeta$ such that $f_\alpha \le_{\mathcal{F}} h$, and now $g \le_{\mathcal{F}} h$ because $g^\bullet$ is the least upper bound of E; but $h(\zeta) < g(\zeta)$ for every $\zeta \in I$, so this is impossible. **XQ**

So $\{\zeta : \hat\theta_\zeta > \lambda\} \in \mathcal{F}$. But this means that if we set

$$\begin{aligned} \theta'_\zeta &= \hat\theta_\zeta \text{ when } \hat\theta_\zeta > \lambda, \\ &= \theta_\zeta \text{ when } \hat\theta_\zeta \le \lambda \end{aligned}$$

and $P' = \prod_{\zeta<\lambda}\theta'_\zeta$, then $P'|\mathcal{F} \cong \hat P|\mathcal{F}$ so $\operatorname{cf}(P'|\mathcal{F}) = \kappa$, as required.

5A2D Definitions (a) Let α, β, γ and δ be cardinals. Following Shelah 92 and Shelah 94, I write

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta)$$

for the least cardinal of any family $\mathcal{E} \subseteq [\alpha]^{<\beta}$ such that for every $A \in [\alpha]^{<\gamma}$ there is a $\mathcal{D} \in [\mathcal{E}]^{<\delta}$ with $A \subseteq \bigcup\mathcal{D}$. In the trivial cases in which there is no such family $\mathcal{E}$ I write $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) = \infty$.

(b) For cardinals α, γ write $\Theta(\alpha,\gamma)$ for the supremum of all cofinalities

$$\operatorname{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta)$$

for families $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ such that $\lambda < \gamma$ is a cardinal, every θ_ζ is a regular infinite cardinal, and $\lambda < \theta_\zeta < \alpha$ for every $\zeta < \lambda$. (This carries some of the same information as the cardinal $\operatorname{pp}_\kappa(\alpha)$ of Shelah 94, p. 41.)

Remarks (i) Immediately from the definitions, we see that

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta',\gamma,\delta') \le \mathrm{cov}_{\mathrm{Sh}}(\alpha',\beta,\gamma',\delta), \quad \Theta(\alpha,\gamma) \le \Theta(\alpha',\gamma')$$

whenever $\alpha \le \alpha'$, $\beta \le \beta'$, $\gamma \le \gamma'$ and $\delta \le \delta'$.

(ii) The definition of Θ demands a moment's thought in trivial cases. If $\gamma = 0$ there is no $\lambda < \gamma$, so we are taking the supremum of an empty set of cofinalities, and $\Theta(\alpha, 0) = 0$ for every α. If $\gamma > 0$ then we are allowed $\lambda = 0$ and $\prod_{\zeta<\lambda}\theta_\zeta = \{\emptyset\}$, so $\Theta(\alpha,\gamma) \ge 1$ for every α.

5A2E Lemma Let α, β, γ, γ' and δ be cardinals.

(a) If $\gamma \le \gamma' \le \beta$ and $\delta \ge 2$ then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \mathrm{cf}[\alpha]^{<\gamma'} \le \#([\alpha]^{<\gamma'}).$$

(b) If either $\omega \le \gamma \le \mathrm{cf}\,\alpha$ or $\omega \le \mathrm{cf}\,\alpha < \mathrm{cf}\,\delta$ then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \max(\alpha, \sup_{\theta<\alpha}\mathrm{cov}_{\mathrm{Sh}}(\theta,\beta,\gamma,\delta)).$$

proof (a) If $\mathcal{E}$ is a cofinal subset of $[\alpha]^{<\gamma'}$ of cardinal $\mathrm{cf}[\alpha]^{<\gamma'}$, then $\mathcal{E}$ witnesses that $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma',\gamma',\delta) \le \mathrm{cf}[\alpha]^{<\gamma}$. Now

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma',\gamma',\delta) \le \mathrm{cf}[\alpha]^{<\gamma'}.$$

(b) Set $\kappa = \max(\alpha, \sup_{\theta<\alpha}\mathrm{cov}_{\mathrm{Sh}}(\theta,\beta,\gamma,\delta))$ and $\lambda = \mathrm{cf}\,\alpha$. Let $\langle\zeta_\xi\rangle_{\xi<\lambda}$ enumerate a cofinal subset of α. For each $\xi < \lambda$, let $\mathcal{E}_\xi \subseteq [\zeta_\xi]^{<\beta}$ be a set of size at most κ such that for every $A \in [\zeta_\xi]^{<\gamma}$ there is a $\mathcal{D} \in [\mathcal{E}_\xi]^{<\delta}$ such that $A \subseteq \bigcup\mathcal{D}$. Set $\mathcal{E} = \bigcup_{\xi<\lambda}\mathcal{E}_\xi$, so that $\mathcal{E} \subseteq [\alpha]^{<\beta}$ has cardinal at most κ. Take $A \in [\alpha]^{<\gamma}$.

If $\omega \le \gamma \le \lambda$ then $\sup A < \alpha$ and there is a $\xi < \lambda$ such that $A \subseteq \zeta_\xi$. Now there is a $\mathcal{D} \in [\mathcal{E}_\xi]^{<\delta} \subseteq [\mathcal{E}]^{<\delta}$ such that $A \subseteq \bigcup\mathcal{D}$.

If $\omega \le \lambda < \mathrm{cf}\,\delta$, then for each $\xi < \lambda$ there is a $\mathcal{D}_\xi \in [\mathcal{E}_\xi]^{<\delta}$ such that $A \cap \zeta_\xi \subseteq \bigcup\mathcal{D}_\xi$. Set $\mathcal{D} = \bigcup_{\xi<\lambda}\mathcal{D}_\xi$; because $\lambda < \mathrm{cf}\,\delta$, $\mathcal{D} \in [\mathcal{E}]^{<\delta}$, while

$$A = \bigcup_{\xi<\lambda} A \cap \zeta_\xi \subseteq \bigcup\mathcal{D}.$$

Thus in either case $\mathcal{E}$ witnesses that $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \kappa$.

5A2F Lemma Let α, γ be cardinals. If $\alpha \le 2^\gamma$, then $\Theta(\alpha,\gamma) \le 2^\gamma$.

proof If $\gamma \le \omega$ then $\Theta(\alpha,\gamma) \le \max(1,\alpha) \le 2^\gamma$. If $\gamma > \omega$, $\lambda < \gamma$ and $\theta_\zeta < 2^\gamma$ for every $\zeta < \lambda$, then

$$\#(\prod_{\zeta<\lambda}\theta_\zeta) \le (2^\gamma)^\gamma = 2^\gamma.$$

5A2G Theorem For any cardinals α and γ,

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \max(\omega,\alpha,\Theta(\alpha,\gamma)).$$

proof (a) To begin with (down to the end of (f) below) let us suppose that we have $\alpha \ge \gamma = \gamma_0^+ > \mathrm{cf}\,\alpha > \omega$, and set $\kappa = \max(\alpha,\Theta(\alpha,\gamma))$.

Take a family $\mathcal{E} \subseteq [\alpha]^{\le\gamma_0}$ such that

(i) $\mathcal{E}$ contains all singleton subsets of α;

(ii) $\mathcal{E}$ contains a cofinal subset of α;

(iii) If $E \in \mathcal{E}$ then $\{\xi : \xi + 1 \in E\} \in \mathcal{E}$;

(iv) if $E \in \mathcal{E}$ then there is an $F \in \mathcal{E}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \le \mathrm{cf}\,\xi \le \gamma_0$;

(v) if $E \in \mathcal{E}$ then $\{\xi : \xi \in E, \mathrm{cf}\,\xi > \gamma_0\} \in \mathcal{E}$;

(vi) if $E \in \mathcal{E}$ and $\mathrm{cf}(\prod_{\eta\in E}\eta) \le \kappa$, then $\{g : g \in \prod_{\eta\in E}\eta,\ g[E] \in \mathcal{E}\}$ is cofinal with $\prod_{\eta\in E}\eta$;

(vii) $\#(\mathcal{E}) \le \kappa$.

To see that this can be done, observe that whenever $E \in [\alpha]^{\le\gamma_0}$ there is an $F \in [\alpha]^{\le\gamma_0}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \le \mathrm{cf}\,\xi \le \gamma_0$; thus condition (iv) can be achieved, like conditions (iii) and (v), by ensuring that $\mathcal{E}$ is closed under suitable functions from $[\alpha]^{\le\gamma_0}$ to itself; while condition (vi) requires that for each $E \in \mathcal{E}$ we have an appropriate family of size at most κ included in $\mathcal{E}$.

Write $\mathcal{J}$ for the σ-ideal of $\mathcal{P}\alpha$ generated by $\mathcal{E}$. Note that if $A \in \mathcal{J}$ then $\{\xi : \xi + 1 \in A\}$ belongs to $\mathcal{J}$, by (iii).

(b) ? If $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) > \kappa$, there must be a set in $[\alpha]^{\le\gamma_0}$ not covered by any sequence from $\mathcal{E}$, that is, not belonging to $\mathcal{J}$; that is, there is a function $f : \gamma_0 \to \alpha$ such that $f[\gamma_0] \notin \mathcal{J}$. Accordingly $\mathcal{I} = \{f^{-1}[E] : E \in \mathcal{J}\}$ is a proper σ-ideal of $\mathcal{P}\gamma_0$. By condition (a-i), $\mathcal{I}$ contains all singletons in $\mathcal{P}\gamma_0$.

Let H be the set of all functions $h:\gamma_0\to\alpha$ such that $f(\xi)\le h(\xi)$ for every $\xi<\gamma_0$ and $h[\gamma_0]\in\mathcal{J}$. Because $\mathcal{E}$ contains a cofinal set $C\subseteq\alpha$ (condition (a-ii)), we can find an $h\in H$; just take $h:\gamma_0\to C$ such that $f(\xi)\le h(\xi)$ for every ξ.

(c) Because $\mathcal{I}$ is a proper σ-ideal, there cannot be any sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in H such that $\{\xi:h_{n+1}(\xi)\ge h_n(\xi)\}\in\mathcal{I}$ for every $n\in\mathbb{N}$. Consequently there is an $h^*\in H$ such that

$$\{\xi:h(\xi)\ge h^*(\xi)\}\notin\mathcal{I}\text{ for every }h\in H.$$

We know that $h^*[\gamma_0]\in\mathcal{J}$; let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ covering $h^*[\gamma_0]$. For $\xi<\gamma_0$ write $\theta_\xi=\operatorname{cf}h^*(\xi)$, so that each θ_ξ is 0 or 1 or a regular infinite cardinal less than α. Set

$$I=\{\xi:\xi<\gamma_0,\,f(\xi)=h^*(\xi)\},$$

$$I'=\{\xi:\xi<\gamma_0,\,f(\xi)<h^*(\xi),\,\theta_\xi=1\},$$

$$I_n=\{\xi:\xi<\gamma_0,\,f(\xi)<h^*(\xi),\,\omega\le\theta_\xi\le\gamma_0,\,h^*(\xi)\in E_n\}\text{ for }n\in\mathbb{N},$$

$$J_n=\{\xi:\xi<\gamma_0,\,f(\xi)<h^*(\xi),\,\gamma_0<\theta_\xi,\,h^*(\xi)\in E_n\}\text{ for }n\in\mathbb{N}.$$

Note that if $\theta_\xi=0$ then $h^*(\xi)=0=f(\xi)$, so I, I', $\langle I_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ constitute a cover of γ_0.

(d) For each $n\in\mathbb{N}$ set $G_n=\{\eta:\eta\in E_n,\,\operatorname{cf}\eta>\gamma_0\}\in\mathcal{E}$; note that $h^*(\xi)\in G_n$ for $\xi\in J_n$. Then $\operatorname{cf}(\prod_{\eta\in G_n}\eta)\le\Theta(\alpha,\gamma)$. **P** For $\eta\in G_n$ set $\theta'_\eta=\operatorname{cf}\eta$; then θ'_η is a regular cardinal and $\#(G_n)\le\gamma_0<\theta'_\eta<\alpha$ for each $\eta\in G_n$. If for each $\eta\in G_n$ we choose a cofinal set $C_\eta\subseteq\eta$ of order type θ'_η, then

$$\operatorname{cf}(\prod_{\eta\in G_n}\eta)=\operatorname{cf}(\prod_{\eta\in G_n}C_\eta)=\operatorname{cf}(\prod_{\eta\in G_n}\theta'_\eta)\le\Theta(\alpha,\gamma)$$

by the definition of $\Theta(\alpha,\gamma)$. **Q**

Consequently, by (a-vi),

$$\{g:g\in\prod_{\eta\in G_n}\eta,\,g[G_n]\in\mathcal{E}\}$$

is cofinal with $\prod_{\eta\in G_n}\eta$.

(e) Define $h:\gamma_0\to\alpha$ as follows.

(i) If $\xi\in I$ set $h(\xi)=h^*(\xi)$.

(ii) If $\xi\in I'$ let $h(\xi)$ be the predecessor of $h^*(\xi)$.

(iii) For each $n\in\mathbb{N}$ take $F_n\in\mathcal{E}$ such that $\eta=\sup(F_n\cap\eta)$ whenever $\eta\in E_n$ and $\omega\le\operatorname{cf}\eta\le\gamma_0$. If $\xi\in I_n\setminus\bigcup_{m<n}I_m$, take $h(\xi)\in F_n$ such that $f(\xi)<h(\xi)<h^*(\xi)$.

(iv) For each $n\in\mathbb{N}$ and $\eta\in G_n$ set

$$g^*(\eta)=\sup\{f(\xi):\xi<\gamma_0,\,h^*(\xi)=\eta\}.$$

Then $g^*(\eta)<\eta$, because $\gamma_0<\operatorname{cf}\eta$. By (d), there is a $g_n\in\prod_{\eta\in G_n}\eta$ such that $g_n[G_n]\in\mathcal{E}$ and $g^*(\eta)<g_n(\eta)$ for every $\eta\in G_n$. So for $\xi\in J_n\setminus\bigcup_{m<n}J_m$ we may set $h(\xi)=g_n(h^*(\xi))$ and see that

$$f(\xi)\le g^*h^*(\xi)<g_nh^*(\xi)=h(\xi)<h^*(\xi),$$

while $h(\xi)\in g_n[G_n]$.

(f) Now we see that

$$h[\gamma_0]\subseteq h^*[\gamma_0]\cup\{\eta:\eta+1\in h^*[\gamma_0]\}\cup\bigcup_{n\in\mathbb{N}}F_n\cup\bigcup_{n\in\mathbb{N}}g_n[G_n]\in\mathcal{J},$$

while $f(\xi)\le h(\xi)$ for every $\xi<\gamma_0$, so $h\in H$. Consequently

$$I=\{\xi:h(\xi)\ge h^*(\xi)\}\notin\mathcal{I}.$$

But also

$$f[I]\subseteq h^*[\gamma_0]\in\mathcal{J},$$

so $I\in\mathcal{I}$, which is absurd. **X**

(g) Thus the special case described in (a) is dealt with, and we may return to the general case. I proceed by induction on α for fixed γ.

(i) To start the induction, observe that if either $\alpha\le\omega$ or $\gamma<\omega$ or $\alpha<\gamma$, then

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1)\le\operatorname{cf}[\alpha]^{<\gamma}\le\max(\alpha,\omega).$$

(ii) For the inductive step to α when *either* $\operatorname{cf}\alpha\ge\gamma\ge\omega$ *or* $\operatorname{cf}\alpha=\omega$, 5A2Eb tells us that

$$\operatorname*{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \max(\alpha, \sup_{\alpha'<\alpha} \operatorname*{cov}_{\mathrm{Sh}}(\alpha',\gamma,\gamma,\omega_1))$$

$$\le \max(\omega,\alpha, \sup_{\alpha'<\alpha} \Theta(\alpha',\gamma)) \le \max(\omega,\alpha,\Theta(\alpha,\gamma))$$

by the inductive hypothesis.

(iii) For the inductive step to α when $\omega < \operatorname{cf}\alpha < \gamma \le \alpha$, observe that

$$[\alpha]^{<\gamma} = \bigcup_{\delta<\gamma}[\alpha]^{\le\delta}.$$

For each $\delta < \gamma$ we have a set $\mathcal{E}_\delta \subseteq [\alpha]^{\le\delta}$ such that $\#(\mathcal{E}_\delta) \le \mathrm{cov}_{\mathrm{Sh}}(\alpha,\delta^+,\delta^+,\omega_1)$ and every member of $[\alpha]^{\le\delta}$ can be covered by a sequence from $\mathcal{E}_\delta$. Set $\mathcal{E} = \bigcup_{\operatorname{cf}\alpha\le\delta<\gamma}\mathcal{E}_\delta$; then $\mathcal{E} \subseteq [\alpha]^{<\gamma}$ and every member of $[\alpha]^{<\gamma}$ can be covered by a sequence from $\mathcal{E}$. So

$$\operatorname*{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \#(\mathcal{E}) \le \max(\gamma, \sup_{\operatorname{cf}\alpha\le\delta<\gamma} \operatorname*{cov}_{\mathrm{Sh}}(\alpha,\delta^+,\delta^+,\omega_1))$$

$$\le \max(\gamma,\alpha, \sup_{\operatorname{cf}\alpha\le\delta<\gamma} \Theta(\alpha,\delta^+))$$

(by (a)-(f) above)

$$\le \max(\alpha,\Theta(\alpha,\gamma)).$$

This completes the proof.

Remark This is taken from SHELAH 94, Theorem II.5.4, where a stronger result is proved, giving an exact description of many of the numbers $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ in terms of cofinalities of reduced products $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}$.

5A2H Lemma Let γ be an infinite regular cardinal and $\alpha \ge \Theta(\gamma,\gamma)$ a cardinal. Then $\Theta(\Theta(\alpha,\gamma),\gamma) \le \Theta(\alpha,\gamma)$.

proof (a) The case $\gamma = \omega$ is elementary, since $\Theta(\alpha,\omega) \le \alpha$ for every cardinal α. So we may suppose that γ is uncountable. If $\Theta(\alpha,\gamma) \le \alpha$ the result is immediate; so we may suppose that $\alpha < \Theta(\alpha,\gamma)$, in which case $\Theta(\gamma,\gamma) < \Theta(\alpha,\gamma)$ and $\gamma < \alpha$.

(b) ? Suppose, if possible, that $\Theta(\alpha,\gamma) < \Theta(\Theta(\alpha,\gamma),\gamma)$. There must be a family $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ of regular infinite cardinals such that $\lambda < \gamma$, $\lambda < \theta_\zeta < \Theta(\alpha,\gamma)$ for every $\zeta < \lambda$, and $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) > \Theta(\alpha,\gamma)$. As $\Theta(\alpha,\gamma) \ge 1$, $\lambda \ne 0$. By 5A2Bc, there is an ultrafilter $\mathcal{F}$ on λ such that $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}) > \Theta(\alpha,\gamma)$. Set $L = \{\zeta : \zeta < \lambda,\ \theta_\zeta < \alpha\}$; then $\operatorname{cf}(\prod_{\zeta\in L}\theta_\zeta) \le \Theta(\alpha,\gamma)$, so $L \notin \mathcal{F}$ and $M = \lambda\setminus L \in \mathcal{F}$. Let $\mathcal{F}'$ be the induced ultrafilter on M, so that $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F} \cong \prod_{\zeta\in M}\theta_\zeta|\mathcal{F}'$, and $\operatorname{cf}(\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}') > \Theta(\alpha,\gamma)$. For each $\zeta \in M$, we have $\theta_\zeta < \Theta(\alpha,\gamma)$, so there must be a family $\langle\theta_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular infinite cardinals with $\lambda_\zeta < \gamma$, $\lambda_\zeta < \theta_{\zeta\eta} < \alpha$ for every $\eta < \lambda_\zeta$ and $\theta_\zeta \le \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta_{\zeta\eta})$. Again by 5A2Bc, there is an ultrafilter $\mathcal{F}_\zeta$ on λ_ζ such that $\theta_\zeta \le \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta_{\zeta\eta}|\mathcal{F}_\zeta)$. Because

$$\lambda_\zeta < \gamma \le \alpha \le \theta_\zeta,$$

5A2C tells us that there is a family $\langle\theta'_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular infinite cardinals such that $\lambda_\zeta < \theta'_{\zeta\eta} \le \theta_{\zeta\eta}$ for every η and $\theta_\zeta = \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta'_{\zeta\eta}|\mathcal{F}_\zeta)$.

(c) Set

$$I = \{(\zeta,\eta) : \zeta \in M,\ \eta < \lambda_\zeta\},$$

$$\mathcal{H} = \{H : H \subseteq I,\ \{\zeta : \{\eta : (\zeta,\eta) \in H\} \in \mathcal{F}_\zeta\} \in \mathcal{F}'\},$$

$$P = \prod_{(\zeta,\eta)\in I}\theta'_{\zeta\eta}.$$

Then $\mathcal{H}$ is an ultrafilter on I, and $\operatorname{cf}(P|\mathcal{H}) \ge \operatorname{cf}(\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}')$. **P** Let $F \subseteq P$ be a set of cardinal $\operatorname{cf}(P|\mathcal{H})$ such that $\{f^\bullet : f \in F\}$ is cofinal with $P|\mathcal{H}$. For $f \in P$ and $\zeta \in M$, define $f_\zeta \in \prod_{\eta<\lambda_\zeta}\theta'_{\zeta\eta}$ by setting $f_\zeta(\eta) = f(\zeta,\eta)$ for each $\eta < \lambda_\zeta$, and let $f^\bullet_\zeta$ be the image of f_ζ in $\prod_{\eta<\lambda_\zeta}\theta'_{\zeta\eta}|\mathcal{F}_\zeta$. For each $\zeta \in M$ let $\langle u_{\zeta\xi}\rangle_{\xi<\theta_\zeta}$ be a strictly increasing cofinal family in the totally ordered set $\prod_{\eta<\lambda_\zeta}\theta'_{\zeta\eta}|\mathcal{F}_\zeta$. Now, for $f \in F$, take a function $g_f \in \prod_{\zeta\in M}\theta_\zeta$ such that $f^\bullet_\zeta \le u_{\zeta,g_f(\zeta)}$ for every $\zeta \in M$.

If $g \in \prod_{\zeta\in M}\theta_\zeta$, then we can find an $h \in P$ such that $h^\bullet_\zeta = u_{\zeta,g(\zeta)}$ for each $\zeta \in M$. Let $f \in F$ be such that $h \le_{\mathcal{H}} f$. Then

$$\{\zeta : g(\zeta) \le g_f(\zeta)\} \supseteq \{\zeta : h^\bullet_\zeta \le f^\bullet_\zeta\} \in \mathcal{F}',$$

so $g \le_{\mathcal{F}'} g_f$. Accordingly $\{g_f : f \in F\}$ is cofinal with $\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}'$ and $\mathrm{cf}(\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}') \le \#(F) = \mathrm{cf}(P|\mathcal{H})$, as claimed. **Q**

(d) Thus $\mathrm{cf}(P|\mathcal{H}) > \Theta(\alpha,\gamma)$. Set $J = \{(\zeta,\eta) : (\zeta,\eta) \in I,\, \theta'_{\zeta\eta} \ge \gamma\}$. Because γ is regular, $\#(J) \le \#(I) < \gamma$, so $\mathrm{cf}(\prod_{(\zeta,\eta)\in J}\theta'_{\zeta\eta}) \le \Theta(\alpha,\gamma)$, and $J \notin \mathcal{H}$. It follows that $K = I \setminus J \in \mathcal{H}$. Set $M' = \{\zeta : \zeta \in M,\, \{\eta : (\zeta,\eta) \in K\} \in \mathcal{F}_\zeta\} \in \mathcal{F}'$.

If $\zeta \in M'$, then $F = \{\eta : \eta < \lambda_\zeta,\, \theta'_{\zeta\eta} < \gamma\}$ belongs to $\mathcal{F}_\zeta$, so

$$\theta_\zeta \le \mathrm{cf}(\textstyle\prod_{\eta\in F}\theta'_{\zeta\eta}) \le \Theta(\gamma,\gamma) \le \alpha \le \theta_\zeta$$

by the definition of M. So in fact $\theta_\zeta = \alpha$ for $\zeta \in M'$ and we have

$$\Theta(\alpha,\gamma) < \mathrm{cf}(\textstyle\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}') \le \mathrm{cf}(\prod_{\zeta\in M'}\theta_\zeta) = \mathrm{cf}(\prod_{\zeta<\delta}\alpha),$$

where $\delta = \#(M')$, while at the same time α is infinite and regular.

But if α is infinite and regular and $\delta < \alpha$,

$$\mathrm{cf}(\textstyle\prod_{\zeta<\delta}\alpha) \le \alpha,$$

so $\Theta(\alpha,\gamma) < \alpha$; which is impossible. **X**

This contradiction completes the proof.

5A2I Lemma Let α and γ be cardinals. Set $\delta = \sup_{\alpha'<\alpha}\Theta(\alpha',\gamma)$.

(a) If $\mathrm{cf}\,\alpha \ge \gamma$ then $\Theta(\alpha,\gamma) \le \max(\alpha,\delta)$.

(b) If $\mathrm{cf}\,\alpha < \gamma$ then $\Theta(\alpha,\gamma) \le \max(\alpha,\delta^{\mathrm{cf}\,\alpha})$, where $\delta^{\mathrm{cf}\,\alpha}$ is the cardinal power.

proof Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family of regular infinite cardinals with $\lambda < \theta_\zeta < \alpha$ for each ζ and $\lambda < \gamma$.

case 1 If $\alpha' = \sup_{\zeta<\lambda}\theta_\zeta$ is less than α, set

$$I = \{\zeta : \zeta < \lambda,\, \theta_\zeta < \alpha'\},$$

$$J = \{\zeta : \zeta < \lambda,\, \theta_\zeta = \alpha'\}.$$

Then

$$\mathrm{cf}(\textstyle\prod_{\zeta\in I}\theta_\zeta) \le \Theta(\alpha',\gamma) \le \delta, \quad \mathrm{cf}(\prod_{\zeta\in J}\theta_\zeta) \le \max(1,\alpha') \le \alpha.$$

If $\lambda = 0$ then $\mathrm{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = 1 \le \alpha$; if $\lambda > 0$ then α is infinite and

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta) \le \max(\omega, \mathrm{cf}(\prod_{\zeta\in I}\theta_\zeta), \mathrm{cf}(\prod_{\zeta\in J}\theta_\zeta)) \le \max(\alpha,\delta).$$

This is enough to deal with (a).

case 2 If $\alpha' = \alpha = 0$ then $\lambda = 0 = \delta$, so

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta) = 1 = \delta^{\mathrm{cf}\,\alpha}.$$

So (b) is true if $\alpha = 0$.

case 3 If $\alpha' = \alpha > 0$ and $\mathrm{cf}\,\alpha < \gamma$, then $\lambda > 0$ and α is a supremum of strictly smaller infinite cardinals, so must be uncountable. Let $\langle\alpha_\xi\rangle_{\xi<\mathrm{cf}\,\alpha}$ be a strictly increasing family of cardinals with supremum α, starting from $\alpha_0 = 0$ and $\alpha_1 = \omega$ and with $\alpha_\xi = \sup_{\eta<\xi}\alpha_\eta$ for non-zero limit ordinals $\xi < \mathrm{cf}\,\alpha$. Set

$$P_\xi = \textstyle\prod_{\zeta<\lambda,\alpha_\xi\le\theta_\zeta<\alpha_{\xi+1}}\theta_\zeta$$

for each $\xi < \mathrm{cf}\,\alpha$. Then

$$\mathrm{cf}\,P_\xi \le \Theta(\alpha_{\xi+1},\gamma) \le \delta$$

for each $\xi < \mathrm{cf}\,\alpha$, so

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta) = \mathrm{cf}(\prod_{\xi<\mathrm{cf}\,\alpha}P_\xi) \le \delta^{\mathrm{cf}\,\alpha}.$$

Putting this together with case 1, we have a proof of (b) when $\alpha > 0$.

5A3 Forcing

My discussion of forcing is based on Kunen 80; in particular, I start from pre-ordered sets rather than Boolean algebras, and the class $V^{\mathbb{P}}$ of terms in a forcing language will consist of subsets of $V^{\mathbb{P}} \times P$. I find however that I wish to diverge almost immediately from standard formulations in a technical respect, which I describe in 5A3A, introducing what I call 'forcing notions'. I do not refer to generic filters or models of ZFC, preferring to express all results in terms of the forcing relation (5A3C). I give some space to the interpretation of names (5A3E, 5A3H) and, in particular, to names for real numbers derived from elements of $L^0(\mathrm{RO}(\mathbb{P}))$ (5A3L).

5A3A Forcing notions (a) A **forcing notion** is a quadruple $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ or $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ where $(P, \le)$ is a pre-ordered set (that is, $\le$ is a transitive reflexive relation on P), $\mathbb{1} \in P$, and

if $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ then $\mathbb{1} \le p$ for every $p \in P$,

if $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ then $p \le \mathbb{1}$ for every $p \in P$.

In this context members of P are commonly called **conditions**.

(b) I had better try to explain what I am doing here. The problem is the following. Consider two of the standard examples of pre-ordered set in this context. For a set I, $\mathrm{Fn}_{<\omega}(I; \{0,1\})$ is the set of functions from finite subsets of I to $\{0,1\}$; for a non-trivial Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^+$ is the set of non-zero elements of $\mathfrak{A}$. In each case, we have a relevant direction. In $\mathrm{Fn}_{<\omega}(I; \{0,1\})$, a condition p is stronger than a condition q if p extends q, that is, if $p \supseteq q$; in $\mathfrak{A}^+$, p is stronger than q if $p \subseteq q$. So the forcing notions, in the terminology I have chosen, are

$$(\mathrm{Fn}_{<\omega}(I; \{0,1\}), \subseteq, \emptyset, \uparrow), \quad (\mathfrak{A}^+, \subseteq, 1_{\mathfrak{A}}, \downarrow).$$

Generally, I will say that a forcing notion $(P, \le, \mathbb{1}, \uparrow)$ is **active upwards**, while $(P, \le, \mathbb{1}, \downarrow)$ is **active downwards**.

(c) Of course this is unconventional. It is much more usual to take all forcing notions to be active in the same direction (usually downwards) and to use local definitions (e.g., saying that '$p \le q$ if p extends q') to ensure that this will be appropriate.

However the great majority of forcing notions, like the two examples in (b) above, come with structures which strongly suggest a natural interpretation of '$\le$'; and these structures are not arbitrary, but are essential to our intuitive conception of the pre- or partial order we are studying. I prefer, therefore, to maintain the notation I would use for the same objects in any other context, and to indicate separately the orientation which is relevant when using them to build a forcing language.

(d) This approach demands further changes in the language. It will no longer be helpful to talk about conditions in P being 'larger' or 'less than' others. Instead, I will use the word '**stronger**'; if $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$, then $p \in P$ will be stronger than $q \in P$ if $p \ge q$; if $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$, then $p \in P$ will be stronger than $q \in P$ if $p \le q$. (So p will be stronger than $\mathbb{1}$ for every $p \in P$.)

Similarly, the words 'cofinal' and 'coinitial' are now inappropriate, and I will turn to the word 'dense', as favoured by most authors discussing forcing; if $\mathbb{P} = (P, \le, \mathbb{1}, \updownarrow)$ is a forcing notion, a subset Q of P is **dense** if for every $p \in P$ there is a $q \in Q$ such that q is stronger than p. In the same way, I can say that two conditions p, q in P are 'compatible' if there is an $r \in P$ stronger than both. We shall have a standard topology on P generated by sets of the form $\{q : q$ is stronger than $p\}$, and a corresponding regular open algebra $\mathrm{RO}(\mathbb{P})$. An antichain for $\mathbb{P}$ will be a set $A \subseteq P$ such that any two distinct conditions in A are incompatible, and $\mathbb{P}$ will be ccc if every antichain for $\mathbb{P}$ is countable. The 'saturation' $\operatorname{sat}\mathbb{P}$ of $\mathbb{P}$ will be the least cardinal κ such that there is no antichain of size κ.

5A3B Forcing languages Let $\mathbb{P} = (P, \le, \mathbb{1}, \updownarrow)$ be a forcing notion.

(a) The class of $\mathbb{P}$**-names**, that is, terms of the forcing language defined by $\mathbb{P}$, is

$$V^{\mathbb{P}} = \{A : A \text{ is a set and } A \subseteq V^{\mathbb{P}} \times P\}$$

(Kunen 80, VII.2.5)[1]. In this context, I will say that the **domain** of a name $A \in V^{\mathbb{P}}$ is the set $\operatorname{dom} A \subseteq V^{\mathbb{P}}$ of first members of elements of A.

(b) For any set X, $\check{X}$ will be the $\mathbb{P}$-name $\{(\check{x}, \mathbb{1}) : x \in X\} \in V^{\mathbb{P}}$ (Kunen 80, VII.2.10).

5A3C The Forcing Relation (Kunen 80, VII.3.3) Suppose that $\mathbb{P} = (P, \le, \mathbb{1}, \updownarrow)$ is a forcing notion, $p \in P$, ϕ, ψ are formulae of set theory, and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$.

[1]For once, I rely on the Axiom of Foundation; to determine whether a set A belongs to $V^{\mathbb{P}}$, we need to induce on the rank of A.

(a) $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$ iff

whenever $(\dot{y}, q) \in \dot{x}_0$ and $r \in P$ is stronger than both p and q, there are a $(\dot{y}', q') \in \dot{x}_1$ and an r' stronger than both r and q' such that $r' \Vdash_{\mathbb{P}} \dot{y} = \dot{y}'$,

whenever $(\dot{y}, q) \in \dot{x}_1$ and $r \in P$ is stronger than both p and q, there are a $(\dot{y}', q') \in \dot{x}_0$ and an r' stronger than both r and q' such that $r' \Vdash_{\mathbb{P}} \dot{y} = \dot{y}'$.

(b) $p \Vdash_{\mathbb{P}} \dot{x}_0 \in \dot{x}_1$ iff

whenever $q \in P$ is stronger than p there are a $(\dot{y}, q') \in \dot{x}_1$ and an r stronger than both q and q' such that $r \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{y}$.

(c) $p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n) \,\&\, \psi(\dot{x}_0, \dots, \dot{x}_n)$ iff

$$p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n) \text{ and } p \Vdash_{\mathbb{P}} \psi(\dot{x}_0, \dots, \dot{x}_n).$$

(d) $p \Vdash_{\mathbb{P}} \neg\phi(\dot{x}_0, \dots, \dot{x}_n)$ iff $q \not\Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n)$ for every q stronger than p.

(e) $p \Vdash_{\mathbb{P}} \exists\, x,\ \phi(x, \dot{x}_0, \dots, \dot{x}_n)$ iff for every q stronger than p there are an r stronger than q and a $\dot{y} \in V^{\mathbb{P}}$ such that $r \Vdash_{\mathbb{P}} \phi(\dot{y}, \dot{x}_0, \dots, \dot{x}_n)$.[2]

(f) In this context I will write $\Vdash_{\mathbb{P}}$ for $\mathbb{1} \Vdash_{\mathbb{P}}$.

5A3D The Forcing Theorem If ϕ is any theorem of ZFC, and $\mathbb{P}$ is any forcing notion, then $\Vdash_{\mathbb{P}} \phi$. (Kunen 80, VII.4.2.)

5A3E More notation In 5A3C I took it for granted that every formula of set theory would have a version in $V^{\mathbb{P}}$. I should perhaps explain some of the versions I have in mind. Let $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ be a forcing notion.

(a) If $\dot{y}_0$, $\dot{y}_1 \in V^{\mathbb{P}}$ then $\dot{x} = \{(\dot{y}_0, \mathbb{1}), (\dot{y}_1, \mathbb{1})\} \in V^{\mathbb{P}}$, and

$$\Vdash_{\mathbb{P}} \dot{x} = \{\dot{y}_0, \dot{y}_1\};$$

so we have a suitable formal expression for pair sets in $V^{\mathbb{P}}$. Similarly, if we think of the formula (x, y) as being an abbreviation for $\{\{x\}, \{x, y\}\}$, we get a $\mathbb{P}$-name

$$\dot{z} = \{(\{(\dot{y}_0, \mathbb{1})\}, \mathbb{1}), (\{(\dot{y}_0, \mathbb{1}), (\dot{y}_1, \mathbb{1})\}, \mathbb{1})\}$$

such that

$$\Vdash_{\mathbb{P}} \dot{z} = (\dot{y}_0, \dot{y}_1).$$

(b) Now let $\langle \dot{x}_i \rangle_{i \in I}$ be a family of $\mathbb{P}$-names. Then

$$\dot{f} = \{((\check{\imath}, \dot{x}_i), \mathbb{1}) : i \in I\}$$

is a $\mathbb{P}$-name, provided that in the formula $((\check{\imath}, \dot{x}_i), \mathbb{1})$ we interpret the inner pair (,) of brackets as an ordered pair in $V^{\mathbb{P}}$, that is, as

$$\{(\{(\check{\imath}, \mathbb{1})\}, \mathbb{1}), (\{(\check{\imath}, \mathbb{1}), (\dot{x}_i, \mathbb{1})\}, \mathbb{1})\}$$

and the outer pair of brackets $(\ , \mathbb{1})$ as an ordered pair in the ordinary universe. In this case,

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is a function with domain } \check{I},$$

and

$$\Vdash_{\mathbb{P}} \dot{f}(\check{\imath}) = \dot{x}_i$$

for every $i \in I$. For obvious reasons I do not wish to spell this procedure out every time, and I will use the rather elliptic formula

$$\langle \dot{x}_i \rangle_{i \in \check{I}}$$

to signify the $\mathbb{P}$-name $\dot{f}$.

(c) Similarly,

[2] This formulation is appropriate if we wish to explore forcing without using the axiom of choice. Subject to AC, we have an alternative condition: $p \Vdash_{\mathbb{P}} \exists\, x,\ \phi(x, \dot{x}_0, \dots, \dot{x}_n)$ iff there is a $\dot{y} \in V^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \phi(\dot{y}, \dot{x}_0, \dots, \dot{x}_n)$.

$$\dot{T} = \{(\dot{x}_i, \mathbb{1}) : i \in I\}$$

is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{x}_i \in \dot{T}$$

for every $i \in I$, and whenever $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{T}$, there are an $i \in I$ and a q stronger than p such that $q \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_i$; I will write $\{\dot{x}_i : i \in \check{I}\}$ for $\dot{T}$.

(d) In the same spirit, if I have a family $\langle \dot{x}_i \rangle_{i \in I}$ of $\mathbb{P}$-names for real numbers between 0 and 1, I will allow myself to write '$\sup_{i \in \check{I}} \dot{x}_i$' to signify a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \sup_{i \in \check{I}} \dot{x}_i = \sup\{\dot{x}_i : i \in \check{I}\},$$

without taking the trouble to spell out any exact formula to represent the supremum. I will do the same for limits of sequences; if $\langle \dot{x}_n \rangle_{n \in \mathbb{N}}$ is a sequence of $\mathbb{P}$-names for real numbers, and

$$\Vdash_{\mathbb{P}} \langle \dot{x}_n \rangle_{n \in \mathbb{N}} \text{ is convergent},$$

then I will write '$\lim_{n \to \infty} \dot{x}_n$' to mean a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash_{\mathbb{P}} \langle \dot{x}_n \rangle_{n \in \mathbb{N}} \to \dot{x} \in \mathbb{R}.$$

Of course this is tolerable only because it is possible to set out a general rule for constructing a suitable name $\dot{x} \in V^{\mathbb{P}}$ from the given sequence $\langle \dot{x}_n \rangle_{n \in \mathbb{N}}$.

5A3F Boolean truth values Let $\mathbb{P}$ be a forcing notion and P its set of conditions.

(a) If ϕ is a formula of set theory, and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$, then

$$\{p : p \in P,\ p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n)\}$$

is a regular open set in P (use 514Md); I will denote it $[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]$.

(b) If ϕ and ψ are formulae of set theory and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$, then

$$[\![\phi(\dot{x}_0, \dots, \dot{x}_n) \,\&\, \psi(\dot{x}_0, \dots, \dot{x}_n)]\!] = [\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!] \cap [\![\psi(\dot{x}_0, \dots, \dot{x}_n)]\!]$$

and

$$[\![\neg\phi(\dot{x}_0, \dots, \dot{x}_n)]\!] = P \setminus \overline{[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]},$$

the complement of $[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]$ in $\mathrm{RO}(\mathbb{P})$.

(c) If ϕ is a formula of set theory, A is a set, and $\dot{x}_0, \dots, \dot{x}_n$ are $\mathbb{P}$-names, then

$$[\![\exists\, x \in \check{A},\ \phi(x, \dot{x}_0, \dots, \dot{x}_n)]\!] = \operatorname{int}\overline{\bigcup_{a \in A} [\![\phi(\check{a}, \dot{x}_0, \dots, \dot{x}_n)]\!]},$$

the supremum of $\{[\![\phi(\check{a}, \dot{x}_0, \dots, \dot{x}_n]\!] : a \in A\}$ in $\mathrm{RO}(\mathbb{P})$. (Use 5A3Ce.)

5A3G Concerning ˇs (a) The reader is entitled to an attempt at consistency on the following point of notation, among others. For any set X and any forcing notion $\mathbb{P}$ there is a corresponding $\mathbb{P}$-name $\check{X}$ (5A3Bb). We start with $\check{\emptyset} = \emptyset$. If $1 = \{\emptyset\}$ is the next von Neumann ordinal, we get a name

$$\check{1} = \{(\check{\emptyset}, \mathbb{1})\} = \{(\emptyset, \mathbb{1})\};$$

and we can check directly from 5A3C that

$$\Vdash_{\mathbb{P}} \check{1} = \{\emptyset\},$$

that is, if you like,

$$\Vdash_{\mathbb{P}} \check{1} = 1,$$

where in this formula the first 1 is interpreted in the ordinary universe and the second is interpreted in the forcing language. Similarly, if we take '2' to be an abbreviation for '$\{\emptyset, \{\emptyset\}\}$', we have

$$\Vdash_{\mathbb{P}} \check{2} = 2,$$

and so on. Indeed we get

$$\Vdash_{\mathbb{P}} \check{\omega} \text{ is the first infinite ordinal},$$

$$\Vdash_{\mathbb{P}} \check{\mathbb{Q}} \text{ is the set of rational numbers},$$

so the same convention would lead to

$$\Vdash_{\mathbb{P}} \check{\omega} = \omega,\ \check{\mathbb{Q}} = \mathbb{Q}.$$

(This formula does not depend on which construction of the set of rational numbers we use, provided that we use the same method both in the ordinary universe and in the forcing language.) Of course it is *not* the case (except for forcing notions of particular types) that

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is the first uncountable ordinal}, \qquad \Vdash_{\mathbb{P}} \check{\mathbb{R}} \text{ is the set of real numbers.}$$

(b) For 'absolute' objects, therefore, like $\omega + 7$, appearing in sentences of a forcing language, I shall have a choice between formulations

$$(\omega + 7)\check{}$$

(working directly from 5A3Bb),

$$\omega + 7$$

(regarding the phrase '$\omega + 7$' as an abbreviation for an expression in set theory which can be evaluated either in the ordinary universe or in the forcing language), or

$$\check{\omega}\,\check{+}\,\check{7}$$

(regarding ω, $+$ and 7 as sets to which the rule of 5A3Bb can be applied, and then interpreting the combination in the forcing language). The least cluttered version, $\omega + 7$, looks better, and this will ordinarily be my choice. But it means that when you see the symbol $\mathbb{Q}$ in a sentence of the forcing language, it is likely to mean two things at once, a superposition of 'the set of rational numbers' and 'the $\mathbb{P}$-name $\check{\mathbb{Q}}$'.

5A3H Names for functions Let $\mathbb{P}$ be a forcing notion, P its set of conditions, and $R \subseteq V^{\mathbb{P}} \times V^{\mathbb{P}} \times P$ a set. Consider the $\mathbb{P}$-names

$$\dot{f} = \{((\dot{x}, \dot{y}), p) : (\dot{x}, \dot{y}, p) \in R\}, \quad \dot{A} = \{(\dot{x}, p) : (\dot{x}, \dot{y}, p) \in R\}.$$

(a) The following are equiveridical:

(i) $\Vdash_{\mathbb{P}} \dot{f}$ is a function;

(ii) whenever $(\dot{x}_0, \dot{y}_0, p_0)$, $(\dot{x}_1, \dot{y}_1, p_1)$ belong to R, $p \in \mathbb{P}$ is stronger than both p_0 and p_1 and $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$, then $p \Vdash_{\mathbb{P}} \dot{y}_0 = \dot{y}_1$.

(b) In this case,

$$\Vdash_{\mathbb{P}} \operatorname{dom} \dot{f} = \dot{A}.$$

Remark In the formula for $\dot{f}$ here, as in that of 5A3Eb, the brackets take different meanings at different points. In the expression $((\dot{x}, \dot{y}), p)$, the inner brackets must be interpreted in the forcing language, while the outer brackets, like the brackets in the expression $(\dot{x}, \dot{y}, p) \in R$, are interpreted in the ordinary universe; Kunen 80 might write

$$\dot{f} = \{(\mathrm{op}(\dot{x}, \dot{y}), p) : (\dot{x}, \dot{y}, p) \in R\}.$$

proof Elementary.

5A3I Regular open algebras If $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ is a forcing notion with regular open algebra $\mathrm{RO}(\mathbb{P})$, then we have a natural map $\iota : P \to \mathrm{RO}(\mathbb{P})^+$ defined by saying that

$$\iota(p) = \operatorname{int} \overline{\{q : q \text{ is stronger than } p\}}$$

for $p \in P$ (Kunen 80, II.3.3); and (allowing for the possible reversal of the direction of $\mathbb{P}$) ι is a dense embedding of the pre-ordered set $(P, \le)$ in the partially ordered set $(\mathrm{RO}(\mathbb{P})^+, \subseteq)$, in the sense of Kunen 80, §VII.7. Consequently, taking $\widehat{\mathbb{P}}$ to be the forcing notion $(\mathrm{RO}(\mathbb{P})^+, \subseteq, P, \downarrow)$, we shall have

$$\Vdash_{\mathbb{P}} \phi \text{ if and only if } \Vdash_{\widehat{\mathbb{P}}} \phi$$

for every sentence ϕ of set theory (Kunen 80, VII.7.11). It follows that if two forcing notions have isomorphic regular open algebras, then they force exactly the same theorems of set theory.

5A3J The following technical device will be useful at one point.

Definition Let $\mathbb{P}$ be a forcing notion. I will say that a $\mathbb{P}$-name $\dot{X}$ is **discriminating** if whenever $(\dot{x}, p)$ and $(\dot{y}, q)$ are distinct members of $\dot{X}$, and r is stronger than both p and q, then $r \Vdash_{\mathbb{P}} \dot{x} \ne \dot{y}$.

5A3K Lemma Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any $\mathbb{P}$-name $\dot{X}$, there is a discriminating $\mathbb{P}$-name $\dot{X}_1$ such that $\Vdash_{\mathbb{P}} \dot{X}_1 = \dot{X}$.

(b) Let $\dot{X}$ be a discriminating $\mathbb{P}$-name, and $f : \dot{X} \to V^{\mathbb{P}}$ a function. Let $\dot{g}$ be the $\mathbb{P}$-name

$$\{((\dot{x}, f(\dot{x}, p)), p) : (\dot{x}, p) \in \dot{X}\}.^3$$

Then

$$\Vdash_{\mathbb{P}} \dot{g} \text{ is a function with domain } \dot{X}.$$

proof (a)(i) Set

$$\dot{X}_0 = \{(\dot{x}, q) : \text{there is some } p \in P \text{ such that } (\dot{x}, p) \in \dot{X} \text{ and } q \in P \text{ is stronger than } p\}.$$

Then $\Vdash_{\mathbb{P}} \dot{X} = \dot{X}_0$. **P** Because $\dot{X} \subseteq \dot{X}_0$, $\Vdash_{\mathbb{P}} \dot{X} \subseteq \dot{X}_0$. In the other direction, if $\dot{x}$ is a $\mathbb{P}$-name and p is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{X}_0$, there are an $(\dot{x}_1, p_1) \in \dot{X}_0$ and a q, stronger than both p and p_1, such that $q \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_1$. Now there is a p_2 such that $(\dot{x}_1, p_2) \in \dot{X}$ and p_1 is stronger than p_2. In this case, q is stronger than p_2 and $p_2 \Vdash_{\mathbb{P}} \dot{x}_1 \in \dot{X}$, so $q \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_1 \in \dot{X}$, while q is stronger than p. As $\dot{x}$ and p are arbitrary, $\Vdash_{\mathbb{P}} \dot{X}_0 \subseteq \dot{X}$. **Q**

Let $\dot{X}_1 \subseteq \dot{X}_0$ be a maximal discriminating name.

(ii) Because $\dot{X}_1 \subseteq \dot{X}_0$, $\Vdash_{\mathbb{P}} \dot{X}_1 \subseteq \dot{X}_0 = \dot{X}$. But we also have $\Vdash_{\mathbb{P}} \dot{X} \subseteq \dot{X}_1$. **P** Suppose that $\dot{x}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{X}$. Then there must be an $(\dot{x}_1, p_1) \in \dot{X}$ and a q stronger than both p and p_1 such that $q \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_1$. In this case, $(\dot{x}_1, q) \in \dot{X}_0$. By the maximality of $\dot{X}_1$, there are a $(\dot{y}, q') \in \dot{X}_1$ and an r stronger than both q and q' such that $r \Vdash_{\mathbb{P}} \dot{x}_1 = \dot{y}$. Now r is stronger than p and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_1 = \dot{y} \in \dot{X}_1.$$

As $\dot{x}$ and p are arbitrary, $\Vdash_{\mathbb{P}} \dot{X} \subseteq \dot{X}_1$. **Q**

So $\Vdash_{\mathbb{P}} \dot{X}_1 = \dot{X}$, as required.

(b) Consider

$$R = \{(\dot{x}, f(\dot{x}, p), p) : (\dot{x}, p) \in \dot{X}\}.$$

If $(\dot{x}_0, p_0)$ and $(\dot{x}_1, p_1)$ belong to $\dot{X}$, p is stronger than both p_0 and p_1, and $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$, then $(\dot{x}_0, p_0) = (\dot{x}_1, p_1)$, because $\dot{X}$ is a discriminating name; so $p \Vdash_{\mathbb{P}} f(\dot{x}_0, p_0) = f(\dot{x}_1, p_1)$. By 5A3H,

$$\Vdash_{\mathbb{P}} \dot{g} \text{ is a function and } \operatorname{dom} \dot{g} = \dot{X}.$$

5A3L Real numbers in forcing languages (a) Let $\mathbb{P}$ be any forcing notion, and P its set of conditions.

(a) I have tried to avoid committing myself to any declaration of what a real number actually 'is'; in fact I believe that at the deepest level this should be regarded as an undefined concept, and that the descriptions offered by Weierstrass and Dedekind are essentially artificial. But if we are to make sense of real analysis in forcing models we must fix on some formulation, so I will say that a real number is the set of strictly smaller rational numbers. (I leave it to you to decide whether a rational number is an equivalence class of pairs of integers, or a coprime pair (m, n) where $m \in \mathbb{Z}$ and $n \in \mathbb{N} \setminus \{0\}$, or something else altogether, provided only that you fix on a construction expressible by a formula of set theory.) Observe that this has the desirable effect that

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is a real number}$$

for every real number α.

(b) Consider the Dedekind complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ and the corresponding space $L^0 = L^0(\mathrm{RO}(\mathbb{P}))$ as defined in 364A.

(i) For every $u \in L^0$, set

$$\vec{u} = \{(\check{\alpha}, p) : \alpha \in \mathbb{Q}, p \in [\![u > \alpha]\!]\}.$$

Then

$$\Vdash_{\mathbb{P}} \vec{u} \text{ is a real number.}$$

P Of course $\Vdash_{\mathbb{P}} \vec{u} \subseteq \mathbb{Q}$. If $p \in P$, there are an $n \in \mathbb{Z}$ and a q stronger than p such that $q \in [\![u > n]\!]$, in which case $(\check{n}, q) \in \vec{u}$ and $q \Vdash_{\mathbb{P}} \vec{u} \neq \emptyset$; accordingly $\Vdash_{\mathbb{P}} \vec{u} \neq \emptyset$. Again, if $p \in P$, there are an $n \in \mathbb{N}$ and a q stronger than p such that $q \in [\![u \le n]\!]$, in which case $\alpha \le n$ whenever $(\check{\alpha}, r) \in \vec{u}$ and r is stronger than q, so that

---

[3] Once again I present a formula in which some ordered pairs are to be interpreted in the ordinary universe, but another is to be interpreted in the forcing language.

$$q \Vdash_{\mathbb{P}} \check{n} \text{ is an upper bound for } \vec{u};$$

accordingly $\Vdash_{\mathbb{P}} \vec{u}$ is bounded above.

If $p \in P$ and $\dot{\alpha}$, $\dot{\beta}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \mathbb{Q},\ \dot{\alpha} \le \dot{\beta} \in \vec{u},$$

then for any q stronger than p there are an r stronger than q and α, $\beta \in \mathbb{Q}$ such that

$$r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha},\ \dot{\beta} = \check{\beta}$$

and $(\check{\beta}, r) \in \vec{u}$. In this case, $\alpha \le \beta$, $r \in [\![u > \beta]\!] \subseteq [\![u > \alpha]\!]$, $(\check{\alpha}, r) \in \vec{u}$ and $r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha} \in \vec{u}$. As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \vec{u};$$

as p, $\dot{\alpha}$ and $\dot{\beta}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \alpha \in \vec{u} \text{ whenever } \alpha \in \mathbb{Q} \text{ and } \alpha \le \beta \in \vec{u}.$$

If $p \in P$ and $\dot{\alpha}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \vec{u},$$

then for any q stronger than p there are an r stronger than q and an $\alpha \in \mathbb{Q}$ such that $r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha}$ and $r \in [\![u > \alpha]\!]$. Now there are a $\beta \in \mathbb{Q}$ and an r' stronger than r such that $\beta > \alpha$ and $r' \in [\![u > \beta]\!]$; in which case $r' \Vdash_{\mathbb{P}} \dot{\alpha} < \check{\beta} \in \vec{u}$. As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \text{ is not the greatest member of } \vec{u};$$

as p and $\dot{\alpha}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{u} \text{ has no greatest member, and is a real number. } \mathbf{Q}$$

(ii) Observe next that $[\![\vec{u} > \check{\alpha}]\!] = [\![u > \alpha]\!]$ for every $\alpha \in \mathbb{Q}$. **P** For $p \in P$,

$$\begin{aligned} p \in [\![\vec{u} > \check{\alpha}]\!] &\iff p \Vdash_{\mathbb{P}} \check{\alpha} < \vec{u} \\ &\iff p \Vdash_{\mathbb{P}} \check{\alpha} \in \vec{u} \\ &\iff \text{ for every } q \text{ stronger than } p \text{ there are } q' \in P,\ \beta \in \mathbb{Q} \\ &\qquad \text{and an } r \text{ stronger than both } q \text{ and } q' \\ &\qquad \text{such that } (\check{\beta}, q') \in \vec{u} \text{ and } r \Vdash_{\mathbb{P}} \check{\beta} = \check{\alpha} \\ &\iff \text{ for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\ &\qquad \text{such that } (\check{\alpha}, r) \in \vec{u} \\ &\iff \text{ for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\ &\qquad \text{such that } r \in [\![u > \alpha]\!] \\ &\iff p \in [\![u > \alpha]\!] \end{aligned}$$

(514Md, because $[\![u > \alpha]\!]$ is a regular open subset of P). **Q**

(iii) In the other direction, if we have a $\mathbb{P}$-name $\dot{x}$ for a real number (that is, a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x}$ is a real number), then there is a unique $u \in L^0$ such that $\Vdash_{\mathbb{P}} \dot{x} = \vec{u}$. **P** For every $\alpha \in \mathbb{Q}$ we have a Boolean value $[\![\dot{x} > \check{\alpha}]\!]$ belonging to $\mathrm{RO}(\mathbb{P})$ (5A3F). It is easy to see that

$$[\![\dot{x} > \check{\alpha}]\!] = \sup_{\beta \in \mathbb{Q}, \beta > \alpha} [\![\dot{x} > \check{\beta}]\!]$$

for every $\alpha \in \mathbb{Q}$,

$$\inf_{n \in \mathbb{Z}} [\![\dot{x} > \check{n}]\!] = 0, \quad \sup_{n \in \mathbb{Z}} [\![\dot{x} > \check{n}]\!] = 1.$$

We therefore have a unique $u \in L^0$ such that

$$[\![u > \alpha]\!] = \sup_{\beta \in \mathbb{Q}, \beta > \alpha} [\![\dot{x} > \check{\beta}]\!] = [\![\dot{x} > \check{\alpha}]\!]$$

for every $\alpha \in \mathbb{R}$. **Q**

(iv) It follows that if $\dot{x}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \mathbb{R}$, then there is a $u \in L^0$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{u}$. (For there is a $\mathbb{P}$-name $\dot{y}$ such that $\Vdash_{\mathbb{P}} \dot{y} \in \mathbb{R}$ and $p \Vdash_{\mathbb{P}} \dot{x} = \dot{y}$.)

(v) If $\alpha \in \mathbb{R}$, then $(\alpha\chi 1)^{\rightharpoonup} = \check{\alpha}$. **P** For any $\beta \in \mathbb{Q}$ and $p \in P$,

$$[\![(\alpha\chi 1)^{\rightharpoonup} > \check\beta]\!] = [\![\alpha\chi 1 > \beta]\!] = 1 \text{ if } \beta < \alpha, 0 \text{ otherwise}$$
$$= [\![\check\alpha > \check\beta]\!] \text{ in either case. } \mathbf{Q}$$

(c) Suppose that u, $v \in L^0$.

(i) $[\![\vec u < \vec v]\!] = [\![v - u > 0]\!]$. $\mathbf{P}$

$$[\![\vec u < \vec v]\!] = [\![\exists\, \alpha \in \mathbb{Q}, \vec u \le \alpha < \vec v]\!]$$
$$= \sup_{\alpha\in\mathbb{Q}} [\![\vec u \le \check\alpha < \vec v]\!]$$

(taking the supremum in RO($\mathbb{P}$), 5A3Fd)

$$= \sup_{\alpha\in\mathbb{Q}} ([\![\vec v > \check\alpha]\!] \setminus [\![\vec u > \check\alpha]\!])$$

(taking the relative complements in RO($\mathbb{P}$))

$$= \sup_{\alpha\in\mathbb{Q}} ([\![v > \alpha]\!] \setminus [\![u > \alpha]\!]) = [\![v - u > 0]\!]. \ \mathbf{Q}$$

(ii) In particular, if $u \le v$ in L^0, then

$$\Vdash_{\mathbb{P}} \vec u \le \vec v \text{ in } \mathbb{R}$$

since $[\![\vec v < \vec u]\!] = 0$. In the same way,

$$[\![\vec u = \vec v]\!] = [\![u = v]\!]$$

for any u, $v \in L^0$.

(iii) $\Vdash_{\mathbb{P}} (u+v)^{\rightharpoonup} = \vec u + \vec v$. $\mathbf{P}$ For any $\alpha \in \mathbb{Q}$,

$$[\![\vec u + \vec v > \check\alpha]\!] = [\![\exists\ \beta \in \mathbb{Q}, \vec u > \beta, \vec v > \check\alpha - \beta]\!]$$
$$= \sup_{\beta\in\mathbb{Q}} [\![\vec u > \check\beta, \vec v > \check\alpha - \check\beta]\!]$$
$$= \sup_{\beta\in\mathbb{Q}} ([\![\vec u > \check\beta]\!] \cap [\![\vec v > (\alpha-\beta)\check{}\,]\!])$$
$$= \sup_{\beta\in\mathbb{Q}} ([\![u > \beta]\!] \cap [\![v > \alpha - \beta]\!])$$
$$= [\![u + v > \alpha]\!]$$

(364E)

$$= [\![(u+v)^{\rightharpoonup} > \check\alpha]\!]. \ \mathbf{Q}$$

(iv) $\Vdash_{\mathbb{P}} (u\times v)^{\rightharpoonup} = \vec u \times \vec v$. $\mathbf{P}$ If u, $v \ge 0$ in L^0 and $\alpha \ge 0$ in $\mathbb{Q}$,

$$[\![\vec u \times \vec v > \check\alpha]\!] = [\![\exists\ \beta \in \mathbb{Q}, \beta > 0, \vec u > \beta, \vec v > \frac{\check\alpha}{\beta}]\!]$$
$$= \sup_{\beta\in\mathbb{Q},\beta>0} [\![\vec u > \check\beta, \vec v > \frac{\check\alpha}{\check\beta}]\!]$$
$$= \sup_{\beta\in\mathbb{Q},\beta>0} [\![\vec u > \check\beta]\!] \cap [\![\vec v > (\frac{\alpha}{\beta})\check{}\,]\!]$$
$$= \sup_{\beta\in\mathbb{Q},\beta>0} [\![u > \beta]\!] \cap [\![v > \frac{\alpha}{\beta}]\!]$$
$$= [\![u \times v > \alpha]\!]$$
$$= [\![(u\times v)^{\rightharpoonup} > \check\alpha]\!].$$

So in this case

$$\Vdash_{\mathbb{P}} (u\times v)^{\rightharpoonup} = \vec u \times \vec v.$$

Since we have an appropriate distributive law in L^0, it follows from (iii) that the same is true for general u, $v \in L^0$. $\mathbf{Q}$

(v) If $\alpha \in \mathbb{R}$, then $\Vdash_{\mathbb{P}} (\alpha u)^{\rightharpoonup} = \check\alpha\vec u$. (Put (iv) and (b-v) together.)

(d) Suppose that $\langle u_i\rangle_{i\in I}$ is a non-empty family in L^0 with supremum $u\in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u}=\sup_{i\in\check{I}}\vec{u}_i \text{ in } \mathbb{R}.$$

P By (c-i),

$$\Vdash_{\mathbb{P}} \vec{u}_i\le\vec{u}$$

for every $i\in I$, so

$$\Vdash_{\mathbb{P}} \sup_{i\in\check{I}}\vec{u}_i\le\vec{u}.$$

In the other direction, **?** suppose, if possible, that

$$\not\Vdash_{\mathbb{P}} \vec{u} \text{ is the least upper bound of } \{\vec{u}_i:i\in\check{I}\}.$$

Then there are a $p\in P$ and an $\alpha\in\mathbb{Q}$ such that

$$p\Vdash_{\mathbb{P}} \check{\alpha}<\vec{u} \text{ is an upper bound for } \{\vec{u}_i:i\in\check{I}\}.$$

In this case,

$$p\in[\![\vec{u}>\check{\alpha}]\!]=[\![u>\alpha]\!]=\sup_{i\in I}[\![u_i>\alpha]\!]=\operatorname{int}\overline{\bigcup_{i\in I}[\![u_i>\alpha]\!]}$$

(364Mb). There are therefore a q stronger than p and an $i\in I$ such that $q\in[\![u_i>\alpha]\!]$; but in this case $q\Vdash_{\mathbb{P}}\vec{u}_i>\check{\alpha}$, which is impossible, because $p\Vdash_{\mathbb{P}}\vec{u}_i\le\check{\alpha}$. **X** So

$$\Vdash_{\mathbb{P}} \vec{u}=\sup_{i\in\check{I}}\vec{u}_i. \quad \mathbf{Q}$$

(e) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, order*-convergent (in the sense of §367) to $u\in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u}=\lim_{n\to\infty}\vec{u}_n.$$

P By 367Hb, $u=\inf_{n\in\mathbb{N}}v_n$ where $v_n=\sup_{m\ge n}u_m$ for every $n\in\mathbb{N}$. Now (d) tells us that

$$\Vdash_{\mathbb{P}} \vec{v}_n=\sup_{m\ge\check{n}}\vec{u}_m$$

for every $n\in\mathbb{N}$, and also that

$$\begin{aligned}\Vdash_{\mathbb{P}} \vec{u}&=-(-\vec{u})=-(-u)^{\rightarrow}=-\sup_{n\in\mathbb{N}}(-v_n)^{\rightarrow}\\&=-\sup_{n\in\mathbb{N}}(-\vec{v}_n)=\inf_{n\in\mathbb{N}}\vec{v}_n=\lim_{n\to\infty}\vec{u}_n. \quad \mathbf{Q}\end{aligned}$$

5A3M Forcing with Boolean algebras Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$. As noted in 5A3Ab, $\mathbb{P}=(\mathfrak{A}^+,\subseteq,1_{\mathfrak{A}},\downarrow)$ is a forcing notion. We have a natural isomorphism between $\mathrm{RO}(\mathbb{P})$ and $\mathfrak{A}$, matching each $G\in\mathrm{RO}(\mathbb{P})$ with $\sup G\in\mathfrak{A}$ (514Sb); by 514Md, as usual, $\sup G$ will belong to G unless $G=\emptyset$. In this context, I will usually identify the two algebras, so that $[\![\phi]\!]$ becomes $\sup\{a:a\in\mathfrak{A}^+,\ a\Vdash_{\mathbb{P}}\phi\}$, and we shall have $[\![\phi]\!]\Vdash_{\mathbb{P}}\phi$ except when $\Vdash_{\mathbb{P}}\neg\phi$.

The identification of $\mathrm{RO}(\mathbb{P})$ with $\mathfrak{A}$ itself simplifies some of the discussion in 5A3L. We have a $\mathbb{P}$-name $\vec{u}$ associated with each $u\in L^0(\mathfrak{A})$, and the formula

$$[\![\vec{u}=\vec{v}]\!]=[\![u=v]\!]$$

of 5A3L(c-ii) turns into

$$u\times\chi a=v\times\chi a\iff a\Vdash_{\mathbb{P}}\vec{u}=\vec{v}$$

whenever u, $v\in L^0(\mathfrak{A})$ and $a\in\mathfrak{A}^+$. **P**

$$\begin{aligned}u\times\chi a=v\times\chi a&\iff(u-v)\times\chi a=0\\&\iff|u-v|\wedge\chi a=0\end{aligned}$$

(because $L^0(\mathfrak{A})$ is an f-algebra, by 364D, so we can use 353Pb, or otherwise)

$$\iff[\![|u-v|>0]\!]\cap[\![\chi a>0]\!]=0$$

(364Nb)

$$\begin{aligned}&\iff a\cap[\![|u-v|>0]\!]=0\\&\iff a\cap[\![|u-v|^{\rightarrow}>0]\!]=0\\&\iff a\cap[\![|\vec{u}-\vec{v}|>0]\!]=0\end{aligned}$$

(assembling facts from 5A3L)

$$\iff a \cap [\![\vec{u} \neq \vec{v}]\!] = 0$$
$$\iff a \Vdash_{\mathbb{P}} \vec{u} = \vec{v}. \text{ } \mathbf{Q}$$

5A3N Ordinals and cardinals Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any ordinal α,

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is an ordinal;}$$

moreover, if $p \in P$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{x} \text{ is an ordinal,}$$

there are a q stronger than p and an ordinal α such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{\alpha}$$

(Jech 03, 14.23).

(b) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is a cardinal}$$

for every cardinal (that is, initial ordinal) κ (Kunen 80, VII.5.6; Jech 03, 14.34). In particular,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal, so is the first uncountable cardinal,}$$

and we can write

$$\Vdash_{\mathbb{P}} \omega_1 = \check{\omega}_1,\ \omega_2 = \check{\omega}_2$$

etc., if we are sure of being understood.

(c) Suppose that $\mathbb{P}$ is ccc, and that we have a set A, a $\mathbb{P}$-name $\dot{X}$ and a cardinal κ such that

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{A} \text{ and } \#(\dot{X}) \leq \check{\kappa}.$$

Then there is a set $B \subseteq A$ such that $\#(B) \leq \max(\omega, \kappa)$ and

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{B}.$$

$\mathbf{P}$ Let $\dot{f}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is an injective function with domain } \dot{X} \text{ and } \dot{f}[\dot{X}] \subseteq \check{\kappa}.$$

For $\xi < \kappa$ set

$$B_\xi = \{a : a \in A \text{ and there is a } p \in P \text{ such that } p \Vdash_{\mathbb{P}} \check{a} \in \dot{X} \ \&\ \dot{f}(\check{a}) = \check{\xi}\}.$$

For each $a \in B_\xi$, choose $p_{\xi a} \in P$ such that

$$p_{\xi a} \Vdash_{\mathbb{P}} \check{a} \in \dot{X} \text{ and } \dot{f}(\check{a}) = \check{\xi};$$

then if a, $b \in B_\xi$ and q is stronger than both $p_{\xi a}$ and $p_{\xi b}$, we have

$$q \Vdash_{\mathbb{P}} \dot{f}(\check{a}) = \dot{f}(\check{b}) \text{ so } \check{a} = \check{b}$$

and $a = b$. Thus $\langle p_{\xi a}\rangle_{a \in B_\xi}$ is an antichain in $\mathbb{P}$ and B_ξ must be countable; setting $B = \bigcup_{\xi<\kappa} B_\xi$, $B \subseteq A$ and $\#(B) \leq \max(\omega, \kappa)$.

Now suppose that $p \in P$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{X}$. Then

$$p \Vdash_{\mathbb{P}} \dot{x} \in \check{A} \text{ and } \dot{f}(\dot{x}) \in \check{\kappa},$$

so there are a q stronger than p, an $a \in A$ and a $\xi < \kappa$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{a} \text{ and } \dot{f}(\dot{x}) = \check{\xi}.$$

Now $a \in B_\xi \subseteq B$, so $q \Vdash_{\mathbb{P}} \dot{x} \in \check{B}$. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{B},$$

as required. $\mathbf{Q}$

(d) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \operatorname{cf}[\check{I}]^{\leq\omega} = (\operatorname{cf}[I]^{\leq\omega})\check{}$$

for every set I. **P** Write κ for $\operatorname{cf}[I]^{\le\omega}$. (i) Let $\mathcal{K}\subseteq[I]^{\le\omega}$ be a cofinal family with $\#(\mathcal{K})=\kappa$. Then $\Vdash_{\mathbb{P}}\check\kappa$ is a cardinal, so

$$\Vdash_{\mathbb{P}}\check{\mathcal{K}}\subseteq[\check I]^{\le\omega}\text{ and }\#(\check{\mathcal{K}})=\check\kappa.$$

If $\dot J$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}}\dot J\in[\check I]^{\le\omega},$$

then by (c) there is a countable set $K\subseteq I$ such that $\Vdash_{\mathbb{P}}\dot J\subseteq\check K$; now there is an $L\in\mathcal{K}$ such that $K\subseteq L$, and

$$\Vdash_{\mathbb{P}}\dot J\subseteq\check K\subseteq\check L\in\check{\mathcal{K}}.$$

As $\dot J$ is arbitrary,

$$\Vdash_{\mathbb{P}}\check{\mathcal{K}}\text{ is cofinal with }[\check I]^{\le\omega}\text{ and }\operatorname{cf}[\check I]^{\le\omega}\le\check\kappa.$$

(ii) **?** If

$$\neg\Vdash_{\mathbb{P}}\check\kappa\le\operatorname{cf}[\check I]^{\le\omega},$$

then there are a $p\in P$ and an ordinal δ such that

$$p\Vdash_{\mathbb{P}}\operatorname{cf}[\check I]^{\le\omega}=\check\delta<\check\kappa.$$

Now there must be a family $\langle\dot J_\xi\rangle_{\xi<\delta}$ of $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}}\{\dot J_\xi:\xi<\check\delta\}\text{ is cofinal with }[\check I]^{\le\omega}.$$

By (c) again, there must be for each $\xi<\kappa$ a countable $K_\xi\subseteq I$ such that $p\Vdash_{\mathbb{P}}\dot J_\xi\subseteq\check K_\xi$. Because $\delta<\operatorname{cf}[I]^{\le\omega}$, there is a $K\in[I]^{\le\omega}$ such that $K_\xi\not\supseteq K$ for every $\xi<\delta$. In this case,

$$p\Vdash_{\mathbb{P}}\check K\in[\check I]^{\le\omega}\text{ so there is a }\xi<\check\delta\text{ such that }\check K\subseteq\dot J_\xi,$$

and there must be a $\xi<\delta$ and a q stronger than p such that

$$q\Vdash_{\mathbb{P}}\check K\subseteq\dot J_\xi\subseteq\check K_\xi.$$

But this implies that $K\subseteq K_\xi$, which isn't so. **X**

We conclude that

$$\Vdash_{\mathbb{P}}\check\kappa\le\operatorname{cf}[\check I]^{\le\omega}\text{ and }\check\kappa=\operatorname{cf}[\check I]^{\le\omega}.\ \mathbf{Q}$$

5A3O Iterated forcing (Kunen 80, VIII.5.2) If $\mathbb{P}$ is a forcing notion and P its set of conditions, and we have a quadruple $\dot{\mathbb{Q}}=(\dot Q,\dot\le,\dot 1,\dot\epsilon)$ of $\mathbb{P}$-names such that $(\dot 1,\mathbb{1}_{\mathbb{P}})\in\dot Q$ and

$\Vdash_{\mathbb{P}}\dot\le$ is a pre-order on $\dot Q$, $\dot\epsilon$ is a direction of activity and every member of $\dot Q$ is stronger than $\dot 1$,

then $\mathbb{P}*\dot{\mathbb{Q}}$ is the forcing notion defined by saying that its conditions are objects of the form $(p,\dot q)$ where

$$p\in P,\quad\dot q\in\operatorname{dom}\dot Q,\quad p\Vdash_{\mathbb{P}}\dot q\in\dot Q,$$

and that $(p,\dot q)$ is stronger than $(p',\dot q')$ if p is stronger than p' and

$$p\Vdash_{\mathbb{P}}\dot q\text{ is stronger than }\dot q'.$$

(Strictly speaking, I should add that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}}=(\mathbb{1}_{\mathbb{P}},\dot 1)$.)[4]

5A3P Martin's axiom Let κ be a regular uncountable cardinal such that $2^\lambda\le\kappa$ for every $\lambda<\kappa$. Then there is a ccc forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}}\mathfrak{m}=\mathfrak{c}=\check\kappa.$$

(Kunen 80, VIII.6.3; Jech 03, 16.13).

5A3Q Countably closed forcings (a) Let $\mathbb{P}$ be a forcing notion, and P its set of conditions. $\mathbb{P}$ is **countably closed** if whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P such that p_{n+1} is stronger than p_n for every n, there is a $p\in P$ which is stronger than every p_n.

(b) If $\mathbb{P}$ is a countably closed forcing notion, then $\Vdash_{\mathbb{P}}\mathcal{P}\mathbb{N}=(\mathcal{P}\mathbb{N})\check{}$. **P** Let P be the set of conditions of $\mathbb{P}$. If $p\in P$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\subseteq\mathbb{N}$, choose $\langle p_n\rangle_{n\in\mathbb{N}}$ inductively in P such that $p_0=p$ and, for each $n\in\mathbb{N}$,

[4] This formulation gives us the freedom to take $\dot\epsilon$ to be non-trivial. I do not mean to suggest that it would be reasonable to take advantage of this.

p_{n+1} is stronger than p_n and either $p_{n+1} \Vdash_{\mathbb{P}} \check{n} \in \dot{x}$ or $p_{n+1} \Vdash_{\mathbb{P}} \check{n} \notin \dot{x}$. Let $q \in P$ be stronger than every p_n, and set $A = \{n : q \Vdash_{\mathbb{P}} \check{n} \in \dot{x}\}$. Then $q \Vdash_{\mathbb{P}} \check{n} \notin \dot{x}$ for every $n \in \mathbb{N} \setminus A$, so $q \Vdash_{\mathbb{P}} \dot{x} = \check{A} \in (\mathcal{P}\mathbb{N})\check{}$. As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} \subseteq (\mathcal{P}\mathbb{N})\check{}$; of course the reverse inequality is trivial. **Q**

Consequently $\Vdash_{\mathbb{P}} \mathbb{R} = \check{\mathbb{R}}$. **P** Of course $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{Q} = (\mathcal{P}\mathbb{Q})\check{}$, and now it is easy to see that

$$\begin{aligned}\Vdash_{\mathbb{P}} \mathbb{R} = \{A : \emptyset \ne A \subseteq \mathbb{Q},\ & A \text{ is bounded above and has no greatest element,} \\ & q \in A \text{ whenever } q \le q' \in A\} \\ = \check{\mathbb{R}}.\ \mathbf{Q} &\end{aligned}$$

Similarly, $\Vdash_{\mathbb{P}} [0,1] = [0,1]\check{}$.

5A3 Notes and comments In terms of the discussion in Kunen 80, §VII.9, you will see that I follow an extreme version of the 'syntactical' approach to forcing. In the first place, this is due to a philosophical prejudice; I do not believe in models of ZF. But it seems to me that quite apart from this there is a fundamental difference between the sentences

$$\mathfrak{m} = \mathfrak{c}$$

and

$$\Vdash_{\mathbb{P}} \mathfrak{m} = \mathfrak{c}$$

associated with the fact that the symbols $\mathfrak{m}$, $\mathfrak{c}$ and even $=$ must be reinterpreted in the second version. I have tried in this section to develop a language which can express and accommodate the difference. It puts a substantial burden on the reader, especially in such formulae as $\sup_{i \in \check{I}} \dot{x}_i$ (5A3E) and $((\dot{y}, f(\dot{y}, p)), p)$ (5A3K), where you may have to read quite carefully to determine which parts of the formulae are supposed to be in the forcing language, and which are in the ordinary language of set theory. There is an additional complication in 5A3L, where I use the same symbol $[\![\ ]\!]$ for two quite different functions; but here at least the objects $[\![u > \alpha]\!]$, $[\![\vec{u} > \check{\alpha}]\!]$ belong to the same set RO($\mathbb{P}$), even if the formulae inside the brackets have to be parsed by very different rules. I hope that the clue of a superscripted letter $\dot{x}$ or $\check{I}$ or $\vec{u}$ will alert you to the need for thought. Once we have grasped this nettle, however, we are in a position to move between the two languages, as in 5A3K; and statements of results such as 5A3P can be shortened by taking it for granted that the preamble '$2^\lambda \le \kappa$ for every $\lambda < \kappa$' refers to the ground universe, while the conclusion '$\mathfrak{m} = \mathfrak{c} = \check{\kappa}$' is to be interpreted in the forcing universe.

Of course a large number of different types of forcing notion have been described and investigated. In 5A3N I mention some basic facts about ccc forcings. Another important class is that of countably closed forcings (5A3Q).

5A4 General Topology

The principal new topological concepts required in this volume are some of the standard cardinal functions of topology (5A4A-5A4B). As usual, there are some particularly interesting phenomena involving compact spaces (5A4C). For special purposes in §513, we need to know some non-trivial facts about metrizable spaces (5A4D, 5A4H). The rest of the section is made up of scraps which are either elementary or standard.

5A4A Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) The **weight** of X, $w(X)$, is the least cardinal of any base for $\mathfrak{T}$.

(b) The **π-weight** of X is $\pi(X) = \mathrm{ci}(\mathfrak{T} \setminus \{\emptyset\})$, the smallest cardinal of any π-base for $\mathfrak{T}$.

(c) The **density** $d(X)$ of X is the smallest cardinal of any dense subset of X.

(d) The **cellularity** of X is

$$c(X) = c^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) = \sup\{\#(\mathcal{G}) : \mathcal{G} \subseteq \mathfrak{T} \setminus \{\emptyset\} \text{ is disjoint}\}.$$

The **saturation** of X is

$$\mathrm{sat}(X) = \mathrm{sat}^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) = \sup\{\#(\mathcal{G})^+ : \mathcal{G} \subseteq \mathfrak{T} \setminus \{\emptyset\} \text{ is disjoint}\},$$

that is, the smallest cardinal κ such that there is no disjoint family of κ non-empty open sets.

(e) The **tightness** of X, $t(X)$, is the smallest cardinal κ such that whenever $A \subseteq X$ and $x \in \overline{A}$ there is a $B \in [A]^{\le\kappa}$ such that $x \in \overline{B}$. (Recall that $[A]^{\le\kappa} = \{B : B \subseteq A,\ \#(B) \le \kappa\}$.)

(f) The **Novák number** $n(X)$ is the smallest cardinal of any family of nowhere dense subsets of X covering X; or ∞ if there is no such family.

(g)(i) The **Lindelöf number** $L(X)$ is the least cardinal κ such that every open cover of X has a subcover of cardinal at most κ.

(ii) The **hereditary Lindelöf number** $\mathrm{hL}(X)$ is $\sup_{Y\subseteq X} L(Y)$.

(h)(i) If $x\in X$, the **character of** x **in** X, $\chi(x,X)$, is the smallest cardinal of any base of neighbourhoods of x.

(ii) The **character** of X is $\chi(X)=\sup_{x\in X}\chi(x,X)$.

(i) The **network weight** of X, $\mathrm{nw}(X)$, is the smallest cardinal of any network for $\mathfrak{T}$.

Remark Recall that X is called 'second-countable' iff $w(X)\le\omega$, 'separable' iff $d(X)\le\omega$, 'ccc' iff $c(X)\le\omega$ (that is, $\mathrm{sat}(X)\le\omega_1$), 'Lindelöf' if $L(X)\le\omega$, 'hereditarily Lindelöf' if $\mathrm{hL}(X)\le\omega$, 'first-countable' if $\chi(X)\le\omega$ and 'countably tight' iff $t(X)\le\omega$.

5A4B Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a)

$$c(X)\le d(X)\le\pi(X)\le w(X),$$

$$\#(\mathfrak{T})\le 2^{\mathrm{nw}(X)},$$

$$\chi(X)\le w(X)\le\max(\#(X),\chi(X)).$$

$\mathrm{sat}(X)=c(X)^+$ unless $\mathrm{sat}(X)$ is a regular uncountable limit cardinal, in which case $\mathrm{sat}(X)=c(X)$.

(b) If Y is a subspace of X, then $w(Y)\le w(X)$, $\mathrm{nw}(Y)\le\mathrm{nw}(X)$ and $\chi(y,Y)\le\chi(y,X)$ for every $y\in Y$.

(c) If a topological space Y is a continuous image of X, then $d(Y)\le d(X)$, $c(Y)\le c(X)$ and $L(Y)\le L(X)$.

(d) If $\mathcal{G}$ is a family of open subsets of X, then there is a subfamily $\mathcal{H}\subseteq\mathcal{G}$ such that $\#(\mathcal{H})<\mathrm{sat}(X)$ and $\overline{\bigcup\mathcal{H}}=\overline{\bigcup\mathcal{G}}$.

(e) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty topological spaces with product X, and λ a cardinal such that $\#(I)\le 2^\lambda$. Then

$$d(X)\le\max(\omega,\lambda,\sup_{i\in I}d(X_i))$$

and

$$c(X)=\sup_{J\subseteq I\text{ is finite}}c(\prod_{i\in J}X_i).$$

(f) If $\mathcal{G}$ is any family of open subsets of X, there is an $\mathcal{H}\subseteq\mathcal{G}$ such that $\#(\mathcal{H})\le\mathrm{hL}(X)$ and $\bigcup\mathcal{H}=\bigcup\mathcal{G}$.

(g) If X is Hausdorff then $\#(X)\le 2^{\max(c(X),\chi(X))}$.

(h) Suppose that X is metrizable.

(i) $d(X)=w(X)$.

(ii) $d(Y)\le d(X)$ for every $Y\subseteq X$. So any discrete subset of X has cardinal at most $d(X)$.

(iii) Let ρ be a metric on X defining its topology. Then X is separable iff there is no uncountable $A\subseteq X$ such that $\inf_{x,y\in A,x\ne y}\rho(x,y)>0$.

proof (a) Let $D\subseteq X$ be a dense set of cardinal $d(X)$. If $\mathcal{G}\subseteq\mathfrak{T}\setminus\{\emptyset\}$ is disjoint, we have a surjection from $D\cap\bigcup\mathcal{G}$ to $\mathcal{G}$, so $\#(\mathcal{G})\le d(X)$; as $\mathcal{G}$ is arbitrary, $c(X)\le d(X)$.

Let $\mathcal{U}$ be a π-base for $\mathfrak{T}$ of cardinal $\pi(X)$. Then there is a set $D\subseteq X$, of cardinal at most $\#(\mathcal{U})$, meeting every non-empty member of $\mathcal{U}$; now D is dense, so $d(X)\le\#(D)\le\pi(X)$.

Any base for $\mathfrak{T}$ is a π-base for $\mathfrak{T}$, so $\pi(X)\le w(X)$.

Let $\mathcal{E}$ be a network for $\mathfrak{T}$ of cardinal $\mathrm{nw}(X)$; then $\mathfrak{T}\subseteq\{\bigcup\mathcal{E}':\mathcal{E}'\subseteq\mathcal{E}\}$, so $\#(\mathfrak{T})\le 2^{\#(\mathcal{E})}=2^{\mathrm{nw}(X)}$.

If $\mathcal{U}$ is a base for $\mathfrak{T}$ of cardinal $w(X)$, and $x\in X$, then $\mathcal{U}_x=\{U:x\in U\in\mathcal{U}\}$ is a base of neighbourhoods of x, so $\chi(x,X)\le\#(\mathcal{U}_x)\le w(X)$; as x is arbitrary, $\chi(X)\le w(X)$.

If X is finite, every point x of X has a smallest neighbourhood V_x, and $\{V_x:x\in X\}$ is a base for $\mathfrak{T}$, so $w(X)\le\#(X)$. If X is infinite, then for each $x\in X$ choose a base $\mathcal{U}_x$ of neighbourhoods of x with $\#(\mathcal{U}_x)=\chi(x,X)\le\chi(X)$. Set $\mathcal{U}=\{\operatorname{int}U:U\in\bigcup_{x\in X}\mathcal{U}_x\}$; then $\mathcal{U}$ is a base for $\mathfrak{T}$ so

$$w(X)\le\#(\mathcal{U})\le\max(\omega,\#(X),\chi(X))=\max(\#(X),\chi(X)).$$

Taking P to be the partially ordered set $(\mathfrak{T}\setminus\{\emptyset\},\subseteq)$, $c(X)=c^{\downarrow}(P)$ and $\mathrm{sat}(X)=\mathrm{sat}^{\downarrow}(P)$, so 513B, inverted, tells us that $\mathrm{sat}(X)=c(X)^+$ unless $\mathrm{sat}(X)$ is a regular uncountable limit cardinal, in which case $\mathrm{sat}(X)=c(X)$.

(b) If $\mathcal{U}$ is a base for $\mathfrak{T}$, then $\{U\cap Y:U\in\mathcal{U}\}$ is a base for the topology of Y, so $w(Y)\le w(X)$. If $\mathcal{E}$ is a network for $\mathfrak{T}$, then $\{E\cap Y:E\in\mathcal{E}\}$ is a network for the topology of Y, so $\mathrm{nw}(Y)\le\mathrm{nw}(X)$. If $y\in Y$ and $\mathcal{V}$ is a base of neighbourhoods of y in X, then $\{V\cap Y:V\in\mathcal{V}\}$ is a base of neighbourhoods of y in Y, so $\chi(y,Y)\le\chi(y,X)$.

(c) Let $f : X \to Y$ be a continuous surjection. If $D \subseteq X$ is dense, then $f[D]$ is dense in Y (3A3Eb), and $d(Y) \le \#(f[D]) \le \#(D)$; as D is arbitrary, $d(Y) \le d(X)$.

If $\mathcal{H}$ is a disjoint family of non-empty open sets in Y, then $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$ is a disjoint family of non-empty open sets in X, so $\#(\mathcal{H}) = \#(\mathcal{G}) \le c(X)$; as $\mathcal{H}$ is arbitrary, $c(Y) \le c(X)$.

If $\mathcal{H}$ is an open cover of Y, then $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$ is an open cover of X; let $\mathcal{G}_0 \in [\mathcal{G}]^{\le L(X)}$ be a subcover; choose $\mathcal{H}_0 \subseteq \mathcal{H}$ such that $\#(\mathcal{H}_0) = \mathcal{G}_0$ and $\mathcal{G}_0 = \{f^{-1}[H] : H \in \mathcal{H}_0\}$; then $\mathcal{H}_0$ covers Y. As $\mathcal{H}$ is arbitrary, $L(Y) \le L(X)$.

(d) Let $\mathcal{V}$ be a maximal disjoint family of non-empty open sets included in members of $\mathcal{G}$. Then $\#(\mathcal{V}) < \operatorname{sat}(X)$. Let $\mathcal{H} \subseteq \mathcal{G}$ be such that $\#(\mathcal{H}) \le \#(\mathcal{V})$ and every member of $\mathcal{V}$ is included in a member of $\mathcal{H}$. If $G \in \mathcal{G}$ then $G \setminus \overline{\bigcup \mathcal{H}}$ meets no member of $\mathcal{V}$, so must be empty; so this $\mathcal{H}$ serves.

(e)(i) By ENGELKING 89, 2.3.15, $d(X) \le \max(\omega, \lambda, \sup_{i\in I} d(X_i))$.

(ii) Set $\kappa = \sup_{J\subseteq I \text{ is finite}} c(\prod_{i\in J} X_i)$. All the finite products $\prod_{i\in J} X_i$ are continuous images of X, so $c(X) \ge \kappa$, by (c). **?** Suppose, if possible, that $c(X) > \kappa$. Let $\mathcal{V}$ be the usual base for the topology of X, consisting of sets of the form $\prod_{i\in I} G_i$ where $G_i \subseteq X_i$ is open for every i and $\{i : G_i \ne X_i\}$ is finite. Let $\langle W_\xi\rangle_{\xi<\kappa^+}$ be a disjoint family of non-empty open sets in X. For each $\xi < \kappa$ let $W'_\xi \subseteq W_\xi$ be a non-empty member of $\mathcal{V}$, so that W'_ξ is determined by a coordinates in a finite subset I_ξ of I. By the Δ-system Lemma (4A1Db) there is a set $A \subseteq \kappa^+$, of cardinal κ^+, such that $\langle I_\xi\rangle_{\xi\in A}$ is a Δ-system with root J say. For $\xi \in A$ express W'_ξ as $U_\xi \cap V_\xi$ where U_ξ is determined by coordinates in J and V_ξ is determined by coordinates in $I_\xi \setminus J$. Now for distinct ξ, $\eta \in A$,

$$\emptyset = W'_\xi \cap W'_\eta = U_\xi \cap U_\eta \cap V_\xi \cap V_\eta.$$

Since V_ξ and V_η and $U_\xi \cap U_\eta$ are determined by coordinates in the disjoint sets $I_\xi \setminus J$, $I_\eta \setminus J$ and J respectively, one of them must be empty, and this can only be $U_\xi \cap U_\eta$. Thus $\langle U_\xi\rangle_{\xi\in A}$ is disjoint. But now observe that each U_ξ is of the form $\pi_J^{-1}[H_\xi]$ where $H_\xi \subseteq \prod_{i\in J} X_i$ is a non-empty open set and $\pi_J(x) = x\restriction J$ for every $x \in X$. So $\langle H_\xi\rangle_{\xi\in A}$ witnesses that $c(\prod_{i\in J} X_i) \ge \kappa^+$, which contradicts the definition of κ. **X**

Thus $c(X) = \sup_{J\subseteq I \text{ is finite}} c(\prod_{i\in J} X_i)$.

(f) This is just because $L(\bigcup \mathcal{G}) \le \operatorname{hL}(X)$.

(g) If X is finite, $c(X) = \#(X)$ and the result is trivial. Otherwise, set $\kappa = \max(c(X), \chi(X))$ and for each $x \in X$ let $\langle U_\xi(x)\rangle_{\xi<\kappa}$ run over a base of neighbourhoods of x consisting of open sets. Let $f : [X]^2 \to [\kappa]^2$ be such that whenever x, $y \in X$ are distinct then there are ξ, $\eta \in f(\{x, y\})$ such that $U_\xi(x)$ and $U_\eta(y)$ are disjoint. **?** If $\#(X) > 2^\kappa$ then by the Erdős-Rado theorem (5A1Ga) there is a $C \subseteq X$ such that $\#(C) > \kappa$ and f is constant on $[C]^2$; let $\{\xi, \eta\}$ be the constant value. For $x \in C$ set $G_x = U_\xi(x) \cap U_\eta(x)$; then $\langle G_x\rangle_{x\in C}$ is a disjoint family of non-empty open sets, so $c(X) \ge \#(C) > \kappa$. **X**

(h) Fix a metric ρ on X defining its topology.

(i) If $d(X) < \omega$ then X is finite and the result is trivial. Otherwise, let D be a dense subset of X of cardinal $d(X)$; setting $U(x, \epsilon) = \{y : \rho(y, x) < \epsilon\}$, $\{U(x, 2^{-n}) : x \in D, n \in \mathbb{N}\}$ is a base for $\mathfrak{T}$, so $w(X) \le \max(\#(D), \omega) = d(X)$. Since we know from (a) that $d(X) \le w(X)$, we have equality.

(ii) Put (i) together with (b) above to see that $d(Y) \le d(X)$. If Y is discrete, then $\#(Y) = d(Y) \le d(X)$.

(iii)(α) If there is an uncountable $A \subseteq X$ such that $\inf_{x,y\in A, x\ne y} \rho(x, y) > 0$, then A is not separable in its subspace topology, so X is not separable, by (ii).

(β) If there is no such A, then for each $n \in \mathbb{N}$ let A_n be a maximal subset of X such that $\rho(x, y) \ge 2^{-n}$ for all distinct x, $y \in A_n$. In this case $\bigcup_{n\in\mathbb{N}} A_n$ is dense, so $d(X) \le \max(\omega, \sup_{n\in\mathbb{N}} \#(A_n)) = \omega$ and X is separable.

5A4C Compactness Let X be a compact Hausdorff space.

(a)(i) $\operatorname{nw}(X) = w(X)$. (ENGELKING 89, 3.1.19.)

(ii) There is a set $Y \subseteq X$, of cardinal at most the cardinal power $d(X)^\omega$, which meets every non-empty G_δ subset of X. **P** Let $D \subseteq X$ be a dense set of cardinal $d(X)$. For each sequence $\omega \in D^{\mathbb{N}}$ choose a cluster point x_ω of $\langle\omega(n)\rangle_{n\in\mathbb{N}}$; set $Y = \{x_\omega : \omega \in D^{\mathbb{N}}\}$. Then $\#(Y) \le \#(D^{\mathbb{N}}) = d(X)^\omega$. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open sets in X with non-empty intersection, take $x \in \bigcap_{n\in\mathbb{N}} G_n$ and choose inductively a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets such that $x \in H_n$ and $\overline{H}_{n+1} \subseteq H_n \cap G_n$ for every n. Let $\omega \in D^{\mathbb{N}}$ be such that $\omega(n) \in H_n$ for every n; then

$$x_\omega \in Y \cap \bigcap_{n\in\mathbb{N}} \overline{H}_n \subseteq \bigcap_{n\in\mathbb{N}} G_n.$$

As $\langle G_n\rangle_{n\in\mathbb{N}}$ is arbitrary, Y is a suitable set. **Q**

(b) If X is perfectly normal it is first-countable. (Every singleton set in X is a zero set (4A2Fi), so is a G_δ set; by 4A2Kf, X is first-countable.)

(c) If $w(X) \le \kappa$, X is homeomorphic to a closed subspace of $[0,1]^\kappa$. (Engelking 89, 3.2.5.)

(d)(i) If Y is a Hausdorff space and $f : X \to Y$ is a continuous irreducible surjection, then $d(X) = d(Y)$. **P** We know that $d(Y) \le d(X)$ (5A4Bc). In the other direction, let $D \subseteq Y$ be a dense set of cardinal $d(Y)$, and $C \subseteq X$ a set of cardinal $\#(D)$ such that $f[C] = D$. If $G \subseteq X$ is open and not empty, $f[X \setminus G]$ is a closed proper subset of Y (because f is irreducible), so $D \not\subseteq f[X \setminus G]$ and $C \not\subseteq X \setminus G$. As G is arbitrary, C is dense, and witnesses that $d(X) \le d(Y)$. **Q**

(ii) If $f : X \to \{0,1\}^\kappa$ is a continuous irreducible surjection, where $\kappa \ge \omega$, then $\chi(x, X) \ge \kappa$ for every $x \in X$. **P** Let $\mathcal{V}$ be a base of neighbourhoods of x of cardinal $\chi(x, X)$. For each $\xi < \kappa$, set $G_\xi = \{y : y \in X,\ f(y)(\xi) = f(x)(\xi)\}$. For $V \in \mathcal{V}$, set $I_V = \{\xi : \xi < \kappa,\ V \subseteq G_\xi\}$; then $f[V] \subseteq \{z : z \in \{0,1\}^\kappa,\ z{\restriction}I_V = f(x){\restriction}I_V\}$; but $f[X \setminus V]$ is a closed proper subset of $\{0,1\}^\kappa$, so $\operatorname{int} f[V]$ is non-empty and I_V is finite. As $\mathcal{V}$ is a base of neighbourhoods of x, $\kappa = \bigcup_{V \in \mathcal{V}} I_V$. As κ is infinite, $\mathcal{V}$ is infinite, and $\kappa \le \#(\mathcal{V}) = \chi(x, X)$). **Q**

(iii) So if there is a continuous surjection from X onto $[0,1]^\kappa$, there is a non-empty closed $K \subseteq X$ such that $\chi(x, K) \ge \kappa$ for every $x \in K$. **P** Let $f : X \to [0,1]^\kappa$ be a continuous surjection. Set $Z = f^{-1}[\{0,1\}]^\kappa$. By 4A2G(i-i), there is a closed $K \subseteq Z$ such that $f{\restriction}K$ is an irreducible surjection onto $\{0,1\}^\kappa$, and we can use (ii). **Q**

(iv) If Y and Z are Hausdorff spaces and $f : X \to Y$, $g : Y \to Z$ are irreducible surjections then $gf : X \to Z$ is irreducible. (If $F \subseteq X$ is a closed proper subset, then $f[F]$ is a closed proper subset of Y and $g[f[F]] \ne Z$.)

(e) If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X with at most one cluster point in X, then $\langle x_n \rangle_{n \in \mathbb{N}}$ is convergent. **P** Because X is compact, $\langle x_n \rangle_{n \in \mathbb{N}}$ has at least one cluster point; let x be such a point. **?** If $\langle x_n \rangle_{n \in \mathbb{N}}$ does not converge to x, let G be an open set containing x such that $J = \{n : n \in \mathbb{N},\ x_n \notin G\}$ is infinite. Then there must be a point y in $\bigcap_{n \in \mathbb{N}} \overline{\{x_i : i \in J \setminus n\}}$; and now y is a cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$ in $X \setminus G$, so cannot be equal to x. **XQ**

(f) Let Y be a Hausdorff space and $f : X \to Y$ a continuous function. If $\mathcal{E}$ is a non-empty downwards-directed family of closed subsets of X, then $f[\bigcap \mathcal{E}] = \bigcap_{F \in \mathcal{E}} f[F]$. **P** Of course $f[\bigcap \mathcal{E}] \subseteq \bigcap_{F \in \mathcal{E}} f[F]$. If $y \in \bigcap_{F \in \mathcal{E}} f[F]$, then $\{F \cap f^{-1}[\{y\}] : F \in \mathcal{E}\}$ is a downwards-directed family of closed subsets of X, so has non-empty intersection; and any point of the intersection witnesses that $y \in f[\bigcap \mathcal{E}]$. **Q**

5A4D Vietoris topologies: Proposition Let X be a separable metrizable space and $\mathcal{K}$ the set of compact subsets of X with the topology induced by the Vietoris topology on the set of closed subsets of X (4A2T).

(a) $\mathcal{K}$ is second-countable.

(b) If Y is a topological space and $R \subseteq Y \times X$ is usco-compact, then $y \mapsto R[\{y\}] : Y \to \mathcal{K}$ is Borel measurable.

(c) There is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of Borel measurable functions from $\mathcal{K} \setminus \{\emptyset\}$ to X such that $\{f_n(K) : n \in \mathbb{N}\}$ is a dense subset of K for every $K \in \mathcal{K} \setminus \{\emptyset\}$.

proof (a) Let $\mathcal{U}$ be a countable base for the topology of X, containing X. Let $\mathcal{V}$ be the family of sets of the form

$$\{K : K \in \mathcal{K},\ K \cap U_i \ne \emptyset \text{ for } i < n,\ K \subseteq \textstyle\bigcup_{i<n} U_i\}$$

where $U_i \in \mathcal{U}$ for $i < n$; then $\mathcal{V}$ is a countable family of open sets in $\mathcal{K}$ and is a base for the topology of $\mathcal{K}$.

(b) If $G \subseteq X$ is open, then

$$\{y : y \in Y,\ R[\{y\}] \subseteq G\} = Y \setminus R^{-1}[X \setminus G]$$

is open. Also G can be expressed as $\bigcup_{n \in \mathbb{N}} F_n$ where every F_n is closed, so

$$\{y : R[\{y\}] \cap G \ne \emptyset\} = \textstyle\bigcup_{n \in \mathbb{N}} R^{-1}[F_n]$$

is F_σ, therefore Borel. Thus

$$\mathcal{W} = \{W : W \subseteq \mathcal{K},\ \{y : R[\{y\}] \in W\} \text{ is Borel}\}$$

includes a subbase for the topology of $\mathcal{K}$. It therefore includes a base; because $\mathcal{K}$ is second-countable, therefore hereditarily Lindelöf, every open set is a countable union of members of $\mathcal{W}$ and belongs to $\mathcal{W}$, that is, $y \mapsto R[\{y\}]$ is Borel measurable.

(c)(i) Note first that if $G \subseteq X$ is open, then $K \mapsto \overline{K \cap G} : \mathcal{K} \to \mathcal{K}$ is Borel measurable. **P** If $H \subseteq X$ is open, then

$$\{K : \overline{K \cap G} \cap H \ne \emptyset\} = \{K : K \cap (G \cap H) \ne \emptyset\}$$

is open. Next, we can express H as the union $\bigcup_{n \in \mathbb{N}} H_n$ of a non-decreasing sequence of open sets such that $\overline{H}_n \subseteq H$ for every n, so

$$\{K : \overline{K \cap G} \subseteq H\} = \textstyle\bigcup_{n \in \mathbb{N}} \{K : K \cap G \subseteq \overline{H}_n\} = \bigcup_{n \in \mathbb{N}} \{K : K \cap (G \setminus \overline{H}_n) = \emptyset\}$$

is F_σ, therefore Borel. As in (b), this is enough. **Q**

(ii) Let $\langle U_n \rangle_{n \in \mathbb{N}}$ run over a base for the topology of X. For each $n \in \mathbb{N}$ define $g_n : \mathcal{K} \to \mathcal{K}$ by setting

$$g_n(K) = \overline{K \cap U_n} \text{ if } K \cap U_n \ne \emptyset,$$
$$= K \text{ otherwise.}$$

Since $\{K : K \cap U_n \ne \emptyset\}$ is open, (i) tells us that g_n is Borel measurable. Set $h_n = g_n \dots g_1 g_0$; then h_n also is Borel measurable, for each n. Now, for each $K \in \mathcal{K} \setminus \{\emptyset\}$, $\langle h_n(K)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty compact sets, so has non-empty intersection. Morover, for each n, $h_n(K)$ is either disjoint from U_n or included in $\overline{U}_n$; so $\bigcap_{n\in\mathbb{N}} h_n(K)$ has exactly one point; call this point $f(K)$. Of course $f(K) \in h_0(K) \subseteq K$.

Now $f : \mathcal{K} \setminus \{\emptyset\} \to X$ is Borel measurable. **P** If $F \subseteq X$ is closed, then

$$f^{-1}[F] = \bigcap_{n\in\mathbb{N}}\{K : F \cap h_n(K) \ne \emptyset\}$$

is a Borel set because every h_n is Borel measurable and $\{K : F \cap K = \emptyset\}$ is open. **Q**

(iii) Set $f_n = fg_n$ for each n. Then $f_n(K) \in K$ for every $n \in \mathbb{N}$ and $K \in \mathcal{K} \setminus \{\emptyset\}$, $f_n : \mathcal{K} \setminus \{\emptyset\} \to X$ is Borel measurable for each n, and $f_n(K) \in \overline{K \cap U_n}$ whenever $K \cap U_n \ne \emptyset$; so $\{f_n(K) : n \in \mathbb{N}\}$ is dense in K for every $K \in \mathcal{K} \setminus \{\emptyset\}$.

5A4E Category and the Baire property Let X be a topological space; write $\widehat{\mathcal{B}}(X)$ for its Baire-property algebra (4A3Q).

(a) Suppose that $\langle G_i\rangle_{i\in I}$ is a disjoint family of open sets, and $\langle E_i\rangle_{i\in I}$ is a family of nowhere dense sets. Then $\bigcup_{i\in I} G_i \cap E_i$ is nowhere dense. (Elementary; see (a-i) of the proof of 4A3R.)

(b) If $A \subseteq X$ and $H = \bigcup\{G : G \subseteq X \text{ is open}, G \cap A \text{ is meager}\}$, then $H \cap A$ is meager. (See the proof of 4A3Ra.)

(c) Let Y be another topological space.

(i) If $A \subseteq X$ is nowhere dense in X, then $A \times Y$ is nowhere dense in $X \times Y$. ($\overline{A \times Y} = \overline{A} \times Y$.) So if $A \subseteq X$ is meager in X, then $A \times Y$ is meager in $X \times Y$.

(ii) $\widehat{\mathcal{B}}(X)\widehat{\otimes}\widehat{\mathcal{B}}(Y) \subseteq \widehat{\mathcal{B}}(X \times Y)$. **P** If $E \in \widehat{\mathcal{B}}(X)$, let $G \subseteq X$ be such that $E\triangle G$ is meager; then

$$E \times Y = (G \times Y)\triangle((E\triangle G) \times Y) \in \widehat{\mathcal{B}}(X \times Y).$$

Similarly, $X \times F \in \widehat{\mathcal{B}}(X \times Y)$ for every $F \in \widehat{\mathcal{B}}(Y)$. Because $\widehat{\mathcal{B}}(X \times Y)$ is a σ-algebra of sets, it includes $\widehat{\mathcal{B}}(X)\widehat{\otimes}\widehat{\mathcal{B}}(Y)$. **Q**

(iii) If Y is compact, Hausdorff and not empty, then a set $A \subseteq X$ is meager in X iff $A \times Y$ is meager in $X \times Y$. **P** (α) If $F \subseteq X$ is nowhere dense in X, then $F \times Y$ is nowhere dense in $X \times Y$. So if A is meager in X, then $A \times Y$ is meager in $X \times Y$. (β) If $A \times Y$ is meager in $X \times Y$, let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence of dense open subsets of $X \times Y$ such that $\bigcap_{n\in\mathbb{N}} W_n$ is disjoint from $A \times Y$. Choose $\langle V_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $V_0 = X \times Y$. Given that V_n is an open subset of $X \times Y$ such that $\pi_1[V_n]$ is dense in X, where π_1 is the projection from $X \times Y$ onto X, then $\pi_1[V_n \cap W_n]$ is dense in X. Let $\mathcal{V}_n$ be a maximal family of open sets $V \subseteq X \times Y$ such that $\overline{V} \subseteq V_n \cap W_n$ for every $V \in \mathcal{V}_n$ and $V \cap V' = \emptyset$ for all distinct V, $V' \in \mathcal{V}_n$; set $V_{n+1} = \bigcup \mathcal{V}_n$. Then $\pi_1[V_{n+1}]$ is dense in X; continue.

If $x \in \bigcap_{n\in\mathbb{N}} \pi_1[V_n]$, $\langle \overline{V_n[\{x\}]}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed subsets of Y, so there is a $y \in \bigcap_{n\in\mathbb{N}} \overline{V_n[\{x\}]}$, because Y is compact. For each n, there is a $V \in \mathcal{V}_n$ such that $x \in \pi_1[V]$, so $V_{n+1}[\{x\}] = V[\{x\}]$ and $(x, y) \in \overline{V} \subseteq V_n \cap W_n$. Thus $x \in \pi_1[\bigcap_{n\in\mathbb{N}} W_n]$ and $x \notin A$. As x is arbitrary, A is disjoint from $\bigcap_{n\in\mathbb{N}} \pi_1[V_n]$ and is meager. **Q**

(d) Suppose that X is completely regular and ccc.

(i) Every nowhere dense subset of X is included in a nowhere dense zero set. **P** If $E \subseteq X$ is nowhere dense, let $\mathcal{G}$ be a maximal disjoint family of cozero sets included in $X \setminus E$. Because X is ccc, $\mathcal{G}$ is countable, and $G = \bigcup \mathcal{G}$ is a cozero set. Because X is completely regular, $X \setminus (G \cup E)$ is nowhere dense and $X \setminus G$ is a nowhere dense zero set including E. **Q**

(ii) Every meager subset of X is included in a meager Baire set. (By (i), it is included in the union of a sequence of nowhere dense zero sets.)

5A4F Normal and paracompact spaces (a) For a normal space X and an infinite set I, the following are equiveridical: (i) there is a continuous surjection from X onto $[0,1]^I$; (ii) there is a continuous surjection from a closed subset of X onto $\{0,1\}^I$. **P** (i)$\Rightarrow$(ii) is elementary, as $\{0,1\}^I$ is a closed subset of $[0,1]^I$. So suppose that (ii) is true. The map $x \mapsto \sum_{n=0}^{\infty} 2^{-n-1}$ is a continuous surjection from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$; there is therefore a continuous surjection from $\{0,1\}^{I\times\mathbb{N}}$ onto $[0,1]^I$; but I is infinite, so $\{0,1\}^I$ is homeomorphic to $\{0,1\}^{I\times\mathbb{N}}$. We therefore have a continuous surjection from $\{0,1\}^I$ onto $[0,1]^I$. Accordingly there is a continuous surjection f from a closed subset F of X onto $[0,1]^I$. Set $f_i(x) = f(x)(i)$ for $x \in F$ and $i \in I$; by Tietze's theorem (4A2F(d-viii)), there is a continuous $g_i : X \to [0,1]$ extending f_i; now $x \mapsto \langle g_i(x)\rangle_{i\in I} : X \to [0,1]^I$ is a continuous surjection, and (i) is true. **Q**

(b) Suppose that X is a paracompact Hausdorff space and $\mathcal{G}$ is an open cover of X. Then there is a continuous pseudometric $\rho : X \times X \to [0, \infty[$ such that whenever $\emptyset \neq A \subseteq X$ and $\sup_{x,y \in A} \rho(x, y) \leq 1$ there is a $G \in \mathcal{G}$ such that $A \subseteq G$. **P** There is an open cover $\mathcal{H}$ of X which is a 'star-refinement' of $\mathcal{G}$, that is, for every $x \in X$ there is a $G \in \mathcal{G}$ including $\bigcup\{H : x \in H \in \mathcal{H}\}$ (Engelking 89, 5.1.12). Next, there is a 'locally finite resolution of the identity subordinate to $\mathcal{H}$', that is, a family $\langle f_i \rangle_{i \in I}$ of continuous functions from X to $[0, 1]$ such that $\langle f_i^{-1}[\,]0, 1]\,] \rangle_{i \in I}$ is a locally finite refinement of $\mathcal{H}$ and $\sum_{i \in I} f_i(x) = 1$ for every $x \in X$ (Engelking 89, 5.1.9). Set $\rho(x, y) = 2 \sum_{i \in I} |f_i(x) - f_i(y)|$. Then ρ is a pseudometric on X, and is continuous because $\langle f_i^{-1}[\,]0, 1]\,] \rangle_{i \in I}$ is locally finite. If $A \subseteq X$ is a non-empty set such that $\rho(x, y) \leq 1$ for all x, $y \in A$, take any $x \in A$. There is a $G \in \mathcal{G}$ such that $H \subseteq G$ whenever $H \in \mathcal{H}$ and $x \in H$. Set $J = \{i : i \in I,\ f_i(x) > 0\}$; then $\sum_{i \in J} f_i(x) = 1$. If $y \in A$, then $\sum_{i \in J} |f_i(x) - f_i(y)| \leq \frac{1}{2}$, so there is an $i \in J$ such that $f_i(y) > 0$. Set $U = f_i^{-1}[\,]0, 1]\}\,]$; then x and y both belong to U. Let $H \in \mathcal{H}$ be such that $U \subseteq H$; then $x \in H$ so $H \subseteq G$, and $y \in H$ so $y \in G$. Thus $A \subseteq G$, as required. **Q**

5A4G Baire σ-algebras Let X be a topological space. Write $\mathcal{B}\mathfrak{a}_0(X)$ for the set of cozero sets in X and for ordinals $\alpha > 0$ set

$$\mathcal{B}\mathfrak{a}_\alpha(X) = \{\bigcup_{n \in \mathbb{N}} (X \setminus E_n) : \langle E_n \rangle_{n \in \mathbb{N}} \text{ is a sequence in } \bigcup_{\beta < \alpha} \mathcal{B}\mathfrak{a}_\beta(X)\}.$$

Then the Baire σ-algebra of X is $\bigcup_{\alpha < \omega_1} \mathcal{B}\mathfrak{a}_\alpha(X)$. **P** Inducing on α, we see that $\mathcal{B}\mathfrak{a}_\alpha(X)$ is included in the Baire σ-algebra of X for every α; and $\bigcup_{\alpha < \omega_1} \mathcal{B}\mathfrak{a}_\alpha(X)$ is a σ-algebra of sets containing every cozero set, so includes the Baire σ-algebra. **Q**

5A4H Blumberg's theorem If X is a metrizable Baire space, Y a second-countable space and $f : X \to Y$ a function, there is a dense subset D of X such that $f{\restriction}D$ is continuous.

proof Fix a metric ρ defining the topology of X.

(a) Let $\mathcal{A}$ be the family of sets $A \subseteq X$ such that $A \cap G$ is non-meager for every non-empty open subset G of X. If $A \in \mathcal{A}$ and $U \subseteq X$ is a non-empty open set, there are an $A_1 \in \mathcal{A}$ and an $x \in U \cap A_1$ such that $A_1 \subseteq A$ and $f{\restriction}A_1$ is continuous at x. **P** Let $\langle V_n \rangle_{n \in \mathbb{N}}$ run over a base for the topology of Y. For each $n \in \mathbb{N}$, set $B_n = A \cap f^{-1}[V_n]$, and let H_n be the largest open subset of X such that $H_n \cap B_n$ is meager (5A4Eb). Set

$$A' = A \setminus \bigcup_{n \in \mathbb{N}} ((H_n \cap B_n) \cup \partial H_n);$$

then $A \setminus A'$ is meager, so $A' \in \mathcal{A}$. In particular, A' is dense; take any $x \in A' \cap U$.

Let $\langle n_i \rangle_{i \in \mathbb{N}}$ be such that $\langle V_{n_i} \rangle_{i \in \mathbb{N}}$ is a non-increasing sequence running over a base of neighbourhoods of $f(x)$. Then $x \in A' \cap B_{n_i}$, so $x \notin \overline{H}_{n_i}$, for every i. Let $\langle U_i \rangle_{i \in \mathbb{N}}$ be a non-increasing sequence, running over a base of neighbourhoods of x, such that $U_i \cap H_{n_i} = \emptyset$ and $\overline{U}_{i+1} \subseteq U_i$ for every i. Set

$$A_1 = \{x\} \cup (A \setminus U_0) \cup \bigcup_{i \in \mathbb{N}} (B_{n_i} \cap U_i \setminus U_{i+1}).$$

If $G \subseteq X$ is open and not empty, then if $G = \{x\}$ we certainly have $G \cap A_1$ non-meager. Otherwise, there is a first i such that $G \setminus \overline{U}_i$ is non-empty. If $i = 0$, then $G \cap A_1 \supseteq A \cap G \setminus \overline{U}_0$ is non-meager. If $i = j + 1$, then $G \setminus \overline{U}_{j+1} \subseteq \overline{U}_j$, so $G' = G \cap U_j \setminus \overline{U}_{j+1}$ is non-empty. Now $G' \cap H_{n_j} = \emptyset$ so $G' \cap B_{n_j}$ and $G \cap A_1 \supseteq G' \cap B_{n_j}$ are non-meager. As G is arbitrary, $A_1 \in \mathcal{A}$.

Of course $x \in A_1$. If V is a neighbourhood of $f(x)$ in Y, let i be such that $V_{n_i} \subseteq V$. If $y \in A_1 \cap U_i \setminus \{x\}$, there is a $j \geq i$ such that $y \in B_{n_j} \cap U_j \setminus U_{j+1}$, in which case $f(y) \in V_{n_j} \subseteq V_{n_i}$. But this means that $f{\restriction}A_1$ is continuous at x. **Q**

(b) Suppose that $A \in \mathcal{A}$, $F \subseteq A$ is a closed subset of X such that $f{\restriction}A$ is continuous at every point of F, $\epsilon > 0$, and $\mathcal{U}$ is a family of open subsets of X such that $\rho(x, y) \geq \epsilon$ whenever x, y belong to different members of $\mathcal{U}$. Then there are an $A' \in \mathcal{A}$ and a closed $F' \subseteq X$ such that $F \subseteq F' \subseteq A' \subseteq A$, $F' \cap U \neq \emptyset$ for every $U \in \mathcal{U}$, and $f{\restriction}A'$ is continuous at every point of F'. **P** Set $\mathcal{U}' = \{U : U \in \mathcal{U},\ F \cap U = \emptyset\}$. For each $U \in \mathcal{U}'$, (a) tells us that we can choose $A_U \in \mathcal{A}$ and $x_U \in U \cap A_U$ such that $A_U \subseteq A$ and $f{\restriction}A_U$ is continuous at x_U. Set $C = \{x_U : U \in \mathcal{U}'\}$; then $\rho(x, y) \geq \epsilon$ for all distinct x, $y \in C$, so C is closed. Set

$$F' = F \cup C, \quad A' = (A \setminus \bigcup \mathcal{U}') \cup \bigcup_{U \in \mathcal{U}'} U \cap A_U.$$

Then F' is closed, $F \subseteq F' \subseteq A' \subseteq A$ and F' meets every element of $\mathcal{U}$.

If $G \subseteq X$ is a non-empty open set, then if $G \cap \bigcup \mathcal{U}' = \emptyset$ we shall have $G \cap A' = G \cap A$, which is non-meager. Otherwise, let $U \in \mathcal{U}'$ be such that $G \cap U$ is non-empty; then $G \cap A' \supseteq G \cap U \cap A_U$ is non-meager. As G is arbitrary, $A' \in \mathcal{A}$.

If $x \in F$, then $f{\restriction}A'$ is continuous at x because $A' \subseteq A$ and $f{\restriction}A$ is continuous at x. If $x \in C$, take $U \in \mathcal{U}'$ such that $x = x_U$. Then $A' \cap U = A_U \cap U$, while $f{\restriction}A_U$ is continuous at x; so $f{\restriction}A'$ is continuous at x. So $f{\restriction}A'$ is continuous at every point of F', and we have a suitable pair (F', A'). **Q**

(c) Let $\mathcal{U}$ be a σ-metrically-discrete base for $\mathfrak{T}$ (4A2L(g-i)); express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where $\rho(x,y)\ge 2^{-n}$ whenever x, y belong to distinct members of any $\mathcal{U}_n$. Now we can choose $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle A_n\rangle_{n\in\mathbb{N}}$ inductively so that

$$F_n \text{ is closed},\quad A_n\in\mathcal{A},\quad F_n\subseteq A_n,$$

$$f\restriction A_n \text{ is continuous at every point in } F_n,$$

$$F_{n+1} \text{ meets every member of } \mathcal{U}_n,\quad F_n\subseteq F_{n+1},\quad A_{n+1}\subseteq A_n$$

for every $n\in\mathbb{N}$. **P** Start with $F_0=\emptyset$, $A_0=X$. (This is where we use the hypothesis that X is a Baire space, to see that $X\in\mathcal{A}$.) Given F_n and A_n, use (b) to find F_{n+1} and A_{n+1}. **Q**

(d) At the end of the induction, set $D=\bigcup_{n\in\mathbb{N}}F_n$. Since $D\cap U\ne\emptyset$ for every $U\in\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$, D is dense. If $x\in D$, there is an $n\in\mathbb{N}$ such that $x\in F_n$; now $D\subseteq A_n$, so $f\restriction D$ is continuous at x. So we're home.

5A4I Proposition If X is a compact metrizable space and (Y,ρ) a complete separable metric space, then $C(X;Y)$, with the topology of uniform convergence, is Polish. (Engelking 89, 4.3.13 and 4.2.18.)

5A4J Old friends (a) The weight of the Stone-Čech compactification $\beta\mathbb{N}$ is $\mathfrak{c}$. (Engelking 89, 3.6.11.)

(b)(i) For any infinite I, there is a continuous surjection from $\{0,1\}^I$ onto $[0,1]^I$. (Immediate from 5A4Fa, or otherwise.)

(ii) There is a continuous surjection from $[0,1]$ onto $[0,1]^{\mathbb{N}}$. (The Cantor set $C\subseteq[0,1]$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$ (4A2Uc), so again 5A4Fa gives the result.)

(c) If X is a non-empty zero-dimensional compact metrizable space without isolated points, it is homeomorphic to $\{0,1\}^{\mathbb{N}}$. **P** Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of X. Because X has no isolated points, $\mathfrak{B}$ is atomless (316Lb). We know that X is second-countable (4A2P(a-ii)); let $\mathcal{U}$ be a countable base for its topology; then every member of $\mathfrak{B}$ is open, so expressible as a union of members of $\mathcal{U}$, and compact, so expressible as the union of a finite subset of $\mathcal{U}$. Accordingly $\mathfrak{B}$ is countable; and as $X\ne\emptyset$, $\mathfrak{B}\ne\{0\}$. By 316M[5], $\mathfrak{B}$ is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$; by 311J, X and $\{0,1\}^{\mathbb{N}}$ are homeomorphic. **Q**

(d) Let X be a non-empty zero-dimensional Polish space in which no non-empty open set is compact. Then X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology. **P** Let ρ be a complete metric on X defining its topology. (i) If $U\subseteq X$ is a non-empty open set and $\epsilon>0$, there is a partition $\langle U_n\rangle_{n\in\mathbb{N}}$ of U into non-empty open-and-closed sets of diameter at most ϵ. To see this, note that as U is not compact, there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in U with no cluster point in U (4A2Le). Let $\mathcal{V}$ be the family of subsets of U, of diameter at most ϵ, which are open-and-closed in X and contain x_i for at most finitely many i. Because X is zero-dimensional, $\mathcal{V}$ is a base for the subspace topology of U. Because U is Lindelöf (4A2P(a-iii)), there is a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ covering U; set $V'_n=V_n\setminus\bigcup_{i<n}V_i$ for each n, so that $\langle V'_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{V}$ covering U. Because no V'_n can contain infinitely many of the x_i, $I=\{n:V'_n\ne\emptyset\}$ is infinite, and we can re-enumerate $\langle V'_n\rangle_{n\in I}$ as $\langle U_n\rangle_{n\in\mathbb{N}}$ to get an appropriate sequence. (ii) Now set $S^*=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ and define $\langle U_\sigma\rangle_{\sigma\in S}$ inductively in such a way that $U_\emptyset=X$ and

$\langle U_{\sigma^\frown<n>}\rangle_{n\in\mathbb{N}}$ is a partition of U_σ into non-empty open-and-closed sets of diameter at most 2^{-k} whenever $k\in\mathbb{N}$ and $\sigma\in\mathbb{N}^k$.

For $\alpha\in\mathbb{N}^{\mathbb{N}}$, $\langle U_{\alpha\restriction k}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed sets and $\operatorname{diam}U_{\alpha\restriction k}\le 2^{-k+1}$ for every $k\ge 1$, so there is exactly one point in $\bigcap_{k\in\mathbb{N}}U_{\alpha\restriction k}$; let $f(\alpha)$ be this point. This defines a function $f:\mathbb{N}^{\mathbb{N}}\to X$. (iii) Because $\langle U_{\sigma^\frown<n>}\rangle_{n\in\mathbb{N}}$ is a partition of U_σ for every σ, f is a bijection. (iv) If $G\subseteq X$ is open, then

$$f^{-1}[G]=\{\alpha:\text{there is some }k\in\mathbb{N}\text{ such that }U_{\alpha\restriction k}\subseteq G\}$$

is open, so f is continuous. (v) If $H\subseteq\mathbb{N}^{\mathbb{N}}$ is open, then

$$f[H]=\bigcup\{U_\sigma:\sigma\in S^*,\ \{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}\subseteq H\}$$

is open, so f is a homeomorphism. **Q**

(e) If X is a non-empty Polish space without isolated points, then it has a dense G_δ set which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology. **P** Let $\mathcal{U}$ be a countable base for the topology of X, and D a countable dense subset of X; set

$$Y=X\setminus(D\cup\bigcup_{U\in\mathcal{U}}\partial U).$$

[5] Formerly 393F.

Then Y is a G_δ set in X, so is Polish (4A2Qd). Because X has no isolated points, Y is comeager in X and is dense and not empty. Because $\{U \cap Y : U \in \mathcal{U}\}$ is a base for the topology of Y consisting of relatively open-and-closed sets, Y is zero-dimensional. If $U \subseteq Y$ is a non-empty relatively open set, let $G \subseteq X$ be an open set such that $G \cap Y = U$; then $D \cap G$ is non-empty, so there is a sequence in U converging (in X) to a point in $D \cap G \subseteq X \setminus U$, and U cannot be compact. By (d), Y is homeomorphic to $\mathbb{N}^\mathbb{N}$. **Q**

5A5 Real analysis

For the sake of an argument in §534 I sketch a fragment of theory.

5A5A Entire functions A real function f is **real-analytic** if its domain is an open subset G of $\mathbb{R}$ and for every $a \in G$ there are a $\delta > 0$ and a real sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ such that $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$ whenever $|x-a| < \delta$. It is **real-entire** if in addition its domain is the whole of $\mathbb{R}$.

We need the following facts: (i) if f and g are real-entire functions so is $f-g$; (ii) if $\langle c_n \rangle_{n\in\mathbb{N}}$ is a real sequence such that $f(x) = \sum_{n=0}^\infty c_n x^n$ is defined in $\mathbb{R}$ for every $x \in \mathbb{R}$, then f is real-entire; (iii) if in this expression not every c_n is zero, then every point of $F = \{x : x \in \mathbb{R},\ f(x) = 0\}$ is isolated in F, so that F is countable. If you have done a basic course in complex functions you should recognise this. If either you missed this out, or you are not sure you understood the proof of Cauchy's theorem, the following is a sketch of a real-variable argument.

(i) is elementary. For (ii), observe first that if $\langle c_n x^n \rangle_{n\in\mathbb{N}}$ is summable then $\lim_{n\to\infty} c_n x^n = 0$ so $\sum_{n=0}^\infty |c_n| t^n$ is finite whenever $0 \le t < |x|$. In the present case, $\sum_{n=0}^\infty |c_n| t^n < \infty$ for every $t \ge 0$. So if a, $x \in \mathbb{R}$,

$$\sum_{n=0}^\infty \sum_{k=0}^n \Bigl|\frac{n!}{k!(n-k)!} c_n x^k a^{n-k}\Bigr| \le \sum_{n=0}^\infty \sum_{k=0}^n \frac{n!}{k!(n-k)!} |c_n| R^n$$

(where $R = \max(|x|, |a|)$)

$$= \sum_{n=0}^\infty |c_n|(2R)^n < \infty.$$

We therefore have

$$f(x+a) = \sum_{n=0}^\infty c_n(x+a)^n = \sum_{n=0}^\infty \sum_{k=0}^n \frac{n!}{k!(n-k)!} c_n x^k a^{n-k} = \sum_{k=0}^\infty c'_k x^k$$

where $c'_k = \sum_{n=k}^\infty \frac{n!}{k!(n-k)!} c_n a^{n-k}$ for each k. Turning this round, $f(x) = \sum_{k=0}^\infty c'_k (x-a)^k$ for every x. This shows that f is real-entire. Next, if not every c_n is zero, there must be some neighbourhood of 0 in which the first non-zero term $c_n x^n$ dominates, so f is not identically zero. (The point is that $\sum_{k=0}^\infty |c_k| < \infty$, so there is some $\delta > 0$ such that $\sum_{k=n+1}^\infty |c_k \delta^{k+1}| < |c_n \delta^n|$.) In this case, not every c'_k can be zero, and there must be some neighbourhood of a in which the first non-zero term $c'_k(x-a)^k$ dominates, so that there can be no zeroes of f in that neighbourhood except perhaps a itself.

5A6 Later editions only

I give bald statements of results which I have interpolated into earlier volumes and which have not yet appeared in a printed version. For the time being they may be found, with proofs, in the online drafts listed in `http://www.essex.ac.uk/maths/staff/fremlin/mtcont.htm`.

133J Proposition Let (X, Σ, μ) be a measure space.

(a) Let f be a real-valued function defined almost everywhere in X.

(i) If $\overline{\int} f$ is finite, then there is an integrable g such that $f \le_{\text{a.e.}} g$ and $\int g = \overline{\int} f$. In this case,

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(x) \le f(x) + g_0(x)\}$$

has full outer measure for every measurable function $g_0 : X \to]0, \infty[$.

(ii) If $\underline{\int} f$ is finite, then there is an integrable h such that $h \le_{\text{a.e.}} f$ and $\int h = \underline{\int} f$. In this case,

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h,\ f(x) \le h(x) + h_0(x)\}$$

has full outer measure for every measurable function $h_0 : X \to]0, \infty[$.

(e) $\mu^* A = \overline{\int} \chi A$ for every $A \subseteq X$.

133L Proposition Let (X,Σ,μ) be a measure space and f a real-valued function defined almost everywhere in X. Suppose that h_1, h_2 are non-negative virtually measurable functions defined almost everywhere in X. Then

$$\overline{\int} f\times(h_1+h_2)=\overline{\int} f\times h_1+\overline{\int} f\times h_2.$$

135I Subspace measures: Proposition Let (X,Σ,μ) be a measure space, and $H\in\Sigma$; write μ_H for the subspace measure on H.

(d) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then

$$\overline{\int}_H(h\restriction H)d\mu_H=\overline{\int}_X h\times\chi H d\mu.$$

214N Upper and lower integralsProposition Let (X,Σ,μ) be a measure space, A a subset of X and f a real-valued function defined almost everywhere in X. Then

(a) if *either* f is non-negative *or* A has full outer measure in X, $\overline{\int}(f\restriction A)d\mu_A\le\overline{\int} f d\mu$;

(b) if A has full outer measure in X, $\underline{\int} f d\mu\le\underline{\int}(f\restriction A)d\mu_A$.

235A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi:D_\phi\to Y$, $J:D_J\to[0,\infty[$ functions defined on conegligible subsets D_ϕ, D_J of X such that

$$\int J\times\chi(\phi^{-1}[F])d\mu \text{ exists }=\nu F$$

whenever $F\in\mathrm{T}$ and $\nu F<\infty$. Then

$$\int_{\phi^{-1}[H]}J\times g\phi\,d\mu \text{ exists }=\int_H g\,d\nu$$

for every ν-integrable function g taking values in $[-\infty,\infty]$ and every $H\in\mathrm{T}$, provided that we interpret $(J\times g\phi)(x)$ as 0 when $J(x)=0$ and $g(\phi(x))$ is undefined. Consequently, interpreting $J\times f\phi$ in the same way,

$$\underline{\int} f d\nu\le\underline{\int} J\times f\phi\,d\mu\le\overline{\int} J\times f\phi\,d\mu\le\overline{\int} f d\nu$$

for every $[-\infty,\infty]$-valued function f defined almost everywhere in Y.

251L Proposition Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2), (Y_1,T_1,ν_1) and (Y_2,T_2,ν_2) be σ-finite measure spaces; let λ_1, λ_2 be the product measures on $X_1\times Y_1$, $X_2\times Y_2$ respectively. Suppose that $f:X_1\to X_2$ and $g:Y_1\to Y_2$ are inverse-measure-preserving functions, and that $h(x,y)=(f(x),g(y))$ for $x\in X_1$, $y\in Y_1$. Then h is inverse-measure-preserving.

272W Theorem Let (Ω,Σ,μ) be a probability space, and $\langle\Sigma_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ. Let $\mathcal{E}\subseteq\Sigma$ be a family of measurable sets, and T the σ-algebra generated by $\mathcal{E}$. Then there is a set $J\subseteq I$ such that $\#(I\setminus J)\le\max(\omega,\#(\mathcal{E}))$ and T, $\langle\Sigma_j\rangle_{j\in J}$ are independent, in the sense that $\mu(F\cap\bigcap_{r\le n}E_r)=\mu F\cdot\prod_{r=0}^n\mu E_r$ whenever $F\in\mathrm{T}$, $j_0,\dots,j_r$ are distinct members of J and $E_r\in\Sigma_{j_r}$ for each $r\le n$.

273J Corollary Let (Ω,Σ,μ) be a probability space, and λ the product measure on $\Omega^{\mathbb{N}}$. If f is a real-valued function such that $\int f$ is defined in $[-\infty,\infty]$, then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\omega_i)=\int f d\mu$$

for λ-almost every $\langle\omega_n\rangle_{n\in\mathbb{N}}\in\Omega^{\mathbb{N}}$.

316P Proposition Let $\mathfrak{A}$ be a homogeneous Boolean algebra. Then its Dedekind completion is homogeneous.

316Q Proposition The free product of any family of homogeneous Boolean algebras is homogeneous.

325M Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ their probability algebra free product. For $J\subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$.

(b)(i) For any $c\in\mathfrak{C}$, there is a unique smallest $J_c\subseteq I$ such that $c\in\mathfrak{C}_{J_c}$, and this J_c is countable.

(ii) If c, $d\in\mathfrak{C}$ and $c\subseteq d$, then there is an $e\in\mathfrak{C}_{J_c\cap J_d}$ such that $c\subseteq e\subseteq d$.

328A Construction Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I.

(a) Set

$$\mathcal{J}=\{\langle a_i\rangle_{i\in I}:\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i,\ \lim_{i\to\mathcal{F}}\bar\mu_i a_i=0\}.$$

Then $\mathcal{J}$ is an ideal in the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$.

(b) Let $\mathfrak{A}$ be the quotient Boolean algebra $\prod_{i\in I}\mathfrak{A}_i/\mathcal{J}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,1]$ defined by saying that

$$\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$$

whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$.

328B Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras and $\mathcal{F}$ an ultrafilter on I, and construct $\mathfrak{A}$ and $\bar\mu$ as in 328A. Then $(\mathfrak{A},\bar\mu)$ is a probability algebra.

328C Definition In the context of 328A/328B, I will call $(\mathfrak{A},\bar\mu)$ the **probability algebra reduced product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ modulo $\mathcal{F}$; I will sometimes write it as $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

If all the $(\mathfrak{A}_i,\bar\mu_i)$ are the same, with common value $(\mathfrak{B},\bar\nu)$, I will write $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$ for $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$, and call it the '**probability algebra reduced power**'.

328D Proposition Let I be a non-empty set, $\le$ a reflexive transitive relation on I, and $\mathcal{F}$ an ultrafilter on I such that $\{j:j\in I,\ j\ge i\}$ belongs to $\mathcal{F}$ for every $i\in I$. Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we are given a family $\langle\pi_{ji}\rangle_{i\le j}$ such that

π_{ji} is a measure-preserving Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}_j$ whenever $i\le j$ in I,

$\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$ in I.

Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

(a) For each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}$ defined by saying that $\pi_i a=\langle a_j\rangle^\bullet_{j\in I}$ whenever $a_j=\pi_{ji}a$ for every $j\ge i$, and $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ in I.

(b) $\langle a_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi_j a_j$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $A\in\mathcal{F}$.

331J Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal.

(a) If there is a family $\langle e_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I}a_\xi=0$ and $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\kappa$, then $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

(b) Let ν_κ be the usual measure on $\{0,1\}^\kappa$ (254J) and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. If there is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{A}$, $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

343C Examples (d) If (X,Σ,μ) is a compact probability space with Maharam type at most $\kappa\ge\omega$, then there is an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X.

344L Theorem Let I be an infinite set, and ν_I the usual measure on $\{0,1\}^I$. If $E\subseteq\{0,1\}^I$ is a measurable set of non-zero measure, the subspace measure on E is isomorphic to $(\nu_I E)\nu_I$.

364B Remarks (e) We have the option of declaring $L^0(\mathfrak{A})$ to be the set of functions $\alpha\mapsto[\![u>\alpha]\!]:\mathbb{Q}\to\mathfrak{A}$ such that

(α'') $[\![u>q]\!]=\sup_{q'\in\mathbb{Q},q'>q}[\![u>q']\!]$ for every $q\in\mathbb{Q}$,

(β') $\inf_{n\in\mathbb{N}}[\![u>n]\!]=0$,

(γ') $\sup_{n\in\mathbb{N}}[\![u>-n]\!]=1$.

364Xw Exercise Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a non-negative finitely additive functional. Let $\int:L^\infty(\mathfrak{A})\to\mathbb{R}$ be the corresponding linear functional. Write U for the set of those $u\in L^0(\mathfrak{A})$ such that $\sup\{\int v:v\in L^\infty(\mathfrak{A}),\ v\le|u|\}$ is finite. Show that $\int$ has an extension to a non-negative linear functional on U.

377B Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product of $\langle\mathfrak{A}_i\rangle_{i\in I}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$. Let W_0 be the subspace of $\prod_{i\in I}L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu_i[\![|u_i|>k]\!]=0$.

(a) W_0 is a solid linear subspace and a subalgebra of $\prod_{i\in I} L^0(\mathfrak{A}_i)$, and there is a unique Riesz homomorphism $T : W_0 \to L^0(\mathfrak{B})$ such that $T(\langle \chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$. Moreover, T is multiplicative.

(b) $[\![Tu > 0]\!] \subseteq \pi(\langle [\![u_i > 0]\!]\rangle_{i\in I})$ whenever $u = \langle u_i\rangle_{i\in I} \in W_0$.

(c) If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for any of the spaces $L^0 = L^0(\mathfrak{A}_i)$, $L^0 = L^0(\mathfrak{B})$, then $\langle \bar h(u_i)\rangle_{i\in I} \in W_0$ and $T(\langle \bar h(u_i)\rangle_{i\in I}) = \bar h(Tu)$ whenever $u = \langle u_i\rangle_{i\in I} \in W_0$.

377D Theorem Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{B}, \bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product $\prod_{i\in I}\mathfrak{A}_i$ and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$. Let $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $T : W_0 \to L^0(\mathfrak{B})$ be as in 377B.

(a) If $u = \langle u_i\rangle_{i\in I} \in W_0$ and $\{i : i \in I,\ u_i = 0\} \in \mathcal{F}$, then $Tu = 0$.

(b) For $1 \le p \le \infty$, write W_p for the set of those families $\langle u_i\rangle_{i\in I} \in \prod_{i\in I} L^p(\mathfrak{A}_i, \bar\mu_i)$ such that $\sup_{i\in I}\|u_i\|_p$ is finite. Then $Tu \in L^p(\mathfrak{B}, \bar\nu)$ and $\|Tu\|_p \le \lim_{i\to\mathcal{F}}\|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I} \in W_p$.

(c) Let W_{ui} be the subspace of $\prod_{i\in I} L^1(\mathfrak{A}_i, \bar\mu_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i| - k\chi 1_{\mathfrak{A}_i})^+ = 0$. Then $\int Tu = \lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1 = \lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u = \langle u_i\rangle_{i\in I} \in W_{ui}$.

(d) Suppose now that $\pi[\mathfrak{A}] = \mathfrak{B}$.

(i) $T[W_0] = L^0(\mathfrak{B})$.

(ii) $T[W_{ui}] = L^1(\mathfrak{B}, \bar\nu)$.

(iii) If $p \in [1, \infty]$, then $T[W_p] = L^p(\mathfrak{B}, \bar\nu)$ and for every $w \in L^p(\mathfrak{B}, \bar\nu)$ there is a $u = \langle u_i\rangle_{i\in I} \in W_p$ such that $Tu = w$ and $\sup_{i\in I}\|u_i\|_p = \|w\|_p$.

377E Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I. Let $\pi : \mathfrak{A}^I \to \mathfrak{B}$ be a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}}\bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}^I$. Let W_0 be the set of families in $L^0(\mathfrak{A})^I$ which are bounded for the topology of convergence in measure on $L^0(\mathfrak{A})$.

(a)(i) W_0 is a solid linear subspace and a subalgebra of $L^0(\mathfrak{A})^I$, and there is a unique multiplicative Riesz homomorphism $T : W_0 \to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}^I$.

(ii) $[\![Tu > 0]\!] \subseteq \pi(\langle [\![u_i > 0]\!]\rangle_{i\in I})$ whenever $u = \langle u_i\rangle_{i\in I} \in W_0$.

(iii) If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for either of the spaces $L^0 = L^0(\mathfrak{A})$, $L^0 = L^0(\mathfrak{B})$, then $\langle\bar h(u_i)\rangle_{i\in I} \in W_0$ and $T(\langle\bar h(u_i)\rangle_{i\in I}) = \bar h(Tu)$ whenever $u = \langle u_i\rangle_{i\in I} \in W_0$.

(b)(i) For $1 \le p \le \infty$ let W_p be the subspace of $L^p(\mathfrak{A}, \bar\mu)^I$ consisting of $\|\ \|_p$-bounded families. Then $T[W_p] \subseteq L^p(\mathfrak{B}, \bar\nu)$, and $\|Tu\|_p \le \lim_{i\to\mathcal{F}}\|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_p.

(ii) Let W_{ui} be the subspace of $L^1(\mathfrak{A}_i, \bar\mu_i)^I$ consisting of uniformly integrablle families. Then $\int Tu = \lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1 = \lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u = \langle u_i\rangle_{i\in I} \in W_{ui}$.

(c)(i) We have a measure-preserving Boolean homomorphism $\tilde\pi : \mathfrak{A} \to \mathfrak{B}$ defined by setting $\tilde\pi a = \pi(\langle a_i\rangle_{i\in I})$, where $a_i = a$ for every $i \in I$, for each $a \in \mathfrak{A}$.

(ii) Let $P_{\tilde\pi} : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{A}, \bar\mu)$ be the conditional-expectation operator corresponding to $\tilde\pi : \mathfrak{A} \to \mathfrak{B}$. If $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A})$, then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}} u_i$ for the weak topology of $L^1(\mathfrak{A}, \bar\mu)$.

(iii) Suppose that $1 < p < \infty$ and that $\langle u_i\rangle_{i\in I}$ is a bounded family in $L^p(\mathfrak{A}, \bar\mu)$. Then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}} u_i$ for the weak topology of $L^p(\mathfrak{A}, \bar\mu)$.

377F Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{A}', \bar\mu')$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{B}, \bar\nu)$ and $(\mathfrak{B}', \bar\nu')$ be the reduced powers $(\mathfrak{A}, \bar\mu)^I|\mathcal{F}$, $(\mathfrak{A}', \bar\mu')^I|\mathcal{F}$, with corresponding homomorphisms $\pi : \mathfrak{A}^I \to \mathfrak{B}$ and $\pi' : \mathfrak{A}' \to \mathfrak{B}'$.

(a) Writing W_0, W_0' for the spaces of topologically bounded families in $L^0(\mathfrak{A})^I$, $L^0(\mathfrak{A}')^I$ respectively, we have unique Riesz homomorphisms $T : W_0 \to L^0(\mathfrak{B})$ and $T' : W_0' \to L^0(\mathfrak{B}')$ such that $T(\langle\chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$, $T'(\langle\chi a_i'\rangle_{i\in I}) = \chi\pi'(\langle a_i'\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}^I$ and $\langle a_i'\rangle_{i\in I} \in (\mathfrak{A}')^I$.

(b) Suppose that $S : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{A}', \bar\mu')$ is a bounded linear operator. Then we have a unique bounded linear operator $\hat S : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{B}', \bar\nu')$ such that $\hat S T(\langle u_i\rangle_{i\in I}) = T'(\langle Su_i\rangle_{i\in I})$ whenever $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A}, \bar\mu)$.

(c) The map $S \mapsto \hat S$ is a norm-preserving Riesz homomorphism from $\mathsf{B}(L^1(\mathfrak{A}, \bar\mu); L^1(\mathfrak{A}', \bar\mu'))$ to $\mathsf{B}(L^1(\mathfrak{B}, \bar\nu); L^1(\mathfrak{B}', \bar\nu'))$.

382Yb Exercise Devise an expression of the ideas of parts (f)-(h) of the proof of 382K which does not involve the Stone representation. (*Hint*: show that there is a non-increasing sequence in $\mathfrak{A}^+$ which makes enough decisions to play the role of the Boolean homomorphism $x : \mathfrak{A} \to \mathbb{Z}_2$.)

391B Definitions (b) I will call a Boolean algebra $\mathfrak{A}$ **chargeable** if there is an additive functional $\nu : \mathfrak{A} \to [0, \infty[$ which is **strictly positive**, that is, $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$.

(c) I will call a Boolean algebra **nowhere measurable** if none of its non-zero principal ideals are measurable algebras.

392J Proposition Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\nu a_n>0$. Then there is an infinite $I\subseteq\mathbb{N}$ such that $\nu(\inf_{i\in I\cap n}a_i)>0$ for every $n\in\mathbb{N}$.

392K Products of submeasures (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with submeasures μ, ν respectively. On the free product $\mathfrak{A}\otimes\mathfrak{B}$, we have a functional $\mu\ltimes\nu$ defined by saying that whenever $c\in\mathfrak{A}\otimes\mathfrak{B}$ is of the form $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, then

$$\begin{aligned}(\mu\ltimes\nu)(c)&=\min_{J\subseteq I}\max(\{\mu(\sup_{i\in J}a_i)\}\cup\{\nu b_i:i\in I\setminus J\})\\&=\min\{\epsilon:\epsilon\in[0,\infty],\,\mu(\sup\{a_i:i\in I,\,\nu b_i>\epsilon\})\le\epsilon\}.\end{aligned}$$

(b) In the context of (a), $\mu\ltimes\nu$ is a submeasure.

393G Proposition Let $\mathfrak{A}$ be a Maharam algebra, and ν and ν' two strictly positive Maharam submeasures on $\mathfrak{A}$. Then the metrics they induce on $\mathfrak{A}$ are uniformly equivalent, so we have a topology and uniformity on $\mathfrak{A}$ which we may call the **Maharam-algebra topology** and the **Maharam-algebra uniformity**.

393I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon>0$ there is a finite partition C of unity in $\mathfrak{A}$ such that $\nu c\le\epsilon$ for every $c\in C$.

393N Proposition Let $\mathfrak{A}$ be a Maharam algebra. Then the Maharam-algebra topology on $\mathfrak{A}$ is the order-sequential topology.

393O Proposition Let $\mathfrak{A}$ be a ccc Dedekind σ-complete Boolean algebra, with its order-sequential topology, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is the smallest order-closed set including $\mathfrak{B}$; $\mathfrak{B}$ is order-closed iff it is topologically closed.

393R Definition Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is **σ-finite-cc** if $\mathfrak{A}$ can be expressed as $\bigcup_{n\in\mathbb{N}}A_n$ where no A_n includes any infinite disjoint set. Observe that in this case $\mathfrak{A}$ is ccc; and that any Boolean algebra with a strictly positive exhaustive submeasure is σ-finite-cc.

393S Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff it is σ-finite-cc, weakly (σ,∞)-distributive and Dedekind σ-complete.

411P Example: Stone spaces (f) Let $(Z,\mathfrak{T},\Sigma,\mu)$ be the Stone space of a semi-finite measure algebra $(\mathfrak{A},\bar\mu)$. Let $W\subseteq Z$ be the union of all the open subsets of Z with finite measure. W has full outer measure, so $(\mathfrak{A},\bar\mu)$ can be identified with the measure algebra of the subspace measure μ_W. μ_W is locally finite. If $(\mathfrak{A},\bar\mu)$ is localizable, then μ_W is a Radon measure.

413Yf Exercise Let $\mathfrak{A}$ be a Boolean algebra and ν is a submeasure on $\mathfrak{A}$ which is *either* supermodular *or* exhaustive and submodular. Show that ν is uniformly exhaustive.

431G Theorem Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I}\subseteq\Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

432I Capacitability (b) A Choquet capacity c on X is **outer regular** if $c(A)=\inf\{c(G):G\supseteq A\text{ is open}\}$ for every $A\subseteq X$.

(c) If P is any lattice, an order-preserving function $c:P\to[0,\infty]$ is **submodular** if $c(p\wedge q)+c(p\vee q)\le c(p)+c(q)$ for all p, $q\in P$.

458L Measure algebras (h) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ be the conditional expectation operator. Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of closed subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$. Then

$$\int_c\prod_{j=0}^n Pu_j\le\int_c\prod_{j=0}^n u_j$$

whenever $c\in\mathfrak{C}$, $i_0,\dots,i_n\in I$ and $u_j\in L^1(\mathfrak{B}_{i_j},\bar\mu\restriction\mathfrak{B}_{i_j})^+$ for each $j\le n$, with equality if $i_0,\dots,i_n$ are distinct.

496B Basic facts (a) Let μ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(i) Set $I = \{a : a \in \mathfrak{A},\ \mu a = 0\}$. I is an ideal of $\mathfrak{A}$; write $\mathfrak{C}$ for the quotient Boolean algebra $\mathfrak{A}/I$. Then we have a strictly positive submeasure $\bar{\mu}$ on $\mathfrak{A}/I$ defined by setting $\bar{\mu}a^\bullet = \mu a$ for every $a \in \mathfrak{A}$.

(ii) If μ is exhaustive, so is $\bar{\mu}$.

(iii) If $\mathfrak{A}$ is Dedekind σ-complete and μ is a Maharam submeasure, then $\mathfrak{C}$ is a Maharam algebra. In this context I will say that $\mathfrak{C}$ is **the Maharam algebra of** μ.

(c) If μ is a submeasure defined on an algebra Σ of subsets of a set X, I will say that the **null ideal** $\mathcal{N}(\mu)$ of μ is the ideal of subsets of X generated by $\{E : E \in \Sigma,\ \mu E = 0\}$. If $\mathcal{N}(\mu) \subseteq \Sigma$ I will say that μ is **complete**. Generally, the **completion** of μ is the functional $\hat{\mu}$ defined by saying that $\hat{\mu}(E \triangle A) = \mu E$ whenever $E \in \Sigma$ and $A \in \mathcal{N}(\mu)$; $\hat{\mu}$ is a complete submeasure.

496C Radon submeasures Let X be a Hausdorff space. A **Radon submeasure** on X is a complete totally finite submeasure μ defined on a σ-algebra Σ of subsets of X such that (i) Σ contains every open set (ii) $\inf\{\mu(E \setminus K) : K \subseteq E$ is compact$\} = 0$ for every $E \in \Sigma$.

In this context I will say that a set $E \in \Sigma$ is **self-supporting** if $\mu(E \cap G) > 0$ whenever $G \subseteq X$ is open and $G \cap E \neq \emptyset$.

496D Proposition Let μ be a Radon submeasure on a Hausdorff space X with domain Σ.

(a) μ is a Maharam submeasure.

(b) $\inf\{\mu(G \setminus E) : G \supseteq E$ is open$\} = 0$ for every $E \in \Sigma$.

(c) If $E \in \Sigma$ there is a relatively closed $F \subseteq E$ such that F is self-supporting and $\mu(E \setminus F) = 0$.

(d) If $E \in \Sigma$ and $\epsilon > 0$ there is a compact self-supporting $K \subseteq E$ such that $\mu(E \setminus K) \leq \epsilon$.

496G Theorem Let $\mathfrak{A}$ be a Maharam algebra, and μ a strictly positive Maharam submeasure on $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a \in \mathfrak{A}$. Then there is a unique Radon submeasure ν on Z such that $\nu\widehat{a} = \mu a$ for every $a \in \mathfrak{A}$. The domain of ν is the Baire-property algebra $\widehat{\mathcal{B}}$ of Z, and the null ideal of ν is the nowhere dense ideal of Z.

4A2B Elementary facts about general topological spaces (f) Open maps (iii) Let X and Y be topological spaces and $f : X \to Y$ a continuous open map. Then $H \mapsto f^{-1}[H]$ is an order-continuous Boolean homomorphism from the regular open algebra of Y to the regular open algebra of X.

4A2I Stone-Čech compactifications (b)(i) Let I be any set, and write βI for its Stone-Čech compactification when I is given its discrete topology. Let Z be the Stone space of the Boolean algebra $\mathcal{P}I$. Then there is a canonical homeomorphism $\phi : \beta I \to Z$ defined by saying that $\phi(i)(a) = \chi a(i)$ for every $i \in I$ and $a \subseteq I$.

Note that if $z : \mathcal{P}I \to \mathbb{Z}_2$ is a Boolean homomorphism, then $\{J : z(J) = 1\}$ is an ultrafilter on I; and conversely, if $\mathcal{F}$ is an ultrafilter on I, we have a Boolean homomorphism $z : \mathcal{P}I \to \mathbb{Z}_2$ such that $\mathcal{F} = z^{-1}[\{1\}]$. So we can identify βI with the set of ultrafilters on I. Under this identification, the canonical embedding of I in βI corresponds to matching each member of I with the corresponding principal ultrafilter on I.

4A2T Topologies on spaces of subsets (g) Suppose that (X, ρ) is a metric space; let $\tilde{\rho}$ be the corresponding Hausdorff metric on the space $\mathcal{C} \setminus \{\emptyset\}$ of non-empty closed subsets of X.

(i) The topology $\mathfrak{S}_{\tilde{\rho}}$ defined by $\tilde{\rho}$ is finer than the Fell topology $\mathfrak{S}_F$ on $\mathcal{C} \setminus \{\emptyset\}$.

(ii) If X is compact, then $\mathfrak{S}_{\tilde{\rho}}$ and $\mathfrak{S}_F$ are the same, and both are compact.

4A4J Inner product spaces (i) Let U be an inner product space over $\overset{\mathbb{R}}{\mathbb{C}}$, and $\langle e_i \rangle_{i \in I}$ an orthonormal family in U. Then $\sum_{i \in I} |(u|e_i)|^2 \leq \|u\|^2$ for every $u \in U$.

References for Volume 5

Argyros S.A. & Kalamidas N.D. [82] 'The $K_{\alpha n}$ property on spaces with strictly positive measure', Canadian J. Math. 34 (1982) 1047-1058. [525U.]

Argyros S.A. & Tsarpalias A. [82] 'Calibers of compact spaces', Trans. Amer. Math. Soc. 270 (1982) 149-162. [525O.]

Babiker A.G. [76] 'On uniformly regular topological measure spaces', Duke Math. J. 43 (1976) 775-789. [533Xd.]

Balcar B. & Franěk F. [82] 'Independent families in complete Boolean algebras', Trans. Amer. Math. Soc. 274 (1982) 607-618. [515H.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [§539 *intro.*, 539E, 539K, 539M, §539 *notes*, 555M.]

Balcar B., Jech T. & Zapletal J. [97] 'Semi-Cohen Boolean algebras', Ann. Pure & Applied Logic 87 (1997) 187-208. [§546 *notes*.]

Balcar B. & Vojtáš P. [77] 'Refining systems on Boolean algebras', pp. 45-58 in Lachlan Srebny & Zarach 77. [515E.]

Bar-Hillel Y. [70] *Mathematical Logic and Foundations of Set Theory.* North-Holland, 1970.

Bartoszyński T. [84] 'Additivity of measure implies additivity of category', Trans. Amer. Math. Soc. 281 (1984) 209-213. [Chap. 52 *intro.*, 522P, §524 *notes*.]

Bartoszyński T. [87] 'Combinatorial aspects of measure and category', Fund. Math. 127 (1987) 225-239. [522R.]

Bartoszyński T. & Judah H. [89] 'On the cofinality of the smallest covering of the real line by meager sets', J. Symbolic Logic 54 (1989) 828-832. [522U.]

Bartoszyński T. & Judah H. [95] *Set Theory: on the structure of the real line.* A.K.Peters, 1995. [522Q, §522 *notes*, 526L, §528 *notes*, 534O, §534 *notes*, 553C, 554Ya.]

Bartoszyński T. & Shelah S. [92] 'Closed measure zero sets', Ann. Pure & Applied Logic 58 (1992) 93-110. [526L.]

Baumgartner J.E., Taylor A.D. & Wagon S. [77] 'On splitting stationary subsets of large cardinals', J. Symbolic Logic 42 (1977) 203-214. [§541 *notes*, §544 *notes*.]

Bell M.G. [81] 'On the combinatorial principle $P(c)$', Fundamenta Math. 114 (1981) 149-157. [§517 *notes*.]

Benedikt M. [98] 'Ultrafilters which extend measures', J. Symbolic Logic 63 (1998) 638-662. [538M, 538Xl, §538 *notes*.]

Benedikt M. [99] 'Hierarchies of measure-theoretic ultrafilters', Annals of Pure and Applied Logic 97 (1999) 203-219. [538H.]

Blair C.E. [77] 'The Baire category theorem implies the principle of dependent choice', Bull. Acad. Polon. Sci. (Math. Astron. Phys.) 25 (1977) 933-934. [§566 *notes*.]

Booth D. [70] 'Ultrafilters on a countable set', Ann. Math. Logic 2 (1970) 1-24. [517R.]

Borel E. [1919] 'Sur la classification des ensembles de mesure nulle', Bull. Soc. Math. France 47 (1919) 97-125. [§534 *intro.*]

Brendle J. [00] *Inner model theory and large cardinals.* Research report, March 2000. [528N, 529F-529H.]

Brendle J. [06] 'Cardinal invariants of the continuum and combinatorics on uncountable cardinals', Annals of Pure and Applied Logic 144 (2006) 43-72. [529F, 529H.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [§535 *notes*.]

Burke M.R. [n96] 'A theorem of Gitik and Shelah on disjoint refinements of sequences of subsets of the real line', note of 23.8.96. [546J, §547 *intro.*]

Burke M.R. [n05] 'Non-null sets of minimal cofinality', note of 2.3.05. [523K.]

Burke M.R. & Magidor M. [90] 'Shelah's pcf theory and its applications', Ann. Pure and Applied Logic 50 (1990) 207-254. [511I, §5A2 *intro.*.]

Burke M.R. & Shelah S. [92] 'Linear liftings for non-complete probability spaces', Israel J. Math. 79 (1992) 289-296. [§535 *notes*.]

Carlson T.J. [84] 'Extending Lebesgue measure to infinitely many sets', Pacific J. Math. 115 (1984) 33-45. [552N, §552 *notes*.]

Carlson T.J. [93] 'Strong measure zero and strongly meager sets', Proc. Amer. Math. Soc. 118 (1993) 577-586. [534O.]

Choksi J.R. & Fremlin D.H. [79] 'Completion regular measures on product spaces', Math. Ann. 241 (1979) 113-128. [532I.]

Cichoń J. & Pawlikowski J. [86] 'On ideals of subsets of the plane and on Cohen reals', J. Symbolic Logic 51 (1986) 560-569. [527F.]

Comfort W.W. & Negrepontis S. [74] *The Theory of Ultrafilters.* Springer, 1974. [538F.]

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [516 *notes*.]

Cummings J. [92] 'A model in which GCH holds at successors but fails at limits', Trans. Amer. Math. Soc. 329 (1992) 115-142. [525Z.]

Dellacherie C., Meyer P.A. & Weil M. (eds.) [73] *Séminaire de Probabilités VII.* Springer, 1973 (Springer Lecture Notes in Mathematics 321).

Devlin K.J. [84] *Constructibility.* Springer, 1984. [5A1S.]

Dow A. & Steprāns J. [94] 'The σ-linkedness of the measure algebra', Canad. Math. Bulletin 37 (1994) 42-45. [524L.]

Drake F.R. & Singh D. [96] *Intermediate Set Theory.* Wiley, 1996. [§5A1.]

Džamonja M. & Kunen K. [93] 'Measures on compact HS spaces', Fundamenta Math. 143 (1993) 41-54. [§531 *notes*.]

Džamonja M. & Plebanek G. [04] 'Precalibre pairs of measure algebras', Topology Appl. 144 (2004) 67-94. [525M.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§5A4.]

Erdős P., Hajnal A., Máté A. & Rado R. [84] *Combinatorial Set Theory: Partition Relations for Cardinals.* Akadémiai Kiadó, 1984 (Disquisitiones Math. Hung. 13). [5A1E, 5A1G, 5A1U.]

Farah I. [00] *Analytic Quotients.* Mem. Amer. Math. Soc. 148 (2000). [5A1W.]

Farah I. [03] 'How many algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?', Illinois J. Math. 46 (2003) 999-1033. [§556 *notes*.]

Farah I. [06] 'Analytic Hausdorff gaps II: the density zero ideal', Israel J. Math. 154 (2006) 235-246. [556S.]

Farah I. & Veličković B. [06] 'Von Neumann's problem and large cardinals', Bull. London Math. Soc. 38 (2006) 907-912. [539P.]

Feferman S. & Lévy A. [6] 'Independence results in set theory by Cohen's method', Notices Amer. Math. Soc. 10 (1963) 53. [561A.]

Fleissner W.G. [91] 'Normal measure axiom and Balogh's theorems', Topology and its Appl. 39 (1991) 123-143. [555P.]

Foreman M. & Wehrung F. [91] 'The Hahn-Banach theorem implies the existence of a non-Lebesgue measurable set', Fundamenta Math. 138 (1991) 13-19. [563A.]

Foreman M. & Woodin W. [91] 'The generalized continuum hypothesis can fail everywhere', Annals of Math. 133 (1991) 1-35. [525Z.]

Fossy J. & Morillon M. [98] 'The Baire category property and some notions of compactness', J. London Math. Soc. (2) 57 (1998) 1-19. [§566 *notes*.]

Freese R. & Nation J.B. [78] 'Projective lattices', Pacific J. Math. 75 (1978) 93-106. [§518 *notes*.]

Freiling C. [86] 'Axioms of symmetry; throwing darts at the real number line', J. Symbolic Logic 51 (1986) 190-200. [537K.]

Fremlin D.H. [74] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [564Xb.]

Fremlin D.H. [77] 'Uncountable powers of **R** can be almost Lindelöf', Manuscripta Math. 22 (1977) 77-85. [533J.]

Fremlin D.H. [84a] *Consequences of Martin's Axiom.* Cambridge U.P., 1984. [§511 *notes*, §517 *notes*, §531 *notes*, 555K.]

Fremlin D.H. [84b] 'On the additivity and cofinality of Radon measures', Mathematika 31 (1984) 323-335. [§524 *notes*.]

Fremlin D.H. [87] *Measure-additive coverings and measurable selectors.* Dissertationes Math. 260 (1987). [551A.]

Fremlin D.H. [88] 'Large correlated families of positive random variables', Math. Proc. Cambridge Phil. Soc. 103 (1988) 147-162. [525T.]

Fremlin D.H. [91a] 'The partially ordered sets of measure theory and Tukey's ordering,' Note di Matematica 11 (1991) 177-214. [523Yc, §524 *notes*, 526B, 526I, 527J, 529C, 529D, 534J.]

Fremlin D.H. [93] 'Real-valued-measurable cardinals', pp. 151-304 in Judah 93. [§537 *notes*, §541 *notes*, §544 *notes*, §545 *notes*, §5A2 *intro.*.]

Fremlin D.H. [97] 'On compact spaces carrying Radon measures of uncountable Maharam type', Fundamenta Math. 154 (1997) 295-304. [531T.]

Fremlin D.H. [03] 'Skew products of ideals', J. Applied Analysis 9 (2003) 1-18. [§527 *notes*.]

Fremlin D.H. [n02] 'Games and the Tukey classification', note of 29.9.02. [§526 *notes*.]

Fremlin D.H. & Grekas S. [95] 'Products of completion regular measures', Fundamenta Math. 147 (1995) 27-37. [532D.]

Fremlin D.H. & Miller A.W. [88] 'On some properties of Hurewicz, Menger and Rothberger', Fund. Math. 129 (1988) 17-33. [534I.]

Fremlin D.H., Natkaniec T. & Recław I. [00] 'Universally Kuratowski-Ulam spaces', Fundamenta Math. 165 (2000) 239-247. [§527 *notes*.]

Friedman S.D. & Koepke P. [97] 'An elementary approach to the fine structure of L', Bull. Symbolic Logic 3 (1997) 453-468. [5A1S.]

Fuchino S., Geschke S., Shelah S. & Soukup L. [01] 'On the weak Freese-Nation property of complete Boolean algebras', Ann. Pure Appl. Logic 110 (2001) 89-105. [518K.]

Fuchino S., Geschke S. & Soukup L. [01] 'On the weak Freese-Nation property of $\mathcal{P}(\omega)$', Arch. Math. Logic 40 (2001) 425-435. [522T, §522 *notes*.]

Fuchino S., Koppelberg S. & Shelah S. [96] 'Partial orderings with the weak Freese-Nation property', Ann. Pure and Applied Logic 80 (1996) 35-54. [518B, 518G, 518Yb, §518 *notes*, 522T.]

Fuchino S. & Soukup L. [97] 'More set theory around the weak Freese-Nation property', Fundamenta Math. 154 (1997) 159-176. [518I, 518K.]

Galvin F. [80] 'Chain conditions and products', Fundamenta Math. 108 (1980) 33-48. [§553 *notes*.]

Galvin F., Mycielski J. & Solovay R.M. [79] 'Strong measure zero sets', Notices Amer. Math. Soc. 26 (1979) A280. [534H.]

Gandy R.O. & Hyland J.M.E. [77](eds.) *Logic Colloquium '76.* North-Holland, 1977.

Geschke S. [02] 'On tightly κ-filtered Boolean algebras', Algebra Universalis 47 (2002) 69-93. [518P, 518S, §535 *notes*.]

Gitik M. & Shelah S. [89] 'Forcings with ideals and simple forcing notions', Israel J. Math. 68 (1989) 129-160. [543E, 546G.]

Gitik M. & Shelah S. [93] 'More on simple forcing notions and forcings with ideals', Annals of Pure and Applied Logic 59 (1993) 219-238. [542E, 543E, 546G.]

Gitik M. & Shelah S. [01] 'More on real-valued measurable cardinals and forcing with ideals', Israel J. Math. 124 (2001) 221-242. [546J, §547 *intro*.]

Główczyński W. [91] 'Measures on Boolean algebras', Proc. Amer. Math. Soc. 111 (1991) 845-849. [555M.]

Goldstern M., Judah H. & Shelah S. [93] 'Strong measure zero sets without Cohen reals', J. Symbolic Logic 58 (1993) 1323-1341. [§534 *notes*.]

Haydon R.G. [77] 'On Banach spaces which contain $\ell^1(\tau)$ and types of measures on compact spaces', Israel J. Math. 28 (1977) 313-324. [531E, 531L.]

Hodges W., Hyland M., Steinhorn C. & Truss J. [96] (eds.) *Logic: from Foundations to Applications, European Logic Colloquium, 1993.* Clarendon, 1996. [ISBN 0-19-853862-6]

Humke P.D. & Laczkovich M. [05] 'Symmetrically approximately continuous functions, consistent density theorems, and Fubini type inequalities', Trans. Amer. Math. Soc. 357 (2005) 31-44. [537Q, §537 *notes*.]

Ihoda J.I. [88] 'Strong measure zero sets and rapid filters', J. Symbolic Logic 53 (1988) 393-402. [534O.]

Jech T. [73] *The Axiom of Choice.* North-Holland, 1973. [561A, 561Xc, 561Yc, 561Yi, 564Yc.]

Jech T. [78] *Set Theory.* Academic, 1978; ISBN [*Intro.*, 553H, 555Q, §555 *notes*, §562 *notes*, §5A1.]

Jech T. [03] *Set Theory.* Springer, 2003; ISBN 3540440852 [§521 *notes*, §555 *notes*, §562 *notes*, §567 *notes*, §5A1, 5A3N, 5A3P.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Judah H. & Repický M. [95] 'Amoeba reals', J. Symbolic Logic 60 (1995) 1168-1185. [528N.]

Just W. [92] 'A modification of Shelah's oracle-c.c., with applications', Trans. Amer. Math. Soc. 329 (1992) 325-326. [527M.]

Just W. & Weese M. [96] *Discovering Modern Set Theory I.* Amer. Math. Soc., 1996 (Graduate Studies in Mathematics 8). [*Intro.*, 5A1E.]

Just W. & Weese M. [97] *Discovering Modern Set Theory II.* Amer. Math. Soc., 1997 (Graduate Studies in Mathematics 18). [5A1G.]

Kanamori A. [03] *The Higher Infinite.* Springer, 2003. [§541 *notes*, §555 *notes*, §567 *notes*, 5A1G, 5A1Q, 5A1U.]

Kechris A.S. [95] *Classical Descriptive Set Theory.* Springer, 1995. [§567 *notes*.]

Keisler H.J. & Tarski A. [64] 'From accessible to inaccessible cardinals', Fundamenta Math. 53 (1964) 225-308; errata Fundamenta Math. 57 (1965) 119. [§541 *intro.*, §541 *notes*.]

Kelley J.L. [50] 'The Tychonoff product theorem implies the axiom of choice', Fundamenta Math. 37 (1950) 75-76. [561D.]

Koppelberg S. [75] 'Homomorphic images of σ-complete Boolean algebras', Proc. Amer. Math. Soc. 51 (1975) 171-175. [515L.]

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [514Yb, §515 *notes*.]

Koppelberg S. & Shelah S. [96] 'Subalgebras of Cohen algebras need not be Cohen', pp. 261-275 in Hodges Hyland Steinhorn & Truss 96. [§546 *notes*.]

Kraszewski J. [01] 'Properties of ideals on the generalized Cantor spaces', J. Symbolic Logic 66 (2001) 1303-1320. [523G-523J, §523 *notes*.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [*Intro.*, 511A, §521 *notes*, §522 *notes*, §541 *notes*, 551Q, §551 *notes*, 556F, §562 *notes*, Appendix *intro.*, §5A1, §5A3.]

Kunen K. [81] 'A compact L-space under CH', Topology and its Appl. 12 (1981) 283-287. [§531 *notes*.]

Kunen K. [84] 'Random and Cohen reals', pp. 887-911 in Kunen & Vaughan 84. [§554 *notes*.]

Kunen K. [n70] 'Π^1_1 reflection at the continuum', note of January 1970. [543C, 544C, 544E, 544F.]

Kunen K. & Mill J.van [95] 'Measures on Corson compact spaces', Fundamenta Math. 147 (1995) 61-72. [§531 *notes*.]

Kunen K. & Vaughan J.E. [84] (eds.) *Handbook of Set-Theoretic Topology.* North-Holland, 1984.

Kuratowski K. [66] *Topology,* vol. I. Academic, 1966. [562F.]

Lachlan A., Srebny M. & Zarach A. [77] (eds.) *Set Theory and Hierarchy Theory.* Springer, 1977 (Lecture Notes in Math. 619).

Laver R. [76] 'On the consistency of Borel's conjecture', Acta Math. 137 (1976) 151-169. [534O.]

Laver R. [87] 'Random reals and Souslin trees', Proc. Amer. Math. Soc. 100 (1987) 531-534. [553M.]

Levinski J.-P., Magidor M. & Shelah S. [90] 'Chang's conjecture for $\aleph_\omega$', Israel J. Math. 69 (1990) 161-172. [5A1U.]

Levy A. [71] 'The sizes of the indescribable cardinals', pp. 205-218 in Scott 71. [§541 *notes*, §544 *notes*.]

Louveau A. & Veličković B. [99] 'Analytic ideals and cofinal types', Ann. Pure and Applied Logic 99 (1999) 171-195. [526K.]

Maharam D. [47] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [539O.]

Makowsky J.A. & Ravve E.V. (eds.) [98] *Logic colloquium '95.* Springer, 1998 (Lecture Notes in Logic 11).

Martin D.A. [70] 'Measurable cardinals and analytic games', Fundamenta Math. 66 (1970) 287-291. [567N.]

Martin D.A. [75] 'Borel determinacy', Ann. of Math. (2) 102 (1975) 363-371. [§567 *notes*.]

Martin D.A. & Solovay R.M. [70] 'Internal Cohen extensions', Ann. Math. Logic 2 (1970) 143-178. [Chap. 52 *intro.*, §528 *intro.*, §528 *notes*.]

Meyer P.-A. [73] 'Limites médiales d'après Mokobodzki', pp. 198-204 in Dellacherie Meyer & Weil 73. [538Yg.]

Meyer P.A. [75] (ed.) *Séminaire de Probabilités IX.* Springer, 1975 (Lecture Notes in Mathematics 465).

Miller A.W. [80] 'There are no Q-points in Laver's model for the Borel conjecture', Proc. Amer. Math. Soc. 78 (1980) 103-106. [§538 *notes*.]

Miller A.W. [81] 'Some properties of measure and category', Trans. Amer. Math. Soc. 266 (1981) 93-114. [522J.]

Miller A.W. [82] 'The Baire category theorem and cardinals of countable cofinality', J. Symbolic Logic 47 (1982) 275-288. [552G, 552Xb.]

Mokobodzki G. [75] 'Rélèvement borélien compatible avec une classe d'ensembles négligeables. Application à la désintégration des mesures', pp. 437-442 in Meyer 75. [535I.]

Mokobodzki G. [7?] 'Désintegration des mesures et relèvements Boreliens de sous-espaces de $L^\infty(X,\mathcal{B},\mu)$', Lab. d'Analyse Fonctionnelle, Univ. Paris VI (?). [535E, §535 *notes*.][1]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Moore J.T. [05] 'Set mapping reflection', J. Math. Logic 5 (2005) 87-98. [517O.]

Moschovakis Y.N. [70] 'Determinacy and prewellorderings of the continuum', pp. 24-62 in Bar-Hillel 70. [567M.]

Mycielski J. [64] 'On the axiom of determinateness', Fund. Math. 53 (1964) 205-224. [567D.]

Mycielski J. & Świerczkowski S. [64] 'On the Lebesgue measurability and the axiom of determinateness', Fund. Math. 54 (1964) 67-71. [567F.]

Naimark M.A. [70] *Normed Rings.* Wolters-Noordhoff, 1970. [§561 *notes*.]

Neumann J.von [31] 'Algebraische Repräsentanten der Funktionen "bis auf eine Menge vom Masse Null"', Crelle's J. Math. 165 (1931) 109-115. [535G.]

Neumann J.von & Stone M.H. [35] 'The determination of representative elements in the residual classes of a Boolean algebra', Fundamenta Math. 25 (1935) 353-378. [§535 *notes*.]

Pawlikowski J. [86] 'Why Solovay real produces Cohen real', J. Symbolic Logic 51 (1986) 957-968. [552G, 552H.]

Plebanek G. [95] 'On Radon measures on first-countable compact spaces', Fundamenta Math. 148 (1995) 159-164. [531P.]

Plebanek G. [97] 'Non-separable Radon measures and small compact spaces', Fundamenta Math. 153 (1997) 25-40. [531L-531N.]

Plebanek G. [00] 'Approximating Radon measures on first-countable compact spaces', Colloquium Math. 86 (2000) 15-23. [533H, 533Yc.]

Pol R. [82] 'Note on the spaces $P(S)$ of regular probability measures whose topology is determined by countable subsets', Pacific J. Math. 100 (1982) 185-201. [533Ya.]

Prikry K. [75] 'Ideals and powers of cardinals', Bull. Amer. Math. Soc. 81 (1975) 907-909. [555P.]

Quickert S. [02] 'CH and the Saks property', Fundamenta Math. 171 (2002) 93-100. [539K.]

Raisonnier J. & Stern J. [85] 'The strength of measurability hypotheses', Israel J. Math. 50 (1985) 337-349. [Chap. 52 *intro.*, 522P.]

[1] I have not been able to locate this paper; I believe it was a seminar report. I made notes from it in 1977.

Rothberger F. [38a] 'Eine Äquivalenz zwischen der kontinuumhypothese unter der Existenz der Lusinschen und Sierpinschischen Mengen', Fundamenta Math. 30 (1938) 215-217. [522G.]

Rothberger F. [38b] 'Eine Verschärfung der Eigenschaft C', Fundamenta Math. 30 (1938) 50-55. [§534 *notes*.]

Rothberger F. [41] 'Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété **C**', Proc. Cambridge Phil. Soc. 37 (1941) 109-126. [534O.]

Scott D.S. [71] (ed.) *Axiomatic Set Theory.* Amer. Math. Soc., 1971 (Proceedings of Symposia in Pure Mathematics XIII, vol. 1).

Shelah S. [82] *Proper forcing.* Springer, 1982 (Lecture Notes in Mathematics 940). [§538 *notes*.]

Shelah S. [83] 'Lifting problem of the measure algebra', Israel J. Math. 45 (1983) 90-96. [§535 *notes*.]

Shelah S. [84] 'Can you take Solovay inaccessible away?', Israel J. Math. 48 (1984) 1-47. [§522 *notes*.]

Shelah S. [92] 'Cardinal arithmetic for skeptics', Bull. Amer. Math. Soc. 26 (1992) 197-210. [5A2D.]

Shelah S. [94] *Cardinal arithmetic.* Oxford U.P., 1994. [5A2D, 5A2G.]

Shelah S. [96] 'Further cardinal arithmetic', Israel J. Math. 95 (1996) 61-114. [542I.]

Shelah S. [98] 'There may be no nowhere dense ultrafilter', pp. 305-324 in Makovsky & Ravve 98. [538H, §538 *notes*.]

Shelah S. [00] 'Covering of the null ideal may have countable cofinality', Fundamenta Math. 166 (2000) 109-136. [§522 *notes*.]

Shelah S. [03] 'The null ideal restricted to some non-null set may be $\aleph_1$-saturated', Fundamenta Math. 179 (2003) 97-129. [§547 *notes*.]

Shelah S. & Steprāns J. [05] 'Comparing the uniformity invariants of null sets for different measures', Advances in Math. 192 (2005) 403-426. [534Za.]

Shipman J. [90] 'Cardinal conditions for strong Fubini theorems', Trans. Amer. Math. Soc. 321 (1990) 465-481. [537K.]

Solecki S. & Todorčević S. [04] 'Cofinal types of topological directed orders', Ann. Inst. Fourier 54 (2004) 1877-1911. [513J, 513N, 513Yf, §526 *notes*.]

Solovay R.M. [66] 'New proof of a theorem of Gaifman and Hales', Bull. Amer. Math. Soc. 72 (1966) 282-284. [514Yb.]

Solovay R.M. [70] 'A model of set theory in which every set of reals is Lebesgue measurable', Annals of Math. 92 (1970) 1-56. [§522 *notes*.]

Solovay R.M. [71] 'Real-valued measurable cardinals', pp. 397-428 in Scott 71. [541J, 541P, 541Ya, §541 *notes*, 555D, 555Q.]

Talagrand M. [08?] 'Maharam's problem', Annals of Math., to appear. [539A.]

Tarski A. [45] 'Ideale in volständigen Mengenkörpen II', Fundamenta Math. 33 (1945) 51-65. [541P.]

Todorčević S. [85] 'Remarks on chain conditions in products', Compositio Math. 55 (1985) 295-302. [537G.]

Todorčević S. [87] 'Partitioning pairs of countable ordinals', Acta Math. 159 (1987) 261-294. [554Yc.]

Todorčević S. [00] 'A dichotomy for P-ideals of countable sets', Fundamenta Math. 166 (2000) 251-267. [5A1V.]

Truss J.K. [77] 'Sets having calibre $\aleph_1$', pp. 595-612 in Gandy & Hyland 77. [522J.]

Truss J.K. [88] 'Connections between different amoeba algebras', Fund. Math. 130 (1988) 137-155. [Chap. 52 *intro.*, 528A, 528D, 528K, §528 *notes*.]

Tukey J.W. [40] *Convergence and Uniformity in Topology.* Princeton U.P., 1940 (Ann. Math. Studies 1). [513F, §513 *notes*.]

Ulam S. [30] 'Zur Masstheorie in der allgemeinen Mengenlehre', Fundamenta Math. 16 (1930) 140-150. [§541 *intro.*, §541 *notes*.]

Veličković B. [92] 'Forcing axioms and stationary sets', Advances in Math. 94 (1992) 256-284. [517O.]

Veličković B. [05] 'CCC forcing and splitting reals', Israel J. Math. 147 (2005) 209-221. [§539 *intro.*, 539E, 539M.]

Vojtáš P. [93] 'Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis', pp. 619-643 in Judah 93. [512A.]

Wimmers E. [82] 'The Shelah P-point independence theorem', Israel J. Math. 43 (1982) 28-48. [§538 *notes*.]

Zakrzewski P. [92] 'Strong Fubini theorems from measure extension axioms', Comm. Math. Univ. Carolinae 33 (1992) 291-297. [544J.]

Index to volumes 1-5

Principal topics and results

The general index below is intended to be comprehensive. Inevitably the entries are voluminous to the point that they are often unhelpful. I have therefore prepared a shorter, better-annotated, index which will, I hope, help readers to focus on particular areas. It does not mention definitions, as the bold-type entries in the main index are supposed to lead efficiently to these; and if you draw blank here you should always, of course, try again in the main index. Entries in the form of mathematical assertions frequently omit essential hypotheses and should be checked against the formal statements in the body of the work.

General index

References in **bold** type refer to definitions; references in *italics* are passing references. Definitions marked with **>** are those in which my usage is dangerously at variance with that of some other author or authors.