K-analytic sets. Analytic Sets

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K-ANALYTIC SETS

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§1. *Introduction.* The classical theory of analytic sets works well in metric spaces, but the analytic sets themselves are automatically separable. The theory of K-analytic sets, developed by Choquet, Sion, Frolík and others, works well in Hausdorff spaces, but the K-analytic sets themselves remain Lindelöf. The theory of k-analytic sets developed by A. H. Stone and R. W. Hansell works well in non-separable metric spaces, especially in the special case, when k is $\aleph_0$, with which we shall be concerned, see [**9**, **10** and **16**–**20**]. Of course the k-analytic sets are metrizable. For accounts of these theories, see, for example, [**15**].

In his thesis, Holický [**12**], and, in a series of papers, Frolík and Holický develop a theory of analytic sets in uniform spaces. They introduce a concept of an analytic uniform space. They say that a completely regular topological space is analytic, if it is an analytic uniform space, when given the fine uniformity. Their theory provides a common generalization of the theory of K-analytic sets in completely regular spaces and of the theory of $\aleph_0$-analytic sets in metric spaces, however, their analytic sets remain paracompact.

In this paper and in a follow-up paper that we hope will be accepted for publication in the next volume of *Mathematika* [**11**], we present the outline of another common generalization that includes the analytic sets of the Frolík–Holický theory. Our theory works well in Hausdorff spaces, but our analytic sets remain subparacompact, in a sense that will be defined below, see Definition 7. Perhaps the most important advantage of our approach is that we work with the natural topological definition of discreteness, while Frolík and Holický have to work with a modified concept defined in terms of uniform structures on the space. It follows from Lemma 9, proved below, that, in a collectionwise normal space, any family of sets that is discrete in the topological sense is also discrete in the fine uniform structure according to the Frolík–Holický definition. In this way, a set A in a collectionwise normal space X is K-analytic in X in our sense, if, and only if, A, with the uniform structure inherited from the fine uniform structure on X, is an analytic uniform space in their sense. We believe that our theory will seem to be the more natural, at least to those not addicted to the study of uniform spaces.

Before we give our definition it will be convenient to introduce some notation and to recall some definitions. We identify each ordinal with the set of smaller ordinals. We identify each cardinal with the smallest ordinal having the given cardinal. We use ω to denote the first infinite cardinal

$$\omega = \aleph_0 = \{0, 1, 2, \ldots\}.$$

When κ is any infinite cardinal we use κ^ω to denote the Baire space, often called $B(\kappa)$, of all sequences

$$\tau = \tau_0, \tau_1, \tau_2, \ldots$$

[MATHEMATIKA, **30** (1983), 189–221]

of elements $\tau_0, \tau_1, \tau_2, \ldots$ chosen from κ, with metric d defined by

$$d(\tau, \rho) = 0, \qquad \text{if } \tau = \rho,$$

and

$$d(\tau, \rho) = 2^{-n}, \qquad \text{if } \tau \neq \rho,$$

and n is the first integer with $\tau_n \neq \rho_n$. If $n \geqslant 1$, and $\tau \in \kappa^\omega$, we use $\tau|n$ to denote the finite sequence

$$\tau_0, \tau_1, \ldots, \tau_{n-1}$$

in κ^n of the first n elements of τ. If $n \geqslant 1$, and $t = t_0, t_1, \ldots, t_{n-1}$ belongs to κ^n, we use $I(t)$ to denote the Baire interval

$$I(t) = \{\tau \in \kappa^\omega : \tau|n = t\}.$$

We recall some definitions.

Definition 1. *A family $\{E_\gamma : \gamma \in \Gamma\}$ of sets in a space X is said to be discrete, if, for each x in X, there is a neighbourhood U of x with $U \cap E_\gamma \neq \varnothing$, for at most one γ in Γ. Such a family is said to be σ-discrete, if it is possible to write*

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma(n),$$

with each family $\{E_\gamma : \gamma \in \Gamma(n)\}$, $n \geqslant 1$, discrete in X. The family is said to be discretely σ-decomposable, [**16**], *if it is possible to write*

$$E_\gamma = \bigcup_{n=1}^{\infty} E_\gamma^{(n)}, \qquad \gamma \in \Gamma,$$

with each family $\{E_\gamma^{(n)} : \gamma \in \Gamma\}$, $n \geqslant 1$, discrete in X.

Note that, in this definition, we count the number of γ in Γ with $U \cap E_\gamma \neq \varnothing$, rather than the number of distinct sets of the family that meet U.

Definition 2. *A set-valued map K from a space X to a space Y is said to be upper semi-continuous if the set $\{x \in X : K(x) \cap H \neq \varnothing\}$ is closed in X whenever H is a closed set in Y.*

We can now introduce our working definition for a K-analytic set A in a Hausdorff space X. As with most good concepts we shall find many equivalent definitions, some more elegant than this.

Definition 3. *A set A in a Hausdorff space X is said to be K-analytic in X, if, for some infinite cardinal κ, and, for some compact-valued upper semi-continuous map K from κ^ω to X, we have $A = K(\kappa^\omega)$, and, for each $n \geqslant 1$, the family $\{K(I(t)) : t \in \kappa^n\}$ is σ-discrete in X.*

We remark that this definition is suggested by Hansell's recognition of the importance of maps that map discrete families of sets in one space to σ-discrete families or to discretely σ-decomposable families in another space.

Note that, when κ takes the value ω, the special K-analytic sets that are obtained coincide with the sets of the Choquet, Sion, Frolík theory; sets that we shall call Lindelöf K-analytic sets. Now Hansell [**10**, Th. 4.1] shows that a k-analytic set, with $k = \aleph_0$, is necessarily the continuous image of κ^ω, for some κ, under a point-valued map that maps discrete families of sets in κ^ω to families in M that have σ-discrete bases, and so have σ-discrete refinements. It follows, by one of the equivalent forms of the definition, see (d) of Theorem 1, below, that such an $\aleph_0$-analytic set in a metric space M is K-analytic in M. Further, each subparacompact Čech complete space is K-analytic, see Corollary 1 to Lemma 5, below.

When $\kappa > \omega$, the concept of a K-analytic set is not intrinsic; it is extrinsic since the concept of an uncountable σ-discrete family essentially involves all the points of X rather than just the points of A.

We develop a theory of K-Lusin sets in parallel with that of the K-analytic sets. The working definition is as follows.

DEFINITION 4. *A set B in a Hausdorff space X is said to be a K-Lusin set in X, if, for some infinite cardinal κ, and for some upper semi-continuous map K from κ^ω to X, taking only mutually disjoint compact values, we have $B = K(\kappa^\omega)$, and, for each $n \geqslant 1$, the family $\{K(I(t)) : t \in \kappa^n\}$ is discretely σ-decomposable in X.*

By a result of Hansell, [**10**, Th. 5.6], it follows that *each Borel subset of a complete metric space M is K-Lusin in M*. Further, *a subset B of a compact Hausdorff space X is K-Lusin in X, if, and only if, B is contained in the smallest family of sets of X that contains the closed sets and that is closed under the operations of countable intersections and of countable disjoint unions*, [**15**, p. 125].

Note the contrast between the use of σ-discrete families in Definition 3 and of discretely σ-decomposable families in Definition 4. We shall see below, in Theorem 1, that in the K-analytic case we could equally well have used discretely σ-decomposable families; we have just chosen to work with the simpler concept. It is possible that we could equally well work with σ-discrete families in the K-Lusin case; but this we cannot prove; we need to work with discretely σ-decomposable families to use the methods that we have available.

In §2, which has its own introduction, we obtain a number of technical results showing that a set is K-analytic or K-Lusin in a space, if, and only if, it has some rather special representation.

In §3 we obtain stability results for K-analytic and K-Lusin sets. Such sets are stable under certain generalized Souslin operations and under certain types of compact-valued upper semi-continuous maps. Recall that, when $\mathscr{S}$ is a family of sets in a space X, a set S is said to be a Souslin-$\mathscr{S}$ set if it has a representation

$$S = \bigcup\left\{\bigcap_{n=1}^{\infty} S(\tau|n) : \tau \in \omega^\omega\right\},$$

where $S(t)$ belongs to $\mathscr{S}$ for all t in ω^n, $n \geqslant 1$. If

$$\left(\bigcap_{n=1}^{\infty} S(\tau|n)\right) \cap \left(\bigcap_{n=1}^{\infty} S(\sigma|n)\right) = \varnothing,$$

whenever τ, σ are distinct in ω^ω, the Souslin representation is said to be disjoint. We use two extended definitions.

DEFINITION 5. *If $\mathscr{S}$ is a class of sets in a space X, we say that a set S is obtained from the family $\mathscr{S}$ by use of the extended Souslin operation in X, and we say that S is an extended Souslin-$\mathscr{S}$ set in X, if*

$$S = \bigcup\left\{\bigcap_{n=1}^{\infty} S(\tau|n) : \tau \in \kappa^{\omega}\right\},$$

where $S(t)$ belongs to $\mathscr{S}$ for all t in κ^n, $n \geq 1$, and where each family

$$\left\{\bigcup\left\{\bigcap_{m=1}^{\infty} S(\tau|m) : \tau|n = t\right\} : t \in \kappa^n\right\},$$

$n \geq 1$, is discretely σ-decomposable in X. We use $\mathbf{ES}(\mathscr{S})$ to denote the family of these sets.

DEFINITION 6. *If $\mathscr{S}$ is a class of sets in a space X, we say that a set S is obtained from the family $\mathscr{S}$ by use of the disjoint extended Souslin operation, and we say that S is a disjoint extended Souslin-$\mathscr{S}$ set in X, if S has the representation described in Definition 5 with the additional requirement that*

$$\left(\bigcap_{n=1}^{\infty} S(\tau|n)\right) \cap \left(\bigcap_{n=1}^{\infty} S(\sigma|n)\right) = \varnothing$$

whenever τ, σ are distinct in κ^{ω}. We use the symbol $\mathbf{ES}_d(\mathscr{S})$ to denote the family of these sets.

Using standard techniques, and the results of §2, we easily verify a number of stability results.

(a) *For any family $\mathscr{S}$ in any space X, the result of applying the extended Souslin operation to the extended Souslin-$\mathscr{S}$ sets is an extended Souslin-$\mathscr{S}$ set*, see Theorem 3.

(b) *For any family $\mathscr{S}$ in any space X, the result of applying the disjoint extended Souslin operation to the disjoint extended Souslin-$\mathscr{S}$ sets is a disjoint extended Souslin-$\mathscr{S}$ set*, see Theorem 4.

(c) *The result of applying the extended Souslin operation to the K-analytic sets in a space X is a K-analytic set in X*, see Theorem 5.

(d) *The result of applying the disjoint extended Souslin operation to the K-Lusin sets in a space X, is a K-Lusin set in X*, see Theorem 6.

These results need to be supplemented by the result, due to Hansell [**9**, Th. 3.1], that *the result of applying the extended Souslin operation to the family $\mathscr{F}$ of closed sets in a topological space X is the class of Souslin-$\mathscr{F}$ sets in X.*

Recall that a family $\{H_\theta : \theta \in \Theta\}$ of sets in a space X is said to be a refinement of a family $\{E_\gamma : \gamma \in \Gamma\}$ of sets of X, if, for each θ in Θ, there is a γ in Γ with $H_\theta \subset E_\gamma$ and also

$$\bigcup\{H_\theta : \theta \in \Theta\} = \bigcup\{E_\gamma : \gamma \in \Gamma\}.$$

Our second type of stability result is illustrated by the following example.

The image in a space Y of a K-analytic set A, in a space X, under a compact-valued upper semi-continuous map K from A to Y, taking each family of subsets of A that is discrete in X to a family in Y having a σ-discrete refinement, is K-analytic in Y, see Theorem 7. The corresponding result for K-Lusin sets is given in Theorem 8.

In §4 we explore topological properties and characterizations of K-analytic sets. When D. K. Burke [1] defined a subparacompact space, one of the mutually equivalent definitions that he used was: a Hausdorff space X is subparacompact if each open cover of X has a closed σ-discrete refinement. We introduce a modified form of this concept.

DEFINITION 7. *A set A is subparacompact in a Hausdorff space X, if each cover of A by relatively open subsets of A has a refinement that is σ-discrete in X.*

If the space X is regular, then the refinement can always be taken to be by relatively closed subsets of A. Thus a regular Hausdorff space is subparacompact in itself, according to Definition 7, if, and only if, it is a subparacompact space, according to Burke's definitions.

We prove that *a set A that is K-analytic in a Hausdorff space X is subparacompact in X*, see Theorem 10.

If a set A, contained in a Hausdorff space X, is K-analytic in itself, then A is not necessarily K-analytic in X. However, in this case, *when A is K-analytic in a Hausdorff space X, it is always possible to express A in the form $G \cap S$, with G a $\mathscr{G}_\delta$-set in X, with S a Souslin-$\mathscr{F}$ set in X, and with A a K-analytic set in G*, see Theorem 12. If X is completely regular, this leads to an elegant characterization of the K-analytic sets in X.

A set A in a completely regular space X is K-analytic in X, if, and only if, A is subparacompact in X and can be expressed as the intersection of a $\mathscr{G}_\delta$-set and a Souslin-$\mathscr{F}$ set in the Stone–Čech compactification of X, see Theorem 14.

A Hausdorff space X is said to be collectionwise normal, if, for each discrete family $\{F_\gamma : \gamma \in \Gamma\}$ of closed sets in X, it is possible to find a discrete family $\{G_\gamma : \gamma \in \Gamma\}$ of open sets with $F_\gamma \subset G_\gamma$ for each γ in Γ. We obtain the following characterization for K-analytic sets in such spaces.

A set A in a collectionwise normal space X is K-analytic in X, if, and only if, it is a Souslin-$\mathscr{F}$ set in some space Z that contains X and that is the product of a compact Hausdorff space with a complete metric space, see Theorem 16.

We also have an elegant characterization in Čech complete spaces.

A set A in a Čech complete space X is K-analytic in X, if, and only if, A is both subparacompact in X and Souslin-$\mathscr{F}$ in X, see Theorem 17.

In general, we have to be content with a characterization that is less direct.

A set A in a Hausdorff space X is K-analytic in X, if, and only if, A is the image in X of a paracompact Čech complete space Z under a continuous point-valued map that *maps each discrete family of sets in Z to a family that is discretely σ-decomposable in* X, see Theorem 13.

When one confines ones attention to a Hausdorff space X that is K-analytic in itself, the theory of K-analytic sets in X takes an especially simple form. Indeed, *if X is a K-analytic space, then the families of K-analytic sets in X, of extended Souslin-$\mathscr{F}$ sets in X, of Souslin-$\mathscr{F}$ sets in X, of sets that are K-analytic in themselves and also subparacompact in X, and of sets that can be expressed as the projection on X of a closed subset of $X \times \omega^{\omega}$, all coincide*, see Theorem 18. Similarly, *if X is a K-Lusin space, then the family of K-Lusin sets in X coincides with the family of extended disjoint Souslin-$\mathscr{F}$ sets in X*, see Theorem 19.

Recall that a space X is said to be paracompact, if X is a Hausdorff space and every open cover of X has a locally finite open refinement. We find it more convenient to work with an alternative defining property. *The Hausdorff space X is paracompact, if, and only if, every open cover of X has a σ-discrete open refinement*, see, for example, [**2**, p. 377]. Note that *a paracompact space is always normal, indeed, it is always collectionwise normal*, see, for example, [**2**, p. 379]. We say that *a set A in a Hausdorff space X is paracompact in X, if each cover of A by relatively open sets has a relatively open refinement that is σ-discrete in X*. We show that *a set A that is K-analytic in a collectionwise normal space X is paracompact in X*, see Theorem 20.

In this introduction we have only stated explicitly a few of the results concerning K-Lusin sets, rather more are considered in the main text.

In the follow-up paper that we have promised, we shall need to introduce classes of extended Borel sets, of extended Borelian-$\mathscr{F}$ sets and of extended Baire sets. Using these classes we obtain natural generalizations of the usual first separation theorems and also of the Novikov approximation theorem [**15**, pp. 55–62 and p. 98]. The first separation theorems have the usual applications, including the Borel nature of K-analytic sets whose complements are also K-analytic sets, and of K-Lusin sets. We shall give a detailed account of K-Lusin sets.

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§2. *The representation of K-analytic sets.* The working definition of a K-analytic set in a Hausdorff space that we stated in the introduction, gives a way of representing a K-analytic set. In this section we give a number of different ways of representing a K-analytic set, concentrating on representations that provide alternative definitions for K-analytic sets and that will enable us to derive their properties. We consider the K-Lusin sets in parallel with the K-analytic sets.

Before we state the main results it is convenient to have some formal definitions.

DEFINITION 8. *Let κ be an infinite cardinal. A set A is said to be κ-Lindelöf if each open cover of A can be reduced to a cover of cardinal at most κ.*

DEFINITION 9. *The family* $\{E_\gamma : \gamma \in \Gamma\}$ *of sets in a space* X *is said to be discretely* σ*-decomposable*, [**16**], *if we can write*

$$E_\gamma = \bigcup_{n=1}^{\infty} E_\gamma^{(n)}, \qquad \gamma \in \Gamma,$$

with each family $\{E_\gamma^{(n)} : \gamma \in \Gamma\}$, $n \geqslant 1$, *discrete in* X.

DEFINITION 10. *The family* $\{R_\theta : \theta \in \Theta\}$ *is said to be a refinement of the family* $\{E_\gamma : \gamma \in \Gamma\}$ *if, for each* θ *in* Θ, *there is a corresponding* $\gamma(\theta)$ *in* Γ *with* $R_\theta \subset E_{\gamma(\theta)}$, *and*

$$\bigcup\{R_\theta : \theta \in \Theta\} = \bigcup\{E_\gamma : \gamma \in \Gamma\}.$$

DEFINITION 11. *The family* $\{B_\theta : \theta \in \Theta\}$ *is said to be a base for the family* $\{E_\gamma : \gamma \in \Gamma\}$ *if* $\{B_\theta : \theta \in \Theta\}$ *is a refinement of* $\{E_\gamma : \gamma \in \Gamma\}$ *and also, for each* γ *in* Γ,

$$E_\gamma = \bigcup\{B_\theta : \theta \in \Theta \text{ and } B_\theta \subset E_\gamma\}.$$

THEOREM 1. *Let* A *be a set in a Hausdorff space* X, *and let* κ *be an infinite cardinal. Each of the following assertions implies all the others.*

(a) *A is a κ-Lindelöf K-analytic set in X.*

(b) *A is the image of κ^ω under a compact-valued upper semi-continuous map K, the family $\{K(I(t)) : t \in \kappa^n\}$ being σ-discrete in X, for each $n \geqslant 1$.*

(c) *A is the image of κ^ω under a compact-valued upper semi-continuous map K, the family $\{K(I(t)) : t \in \kappa^n\}$ having a σ-discrete base in X, for each $n \geqslant 1$.*

(d) *A is the image of κ^ω under a compact-valued upper semi-continuous map K, the family $\{K(I(t\,; t_n)) : t_n \in \kappa\}$ having a σ-discrete refinement in X for each t in κ^n and each $n \geqslant 0$.*

(e) *A is the image of $\omega^\omega \times \kappa^\omega$ under a compact-valued upper semi-continuous map K, the family $\{K(I(s) \times I(t)) : t \in \kappa^n\}$ being discrete in X for each fixed s in ω^n, and for each $n \geqslant 1$.*

(f) *A is the image of a κ-Lindelöf paracompact Čech complete space G under a continuous map that maps discrete families in G to discretely σ-decomposable families in X.*

THEOREM 2. *Let* A *be a set in a Hausdorff space* X *and let* κ *be an infinite cardinal. Each of the following assertions implies the others.*

(a) *A is a κ-Lindelöf K-Lusin set in X.*

(b) *A is the image of κ^ω under a compact-valued upper semi-continuous map K, with $K(\tau) \cap K(\tau') = \varnothing$, whenever τ and τ' are distinct points of κ^ω, and with the family $\{K(I(t)) : t \in \kappa^n\}$ discretely σ-decomposable in X for each $n \geqslant 1$.*

(f) *A is the injective image of a κ-Lindelöf paracompact Čech complete space G under a continuous map that maps discrete families in G to discretely σ-decomposable families in X.*

It might seem to be more natural in condition (d) of Theorem 1, merely to insist that each family $\{K(I(t)) : t \in \kappa^n\}$, $n = 1, 2, 3, \dots$ should have a σ-discrete refinement. However this would not be enough, as we can show by adapting an example given by Hansell [**9**, p. 407]. Let S be any non-analytic subset of $[0, 1]$. Let S have cardinal κ, and let

$$s_0, s_1, s_2, \dots, s_\alpha, \dots \quad , \qquad \alpha < \kappa ,$$

be a wellordering of S. For t in κ^ω, define $K(t)$ to be $\{s_\gamma\}$, if, for some $n \geqslant 1$,

$$t_0 = t_1 = \dots = t_{n-1}, \quad t_n = t_{n+1} = \dots = \gamma \neq n ,$$

and define $K(t)$ to be $\varnothing$, if t is not of this form. It is easy to verify that K is an upper semi-continuous point-valued map from κ^ω onto S, and that each family $\{K(I(t)) : t \in \kappa^n\}$ has a σ-discrete refinement. Indeed the two sets

$$K(I(n, n, \dots, n)) = S \setminus \{s_n\} ,$$

and

$$K(I(1, n, n, \dots, n)) = \{s_n\}$$

yield a two-set refinement of this family.

The casual reader of the proofs of Theorems 1 and 2 below might believe, as we indeed did all believe at one time, that had we defined K-Lusin sets using σ-discrete families, then we could have established an analogue of Theorem 1 showing the equivalence of the definitions. The difficulty that we have not been able to overcome is the need in the proof of Lemma 3 to work with the closures $\operatorname{cl} C(\sigma_{n-1} ; \tau|n)$ of the sets $C(\sigma_{n-1} ; \tau|n)$, in order to ensure that $H(\sigma ; \tau)$ is compact-valued and upper semi-continuous, coupled with the requirement, in the K-Lusin case, that

$$H(\sigma ; \tau) \cap H(\sigma' ; \tau) = \varnothing$$

when σ, σ' are distinct points of ω^ω and $\tau \in \kappa^\omega$.

Proof of Theorem 1. We establish the logical connections between the assertions (a) to (f) shown in the following diagram. The unlabelled implications are immediate; the others are labelled with the numbers of the lemmas that establish them.

$$\begin{tikzcd}
 & \text{(a)} \arrow[d, Leftrightarrow, "1"] & \\
\text{(e)} \arrow[r, Rightarrow, "4"] & \text{(b)} \arrow[r, Leftrightarrow, "6"] \arrow[d, Rightarrow] & \text{(f)} \\
\text{(d)} \arrow[u, Rightarrow, "3"] & \text{(c)} \arrow[l, Rightarrow] &
\end{tikzcd}$$

LEMMA 1. *The assertions* (a) *and* (b) *of Theorem* 1 *imply each other.*

Proof. First suppose that A is a κ-Lindelöf K-analytic set in X. Then we can choose a cardinal, ξ say, and a compact-valued upper semi-continuous map K, with

$A = K(\xi^\omega)$, the family $\{K(I(y)) : y \in \xi^n\}$ being σ-discrete in X for each $n \geqslant 1$. Suppose that $\xi > \kappa$, otherwise (b) holds. Let Ξ be the set of those η in ξ^ω for which $K(\eta) \neq \varnothing$. Then as K is upper semi-continuous, Ξ is closed in ξ^ω. Suppose that for some $n \geqslant 1$, the set $K(I(y))$ contains a point, say $k(y)$, for more than κ choices of y in ξ^n. As the family $\{K(I(y)) : y \in \xi^n\}$ is σ-discrete in X, there will be a subset, Q say, of ξ^n with $\{k(y) : y \in Q\}$ a discrete family of points in A of cardinality exceeding κ. This is impossible as A is κ-Lindelöf. We conclude that, for each $n \geqslant 1$, $I(y) \cap \Xi$ is non-empty for at most κ choices of y in ξ^n.

Now, for some $\kappa(\varnothing) \leqslant \kappa$, we can choose a wellordering

$$\xi_0(\tau_0), \qquad 0 \leqslant \tau_0 < \kappa(\varnothing),$$

of those η in Ξ, for which $I(\eta) \cap \Xi \neq \varnothing$. When $\tau_0, \tau_1, \ldots, \tau_n$ are fixed, within suitable ranges of variation, for some $\kappa(\tau_0, \tau_1, \ldots, \tau_n) \leqslant \kappa$, we can choose a wellordering

$$\xi_{n+1}(\tau_0, \tau_1, \ldots, \tau_n, \tau_{n+1}), \qquad 0 \leqslant \tau_{n+1} < \kappa(\tau_0, \tau_1, \ldots, \tau_n),$$

of those η in ξ such that

$$I(\xi_0(\tau_0), \ldots, \xi_n(\tau_0, \ldots, \tau_n), \eta) \cap \Xi \neq \varnothing .$$

Let T be the set of all $\tau = \tau_0, \tau_1, \ldots$ in κ^ω satisfying

$$0 \leqslant \tau_0 < \kappa(\varnothing),$$

$$0 \leqslant \tau_1 < \kappa(\tau_0),$$

$$\ldots$$

$$0 \leqslant \tau_{n+1} < \kappa(\tau_0, \tau_1, \ldots, \tau_n),$$

$$\ldots .$$

Let ϕ denote the map of T to ξ^ω defined by

$$\phi : \tau \mapsto \xi_0(\tau_0), \xi_1(\tau_1), \ldots .$$

It is now easy to verify that ϕ maps T homeomorphically onto Ξ. Further, defining a compact-valued map H from κ^ω to X by taking

$$H(\tau) = K(\phi(\tau)), \qquad \text{if } \tau \in T,$$

$$H(\tau) = \varnothing, \qquad \text{if } \tau \notin T,$$

it is easy to verify that $A = H(B)$, that H is upper semi-continuous, and that each family $\{H(I(t)) : t \in \kappa^n\}$, $n \geqslant 1$, coincides with the corresponding family $\{K(I(y)) : y \in \xi^n\}$, except for the elimination of some empty sets, and so is σ-discrete in X. Thus (b) follows as a consequence of (a).

Now suppose that A is the image of κ^ω under a compact-valued upper semi-continuous map. As κ^ω is κ-Lindelöf, it follows, without difficulty, that A is also κ-Lindelöf. Thus the assertion (b) implies the assertion (a).

LEMMA 2. *Let K be a compact-valued upper semi-continuous map from κ^ω to a Hausdorff space X. For each $n \geqslant 1$, and each t in κ^n, let $E(t)$ be a given relatively closed subset of $K(I(t))$, and define a set-valued map from κ^ω to X, by taking*

$$H(\tau) = \bigcap_{n=1}^{\infty} E(\tau|n),$$

for each τ in κ^ω. Then H is compact-valued and upper semi-continuous.

Moreover, if

$$\bigcup\{E(t, t_n) : t_n \in \kappa\} = \bigcup\{K(I(t, t_n)) : t_n \in \kappa\},$$

for each t in κ^n and each $n \geqslant 0$, then $H(\kappa^\omega) = K(\kappa^\omega)$.

Proof. For each $s = s_0, s_1, \ldots, s_{n-1}$, in κ^n, and for each $n \geqslant 1$, it will be convenient to write

$$A(s) = E(s_0) \cap E(s_0, s_1) \cap \ldots \cap E(s_0, s_1, \ldots, s_{n-1}).$$

We fix σ in κ^ω and show that H is compact-valued and upper semi-continuous at σ. As K is compact-valued and upper semi-continuous and X is Hausdorff, it is easy to verify that

$$\bigcap_{n=1}^{\infty} K(I(\sigma|n)) = K(\sigma).$$

As $E(\sigma|n)$ is relatively closed in $K(I(\sigma|n))$ for each $n \geqslant 1$, we have

$$\begin{aligned} H(\sigma) &= \bigcap_{n=1}^{\infty} E(\sigma|n) \\ &= \bigcap_{n=1}^{\infty} \{(\mathrm{cl}\, E(\sigma|n)) \cap K(I(\sigma|n))\} \\ &= \left\{\bigcap_{n=1}^{\infty} \mathrm{cl}\, E(\sigma|n)\right\} \cap K(\sigma), \end{aligned}$$

so that $H(\sigma)$ is a compact subset of $K(\sigma)$. Suppose that $H(\sigma) \subset U$, for some set U, open in X. Then

$$\left\{\bigcap_{n=1}^{\infty} \mathrm{cl}\, E(\sigma|n)\right\} \cap K(\sigma) \cap \{X \setminus U\} = H(\sigma) \setminus U = \emptyset.$$

Hence, for some integer $m \geqslant 1$,

$$\left\{\bigcap_{n \leqslant m} \mathrm{cl}\, E(\sigma|n)\right\} \cap K(\sigma) \cap \{X \setminus U\} = \emptyset,$$

so that

$$K(\sigma) \subset U \cup \left\{X \setminus \bigcap_{n \leqslant m} \operatorname{cl} E(\sigma|n)\right\}.$$

By the upper semi-continuity of K, there is an $l \geqslant m$, with

$$K(I(\sigma|l)) \subset U \cup \left\{X \setminus \bigcap_{n \leqslant m} \operatorname{cl} E(\sigma|n)\right\}.$$

Hence

$$\begin{aligned} A(\sigma|l) &= \bigcap_{n \leqslant l} E(\sigma|n) \\ &= \bigcap_{n \leqslant l} (\operatorname{cl} E(\sigma|n)) \cap K(I(\sigma|n)) \\ &\subset U. \end{aligned}$$

Thus H is compact-valued and upper semi-continuous at σ.

Now suppose that

$$\bigcup\{E(t, t_n) : t_n \in \kappa\} = \bigcup\{K(I(t, t_n)) : t_n \in \kappa\},$$

for each t in κ^n and each $n \geqslant 0$. Consider any point x of $K(\kappa^\omega)$. It is clear that we can choose $\tau_0, \tau_1, \tau_2, \ldots$ inductively so that

$$x \in E(\tau_0, \tau_1, \ldots, \tau_n) \subset K(I(\tau_0, \tau_1, \ldots, \tau_n)),$$

for $n = 0, 1, 2, \ldots$. Now $\tau = \tau_0, \tau_1, \ldots$ is an element of κ^ω with

$$x \in H(\tau) = \bigcap_{n=0}^{\infty} E(\tau_0, \tau_1, \ldots, \tau_n).$$

Thus $K(\kappa^\omega) \subset H(\kappa^\omega)$. As $H(\kappa^\omega)$ is trivially contained in $K(\kappa^\omega)$ we have $H(\kappa^\omega) = K(\kappa^\omega)$, as required.

Lemma 3. *Assertion* (d) *implies assertion* (e).

Proof. Using assertion (d) we may suppose that, for each $n \geqslant 0$ and for each t in κ^n, $\{B(t; \gamma) : \gamma \in \Gamma(t)\}$ is a σ-discrete refinement of $\{K(I(t, t_n)) : t_n \in \kappa\}$. For each $n \geqslant 0$ and each t in κ^n, we write

$$\Gamma(t) = \bigcup\{\Gamma(s_n; t) : s_n \in \omega\},$$

with each family

$$\{B(t; \gamma) : \gamma \in \Gamma(s_n; t)\},$$

$s_n \in \omega$, discrete in X. Further, for each γ in $\Gamma(t)$ we can choose a $\tau_n(\gamma)$ in κ with

$$B(t; \gamma) \subset K\Big(I(t, \tau_n(\gamma))\Big).$$

For each s_n in ω and each t_n in κ, write

$$C(s_n; t, t_n) = \bigcup\{B(t; \gamma) : \gamma \in \Gamma(s_n; t) \text{ and } \tau_n(\gamma) = t_n\}.$$

Then, for each t in κ^n and each s_n in ω, the family

$$\{C(s_n; t, t_n) : t_n \in \kappa\}$$

is discrete in X; the family

$$\{C(s_n; t, t_n) : s_n \in \omega \text{ and } t_n \in \kappa\}$$

is a refinement of the family

$$\{K(I(t, t_n)) : t_n \in \kappa\}$$

with

$$C(s_n; t, t_n) \subset K(I(t, t_n)), \qquad \text{for } s_n \in \omega;$$

and

$$\bigcup\{C(s_n; t, t_n) : s_n \in \omega \text{ and } t_n \in \kappa\} = \bigcup\{K(I(t, t_n)) : t_n \in \kappa\}$$
$$= K(I(t)).$$

Now, for (σ, τ) in $\omega^\omega \times \kappa^\omega$ and $n \in \omega$, we take

$$K^*(\sigma; \tau) = K(\tau),$$

and

$$E(\sigma|n; \tau|n) = K(I(\tau|n)) \cap \operatorname{cl} C(\sigma_{n-1}; \tau|n).$$

Then K^* is a compact-valued upper semi-continuous map from $\omega^\omega \times \kappa^\omega$ to X. We identify $\omega^\omega \times \kappa^\omega$ with $(\omega \times \kappa)^\omega$ through the natural homeomorphism with

$$(\sigma_0, \sigma_1, \dots; \tau_0, \tau_1, \dots)$$

corresponding to

$$(\sigma_0, \tau_0), (\sigma_1, \tau_1), \dots.$$

Then, for each (s, t) in $\omega^n \times \kappa^n$, the set $E(s; t)$ is a relatively closed subset of

$$K^*(I(s) \times I(t)) = K(I(t))$$

and

$$\{E(s, s_n; t, t_n) : (s_n, t_n) \in \omega \times \kappa\}$$

is a refinement of the family

$$\{K(I(t, t_n)) : t_n \in \kappa\}$$

and so also of the family

$$\{K^*(I(s, s_n) \times I(t, t_n)) : (s_n, t_n) \in \omega \times \kappa\}.$$

Since $\kappa \geqslant \omega$, the product $\omega \times \kappa$, regarded as a discrete set, is homeomorphic to κ. This homeomorphism generates a corresponding homeomorphism between $(\omega \times \kappa)^\omega$

and κ^ω. Using this homeomorphism, we can apply Lemma 2 to K^* and E. Taking

$$H(\sigma\,;\tau) = \bigcap_{n=1}^{\infty} E(\sigma|n\,;\tau|n)$$

for $(\sigma, \tau) \in \omega^\omega \times \kappa^\omega$, we find that H is a compact-valued upper semi-continuous map from $\omega^\omega \times \kappa^\omega$ to X, and moreover

$$A = K(\kappa^\omega) = K^*(\omega^\omega \times \kappa^\omega) = H(\omega^\omega \times \kappa^\omega).$$

It remains to prove that, if $n \geqslant 1$, and $s \in \omega^n$, then the family

$$\{H(I(s) \times I(t)) : t \in \kappa^n\}$$

is discrete in X. As

$$H(I(s) \times I(t)) \subset \bigcap_{m \leqslant n} E(s|m\,;t|m),$$

it suffices to prove that the family

$$\left\{\bigcap_{m \leqslant n} E(s|m\,;t|m) : t \in \kappa^n\right\}$$

is discrete in X.

Now, if $s_0, s_1, \ldots, s_n$ and $t_0, t_1, \ldots, t_{n-1}$ are fixed,

$$E(s_0, \ldots, s_n\,; t_0, \ldots, t_n) = K(t_0, \ldots, t_n) \cap \mathrm{cl}\, C(s_n\,; t_0, \ldots, t_n),$$

so that the family

$$\{E(s_0, \ldots, s_n\,; t_0, \ldots, t_n) : t_n \in \kappa\}$$

is discrete in X. It follows inductively that the family

$$\left\{\bigcap_{m \leqslant n} E(s|m\,;t|m) : t \in \kappa^n\right\}$$

is discrete in X for each fixed $s|n$ in ω^n. This completes the proof.

LEMMA 4. *The assertion* (e) *implies the assertion* (b).

Proof. Let d be a homeomorphism mapping κ to $\omega \times \kappa$ with

$$d(r) = (s(r), t(r))$$

for $r \in \kappa$. Let

$$\delta(\rho) = (\sigma(\rho), \tau(\rho))$$

be the induced homeomorphism mapping κ^ω to $\omega^\omega \times \kappa^\omega$. Let K be the compact-valued map provided by assertion (e) and write

$$H(\rho) = K(\sigma(\rho), \tau(\rho)).$$

It is easy to verify that H satisfies the requirements (for K) in assertion (b).

To discuss the assertion (f) we need a lemma of a rather different character.

LEMMA 5. *If a $\mathscr{G}_\delta$-set in a regular K-analytic space X is subparacompact in itself, then it is K-analytic in itself.*

Proof. Let $G = \bigcap_{i=0}^{\infty} G_i$, with each G_i open in X. As X is regular, for each $i \geqslant 0$, and for each point g of G we can choose an open set $N(g, i)$ containing g and with its closure contained in G_i. As G is subparacompact in itself, for each $i \geqslant 0$, we can choose a family, say $\{E(i, y) : 0 \leqslant y < \xi\}$, with ξ a sufficiently large cardinal, of subsets of G refining the family $\{N(g, i) \cap G : g \in G\}$ and σ-discrete in G. For $i \geqslant 0$, and $0 \leqslant y < \xi$, write

$$F(i, y) = \operatorname{cl} E(i, y).$$

Then, for some g in G, we have

$$\begin{aligned} F(i, y) &= \operatorname{cl} E(i, y) \\ &\subset \operatorname{cl} N(g, i) \\ &\subset G_i . \end{aligned}$$

As

$$G \subset \bigcup\{F(i, y) : 0 \leqslant y < \xi\},$$

for each $i \geqslant 0$, it is easy to verify that G has the representation

$$G = \bigcup\{\bigcap\{F(n, \eta_n) : n \in \omega\} : \eta \in \xi^\omega\}.$$

Further, each family

$$\{\bigcap\{F(m, \eta_m) : 0 \leqslant m < n\} : \eta \in \xi^n\},$$

$n \geqslant 1$, is σ-discrete in G.

The space X is itself K-analytic. Following one of the proofs that the intersection of a Souslin-$\mathscr{F}$ set and a Lindelöf K-analytic set is a Lindelöf K-analytic set, see, for example, [**15**, pp. 26–27], and using the σ-discreteness of the various families in G, it follows that G is K-analytic in itself.

COROLLARY 1. *A Čech complete space is K-analytic, if, and only if, it is subparacompact.*

Proof. Let A be a Čech complete space. Then A is a $\mathscr{G}_\delta$-set in some compact Hausdorff space X.

If A is subparacompact, in itself, it is K-analytic, in itself, by Lemma 5.

If A is K-analytic, in itself, then it is subparacompact, in itself, by Theorem 10, proved below.

COROLLARY 2. *If a $\mathscr{G}_\delta$-set is subparacompact in a regular Hausdorff space X, then it is an $\mathscr{F}_{\sigma\delta}$-set in X.*

Proof. Follow the first paragraph of the proof of Lemma 5, choosing the families $\{E(i, y) : 0 \leqslant y < \xi\}$ with $i \in \omega$ to be σ-discrete in X. In this case, it is easy to verify that G has the representation

$$G = \bigcap\{\bigcup\{F(i, y) : 0 \leqslant y < \xi\} : i \in \omega\}$$

with each set

$$\bigcup\{F(i, y) : 0 \leqslant y < \xi\},$$

$i \in \omega$, an $\mathscr{F}_\sigma$-set in X.

Lemma 6. *The assertions* (b) *and* (f) *of Theorem* 1 *imply each other.*

Proof. First suppose that the assertion (b) of Theorem 1 holds. We may suppose that A is the image of κ^ω under a compact-valued upper semi-continuous map K, the family $\{K(I(t)) : t \in \kappa^n\}$ being σ-discrete in X for each $n \geqslant 1$.

Write

$$G = \bigcup\{K(\tau) \times \{\tau\} : \tau \in \kappa^\omega\},$$

so that G is the graph of K.

We first verify that the projection map $p : (x, \tau) \mapsto x$ maps any discrete family of sets in G to a discretely σ-decomposable family in X. Let $\{F_\theta : \theta \in \Theta\}$ be any discrete family of sets in G. We suppose, as we may, that each set F_θ, $\theta \in \Theta$, is closed in G.

Consider a fixed τ in κ^ω. Write

$$F(\theta, \tau) = p(F_\theta \cap [K(\tau) \times \{\tau\}]),$$

for θ in Θ. Then $\{F(\theta, \tau) : \theta \in \Theta\}$ is a discrete family of closed sets contained in the compact set $K(\tau)$. So the set $\Theta(\tau)$ of those θ with $F(\theta, \tau) \neq \emptyset$ is necessarily finite. Now the set

$$H(\tau) = \bigcup\{F_\theta : \theta \in \Theta \setminus \Theta(\tau)\}$$

is a closed subset of G, disjoint from $K(\tau) \times \{\tau\}$. Thus the function

$$H(\tau, \sigma) = p(H(\tau) \cap [K(\sigma) \times \{\sigma\}])$$

is a compact-valued upper semi-continuous map from κ^ω to X with

$$H(\tau, \tau) = \emptyset.$$

So we can choose an integer $n(\tau)$ so that

$$H\Big(\tau, I(\tau|n(\tau))\Big) = \emptyset.$$

Hence $\Theta(\sigma) \subset \Theta(\tau)$ for all σ in the Baire interval $I(\tau|n(\tau))$.

Thus each τ in κ^ω belongs to a Baire interval J_τ with the property that

$$X \times J_\tau$$

meets at most a finite number of the sets of the family $\{F_\theta : \theta \in \Theta\}$. Let $\{I_\gamma : \gamma \in \Gamma\}$ be the family of all maximal Baire intervals of κ^ω having this property. Then $\{I_\gamma : \gamma \in \Gamma\}$

is a disjoint family of Baire intervals covering κ^{ω}. For each γ in Γ, let $\Phi(\gamma)$ be the finite subset, say
$$\phi_1(\gamma), \phi_2(\gamma), \ldots, \phi_{m(\gamma)}(\gamma),$$
of those θ in Θ with
$$F_\theta \cap [X \times I_\gamma] \neq \varnothing .$$

As K maps each family $\{I(t) : t \in \kappa^n\}$, $n \geqslant 1$, to a σ-discrete family in X, it maps the family $\{I_\gamma : \gamma \in \Gamma\}$ to a σ-discrete family in X. So we can write
$$\Gamma = \bigcup_{n=0}^{\infty} \Gamma(n),$$
with each family
$$\{K(I_\gamma) : \gamma \in \Gamma(n)\},$$
$n \geqslant 0$, discrete in X. Now, once $n \geqslant 0$, $m \geqslant 1$ and θ in Θ are chosen, there is at most one γ in Γ with
$$\gamma \in \Gamma(n), \ \phi_m(\gamma) = \theta \text{ and } F_\theta \cap [X \times I_\gamma] \neq \varnothing .$$
So
$$p(F_\theta) = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{\gamma \in \Gamma} \{p(F_\theta \cap [X \times I_\gamma]) : \gamma \in \Gamma(n) \text{ and } \phi_m(\gamma) = \theta\},$$
with each family
$$\left\{\bigcup_{\gamma \in \Gamma} \{p(F_\theta \cap [X \times I_\gamma]) : \gamma \in \Gamma(n) \text{ and } \phi_m(\gamma) = \theta\} : \theta \in \Theta\right\},$$
$n \geqslant 0$, $m \geqslant 1$, discrete in X. Thus p maps $\{F_\theta : \theta \in \Theta\}$ into a discretely σ-decomposable family in X.

As A is κ-Lindelöf and κ^{ω} has a base of κ sets for its topology, it is easy to verify that G is κ-Lindelöf.

As K is compact-valued and upper semi-continuous, it is easy to verify that the restriction r, of the projection map $q : (x, \tau) \mapsto \tau$ to G, is a proper map of G to a closed subset F of κ^{ω}, see, for example, [**15**, pp. 37–39]. By the preservation of paracompactness under the inverses of proper maps, see [**2**, p. 386], the set G is necessarily paracompact. Now the proper map r extends to a continuous map $\hat{r}$ of βG to βF, and $\hat{r}$ maps G to F and $\beta G \setminus G$ to $\beta F \setminus F$, see, for example, [**15**, p. 101]. As F is a complete metric space, F is a $\mathscr{G}_\delta$-set in βF. Thus $\beta F \setminus F$ is an $\mathscr{F}_\sigma$-set in βF. As $\hat{r}$ is continuous, $\beta G \setminus G$ is an $\mathscr{F}_\sigma$-set and G is a $\mathscr{G}_\delta$-set in βG. Hence G is a paracompact Čech complete space. Thus (f) holds.

Now suppose that assertion (f) holds so that A is the continuous image in the Hausdorff space X of a paracompact Čech complete space G under a map ϕ that maps discrete families in G to families in X that are discretely σ-decomposable. Then G, being Čech complete, is a $\mathscr{G}_\delta$-set in some compact Hausdorff space Z. Further, as G is paracompact, it is subparacompact in itself. Hence, by Lemma 5, G is K-analytic, in itself. It follows by the stability result Theorem 7, proved below, using only the results in the present theorem that have been proved above, that A is K-analytic in X. As G is κ-Lindelöf, so is A. Thus (a) and so also (b) holds.

Proof of Theorem 2. With one exception the results follow from the relevant parts of the proof of Theorem 1, making the modifications that are obviously necessary. The exception arises as it is not clear that the methods used in the proof of Theorem 1 enable us to prove that a paracompact Čech complete space is K-Lusin. However, by a result of Frolík, [**3**], see, for example [**2**, p. 422], a paracompact Čech complete space G admits a proper map p onto a complete metric space M. By the result of Hansell, quoted in the introduction, this space M is a K-Lusin space. Now p^{-1} is an upper semi-continuous map of M onto G, taking mutually disjoint compact values. Further, as p is continuous, p^{-1} maps discrete families in M to discrete families in G. Hence, by Theorem 8, proved below using neither Theorem 1 nor Theorem 2, it follows that G is a K-Lusin space. By this same theorem, it then follows that A is K-Lusin in X.

§3. *The stability of K-analytic sets and of K-Lusin sets.* In this section we outline the proofs of the stability results announced in the introduction. We prove the following theorems.

THEOREM 3. *For any family $\mathscr{S}$ of sets in any space X,*

$$\mathbf{ES}(\mathbf{ES}(\mathscr{S})) = \mathbf{ES}(\mathscr{S}).$$

THEOREM 4. *For any family $\mathscr{S}$ of sets in any space X,*

$$\mathbf{ES}_d(\mathbf{ES}_d(\mathscr{S})) = \mathbf{ES}_d(\mathscr{S}).$$

THEOREM 5. *Using $\mathscr{A}$ to denote the family of K-analytic sets in a space X, we have*

$$\mathbf{ES}(\mathscr{A}) = \mathscr{A}.$$

Note that, as the intersection of a closed set in X with a K-analytic set in X is clearly K-analytic in X, it follows immediately from this theorem that *the intersection of an extended Souslin-$\mathscr{F}$ set in X with a K-analytic set in X is a K-analytic set in X.*

THEOREM 6. *Using $\mathscr{L}$ to denote the family of K-Lusin sets in a space X, we have*

$$\mathbf{ES}_d(\mathscr{L}) = \mathscr{L}.$$

THEOREM 7. *Let A be a set that is K-analytic in a Hausdorff space X, and let K be a compact-valued upper semi-continuous map from A to a Hausdorff space Y, taking each family of subsets of A that is discrete in X to a family in Y having a refinement that is σ-discrete in Y. Then $K(A)$ is K-analytic in Y.*

THEOREM 8. *Let L be a set that is K-Lusin in a Hausdorff space X, and let K be a compact-valued upper semi-continuous map, taking only mutually disjoint values, from L to a Hausdorff space Y, with the property that each family of subsets of L that is discrete in X maps to a family in Y that is discretely σ-decomposable. Then $K(L)$ is a K-Lusin set in Y.*

THEOREM 9. *If A_i is K-analytic in a Hausdorff space X_i, for $i \geqslant 1$, then*

$$\prod_{i=1}^{\infty} A_i$$

is K-analytic in

$$\prod_{i=1}^{\infty} X_i .$$

The corresponding result also holds for K-Lusin sets.

Proof of Theorem 3. We follow the classical proof of the stability of the Souslin operation closely, using the version in [**15**, pp. 12–16]. To facilitate comparison we temporarily replace the use of ω to denote the finite ordinals (including zero) by the use of $\mathbb{N}$ to denote the natural numbers (excluding zero). We note that the map $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by

$$\nu(m, n) = 2^{m-1}(2n-1),$$

with its inverse

$$\nu^{-1}(l) = (\phi(l), \psi(l)),$$

defined by the solution of the Diophantine equation

$$l = 2^{\phi(l)}(2\psi(l)-1),$$

is used for two quite distinct purposes. One is to recode the integral suffices of the sequences

$$\sigma = \sigma_1, \sigma_2, \ldots, \qquad \tau(m) = \tau_1^{(m)}, \tau_2^{(m)}, \ldots ;$$

and the other is to recode the integral values σ_n, $\tau_n^{(m)}$ in these sequences. We retain the functions ν, ϕ, ψ for the first of these purposes, but we introduce new functions μ, ξ, η to recode the values of σ_n, $\tau_n^{(m)}$, which will now be ordinals less than a suitable sufficiently large ordinal κ. We choose μ to be any bijection mapping $\kappa \times \kappa$ to κ, and we define ξ, η so that

$$\mu(\xi(\rho), \eta(\rho)) = \rho$$

for $0 \leqslant \rho < \kappa$. Following [**15**, Lemma 2.3.2] we take

$$\chi(\sigma ; \tau(1), \tau(2), \ldots) = \mu(\sigma_1, \tau_{\psi(1)}^{(\phi(1))}), \mu(\sigma_2, \tau_{\psi(2)}^{(\phi(2))}), \ldots, \mu(\sigma_l, \tau_{\psi(l)}^{(\phi(l))}), \ldots,$$

for $\sigma ; \tau(1), \tau(2), \ldots$ in $\kappa^{\mathbb{N}} \times (\kappa^{\mathbb{N}})^{\mathbb{N}}$, and find that the equation

$$\chi(\sigma ; \tau(1), \tau(2), \ldots) = \rho$$

has the unique solution

$$\sigma_m = \xi(\rho_m), \qquad m \geqslant 1,$$

$$\tau_n^{(m)} = \eta(\rho_{\nu(m,n)}), \qquad m \geqslant 1, \quad n \geqslant 1 .$$

As before, a knowledge of the first $\nu(m, n)$ components of $\rho = \chi(\sigma ; \tau(1), \tau(2), \ldots)$ determines $\sigma|m$ and $\tau(m)|n$ uniquely. Indeed, a knowledge of the first l components of

ρ corresponds precisely to a knowledge of the first l components of σ and of the first $t(m, l)$ components of $\tau(m)$, where $t(m, l)$ is the largest integer t with

$$2^{m-1}(2t-1) \leqslant l.$$

Note that $t(m, l) = 0$ when $2^{m-1} > l$.

Now suppose that T belongs to $\mathbf{ES}(\mathbf{ES}(\mathscr{S}))$. Then

$$T = \bigcup_\sigma \bigcap_m T(\sigma|m)$$

with each family

$$\left\{\bigcup\left\{\bigcap_{n=1}^{\infty} T(\sigma|n) : \sigma|m = s\right\} : s \in \kappa^m\right\}, \qquad m \geqslant 1,$$

discretely σ-decomposable in X, and

$$T(\sigma|m) = \bigcup_\tau \bigcap_n S(\sigma|m ; \tau|n),$$

with each family

$$\left\{\bigcup\left\{\bigcap_{l=1}^{\infty} S(\sigma|m ; \tau|l) : \tau|n = t\right\} : t \in \kappa^n\right\},$$

for $m \geqslant 1$, $\sigma|m \in \kappa^m$ and $n \geqslant 1$, discretely σ-decomposable in X and with each set

$$S(\sigma|m ; \tau|l)$$

in $\mathscr{S}$.

It will be convenient to write

$$T^*(\sigma) = \bigcap_{m=1}^{\infty} T(\sigma|m)$$

for σ in $\kappa^{\mathbb{N}}$, and

$$S^*(\sigma|m ; \tau) = \bigcap_{n=1}^{\infty} S(\sigma|m, \tau|n)$$

for $\sigma|m$ in κ^m and τ in $\kappa^{\mathbb{N}}$. Then the families

$$\{T^*(I(s)) : s \in \kappa^m\}, \qquad m \geqslant 1,$$

and

$$\{S^*(\sigma|m ; I(t)) : t \in \kappa^n\}, \qquad m \geqslant 1, \quad \sigma|m \in \kappa^m, \quad n \geqslant 1,$$

are all discretely σ-decomposable in X.

As before, we find that T has the representation

$$T = \bigcup_\rho \bigcap_l R(\rho|l),$$

with

$$R\Big(\chi(\sigma ; \tau(1), \tau(2), \ldots)|l\Big) = S\Big(\sigma|\phi(l) ; \tau(\phi(l))|\psi(l)\Big).$$

Writing

$$\boldsymbol{\xi}(\rho) = \xi(\rho_1), \xi(\rho_2), \dots,$$

$$\boldsymbol{\eta}^{(m)}(\rho) = \eta(\rho_{\nu(m,1)}), \eta(\rho_{\nu(m,2)}), \dots,$$

we obtain the explicit formula

$$R(\rho|l) = S(\boldsymbol{\xi}(\rho)|\phi(l)\,;\boldsymbol{\eta}^{(\phi(l))}(\rho)|\psi(l)).$$

Write

$$R^*(\rho) = \bigcap_{l=1}^{\infty} R(\rho|l)$$

for $\rho \in \kappa^{\mathbb{N}}$. Then, for r in κ^l, $l \geqslant 1$,

$$\begin{aligned}
R^*(I(r)) &= \bigcup\{R^*(\rho) : \rho|l = r\} \\
&= \bigcup\left\{\bigcap_{k=1}^{\infty} R(\rho|k) : \rho|l = r\right\} \\
&= \bigcup\left\{\bigcap_{k=1}^{\infty} S(\boldsymbol{\xi}(\rho)|\phi(k)\,;\boldsymbol{\eta}^{(\phi(k))}(\rho)|\psi(k)) : \rho|l = r\right\} \\
&= \bigcup\left\{\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} S(\boldsymbol{\xi}(\rho)|m\,;\boldsymbol{\eta}^{(m)}(\rho)|n) : \rho|l = r\right\} \\
&\subset \bigcup\left\{T^*(\boldsymbol{\xi}(\rho)) \cap \bigcap_{m=1}^{\infty} S^*(\boldsymbol{\xi}(\rho)|m\,;\boldsymbol{\eta}^{(m)}(\rho)) : \rho|l = r\right\} \\
&\subset T^*\Big(I(\boldsymbol{\xi}(\rho)|l)\Big) \cap \bigcap_{m=1}^{l} S^*\Big(\boldsymbol{\xi}(\rho)|m\,; I(\boldsymbol{\eta}^{(m)}(\rho)|t(m,l))\Big),
\end{aligned}$$

this last inclusion holding for all ρ with $\rho|l = r$. Writing

$$\boldsymbol{\xi}(r) = \xi(r_1), \xi(r_2), \dots, \xi(r_l),$$

$$\boldsymbol{\eta}^{(m)}(r) = \eta(r_{\nu(m,1)}), \eta(r_{\nu(m,2)}), \dots, \eta(r_{\nu(m,t(m,l))}),$$

for $m \geqslant 1$, we have

$$R^*(I(r)) \subset T^*\Big(I(\boldsymbol{\xi}(r))\Big) \cap \bigcap_{m=1}^{l} S^*\Big(\boldsymbol{\xi}(r)|m\,; I(\boldsymbol{\eta}^{(m)}(r))\Big),$$

with the convention that any factor on the right is omitted if $\boldsymbol{\eta}^{(m)}(r)$ reduces to a vector of zero length.

So the sets of the family

$$\{R^*(I(r)) : r \in \kappa^l\}$$

are each contained in the corresponding set of the family

$$\left\{T^*(I(\xi)) \cap \bigcap_{m=1}^{l} S^*(\xi|m\,;I(\eta^{(m)})) : \xi \in \kappa^l \text{ and } \eta^{(m)} \in \kappa^{t(m,l)} \text{ for } 1 \leqslant m \leqslant l\right\}.$$

It is now easy to verify that this family is discretely σ-decomposable. Hence T belongs to $\mathbf{ES}(\mathscr{S})$ as required.

Remark. The result also holds when the condition "is discretely σ-decomposable" is replaced by "is σ-discrete", both in the definition of the extended Souslin operation and also throughout the proof.

Proof of Theorem 4. This follows by the method used to prove Theorem 3, provided care is taken over the trivial difficulty discussed in [**15**, pp. 15–16].

Proof of Theorem 5. Consider a set

$$T = \bigcup_{\sigma} \bigcap_{m} T(\sigma|m)$$

with each family

$$\left\{\bigcup \left\{\bigcap_{n=1}^{\infty} T(\sigma|n) : \sigma|m = s\right\} : s \in \kappa^m\right\},$$

$m \geqslant 1$, discretely σ-decomposable in X, and with each set $T(\sigma|m)$ a κ-Lindelöf K-analytic set in X, with representation

$$T(\sigma|m) = K(\sigma|m\,;\kappa^{\mathbb{N}}),$$

for $\sigma|m$ in κ^m and $m \geqslant 1$. Write

$$S(\sigma|m\,;\tau|n) = K(\sigma|m\,;I(\tau|n))$$

for $\sigma|m$ in κ^m and $\tau|n$ in κ^n. Then we have

$$T(\sigma|m) = \bigcup_{\tau} \bigcap_{n} S(\sigma|m\,;\tau|n)$$

for $\sigma|m$ in κ^m, and each family

$$\left\{\bigcup \left\{\bigcap_{l=1}^{\infty} S(\sigma|m\,;\tau|l) : \tau|n = t\right\} : t \in \kappa^n\right\} = \left\{K(\sigma|m\,;I(t)) : t \in \kappa^n\right\}$$

σ-discrete in X for fixed $\sigma|m$ in κ^m and fixed $n \geqslant 1$. We are now essentially in the situation discussed in the proof of Theorem 3, with $\mathscr{S}$ taken to be the set $\mathscr{A}$ of K-analytic sets. We adopt all the notation introduced in the proof of Theorem 3. We find that

$$T = R^*(\kappa^{\mathbb{N}}),$$

with each family

$$\left\{R^*(I(r)) : r \in \kappa^l\right\},$$

$l \geqslant 1$, discretely σ-decomposable in X, and with

$$\begin{aligned} R^*(\rho) &= \bigcap_{l=1}^{\infty} R(\rho|l) \\ &= \bigcap_{l=1}^{\infty} S(\xi(\rho)|\phi(l)\,;\boldsymbol{\eta}^{(\phi(l))}(\rho)|\psi(l)) \\ &= \bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} S(\xi(\rho)|m\,;\boldsymbol{\eta}^{(m)}(\rho)|n) \\ &= \bigcap_{m=1}^{\infty} K(\xi(\rho)|m\,;\boldsymbol{\eta}^{(m)}(\rho)), \end{aligned}$$

for $\rho \in \kappa^{\mathbb{N}}$. Clearly R^* is compact-valued. It is easy to verify that R^* is upper semi-continuous. Hence T is K-analytic in X, using Theorem 1.

Proof of Theorem 6. The result follows by the method used to prove Theorem 5.

Proof of Theorem 7. Let the K-analytic set A have the representation

$$A = H(\kappa^{\omega}),$$

with H a compact-valued upper semi-continuous map of κ^{ω} into X with each family

$$\{H(I(t)) : t \in \kappa^n\},$$

$n \geqslant 1$, σ-discrete in X. Now the map $L : \kappa^{\omega} \to Y$, defined by

$$L(\tau) = K(H(\tau)), \qquad \tau \in \kappa^{\omega},$$

is easily verified to be a compact-valued upper semi-continuous map of κ^{ω} onto $K(A)$. Further, each family

$$\{L(I(t\,;t_n)) : t_n \in \kappa\} = \Big\{K\Big(H(I(t\,;t_n))\Big) : t_n \in \kappa\Big\},$$

$t \in \kappa^n$, $n \geqslant 0$, has a refinement that is σ-discrete in Y. By the first part of Theorem 1, whose proof does not depend on the use of Theorem 7, it follows that $K(A)$ is K-analytic in Y.

Proof of Theorem 8. The result follows by the methods used to prove Theorem 7. In this case the map takes families that are discretely σ-decomposable in X to families that are discretely σ-decomposable in Y and there is no need to use an analogue of Theorem 1.

Proof of Theorem 9. The result follows using the method of the standard proof of the corresponding result for Lindelöf K-analytic sets, see, for example, [**15**, pp. 28–29].

§4. *Topological properties and characterizations of K-analytic sets.* In this section we prove the results of the above nature described in the introduction, in the form of the following theorems.

THEOREM 10. *If A is a K-analytic set in a Hausdorff space X, then A is subparacompact in X.*

THEOREM 11. *If A is a K-analytic set in a Hausdorff space X, then A is a Souslin-$\mathscr{F}$ set in X. If A is a K-Lusin set in X, then A has a disjoint extended Souslin-$\mathscr{F}$ representation in X.*

THEOREM 12. *Let A be a set in a Hausdorff space X and suppose that A is K-analytic in itself. Then A is of the form $G \cap S$, with G a $\mathscr{G}_\delta$-set in X, with S a Souslin-$\mathscr{F}$ set in X, and with A a K-analytic set in G. Further, A is K-analytic in X, if, and only if, it is subparacompact in X. If B is a set that is K-Lusin in itself and is contained in X, then B is of the form $G \cap S$, with G a $\mathscr{G}_\delta$-set in X, with S a disjoint extended Souslin-$\mathscr{F}$ set in X, and with B a K-Lusin set in G. Further, B is K-Lusin in X, if, and only if, it is subparacompact in X.*

THEOREM 13. *If a set A is K-analytic in a Hausdorff space X, then A is the image in X of a paracompact Čech complete space Z under a continuous point-valued map that maps each discrete family of sets in Z to a family of sets in X that is discretely σ-decomposable. On the other hand, if A is the image, in a Hausdorff space X, of a set G that is subparacompact in itself and is a $\mathscr{G}_\delta$-set in a regular K-analytic space, under a compact-valued upper semi-continuous map, taking discrete families in G to families in X, having σ-discrete refinements in X, then A is K-analytic in X.*

THEOREM 14. *A set A in a completely regular space X is K-analytic in X, if, and only if, it is subparacompact in X and can be expressed as the intersection of a $\mathscr{G}_\delta$-set and a Souslin-$\mathscr{F}$ set in the Stone–Čech compactification of X. A set A in a completely regular space X is a K-Lusin set in X, if, and only if, it is subparacompact in X and can be expressed as the intersection $G \cap S$, with G a $\mathscr{G}_\delta$-set in βX, with S a Souslin-$\mathscr{F}$ set in βX, and with $S \cap G$ a disjoint extended Souslin-$\mathscr{F}$ set in G.*

THEOREM 15. *A set A that is K-analytic in a collectionwise normal space X is also K-analytic in a space Z that contains X and that is the cartesian product of a compact Hausdorff space with a complete metric space. The compact Hausdorff space can be taken to be homeomorphic to the Stone–Čech compactification of X and the complete metric space can be taken to be the completion of the quotient space of X with respect to the decomposition of X induced by a suitable continuous pseudometric on X depending on the set A. The same statements hold with K-analytic replaced by K-Lusin.*

COROLLARY. *A set A that is K-analytic in a collectionwise normal space X can be expressed as the intersection of a Souslin-$\mathscr{F}$ set, in a compact Hausdorff space Z containing X, with a paracompact $\mathscr{G}_\delta$-set also contained in Z.*

Note that, in general, the $\mathscr{G}_\delta$-set is paracompact in itself, in the usual way, rather than 'paracompact in Z'. Indeed, if the $\mathscr{G}_\delta$-set is even subparacompact in Z, then A has to be a Lindelöf K-analytic set.

THEOREM 16. *A set A in a collectionwise normal space X is K-analytic in X, if, and only if, it is a Souslin-$\mathscr{F}$ set in some space Z that contains X and that is the cartesian product of a compact Hausdorff space with a complete metric space. A set A in a collectionwise normal space X is K-Lusin in X, if, and only if, it is a disjoint extended Souslin-$\mathscr{F}$ set in some such space Z.*

THEOREM 17. *A set A in a Čech complete space X is K-analytic in X, if, and only if, it is subparacompact in X and Souslin-$\mathscr{F}$ in X.*

Turning our attention to spaces that are K-analytic in themselves we find that the theory takes an especially simple form.

THEOREM 18. *Let X be a K-analytic space. Then the following families of sets all coincide.*

(a) *The sets that are K-analytic in X.*

(b) *The sets that are extended Souslin-$\mathscr{F}$ sets in X.*

(c) *The sets that are Souslin-$\mathscr{F}$ sets in X.*

(d) *The sets in X that can be expressed as the projection on X of a closed set in $X \times \omega^\omega$.*

If X is also a regular Hausdorff space, then these families coincide with the family (e).

(e) *The sets in X that are K-analytic in themselves and that are subparacompact in X.*

THEOREM 19. *Let X be a K-Lusin space. Then the family of sets that are K-Lusin in X coincides with the disjoint extended Souslin-$\mathscr{F}$ sets in X.*

In a collectionwise normal space we obtain one extra result.

THEOREM 20. *A set A that is K-analytic in a collectionwise normal space X, is paracompact in X.*

Proof of Theorem 10. Let the set A in the Hausdorff space X have the representation $A = K(\kappa^\omega)$ with K a compact-valued upper semi-continuous map and with each family
$$\{K(I(t)) : t \in \kappa^n\},$$
$n \geqslant 1$, σ-discrete in X. Let $\{G_\theta : \theta \in \Theta\}$ be any cover of A by open sets, so that $\{A \cap G_\theta : \theta \in \Theta\}$ may be any cover of A by relatively open sets. Then, for each τ in κ^ω, the compact set $K(\tau)$ is covered by some finite sub-family, say $\{G_\phi : \phi \in \Phi(\tau)\}$ of the family $\{G_\theta : \theta \in \Theta\}$. By the upper semi-continuity of K, for each τ in κ^ω we can choose a Baire interval $I(t(\tau))$ with
$$K\Big(I(t(\tau))\Big) \subset \bigcup\{G_\phi : \phi \in \Phi(\tau)\}.$$
Now we can choose a subset T of κ^ω so that $\{I(t(\tau)) : \tau \in T\}$ is a disjoint cover of κ^ω.

Then the family

$$\Big\{K\Big(I(t(\tau))\Big) : \tau \in T\Big\}$$

is σ-discrete in X. Hence the family

$$\Big\{G_\phi \cap K\Big(I(t(\tau))\Big) : \phi \in \Phi(\tau) \text{ and } \tau \in T\Big\}$$

is also σ-discrete in X and is a refinement of $\{G_\theta \cap A : \theta \in \Theta\}$. Thus A is subparacompact in X.

Proof of Theorem 11. Theorem 1 enables us to express the K-analytic set A in the Hausdorff space X in the form $A = K(\omega^\omega \times \kappa^\omega)$, where K is a compact-valued upper semi-continuous map, and the family

$$\{K(I(s) \times I(t)) : t \in \kappa^n\}$$

is discrete in X, for each s in ω^n, and for each $n \geqslant 1$.

For each $n \geqslant 1$, each s in ω^n and each t in κ^n, write

$$F(s, t) = \operatorname{cl} K(I(s) \times I(t)),$$

and

$$F(s) = \bigcup\{F(s, t) : t \in \kappa^n\}.$$

Then $F(s)$ is closed, for each $n \geqslant 1$, and for each s in ω^n. By the standard method of proving that a Lindelöf K-analytic set is always a Souslin-$\mathscr{F}$ set (see, for example, [**15**, p. 25]) it follows that A has the Souslin-$\mathscr{F}$ representation

$$A = \bigcup\{\bigcap\{F(\sigma|n) : n \in \omega\} : \sigma \in \omega^\omega\}.$$

Now suppose that A is a K-Lusin set of the form $A = K(\kappa^\omega)$, with K a compact-valued upper semi-continuous map from κ^ω to X, taking only mutually disjoint values, with each family $\{K(I(t)) : t \in \kappa^n\}$, $n \geqslant 1$, discretely σ-decomposable in X. Write

$$F(t) = \operatorname{cl} K(I(t))$$

for $t \in \kappa^n$, and $n \geqslant 1$. By the same standard methods,

$$K(\tau) = \bigcap_{n=1}^{\infty} F(\tau|n)$$

for each τ in κ^ω, and A has the disjoint extended Souslin-$\mathscr{F}$ representation

$$A = \bigcup\{\bigcap\{F(\tau|n) : n \geqslant 1\} : \tau \in \kappa^\omega\}.$$

Before we prove Theorem 12, it will be convenient to prove two lemmas.

LEMMA 7. *If A is a set that is subparacompact in a Hausdorff space X, and $\{B_\gamma : \gamma \in \Gamma\}$ is a family of subsets of A that is discrete in A, then $\{B_\gamma : \gamma \in \Gamma\}$ is discretely σ-decomposable in X.*

Proof. For each a in A, we can choose an open set U_a that contains a and meets B_γ for at most one γ in Γ. Then $\{U_a : a \in A\}$ is an open cover of A. As A is subparacompact in X, the family $\{A \cap U_a : a \in A\}$ has a refinement, say $\{F_\theta : \theta \in \Theta\}$, that is σ-discrete in X. Write

$$\Theta = \bigcup_{n=1}^{\infty} \Theta(n)$$

with each family $\{F_\theta : \theta \in \Theta(n)\}$, $n \geqslant 1$, discrete in X. For each θ in Θ there is an $a(\theta)$ in A with $F_\theta \subset A \cap U_{a(\theta)}$ and so with $F_\theta \cap B_\gamma$ non-empty for at most one γ in Γ. Let Θ^* be the set of θ in Θ, for which $F_\theta \cap B_\gamma$ is non-empty for some γ, say $\gamma(\theta)$, in Γ. Then $\gamma(\theta)$ is uniquely defined on Θ^*. As $\{F_\theta : \theta \in \Theta\}$ refines $\{A \cap U_a : a \in A\}$, we have

$$\bigcup\{F_\theta : \theta \in \Theta\} = \bigcup\{A \cap U_a : a \in A\} = A .$$

Hence, for each γ in Γ, we have

$$B_\gamma = \bigcup\{B_\gamma \cap F_\theta : \theta \in \Theta^* \text{ and } \gamma(\theta) = \gamma\} .$$

Thus

$$B_\gamma = \bigcup_{n=1}^{\infty} B_\gamma(n) ,$$

with

$$B_\gamma(n) = \bigcup\{B_\gamma \cap F_\theta : \theta \in \Theta^* \cap \Theta(n) \text{ and } \gamma(\theta) = \gamma\} ,$$

for each γ in Γ. If $\gamma \neq \gamma'$, then the set of θ in $\Theta^* \cap \Theta(n)$ with $\gamma(\theta) = \gamma$ is disjoint from the set of θ in $\Theta^* \cap \Theta(n)$ with $\gamma(\theta) = \gamma'$. As $\{F_\theta : \theta \in \Theta(n)\}$ is discrete in X for each $n \geqslant 1$, it follows that $\{B_\gamma(n) : \gamma \in \Gamma\}$ is also discrete in X, for each $n \geqslant 1$. Thus $\{B_\gamma : \gamma \in \Gamma\}$ is discretely σ-decomposable.

LEMMA 8. *Let A be a set that is K-analytic in a set S that is subparacompact in a Hausdorff space X. Then A is K-analytic in X. Similarly, if A is K-Lusin in a set S that is subparacompact in a Hausdorff space X, then A is K-Lusin in X.*

Proof. In the K-analytic case the result follows immediately from Lemma 7 and Theorem 1. In the K-Lusin case, the result follows immediately from Lemma 7 and the definition of a K-Lusin set.

Proof of Theorem 12. As A is K-analytic in itself, we have $A = K(\kappa^\omega)$ with K a compact-valued upper semi-continuous map, with $\{K(I(t)) : t \in \kappa^n\}$ a σ-discrete family in A for each $n \geqslant 1$. For each $n \geqslant 1$, write

$$\kappa^n = \bigcup_{m=1}^{\infty} \Gamma(n, m)$$

with

$$\{K(I(t)) : t \in \Gamma(n, m)\}$$

discrete in A for $m \geqslant 1$. Now, for each n, m, and, for each point a of A, we can choose an open set $G(a, n, m)$ that contains a and meets at most one set of the family

$$\{K(I(t)) : t \in \Gamma(n, m)\}.$$

Hence, this family is discrete in the open set

$$G(n, m) = \bigcup\{G(a, n, m) : a \in A\}.$$

It is now easy to verify that A is contained in and is K-analytic in the $\mathscr{G}_\delta$-set

$$G = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} G(n, m).$$

Further, as A is K-analytic in G, it is a Souslin-$\mathscr{F}$ set in G, by Theorem 11, and so $A = G \cap S$ with G a $\mathscr{G}_\delta$-set in X and with S a Souslin-$\mathscr{F}$ set in X.

If A is, in fact, K-analytic in X, then it is subparacompact in X, by Theorem 10. Now suppose that A is both K-analytic in itself and also subparacompact in X. By Lemma 8, with $S = A$, it follows that A is K-analytic in X.

The K-Lusin results follow by the same methods.

Proof of Theorem 13. The fact that a K-analytic set in a Hausdorff space X has a representation in the required form has been established in Theorem 1. Now suppose that A is the image, in a Hausdorff space X, of a set G that is subparacompact in itself and is a $\mathscr{G}_\delta$-set in a regular K-analytic space Z, under a compact-valued upper semi-continuous map, taking discrete families in G to families in X having σ-discrete refinements in X. By Lemma 5, the set G is K-analytic in itself. It follows by Theorem 7 that A is K-analytic in X.

Proof of Theorem 14. First suppose that a set A is K-analytic in a completely regular space X. Then A is K-analytic in itself as a subset of the Stone–Čech compactification βX. By Theorem 12, the set A is K-analytic in some $\mathscr{G}_\delta$-set, G say, in βX. By Theorem 11, the set A is a Souslin-$\mathscr{F}$ set in G. Hence A is the intersection of the $\mathscr{G}_\delta$-set G with the Souslin-$\mathscr{F}$ set in βX, obtained by replacing the closed subsets of G, used in representing A as a Souslin-$\mathscr{F}$ set, by their closures in βX. By Theorem 10, the set A is subparacompact in X.

Now suppose that A is subparacompact in a completely regular space X, and that A is the intersection of a $\mathscr{G}_\delta$-set G with a Souslin-$\mathscr{F}$ set S in βX. The set S, being a Souslin-$\mathscr{K}$ set in βX, is a completely regular K-analytic space. Now A is subparacompact in itself and is a $\mathscr{G}_\delta$-subset of S. Hence, by Lemma 5, the set A is K-analytic in itself. But A is subparacompact in X. So, by Theorem 12, the set A is K-analytic in X.

The K-Lusin result follows by use of the same method.

Before proving Theorem 15 it is convenient to prove a lemma.

LEMMA 9. *Let* $\{F_\gamma : \gamma \in \Gamma(n)\}$, $n = 1, 2, \ldots$, *be a sequence of discrete families of closed sets in a collectionwise normal space* X. *Then it is possible to choose a continuous pseudometric* ρ *on* X *with the property that the projection map* p *from* X *to* *the metric quotient space X/R, with R the relation of equivalence of points under ρ, maps each family $\{F_\gamma : \gamma \in \Gamma(n)\}$, $n = 1, 2, \ldots$, to a family that is discrete in the completion of X/R under its metric.*

Proof. We first consider a single discrete family, say $\{F_\gamma : \gamma \in \Gamma\}$, of closed sets in X. As X is collectionwise normal, we can choose a discrete family $\{G_\gamma : \gamma \in \Gamma\}$ of open sets, with

$$F_\gamma \subset G_\gamma \qquad \text{for } \gamma \in \Gamma .$$

For each γ in Γ, we can choose a continuous map f_γ from X to $[0, 1]$ taking the value 1 on F_γ and the value 0 on $X \setminus G_\gamma$. As the family $\{G_\gamma : \gamma \in \Gamma\}$ is discrete, each point x has a neighbourhood on which f_γ takes non-zero values for at most one γ in Γ.

Consider the function ρ defined on $X \times X$ by taking

$$\rho(x, y) = \sum_{\gamma \in \Gamma} |f_\gamma(x) - f_\gamma(y)| .$$

As $f_\gamma(x)$ and $f_\gamma(y)$ can take non-zero values for at most one value of γ each, this sum contains at most two non-zero terms, so that

$$0 \leqslant \rho(x, y) \leqslant 2 .$$

Further, each point (x_0, y_0) in $X \times X$ has a neighbourhood in $X \times X$ on which

$$|f_\gamma(x) - f_\gamma(y)|$$

is non-zero for at most two values of γ in Γ. Hence ρ is a continuous map from $X \times X$ to $\mathbb{R}$. It is now easy to verify that ρ is a continuous pseudometric on X.

Changing the notation slightly, for each $n \geqslant 1$, let $\rho^{(n)}$ be the pseudometric constructed, in this way, from the discrete family $\{F_\gamma : \gamma \in \Gamma(n)\}$. Now take

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \rho^{(n)}(x, y) .$$

Clearly ρ is a continuous pseudometric on X. Let R be the corresponding equivalence relation identifying points x, y in X, when $\rho(x, y) = 0$, and let p be the projection map from X to X/R mapping each point of x to the equivalence class of all points at ρ-distance zero from x. We use the natural metric σ on X/R, defining

$$\sigma(\xi, \eta) = \rho(x, y)$$

for any choice of x and y as representatives of the equivalence classes ξ and η in X/R. For a detailed discussion of the construction of X/R see, for example [**14**, pp. 96–97].

It remains to prove that each family $\{p(F_\gamma) : \gamma \in \Gamma(n)\}$, $n = 1, 2, \ldots$, is discrete in the completion of X/R under its metric. Consider a fixed integer $n^* \geqslant 1$, and consider any point ξ^* of X/R. Suppose that the sphere in X/R, of all points ξ with

$$\sigma(\xi, \xi^*) < 2^{-n^*} ,$$

were to meet both

$$p(F_\gamma) \quad \text{and} \quad p(F_\delta)$$

with γ and δ distinct in $\Gamma(n^*)$. Then there would be points x_γ in F_γ and x_δ in F_δ with

$$\sigma(p(x_\gamma), \xi^*) < 2^{-n^*}, \quad \text{and} \quad \sigma(p(x_\delta), \xi^*) < 2^{-n^*}.$$

Hence we should have

$$\rho(x_\gamma, x_\delta) < 2 \, . \, 2^{-n^*},$$

and

$$\sum_{n=1}^{\infty} 2^{-n} \rho^{(n)}(x_\gamma, x_\delta) < 2 \, . \, 2^{-n^*},$$

so that

$$\rho^{(n^*)}(x_\gamma, x_\delta) < 2 \, .$$

As $x_\gamma \in F_\gamma$, $x_\delta \in F_\delta$ and γ and δ are distinct in $\Gamma(n^*)$, we should have

$$\begin{aligned} \rho^{(n^*)}(x_\gamma, x_\delta) &= \sum_{\varepsilon \in \Gamma(n^*)} |f_\varepsilon(x_\gamma) - f_\varepsilon(x_\delta)| \\ &= |f_\gamma(x_\gamma) - f_\gamma(x_\delta)| + |f_\delta(x_\gamma) - f_\delta(x_\delta)| \\ &= 2 \, , \end{aligned}$$

and this would be a contradiction. Thus the open ball in X/R with centre ξ^* and radius 2^{-n^*} meets at most one of the sets of the family $\{p(F_\gamma) : \gamma \in \Gamma(n^*)\}$. Thus this family, being discrete in this uniform way, remains discrete in the completion of X/R in its metric.

Proof of Theorem 15. Let A be a set that is analytic in a collectionwise normal space X. Let A have the representation $A = K(\kappa^\omega)$ with K a compact-valued upper semi-continuous map from κ^ω to X, each family

$$\{K(I(t)) : t \in \kappa^n\}, \qquad n \geqslant 1,$$

being σ-discrete in X. Write

$$\kappa^n = \bigcup_{m=1}^{\infty} \Gamma(n, m), \qquad n \geqslant 1,$$

with each family

$$\{K(I(t)) : t \in \Gamma(n, m)\}, \qquad n, m \geqslant 1,$$

discrete in X. Let ρ be the pseudometric provided by the application of Lemma 9 to the countable system of discrete families

$$\{\operatorname{cl} K(I(t)) : t \in \Gamma(n, m)\}, \qquad m, n \geqslant 1;$$

let p denote the corresponding projection map; and let M denote the completion of the quotient space of X under the identification relation induced by ρ.

Consider the set X^* of all points in $\beta X \times M$ of the form $(x, p(x))$ with $x \in X$, and the corresponding set A^* of the points $(x, p(x))$ with $x \in A$. As p is continuous, the map

$$h : x \mapsto (x, p(x))$$

maps X and A homeomorphically to X^* and A^* in $\beta X \times M$. Hence the map H, defined by

$$H(\tau) = h(K(\tau)),$$

is a compact-valued upper semi-continuous map from κ^ω onto A^* in $\beta X \times M$. The choice of M ensures that each family

$$\left\{p\Big(\operatorname{cl} K(I(t))\Big) : t \in \Gamma(n, m)\right\}, \qquad m, n \geqslant 1,$$

is discrete in M. Hence, given any $n^* \geqslant 1$, any $m^* \geqslant 1$ and any point (x^*, ξ^*) in $\beta X \times M$, there is a neighbourhood N of ξ^* in M that meets $p(\operatorname{cl} K(I(t)))$ for at most one t in $\Gamma(n^*, m^*)$. Now $\beta X \times N$ is a neighbourhood of (x^*, ξ^*) in $\beta X \times N$ that meets

$$H(I(t)) \subset \beta X \times p\Big(\operatorname{cl} K(I(t))\Big)$$

for at most one t in $\Gamma(n^*, m^*)$. Thus A^* is K-analytic in the space $\beta X \times M$. Hence A is K-analytic in a space Z containing X, with Z homeomorphic to $\beta X \times M$.

The results for K-Lusin sets follows in the same way, with a few obvious modifications.

Proof of the Corollary to Theorem 15. By Theorem 15, the set A becomes K-analytic in Z, when X is embedded in a space $Z = C \times M$, with C homeomorphic to βX and with M a complete metric space. Now, by Theorem 11, A is a Souslin-$\mathscr{F}$ set in Z. Further, M, being a complete metric space, is a $\mathscr{G}_\delta$-set in its Stone–Čech compactification βM. Hence $C \times M$ is a $\mathscr{G}_\delta$-set in $C \times \beta M$. Thus A is the intersection of some Souslin-$\mathscr{F}$ set in $C \times \beta M$ with the $\mathscr{G}_\delta$-set $C \times M$ in $C \times \beta M$. As M is a metric space, the product $C \times M$ is the product of a compact space with a paracompact space, and so is paracompact, see, for example, [**2**, p. 387]. Thus A has the required form.

Proof of Theorem 16. If A is K-analytic in the collectionwise normal space X, then, by Theorem 15, A is also K-analytic in a space Z that contains X and that is the product of a compact Hausdorff space with a complete metric space. By Theorem 11, the set A is a Souslin-$\mathscr{F}$ set in Z, as required.

Now suppose that A is a Souslin-$\mathscr{F}$ set in a space Z that contains X and that is the cartesian product of a compact Hausdorff space C with a complete metric space M. As we remarked in the introduction, Hansell shows, in particular, that a complete metric space is K-Lusin in itself, and so also K-analytic in itself. By Theorem 9, the product $Z = C \times M$ is K-Lusin and K-analytic in itself. Hence, by the remark after the statement of Theorem 5, the set A is K-analytic in Z and so also in X.

The K-Lusin result follows in the same way.

Proof of Theorem 17. By Theorems 10 and 11, if A is K-analytic in X, then A is subparacompact and Souslin-$\mathscr{F}$ in X.

Now suppose that A is subparacompact and Souslin-$\mathscr{F}$ in X. As X is Čech complete it is a $\mathscr{G}_\delta$-set in its Stone–Čech compactification βX. Thus A is of the form $X \cap S$, with X a $\mathscr{G}_\delta$-set in βX and with S a Souslin-$\mathscr{F}$ set in βX. By Theorem 14, it follows that A is K-analytic in X.

Proof of Theorem 18. If A is K-analytic in X, then A is a Souslin-$\mathscr{F}$ set in X by Theorem 11. If A is a Souslin-$\mathscr{F}$ set in X, then A is trivially an extended Souslin-$\mathscr{F}$ set in X. If A is an extended Souslin-$\mathscr{F}$ set in the K-analytic space X, then A is K-analytic in X by Theorem 5. Thus the families (a), (b) and (c) all coincide.

Now, if A in X has the Souslin-$\mathscr{F}$ representation

$$A = \bigcup\{\bigcap\{F(\sigma|n) : n \geqslant 1\} : \sigma \in \omega^\omega\}$$

with $F(s)$ a closed subset of X for each s in ω^n, $n \geqslant 1$, it is easy to verify that A is the projection on X of the closed set

$$\bigcap\{\bigcup\{F(s) \times I(s) : s \in \omega^n\} : n \geqslant 1\}$$

in $X \times \omega^\omega$. On the other hand, as ω^ω is a separable K-analytic space, if A is the projection on X of a closed set, or of a Souslin-$\mathscr{F}$ set, in $X \times \omega^\omega$, then A is a Souslin-$\mathscr{F}$ set in X, see, for example, [**15**, p. 30, Th. 2.6.6]. Thus the family (d) coincides with the family (c).

If A is a K-analytic set in X, it follows immediately that A is K-analytic in itself, and, by Theorem 10, that A is subparacompact in X. Finally suppose that X is regular and that A is K-analytic in itself and subparacompact in X. By Theorem 12, the set A is of the form $G \cap S$, with G a $\mathscr{G}_\delta$-set in X and with S a Souslin-$\mathscr{F}$ set in X. Now S is a regular Hausdorff space, and $A = G \cap S$, being subparacompact in X, is a subparacompact $\mathscr{G}_\delta$-set in S. By Corollary 2 to Lemma 5, the set A is an $\mathscr{F}_{\sigma\delta}$-set in S and so is a Souslin-$\mathscr{F}$ set in X. This completes the proof.

Proof of Theorem 19. The result follows immediately from Theorems 11 and 6.

Before we prove Theorem 20, we prove two lemmas.

LEMMA 10. *Let E be a set that is subparacompact in a regular Hausdorff space X. Then, given any open set U with $E \subset U$, it is possible to find an $\mathscr{F}_\sigma$-set F with $E \subset F \subset U$.*

LEMMA 11. *Let E be a set in a paracompact space X, with the property that, given any open set U with $E \subset U$, it is possible to find an $\mathscr{F}_\sigma$-set F with $E \subset F \subset U$. Then E is paracompact in X.*

Proof of Lemma 10. Let E be subparacompact in X and let U be any open set in X containing E. As X is regular, for each point e of E we can choose an open neighbourhood $V(e)$ of e, with $\operatorname{cl} V(e) \subset U$. Now the family $\{E \cap V(e) : e \in E\}$ is a cover of E by relatively open sets. As E is subparacompact in X, this family has a refinement, say $\{F(\theta) : \theta \in \Theta\}$, that is σ-discrete in X. Write $\Theta = \bigcup_{n=1}^{\infty} \Theta(n)$ with each family $\{F(\theta) : \theta \in \Theta(n)\}$, $n \geqslant 1$, discrete in X. Then each set

$$F_n = \bigcup\{\mathrm{cl}\, F(\theta) : \theta \in \Theta(n)\}, \qquad n \geqslant 1,$$

is closed in X and

$$F = \bigcup_{n=1}^{\infty} F_n = \bigcup\{\mathrm{cl}\, F(\theta) : \theta \in \Theta\}$$

is an $\mathscr{F}_\sigma$-set in X containing E. Now, for each θ in Θ, there is an $e(\theta)$ in E with $F(\theta) \subset V(e(\theta))$, so that

$$\mathrm{cl}\, F(\theta) \subset \mathrm{cl}\, V(e(\theta)) \subset U.$$

Hence $E \subset F \subset U$, as required.

Proof of Lemma 11. Let E be a set in the paracompact space X, and suppose that E has the given property of approximation from without by $\mathscr{F}_\sigma$-sets. Let $\{G(\gamma) : \gamma \in \Gamma\}$ be any open cover of E, so that $\{E \cap G(\gamma) : \gamma \in \Gamma\}$ can be any cover of E by relatively open sets. Choose an $\mathscr{F}_\sigma$-set

$$F = \bigcup_{n=1}^{\infty} F_n,$$

with F_n closed for each $n \geqslant 1$, with $E \subset F \subset \bigcup\{G(\gamma) : \gamma \in \Gamma\}$. Then

$$\{X \setminus F_n : n \geqslant 1\} \cup \{G(\gamma) : \gamma \in \Gamma\}$$

is an open cover of the paracompact space X. Hence we can choose an open, σ-discrete refinement, say $\{H(\theta) : \theta \in \Theta\}$, of this family. Let Θ' be the set of all θ with

$$H(\theta) \cap E \neq \varnothing.$$

Then $\{H(\theta) : \theta \in \Theta'\}$ is a family of open sets, σ-discrete in X, covering E, and

$$\{E \cap H(\theta) : \theta \in \Theta'\}$$

is a relatively open family that refines the family $\{E \cap G(\gamma) : \gamma \in \Gamma\}$, and that is σ-discrete in X. Thus E is paracompact in X.

Proof of Theorem 20. Let A be a set that is K-analytic in a collectionwise normal space X. By Theorem 16, the set A is K-analytic in a space Z that contains X and that is the product of a compact Hausdorff space with a complete metric space. As in the proof of Theorem 16, this space Z is paracompact. By Theorem 10, the set A is subparacompact in Z. By Lemmas 10 and 11, it follows that A is paracompact in Z and so also in X.

Note added in proof. Let A be a set that is subparacompact in a completely regular space X. The first part of Theorem 14 asserts that A will be K-analytic in X, if, and only if, A can be expressed as the intersection of a $\mathscr{G}_\delta$-set and a Souslin-$\mathscr{F}$ set in the Stone–Čech compactification of X. The proof of Theorem 14, in fact, shows that

A will be K-analytic in X, if, and only if, it is a Souslin-$\mathscr{F}$ set in some Čech complete space containing X. Similarly, A will be K-Lusin in X, if, and only if, it is a disjoint extended Souslin-$\mathscr{F}$ set in some Čech complete space containing X.

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