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XXVIII

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Non-separable Borel sets

WARSZAWA 1962
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

1. INTRODUCTION

The object of this paper is to extend part of the structure theory of Borel sets, from the "classical" separable metric case, to general metric spaces. Some results of this nature have long been known—notably the "absolute" properties of Borel subsets of complete spaces, and properties of "locally Borel" sets [20]. The reader will find an extended account of the standard theory in [14], to which we also refer for standard terminology(1). It has also long been known that certain theorems depend upon separability, and cannot be extended to non-separable spaces(2). We shall see, however, that a significant part of the theory, especially properties involving cardinality, can be extended; and we apply these results to a study of "Borel isomorphism"(3). It is a classical result that two *separable* absolute Borel sets are Borel isomorphic if (and only if) they have the same cardinal, which must be finite, $\aleph_0$, or $\mathfrak{c}$. In the general case, the results are much less simple. There are really two problems: that of *classifying* the Borel isomorphism types, and that of *characterizing* the distinct types by intrinsic topological invariants. We shall be able to make fair progress in the first of these problems, by giving a class of simple standard spaces such that every (metrisable) absolute Borel set is Borel isomorphic to at least one of them (Theorem 10, 5.4). The characterization problem is solved only in some special cases (Theorems 12 and 13, 6.2 and 6.4).

In more detail, the paper proceeds as follows. We begin (§ 2) by defining and discussing the "Baire spaces" $B(k)$ and $C(k)$; these are natural generalizations of the two special spaces—the set of irrational numbers and the Cantor set—which play a dominant role in the separable case, and these generalizations will be correspondingly important for

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(1) For further notation see 2.1 and 3.1 below.

(2) See 7.6 and 8.6 below for two examples.

(3) A Borel isomorphism is a 1-1 correspondence f such that both f and f^{-1} take Borel sets into Borel sets. The author is indebted to Prof. G. W. Mackey for calling his attention to the problem of Borel isomorphism for non-separable spaces. This problem is relevant to the theory of topological groups; see [15, 16].

spaces of (infinite) weight k [4]. It unexpectedly turns out that, for $k > \aleph_0$, $B(k)$ and $C(k)$ are homeomorphic. In § 3 we extend a classical result to the theorem, fundamental to what follows, that every absolute Borel set of weight k is Borel isomorphic to some closed subset of $B(k)$. In § 4 we first show (Theorem 5) that, roughly speaking, every absolute Borel set contains $B(k)$ as a closed subspace for some not too small value of k; from this we then deduce results about the cardinals of absolute Borel spaces of given weights, and how many of them there are. One significant consequence is that the weight is invariant under Borel isomorphism, at least for absolute Borel spaces (Theorem 8). Next, § 5 is concerned with the classification problem; we show (Theorem 10) that every absolute Borel space of weight k is Borel isomorphic to a discrete union of k or fewer spaces $B(p_\lambda)$ $(p_\lambda \leqslant k)$. The characterization problem is studied in § 6; we obtain topological characterizations of spaces which are Borel isomorphic to discrete spaces, or to $B(k)$ for certain values of k. In § 7 we consider the relation between general Borel isomorphism and the particular kind called "generalized homeomorphism". All the Borel isomorphisms constructed in this paper are in fact generalized homeomorphisms, and it is natural to conjecture that the two notions coincide. For separable spaces this is trivially so; we show that it is also true in some other cases, but the general problem remains open. As a by-product, we are able to improve one of the classical theorems about separable spaces (Theorem 16). Finally, in § 8, we sketch a corresponding generalization of the theory of analytic sets, showing that every absolute Borel set of weight k is "absolutely k-analytic", and that the cardinality properties of Borel sets extend to k-analytic sets. The theory of k-analytic sets turns out to be fully significant only if $k < k^{\aleph_0}$; there are, however, arbitrarily large cardinals satisfying this restriction [5].

The methods we use are, of course, to a large extent straightforward generalizations of methods well-known for separable spaces. But those methods make frequent use of the fact that separable spaces have countable bases; each such use must be replaced by a new argument, and these replacements do not always seem to be easy.

Throughout the paper, all spaces considered are assumed to be metrizable, except where the contrary is stated. When they are required to be absolutely Borel, we shall say so. We assume the axiom of choice, but not the continuum hypothesis.

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(4) The "weight" of a space is the least cardinal of a basis of open sets. In metric spaces it coincides with the least cardinal of a dense subset.

(5) For example, all sequential cardinals. On the generalized continuum hypothesis, there are no others.

2. BAIRE SPACES

2.1. Notation. A "Baire" space(6) is a product $T = \prod T_n$ ($n = 1, 2, \ldots$) of countably many discrete spaces T_n, each of arbitrary cardinal; we shall usually suppose, however, that each $\|T_n\| > 1$, where $\|X\|$ denotes the cardinal of X. The typical basic neighbourhood $\{s \mid s \in T,\ s_1 = t_1, \ldots, s_n = t_n\}$ of the point $t = (t_1, t_2, \ldots)$ of T, where $s = (s_1, s_2, \ldots)$, will be denoted by $V_n(t)$, or by $V(t_1, \ldots, t_n)$. We suppose T given the metric ϱ for which $\varrho(s, t) = 1/n$ if $s_1 = t_1, \ldots, s_{n-1} = t_{n-1}$, $s_n \neq t_n$; T is complete in this metric, and so is an absolute G_δ.

Given an infinite cardinal k, we shall be concerned with two special Baire spaces, which we denote by $B(k)$ and $C(k)$. $B(k)$ is defined to be $\prod T_n$ where each $\|T_n\| = k$. $C(k)$ is defined only when k is sequential (that is, the sum of $\aleph_0$ smaller cardinals). Choose any sequence of non-zero cardinals $p_n < k$ such that $\sup p_n = k$; we define $C(k) = \prod T_n$ where $\|T_n\| = p_n$. It is not hrad to see that, to within homeomorphism, $C(k)$ depends only on k, and not on the sequence $\{p_n\}$. However, we do not need to prove this here, since it turns out that, for $k > \aleph_0$, $C(k)$ is homeomorphic to $B(k)$ (see 2.4 (1) below). Thus the notation $C(k)$ is logically superfluous; but it is convenient to keep it in order to cover the case $k = \aleph_0$. Of course, $C(\aleph_0)$ is simply the Cantor set, and $B(\aleph_0)$ is the space of irrational numbers. We shall sometimes find the notation $B(1)$ useful to denote a 1-point space.

Remark. Every Baire space can be shown to be of the form $B(k) \times D$ or $C(\aleph_0) \times D$, where D is a discrete space and $k = 1$ or $k \geqslant \aleph_0$.

2.2. Elementary properties. In what follows, k denotes an infinite cardinal. We shall later need the obvious properties:

(1) *$B(k)^{\aleph_0}$ is homeomorphic to $B(k)$, and* $\|B(k)\| = k^{\aleph_0}$.

Clearly $B(k)$ has a discrete subset (7) of cardinal k, and a dense

(6) Here our terminology follows Morita [21]; the term is used in a slightly different sense by Hausdorff [6].

(7) See [27]. A subset A of a space X is said to be discrete (in X) if each point of X has a neighbourhood containing at most one point of A; this implies that A is closed. If A is discrete in A, we say that A is "relatively discrete".

subset of cardinal k; hence its weight is precisely k, and $B(k)$ contains no relatively discrete subset of cardinal greater than k (see [27]). Again,

(2) *$B(k)$ is 0-dimensional* (in the covering sense)([8]).

In fact, every product of countably many metrizable 0-dimensional spaces is 0-dimensional. A simple direct proof of (2) is as follows. Given any (open) covering $\mathscr{U}$ of $B(k)$, take a refinement $\mathscr{V}$ of $\mathscr{U}$ consisting of sets of the form $V(t_1, \ldots, t_n)$. From $\mathscr{V}$, omit all sets which are proper subsets of other sets of $\mathscr{V}$. The remaining sets are disjoint. Further, they still cover $B(K)$ because each $x \in B(k)$ is in some $V(t_1, \ldots, t_n)$ for which n is as small as possible (depending on x), and this set cannot have been omitted from $\mathscr{V}$.

Finally, each $t \in B(k)$ has arbitrarily small neighbourhoods $V_n(t)$ which are homeomorphic to $B(k)$, and which therefore also have the above properties.

2.3. Characterization. We now show that the properties already mentioned suffice to characterize $B(k)$ topologically.

THEOREM 1. *If X is a metrizable, 0-dimensional absolute G_δ with a dense set of cardinal k, and if every non-empty open subset of X contains a discrete subset of cardinal k, then X is homeomorphic to $B(k)$.*

The case $k = \aleph_0$ is due to Hausdorff [7].

We may suppose that X has a complete metric.

LEMMA. *If G is a non-empty open-closed subset of X, and $\varepsilon > 0$ is given, then G can be expressed as the union of a family of k disjoint non-empty open-closed sets, each of diameter $< \varepsilon$.*

By hypothesis, G contains a (closed) discrete set A with $\|A\| = k$. Each $x \in G$ has an open neighbourhood $U(x) \subset G$, of diameter $< \varepsilon$, such that $\|U(x) \cap A\| \leqslant 1$. The family $\{U(x)\}$ $(x \in G)$ is a covering of G, and has a refinement by disjoint open-closed sets; let the distinct sets occurring here be $\{U_\lambda \mid \lambda \in L\}$. As $\|U_\lambda \cap A\| \leqslant 1$, $\|L\| \geqslant k$; but X has a dense set of k points, so that $\|L\| \leqslant k$; and $\{U_\lambda\}$ is the family required.

Proof of Theorem 1. Applying the Lemma to $G = X$, $\varepsilon = 1$, we write $X = \bigcup \{U(t_1) \mid t_1 \in T_1\}$, where the sets $U(t_1)$ are disjoint, non-empty, open-closed and of diameters < 1, and where $\|T_1\| = k$. Applying the Lemma to $U(t_1)$, with $\varepsilon = 1/2$, we obtain $U(t_1) = \bigcup \{U(t_1, t_2) \mid t_2 \in T_2\}$, where the sets $U(t_1, t_2)$ are again disjoint (for fixed t_1, and hence also without this restriction), non-empty, open-closed and of diameters $< \frac{1}{2}$, and where $\|T_2\| = k$. In this way, for each positive integer n, we express

([8]) In fact, the 0-dimensional metric spaces coincide with the subsets of the spaces $B(k)$; see [11, 21]. Throughout this paper, we use "dimension" in the convering sense.

$U(t_1, \ldots, t_n)$ as the union of k disjoint open-closed sets $U(t_1, \ldots, t_n, t_{n+1})$ of diameters $< 1/(n+1)$, where each $t_i \in T_i$ and $\|T_i\| = k$. Let $B = \prod T_n$ $(n = 1, 2, \ldots)$, each T_n being given the discrete topology; B is homeomorphic to $B(k)$. For each $t = (t_1, t_2, \ldots)$ of B, the closed sets $U(t_1)$, $U(t_1, t_2)$, $\ldots$, form a decreasing sequence, with diameters tending to 0; they intersect in a single point of X, which we denote by $f(t)$. The mapping f is easily seen to be a homeomorphism of B onto X.

2.4. Corollaries.

(1) *If $C(k)$ is defined, and $k > \aleph_0$, $C(k)$ is homeomorphic to $B(k)$.*

Given $C(k) = \prod T_n$, where $\|T_n\| = p_n$, $1 \leqslant p_n < k$ and $\sup p_n = k$, we first verify that each neighbourhood $V(t_1, \ldots, t_{r-1})$ in $C(k)$ contains a (closed) discrete set of k points. For some $m \geqslant r$ we have $p_m \geqslant \aleph_0$, and by renumbering the factors T_n we may assume that $p_r \geqslant \aleph_0$. Choose $\aleph_0$ distinct elements of T_r, which we denote by $1, 2, \ldots$, and pick one element, say 0_i, in each T_i. For each n $(= 1, 2, \ldots)$ put $A_n = \{s \mid s_1 = t_1, \ldots, s_{r-1} = t_{r-1},\ s_r = n,\ s_{r+i} = 0_{r+i}$ if $i \neq n$ $(i = 1, 2, \ldots)\}$, and let $A = \bigcup A_n$. Clearly $\|A_n\| = p_{r+n}$, whence $\|A\| = k$; and clearly $A \subset V(t_1, \ldots, t_{r-1})$. Each $u \in C(k)$ has a neighbourhood $V_r(u)$ meeting at most one A_n; say $V_r(u) \cap A_n = \emptyset$ unless $n = n(u)$. The neighbourhood $V_{r+n(u)}(u)$ then meets A in at most one point. Thus A is discrete in $C(k)$.

Clearly $C(k)$ is homeomorphic to a subset of $B(k)$, and is therefore ([11, 21]) 0-dimensional. Thus Theorem 1 applies to $C(k)$.

(2) *Every non-empty open subset of $B(k)$ is homeomorphic to $B(k)$.*

(3) *Every dense G_δ subset of $B(k)$ is homeomorphic to $B(k)$.*

It is enough to show that, if X is dense in $B(k)$, then each intersection $V_n(t) \cap X$ (where $t \in B(k)$) has a discrete subset of k points. Put $S = \{s \mid s \in B(k),\ s_i = t_i \text{ if } i \neq n+1\}$. Clearly $\|S\| = k$, and $\varrho(s, s') = 1/(n+1)$ if s, s' are distinct points of S. For each $s \in S$, pick $x(s) \in X$ so close to s that (i) $x(s) \in V_n(t)$, (ii) $\varrho(s, x(s)) < 1/3(n+1)$. Then if $s \neq s'$, $\varrho(x(s), x(s')) > 1/3(n+1)$, and the set $\{x(s) \mid s \in S\}$ is the required discrete set of k points.

2.5. Further properties. We shall later need the following two properties of $B(k)$.

THEOREM 2. *Every G_δ subset of $B(k)$ is homeomorphic to a closed subset of $B(k)$.*

Say $H = \bigcap G_n$ $(n = 1, 2, \ldots)$, where G_n is open in $B(k)$. For each n we take a copy $B_n(k)$ of $B(k)$ and a homeomorphism f_n of $B_n(k)$ onto G_n; this is possible, from 2.4 (2), since we may evidently assume $H \neq \emptyset$.

Let A denote the subset of $\prod B_n(k)$ consisting of those points $a = (a_1, a_2, \ldots)$ for which $f_1(a_1) = f_2(a_2) = \ldots$ Then A is closed in $\prod B_n(k)$, which is homeomorphic to $B(k)$ by 2.2 (1). Define $f(a) = f_1(a_1)$ $(a \in A)$; thus $f(a) = f_n(a_n) \in G_n$ for all n, so that $f(a) \in H$. It is easily verified that f is a homeomorphism of A onto H, proving the theorem.

Remark. Conversely, every subset of $B(k)$ which is homeomorphic to a closed subset of $B(k)$—or of any other complete metric space—must be a G_δ in $B(k)$; further, such sets are characterized as being metrizable 0-dimensional absolute G_δ's of weight $\leqslant k$.

THEOREM 3. *Every closed non-empty subset A of $B(k)$ is a retract of $B(k)$, under a retraction which maps $B(k)-A$ onto a σ-discrete subset of A* (9).

We deduce this from a more general theorem, which for separable spaces is essentially due to Sierpiński [23]:

THEOREM 3'. *A necessary and sufficient condition that every non-empty closed subset A of a (metric) space X be a retract of X is that X be 0-dimensional. If the condition is satisfied, the retraction can be chosen so that the image of $X-A$ is σ-discrete.*

To prove sufficience, let F be a given non-empty closed subset of X, and let U be any open set containing F. Put $A = F \cup (X-U)$, and let f retract X onto the closed set A. Since F is open-closed in A, $f^{-1}(F)$ is open-closed in X; thus, as $F \subset f^{-1}(F) \subset U$, F has arbitrarily small open-closed neighbourhoods in X, and X is 0-dimensional [11, 21].

Conversely, if X is 0-dimensional and A is a non-empty closed subset, we easily construct a sequence $U_1, U_2, \ldots$ of open-closed neighbourhoods of A such that $U_1 \supset U_2 \supset \ldots$ and each point of U_n is distant less than $1/n$ from A. Put $W_n = U_n - U_{n+1}$ $(n = 1, 2, \ldots)$; W_n is open-closed, and is the union of a family $\{W_{n\lambda} \mid \lambda \in \Lambda_n\}$ of disjoint open-closed sets, each of diameter $< 1/n$. Let D_n $(n = 1, 2, \ldots)$ be a maximal $1/n$-discrete subset of A (10). Choose points $a \in D_1, p_{n\lambda} \in W_{n\lambda}$, arbitrarily; because $p_{n\lambda} \in U_n$, there exists $a_{n\lambda} \in A$ such that $\varrho(p_{n\lambda}, a_{n\lambda}) < 1/n$; and, from the maximality of D_n, there exists $d_{n\lambda} \in D_n$ such that $\varrho(a_{n\lambda}, d_{n\lambda}) < 1/n$. Define $f\colon x \to A$ by: $f(x) = x$ if $x \in A$, $f(x) = a$ if $x \notin U_1$, and $f(x) = d_{n\lambda}$ if $x \in W_{n\lambda}$. It is easy to check that f is continuous; hence f retracts X onto A. Further, $f(X-A) \subset \bigcup D_n$, and is therefore the union of countably many discrete subsets of A.

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(9) A subset A of a space X is "σ-discrete" if it is the union of countably many discrete (closed) subsets.

(10) A subset A of a space X, with metric ϱ, is "ε-discrete" if, whenever x, y are distinct points of A, $\varrho(x, y) \geqslant \varepsilon$.

3. THE BASIC THEOREM

3.1. Notation and preliminaries. We recall that all the spaces considered in this paper are to be metrisable. If X is a space with metric ϱ, and $\varepsilon > 0$, we write $S(x, \varepsilon)$ for the neighbourhood $\{y \mid \varrho(x, y) < \varepsilon\}$ of x.

We refer to [14] for the definitions and fundamental properties of Borel sets, but here we briefly summarise some which will be especially useful. The open (closed) subsets of X are said to be of additive (multiplicative) class 0 in X. If α is a countable ordinal > 0, a subset of X is said to be of additive (multiplicative) class α in X if it is the union (intersection) of a sequence of subsets of X of classes $< \alpha$ in X; note that it is also of (both additive and multiplicative) class β, for all $\beta > \alpha$. The family of all sets of classes $< \omega_1$ in X is the family of *Borel subsets* of X. We have:

(1) If E is of additive class $\alpha \geqslant 1$ in X, E is the union of a sequence of *disjoint* sets of classes $< \alpha$ in X.

(2) If $H \subseteq Y \subset X$, H is of additive (multiplicative) class α in Y if and only if it is of the form $Y \cap K$ where K is of the corresponding class in X.

(3) If $H \subset Y \subset X$, where H is of additive class α in Y, and Y is of additive class β in X, then H is of additive class $\max(\alpha, \beta)$ in X; and a similar statement applies to multiplicative classes.

(4) If $E \subset X$, and E is homeomorphic to a set of additive class > 1, or of multiplicative class $\geqslant 1$, in some complete metric space, then E is of the same class in X.

Sets E of the kind mentioned in (4) are called *absolute Borel sets*. Evidently (from (3)) every Borel subset of an absolute Borel set is itself an absolute Borel set.

A mapping f (not necessarily continuous) of a space X in a space Y is said to be of *class* α if, for every closed $F \subset Y$, $f^{-1}(F)$ is of multiplicative class α in X, or equivalently if, for every open $G \subset Y$, $f^{-1}(G)$ is of additive class α in X. We shall often use the fact ([14], p. 283):

(5) If f is of class α, and $E \subset Y$ is of additive (multiplicative) class β in Y, then $f^{-1}(E)$ is of additive (multiplicative) class $\alpha + \beta$ in X.

A 1-1 mapping f of X onto Y which is of class α, and for which f^{-1} is of class β, is called a *generalized homeomorphism of class* (α, β) of X onto Y. From (5), we have

(6) If f is a generalized homeomorphism of class (α, β) of X onto Y, and g is a generalized homeomorphism of class (γ, δ) of Y onto Z, then gf is a generalized homeomorphism of class $(\alpha+\gamma, \delta+\beta)$ of X onto Z.

An ordinary homeomorphism is simply a generalized homeomorphism of class $(0, 0)$. From (5), every generalized homeomorphism is a Borel isomorphism in the sense of § 1; we shall discuss the converse question later (§ 7).

3.2. THEOREM 4. *If Y is an absolute Borel set of class $\alpha \geqslant 1$, and of weight $\leqslant k$, there exists a generalized homeomorphism f, of class $(0, \omega_0^\alpha)$, of some closed subset A of $B(k)$ onto Y.*

If $k = \aleph_0$, this theorem is well known ([14], p. 355), and one can then take f of class $(0, \alpha)$. The proof in the general case follows the same plan as in the separable case; but the details become more complicated in the first and third steps, especially the latter (3.5 below). The new argument in 3.5 leads only to ω_0^α, instead of α, in the class of f; I do not know whether the bound ω_0^α can be reduced to α in general (it can be slightly improved). It is of course assumed throughout that k is infinite.

3.3. LEMMA. *If X is a complete metric space of weight $\leqslant k$, there exists a generalized homeomorphism f, of class $(0, 1)$, of a G_δ subset A of $B(k)$ onto X.*

We first note that, in any (metric) space of weight $\leqslant k$,

(1) every point-countable system $\{U_\lambda\}$ of non-empty open sets has at most k sets.

For there is a dense set D of $\leqslant k$ points; each U_λ contains at least one point of D, and each point of D belongs to at most $\aleph_0$ sets U_λ.

Now, if X is the space in the Lemma, for each $n = 1, 2, \ldots$ we use the paracompactness of X to cover X by a locally finite system of non-empty open sets $U_n(t_n)$ $(t_n \in T_n)$ of diameters $< 1/n$. By (1), $\|T_n\| \leqslant k$; by repeating sets if necessary we ensure that $\|T_n\| = k$. Let C be the subset of $\prod T_n = B(k)$ consisting of those points $t = \{t_n\}$ for which $\bigcap \overline{U}_n(t_n) \neq \emptyset$ (the bar denoting closure). If $t \notin C$, then $\bigcap \overline{U}_n(t_n) = \emptyset$, and because X is complete this implies that $\overline{U}_1(t_1) \cap \ldots \cap \overline{U}_N(t_N) = \emptyset$ for some (finite) N, so that the neighbourhood $V_N(t)$ is disjoint from C. Thus C is closed in $B(k)$. If $t \in C$, the set $\bigcap \overline{U}_n(t_n)$ is a single point, which we call $f(t)$; it is easily seen that f is a (uniformly) continuous mapping of C onto X.

Now well-order each T_n, and map X in C by the rule: $g(x) = \{t_n\}$ where, for each n, t_n is the *first* element of T_n for which $x \in \overline{U}_n(t_n)$. Clearly $f(g(x)) = x$, for all $x \in X$. Write $A = g(X) \subset C$. We shall show that A is a G_δ in C (and so also in $B(k)$).

For each positive integer m, let

$$A_m = \Big\{t \mid t \in B(k) \text{ and, for some } N, \\ \bigcap\{\overline{U}_n(t_n) \mid n \leqslant N\} \cap \bigcup\{\overline{U}_m(s_m) \mid s_m < t_m\} = \emptyset\Big\}.$$

Then

(2) A_m is open (in $B(k)$).

For we may clearly suppose that $N \geqslant m$ in the definition of A_m, and then for each $t \in A_m$ we have $V_N(t) \subset A_m$.

Next we assert:

(3) $A = C \cap \bigcap A_m \qquad (m = 1, 2, \ldots)$.

For if $t \notin \bigcap A_m$, then for some m the set

$$E_N = \bigcap\{\overline{U}_n(t_n) \mid n \leqslant N\} \cap \bigcup\{\overline{U}_m(s_m) \mid s_m < t_m\}$$

is non-empty for every N. But the system $\{U_m(s_m)\}$ is locally finite; hence E_N is closed. Also (m and t remaining fixed) $E_1 \supset E_2 \supset \ldots$ and E_N has diameter $< 1/N$. Thus there exists a point $x \in \bigcap E_N$; and $x \in \overline{U}_m(s_m)$ for some $s_m < t_m$, so that $g(x) \neq t$. But if also $t \in A$, then $t = g(x')$ for some $x' \in X$; each of the sets $\overline{U}_n(t_n)$ $(n = 1, 2, \ldots)$ contains x' as well as x, so that $x = x'$ and $g(x) = t$. This proves that $A \subset C \cap \bigcap A_m$. Conversely, if $t \in C \cap \bigcap A_m$, let $f(t) = x$ (defined because $t \in C$); then, because $t \in A_m$, we cannot have $x \in \overline{U}_m(s_m)$ if $s_m < t_m$. But $x \in \overline{U}_m(t_m)$, from the definition of f; hence $g(x) = t$, proving $t \in A$.

From (2) and (3), A is a G_δ in $B(k)$. The mapping $f \mid A$ (*i.* e., f restricted to A) is easily verified to be 1-1 and onto X, with inverse g. Of course, $f \mid A$ is continuous; the Lemma will thus be proved when we have shown that, for each open $G \subset B(k)$, $f(A \cap G)$ is F_σ in X.

We express G as a union of neighbourhoods, each of the form $V_{N(t)}(t)$ $(t \in G)$, and for each $N = 1, 2, \ldots$ we write $G_N = \bigcup\{V_{N(t)}(t) \mid t \in G, N(t) = N\}$, so that $G = G_1 \cup G_2 \cup \ldots$ It will suffice to prove that each $f(A \cap G_N)$ is F_σ.

Now $f(A \cap V_N(t)) = g^{-1}(V_N(t)) = H(t_1, \ldots, t_N) - K(t_1, \ldots, t_N)$ where $H(t_1, \ldots, t_N) = \overline{U}_1(t_1) \cap \ldots \cap \overline{U}_N(t_N)$ and $K(t_1, \ldots, t_N) = \bigcup\{\overline{U}_n(s_n) \mid s_n < t_n, n = 1, 2, \ldots, N\}$. Each of $H(t_1, \ldots, t_N)$ and $K(t_1, \ldots, t_N)$ is closed; their difference is therefore F_σ in X. That is, we may write $f(A \cap V_N(t)) = \bigcup\{F(t_1, \ldots, t_N; m) \mid m = 1, 2, \ldots\}$, where each $F(t_1, \ldots, t_N; m)$ is closed.

Keeping N fixed, let t vary in G, subject to $N(t) = N$. The distinct sets $V_N(t)$ correspond to distinct sequences $(t_1, \ldots, t_N)$. Now the sets $\bar{U}_1(t_1) \cap \ldots \cap \bar{U}_N(t_N)$ form a locally finite system; hence the sets $F(t_1, \ldots, t_N; m)$ form, for fixed m, a locally finite system likewise, and their union, say F_{Nm}, is closed. We have $f(A \cap G_N) = \bigcup \{F_{Nm} \mid m = 1, 2, \ldots\}$, an F_σ as required.

COROLLARY. *In the Lemma, we may take A to be a closed subset of $B(k)$.*

From Theorem 2 (2.5).

3.4. In proving Theorem 4, it may be assumed that Y is a Borel subset of a *fixed* complete metric space X of weight $\leqslant k$; for every absolute Borel set of weight $\leqslant k$ arises in this way from a suitable X (e.g., its completion). The preceding Corollary shows that the theorem is true when Y is of multiplicative class 1 (for every absolute G_δ has a complete metric). By 3.1 (1), every F_σ in X is the union of a sequence of disjoint G_δ sets; thus the theorem follows for Y of additive class 1, by the same reasoning as in the lemma which follows. We may therefore assume Theorem 4 true for all Borel subsets of X of classes $< a$, and have only to deduce it for those of class a, where $1 < a < \omega_1$. For additive class a, this is easy:

LEMMA. *Theorem 4 follows when Y is of additive class a.*

By 3.1 (1) we can write $Y = \bigcup F_n$ $(n = 1, 2, \ldots)$, where F_n is of multiplicative class $a_n < a$, and where the sets F_n are disjoint. We may assume without loss that $a_n \geqslant 1$. From the induction hypothesis, there exist closed subsets A_n of $B(k)$ and generalized homeomorphisms f_n of A_n onto F_n, f_n being of class $(0, \omega_0^{a_n})$. Let I denote the discrete space formed by the positive integers, and write $A = \bigcup (A_n \times n)$ $(n = 1, 2, \ldots)$, a subset of $B(k) \times I$. Define $f: A \to Y$ by $f(a_n, n) = f_n(a_n)$ $(a_n \in A_n)$. Clearly A is closed in $B(k) \times I$, which is homeomorphic to $B(k)$, and f is a continuous 1-1 mapping of A onto Y. Further, if G is open in A, we have $f(G) = \bigcup f_n(G_n)$ where G_n is open in A_n. Now $f_n(G_n)$ is a Borel set of class $\omega_0^{a_n}$ in F_n, and therefore (3.1 (3)) of (additive) class $\omega_0^{a_n} < \omega_0^{a}$ in X also. Hence $f(G)$ is Borel of additive class ω_0^{a} in X, and the lemma is proved.

3.5. The proof of Theorem 4 is concluded by establishing the corresponding deduction for multiplicative class a; this is decidedly less easy.

LEMMA. *Theorem 4 follows when Y is of multiplicative class a.*

We can write $Y = \bigcap F_n$ $(n = 1, 2, \ldots)$ where F_n is of additive class $a_n < a$ in X, and where $F_1 \supset F_2 \supset \ldots$ From the induction hypothesis, there exist, for each n, a closed subset A_n of a copy B_n of $B(k)$, and a generalized homeomorphism f_n of A_n onto F_n, of class $(0, \omega_0^{a_n})$. As in ([14], p. 354), let A denote the subset of $\prod B_n$ $(n = 1, 2, \ldots)$ consisting of all points $a = \{a_n\}$ for which $a_n \in A_n$ and $f_n(a_n) = f_1(a_1)$, for all n. Because each f_n is continuous, A is a closed subset of $\prod B_n$, which in turn is homeomorphic to $B(k)$. Define $f\colon A \to X$ by $f(a) = f_1(a_1)$ $(= f_n(a_n))$; one easily sees that f is a continuous 1-1 mapping of A onto $\bigcap F_n = Y$. (This argument has already been used in proving Theorem 2, 2.5). We have only to prove that, given any open set G in $\prod A_n$, $f(A \cap G)$ is Borel of additive class ω_0^a in Y.

Write $\mathscr{V}_{nr}$ for the family of all neighbourhoods $A_n \cap V_r(a_n)$ in A_n; here $a_n \in A_n$ and $V_r(a_n)$ is the neighbourhood in $B_n = B(k)$ which fixes the first r co-ordinates (as in 2.1). Then $\prod A_n$ has a basis of open sets of the form $U_1 \times U_2 \times \ldots \times U_r \times A_{r+1} \times \ldots$, where $U_i \in \mathscr{V}_{ir}$ $(1 \leqslant i \leqslant r$, $r = 2, 3, \ldots)$. Note that, when i and r are fixed, the distinct sets U_i here are disjoint, and in fact form a discrete collection of open-closed sets in A_i. For each r, let G_r be the union of the members of the above basis (of $\prod A_n$) which are contained in G and for which r has the given value. As $G = \bigcup G_r$, it will suffice to prove the assertion for G_r instead of G. Accordingly we keep r fixed throughout the rest of the argument.

Let π_r denote the projection of $\prod A_n$ onto $A_1 \times A_2 \times \ldots \times A_r$. We can express $\pi_r(G_r)$ in the form $\bigcup \{U_1(\lambda) \times U_2(\lambda) \times \ldots \times U_r(\lambda) \mid \lambda \in L\}$, where $U_i(\lambda) \in \mathscr{V}_{ir}$ $(1 \leqslant i \leqslant r)$ and λ runs over some index set L. Consider the *distinct* sets $U_1(\lambda)$ arising here; we select $L_1 \subset L$ so that each distinct set $U_1(\lambda)$ $(\lambda \in L)$ arises exactly once as $U_1(\lambda_1)$ for some $\lambda_1 \in L_1$. The sets $U_1(\lambda_1)$ $(\lambda_1 \in L_1)$ then form a discrete collection in A_1, and we have

(1) $\pi_r(G_r) = \bigcup \{U_1(\lambda_1) \times P_2(\lambda_1) \mid \lambda_1 \in L_1\}$, where

(2) $P_2(\lambda_1) = \bigcup \{U_2(\lambda) \times \ldots \times U_r(\lambda) \mid \lambda \in L$ such that $U_1(\lambda) = U_1(\lambda_1)\}$.

Again, the distinct sets $U_2(\lambda)$ arising in (2) can be written as a discrete collection $\{U_2(\lambda_2) \mid \lambda_2 \in L_2(\lambda_1)\}$, and then $P_2(\lambda_1) = \bigcup \{U_2(\lambda_2) \times P_3(\lambda_1, \lambda_2) \mid \lambda_2 \in L_2(\lambda_1)\}$. Repeating this process, we obtain for each $i < r$,

(3) $P_i(\lambda_1, \ldots, \lambda_{i-1}) = \bigcup \{U_i(\lambda_i) \times P_{i+1}(\lambda_1, \ldots, \lambda_i) \mid \lambda_i \in L_i(\lambda_1, \ldots, \lambda_{i-1})\}$,

where the sets $U_i(\lambda_i)$ $(\lambda_i \in L_i(\lambda_1, \ldots, \lambda_{i-1})$ form (for fixed i) a discrete collection in A_i. Finally the construction terminates with

(3′) $P_r(\lambda_1, \ldots, \lambda_{r-1}) = \bigcup \{U_r(\lambda_r) \mid \lambda_r \in L_r(\lambda_1, \ldots, \lambda_{r-1})\}$.

For uniformity of notation, we put $P_1 = \pi_r(G_r)$.

Next we define sets $Q_i(\lambda_1, \ldots, \lambda_{i-1}) \subset A_i$ for $i = r, r-1, \ldots, 1$, by "backwards induction" as follows:

(4_r) $Q_r(\lambda_1, \ldots, \lambda_{r-1}) = P_r(\lambda_1, \ldots, \lambda_{r-1})$,

(4_i) $Q_i(\lambda_1, \ldots, \lambda_{i-1})$
$$= \bigcup \Big\{U_i(\lambda_i) \cap f_i^{-1}\Big(f_{i+1}\big(Q_{i+1}(\lambda_1, \ldots, \lambda_i)\big)\Big) \;\Big|\; \lambda_i \in L_i(\lambda_1, \ldots, \lambda_{i-1})\Big\},$$

for $i = r-1, r-2, \ldots$, and finally

(4_1) $Q_1 = \bigcup \Big\{U_1(\lambda_1) \cap f_1^{-1}\Big(f_2\big(Q_2(\lambda_1)\big)\Big) \;\Big|\; \lambda_1 \in L_1\Big\}$.

We shall show that

(5) $Q_i(\lambda_1, \ldots, \lambda_{i-1})$ is a Borel subset of A_i of additive clsas

$$\omega_0^{a_{i+1}} + \omega_0^{a_{i+2}} + \ldots + \omega_0^{a_r}.$$

In fact, for $i = r$ we have that $Q_r(\lambda_1, \ldots, \lambda_{r-1}) = P_r(\lambda_1, \ldots, \lambda_{r-1})$, open in A_r (from (3′)), and so of additive class 0 in A_r. Assuming (5) for $i+1$ instead of i, we have that $f_{i+1}\big(Q_{i+1}(\lambda_1, \ldots, \lambda_i)\big)$ is of additive class $\omega_0^{a_{i+1}} + \omega_0^{a_{i+2}} + \ldots + \omega_0^{a_r}$ in F_{i+1}, from 3.1 (5). Now F_{i+1} is of additive class $a_{i+1} \leqslant \omega_0^{a_{i+1}}$ in F_i, from 3.1 (2), so it follows from 3.1 (3) that $f_{i+1}\big(Q_{i+1}(\lambda_1, \ldots, \lambda_i)\big)$ is of additive class $\omega_0^{a_{i+1}} + \ldots + \omega_0^{a_r}$ in F_i. Hence $f_i^{-1}\Big(f_{i+1}\big(Q_{i+1}(\lambda_1, \ldots, \lambda_i)\big)\Big)$ is of this same additive class in A_i. The open sets $U_i(\lambda_i)$ $(\lambda_i \in L_i(\lambda_1, \ldots, \lambda_{i-1}))$ form a discrete collection, so (4_i) shows that $Q_i(\lambda_1, \ldots, \lambda_{i-1})$ is Borel in A_i of the required class. Thus (5) is established by induction; and in particular

(6) $f_1(Q_1)$ is a Borel subset of F_1 of additive class

$$\omega_0^{a_1} + \omega_0^{a_2} + \ldots + \omega_0^{a_r}.$$

We next show:

(7) $Y \cap f_1(Q_1) = f(A \cap G_r)$.

For suppose $a \in A \cap G_r$, say $a = \{a_n\}$ where $a_n \in B_n$; thus $f(a) = f_1(a_1) = f_2(a_2) = \ldots \in \bigcap F_n = Y$. Because $a \in G_r$, $(a_1, \ldots, a_r) \in \pi_r(G_r) = P_1$, and (1) shows that $a_1 \in U_2(\lambda_1^*)$ for some $\lambda_1^* \in L_1$, and that $(a_2, \ldots, a_r) \in P_2(\lambda_1^*)$. Repetition of this argument, using (3), shows successively that $a_2 \in U_2(\lambda_2^*)$ for some $\lambda_2^* \in L_2(\lambda_1^*)$ and so on: finally $a_r \in U_r(\lambda_r^*)$ for some $\lambda_r^* \in L_r(\lambda_1^*, \ldots, \lambda_{r-1})$ and (from (3′)) $a_r \in P_r(\lambda_1^*, \ldots, \lambda_{r-1}^*) = Q_r(\lambda_1^*, \ldots, \lambda_{r-1}^*)$. Now we apply "backwards induction". When it has been shown that $a_{i+1} \in Q_{i+1}(\lambda_1^*, \ldots, \lambda_i^*)$, we note that $f_i(a_i) = f_{i+1}(a_{i+1})$ so that $a_i \in f_i^{-1}\Big(f_{i+1}\big(Q_{i+1}(\lambda_1^*, \ldots, \lambda_i^*)\big)\Big)$; but $a_i \in U_i(\lambda_i^*)$ and $\lambda_i^* \in L_i(\lambda_1^*, \ldots, \lambda_{i-1}^*)$, so that (4_i) gives $a_i \in Q_i(\lambda_i^*, \ldots, \lambda_{i-1}^*)$. Finally we obtain $a_1 \in Q_1$, so that $f(a) = f_1(a_1) \in Y \cap f_1(Q_1)$. To complete the proof of (7), suppose a point y of $Y \cap f_1(Q_1)$ to be given. Then there exists $a_1 \in Q_1$ such that $f_1(a_1) = y$. Because $a_1 \in Q_1$, (4_1) shows that $a_1 \in U_1(\lambda_1^*)$ for some $\lambda_1^* \in L_1$, and also that $f_1(a_1) \in f_2\big(Q_2(\lambda_1^*)\big)$; that is, there exists $a_2 \in Q_2(\lambda_1^*)$ such that $f_1(a_1) = f_2(a_2)$. Repetition of this argument (using (4_i)) leads to $a_i \in U_i(\lambda_i^*)$, where $\lambda_i^* \in L_i(\lambda_i^*, \ldots, \lambda_{i-1}^*)$, such that $f_i(a_i) = y$ $(i = 1, 2, \ldots, r)$, and also $a_i \in Q_i(\lambda_1^*, \ldots, \lambda_{i-1}^*)$. Thus $a_r \in Q_r(\lambda_1^*, \ldots, \lambda_{r-1}^*) = P_r(\lambda_1^*, \ldots, \lambda_{r-1}^*)$. An easy "backwards induction" argument, using the sets P_i, now shows that $(a_1, a_2, \ldots, a_r) \in P_1 = \pi_r(G_r)$. Further, we have $y \in Y \subset F_n$ for each $n > r$, and may therefore take $a_n \in A_n$ such that $f_n(a_n) = y$ $(n = r+1, \ldots)$. The point $a = (a_1, \ldots, a_r, a_{r+1}, \ldots)$ is then in $A \cap G_r$ and satisfies $f(a) = y$.

From (6) and (7) we see that $f(A \cap G_r)$ is a Borel subset of Y, of additive class $\leqslant \omega_0^\beta \cdot r$ (where $\beta = \max(\alpha_1, \alpha_2, \ldots, \alpha_r)) < \omega_0^\alpha$. Hence finally $f(A \cap G) = \bigcup f(A \cap G_r)$ $(r = 2, 3, \ldots)$ is of additive class ω_0^α in Y, and Theorem 4 is proved.

3.6. Corollary. *Every non-empty absolute Borel set of weight $\leqslant k$ is a continuous image of $B(k)$, under a mapping which takes every Borel set into a Borel set.*

(When $k = \aleph_0$, this gives a sharpened form of a well-known theorem; see [14, p. 353].)

For every such absolute Borel set is of the form $f(A)$, where f is the generalized homeomorphism of Theorem 5 and A is a closed subset of $B(k)$; and there is a retraction g of $B(k)$ onto A such that $g(B(k)-A)$ is σ-discrete, by Theorem 3 (2.5). If E is any Borel set, say of additive class $\alpha \geqslant 1$, in $B(k)$, then $g(E) = (A \cap E) \cup g(E-A)$ where $A \cap E$ is of additive class α and $g(E-A)$ is σ-discrete and therefore F_σ. Hence the continuous mapping $f(g)$ has the stated property.

Of course, even when $k = \aleph_0$, not every continuous image of $B(k)$ is an absolute Borel set.

4. CARDINALITY PROPERTIES; INVARIANCE OF WEIGHT

4.1. THEOREM 5. *Let X be a complete (metric) space of weight $\leqslant k$, and let f be a continuous mapping of X in any (metric) space Y. If $\|f(X)\| > k$, then X contains a closed subset C^* such that* (i) *C^* is homeomorphic to a Baire space $B(p)$ or $C(\aleph_0)$ of cardinal $k^{\aleph_0}$,* (ii) *$f(C^*)$ is closed in Y,* (iii) *$f \mid C^*$ is a homeomorphism.*

The theorem is trivial when k is finite, and well known when $k = \aleph_0$ [14, p. 351]. The proof of the general case is in the main a natural generalization of the standard proof for $k = \aleph_0$. We assume k infinite. Let ϱ, σ denote the metrics in X, Y respectively.

Let p be the smallest cardinal such that $p^{\aleph_0} \geqslant k^{\aleph_0}$; thus $p \leqslant k$, $p^{\aleph_0} = k^{\aleph_0}$, and if $q < p$ then $q^{\aleph_0} < k$ and so also $q^{\aleph_0} < p$. We suppose first that $p \neq 2$ (so that $p > \aleph_0$) and that p is not sequential.

The construction will be such that $f(C^*)$ is complete, and so automatically closed in Y; hence we may simplify the notation by replacing Y by $f(X)$—i.e., we may assume $Y = f(X)$. By hypothesis, $\|Y\| \geqslant k^+$ (11). Pick for each $y \in Y$ a point $x_y \in f^{-1}(y)$; let $P = \{x_y \mid y \in Y\}$, and pick $Q \subset P$ so that $\|Q\| = k^+$. Because X has weight $\leqslant k$, Q has no metrically discrete subset of cardinal k^+ (12), and therefore ([27], p. 100) all but less than k^+ points of Q are complete limit points of Q. Thus we obtain $R \subset Q$ such that $\|R\| = k^+$ and every point of R is a complete limit point of R.

We start an inductive construction by picking $c_0 \in R$. The open set $f^{-1}\big(S(f(c_0), 1)\big)$ (see 3.1 for this notation) contains a neighbourhood $U(c_0) = S(c_0, \varepsilon_0)$ for some positive $\varepsilon_0 < 1$. We are going to construct disjoint subsets $C_0 = (c_0)$, $C_1, \ldots, C_i, \ldots$ of R, subject to the requirements stated below, in such a way that each C_i $(i \geqslant 1)$ is the union of disjoint sets $C_i(c_{i-1})$, one for each $c_{i-1} \in C_{i-1}$; this c_{i-1} thus belongs to $C_{i-1}(c_{i-2})$ for just one $c_{i-2} \in C_{i-2}$, and so on; finally $c_1 \in C_1 = C_1(c_0)$. Each $c_i \in C_i$ will thus determine a unique sequence $c_{i-1}, c_{i-2}, \ldots, c_0$ such that $c_j \in C_j(c_{j-1}) \subset C_j$ $(j = 1, \ldots, i)$; we refer to $c_{i-1}, c_{i-2}, \ldots, c_0$ here as the "ancestors" of c_i, and say that c_i is a "descendant" of each of them.

(11) If m is any cardinal, m^+ denotes the successor of m.

(12) A set is "metrically discrete" if it is ε-discrete for some $\varepsilon > 0$.

Assume, then, that $C_0, \ldots, C_h$ have been defined; further, suppose that open sets $U(c_i) \ni c_i$ and positive numbers $\varepsilon_j(c_{j-1}) < 1/j$ have been defined for each $c_i \in C_i$ and $c_{j-1} \in C_{j-1}$, where $0 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant h$ (note that $\varepsilon_{h+1}(c_h)$ has not yet been defined), in such a way that

(1_h) $C_i \cap C_j = \emptyset$ if $0 \leqslant i < j \leqslant h$; $C_i \subset R$ $(0 \leqslant i \leqslant h)$, $C_0 = (c_0)$;

(2_h) If $1 \leqslant j \leqslant h$, $C_j = \bigcup\{C_j(c_{j-1}) \mid c_{j-1} \in C_{j-1}\}$; the sets $C_j(c_{j-1})$ here are pairwise disjoint, and each has cardinal p;

(3_h) If c_i is an ancestor of c_j $(0 \leqslant i < j \leqslant h)$, $U(c_j) \subset U(c_i)$;

(4_h) If $c_j \in C_j(c_{j-1})$ $(1 \leqslant j \leqslant h)$, both $U(c_j)$ and $f\big(U(c_j)\big)$ have diameters $< \varepsilon_j(c_{j-1})$; further, $\varrho\big(c_{j-1}, U(c_j)\big) > \varepsilon_j(c_{j-1})$ and $\sigma\Big(f(c_{j-1}), f\big(U(c_j)\big)\Big) > > \varepsilon_j(c_{j-1})$;

(5_h) If c_j, c_j' are distinct elements of $C_j(c_{j-1})$ $(1 \leqslant j \leqslant h)$ then $\varrho\big(U(c_j), U(c_j')\big) > \varepsilon_j(c_{j-1})$ and $\sigma\Big(f\big(U(c_j)\big), f\big(U(c_j')\big)\Big) > \varepsilon_j(c_{j-1})$;

(6_h) $\varepsilon_1(c_0) < \varepsilon_0$, and if $c_i \in C_i(c_{i-1})$ $(1 \leqslant i < h)$, then $\varepsilon_{i+1}(c_i) \leqslant \frac{1}{3}\varepsilon_i(c_{i-1})$.

We have to define sets $C_{h+1}(c_h)$, $U(c_{h+1})$ and positive numbers $\varepsilon_{h+1}(c_h)$ so that (1_{h+1})-(6_{h+1}) will be satisfied.

Let c_h be an element, fixed for the present, of C_h. From (1_h), $c_h \in R$, and is therefore a complete limit point of R; hence $\|R \cap U(c_h)\| = k^+$. Now f is 1-1 on R, so $\big\|f\big(R \cap U(c_h)\big)\big\| = k^+$. For each n $(=1, 2, \ldots)$ oet K_n be a maximal $1/n$-discrete subset of $f\big(R \cap U(c_h)\big)$, say consisting lf q_n points. The set $\bigcup K_n$ has $q = \sup q_n$ points, and is dense in $f\big(R \cap U(c_h)\big)$; hence $q^{\aleph_0} \geqslant k$, and therefore $q \geqslant p$. By hypothesis, p is not sequential, so we have $q_n \geqslant p$ for some n, where we may clearly assume $n > h+1$. In what follows, n has this fixed value; note that $\|K_n\| \geqslant p$. At most one point of K_n can have distance $< 1/2n$ from $f(c_h)$; omitting this point from K_n, if there is one, we ensure that $\sigma\big(f(c_h), K_n\big) \geqslant 1/2n$. Let $H = = R \cap f^{-1}(K_n) = \{x_y \mid y \in K_n\}$. The sets $f^{-1}\Big(S\big(f(x), 1/3n\big)\Big)$ $(x \in H)$ are disjoint; by omitting at most one point from H we ensure that none of them contains c_h. For each $x \in H$ there is some $\delta(x) > 0$ such that $S\big(x, \delta(x)\big) \subset f^{-1}\Big(S\big(f(x), 1/3n\big)\Big)$; let $H_m = \{x \mid x \in H, \delta(x) > 1/m\}$. Because $\|H\| \geqslant p$ and p is not sequential, we have $\|H_m\| \geqslant p$ for some m. We take $C_{h+1}(c_h)$ to be any subset of H_m with p points. Further, if $h \geqslant 1$, we put $\varepsilon_{h+1}(c_h) = \min\{1/7n, 1/3m, \varepsilon_h(c_{h-1})/3\}$, where c_{h-1} is the immediate ancestor of c_h; if $h = 0$, we replace $\varepsilon_h(c_{h-1})/3$ by ε_0. Because f is 1-1 on R, we have $C_{h+1}(c_h) \subset U(c_h)$; hence, if η is positive and small enough (depending on c_{h+1}) $S(c_{h+1}, \eta) \subset U(c_h)$. We may suppose $\eta < \varepsilon_{h+1}(c_h)/3$, and then put $U(c_{h+1}) = S(c_{h+1}, \eta)$.

Finally we write $C_{h+1} = \bigcup\{C_{h+1}(c_h) \mid c_h \in C_h\}$. It is a straightforward matter to verify that properties (1_{h+1})-(6_{h+1}) hold. Thus the construction is validated for all h.

Put $C = \bigcup C_i$ $(i = 0, 1, \ldots)$, $C^* = \bar{C} - C$. We show that C^* has the properties asserted in the theorem.

From (3) and (5) we see that, if $c_i \neq c_j$ $(c_i \in C_i,\ c_j \in C_j)$, then $U(c_i)$ and $U(c_j)$ meet only if one of c_i, c_j is an ancestor of the other. Further, from (4), $U(c_i)$ contains no ancestor of c_i, and $S\big(c_i, \varepsilon_{i+1}(c_i)\big)$ contains no descendant of c_i. Thus each point of C is isolated (i.e., open) in C and so also in $\bar{C}$; hence C^* is closed in X. (In fact, $C^* = C'$ = set of limit points of C). Similarly, if we write $D_i = f(C_i)$, $D = \bigcup D_i$, and $D^* = \bar{D} - D$, then $D^* = D'$ and is closed in Y.

Now let $c_0, c_1, \ldots, c_i, \ldots$ be any "descendant sequence"; that is, let $c_1 \in C_1(c_0), \ldots, c_i \in C_i(c_{i-1}), \ldots$ If $j > i$ we have $c_j \in U(c_j) \subset U(c_i)$, of diameter $< 1/i$; hence the descendant sequence is a Cauchy sequence, and converges to a point of C^*. Conversely, if $c_{1'}, c_{2'}, \ldots, c_{i'}, \ldots$ is any sequence of points of C converging to a point x of C^*, let $c_{1(i')}$ be the ancestor of $c_{i'}$ in C_1 (here we count a point as one of its own ancestors). Then $c_{i'} \in U(c_{i'}) \subset U(c_{1(i')})$; and (5) now shows that if $c_{1(i')} \neq c_{1(j')}$ then $\varrho(c_{i'}, c_{j'}) > \varepsilon_1(c_0)$. Because the sequence $\{c_{i'}\}$ is Cauchy, we must have $c_{1(i')} = c_{1(j')}$ for all large enough i', j'; that is, the points $c_{i'}$ ultimately have a common ancestor c_1 (say) in C_1. Because $x \notin C$, the points $c_{i'}$ ultimately have ancestors in C_2, which (by reasoning similar to the preceding) must ultimately coincide with a fixed $c_2 \in C_2(c_1)$; and so on. This argument constructs a descendant sequence $c_0, c_1, c_2, \ldots$ $(c_i \in C_i(c_{i-1}))$, which must also converge to x since, for every i, $c_{j'}$ is ultimately in $U(c_i)$. Thus C^* coincides with the set of limits of descendant sequences. We have seen that each such sequence does converge; moreover, two distinct descendant sequences must have distinct limits (for if $\{c_i\}$ and $\{c_i'\}$ are two such sequences, which first differ at the i'th terms, their respective limits are in the disjoint sets $\bar{U}(c_i)$, $\bar{U}(c_{i'})$).

Similar considerations apply to D^*. First, each "descendant sequence" $d_0, d_1, \ldots,$ where $d_i = f(c_i)$ and $c_i \in C_i(c_{i-1})$, is convergent because, as shown above, $\{c_i\}$ converges to some $x \in C^*$, so $d_i \to f(x) \in \bar{D}$. As the points d_i are all different (because f is 1-1 on $R \supset C$), we have $f(x) \in D^*$. The same reasoning as before shows that D^* coincides with the set of limits of descendant sequences, and that distinct descendant sequences $\{d_i\}$ have different limits. It follows at once that D^* is complete, and that $f|\ C^*$ is a 1-1 mapping of C^* onto D^*; and it is easily verified that $f^{-1}|\ D^*$ is continuous, so that $f|\ C^*$ is a homeomorphism.

We take a sequence of index sets T_n $(n = 1, 2, \ldots)$, each of cardinal p, and set up an arbitrary 1-1 correspondence $\varphi(\ ; c_{n-1})$ between T_n and $C_n(c_{n-1})$ for each $c_{n-1} \in C_{n-1}$ (so that $\varphi(t_n;\ c_{n-1}) \in C_n(c_{n-1})$ for each $t_n \in T_n$). Put $B(p) = \prod T_n$, each T_n being discrete. There is an obvious mapping Φ of $B(p)$ onto C^*; explicitly, given $t_1 \in T_1, t_2 \in T_2, \ldots$, we define $\Phi(t_1, t_2, \ldots)$ to be the limit of the descendant sequence $c_1 = \Phi(t_1; c_0)$, $c_2 = \varphi(t_2; c_1), \ldots, c_n = \varphi(t_n; c_{n-1}), \ldots$ It is easily verified that Φ is a homeomorphism of $B(p)$ onto C^*.

This completes the proof of the theorem when $p > 2$ and is not sequential. If p is sequential, then (since $p > \aleph_0$) we may write $p = \sup p_n$ where $\aleph_0 \leqslant p_1 < p_2 < \ldots$ We modify the above construction slightly, replacing the condition $\|C_i(c_{i-1})\| = p$ by $\|C_i(c_{i-1})\| = p_i^+$, and taking T_n to have cardinal p_n^+; this produces C^* and D^* homeomorphic to $C(p)$, and thus (by 2.4 (1)) to $B(p)$. Finally, if $p = 2$, the result is well known; the construction simplifies greatly, as we need only take each $C_i(c_{i-1})$ to have 2 points.

4.2. Corollary. *Under the hypotheses of Theorem* 5, $\|Y\| \geqslant \|f(X)\| = k^{\aleph_0}$.

It cannot, however, be asserted that the weight of X is k.

4.3. Theorem 6. *If Y is an absolute Borel set of weight $\leqslant k$, and if $\|Y\| > k$, then $\|Y\| = k^{\aleph_0}$, and Y contains a closed set homeomorphic to a Baire space ($B(p)$ or $C(\aleph_0)$) of cardinal $k^{\aleph_0}$.*

For if Y has weight $m \leqslant k$, then (3.6) Y is a continuous image of $B(k)$, a space of weight k. Apply Theorem 5 (4.1), taking $X = B(k)$.

Remark. Theorems 5 and 6 say nothing unless $k^{\aleph_0} > k$ (a condition which, under the generalized continuum hypothesis, is equivalent to requiring k to be sequential). For such cardinals Theorem 6 can be extended to more general sets; see 8.4 below.

4.4. Theorem 7. *Let X be any (metric) space of cardinal m and infinite weight k. Then X has exactly 2^k Borel subsets. Further, the number of Borel subsets of cardinal n of X is* (i) m^n *if* $n \leqslant k$, (ii) $2^k = m^k$ *if* $n = m$, (iii) 0 *or* 2^k *if* $k < n < m$, (iv) 0 *if* $k < n < m$ *and X is absolutely Borel.*

From ([27], Th. 3.6), X has 2^k closed subsets (exactly). An easy transfinite induction argument proves that X has at most 2^k Borel sets of class α, for each $\alpha < \omega_1$, and hence at most $\aleph_1 2^k = 2^k$ Borel sets altogether. But X has 2^k closed subsets, and therefore has exactly 2^k Borel subsets.

The assertion (i) follows from the fact that X has only m^n subsets of cardinal n, and at least m^n of them are closed [27, 3.6]. Similarly (ii) follows because there are "enough" closed sets. To prove (iv), we note that if Y is Borel in X and of cardinal $n > k$, Theorem 6 (4.3) shows that $n \geqslant k^{\aleph_0} \geqslant m$. Finally, to prove (iii), suppose that X has at least one Borel subset Y of n points, where $k < n < m$; it will suffice to prove that X has at least 2^k such subsets. From ([27], 3.4), $Y = A \cup B$ where

A, B are closed in Y (hence Borel in X), $A \cap B$ is a single point, say a, and $\|A\| = \|B\| = n$. Since $X-(a) = (X-A)\cup(X-B)$, either $X-A$ or $X-B$ has weight k; say the former. Then $X-A$ has 2^k distinct Borel subsets F of cardinal k, by (i), and the sets $F \cup A$ are all Borel in X, all of cardinal n, and all distinct.

Remark. If the generalized continuum hypothesis is assumed, the alternatives (iii) and (iv) cannot arise, for $m = k$ or k^+ (because in any case $k \leqslant m \leqslant k^{\aleph_0}$).

4.5. Similarly one can prove:

THEOREM 7′. *Under the hypotheses of Theorem 7, X has exactly 2^k absolute Borel subsets. Further, the number of absolute Borel subsets of cardinal n of X is* (i) m^n *if* $n \leqslant k$, (ii) 0 *if* $k < n < k^{\aleph_0}$, (iii) 0 *or* 2^k *if* $n = k^{\aleph_0}$.

The proof requires a closer examination of the argument in ([27], 3.6), to see that "enough" Borel sets are provided by suitable combinations of metrically discrete subsets of X; they are then absolute Borel sets because a metrically discrete set, being complete, is an absolute G_δ.

4.6. THEOREM 8. *If two absolute Borel sets are Borel isomorphic, they have the same weight (and, of course, the same cardinal)*(13).

Suppose f is a Borel isomorphism of X, with m points and weight k, onto Y, with m' points and weight k'. Obviously $m = m'$; suppose $k < k'$. By Theorem 7, Y has a Borel subset E of cardinal k'; in fact, E may even be taken to be closed. Then $f^{-1}(E)$ is a Borel subset of X, of cardinal $k' > k$. Hence, if X is absolutely Borel, $k' = k^{\aleph_0}$ by Theorem 6 (4.3). Since $k^{\aleph_0} \geqslant m \geqslant k'$, it follows that $k^{\aleph_0} = m$.

Because $k' = k^{\aleph_0}$, k' is not sequential (König's theorem; see e.g. ([26], p. 181)), so that ([27], Th. 2.7) Y has a metrically discrete subset A of cardinal k'. Every subset of A is closed in Y, and therefore every subset of $f^{-1}(A)$ is Borel in X. In particular, $f^{-1}(A)$ is an absolute Borel set of cardinal k' and weight $\leqslant k < k'$. By Theorem 6, $f^{-1}(A)$ contains a homeomorph of some Baire space $B(p)$ or $C(\aleph_0)$, and therefore contains a homeomorph of the Cantor set $C(\aleph_0)$. This in turn has a non-Borel subset, contradicting the assertion that every subset of $f^{-1}(A)$ is Borel.

Remark. I do not know whether an absolute Borel set X can be Borel isomorphic to a space Y which is *not* absolutely Borel. But if this can happen, the above argument shows that (weight of X) $\geqslant$ (weight of Y). It seems possible that Theorem 8 applies even without the hypothesis that the sets be *absolutely* Borel, but I can prove this only if the generalized continuum hypothesis is assumed:

(13) This seems to be new even when one of the sets is separable. Of course, the converse of Theorem 8 is true for separable spaces, but not in general.

4.7. THEOREM 8′. *Assuming the generalized continuum hypothesis, if any two (metrisable) spaces are Borel isomorphic, they have the same weight (and, of course, the same cardinal).*

In the notation of 4.6, f sets up a 1-1 correspondence between the family of Borel subsets of X and the family of Borel subsets of Y; hence, by Theorem 7 (4.4), $2^k = 2^{k'}$. Because of the generalized continuum hypothesis, it follows that $k = k'$.

5. CLASSIFICATION OF ABSOLUTE BOREL SETS

5.1. THEOREM 9. *Let A_2, A_3, be Borel subsets, of additive and multiplicative class γ, of an arbitrary space A_1; suppose that $A_2 \supset A_3$ and that there is a generalized homeomorphism of class (α, β) of A_1 onto A_3. Then there is a generalized homeomorphism, of class depending only on α, β, γ, of A_1 onto A_2.*

This is an analogue of the Schröder-Bernstein theorem, and is proved in the same way; we sketch the argument briefly. Let f be the generalized homeomorphism between A_1 and A_3. Put $A_{n+2} = f(A_n)$ $(n = 2, 3, \ldots)$, $A_\infty = \bigcap A_n$, $B_n = A_n - A_{n+1}$ $(n = 1, 2, \ldots)$. It is easily seen that $A_n \supset A_{n+1}$ and that each of the sets B_n, A_∞, is Borel of additive and multiplicative class $\beta\omega_0 + \gamma$ in A_1. Define a mapping g of A_1 in A_2 by: if $x \in B_n$, n odd, $g(x) = f(x)$; if $x \in B_n$, n even, or if $x \in A_\infty$, $g(x) = x$. Then g is a 1-1 mapping of A_1 onto A_2. If E is an open subset of A_1, we have $g(E) = \bigcup \{f(E \cap B_n) \mid n \text{ odd}\} \cup \bigcup \{(E \cap B_n) \mid n \text{ even}\} \cup E \cap A_\infty$, of Borel additive class $\beta\omega_0 + \gamma$ in A_1 and therefore also in A_2. Similarly we show that if G is open in A_2, $g^{-1}(G)$ is of Borel additive class $\alpha + \beta\omega_0 + \gamma$ in A_1. That is, g is a generalized homeomorphism, of class $(\alpha + \beta\omega_0 + \gamma, \beta\omega_0 + \gamma)$, between A_1 and A_2.

5.2. Corollaries.

(1) *If each of two spaces X, Y, is generalized homeomorphic, or Borel isomorphic, to a Borel subset of the other, then X and Y are, respectively, generalized homeomorphic or Borel isomorphic to each other.*

For "generalized homeomorphic" this follows from Theorem 9; the result for Borel isomorphism follows by an entirely analogous argument. In particular, we obtain the well-known result that $B(\aleph_0)$ and $C(\aleph_0)$ are generlaized homeomorphic.

(2) *If an absolute Borel set X, of weight k, contains a subspace homeomorphic to $B(k)$ (or, more generally, contains a Borel subset which is generalized homeomorphic to $B(k)$), then X is generalized homeomorphic to $B(k)$.*

For, in view of Theorem 4 (3.2), we can apply the preceding corollary to X and $B(k)$.

5.3. LEMMA. *Let k be an infinite cardinal, and let S be a non-empty complete metric space in which every non-empty open set has weight at least k. Then S has a closed subset homeomorphic to $B(k)$ or $C(k)$.*

This is a slight generalization ("at least k" replacing "k") of ([27], 3.1), and the same proof applies. Of course, if $k > \aleph_0$ we may replace $C(k)$ by $B(k)$, from 2.4 (1).

COROLLARY. *If S is a non-empty absolute G_δ in which every set of cardinal less than k is nowhere dense, and if S has weight k, then S is generalized homeomorphic to $B(k)$.*

5.4. THEOREM 10. *Every (metrizable) absolute Borel set X is generalized homeomorphic to a discrete union Z of Baire spaces. More precisely, if X has weight k, we may take Z to be the discrete union of at most k spaces $B(p_a)$, where a runs over an index-set of cardinal $\leqslant k$, each $p_a \leqslant k$, and each p_a is either* 1 *or infinite.* (We recall that $B(1)$ is a single point.) *The class of the generalized homeomorphism required depends only on the class of X.*

The proof goes by transfinite induction over k, the weight of X. For finite k, the theorem is trivial; we suppose, therefore, that k is infinite, and that the theorem is true for all smaller cardinals.

By Theorem 4 (3.2), X is generalized homeomorphic to a closed subset A of $B(k)$. Let U_0 denote the union of all open subsets G (relative to A) of A of weight $< k$ (of course $\emptyset$ is such a G), and let $A_0 = A - U_0$, a closed subset of A. When U_β and A_β have been defined for all ordinals $\beta < a$, we define U_a and A_a as follows. If a is a limit ordinal, put $U_a = \emptyset$, $A_a = \bigcap \{A_\beta | \ \beta < a\}$. If $a = \gamma + 1$, let $U_a =$ union of all open subsets G (relative to A_γ) of A_γ of weight $< k$, and put $A_a = A_\gamma - U_a$. It follows that A_a is closed in A (and so in $B(k)$), and that $A_0 \supset A_1 \supset \ldots \supset A_a \supset \ldots$ For some large enough ordinal a^* (not necessarily countable) we have $A_{a^*} = A_{a^*+1} = \ldots = K$, say. We distinguish two cases.

(1) If $K \neq \emptyset$, the fact that $U_{a^*+1} = \emptyset$ shows that every non-empty open subset of the subspace K has weight $\geqslant k$.

By 5.3, Corollary, K is generalized homeomorphic to $B(k)$. The same must be true of A, from 5.2 (2); hence X is generalized homeomorphic to $B(k)$ in this case, and the theorem holds with $Z = B(k)$.

(2) If $K = \emptyset$, we have $A = \bigcup U_a$ $(a < a^*)$.

Form a collection of disjoint sets V_a $(a < a^*)$, each V_a being in 1-1 correspondence with U_a, and topologize $Y = \bigcup V_a$ so that each V_a is open in Y and homeomorphic to U_a. (That is, Y is the discrete union of copies V_a of U_a). Clearly Y is metrisable; also Y is absolutely Borel, because each U_a, being the difference between two closed subsets of $B(k)$, is itself an absolute G_δ. We verify that the obvious mapping f of Y onto A (in which each $y \in V_a$ is mapped onto the corresponding point of U_a) is a generalized homeomorphism. In fact, since f is continuous, we have only to prove that if H is open in Y, then $f(H)$ is Borel (of class independent of H) in A. Now $f(H) = \bigcup\{f(H \cap V_a) \mid a < a^*\} = \bigcup H_a$ where H_a is a relatively open subset of U_a. There exists G_a, open in A, such that $H_a = G_a \cap U_a$. Put $W_a = A - A_a$; thus W_a is open in A, $W_0 \subset W_1 \subset \ldots$, $\bigcup W_a = A$, and we readily verify that $U_a = W_a - - \bigcup\{W_\beta \mid \beta < a\}$. It follows that $\bigcup H_a = \bigcup G_a \cap \{W_a - \bigcup W_\beta \mid B < a\}$ $(a < a^*)$, the result of Montgomery's "operation $\mathscr{M}$" on the open sets G_a (see [14, p. 265]). Hence $\bigcup H_a$ is Borel (in fact F_σ) in A, and f is a generalized homeomorphism (of class $(0,1)$).

If, then, the assertion of the theorem is proved for Y, it will follow for A and so also for X. As Y is the discrete union of the *non-empty* spaces V_a, and there are at most k of them (because Y must also have weight k, by Theorem 8 (4.6)), it will suffice to prove the assertion of the theorem for a single set V_a—i.e., for U_a. Accordingly we keep a fixed throughout what follows(14).

By definition, U_a has a (relatively) open covering by sets G_λ each of weight $< k$. As U_a is paracompact, this covering has a σ-discrete open refinement $\{N_{n\mu}\}$ $(n = 1, 2, \ldots)$ where for fixed n the collection of open sets $N_{n\mu}$ is discrete, and where each $N_{n\mu} \subset$ some G_λ, and hence has weight $k_{n\mu} < k$. Put $P_n = \bigcup_\mu N_{n\mu}$, $Q_1 = P_1$, $Q_n = P_n - (P_1 \cup \ldots \cup P_{n-1})$ $(n = 2, 3, \ldots)$. The sets $Q_1, Q_2, \ldots$, are disjoint G_δ sets covering U_a, so U_a is evidently generalized homeomorphic to the discrete union of the sequence of sets Q_n. Now each Q_n is itself the discrete union of the relatively open sets $Q_n \cap N_{n\mu}$, each of weight $< k$. By the hypothesis of induction, each $Q_n \cap N_{n\mu}$ is generalized homeomorphic to a discrete union of $< k$ sets each of the form $B(p)$, $p < k$; and this gives the desired generalized homeomorphism of Q_n, and thus of U_a, onto a space Z of the kind described in the theorem.

5.5. It follows from 5.2 (2) that if any one of the summands $B(p_a)$, in the above representation of X, is $B(k)$, then we may omit all the rest. Similarly, if any $B(m)$ occurs, we may omit any family of not more than m spaces $B(p_a)$ with $p_a \leqslant m$. We may, of course, carry out such omissions any finite number of times.

In view of these remarks, Theorem 10 includes the "classical" theo-

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(14) To simplify the details in what follows, we omit part of the argument (easily supplied) showing that the generalized homeomorphisms to be constructed are all of fixed class. Without this, they could not be combined to give a generalized homeomorphism (or even Borel isomorphism) of all Y.

rem that an infinite separable absolute Borel set is Borel isomorphic to either $B(\aleph_0)$ or to the discrete union of $\aleph_0$ 1-point spaces $B(1)$. When $k = \aleph_1$, we obtain:

THEOREM 11. *Every absolute Borel set of weight* $\aleph_1$ *is generalized homeomorphic to one of the following* 4 *spaces*:

(1) *A discrete set of* $\aleph_1$ *points.*

(2) *The discrete union of* (1) *with* $B(\aleph_0)$.

(3) *The discrete union of* $\aleph_1$ *copies of* $B(\aleph_0)$.

(4) $B(\aleph_1)$.

It is easy to draw up similar, but longer, lists for $k = \aleph_2, \aleph_3, \ldots$; from $\aleph_{\omega_0}$ on, the lists are infinite.

The *classification problem* would be completely solved if it could be determined just which spaces on these lists are generalized homeomorphic, or Borel isomorphic. For the case $k = \aleph_1$, we observe that, in Theorem 11, (1) *and* (2) *are not Borel isomorphic to each other, or to* (3) *or* (4). For (1) is distinguished from the others by the property, invariant under Borel isomorphism, that every subset of (1) is Borel. The space (2) is similarly characterized by the property that, though it has non-Borel subsets, it has a separable Borel subset S such that every subset disjoint from S is Borel. (This property is invariant under Borel isomorphism because separability is, by Theorem 8, 4.6). All that remains is to decide whether or not (3) and (4) are generalized homeomorphic, or Borel isomorphic; we shall see below (Theorem 15, 7.5) that these questions are equivalent, but they remain unanswered.

6. CHARACTERIZATIONS

6.1. We have just seen that the absolute Borel sets of a given weight k fall into a relatively small number of types (under generalized homeomorphism or Borel isomorphism), of which the "thinnest" is the discrete set of cardinal k, and the "thickest" is $B(k)$. We now characterize these two extreme cases topologically, the second only for special values of k. It would be very desirable to have topological characterizations of the other types.

6.2. THEOREM 12. *The following statements about an absolute (metrizable) Borel set X are equivalent:*

(1) *Every subset of X is Borel.*

(2) *X is Borel isomorphic to a discrete set.*

(3) *X is generalized homeomorphic to a discrete set.*

(4) *Every separable subset of X is countable.*

Further, if these conditions hold, then, for every $Y \subset X$, $\|\overline{Y}\| = \|Y\|$; and if X is of additive class $\alpha \geqslant 1$, every subset of X is of class $\omega_0^\alpha + 1$.

Proof. (1) implies (4). For if (4) is false, X has a countable set $Y \subset X$ such that $\overline{Y}$ is uncountable. By Theorem 6 (4.3), $\overline{Y}$ contains a homeomorph of $B(\aleph_0)$ or $C(\aleph_0)$, and therefore has non-Borel subsets, in contradiction to (1).

Next, (4) implies (3). Suppose X is of additive class $\alpha \geqslant 1$ and of weight k; we may clearly assume $k \geqslant \aleph_0$. By Theorem 4 (3.2), there is a generalized homeomorphism f, of class $(0, \omega_0^\alpha)$, taking a closed subset A of $B(k)$ onto X. Because f is 1-1 and continuous, (4) implies that every separable subset of A is also countable. We form the transfinite sequence $\{A_\lambda\}$ of successive derived sets of A; that is, $A_0 = A$, for each ordinal λ (of a sufficiently large section of ordinals) $A_{\lambda+1} = (A_\lambda)'$, the set of limit points of A_λ, and for each limit ordinal λ, $A_\lambda = \bigcap \{A_\mu | \mu < \lambda\}$. The sets A_λ are closed, and $A = A_0 \supset A_1 \supset \ldots$; if λ is large enough, say for $\lambda \geqslant \lambda^*$, we have $A_\lambda = A_{\lambda^*} = K$, say. K is the largest perfect subset of A (but need not in general coincide with the set of complete limit points of A; moreover, λ^* need not be countable). We show that here $K = \emptyset$. In fact, being complete and dense-in-itself, K would otherwise contain a subset homeomorphic to the Cantor set $C(\aleph_0)$ (by a well-known special case of the Lemma in 5.3), and A would thus have an uncountable separable subset.

Write $U_\lambda = A - A_\lambda$ ($\lambda \leqslant \lambda^*$); because $K = \emptyset$, the increasing transfinite sequence $\{U_\lambda\}$ forms an open covering of the space A. Put $V_\lambda = U_\lambda - \bigcup\{U_\mu | \mu < \lambda\}$; V_λ is either $\emptyset$ (if λ is a limit ordinal) or of the form $A_\mu - A_{\mu+1}$ (if $\lambda = \mu + 1$), so that in any case V_λ is relatively discrete. It follows that every subset of V_λ is closed in V_λ, and hence F_σ in A. If Z is any subset of A, we have $Z = \bigcup(Z \cap V_\lambda)$, the result of the "operation $\mathscr{M}$" [14, p. 267] on the F_σ sets $Z \cap V_\lambda$; hence Z is also F_σ in A. Because f is of class $(0, \omega_0^a)$, it follows that every subset of X is of additive class $\omega_0^a + 1$ in X. Thus the identity mapping from X to X with the discrete topology is a generalized homeomorphism, of class $(\omega_0^a + 1, 0)$, and (3) is established.

The implications (3) $\to$ (2) $\to$ (1) are trivial, and the other assertions of the theorem follow by arguments very similar to those given. It would be interesting to know if the bound $\omega_0^a + 1$ can be sharpened.

6.3. Corollary. *If an absolute Borel set X has Borel sets of arbitrarily high classes (i.e., if for each $a < \omega_1$ X has a Borel subset Y_a which is not of class a), then X has a non-Borel subset.*

We shall see later (7.4) that this corollary applies even to non-absolutely Borel sets X, provided that the continuum hypothesis is assumed.

6.4. Theorem 13. *Let k be an infinite cardinal such that* (i) $k < k^{\aleph_0}$, (ii) $p^{\aleph_0} < k$ *whenever* $\aleph_0 \leqslant p < k$. *Then the following statements about an absolute Borel set X are equivalent:*

(1) *X has weight $\leqslant k$ and cardinal $> k$,*

(2) *X is Borel isomorphic to $B(k)$,*

(3) *X is generalized homeomorphic to $B(k)$.*

Remark. The hypotheses on k are satisfied by $k = \aleph_0$, and also by arbitrarily large cardinals—for example, the sum of any sequence of cardinals $p_1, p_2, \ldots$, such that $p_{n+1} > p_n^{\aleph_0}$. On the generalized continuum hypothesis, the cardinals satisfying (i) and (ii) are precisely the sequential cardinals.

Proof of Theorem 13. It is trivial that (3) implies (2). If (2) holds, then X has cardinal $k^{\aleph_0} > k$, and X has weight k because of Theorem 8 (4.6); thus (2) implies (1). Finally, suppose that (1) holds. By Theorem 10 (5.4), X is generalized homeomorphic to a discrete union Z of spaces $B(p_\lambda)$ ($\lambda \in \Lambda$), where $\|\Lambda\| \leqslant k$ and each $p_\lambda \leqslant k$. If each $p_\lambda < k$ here, then each $\|B(p_\lambda)\| = p_\lambda^{\aleph_0} < k$, by (ii); hence $\|X\| \leqslant k$, contradicting (1). Hence Z has a subset homeomorphic to $B(k)$, and it follows from 5.2 (2) that X is generalized homeomorphic to $B(k)$, completing the proof.

It would be interesting to have a topological characterization of the absolute Borel sets which are generalized homeomorphic to $B(k)$ for other values of k. By Theorem 4 (3.2) the problem is reduced to deciding when a closed subset A of $B(k)$ is generalized homeomorphic to $B(k)$. A sufficient condition is given by 5.3, Corollary; but it is of course too strong.

7. BOREL ISOMORPHISM AND GENERALIZED HOMEOMORPHISM

7.1. We have already encountered two very natural questions:

(a) Is every Borel isomorphism (between two metric spaces) necessarily a generalized homeomorphism? (The converse is true and trivial).

(b) If Y is Borel isomorphic to X, and X is absolutely Borel, need Y be absolutely Borel?

Both are unanswered. The answer to (a) is trivially "yes" if the spaces are separable; Theorems 12 and 13 provide other cases in which (a) has an affirmative answer. In this section we shall prove some further results in this direction (Theorems 14, 15), and show that the answer to (b) is also "yes" when X is separable (Theorem 16; even this seems to be new). It may be as well to point out first one difficulty which arises in studying (a) for non-separable spaces. It can happen that a 1-1 mapping f between two (metric) spaces X, Y is such that, for each member U of a base of open sets of X, $f(U)$ is Borel of class β in Y, and for each member V of a base of open sets of Y, $f^{-1}(V)$ is Borel of class α in X, and nevertheless f is not even a Borel isomorphism (even though α and β are fixed). An example is obtained by taking $X = B(k)$, $Y =$ any discrete set of $k^{\aleph_0}$ points, and f any 1-1 mapping between them; $f(U)$ is closed in Y for every $U \subset X$, and if $\mathscr{V}$ is the family of 1-point subsets of Y, then $\mathscr{V}$ is a base of open sets of Y, and $f^{-1}(V)$ is closed in X for each $V \in \mathscr{V}$. Yet f is not a Borel isomorphism, because $B(k)$ contains a non-Borel subset (because $B(k)$ has a subspace homeomorphic to $B(\aleph_0)$). For *separable* X and Y, of course, this difficulty disappears and it is enough to consider the effects of f and f^{-1} on members of *bases* of open sets.

7.2. THEOREM 14. *Let f be a Borel isomorphism between two (metric) spaces X, Y, and assume further that either* (i) *the continuum hypothesis holds, or* (ii) *Y is absolutely Borel. Then, if f is "generalized continuous", it is a generalized homeomorphism.*

Here "generalized continuous" means that there is some $\alpha < \omega_1$ such that, for every open set $H \subset Y$, $f^{-1}(H)$ is Borel of class α in X.

We have to deduce that there is some $\beta < \omega_1$ such that, for every open set $G \subset X$, $f(G)$ is Borel of class β in Y. The proof needs three lemmas, of which the first is a well-known theorem of Montgomery [20]:

LEMMA 1 (Montgomery). *If $E \subset Y$ and each $y \in E$ has an open neighbourhood $V(y)$ in Y such that $V(y) \cap E$ is Borel of (fixed) additive class β in Y, where $\beta \geqslant 1$, then E is Borel of additive class β in Y.*

LEMMA 2. *Under the hypotheses of Theorem 14, if $\{U_\lambda\}$ is a discrete collection of open subsets of X, the Borel sets $f(U_\lambda)$ are all of additive class β in Y, for some $\beta < \omega_1$.*

Suppose not; then for each $\beta < \omega_1$ we can find a set $U_{\lambda(\beta)}$ for which $f(U_{\lambda(\beta)})$ is *not* of additive class $\beta+1$ in Y; and we may evidently suppose the selected suffixes $\lambda(\beta)$ to be all distinct. By Lemma 1, there is a point $q_\beta \in f(U_{\lambda(\beta)})$ such that, for every open neighbourhood $V(q_\beta)$ of q_β in Y, $V(q_\beta) \cap U_{\lambda(\beta)}$ is not of class β. Let $p_\beta = f^{-1}(q_\beta)$, and put $P = \{p_\beta \mid \beta < \omega_1\}, Q = \{q_\beta \mid \beta < \omega_1\}$. Because f is 1-1, $\|Q\| = \|P\| = \aleph_1$. Now P is discrete in X, so that every subset of P is Borel (in fact closed) in X. Hence every subset of Q is Borel in Y. We deduce that Q ***is not separable.*** In fact, if Q were separable, it could have only c Borel subsets (by a well-known special case of Theorem 7, 4.4); but Q has just been shown to have $2^{\aleph_1}$ Borel subsets, and on the continuum hypothesis this gives a contradiction. If, instead of the continuum hypothesis, we assume that Y is absolutely Borel, then Q is also absolutely Borel, and Theorem 12 (6.2) shows that every separable subset of Q is countable. Hence in either case Q is not separable.

Let Q_n be a maximal $1/n$-discrete subset of Q ($n = 1, 2, \ldots$). Since $\bigcup Q_n$ is dense in Q, at least one Q_n has $\aleph_1$ points and therefore contains q_β for a cofinal set of β's. Fixing such an n, we write $V_\beta = S(q_\beta, 1/2n)$ ($q_\beta \in Q_n$); the sets V_β are open and disjoint. Since f is generalized continuous, there is some $\alpha < \omega_1$ such that all the sets $f^{-1}(V_\beta)$ ($q_\beta \in Q_n$) are of class α in X. Now the collection $\{U_\lambda\}$ is discrete; hence the set $Z = \bigcup \{U_{\lambda(\beta)} \cap f^{-1}(V_\beta) \mid q_\beta \in Q_n\}$ is also Borel of class α in X. Hence $f(Z)$ is Borel in Y, say of additive class γ ($< \omega_1$). By increasing γ, if necessary, we may assume $q_\gamma \in Q_n$. But V_γ is open; hence $V_\gamma \cap f(Z)$ is also Borel of class γ in Y. However, $V_\gamma \cap f(Z) = V_\gamma \cap f(U_{\lambda(\gamma)})$ which, by choice of q_γ, is *not* of class γ; and this contradiction proves the lemma.

LEMMA 3. *Under the hypotheses of Theorem 14, given any discrete collection $\{U_\lambda\}$ of open subsets of X, there exists $\beta < \omega_1$ such that every union of (some or all) sets $f(U_\lambda)$ is Borel of class β in Y.*

Suppose not; then, by transfinite induction, we can find for each $\beta < \omega_1$ a union $G_\beta = \bigcup \{U_\lambda \mid \lambda \in \Lambda_\beta\}$ such that the exact (i.e., smallest) additive class α_β of the Borel set $f(G_\beta)$ in Y satisfies both $\alpha_\beta > \beta$ and (if $\beta > 0$) $\alpha_\beta > \gamma_\beta$, where $\gamma_\beta = \sup\{\alpha_{\beta'} \mid \beta' < \beta\}$. Put $H_\beta = G_\beta -$

$-\bigcup\{G_{\beta'} \mid \beta' < \beta\}$; then $\{H_\beta\}$ is a discrete collection of open sets in X. Further, since $\bigcup\{f(G_{\beta'}) \mid \beta' < \beta\}$ is a countable union of Borel sets each of class γ_β, while $f(G_\beta)$ has exact class $\alpha_\beta > \gamma_\beta$, $f(H_\beta)$ has exact class $\alpha_\beta > \beta$. This contradicts Lemma 2.

7.3. Proof of Theorem 14. Being metrisable, X has a σ-discrete base $\{B_{n\lambda}\}$ of open sets, the sets $B_{n\lambda}$ for fixed n forming a discrete system ($n = 1, 2, \ldots$). By Lemma 3 there is some $\beta_n < \omega_1$ such that every union of sets $f(B_{n\lambda})$ with n fixed is of additive class β_n in Y. Let $\beta = \sup \beta_n$; β is also $< \omega_1$. If G is any open subset of X, we have $G = \bigcup G_n$ where, for each fixed n, $G_n = \bigcup\{B_{n\lambda} \mid B_{n\lambda} \subset G\}$. Hence $f(G) = \bigcup f(G_n)$ is of class β in Y, and the theorem is proved.

7.4. COROLLARY. *Assuming the continuum hypothesis, if a* (*metric*) *space Y has Borel subsets of arbitrarily high* (*exact*) *classes, it has non-Borel subsets.* (Cf. 6.3).

For let X be a discrete space with $\|X\| = \|Y\|$, and let f be any 1-1 mapping of X onto Y. If every subset of Y is Borel in Y, then f is a Borel isomorphism. But f is continuous; hence, by Theorem 14, f is a generalized homeomorphism, say of class $(0, \beta)$. Then every subset of Y is of class β.

7.5. When X is locally separable, more can be said. Note, however, that *local* separability is *not* invariant under Borel isomorphism, or even generalized homeomorphism. For example, if X is the discrete union of $\aleph_1$ unit intervals, and Y is the union of $\aleph_1$ unit intervals with a common end-point, metrized in an obvious way, then X and Y are generalized homeomorphic absolute G_δ's; but X is locally separable and Y is not.

THEOREM 15. *Let f be a Borel isomorphism between two spaces X, Y, and assume further that either* (i) *the continuum hypothesis holds, or* (ii) *X and Y are absolutely Borel. Then, if X is locally separable, f is a generalized homeomorphism.*

By Theorem 14, it is enough to prove that f is generalized continuous; and for this, by the same reasoning as in the proof of Theorem 14, it will suffice to prove the analogue of Lemma 2 (7.2), namely:

LEMMA 2'. *Under the hypotheses of Theorem 15, if $\{V_\lambda\}$ is a discrete collection of open subsets of Y, the Borel sets $f^{-1}(V_\lambda)$ are of bounded class in X.*

Suppose not; then we readily obtain a subcollection of distinct sets $V_{\lambda(\alpha)}$ ($\alpha < \alpha_1$) such that $f^{-1}(V_{\lambda(\alpha)}) = Q_\alpha$ say is not of additive class α in X. By Lemma 1 (7.2) there exists $x_\alpha \in Q_\alpha$ each (open) neighbourhood of which meets Q_α in a set not of class α in X. Pick $\alpha(1) > 1$, and let U_1 be any separable open neighbourhood of $x_{\alpha(1)}$ in X. Let $U_1 \cap Q_{\alpha(1)}$

have exact Borel class β_1 in X; thus $\beta_1 > \alpha(1)$. Because f is a Borel isomorphism, $f(U_1)$ is separable (Theorems 8 and 8′, 4.6 and 4.7), and so meets only countably many of the disjoint open sets V_λ; hence there exists $\alpha(2) < \omega_1$ such that $\alpha(2) > \beta_1$ and $U_1 \cap Q_\alpha = \emptyset$ whenever $\alpha > \alpha(2)$. When $\alpha(\gamma)$ and U_γ have been defined for all $\gamma < \delta$ $(< \omega_1)$, with $U_\gamma \cap Q_{\alpha(\gamma)}$ of exact class $\beta_\gamma > \alpha(\gamma)$, we pick $\alpha(\delta) < \omega_1$ so large that (a) each of the separable open sets U_γ $(\gamma < \delta)$ is disjoint from Q_α whenever $\alpha > \alpha(\delta)$, (b) $\alpha(\delta) > \max(\delta, \xi_\delta + 1)$, where $\xi_\delta = \sup\{\beta_\gamma \mid \gamma < \delta\}$. Then we take U_δ to be a separable open neighbourhood of $x_{\alpha(\delta)}$, and we put $\beta_\delta =$ exact class of $Q_{\alpha(\delta)} \cap U_\delta$. This defines $\alpha(\delta)$ and U_δ for all $\delta < \omega_1$.

Now $\bigcup\{Q_{\alpha(\delta)} \mid \delta < \omega_1\} = f^{-1}(\bigcup V_{\lambda(\alpha(\delta))})$, and is therefore Borel in X, say of additive class $\eta < \omega_1$. Hence the set $E = U_\eta \cap \bigcup Q_{\alpha(\delta)}$ is also of class $(\leqslant)\eta$. But, because of condition (a) in the above construction, $E = U_\eta \cap \bigcup\{Q_{\alpha(\delta)} \mid \delta \leqslant \eta\}$ and therefore $U_\eta \cap Q_\alpha(\eta) = E - \bigcup\{U_\eta \cap Q_{\alpha(\delta)} \mid \delta < \eta\} = E - F$, say, where F, being a countable union of sets each of class $\sup\{\beta_\delta \mid \delta < \eta\}$, has additive class ξ_η. Thus $U_\eta \cap Q_{\alpha(\eta)}$ is of class $\max(\eta, \xi_\eta + 1)$; but its exact class is $\beta_\eta > \alpha(\eta) > \max(\eta, \xi_\eta + 1)$ by (b). This contradiction proves the theorem.

7.6. In the *separable* absolute Borel case, much more is true. For completeness, we state the following theorem in full, though most of it is known ([14], pp. 397, 398); the new ingredient (the separability of Y) answers a question of Kuratowski (loc. cit. p. 399, Remark 3). In stating the theorem, we have interchanged X and Y as they appear in Theorems 14 and 15, and replaced f by f^{-1}, to obtain a more natural statement.

THEOREM 16. *Let X be a separable absolute Borel set, and let f be any 1-1 Borel measurable mapping of X onto any (metric) space Y* (15). *Then f is a generalized homeomorphism, and Y is separable and absolutely Borel.*

Suppose Y is not separable; then, for some $\varepsilon > 0$, Y contains an uncountable ε-discrete set D. Let $C = f^{-1}D)$, an uncountable subset of X. Because every subset of D is closed in Y, every subset of C is Borel in X. By Theorem 12 (6.2), C cannot be separable (else it would have to be countable); and we have a contradiction. Hence Y is separable. The rest of the theorem now follows from known results; for the convenience of the reader, we summarize the argument. Because Y has a countable base, f is generalized continuous, say of class α. Let Z denote the completion of Y, and consider the "graph" $I = \{(x, y) \mid y = f(x)\}$ in $X \times Z$. One shows ([14], p. 291) that I is Borel of multiplicative class α in $X \times Z$. Because the projection of I onto Y is a 1-1 continuous image of a *separable* absolute Borel set, Y is absolutely Borel. The same argument

(15) The requirement that f is "Borel measurable" means that, for each Borel set $E \subset Y$, $f^{-1}(E)$ is Borel in X.

shows that, for each Borel subset E of X, $f(E)$ is Borel in Y, and f is a generalized homeomorphism.

Remark. One could replace the hypothesis that X is *absolutely* Borel by the continuum hypothesis in proving that Y is separable, but not in the remainder of the theorem. To see this, let X be the graph of a non-Borel-measurable real function g defined on the real line, and let f be the projection sending $(y, g(y))$ to y. Then f is continuous and so Borel measurable. Nevertheless, f cannot be a generalized homeomorphism, since otherwise Theorem 16 would apply to f^{-1}, and X would be absolutely Borel, making g Borel measurable after all.

We remark also that Theorem 16 cannot be extended as it stands to non-separable spaces; for an arbitrary space is a 1-1 continuous image of a discrete space X.

7.7. Finally we observe that *if two absolutely Borel spaces of weights* $\leqslant \aleph_1$ *are Borel isomorphic, they are generalized homeomorphic.* For if the spaces are X and Y, and if one of them is separable, this follows from Theorem 16 (7.6). Hence we may assume that both X, Y have weight $\aleph_1$, and so each of them is generalized homeomorphic to one of the 4 types of space listed in Theorem 11 (5.5). If they are both generalized homeomorphic to the same space, there is nothing to prove. From the remarks at the end of 5.5, the only remaining case is that in which X (say) is generalized homeomorphic to the discrete union X_0 of $\aleph_1$ copies of $B(\aleph_0)$, while Y is generalized homeomorphic to $B(\aleph_1)$. In this case, $B(\aleph_1)$ must be Borel isomorphic to the locally separable space X_0, and therefore (by Theorem 15, 7.4) is generalized homeomorphic to it; and the result follows.

Note, however, that this argument does *not* show that every Borel isomorphism f between X and Y is *itself* a generalized homeomorphism. This would follow providing every Borel isomorphism of $B(\aleph_1)$ onto itself is a generalized homeomorphism; but this problem also remains open.

8. k-ANALYTIC SETS

8.1. We conclude by sketching a generalization of the theory of analytic sets. Several such generalizations have been given already (see e. g. [3, 4, 18, 19]); that adopted here is quite different from Choquet's "$\mathscr{K}$-analytic" [3, 4] and from Menger's "k-analytic" [19], but has features in common with a definition due to Maximoff [18] and closely resembles Menger's "k-separable" [19]. It has the advantage that it fits in well with the theory of Borel sets developed above; against this is the drawback that, as we shall see, the present theory is fully significant only if $k^{\aleph_0} > k$. When $k = \aleph_0$, we obtain the classical notion of analytic set.

Let k be a given infinite cardinal, X any topological space (later to be metric) and $\mathscr{M}$ any family of subsets of X (later to be the closed sets) containing $\emptyset$ and X. We recall that $B(k) = \prod T_n$ $(n = 1, 2, \ldots)$ where each T_n is discrete and of cardinal k (cf. 2.1). For each finite sequence $t_1, \ldots, t_n$ $(t_i \in T_i)$ suppose a set $F(t_1, \ldots, t_n) \in \mathscr{M}$ is given, and write, for each $t = (t_1, t_2, \ldots) \in B(k)$.

$H(t) = F(t_1) \cap F(t_1, t_2) \cap \ldots \cap F(t_1, \ldots, t_n) \cap \ldots$ Then $A = \bigcup\{H(t) | t \in B(k)\}$ is said to be obtained from $\mathscr{M}$ by the "operation kA", and the family of all such sets A is written $kA(\mathscr{M})$. When $\mathscr{M}$ consists of the closed sets of X, $kA(\mathscr{M})$ is defined to be the family of *k-analytic* subsets of X. In any case, the following properties are immediate:

(1) $kA(\mathscr{M}) \supset \mathscr{M}$.

(2) The intersection of every countable family of sets of $\mathscr{M}$ is in $kA(\mathscr{M})$.

(3) The union of every family of $\leqslant k$ sets of $\mathscr{M}$ is in $kA(\mathscr{M})$.

THEOREM 17. $kA(kA(\mathscr{M})) = kA(\mathscr{M})$.

The proof is a fairly straightforward generalization of the proof in the classical case $k = \aleph_0$; see e.g., [5 p. 92], or [18].

COROLLARY. *The intersection of every countable family of sets of $kA(\mathscr{M})$, and the union of every family of at most k sets of $kA(\mathscr{M})$, are in $kA(\mathscr{M})$.*

For they are in $kA(kA(\mathscr{M}))$, by (2) and (3).

8.2. Throughout what follows, $\mathscr{M}$ will be the family of closed sets in a *metric* space X; thus $kA(\mathscr{M})$ is the family of k-analytic subsets of X [16]. From the preceding, every Borel set is k-analytic, and if $\mathscr{M}$ is replaced by the family of open, or Borel, or k-analytic sets, $kA(\mathscr{M})$ is still the family of k-analytic sets. Of course, every 1-point set is k-analytic; hence

(4) every set of $\leqslant k$ points is k-analytic.

If Y is a subspace of X, the closed sets in Y are just the sets $Y \cap M$ where M is closed in X. Hence

(5) the k-analytic subsets of Y are the sets $Y \cap A$, where A is k-analytic in X.

It follows at once that

(6) If Z is a k-analytic subset of Y, and Y is a k-analytic subset of X, then Z is a k-analytic subset of X.

We mention the obvious facts:

(7) If $\aleph_0 \leqslant k < k'$, every k-analytic subset of X is also k'-analytic. "$\aleph_0$-analytic" coincides with "analytic" in the usual sense.

We have also the following generalization of Montgomery's theorem (see [20]):

THEOREM 18. *If $Y \subset X$ is locally k-analytic, it is k-analytic.*

By hypothesis, Y can be covered by open sets U_a of X such that $Y \cap U_a$ is k-analytic (in X). Put $G = \bigcup U_a$; it is enough to prove that Y is k-analytic in G, and we may therefore assume $X = G$. The covering $\{U_a\}$ of X has a σ-discrete open refinement $\{V_{n\lambda}\}$, $n = 1, 2, \ldots$, discrete for fixed n, such that each $\overline{V}_{n\lambda} \subset$ some U_a. Fix n till the end of the argument. Each $Y \cap \overline{V}_{n\lambda}$ is k-analytic, and so is expressible as $\bigcup_t F_\lambda(t_1) \cap F_\lambda(t_1, t_2) \cap \ldots$, the F's being closed and the union being taken over all $t = (t_1, t_2, \ldots)$ in $B(k)$. Put $E_\lambda(t_1, \ldots, t_m) = F_\lambda(t_1, \ldots$ $\ldots, t_m) \cap \overline{V}_{n\lambda}$, $D(t_1, \ldots, t_m) = \bigcup_\lambda E_\lambda(t_1, \ldots, t_m)$; these sets are all closed, because $\{V_{n\lambda}\}$ is discrete. Let $Y_n = Y \cap \bigcup_\lambda \overline{V}_{n\lambda}$. Using the fact that the sets $\overline{V}_{n\lambda}$ are disjoint, one easily verifies that $Y_n = \bigcup_t D(t_1) \cap D(t_1, t_2) \cap \ldots$ Thus Y_n is k-analytic; hence so is $\bigcup Y_n = Y$.

8.3. Absolutely k-analytic sets.

THEOREM 19. *Every continuous image of $B(k)$ in any (metric) space X is k-analytic in X. Conversely, if X is complete, every non-empty k-analytic set in X of weight $\leqslant k$ is a continuous image of $B(k)$.*

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(16) For most of the results in this section, it would suffice that X, instead of being metric, was perfectly normal, or more generally was a regular space in which every open set is the union of k or fewer closed sets.

Let $f: B(k) \to X$ be continuous. For each $t = (t_1, t_2, \ldots) \in B(k)$, we define (using the notation of 2.1) $F(t_1, \ldots, t_n) =$ closure of $f\big(V(t_1, \ldots, \ldots, t_n)\big)$ and $H(t) = F(t_1) \cap F(t_1, t_2) \cap \ldots$ One easily sees that $H(t)$ is the single point $f(t)$; hence $f\big(B(k)\big) = \bigcup H(t)$, and is k-analytic.

Conversely, let A be a non-empty k-analytic subset of X, of weight $\leqslant k$. By replacing X by $\bar{A}$, we may assume that X has weight $\leqslant k$. Say $A = \bigcup \{H(t) \mid t \in B(k)\}$ where $H(t) = F(t_1) \cap F(t_1, t_2) \cap \ldots$ By intersecting each $F(t_1, \ldots, t_n)$ here with the members of a covering of X by $\leqslant k$ closed sets of diameters $< 1/n$ (cf. 3.3 (1)), we write

$$F(t_1, \ldots, t_n) = \bigcup E(t_1, \ldots, t_n, s_n),$$

where s_n runs over a discrete set S_n of cardinal k and $E(t_1, \ldots, t_n, s_n)$ is closed and of diameter $< 1/n$. Put

$$D(t_1, \ldots, t_n, s_1, \ldots, s_n) = E(t_1, s_1) \cap \ldots \cap E(t_1, \ldots, t_n, s_n);$$

this is closed, and

$$H(t) = \bigcup D(t_1, s_1) \cap \ldots \cap D(t_1, \ldots, t_n, s_1, \ldots, s_n),$$

the union being taken over all $s = (s_1, \ldots, s_n, \ldots) \in \prod S_n$. Now the ordered pair $r = (s, t)$ also runs over a space homeomorphic to $B(k)$; hence, with an obvious change of notation, we may write $A = \bigcup \{K(r) \mid r \in \in B(k)\}$ where

$$K(r) = D(r_1) \cap \ldots \cap D(r_1, \ldots, r_n) \cap \ldots$$

Now $D(r_1, \ldots, r_n)$ is closed and of diameter $< 1/n$; further $D(r_1) \supset \supset D(r_1, r_2) \supset \ldots$ Hence $K(r)$ consists of at most one point.

Let $C = \{r \mid r \in B(k),\ K(r) \neq \emptyset\}$. The map $r \to K(r)$ maps C onto A; and it is uniformly continuous because if $r' \in V(r_1, \ldots, r_n)$ then both $K(r')$ and $K(r)$ are contained in $D(r_1, \ldots, r_n)$, of diameter $< 1/n$. Now X is complete; hence if $K(r) = \emptyset$ we have $D(r_1, \ldots, r_n) = \emptyset$ for some n, and $B(k) - C$ is therefore open. That is, C is closed. Also $C \neq \emptyset$ (else $A = \emptyset$). By Theorem 3 (2.5), C is a retract of $B(k)$, and A is therefore a continuous image of $B(k)$.

THEOREM 20. *If Y is k-analytic in some one complete (metric) space, it is k-analytic in every (metric) space containing Y topologically.*

Suppose Y is k-analytic in the complete space X, and is homeomorphic to $Y_1 \subset X_1$. By Lavrentieff's theorem (see [14, p. 335]), the homeomorphism is extensible to one between two G_δ sets, say Z and Z_1, where $Y \subset Z \subset X$ and $Y_1 \subset Z_1 \subset \tilde{X}_1$; here $\tilde{X}_1$ denotes the completion of X_1. We have $Y = \bigcup_t F(t_1) \cap F(t_1, t_2) \cap \ldots$ where each F is closed in X; hence $Y = \bigcup_t E(t_1) \cap E(t_1, t_2) \cap \ldots$ where $E(t_1, \ldots, t_n) = Z \cap F(t_1, \ldots, t_n)$, a Borel subset of Z. On applying the homeomorphism we express $Y_1 = \bigcup_t E_1(t_1) \cap E_1(t_1, t_2) \cap \ldots$, where each E_1 is Borel in Z_1, and so also in $\tilde{X}_1$. Thus $Y_1 \epsilon kA(\mathscr{M})$, where $\mathscr{M}$ is the family of Borel subsets of $\tilde{X}_1$; but this means that Y_1 is k-analytic in $\tilde{X}_1$ (by Theorem 17, 8.1) and therefore also in X_1 (by 8.2(5)).

A space Y having the properties of Theorem 20 will be called *absolutely k-analytic*. Clearly the analogues of (6), (7) and Theorem 18 (8.2) apply to absolutely k-analytic spaces. Moreover, as an easy consequence of Theorem 19, we have:

THEOREM 21. *A necessary and sufficient condition that a space be a continuous image of $B(k)$, is that it be non-empty, of weight $\leqslant k$, and absolutely k-analytic.*

COROLLARY. *If X is absolutely k-analytic and of weight $\leqslant k$, then so is every continuous image of X.*

Remark. If, in this corollary, we drop the restriction that X be of weight $\leqslant k$, then a continuous image of X need no longer be absolutely k-analytic. For example, let Y be a non-analytic subset of the real line, and let X be a discrete set of $\|Y\|$ points; Y is a continuous image of X, and X is absolutely Borel (being complete) and therefore absolutely analytic. This gives a counterexample when $k = \aleph_0$; there is no difficulty in finding similar examples with larger values of k.

8.4. The next theorems extend Theorems 6, 7 and 7′ (4.3-4.5) to k-analytic sets.

THEOREM 22. *If is an absolutely k-analytic set of weight $\leqslant k$, and if $\|X\| > k$, then $\|X\| = k^{\aleph_0}$, and X contains a closed subset homeomorphic to a Baire space ($B(p)$ or $C(\aleph_0)$) of cardinal $k^{\aleph_0}$.*

For, by Theorem 21, X is a continuous image of $B(k)$; we now apply Theorem 5 (4.1). Like Theorem 6, Theorem 22 says nothing unless $k^{\aleph_0} > k$.

THEOREM 23. *Let X be any (metric) space of cardinal m and infinite weight k. Then X has exactly 2^k k-analytic subsets. Further, the number of k-analytic subsets of cardinal n of X is* (i) m^n *if* $n \leqslant k$, (ii) $2^k = m^k$ *if* $n = m$, (iii) 0 *or* 2^k *if* $k < n < m$, (iv) 0 *if* $k < n < m$ *and X is absolutely k-analytic.*

By Theorem 7 (4.4) it is enough to prove that X has no more k-analytic subsets than stated. By 8.2(5), each k-analytic subset of X is of the form $X \cap A$, where A is k-analytic in the completion $\tilde{X}$ of X. Thus the first assertion follows from Theorem 19 and the observation that (each map being determined by its effect on a dense set of k points) the number of continuous mappings of $B(k)$ in X is at most $(k^{\aleph_0})^k = k^k = 2^k$. The inequalities to be proved in cases (i)-(iv) follow by the same arguments as in Theorem 7.

Remark. In view of 8.2 (7), Theorem 23 applies a fortiori to the k'-analytic subsets of X, for each infinite $k' < k$. A similar remark applies to the next theorem.

THEOREM 23′. *Under the hypotheses of Theorem 23, X has exactly 2^k absolutely k-analytic subsets. Further, the number of absolutely k-analytic subsets of cardinal n of X is* (i) m^n *if* $n \leqslant k$, (ii) 0 *if* $k < n < k^{\aleph_0}$, (iii) 0 *or* 2^k *if* $n = k^{\aleph_0}$.

This is deduced from Theorem 7′ (4.5) in a similar way.

8.5. We remark that (from 8.2 (4)), if $k^{\aleph_0} = k$, then the k-analytic subsets of X of weight k are simply the subsets of X of cardinal k; and similarly the absolutely k-analytic sets of weight k are simply the metric spaces of k points. Thus the results of 8.3 and 8.4 are trivial unless $k < k^{\aleph_0}$—i.e. (on the generalized continuum hypothesis) unless k is sequential. We conclude with an analogue of Theorem 12 (6.2) which illustrates the same point.

THEOREM 24. *Assuming the generalized continuum hypothesis, the following statements about any (metric) space X of weight $\leqslant k$ are equivalent:*

(1) *Every subset of X is k-analytic (in X).*

(2) *Every subset of X is absolutely k-analytic.*

(3) $\|X\| \leqslant k$.

For (3) implies (2) from 8.2 (4); and (2) implies (1), trivially. If (1) is true, and $\|X\| = n$, then $2^n = 2^k$ from Theorem 23; hence $n = k$ on the generalized continuum hypothesis.

It would be interesting to know which spaces (not necessarily of weight $\leqslant k$) have the property that all their subsets are k-analytic; they include, of course, the spaces described in Theorem 12 (6.2).

8.6. The deeper properties of analytic sets in separable spaces (such as Souslin's theorem, that if a set and its complement are both analytic, they are Borel; cf. [14, p. 395]) do not generalize so readily; trivial counterexamples show that their extensions, if any, to k-analytic sets will require careful reformulation. It seems that some of the properties of k-analytic sets will relate, not to the Borel sets, but to the "k-hyperborel sets" (see [22]; a related idea is in [18]), defined as constituting the smallest family of sets containing all closed sets and closed under intersections of $\aleph_0$ and unions of k of them, and their duals.

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