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NON-SEPARABLE BOREL SETS, II *

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Abstract: For each infinite cardinal $\mathfrak{t}$, the property of being the union of countably many sets each locally of weight less than $\mathfrak{t}$ (in symbols, $\sigma LW(<\mathfrak{t})$) is shown to be preserved by analytic isomorphisms of absolutely analytic metric spaces. From this, necessary and sufficient topological conditions are obtained for two absolutely Borel metric spaces to be Borel isomorphic or generalized homeomorphic. Normal forms are found for the corresponding isomorphism classes; and the number of them, given the weight of the space, is determined.

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absolute Borel space	Borel isomorphism	generalized homeomorphism
Baire kernel	weight function	Borel type
analytic set	$\mathfrak{t}$-analytic set	analytic isomorphism
local weight	σ-locally of weight less than $\mathfrak{t}$	

1. Introduction

This paper is concerned with the classification and characterization of the absolute Borel metric spaces, not necessarily separable. "Classification" here has two possible interpretations: classification under Borel isomorphism, or under generalized homeomorphism. (These terms are defined in section 2.3.) It conveniently turns out that these two interpretations are in fact equivalent. The main result is Theorem 9 (4.5), giving necessary and sufficient conditions, in topological terms, for two absolute Borel spaces to be Borel isomorphic (or generalized homeomorphic). We also obtain (Theorem 10) a class of simple standard spaces such that every absolute Borel space is Borel isomorphic (and hence generalized

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homeomorphic) to exactly one of them. However, the problem of whether every Borel isomorphism is itself a generalized homeomorphism (the converse is trivial) remains open in general, though the answer is known to be affirmative in some special cases [10, §7]. Throughout, all spaces considered are assumed to be metrizable. They will usually be absolutely Borel too, but we shall occasionally have to consider non-Borel sets, so we do not impose this requirement without saying so.

The paper is a sequel to [10], to which we refer for notation and background. It was there shown that every absolute Borel space is generalized homeomorphic to one of a simple type, so that (roughly speaking) what remains is to prove that "different" spaces are *not* Borel isomorphic. That is, we must find topological properties which are invariant under Borel isomorphism. One such property is having a given weight [10, §4]. The *local* weight of a space is *not* an invariant in the present sense [11, 2.4], but we shall see that the property of being σ-locally of weight less than a prescribed cardinal $\mathfrak{k}$, which we denote for short by σLW($< \mathfrak{k}$), provides the essential invariant. The case $\mathfrak{k} = \aleph_0$, when σLW($< \mathfrak{k}$) reduces to σ-discreteness, was studied in [11], and the present paper is in part a generalization (though hardly a straightforward one) of [11].

The organization of the paper is as follows. We begin (in section 2) with some lemmas about σLW($< \mathfrak{k}$), Borel isomorphism and related notions. In section 3 we prove the invariance of the σLW($< \mathfrak{k}$) property (Theorem 1), using an argument which was kindly pointed out to me by R.W. Hansell. Not only is this argument much shorter than my original proof; it proves more – the σLW($< \mathfrak{k}$) property is invariant under analytic isomorphisms of absolutely analytic sets. Some further properties of σLW($< \mathfrak{k}$) absolute Borel spaces are then deduced; for instance (Theorem 2), for each infinite cardinal $\mathfrak{h}$, every absolute Borel space is either σLW($< \mathfrak{h}$) or contains a subset homeomorphic to the Baire space $\mathcal{B}(\mathfrak{h})$ (and these possibilities are mutually exclusive). In section 4 we introduce the notion of the *weight function* of a space, and use it to complete the classification of absolute Borel spaces under Borel isomorphism and generalized homeomorphism (Theorem 9). Finally, in section 5, we provide *normal forms* for absolute Borel spaces under Borel isomorphism, and determine how many different types there are of a given weight.

The axiom of choice is assumed throughout, but not the continuum hypothesis. We repeat that all spaces considered are to be metric, though not necessarily absolutely Borel. Throughout, $\mathfrak{k}$ denotes an infinite cardinal number.

2. Some lemmas

Throughout this section, X denotes an arbitrary metrizable space, and ρ denotes a metric on X (agreeing with the topology).

2.1. *Weight, local weight, σ-local weight*. We recall that the *weight* of X, denoted here by $w(X)$, is the least cardinal of a base of open sets for X; since X is metric this is also the least cardinal of a dense subset of X. Thus:

(1). If $E \subset X$, then $w(E) = w(\bar{E}) \leq w(X)$.
(2). If $f : X \to Y$ is continuous, then $w(f(X)) \leq w(X)$.

We say that X is *locally of weight* $< \mathfrak{k}$ (in symbols, $\mathrm{LW}(< \mathfrak{k})$) provided each $x \in X$ has a neighborhood $U(x)$ of weight $< \mathfrak{k}$. From (1), we may assume that $U(x)$ is open. If $\epsilon > 0$ is such that each spherical neighborhood $S(x, \epsilon)$ $(= \{y \in X \mid \rho(x, y) < \epsilon\})$ has weight $< \mathfrak{k}$, X is said to be ϵ-$\mathrm{LW}(< \mathfrak{k})$ (with respect to ρ). We say X is *uniformly* $\mathrm{LW}(< \mathfrak{k})$ if it is ϵ-$\mathrm{LW}(< \mathfrak{k})$ for some $\epsilon > 0$.

Clearly X is $\mathrm{LW}(< \aleph_0)$ if and only if it is discrete, and $\mathrm{LW}(< \aleph_1)$ if and only if it is locally separable.

We say that X is *σ-locally of weight* $< \mathfrak{k}$ (in symbols, $\sigma\mathrm{LW}(< \mathfrak{k})$) provided X can be expressed as $\bigcup\{X_n \mid n = 1, 2, ...\}$, where each X_n (in its relative topology) is $\mathrm{LW}(< \mathfrak{k})$. Clearly

(3). If $E \subset X$ and X is $\sigma\mathrm{LW}(< \mathfrak{k})$, then so is E.

(However, if E is $\sigma\mathrm{LW}(< \mathfrak{k})$, this need not be true of $\bar{E}$.)

(4). If $X = \bigcup\{X_n \mid n = 1, 2, ...\}$, where each X_n is $\sigma\mathrm{LW}(< \mathfrak{k})$, then X is $\sigma\mathrm{LW}(< \mathfrak{k})$.

Now we show:

(5). X is $\sigma\mathrm{LW}(< \mathfrak{k})$ if and only if $X = \bigcup\{A_n \mid n = 1, 2, ...\}$ where each A_n is closed (in X) and is uniformly $\mathrm{LW}(< \mathfrak{k})$ (with respect to ρ).

Proof. To prove the non-trivial implication, suppose X is $\sigma\mathrm{LW}(< \mathfrak{k})$; then $X = \bigcup_n X_n$ where X_n is $\mathrm{LW}(< \mathfrak{k})$. Each $x \in X_n$ has an open neighborhood $U_n(x)$ (relative to X_n) of weight $< \mathfrak{k}$, and the cover

$\{U_n(x) \mid x \in X_n\}$ of X_n has a σ-discrete (relatively) open refinement $\{V_{nm\lambda} \mid m = 1, 2, ..., \lambda \in \Lambda_{nm}\}$, each of the families $\{V_{nm\lambda} \mid \lambda \in \Lambda_{nm}\}$ being discrete in X_n. Since each $V_{nm\lambda}$ is contained in some $U_n(x)$, it has weight $< \mathfrak{k}$. For $n, m, p = 1, 2, ...$ and $\lambda \in \Lambda_{nm}$, let $E_{nmp\lambda} = \{x \in X_n \mid X_n \cap S(x, 1/p) \subset V_{nm\lambda}\}$; thus $E_{nmp\lambda} \subset V_{nm\lambda}$ and has weight $< \mathfrak{k}$. It is easy to check that, if $\lambda \neq \mu$, then $\rho(E_{nmp\lambda}, E_{nmp\mu}) \geq 1/p$. Hence, if we write $F_{nmp\lambda} = \overline{E}_{nmp\lambda}$ (the bar denoting closure in X), we have also $w(F_{nmp\lambda}) < \mathfrak{k}$ and (if $\lambda \neq \mu$) $\rho(F_{nmp\lambda}, F_{nmp\mu}) \geq 1/p$. Now put $A_{nmp} = \bigcup\{F_{nmp\lambda} \mid \lambda \in \Lambda_{nm}\}$; then A_{nmp} is closed and uniformly LW($< \mathfrak{k}$) (we can take $\epsilon = 1/p$). All that remains is to renumber the sets A_{nmp} into a single sequence.

As a corollary to the proof of (5), we have:

(6). Each σLW($< \mathfrak{k}$) subset of X is contained in a countable union of closed LW($< \mathfrak{k}$) subsets of X (and hence in an F_σ subset of X which is also σLW($< \mathfrak{k}$)).

We recall that $B(\mathfrak{k})$ is the product space $T^{\aleph_0}$, where T is a discrete set of cardinal $\mathfrak{k}$.

(7). $B(\mathfrak{k})$ is not σLW($< \mathfrak{k}$).
Proof. This follows from Baire's theorem, in view of (5) and the fact that each non-empty open subset of $B(\mathfrak{k})$ has weight $\mathfrak{k}$.

(8) If X is locally σLW($< \mathfrak{k}$), it is σLW($< \mathfrak{k}$).
Proof. The hypothesis here means that each point of X has some σLW($< \mathfrak{k}$) neighborhood, and hence has arbitrarily small open ones. The result follows from [7, Theorem 3.6(a)]; alternatively it can be proved by an argument similar to that proving (5).

As an immediate corollary, we have:

(9). If X is σ-locally σLW($< \mathfrak{k}$), it is σLW($< \mathfrak{k}$).

2.2. *Kernels.* In this section we suppose that a metric space X and an infinite cardinal $\mathfrak{k}$ are given.

(1). There is a closed subset $K(\mathfrak{k})$ of X such that

(a) every nonempty relatively open subset of $K(\mathfrak{k})$ has weight $\geq \mathfrak{k}$,

(b) $K(\mathfrak{k})$ is the largest subset of X for which (a) is true,

(c) $X \backslash K(\mathfrak{k})$ is σLW($< \mathfrak{k}$).

Proof. This follows from [9, Theorems 1, 4] applied to the completely hereditary property *having weight* $< \mathfrak{k}$.

We call $K(\mathfrak{k})$ the *nowhere* LW($< \mathfrak{k}$) *kernel* of X.

The following consequence of (1) will be required later (and considerably generalized later still, in Theorem 2).

(2). If A is a complete metric space, then either A is σLW($< \mathfrak{k}$) or A contains a homeomorph of $B(\mathfrak{k})$.

Proof. Apply (1) to the space A, obtaining its nowhere LW($< \mathfrak{k}$) kernel K. If $K = \emptyset$, (c) shows that X is σLW($< \mathfrak{k}$). If $K \neq \emptyset$, then K is a complete metric space having property (a), and it follows [11, Corollary 4.3] that K contains a homeomorph of $B(\mathfrak{k})$.

Remark. By 2.1(7), the alternatives in (2) are mutually exclusive.

(3). Let $H(\mathfrak{k})$ be the union of all subsets Y of X which are homeomorphic to $B(\mathfrak{k})$. Then $H(\mathfrak{k})$ is closed, and $X \backslash H(\mathfrak{k})$ contains no homeomorph of $B(\mathfrak{k})$.

Proof. Suppose $a \in \overline{H(\mathfrak{k})}$; we show that $a \in H(\mathfrak{k})$. Suppose not. Put $\delta_1 = 1$ and pick $b_1 \in S(a, \delta_1) \cap H(\mathfrak{k})$ (thus $b_1 \neq a$). Then $b_1 \in$ some Y_1 homeomorphic to $B(\mathfrak{k})$. Each point of $B(\mathfrak{k})$ has arbitrarily small open-closed neighborhoods which are homeomorphic to $B(\mathfrak{k})$ itself (fix the first n coordinates for suitably large n), so b_1 has a neighborhood Z_1 in Y_1, homeomorphic to $B(\mathfrak{k})$, and so small that $Z_1 \subset S(a, \delta_1)$ and $\rho(a, Z_1) > 0$. Take $\delta_2 = \min\{\rho(a, Z_1), 1\}/2$; pick $b_2 \in S(a, \delta_2) \cap H(\mathfrak{k})$ and use it to obtain $Z_2 \subset S(a, \delta_2)$ such that Z_2 is homeomorphic to $B(\mathfrak{k})$ and $\rho(a, Z_2) > 0$. Proceeding in this way we obtain $Z_3, Z_4, \ldots$ each at positive distance from the others, each homeomorphic to $B(\mathfrak{k})$, and such that $Z_n \subset S(a, 1/n)$ and $\rho(a, Z_n) > 0$. Put $Q = \{a\} \cup \bigcup\{Z_n \mid n = 1, 2, \ldots\}$; we show that Q is homeomorphic to $B(\mathfrak{k})$. Since Z_n is homeomorphic to $B(\mathfrak{k})$, it has a complete metric ρ_n of diameter $< n^{-2}$. Define a metric ρ' on Q as follows:

$$\begin{aligned}
\rho'(x,y) &= \rho_n(x,y) \text{ if } x, y \text{ both} \in Z_n;\\
\rho'(x,y) &= \rho_n(x,b_n) + |n^{-1} - m^{-1}| + \rho_m(y,b_m)\\
&\qquad\qquad\qquad\qquad \text{if } x \in Z_n \text{ and } y \in Z_m \text{ with } m \neq n;\\
\rho'(a,x) &= \rho_n(x,b_n) + n^{-1} \quad \text{if } x \in Z_n;\\
\rho'(a,a) &= 0\,.
\end{aligned}$$

It is easily verified that ρ' is a metric on Q, agreeing with the topology of Q (as a subspace of X), and that ρ' is complete. Clearly Q is 0-dimensional (in the covering sense) and of weight $\mathfrak{k}$, and each non-empty open subset of Q contains a discrete subset of cardinal $\mathfrak{k}$. Thus [10, Theorem 1] Q is homeomorphic to $B(\mathfrak{k})$. But then $a \in Q \subset H(\mathfrak{k})$, giving a contradiction. Thus $H(\mathfrak{k})$ is closed; and the remaining assertion is trivial.

We shall refer to $H(\mathfrak{k})$ as the *Baire kernel* of X, of order $\mathfrak{k}$.

(4). $H(\mathfrak{k}) \subset K(\mathfrak{k})$. If X has a complete metric, then $H(\mathfrak{k}) = K(\mathfrak{k})$.
Proof. Suppose $x \in H(\mathfrak{k}) \setminus K(\mathfrak{k})$; then x belongs to some homeomorph Y of $B(\mathfrak{k})$. The neighborhood $(X \setminus K(\mathfrak{k})) \cap Y$ of x in Y then contains a homeomorph Z of $B(\mathfrak{k})$. Because of (1), Z must be σLW($< \mathfrak{k}$), contradicting 2.1(7). Thus $H(\mathfrak{k}) \subset K(\mathfrak{k})$.

Now suppose X has a complete metric. If $K(\mathfrak{k}) \setminus H(\mathfrak{k})$ is not empty, the argument used to prove (2) above now shows that $K(\mathfrak{k}) \setminus H(\mathfrak{k})$ contains a homeomorph of $B(\mathfrak{k})$, in contradiction to (3).

Remark. Without the assumption that X is complete, the kernels $H(\mathfrak{k})$ and $K(\mathfrak{k})$ can be different, even for "nice" spaces X. For instance, let X be the subset of $B(\mathfrak{k}) = T^{\aleph_0}$ defined as follows: pick $t_0 \in T$ and let X consist of all points $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that $x_n = t_0$ for all but finitely many values of n. Then every neighborhood in X has weight exactly $\mathfrak{k}$, so $K(\mathfrak{k}) = X$; but X is σ-discrete, so $H(\mathfrak{k}) = \emptyset$.

We shall need to consider yet a third kernel. Let $L(\mathfrak{k})$ be the kernel produced just as in (1) but by applying [9, Theorems 1 and 4] to the completely hereditary property of being σLW($< \mathfrak{k}$). That is, $L(\mathfrak{k})$ is the "nowhere σLW($< \mathfrak{k}$) kernel" of X. Thus $L(\mathfrak{k})$ is closed; no relatively open subset of $L(\mathfrak{k})$, other than $\emptyset$, is σLW($< \mathfrak{k}$); and (from 2.1(9))

(5) $X \setminus L(\mathfrak{k})$ is the largest open subset of X which is σLW($< \mathfrak{k}$).

It follows easily that

(6) $H(\mathfrak{t}) \subset L(\mathfrak{t}) \subset K(\mathfrak{t})$.

Thus, from (4),

(7) if X has a complete metric, then $H(\mathfrak{t}) = L(\mathfrak{t}) = K(\mathfrak{t})$.

We shall later see, as a corollary to Theorem 5 (3.8), that $L(\mathfrak{t}) = H(\mathfrak{t})$ whenever X is absolutely Borel.

2.3. *Borel isomorphism and generalized homeomorphism.* We recall that a 1-1 map f of a space X onto a space Y is a *Borel isomorphism* [6] if both f and f^{-1} take Borel sets to Borel sets; it is a *generalized homeomorphism* of class (α, β) (where α, β are countable ordinals) if, for all open subsets U of X and V of Y, $f^{-1}(V)$ is Borel of additive class α in X and $f(U)$ is Borel of additive class β in Y. As was mentioned in the introduction, every generalized homeomorphism is a Borel isomorphism, but the converse is an open question in general.

We use the notation $\Sigma\{X_\lambda \mid \lambda \in \Lambda\}$, or $\Sigma_\lambda X_\lambda$ for short, to denote the *discrete union* (topological sum, coproduct) of the family of spaces $\{X_\lambda \mid \lambda \in \Lambda\}$; and $\mathfrak{t}X$ denotes the discrete union of $\mathfrak{t}$ copies of the space X. The following observations will be useful later.

(1). The spaces $B(\mathfrak{t})$, $\mathfrak{t}B(\mathfrak{t})$, $\Sigma\{B(\mathfrak{h}) \mid \mathfrak{h} \leq \mathfrak{t}\}$, $\Sigma\{\mathfrak{t}B(\mathfrak{h}) \mid \mathfrak{h} \leq \mathfrak{t}\}$, are all generalized homeomorphic to each other, under maps of class $(1, 1)$.

This follows from the fact that each is homeomorphic to a closed subset of each other, plus the "Schröder-Bernstein principle" [10, Theorem 9].

(2). If $X = \bigcup\{A_n \mid n = 1, 2, \ldots\}$, where the sets A_n are Borel (in X) and pairwise disjoint, then X is generalized homeomorphic to $\Sigma_n A_n$.

(3). If A_n is generalized homeomorphic to B_n $(n = 1, 2, \ldots)$, then $\Sigma_n A_n$ is generalized homeomorphic to $\Sigma_n B_n$.

(4). If, for each λ in an arbitrary index set Λ, A_λ is generalized homeomorphic to B_λ by a map of fixed class (α, β) (independent of λ), then $\Sigma_\lambda A_\lambda$ is generalized homeomorphic to $\Sigma_\lambda B_\lambda$.

The analogous results for Borel isomorphism hold for (2) and (3), but not for (4).

2.4. *Analytic sets and isomorphisms.* We recall that a subset A of a space X is *Souslin* or *analytic* in X (called $\aleph_0$-*analytic* in [10]) provided there exists a map assigning to each finite sequence $(n_1, n_2, ..., n_p)$ of positive integers a closed set $A_{n_1 n_2 ... n_p}$ in X such that

$$(1) \qquad A = \bigcup_{\xi} \bigcap_{p=1}^{\infty} A_{\xi|p} ,$$

where ξ runs over the set of all infinite sequences $(n_1, n_2, ...)$ of positive integers, and $\xi|p$ consists of the first p terms of ξ. As is well known, if A is Borel in X it is analytic in X. Further, if A is analytic in some complete metric space it is analytic in every (metric) space containing A; such sets A are said to be *absolutely analytic*. (For details, we refer to [5].)

A map $f: X \to Y$ such that, for every open set V in $Y, f^{-1}(V)$ is analytic in X, will be said to be *analytically measurable*. Clearly every Borel measurable map is analytically measurable; and if $f: X \to Y$ is analytically measurable and B is analytic in Y, it is not hard to see that $f^{-1}(B)$ is analytic in X. A 1-1 map f of X onto Y such that both f and f^{-1} are analytically measurable will be called an *analytic isomorphism*.

We shall need the following refinement of [10, Theorem 8]:

(2). If f is an analytically measurable 1-1 (or countable-to-one) map of an absolutely analytic space X onto an arbitrary (metric) space Y, then $w(Y) \leq w(X)$.

Proof. To prove this, write $w(X) = \mathfrak{k}$, $w(Y) = \mathfrak{k}'$ and $\|X\| = \|Y\| = \mathfrak{m}$ (where $\|X\|$ denotes the cardinal of X). We may assume both $\mathfrak{k}$ and $\mathfrak{k}'$ to be infinite. Suppose $\mathfrak{k}' > \mathfrak{k}$. From [8], Y has a Borel (in fact closed) subset B with $\|B\| = \mathfrak{k}'$; then $f^{-1}(B)$ is an analytic subset of X of cardinal $\mathfrak{k}' > \mathfrak{k}$. But X, being absolutely $\aleph_0$-analytic, is a fortiori absolutely $\mathfrak{k}$-analytic [10, 8.2(7)]; hence [10, Theorem 23] $\mathfrak{k}' = \mathfrak{m}$. Thus $\mathfrak{m} > \mathfrak{k}$, so [10, Theorem 22] $\mathfrak{m} = \mathfrak{k}^{\aleph_0}$. This proves that $\mathfrak{k}' = \mathfrak{k}^{\aleph_0}$, so that $\mathfrak{k}'$ is not sequential. Thus, from [8] again, Y has a discrete subset F of cardinal $\mathfrak{k}'$. Hence $f^{-1}(F)$ is an analytic subset of X, every subset of which is analytic in X. But, from [10, 8.2(7) and 8.2(6)], $f^{-1}(F)$ is an absolutely

$\mathfrak{t}$-analytic set of weight $\leq \mathfrak{t}$ and cardinal $> \mathfrak{t}$; so [10, Theorem 22] it contains a subset homeomorphic to the Cantor set; and this in turn contains a non-analytic set, giving a contradiction.

In particular,

(3) if X and X are Borel isomorphic (or analytically isomorphic) absolutely analytic spaces, $w(X) = w(Y)$.

The key lemma for what follows is the following theorem of Hansell [3, Theorem 4]:

(4). If X is absolutely analytic, and $A_\lambda (\lambda \in \Lambda)$ are pairwise disjoint subsets of X such that every union of them is analytic in X, then the family $\{A_\lambda \mid \lambda \in \Lambda\}$ is *σ-discretely decomposable*; that is, there exist sets $A_{\lambda n} (\lambda \in \Lambda, n = 1, 2, \ldots)$ such that
(i) each $A_{\lambda n}$ is closed in A_λ,
(ii) for each $\lambda \in \Lambda$, $\bigcup_n A_{\lambda n} = A_\lambda$,
(iii) for each $n = 1, 2, \ldots$, there exists $\epsilon_n > 0$ such that if $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$ then $\rho(A_{\lambda n}, A_{\mu n}) > \epsilon_n$.

3. Invariance of the σLW($< \mathfrak{t}$) property

3.1. The main result in this section is:

Theorem 1. If X and Y are absolute Borel sets which are Borel isomorphic, and if X is σLW($< \mathfrak{t}$), then so is Y.

My original proof of Theorem 1 was long and complicated; I am grateful to R.W. Hansell for the following short argument, which proves the following stronger theorem:

Theorem 1′. If X and Y are absolutely analytic spaces which are analytically isomorphic, and if X is σLW($< \mathfrak{t}$), then so is Y.
Proof. Suppose f is an analytic isomorphism of X onto Y. By 2.1(5) we have $X = \bigcup \{X_n \mid n = 1, 2, \ldots\}$ where X_n is closed and LW($< \mathfrak{t}$). Thus X_n can be covered by open sets (relative to X_n) each of weight $< \mathfrak{t}$; take a σ-discrete (relatively) open refinement of this covering, say $\{U_{n\lambda m} \mid \lambda \in$

Λ_{nm}, $m = 1, 2, \ldots\}$ where for fixed n and m the system $\{U_{n\lambda m} \mid \lambda \in \Lambda_{nm}\}$ is discrete. By construction, each $U_{n\lambda m}$ has weight $< \mathfrak{k}$. Now $U_{n\lambda m}$ (being open relative to the closed set X_n) is absolutely analytic; hence (by 2.4(2)) $w(f(U_{n\lambda m})) \leq w(U_{n\lambda m}) < \mathfrak{k}$.

Again, for fixed n and m, every union of $U_{n\lambda m}$'s is open in X_n and so Borel in X. Hence every union of the form $\bigcup\{f(U_{n\lambda m}) \mid \lambda \in \Lambda'\}$ $(\Lambda' \subset \Lambda)$ is analytic in Y. By Hansell's theorem (2.4(4)) we may therefore write $f(U_{n\lambda m}) = \bigcup\{V_{n\lambda mp} \mid p = 1, 2, \ldots\}$ where (for fixed n, m, p) the sets $V_{n\lambda mp}$ $(\lambda \in \Lambda)$ form a discrete system in Y. Put $Y_{nmp} = \bigcup\{V_{n\lambda mp} \mid \lambda \in \Lambda\}$. Since $V_{n\lambda mp}$ is open in Y_{nmp}, and of weight $< \mathfrak{k}$ (being a subset of $f(U_{n\lambda m})$), Y_{nmp} is LW($< \mathfrak{k}$). Hence $Y = \bigcup_{n,m,p} Y_{nmp}$ is σLW($< \mathfrak{k}$), as required.

3.2. Corollary. *$B(\mathfrak{k})$ is not Borel isomorphic to any σLW($< \mathfrak{k}$) absolute Borel (or absolutely analytic) space.*

From 2.1(7).

3.3. Remark. It would be interesting to know to what extent Theorems 1 and 1′ apply to spaces X, Y which need not be absolutely Borel (or analytic). Some restrictions are necessary, for it is reportedly consistent with standard set theory that there exists an uncountable subset Y of the real line $\mathbf{R}$ such that every subset of Y is F_σ in Y. Then if X is a discrete space with $\|X\| = \|Y\|$, X and Y are Borel isomorphic (even generalized homeomorphic), and X is absolutely Borel; but X is σLW($< \aleph_0$) and Y is not.

In this direction we have the following results.

(1). *If $f : X \to Y$ is a Borel (or analytic) isomorphism of X onto Y, where X and Y are absolutely analytic, and Z is an arbitrary subset of Y, then if Z is σLW($< \mathfrak{k}$), so is $f^{-1}(Z)$.*

Proof. By 2.1(6) there is an F_σ subset Z' of Y which contains Z and is also σLW($< \mathfrak{k}$). Applying Theorem 1′ to Z' and $f^{-1}(Z')$ we see that $f^{-1}(Z')$ is σLW($< \mathfrak{k}$); hence (by 2.1(3)) so is $f^{-1}(Z)$.

(2). *If $f : X \to Y$ is a continuous analytic isomorphism of X onto Y, where Y is absolutely analytic, and $Z \subset Y$, then if Z is σLW($< \mathfrak{k}$), so is $f^{-1}(Z)$.*

Proof. Again by 2.1(6) we may assume Z is closed (in Y) and LW($< \mathfrak{k}$).

For each $x \in f^{-1}(Z)$, $f(x)$ has an open neighborhood V in Y such that $w(V \cap Z) < \mathfrak{k}$. Then $f^{-1}(V \cap Z) = U$, say, is a neighborhood of x in $f^{-1}(Z)$, and the restriction $f \mid U$ is an analytic isomorphism of U onto the absolutely analytic set $V \cap Z$. By 2.4(2), $w(U) \leq w(V \cap Z) < \mathfrak{k}$; thus $f^{-1}(Z)$ is LW($< \mathfrak{k}$), and the result follows.

When $\mathfrak{k} = \aleph_0$, (2) holds without the requirement that f be an analytic isomorphism; it suffices that f be continuous and 1-1 [11, Lemma 3]. But in general the assumption on f cannot be weakened in this way. For example, take $X = B(\mathfrak{c}) = T^{\aleph_0}$ where T is a discrete set of cardinal $\mathfrak{c}$, and let g be any 1-1 map of T onto **R**. Then $f = g^{\aleph_0}$ is a continuous bijection of X onto $Y = \mathbf{R}^{\aleph_0}$. Now both X and Y are absolutely Borel, and Y is σLW($< \mathfrak{c}$), but X is not.

3.4. Theorem 2. If Y is an absolute Borel space, and $\mathfrak{k}$ is any infinite cardinal, then either Y is σLW($< \mathfrak{k}$) or Y contains a subset homeomorphic to $B(\mathfrak{k})$; and these alternatives are mutually exclusive.

Proof. Suppose $w(Y) = \mathfrak{h}$. If $\mathfrak{h} < \mathfrak{k}$ then Y is trivially σLW($< \mathfrak{k}$), so assume $\mathfrak{h} \geq \mathfrak{k}$. By [10, Theorem 4] there exists a continuous Borel isomorphism f mapping A onto Y, where A is some closed subset of $B(\mathfrak{h})$. If A is σLW($< \mathfrak{k}$), then so is Y, by Theorem 1. And if A is not σLW($< \mathfrak{k}$), then by 2.2(2) A contains a set C homeomorphic to $B(\mathfrak{k})$ (hence necessarily a G_δ set in A). By [11, Theorem 4], $f(C)$ contains a subspace homeomorphic to $B(\mathfrak{k})$. Thus at least one of the stated alternatives applies; and they cannot both apply, by Corollary 3.2.

Remark. It can be shown that Theorem 2 continues to hold when Y is merely required to be absolutely analytic instead of absolutely Borel. (When $\mathfrak{k} = \aleph_0$ this is a theorem of El'kin [1], and his method can be adapted to general $\mathfrak{k}$.) One might conjecture that, still more generally, Theorem 2 might hold whenever Y is merely absolutely $\mathfrak{k}$-analytic (in the sense of [10, §8]); but this is false. The set E constructed in [11, §5] is easily seen to be absolutely $\aleph_1$-analytic and not to contain any homeomorph of $B(\aleph_1)$ (or of $B(\aleph_0)$ either); and it can be shown that E is not σLW($< \aleph_1$). A valid extension of the theorem to the case when Y is absolutely $\mathfrak{k}$-analytic is due to Freiwald [2, Theorem 4.18]: if $\mathfrak{k}^{\aleph_0} > \mathfrak{k}$, then either Y contains a subset homeomorphic to $B(\mathfrak{p})$ for some cardinal $\mathfrak{p}$ such that $\mathfrak{p}^{\aleph_0} = \mathfrak{k}^{\aleph_0}$, or Y is the union of $\mathfrak{k}$ discrete subsets, and these alternatives are mutually exclusive.

3.5. Theorem 2 enables us to show in the next section that, under certain conditions, the σLW($<\mathfrak{k}$) property is preserved by quite large unions. First we need some lemmas about $B(\mathfrak{k})$, which it is convenient to regard as $T^{\aleph_0}$ where T is the set of all ordinals less than the initial ordinal ω^* of cardinal $\mathfrak{k}$, topologized discretely. Let A be the set of those limit ordinals $< \omega^*$ which are limits of sequences of smaller ordinals, and choose for each $\alpha \in A$ a sequence $\beta(\alpha) = \{\beta_1(\alpha), \beta_2(\alpha), \ldots\}$ of ordinals less than α such that $\sup_n \beta_n(\alpha) = \alpha$. Define

$$E(\mathfrak{k}) = \{\beta(\alpha) \mid \alpha \in A\} \subset B(\mathfrak{k}) .$$

(When $\mathfrak{k} = \aleph_0$, this set reduces to the set E considered in [11, §5].) We use the notation $\mathfrak{h}^+$ for the successor of the infinite cardinal $\mathfrak{h}$.

Lemma 1. $E(\mathfrak{h}^+)$ is not the union of $\mathfrak{h}$ (relatively) discrete subsets.

The argument in [11, 5.2] generalizes straightforwardly to prove this: we replace $\aleph_0$ by $\mathfrak{h}$, ω_1 by ω^* and "countable" by "of cardinal at most $\mathfrak{h}$". (But the ordinal α^* is still defined to be $\sup\{\alpha_r \mid r = 1, 2, \ldots\}$.)

Lemma 2. If $Z \subset E(\mathfrak{h}^+)$ and $w(Z) \leq \mathfrak{h}$, then $\|Z\| \leq \mathfrak{h}$.

This is a straightforward generalization of [11, Lemma 6].

3.6.

Theorem 3. Suppose that $\mathfrak{k}$ is a non-limit cardinal and that $\mathfrak{h} < \mathfrak{k}$. If an absolute Borel space Y is the union of $\mathfrak{h}$ subsets each σLW($<\mathfrak{k}$), then Y is σLW($<\mathfrak{k}$).

Proof. Suppose not; then, by Theorem 2, Y contains $B(\mathfrak{k})$ topologically. Thus $B(\mathfrak{k})$ is the union of $\mathfrak{h}$ sets each σLW($<\mathfrak{k}$). We may suppose (by replacing $\mathfrak{h}$ by a larger cardinal if necessary) that $\mathfrak{h}$ is infinite and $\mathfrak{h}^+ = \mathfrak{k}$. Then $B(\mathfrak{k})$ is the union of $\mathfrak{h}$ sets each LW($<\mathfrak{k}$), say of sets X_λ, $\lambda \in \Lambda$, where $\|\Lambda\| = \mathfrak{h}$. Each X_λ is the union of a σ-discrete family of relatively open sets, say $X_\lambda = \bigcup\{X_{\lambda m \mu} \mid \mu \in M_{\lambda m}, m = 1, 2, \ldots\}$, where for fixed λ and m these sets form a discrete family, and where $w(X_{\lambda m \mu}) < \mathfrak{k}$ (and therefore $\leq \mathfrak{h}$) for all λ, m, μ. Using the set $E(\mathfrak{k})$ constructed in section 3.5, we write $E_{\lambda m \mu} = E(\mathfrak{k}) \cap X_{\lambda m \mu}$, a subset of $E(\mathfrak{k})$ of weight $\leq \mathfrak{h}$. By Lemma 2 we have $\|E_{\lambda m \mu}\| \leq \mathfrak{h}$.

For each $\lambda \in \Lambda$ and $m = 1, 2, \ldots$ put

$$M'_{\lambda m} = \{\mu \in M_{\lambda m} \mid E_{\lambda m \mu} \neq \emptyset\} .$$

Take an index set N of cardinal $\mathfrak{h}$; and for each $\lambda \in \Lambda$, $m = 1, 2, ...,$ and $\mu \in M'_{\lambda m}$ choose a map $g_{\lambda m \mu}$ of N onto $E_{\lambda m \mu}$. Define

$$C_{\lambda m \nu} = \{g_{\lambda m \mu}(\nu) \mid \mu \in M'_{\lambda m}\} ;$$

this is a subset of X_λ and is discrete (having at most one point in each of the disjoint open subsets $X_{\lambda m \mu}$, $\mu \in M_{\lambda m}$, of X_λ). Thus $E(\mathfrak{k})$ is the union of the $\leq \aleph_0 \mathfrak{h}^2 = \mathfrak{h}$ discrete sets $C_{\lambda m \nu}$, contradicting Lemma 1.

Remark. Theorem 3 would not be true if $\mathfrak{k}$ were allowed to be a (non-sequential) limit cardinal. For instance, consider $B(\aleph_{\omega_1}) = T^{\aleph_0}$ where T consists of the ordinals $< \omega_{\omega_1}$, topologized discretely. This is an absolute Borel space and it is not σLW($< \aleph_{\omega_1}$), by 2.1(7). However it is the union of $\aleph_1$ closed LW($< \aleph_{\omega_1}$) subsets C_α ($\alpha < \omega_1$), where C_α is the set of points all of whose coordinates are less than ω_α.

Nor would Theorem 3 be true if we allowed $\mathfrak{h} = \mathfrak{k}$. For instance, $B(\mathfrak{c})$ is not σLW($< \mathfrak{c}$), but it is the union of $\mathfrak{c}$ singletons.

It would be interesting, however, to know whether Theorem 3 holds when $\mathfrak{k}$ is sequential. It can be shown that the answer is yes, even with $\mathfrak{h} = \mathfrak{k}$, if the generalized continuum hypothesis is assumed, or more generally if the sequential cardinal $\mathfrak{k}$ is such that $\mathfrak{p} < \mathfrak{k}$ implies $\mathfrak{p}^{\aleph_0} < \mathfrak{k}$.

3.7. We are now in a position to answer a question raised, but only partially settled, in [10, Theorem 13].

Theorem 4. For each infinite cardinal $\mathfrak{k}$, the following assertions about an absolute Borel space X are equivalent.

(a). X is Borel isomorphic to $B(\mathfrak{k})$.

(b). X is generalized homeomorphic to $B(\mathfrak{k})$.

(c). X has weight $\leq \mathfrak{k}$, and is not σLW($< \mathfrak{k}$).

Proof. The implication (b) → (a) is trivial, and (a) → (c) is contained in [10, Theorem 8] and Theorem 1. We must prove (c) → (b).

Assume (c). By [10, Theorem 10], X is generalized homeomorphic to a space of the form $Y = \Sigma\{B(\mathfrak{p}_\lambda) \mid \lambda \in \Lambda\}$, where $\|\Lambda\| \leq \mathfrak{k}$ and each $\mathfrak{p}_\lambda \leq \mathfrak{k}$. Some $\mathfrak{p}_\lambda$ must equal $\mathfrak{k}$ here, since otherwise Y is LW($< \mathfrak{k}$), contradicting Theorem 1. So Y contains $B(\mathfrak{k})$ as a closed subset. On the other hand, Y is easily seen to be homeomorphic to a closed subset of

$B(\mathfrak{k})$. Hence, by [10, Corollary 5.2(1)], Y is generalized homeomorphic to $B(\mathfrak{k})$; and (b) follows.

3.8.

Theorem 5. If X is absolutely Borel, and $H(\mathfrak{h})$ (in the notation of section 2.2) is its Baire kernel of order $\mathfrak{h}$, where $\mathfrak{h}$ is an infinite cardinal, then $X \backslash H(\mathfrak{h})$ is the largest open subset of X which is $\sigma\mathrm{LW}(< \mathfrak{h})$.
Proof. By 2.2(3), $X \backslash H(\mathfrak{h})$ is open, and hence absolutely Borel. If it is not $\sigma\mathrm{LW}(< \mathfrak{h})$, it contains (by Theorem 2) a subset B homeomorphic to $B(\mathfrak{h})$; but then $B \subset H(\mathfrak{h})$ by definition of $H(\mathfrak{h})$, giving a contradiction. On the other hand, if U is any open $\sigma\mathrm{LW}(< \mathfrak{h})$ subset of X, we must have $U \subset X \backslash H(\mathfrak{h})$ since otherwise U meets some subset B of $H(\mathfrak{h})$ that is homeomorphic to $B(\mathfrak{h})$; but then $U \cap B$ also contains a set B' homeomorphic to $B(\mathfrak{h})$, and B', as a subset of U, is $\sigma\mathrm{LW}(< \mathfrak{h})$, contradicting 2.1(7).

Corollary (Compare 2.2(7)). If X is absolutely Borel, then $H(\mathfrak{h}) = L(\mathfrak{h})$.
This follows from the preceding and 2.2(5).

4. The weight function

4.1. In this section, $\mathfrak{h}$ denotes a cardinal which is either infinite or 1, with the convention that $B(\mathfrak{h})$ denotes a single point if $\mathfrak{h} = 1$. Let X be an arbitrary (metric) space of infinite weight $\mathfrak{k}$, and consider the Baire kernels $H(\mathfrak{h})$ of X. If $\mathfrak{h} > \mathfrak{k}$, clearly $H(\mathfrak{h}) = \emptyset$, so we assume $\mathfrak{h} \leq \mathfrak{k}$ throughout what follows. We define the *weight function of* X, W_X, or W for short, to be the function which assigns to each $\mathfrak{h}$ $(\leq \mathfrak{k})$ the value $W_X(\mathfrak{h}) = W(\mathfrak{h}) = w(H(\mathfrak{h}))$. Thus W_X can be thought of as the transfinite sequence $\{W(1), W(\aleph_0), W(\aleph_1), \ldots, W(\mathfrak{k})\}$ where $\mathfrak{k} = w(X)$. Clearly $X = H(1) \supset H(\aleph_0) \supset \ldots \supset H(\mathfrak{k})$, so that

(1) $k = W(1) \geq W(\aleph_0) \geq \ldots \geq W(\mathfrak{k})$.

Again, if $H(\mathfrak{h}) \neq \emptyset$ it contains $B(\mathfrak{h})$ topologically, so that

(2) either $W(h) \geq \mathfrak{h}$, or $W(\mathfrak{h}) = 0$.

Note that, from (1), W_X takes only finitely many distinct values.

4.2. Conversely, suppose we are given a function W, defined for all $\mathfrak{h} \leq \mathfrak{k}$, whose values $W(\mathfrak{h})$ are cardinals satisfying (1) and (2). We define the *model space* of W, M_W, to be the discrete union $\Sigma\{W(\mathfrak{h})B(\mathfrak{h}) \mid \mathfrak{h} \leq \mathfrak{k}\}$. Clearly M_W is an absolute G_δ (having a complete metric); further its weight is $\mathfrak{k}$ because, on the one hand, $w(M_W) \geq w(W(1)B(1)) = \mathfrak{k}$, and on the other $w(M_W) \leq \Sigma_\mathfrak{h} W(\mathfrak{h})w(B(\mathfrak{h})) \leq \Sigma_\mathfrak{h} \mathfrak{k}^2 = \mathfrak{k}$.

Theorem 6. The weight function of M_W is precisely W.
Proof. We first show that the Baire kernels of M_W are given by

(1) $H(\mathfrak{h}) = \Sigma\{W(\mathfrak{h}')B(\mathfrak{h}') \mid \mathfrak{h} \leq \mathfrak{h}' \leq \mathfrak{k}\}$ (where $\mathfrak{h} \leq \mathfrak{k}$).

For by the corollary to Theorem 5 (or 2.2(7)) $H(\mathfrak{h}) = L(\mathfrak{h})$ and $M_W \backslash H(\mathfrak{h})$ is the largest open subset of M_W which is σLW$(< \mathfrak{h})$. This certainly includes the subspace $\Sigma\{W(\mathfrak{h}')B(\mathfrak{h}') \mid \mathfrak{h}' < \mathfrak{h}\}$, and is disjoint from each summand $B(\mathfrak{h}')$ with $\mathfrak{h}' \geq \mathfrak{h}$ because each non-empty open subset of $B(\mathfrak{h}')$ contains a homeomorph of $B(\mathfrak{h}')$ and is therefore not σLW$(< \mathfrak{h}')$. Thus (1) follows.

Next,

(2) $w(H(\mathfrak{h})) \geq W(\mathfrak{h})$.

For (1) gives $w(H(\mathfrak{h})) \geq w(W(\mathfrak{h})B(\mathfrak{h})) = W(\mathfrak{h})\mathfrak{h}$. Since W is assumed to satisfy 4.1(2), we have $W(\mathfrak{h}) \geq \mathfrak{h}$ unless $W(\mathfrak{h}) = 0$; in either case (2) follows.

To complete the proof, we show

(3) $w(H(\mathfrak{h})) \leq W(\mathfrak{h})$.

For (1) gives

$$w(H(\mathfrak{h})) \leq \Sigma\{W(\mathfrak{h}')\mathfrak{h}' \mid \mathfrak{h} \leq \mathfrak{h}' \leq \mathfrak{k}\} = \Sigma\{W(\mathfrak{h}') \mid \mathfrak{h} \leq \mathfrak{h}' \leq \mathfrak{k}\}.$$

In this sum, we note that each term $W(\mathfrak{h}')$ with $\mathfrak{h}' > W(\mathfrak{h})$ must be 0, since otherwise 4.1(2) gives $W(\mathfrak{h}') \geq \mathfrak{h}' > W(\mathfrak{h})$, where $\mathfrak{h}' \geq \mathfrak{h}$, in contradiction to 4.1(1). Thus $w(H(\mathfrak{h})) \leq \Sigma\{W(\mathfrak{h}') \mid \mathfrak{h} \leq \mathfrak{h}' \leq W(\mathfrak{h})\}$ and in this sum each $W(\mathfrak{h}') \leq W(\mathfrak{h})$, by 4.1(1); thus $w(H(\mathfrak{h})) \leq (W(\mathfrak{h}))^2 = W(\mathfrak{h})$ since $W(\mathfrak{h})$ is infinite or 0.

Corollary. Conditions 4.1(1) and 4.1(2) are necessary and sufficient for a cardinal-valued function W, defined for all $\mathfrak{h} \leq \mathfrak{k}$, to be the weight function of some (metric) space. If they are satisfied, W is the weight function of a complete metric (hence absolute G_δ) space.

4.3.

Theorem 7. If two absolute Borel spaces X, Y are Borel isomorphic, they have the same weight function.

Proof. From [10, Theorem 8], X and Y have the same weight, say $\mathfrak{k}$. If $\mathfrak{k}$ is finite, W_X and W_Y will both be defined on the single cardinal 1, and both will have the value $\mathfrak{k}$ there; thus we may assume that $\mathfrak{k}$ is infinite. We must prove that for $\mathfrak{h} \leq \mathfrak{k}$ the Baire kernels $H(\mathfrak{h})$ of X, $H'(\mathfrak{h})$ of Y, have equal weights; this is trivial if $\mathfrak{h} = 1$, so we assume $\mathfrak{h} \geq \aleph_0$. Write $w = w(H(\mathfrak{h}))$, $w' = w(H'(\mathfrak{h}))$; we first prove $w' \geq w$. In doing this we may of course assume $H(\mathfrak{h}) \neq \emptyset$.

Let f be a Borel isomorphism of X onto Y. Put $Z = H(\mathfrak{h}) \cap f^{-1}(H'(\mathfrak{h}))$. We show that $w(Z) \geq w$, considering two cases:

(a). If w is not sequential, there exists [8, p. 99] a 3ϵ-discrete set $A \subset H(\mathfrak{h})$, for some $\epsilon > 0$, such that $\|A\| = w$. Each $a \in A$ is in some set $B_a \subset H(\mathfrak{h})$ homeomorphic to $B(\mathfrak{h})$. The ϵ-neighborhood $S(a, \epsilon)$ intersects B_a in a set containing a homeomorph of $B(\mathfrak{h})$; hence $S(a, \epsilon) \cap B_a$ is not $\sigma\mathrm{LW}(< \mathfrak{h})$. But $Y \backslash H'(\mathfrak{h})$ is $\sigma\mathrm{LW}(< \mathfrak{h})$, by Theorem 5 (3.8), and therefore, by Theorem 1 (3.1), the same is true of $X \backslash f^{-1}(H'(\mathfrak{h}))$. Thus $S(a,\epsilon) \cap B_a$ is not contained in $X \backslash f^{-1}(H'(\mathfrak{h}))$, so that we may pick $x_a \in S(a, \epsilon) \cap Z$. Then $\{x_a \mid a \in A\}$ is an ϵ-discrete subset of Z, of cardinal $\|A\| = w$, and this proves $w(Z) \geq w$.

(b). If w is sequential, say $w = \sup\{v_n \mid n = 1, 2, \ldots\}$ where each $v_n < w$, the argument in case (a) applies if w is replaced by v_n^+ throughout. Thus $w(Z) > v_n$ for each $n = 1, 2, \ldots$, and therefore again $w(Z) \geq w$.

Now $w \leq w(Z) \leq w(f^{-1}(H'(\mathfrak{h}))) = w(H'(\mathfrak{h})) = w'$. Similarly $w \geq w'$, and Theorem 7 follows.

4.4.

Theorem 8. If X is absolutely Borel, than X is generalized homeomorphic to the model space M_W constructed (as in section 4.2) from the weight function W of X.

Proof. From [10, Theorem 10] we know that X is generalized homeomorphic to $Y = \Sigma\{p(\mathfrak{h})B(\mathfrak{h}) \mid \mathfrak{h} \leq \mathfrak{k}\}$, where $\mathfrak{k} = w(X)$ and the coefficients satisfy $0 \le p(\mathfrak{h}) \le \mathfrak{t}$. By 2.3(1) and 2.3(4) we may replace each $B(\mathfrak{h})$ here by $\Sigma\{\mathfrak{h} B(\mathfrak{i}) | \mathfrak{i} \le \mathfrak{t}\}$; we make the convention that $\mathfrak{i}$, like $\mathfrak{h}$, takes values which are 1 or infinite. Thus Y is generalized homeomorphic to $Z = \Sigma_{\mathfrak{h} \le \mathfrak{t}} \Sigma_{\mathfrak{i} \le \mathfrak{h}} \mathfrak{h}p(\mathfrak{h})B(\mathfrak{i})$ which, on rearranging, gives

$$(1) \quad Z = \Sigma\{w(\mathfrak{i})B(\mathfrak{i}) | \mathfrak{i} \le \mathfrak{t}\} \quad \text{where} \quad w(\mathfrak{i}) = \Sigma\{\mathfrak{h}p(\mathfrak{h}) | \mathfrak{i} \le \mathfrak{h} \le \mathfrak{t}\}.$$

Consider the function w arising here, which assigns to each $\mathfrak{i} = 1, \aleph_0, \dots$ $\dots, \mathfrak{t}$ the value $w(\mathfrak{i})$. Clearly $w(\mathfrak{i})$ is either 0 or $\ge \mathfrak{i}$, and $w(1) \ge w(\aleph_0) \ge \dots \ge w(\mathfrak{t})$; also $\mathfrak{t}$ = weight of $Y = \Sigma\{\mathfrak{h}p(\mathfrak{h}) | \mathfrak{h} \le \mathfrak{t}\} = w(1)$. Thus, by Theorem 6, w is the weight function of the model space M_w. But clearly $M_w = Z$, which is generalized homeomorphic to Y and therefore to X. By Theorem 7 we must have $w = W$, so $Z = M_W$ and the theorem is proved.

4.5. The foregoing results combine to give the following, which is the main result of this paper:

Theorem 9. If X and Y are absolute Borel sets, the following statements are equivalent:

(a). X and Y are Borel isomorphic.
(b). X and Y are generalized homeomorphic.
(c). X and Y have the same weight functions.

Proof. In fact, (b) → (a) trivially, (a) → (c) by Theorem 7, and (c) → (b) by Theorem 8.

We remark that this theorem includes Theorem 4 (3.7) as a special case. In fact, condition (c) in Theorem 4 says (in view of Theorem 5) that $H(\mathfrak{h}) = X$ for all $\mathfrak{h} \le \mathfrak{t}$, so the weight function W_X is the constant function with value $\mathfrak{t}$. This is the same as the weight function of $B(\mathfrak{t})$.

Theorem 9 also completes the classification, in [10, Theorem 11], of the absolute Borel spaces of weight $\aleph_1$, by showing that no two of the four spaces listed there are Borel isomorphic. In the next section we generalize this to absolute Borel spaces of arbitrary weights.

5. Normal forms for the Borel types

5.1. *Notation.* Let $\mathfrak{X}_\alpha$ denote the class of all absolute Borel spaces of weight $\mathfrak{t} = \aleph_\alpha$. (If preferred, we may regard $\mathfrak{X}_\alpha$ as a set by restricting attention to subspaces of a fixed space which is "universal" for metric spaces of weight $\leq \mathfrak{k}$; such a space is provided, for instance, by [4].) We have seen that Borel isomorphism and generalized homeomorphism produce the same equivalence relation on $\mathfrak{X}_\alpha$; we shall refer to this simply as *equivalence*, and denote it by $\sim$. However, though we shall not say so explicitly, all the equivalences constructed in the following arguments will in fact be generalized homeomorphisms of fixed class, and we therefore apply 2.3(4) freely. We call the equivalence classes produced on $\mathfrak{X}_\alpha$ by $\sim$ the *Borel types* of weight $\aleph_\alpha$. Our present object is to select from each such Borel type a simple representative space to act as a *normal form*. One obvious way of doing this would be to use as normal forms the *model spaces* M_W for the various weight functions W. However, the spaces M_W are usually far from being the simplest representatives of their types; the spaces described in Theorem 10 below seem to be as simple as possible.

It will be convenient to introduce a more condensed notation. We abbreviate $B(\aleph_\beta)$ to B_β; and if W is a weight function we abbreviate $W(\aleph_\beta)$ to W_β. We count -1 as an ordinal number, and take $\aleph_{-1}$ to mean 1; thus B_{-1} is the 1-point space, and $B_0 = B(\aleph_0)$ is the usual space of irrational numbers. For each limit ordinal α we define

(1) $A_\alpha = \Sigma\{B_\beta \mid \beta < \alpha\}$ (α a limit ordinal).

For non-limit α, the symbol A_α is left undefined. We observe:

(2). If α is a limit ordinal and $\gamma < \alpha$, then

$$A_\alpha \sim \Sigma\{B_\beta \mid \gamma < \beta < \alpha\} \sim \aleph_\alpha A_\alpha .$$

For, from 2.3(1) and 2.3(4),

$$\begin{aligned}\Sigma\{B_\beta \mid \gamma < \beta < \alpha\} &\sim \Sigma\{\Sigma\{\aleph_\beta B_\delta \mid \delta \leq \beta\} \mid \gamma < \beta < \alpha\}\\ &= \Sigma\{\Sigma\{\aleph_\beta B_\delta \mid \max(\delta, \gamma + 1) \leq \beta < \alpha\} \mid \delta < \alpha\}\\ &= \Sigma\{\aleph_\alpha B_\delta \mid \delta < \alpha\} = \aleph_\alpha A_\alpha .\end{aligned}$$

An entirely similar argument proves $A_\alpha \sim \aleph_\alpha A_\alpha$, and (2) follows.

For each ordinal α we use C_α to stand for either A_α or B_α (the former possibility arising only if α is a limit ordinal, of course). Clearly C_α is in any case of weight $\aleph_\alpha$.

5.2. With the foregoing notation, we now have:

Theorem 10. Each absolute Borel metric space of weight $\aleph_\alpha$ (where $\alpha \geq 0$) is equivalent to one and only one of the following *normal form* spaces:

(a) C_α,
(b) $\aleph_\alpha C_\beta$ for some $\beta < \alpha$,
(c) $\mathfrak{p}_1 C_{\gamma_1} + \mathfrak{p}_2 C_{\gamma_2} + \ldots + \mathfrak{p}_n C_{\gamma_n}$ where
  (i) n is an integer > 1,
  (ii) $\gamma_1, \ldots, \gamma_n$ are ordinals and $-1 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n < \alpha$, and if $\gamma_i = \gamma_{i+1}$ then C_{γ_i} is A_{γ_i} and $C_{\gamma_{i+1}}$ is $B_{\gamma_{i+1}}$,
  (iii) $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are cardinals and $\aleph_\alpha = \mathfrak{p}_1 > \mathfrak{p}_2 > \ldots > \mathfrak{p}_n \geq 1$,
  (iv) $\mathfrak{p}_{n-1} > \aleph_{\gamma_n}$,
  (v) either $\mathfrak{p}_n > \aleph_{\gamma_n}$ or $\mathfrak{p}_n = 1$.

Proof. Let X be an absolute Borel space of weight $\aleph_\alpha$; we first show X equivalent to at least one of the spaces listed in the theorem. By Theorem 8 (4.4), we have $X \sim Y = \Sigma\,\{W_\beta B_\beta \mid \beta \leq \alpha\}$, where $W_\beta = W_X(\aleph_\beta)$; and from section 4.1 we have

(1) $\aleph_\alpha = W_{-1} \geq W_0 \geq \ldots \geq W_\alpha$ and
(2) if $-1 \leq \beta \leq \alpha$, either $W_\beta \geq \aleph_\beta$ or $W_\beta = 0$.

If $W_\alpha \neq 0$, the argument proving 2.3(1) shows that $X \sim B_\alpha$. Thus we may assume $W_\alpha = 0$. The decreasing transfinite sequence $\{W_\beta \mid -1 \leq \beta \leq \alpha\}$ of cardinals can take only a finite number, say n, of distinct non-zero values; say these values are $\aleph_\alpha = \mathfrak{p}_1 > \mathfrak{p}_2 > \ldots > \mathfrak{p}_n > 0$. We put $\mathfrak{p}_{n+1} = 0$ and define $\beta_i =$ first β for which $W_\beta = \mathfrak{p}_{i+1}$ $(i = 0, 1, \ldots, n)$; then

(3) $-1 = \beta_0 < \beta_1 < \ldots < \beta_n \leq \alpha$,
(4) $W_\beta = \mathfrak{p}_i$ for $\beta_{i-1} \leq \beta < \beta_i$,
  $W_\beta = 0$ for $\beta_n \leq \beta \leq \alpha$.

Thus

(5) $Y = \mathfrak{p}_1 R_1 + \ldots + \mathfrak{p}_n R_n$, where $R_i = \Sigma\{B_\beta \mid \beta_{i-1} \leq \beta < \beta_i\}$.

If β_i has an immediate predecessor, we put $\beta_i = \gamma_i + 1$ and note that $R_i \sim B_{\gamma_i}$ because $\Sigma\{B_\beta \mid \beta \leq \gamma_i\} \sim B_{\gamma_i}$ by 2.3(1), and R_i is bracketted between them (so that [10, Theorem 9] applies). If instead β_i is a limit ordinal, we denote it by γ_i and note that $R_i \sim A_{\gamma_i}$ by 5.1(2). Thus in any case

(6) $Y \sim Z = \mathfrak{p}_1 C_{\gamma_1} + \ldots + \mathfrak{p}_n C_{\gamma_n}$, and
(7) $-1 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \alpha$; and if $\gamma_i = \gamma_{i+1}$ then C_{γ_i} is A_{γ_i} and $C_{\gamma_{i+1}}$ is $B_{\gamma_{i+1}}$.

Suppose first that $\gamma_n = \alpha$. Then C_{γ_n} is either B_α or A_α. In the first case, we see that Z (containing B_α and contained in $\aleph_\alpha B_\alpha$) is equivalent to B_α (compare 2.3(1)); in the second case, a similar argument using 5.1(2) shows $Z \sim A_\alpha$. Thus $X \sim C_\alpha$, as in case (a) of the theorem. We may now assume that $\gamma_n < \alpha$.

If $n = 1$, (6) gives $X \sim Z = \aleph_\alpha C_{\gamma_1}$ where $\gamma_1 < \alpha$, as in case (b) of the theorem. Now assume $n > 1$.

From (2) and (4) we have $\mathfrak{p}_i \geq \sup\{\aleph_\beta \mid \beta < \beta_i\} = \aleph_{\gamma_i}$ ($i = 1, 2, \ldots, n$), whence $\mathfrak{p}_{n-1} > \mathfrak{p}_n \geq \aleph_{\gamma_n}$. Thus Z satisfies all but one of the requirements for case (c) of the theorem, the only requirement missing being (v); we have instead that $\mathfrak{p}_n \geq \aleph_{\gamma_n}$. However, if $\mathfrak{p}_n = \aleph_{\gamma_n}$, we observe that the last term $\mathfrak{p}_n C_{\gamma_n}$ of Z is equivalent to C_{γ_n} itself (from 2.3(1) and 5.1(2)), so we can then replace $\mathfrak{p}_n$ by 1 and have fulfilled all the requirements.

To see that X is equivalent to only one space Z of the kind listed in the theorem, we observe that the weight function W (as reconstructed from Z) determines n (the number of different values of W), the ordinals $\gamma_1, \gamma_2, \ldots, \gamma_n$ (the end-points, roughly speaking, of the intervals of constancy of W, in increasing order), the cardinals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ (the different non-zero values taken by W, in decreasing order) and the nature of each C_{γ_i}. Thus W determines Z completely; but $W = W_X$, showing that X determines Z uniquely.

5.3.

Theorem 11. The number of Borel types of weight $\aleph_\alpha$ is exactly $\|\alpha\|$ if α is infinite, $2^{\alpha+1}$ if α is finite.

Proof. If α is infinite, we see from Theorem 10 that there are at most $\|\alpha\|$ such types, and also that there are at least $\|\alpha\|$ of them (provided, for instance, by the spaces $\aleph_\alpha B_\beta$, $\beta < \alpha$), hence exactly $\|\alpha\|$.

If α is finite, we count the number of normal forms provided by Theorem 10. In doing this it is convenient to reverse the final step in the reduction to normal form given above; this is the step which replaced $\mathfrak{p}_n C_{\gamma_n}$ by C_{γ_n} if $\mathfrak{p}_n = \aleph_{\gamma_n}$, in effect replacing $\mathfrak{p}_n$ by 1 in that case. By omitting this step, we get a slightly more complicated normal form, but one with a simpler description: in Theorem 10(c), the conditions (iv) and (v) are replaced by the single condition

(iv)': $\mathfrak{p}_n \geq \aleph_{\gamma_n}$.

If we now modify (i) to read "$n \geq 1$", cases (a) and (b) are included in the modified (c). Again, because α is finite, all C_γ's occurring are B_γ's. Thus, on writing $\mathfrak{p}_i = \aleph_{\alpha_i}$ ($i = 1, 2, ..., n$) we merely have to count the number of different sequences

$$\alpha = \alpha_1 > \alpha_2 > ... > \alpha_n \geq \gamma_n > \gamma_{n-1} > ... > \gamma_1 \geq -1$$

for $n = 1, 2, ...$

For given n and $\gamma_n = \gamma$, this number is evidently $\binom{\alpha-\gamma}{n-1}\binom{\gamma+1}{n-1}$; and the total number of Borel types of weight $\aleph_\alpha$ is thus

$$(1) \qquad \sum_{\gamma=-1}^{\alpha} \sum_{n=1}^{\infty} \binom{\alpha-\gamma}{n-1}\binom{\gamma+1}{n-1}$$

(with the usual convention that a meaningless binomial coefficient is interpreted as 0). The inner sum here is well known to be $\binom{\alpha+1}{\gamma+1}$, as can be verified (for instance) by equating the coefficients of $x^{\gamma+1}$ in the identity $(1+x)^{\alpha-\gamma}(1+x)^{\gamma+1} = (1+x)^{\alpha+1}$. Thus the total number in (1) is $\Sigma_{\gamma=-1}^{\alpha}\binom{\alpha+1}{\gamma+1} = (1+1)^{\alpha+1} = 2^{\alpha+1}$.

Remark. When $\alpha = 0$, Theorems 10 and 11 reduce to the classical theorem that every separable infinite absolute Borel space is equivalent either to the integers ($\aleph_0 B(1)$) or to the irrationals ($B(\aleph_0)$). When $\alpha = 1$ we similarly obtain the four types of weight $\aleph_1$ listed in [10, Theorem 11]; in the present notation these are respectively

$$\aleph_1 B_{-1}, \quad \aleph_1 B_{-1} + B_0, \quad \aleph_1 B_0, \quad B_1 .$$

5.4.

Corollary. The number n of terms in a normal form (as in Theorem 10) for the Borel types of weight $\aleph_\alpha$, α finite, satisfies $1 \leq n \leq [\frac{1}{2}(\alpha+3)]$; and the number of normal forms having exactly n terms is $\binom{\alpha+2}{2n-1}$.

Proof. The reasoning leading to 5.3(1) above shows that the number of normal forms with exactly n terms is $\Sigma_\gamma \binom{\alpha-\gamma}{n-1}\binom{\gamma+1}{n-1}$; this is well known to equal $\binom{\alpha+2}{2n-1}$ as can be verified by equating coefficients of $x^{\alpha+3-2n}$ in the identity $(1-x)^{-n}(1-x)^{-n} = (1-x)^{-2n}$. And $\binom{\alpha+2}{2n-1}$ is non-zero only if $1 \leq n \leq [\frac{1}{2}(\alpha+3)]$.

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