$L_g(\phi)=\int_\Omega g\delta(\phi)|\nabla\phi|\,dxdy$

令 $F(\phi)=g\delta(\phi)|\nabla\phi|$. 取小变量 ε 和随机函数 h . 满足 $h|_{\partial\Omega}=0$

可得 $F(\phi+\varepsilon h)=g\delta(\phi+\varepsilon h)\sqrt{(\phi+\varepsilon h)_x^2+(\phi+\varepsilon h)_y^2}$

$$\frac{\partial F(\phi+\varepsilon h)}{\partial\varepsilon}=gh\delta'(\phi+\varepsilon h)\sqrt{(\phi+\varepsilon h)_x^2+(\phi+\varepsilon h)_y^2}+g\delta(\phi+\varepsilon h)\frac{\nabla h\cdot\nabla(\phi+\varepsilon h)}{\sqrt{(\phi+\varepsilon h)_x^2+(\phi+\varepsilon h)_y^2}}$$

$$\left.\frac{\partial F(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to0}=gh\delta'(\phi)|\nabla\phi|+g\delta(\phi)\frac{h_x\phi_x+h_y\phi_y}{\sqrt{\phi_x^2+\phi_y^2}}$$

$$\therefore\left.\frac{\partial L_g(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to0}=\int_\Omega gh\delta'(\phi)|\nabla\phi|\,dxdy+\int_\Omega g\delta(\phi)\cdot\frac{h_x\phi_x+h_y\phi_y}{\sqrt{\phi_x^2+\phi_y^2}}\,dxdy$$

$$\frac{\partial}{\partial x}\left[g\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h\right]=g_x\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h+g\frac{\partial}{\partial x}[\delta(\phi)]\frac{\phi_x}{|\nabla\phi|}h+g\delta(\phi)\frac{\partial}{\partial x}\left[\frac{\phi_x}{|\nabla\phi|}\right]h+g\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h_x$$

$$\frac{\partial}{\partial y}\left[g\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h\right]=g_y\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h+g\frac{\partial}{\partial y}[\delta(\phi)]\frac{\phi_y}{|\nabla\phi|}h+g\delta(\phi)\frac{\partial}{\partial y}\left[\frac{\phi_y}{|\nabla\phi|}\right]h+g\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h_y$$

$$\frac{\partial}{\partial x}[\delta(\phi)]=\delta'(\phi)\phi_x\quad,\quad\frac{\partial}{\partial y}[\delta(\phi)]=\delta(\phi)\phi_y$$

$$\therefore\left.\frac{\partial L_g(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to0}=\int_\Omega gh\delta'(\phi)|\nabla\phi|\,dxdy+\int_\Omega\frac{\partial}{\partial x}\left[g\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h\right]dxdy+$$

$$\int_\Omega\frac{\partial}{\partial y}\left[g\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h\right]dxdy-\int_\Omega\nabla g\cdot\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}h\,dxdy-\int_\Omega gh\delta'(\phi)|\nabla\phi|\,dxdy-$$

$$\int_\Omega g\delta(\phi)\frac{\partial}{\partial x}\left[\frac{\phi_x}{|\nabla\phi|}\right]h+g\delta(\phi)\frac{\partial}{\partial y}\left[\frac{\phi_y}{|\nabla\phi|}\right]h\,dxdy$$

$$=\int_\Omega\frac{\partial}{\partial x}\left[g\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h\right]+\frac{\partial}{\partial y}\left[g\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h\right]dxdy-\int_\Omega g\delta(\phi)\frac{\partial}{\partial x}\left[\frac{\phi_x}{|\nabla\phi|}\right]h$$

$$+g\delta(\phi)\frac{\partial}{\partial y}\left[\frac{\phi_y}{|\nabla\phi|}\right]h\,dxdy-\int_\Omega\nabla g\cdot\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}h\,dxdy$$

由格林公式及 $h|_{\partial\Omega}=0$. 有

$$\int_\Omega \frac{\partial}{\partial x}\left[g\delta(\phi)\frac{\phi_x}{|\nabla\phi|}h\right]+\frac{\partial}{\partial y}\left[g\delta(\phi)\frac{\phi_y}{|\nabla\phi|}h\right]dxdy$$

$$= \oint_{\partial\Omega} h\left(g\delta(\phi)\phi_x\,dy - g\delta(\phi)\phi_y\,dx\right) = 0$$

$$\therefore \left.\frac{\partial L_g(\phi+\varepsilon h)}{\partial \varepsilon}\right|_{\varepsilon\to 0} = -\int_\Omega h\delta(\phi)\left(g\cdot\nabla\left[\frac{\nabla\phi}{|\nabla\phi|}\right]+\nabla g\cdot\frac{\nabla\phi}{|\nabla\phi|}\right)dxdy$$

$$= -\int_\Omega h\delta(\phi)\,\mathrm{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right)dxdy$$

$$\left.\frac{\partial A_g(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to 0} = \left.\int_\Omega g\frac{\partial H(-\phi-\varepsilon h)}{\partial\varepsilon}dxdy\right|_{\varepsilon\to 0} = \left.\int_\Omega g(-h)H'(-\phi-\varepsilon h)dxdy\right|_{\varepsilon\to 0}$$

$$= -\int_\Omega ghH'(-\phi)\,dxdy = -\int_\Omega gh\,\delta(-\phi)\,dxdy$$

$$= -\int_\Omega gh\,\delta(\phi)\,dxdy \quad 占 ⿻$$

$$\therefore 有 \left.\frac{\partial E(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to 0} = -\int_\Omega h\left[\mu\left(\Delta\phi-\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right)+\lambda\delta(\phi)\,\mathrm{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right)+\nu g\delta(\phi)\right]dxdy$$

令 $\left.\frac{\partial E(\phi+\varepsilon h)}{\partial\varepsilon}\right|_{\varepsilon\to 0} = 0$. 则有

$$\mu\left[\Delta\phi-\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right]+\lambda\delta(\phi)\,\mathrm{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right)+\nu g\delta(\phi)$$