

EECS 559 HW 1

QUESTION 1:

(a) Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{y} \in \mathbb{R}^m$, show that the basis pursuit problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{y}$$

is a convex problem, and recast it as a *linear program* in the standard form.

SOLUTION: Firstly, we can show that this is a convex optimization problem by showing that both the objective function and constraints are convex. We start by examining the objective function:

$$\begin{aligned} \|\mathbf{x}\|_1 &= \sum_i |x_i| \\ &= \sum_i \max\{x_i, -x_i\} \end{aligned}$$

We know that x_i and $-x_i$ are linear functions, therefore convex, thus the pointwise maximum of these two functions is convex. Since the objective function is a nonnegative sum of these pointwise maxima, it is also convex. The constraint:

$$\mathbf{Ax} = \mathbf{y}$$

is linear, therefore convex. which means that this *is a convex optimization problem*.

We can now recast this as a linear program in the standard form. We start by defining \mathbf{x}_+ and \mathbf{x}_- as follows:

$$\begin{aligned} \mathbf{x}_+ &\triangleq \sum_i x_i \mathbb{1}_{\{x_i \geq 0\}} \mathbf{e}_i \\ \mathbf{x}_- &\triangleq \sum_i -x_i \mathbb{1}_{\{x_i < 0\}} \mathbf{e}_i \end{aligned}$$

Where \mathbf{e}_i is the i^{th} canonical basis vector of \mathbb{R}^n . Notice that:

$$\mathbf{x} = \mathbf{x}_+ - \mathbf{x}_-$$

QUESTION 2: Consider the following unconstrained optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

Where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function.

- (a) Show that if \mathbf{x}_0 is a local minimizer of f , then it is also a global solution (need not to be unique). Moreover, if f is *strictly* convex, then show that the solution is unique (if exists).

PROOF: (By contradiction) Let \mathbf{x}_0 be a local minimizer of f , meaning that $\exists \epsilon > 0$ such that:

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{B}(\mathbf{x}_0, \epsilon) \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}_0 - \mathbf{x}\|_2 \leq \epsilon\}$$

Assume $\exists \mathbf{x}_* \in \mathbb{R}^n$ such that $f(\mathbf{x}_*) < f(\mathbf{x}_0)$. $\forall \epsilon$ sufficiently small, $\exists \alpha \in (0, 1)$ such that:

$$\mathbf{y} \triangleq \alpha \mathbf{x}_0 + (1 - \alpha) \mathbf{x}_* \in \mathcal{B}(\mathbf{x}_0, \epsilon)$$

And by the convexity of f :

$$f(\mathbf{y}) \leq \alpha f(\mathbf{x}_0) + (1 - \alpha) f(\mathbf{x}_*)$$

Since $f(\mathbf{x}_*) < f(\mathbf{x}_0)$, we can write:

$$f(\mathbf{y}) < \alpha f(\mathbf{x}_0) + (1 - \alpha) f(\mathbf{x}_0) = f(\mathbf{x}_0)$$

Which is when we arrive at our contradiction. Since for any arbitrary $\epsilon > 0$ there is an element $\mathbf{y} \in \mathcal{B}(\mathbf{x}_0, \epsilon)$ such that $f(\mathbf{x}_0) > f(\mathbf{y})$, our assumption is false. Therefore, any local minimum is a global minimum. \square

Let's now assume that f is *strictly convex*. Assuming it exists, we can show the uniqueness of the global minimizer as follows.

PROOF: (By contradiction) Let \mathbf{x}_0 is a local minimizer of f . Since strict convexity implies convexity, we know that \mathbf{x}_0 is also a global minimizer (from the previous argument).

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n$$

Assume $\exists \mathbf{x}_* \in \mathbb{R}^n$ such that $f(\mathbf{x}_0) = f(\mathbf{x}_*)$. By strict convexity of f , $\forall \alpha \in (0, 1)$:

$$f(\alpha \mathbf{x}_0 + (1 - \alpha) \mathbf{x}_*) < \alpha f(\mathbf{x}_0) + (1 - \alpha) f(\mathbf{x}_*) = f(\mathbf{x}_0)$$

Therefore there is an element in the domain that achieves a lower value than our global minimizer, which is our contradiction. Therefore our assumption is wrong. Given that f is *strongly convex* the local minimizer \mathbf{x}_0 is a *unique global* minimizer. \square

QUESTION 2: Continued

- (b) Furthermore, suppose $f \in \mathcal{C}^1$ (i.e., continuously differentiable). Then show that \mathbf{x}_0 is a *global* minimizer of f iff:

$$\nabla f(\mathbf{x}_0) = \mathbf{0}$$

PROOF: (\Rightarrow) Let \mathbf{x}_0 be a global minimum of f . Since f is convex, it satisfies the first order condition, $\forall \mathbf{y} \in \mathbb{R}^n$:

$$f(\mathbf{y}) \geq f(\mathbf{x}_0) + (\mathbf{y} - \mathbf{x}_0)^T \nabla f(\mathbf{x}_0)$$

TODO: FINISH FORWARD

PROOF: (\Leftarrow) Assume $\nabla f(\mathbf{x}_0) = \mathbf{0}$. Since f is convex, it satisfies the first order condition, $\forall \mathbf{y} \in \mathbb{R}^n$:

$$\begin{aligned} f(\mathbf{y}) &\geq f(\mathbf{x}_0) + (\mathbf{y} - \mathbf{x}_0)^T \nabla f(\mathbf{x}_0) \\ &= f(\mathbf{x}_0) + (\mathbf{y} - \mathbf{x}_0)^T \mathbf{0} \\ &= f(\mathbf{x}_0) \end{aligned}$$

So $\forall \mathbf{y} \in \mathbb{R}^n$, $\nabla f(\mathbf{x}_0) = \mathbf{0}$ implies that \mathbf{x}_0 is a global minimum of f .

QUESTION 3: Given $f(\mathbf{x}), f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})$ are convex on \mathbb{R}^m , show that:

- (a) If $\alpha_i \geq 0 \ \forall i \in [n]$, then $g(\mathbf{x}) \triangleq \sum \alpha_i f_i(\mathbf{x})$ is a convex function.

Consider some $\beta \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By the convexity of f_i :

$$\beta f_i(\mathbf{x}) + (1 - \beta) f_i(\mathbf{y}) \geq f_i(\beta \mathbf{x} + (1 - \beta) \mathbf{y})$$

Which implies:

$$\begin{aligned} \sum \alpha_i [\beta f_i(\mathbf{x}) + (1 - \beta) f_i(\mathbf{y})] &\geq \sum \alpha_i f_i(\beta \mathbf{x} + (1 - \beta) \mathbf{y}) \\ \beta \sum \alpha_i f_i(\mathbf{x}) + (1 - \beta) \sum \alpha_i f_i(\mathbf{y}) &\geq g(\beta \mathbf{x} + (1 - \beta) \mathbf{y}) \\ \beta g(\mathbf{x}) + (1 - \beta) g(\mathbf{y}) &\geq g(\beta \mathbf{x} + (1 - \beta) \mathbf{y}) \quad \forall \beta \in [0, 1] \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \end{aligned}$$

Therefore g is a convex function.

- (b) $g(\mathbf{x}) \triangleq \max\{f_1(\mathbf{x}), \dots, f_n(\mathbf{x})\}$ is convex:

Consider some $\beta \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We will construct two sets:

$$\begin{aligned} A &\triangleq \{\beta f_i(\mathbf{x}) + (1 - \beta) f_i(\mathbf{y}) | i \in [n]\} \\ B &\triangleq \{f_i(\beta \mathbf{x} + (1 - \beta) \mathbf{y}) | i \in [n]\} \end{aligned}$$

By the convexity of f_i , each element in B is upper bounded by at least one element in A which means that:

$$\max A \geq \max B$$

TODO: FINISH THIS ONE

- (c) Given $\mathbf{A} \in \mathbb{R}^{n \times m}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$, $g(\mathbf{x}) \triangleq f(\mathbf{Ax} + \mathbf{b})$ is convex.

QUESTION 4: Show that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex iff for all integers $m \geq 2$:

$$f\left(\sum_{i=1}^m \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^m \lambda_i f(\mathbf{x}_i) \quad (\star)$$

Where $\mathbf{x}_i \in \mathbb{R}^n$ and $\lambda_i \geq 0 \ \forall i \in [m]$ satisfying $\sum \lambda_i = 1$.

PROOF: (by induction) Consider the base case ($m = 2$). The expression above boils down to the following:

$$f(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) \leq \lambda_1 f(\mathbf{x}_1) + \lambda_2 f(\mathbf{x}_2)$$

Notice that $\lambda_1 + \lambda_2 = 1$ is equivalent to $\lambda_2 = 1 - \lambda_1$. And since $\lambda_2 \geq 0$ must be true, we know that $\lambda_1 \in [0, 1]$. After substituting we get the definition of convexity:

$$f(\lambda_1 \mathbf{x}_1 + (1 - \lambda_1) \mathbf{x}_2) \leq \lambda_1 f(\mathbf{x}_1) + (1 - \lambda_1) f(\mathbf{x}_2)$$

So the base case ($m = 2$) the inequality (\star) is true iff f is convex.

Now let's assume (\star) is true for some fixed value ($m = k \geq 2$):

$$f\left(\sum_{i=1}^k \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^k \lambda_i f(\mathbf{x}_i)$$

W.l.o.g., we can express \mathbf{x}_k as follows:

$$\mathbf{x}_k = \alpha \mathbf{y}_k + (1 - \alpha) \mathbf{y}_{k+1} \quad \text{where } \alpha \in [0, 1]$$

Where \mathbf{y}_k and \mathbf{y}_{k+1} are arbitrary vectors in \mathbb{R}^n . We can rewrite (\star) as follows:

$$f\left(\sum_{i=1}^{k-1} \lambda_i \mathbf{x}_i + \lambda_k \alpha \mathbf{y}_k + \lambda_k (1 - \alpha) \mathbf{y}_{k+1}\right) \leq \sum_{i=1}^{k-1} \lambda_i f(\mathbf{x}_i) + \lambda_k \alpha f(\mathbf{y}_k) + \lambda_k (1 - \alpha) f(\mathbf{y}_{k+1})$$

And by the convexity of f :

$$\lambda_k f(\alpha \mathbf{y}_k + (1 - \alpha) \mathbf{y}_{k+1}) \leq \lambda_k \alpha f(\mathbf{y}_k) + \lambda_k (1 - \alpha) f(\mathbf{y}_{k+1})$$

Which leaves us with:

$$f\left(\sum_{i=1}^{k-1} \lambda_i \mathbf{x}_i + \lambda_k \alpha \mathbf{y}_k + \lambda_k (1 - \alpha) \mathbf{y}_{k+1}\right) \leq \sum_{i=1}^{k-1} \lambda_i f(\mathbf{x}_i) + \lambda_k \alpha f(\mathbf{y}_k) + \lambda_k (1 - \alpha) f(\mathbf{y}_{k+1})$$

And since $\sum_{i=1}^{k-1} \lambda_i + \lambda_k (\alpha + 1 - \alpha) = 1$, the $m = k$ case being true implies that the $m = k + 1$ case is also true. Therefore by mathematical induction, $\forall m \geq 2, \forall i \in [m], \forall \lambda_i \geq 0, \forall \mathbf{x}_i \in \mathbb{R}^n$:

$$f : \mathbb{R}^n \mapsto \mathbb{R} \text{ convex} \iff f\left(\sum_{i=1}^m \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^m \lambda_i f(\mathbf{x}_i) \quad \text{where } \sum \lambda_i = 1$$

□

QUESTION 5:

- (a) Show that the sequence $x_k = 1 + \left(\frac{1}{2}\right)^{2^k}$ is Q-quadratically convergent to 1.

SOLUTION: We know that $\lim_{k \rightarrow \infty} x_k = 1$, consider:

$$\begin{aligned} \left\| 1 + \left(\frac{1}{2}\right)^{2^k} - 1 \right\|_2 &\geq \gamma \left\| 1 + \left(\frac{1}{2}\right)^{2^{k-1}} - 1 \right\|_2^2 \\ \left(\frac{1}{2}\right)^{2^k} &\geq \gamma \left[\left(\frac{1}{2}\right)^{2^{k-1}} \right]^2 \\ \left(\frac{1}{2}\right)^{2^k} &\geq \gamma \left(\frac{1}{2}\right)^{2^k} \end{aligned}$$

Which is true $\forall \gamma \leq 1, \forall k \geq 1$, therefore the sequence x_k is Q-quadratically convergent.

- (b) Does the sequence $x_k = \frac{1}{k!}$ converge Q-superlinearly? Q-quadratically?
- (c) Consider the sequence x_k defined by

$$x_k = 5$$

Is this sequence Q-superlinearly convergent? Q-quadratically convergent? R-quadratically convergent? Justify your answer.