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Charilaos Lyritsakis

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The dissertation of Charilaos Lyrtsakis was reviewed and approved* by the following:

Kostas Papakonstantinou
Associate Professor of Civil & Environmental Engineering
Dissertation Advisor
Chair of Committee

Gordon Warn
Associate Professor of Civil & Environmental Engineering

Pinlei Chen
Assistant Professor of Civil & Environmental Engineering

Tong Qiu
Professor of Civil & Environmental Engineering

Reuben Kraft
Associate Professor of Mechanical Engineering

Patrick Fox
Professor of Civil & Environmental Engineering
Head of Department of Civil & Environmental Engineering

* Signatures are on file in the Graduate School.

Abstract

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To my parents

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Nomenclature

\vec{E}_i	Spatial basis unit vectors, $i = 1, 2, 3$
\vec{e}_i	Material basis unit vectors, $i = 1, 2, 3$
X_i	Initial configuration coordinates, $i = 1, 2, 3$
\vec{X}	Position vector of a material point in the initial configuration
x_i	Current configuration coordinates, $i = 1, 2, 3$
\vec{x}	Position vector of a material point in the current configuration
\vec{r}_o	Position vector of a material point on the centerline in the initial configuration
\vec{r}	Position vector of a material point on the centerline in the current configuration
\vec{u}	Displacement vector of points on the centerline with respect to the initial configuration.
\vec{t}	Position vector of a material point on the local system of a cross-section.
ϕ	Angle between \vec{E}_i and \vec{e}_i
$\mathbf{R}(\phi)$	Rotation matrix
κ	Curvature of the beam centerline.
ϵ	Axial strain of the beam centerline
γ	Shear strain of the beam centerline
\vec{s}	Vector containing ϵ and γ but not κ
u	Displacement component along X_1
v	Displacement component along X_2
\vec{d}	Displacement vector containing u , v and ϕ
\vec{q}	Strain vector containing ϵ , γ and κ

f	Generic objective function
\vec{h}	Vector of discretized constraints
\mathcal{S}	Constrained space for \vec{x}
\mathcal{L}	Lagrangian function associated with the NLP
$\vec{\lambda}$	Vector of Lagrange multipliers
Π	Total potential energy function
U	Strain energy function
W	Potential energy of external loads
\mathcal{W}_s	Cross-section strain energy density
\vec{P}	Vector of external loads
\vec{F}_{sec}	Cross-section stress resultants
N	Section axial force
V	Section shear force
M	Section internal moment
E	Young's modulus
G	Shear modulus
I	Cross-section moment of inertia
A	Cross-section area
A_s	Effective area of the cross-section
ℓ	Element length
n_{nod}	Number of non-restrained nodes
n	Number of quadrature points in the element
w_i	Quadrature weight for the i -th quadrature point
ξ	Normalized coordinate associated with X_1
L_i	Lagrange cardinal function
\mathbf{T}	Curvature mapping matrix
Θ	Vandermonde matrix

\vec{y}	Internal field variable vector, associated with unknowns at quadrature points
Λ	Element orientation matrix
θ	Angle of element centerline in the initial configuration with \vec{E}_1
\mathcal{E}^e	Energy contribution of element e in the Lagrangian function
\mathbf{H}	Hessian matrix
\vec{z}	Vector of all unknown state variables of the system
\mathbf{k}_s	Section stiffness matrix
\mathbf{K}	Stiffness matrix
\vec{F}_{int}	Internal nodal force vector
ΔS	Incremental arc-length parameter
$\vec{\sigma}_f$	Active stress components on a fiber
$\vec{\epsilon}_f$	Strain components corresponding to $\vec{\sigma}_f$
\mathbf{N}_s	Cross-section shape function matrix
φ	Shear strain distribution function along the cross-section height
\mathbf{C}	Material tangent modulus
\vec{d}_{lorr}	Element nodal displacement vector for left (l) or right (r) element edge
\vec{P}_{lorr}	Element externally applied nodal force vector for left (l) or right (r) element edge

Acronyms

DOF Degree of Freedom.

FEM Finite Element Method.

NLP Non-Linear Programming.

TPE Total Potential Energy.

Acknowledgements

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CHAPTER 1
INTRODUCTION AND BACKGROUND

CHAPTER 2

HYBRID BEAM ELEMENT FORMULATION

2.1 Introduction

In this chapter we present the formulation of a novel hybrid beam element which is based on NLP principles. The kinematic assumptions adopted fall under the category of geometrically exact or Simo-Reissner beam theory[1, 2, 3, 4], whereby no simplifying approximation is made with respect to the strain-displacement equations. This allows for capturing arbitrarily large displacements and rotations, as well as accounting for the effect of shear deformation at the section level.

As opposed to deriving the system equations from the Galerkin form,, we recast the problem in an NLP framework by utilizing the underlying variational structure. The total potential energy (TPE) functional is augmented with all relevant conditions that enforce the exact kinematics, by introducing a set of Lagrange multipliers that act as conjugate force quantities. The resulting modified functional is then approximated by employing a Gauss-Legendre quadrature rule, which yields the objective function to be minimized. With this particular approach, the primary variables in the element interior contributing to the elastic strain energy are the generalized strain measures of the centroid, which are the unknown quantities at the quadrature points. Displacement measures at the edge nodes of the element, namely, the translations along coordinate axes and the rotation of cross sections, are only associated with the external work. Kinematic consistency between the rotational measures of displacement and strains is enforced by using a Lagrange interpolation scheme for the curvature field, similar to the one used by Neuenhofer and Filippou [5] and Schulz & Filippou [6] for force-based elements. In this work, the points used for the interpolation coincide with the integration points of the quadrature rule used for approximating the en-

ergy functional. To avoid ill-conditioning issues of the linearized operator, we also outline a block-elimination procedure for the linear system involving the Hessian matrix so that the reduced system includes only displacement components. The corresponding stiffness operator is well-conditioned, sparse, banded and symmetric. This renders standard FEM routines from existing codes reusable and, in conjunction with the capability for accuracy even with crude discretization, provides an attractive formulation for fast and accurate computation. Additionally, accuracy and locking free performance are guaranteed with just one element per structural member, even in the presence of arbitrarily large displacements and rotations. Accordingly, the element can also capture high curvature gradients due to plastic hinge formation, in the case of inelastic analysis.

This chapter starts with a brief outline of the beam geometric description while adhering to the derivations by Cardona & Geradin[7], followed by the hybrid NLP formulation for the element. The chapter concludes with implementation details, where special attention is given to the block elimination technique used for the reduction of the linear system involving the Hessian matrix.

2.2 Geometrically-Exact Model

2.2.1 Beam Kinematics

Let $\{\mathbf{E}\}$ be a fixed orthonormal coordinate frame with unit vectors $\{\vec{\mathbf{E}}_1, \vec{\mathbf{E}}_2, \vec{\mathbf{E}}_3\}$ along the axes X_1, X_2, X_3 . For an initially straight beam of length ℓ with its centerline coinciding with X_1 , the undeformed configuration is completely described by:

$$\vec{\mathbf{X}}(X_1, X_2) = X_1 \vec{\mathbf{E}}_1 + X_2 \vec{\mathbf{E}}_2 = \vec{\mathbf{r}}_o(X_1) + \vec{\mathbf{t}}(X_2) \quad (2.1)$$

where $\vec{\mathbf{r}}_o$ traces the centerline in the reference configuration and $\vec{\mathbf{t}}$ locates a material point on the cross section. Assuming a rectangular cross-section of height h , without loss of generality, we have that $X_1 \in [0, \ell]$ and $X_2 \in [-h/2, h/2]$. We should note here that X_3 is

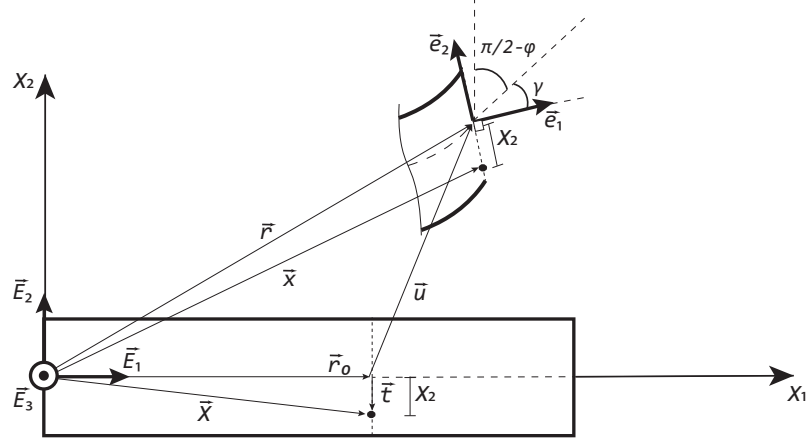


Figure 2.1: Undeformed and deformed configurations of the beam.

omitted as it does not explicitly come into the expressions. We nevertheless hold on to the \mathbb{R}^3 vector formalism to maintain consistency with the representation of rotation as a linear operator. After the application of a displacement field $\vec{u}(X_1)$, the deformed configuration is given by vector \vec{x} such that:

$$\vec{r} = \vec{r}_o + \vec{u} \quad (2.2)$$

$$\vec{x} = \vec{r} + X_2 \vec{e}_2 \quad (2.3)$$

where $\{\mathbf{e}\}$ is a local coordinate system attached at cross-sections that completely describes their orientation. For the unit vectors of the local base \vec{e}_1 , \vec{e}_2 , \vec{e}_3 , we let \vec{e}_1 be normal to the cross-section, but not necessarily tangent to the deformed centerline, and \vec{e}_2 (and \vec{e}_3) coincide with the cross-section principal axes of inertia, as shown in Fig. (2.1).

If \mathbf{R} is the rotation operator that rotates $\{\mathbf{E}\}$ to $\{\mathbf{e}\}$, then:

$$\vec{e}_i = \mathbf{R} \vec{E}_i, \quad i = 1, 2, 3 \quad (2.4)$$

For the plane case, the component form of \mathbf{R} reduces to:

$$\mathbf{R} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.5)$$

Then, using Eq. (2.4), Eq. (2.3) becomes:

$$\vec{x} = \vec{r} + \mathbf{R}\vec{t} \quad (2.6)$$

For Eqs. (2.3),(2.6) to hold in this manner, we tacitly assume that cross-sections remain rigid during deformation, that is, the position of a material point on a cross-section with respect to the centroid does not change.

2.2.2 Strain Measures, Strain-Displacement Relations

Within the context of geometrically-exact formulations, it is common to adopt as a strain measure \vec{s} the difference of the position vector gradients, \vec{r} and \vec{r}_o , with respect to the local frame (e.g. see [2, 7, 8]). Denoting derivatives with respect to X_1 by $(\cdot)'$, we get:

$$\vec{s} = \mathbf{R}^T \vec{r}' - \vec{r}_o' \quad (2.7)$$

whereas the cross-section curvature is given as:

$$\kappa = \phi' \quad (2.8)$$

The axial and shear strain of the centroid, ϵ and γ respectively, can then be retained as

follows:

$$\epsilon = \vec{e}_1^T \vec{s} \quad (2.9)$$

$$\gamma = \vec{e}_2^T \vec{s} \quad (2.10)$$

where $\vec{s} = \begin{bmatrix} \epsilon & \gamma & 0 \end{bmatrix}^T$, with the third component, representing the shear strain along the out-of-plane axis, being by assumption zero.

Using Eq. (2.2) and performing the differentiations in Eq. (2.7) with respect to X_1 , we get:

$$\vec{s} = \mathbf{R}^T [\vec{u}' + \vec{E}_1] - \vec{E}_1 \quad (2.11)$$

Deriving the exact differential strain-displacement relations as presented in Reissner¹ is straightforward if we solve Eq. (2.11) for \vec{u}' :

$$\vec{u}' = \mathbf{R} \vec{s} + \vec{e}_1 - \vec{E}_1 \quad (2.12)$$

If we write $\vec{u} = \begin{bmatrix} u & v & 0 \end{bmatrix}^T$, then the explicit expressions for u', v' along with Eq. (2.8) are:

$$u' = \vec{E}_1^T \vec{u}' = (1 + \epsilon) \cos \phi - \gamma \sin \phi - 1 \quad (2.13a)$$

$$v' = \vec{E}_2^T \vec{u}' = (1 + \epsilon) \sin \phi + \gamma \cos \phi \quad (2.13b)$$

$$\phi' = \kappa \quad (2.13c)$$

By introducing vectors \vec{d} and \vec{q} , such that $\vec{d} = \begin{bmatrix} u & v & \phi \end{bmatrix}^T$ and $\vec{q} = \begin{bmatrix} \epsilon & \gamma & \kappa \end{bmatrix}^T$, we can maintain the structure of Eq. (2.12) and express all fields involved in matrix notation, as

¹See equations (15) in [1]

follows:

$$\vec{d}' = \mathbf{R}\vec{q} + \vec{e}_1 - \vec{E}_1 \quad (2.14)$$

Strain-displacement equations as represented in Eq. (2.14) will be subsequently utilized in the formulation of the beam model.

2.3 Nonlinear Programming Formulation

The hybrid beam element discretization is presented by utilizing the underlying variational structure of the static problem. The minimizing principle for the TPE is recast here in an NLP formulation and the strain-displacement equations in (2.14) are incorporated, representing the equality constraints of the problem.

2.3.1 NLP framework

The NLP problem adopted herein is of the following form:

$$\begin{aligned} &\text{minimize} && f(\vec{x}) \\ &\text{subject to} && \vec{h}(\vec{x}) = \vec{0} \\ &&& \vec{x} \in \mathcal{S} \end{aligned} \quad (2.15)$$

The objective function $f(\vec{x})$ represents the total potential energy of the structure, whereas vector $\vec{h}(\vec{x})$ includes the active constraints of the program. The corresponding Lagrangian is:

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \vec{\lambda}^T \vec{h}(\vec{x}) \quad (2.16)$$

where $\vec{\lambda}$ are the Lagrange multipliers. The first-order optimality conditions are given from

the following expression:

$$\nabla \mathcal{L} = \vec{0} \quad (2.17)$$

This is the standard form of a nonlinear constrained minimization program with equality constraints, as encountered in the literature (e.g. [9]).

2.3.2 Total potential energy

We consider the following decomposition of the total potential energy of a beam:

$$\Pi = U - W \quad (2.18)$$

where U and W are the stored strain energy and potential energy associated with external loading respectively. For the case of concentrated external loads, Eq. (2.18) can be expressed as:

$$\Pi = \int_0^\ell \mathcal{W}_S(X_1, \vec{q}) dX_1 - \sum_{i=1}^2 \vec{P}_i^T \vec{u}_i \quad (2.19)$$

where \mathcal{W}_S is the strain energy density of a cross section at X_1 and \vec{q} denotes the neutral axis strain vector, \vec{P}_i, \vec{u}_i the element nodal force and displacement vectors, respectively. It can be proven[10] that for strain hardening materials within the premises of small strain elastoplasticity, the exact solution to the problem renders Π an absolute minimum. Equivalently, the exact solution solves the program in Eq. (2.15) for $f = \Pi$ with \vec{h} acting as constraints. Assuming a stable work-hardening material, the corresponding Euler-Lagrange equations yield the constitutive equations of elastoplasticity (e.g. see discussion in Simo and Hughes [11]). As it will be shown in subsequent sections, since the constitutive update is carried out locally at the section fiber level.

Section stress resultants are then given by the gradient of \mathcal{W}_S :

$$\vec{\mathbf{F}}_{sec} = \nabla_{\mathbf{q}} \mathcal{W}_S \quad (2.20)$$

with $\vec{\mathbf{F}}_{sec} = \begin{bmatrix} N & V & M \end{bmatrix}^T$. In the case of linear elasticity, the energy density is given by $\mathcal{W}_S = \frac{1}{2}[EA\epsilon^2 + GA_s\gamma^2 + EI\kappa^2]$ and the section resultants are:

$$\vec{\mathbf{F}}_{sec} = \begin{bmatrix} EA\epsilon \\ GA_s\gamma \\ EI\kappa \end{bmatrix}$$

where A_s is the effective shear area of the cross-section. Numerical approximation of Eq. (2.19) by an appropriate quadrature yields:

$$f(\vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_n, \vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_N) = \sum_{i=1}^n w_i \mathcal{W}_{S,i} - \sum_{i=1}^2 \vec{\mathbf{P}}_i^T \vec{\mathbf{u}}_i \quad (2.21)$$

with w_i and n being the weights and the number of integration points respectively, and $\mathcal{W}_{S,i} = \mathcal{W}_S(\vec{\mathbf{q}}_i)$. Eq. (2.21) serves as the objective function of the NLP of Eq. (2.15).

2.3.3 Strain-Displacement Constraints

The constrained conditions are derived by applying the same quadrature rule on the integral form of the strain-displacement equations (2.14):

$$\vec{\mathbf{d}}(\ell) - \vec{\mathbf{d}}(0) = \int_0^\ell \vec{\mathbf{d}}' dX_1 = \int_0^\ell \mathbf{R}\vec{\mathbf{q}} + \vec{\mathbf{e}}_1 - \vec{\mathbf{E}}_1 dX_1 \quad (2.22)$$

Using Eq. (2.4), numerical approximation of the right-hand side integral leads to the ele-

ment specific constraints:

$$\vec{h}^A = \vec{d}(\ell) - \vec{d}(0) - \sum_{i=1}^n w_i \mathbf{R}_i(\vec{q}_i + \vec{E}_1) + \ell \vec{E}_1 = \vec{0} \quad (2.23)$$

The component form of Eq. (2.23) can be given as:

$$\vec{h}^A = \begin{bmatrix} u(\ell) - u(0) - \sum_{i=1}^n w_i \left[(\epsilon_i + 1) \cos \phi_i - \gamma_i \sin \phi_i \right] + \ell \\ v(\ell) - v(0) - \sum_{i=1}^n w_i \left[(\epsilon_i + 1) \sin \phi_i + \gamma_i \cos \phi_i \right] \\ \phi(\ell) - \phi(0) - \sum_{i=1}^n w_i \kappa_i \end{bmatrix} \quad (2.24)$$

Although the strain fields appear explicitly only in the weighted evaluation points of the quadrature, the rotational field ϕ is involved in both the integration points and the element edge nodes. As such, we introduce a Lagrange interpolation scheme for the curvature field in the same fashion as in [12]:

$$\kappa(\xi) = \sum_{i=1}^n L_i(\xi) \kappa_i \quad (2.25)$$

where L_i are the Lagrange cardinal functions:

$$L_i(\xi) = \frac{\prod_{j=1, j \neq i}^n (\xi - \xi_j)}{\prod_{j=1, j \neq i}^n (\xi_i - \xi_j)}, \quad \xi = \frac{X_1}{\ell}$$

Substitution of Eq. (2.25) to Eq. (2.13c) yields the expression for rotations ϕ_i :

$$\phi_i - \phi(0) = \sum_{j=1}^n \left(\int_0^{\xi_i} L_j(x) dx \right) \kappa_j = \sum_{i=1}^n T_{ij} \kappa_j \quad (2.26)$$

with derivation of T_{ij} illustrated in Fig. (2.2). In matrix form, the above equation can be

directly restated as a linear equality constraint set as:

$$\vec{h}^B = \vec{\phi} - \phi(0)\vec{1} - \mathbf{T}\vec{\kappa} = \vec{0} \quad (2.27)$$

where

$$\vec{\phi} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}^T, \vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^n$$

$$\mathbf{T} = L \begin{bmatrix} \xi_1 & \frac{\xi_1^2}{2} & \cdots & \frac{\xi_1^n}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n & \frac{\xi_n^2}{2} & \cdots & \frac{\xi_n^n}{n} \end{bmatrix} \Theta^{-1}, \Theta = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \xi_n & \xi_n^2 & \cdots & \xi_n^{n-1} \end{bmatrix}$$

We can then collect all active constraints, Eqs. (2.23),(2.27), in one vector \vec{h} :

$$\vec{h} = \begin{bmatrix} \vec{h}^A \\ \vec{h}^B \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} \quad (2.28)$$

containing all constraints pertaining to one element. While vector \vec{h}^A of strain-displacement constraints will always contain three components for each element, the number of components in vector \vec{h}^B will depend on the chosen quadrature rule.

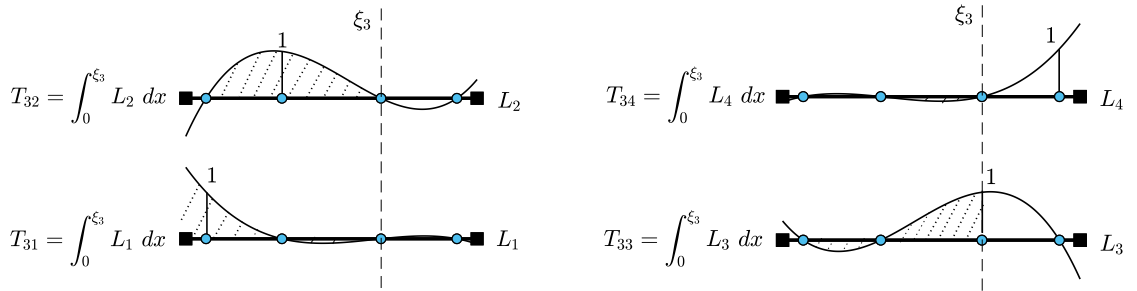


Figure 2.2: Integration of curvature shape functions.

2.3.4 Lagrangian function

In order to derive the Lagrangian function in the form of Eq. (2.16), we introduce a vector $\vec{\lambda}$ of additional Lagrange multiplier variables and augment Eq. (2.21) with Eq. (2.28):

$$\mathcal{L} = f + \vec{\lambda}^T \vec{h} \quad (2.29)$$

A more convenient form for the Lagrangian function can also be achieved by expanding f into its constituents, namely, the element stored energy and the potential energy:

$$\mathcal{L} = \sum_{e=1}^{n_{el}} U^e - W + \vec{\lambda}^T \vec{h} \quad (2.30)$$

where:

$$U^e = \sum_{i=1}^n w_i \mathcal{W}_{S,i}^e, \quad W = \sum_{i=1}^N \vec{P}_i^T \vec{u}_i$$

Thereby, additional elements can be simply incorporated by adding their stored energy in the corresponding sum, and considering element constraints by augmenting vectors $\vec{\lambda}$, \vec{h} , after transforming them to the global system. Potential function W is “global” in the sense that external loads and the corresponding work-conjugate displacement degrees of freedom are directly added to the expression and are not influenced by adding nodal contributions from adjacent elements. As a result, no additional connectivity constraints at the element interfaces are needed, and Eq. (2.30) is thus a global function for the whole structure.

2.4 Implementation

The set of vectors associated with internal field variables, $\{\vec{y}_i\}_{i=1}^n$, is defined as $\vec{y}_i = \begin{bmatrix} \vec{q}_i & \phi_i \end{bmatrix}^T = \begin{bmatrix} \epsilon_i & \gamma_i & \kappa_i & \phi_i \end{bmatrix}^T$. Moreover, and according to Fig. (2.3), we designate the local element edge degrees of freedom in vector form as $\hat{d}_l = \vec{d}(0)$ and $\hat{d}_r = \vec{d}(\ell)$. The ele-

ment displacement vector is thus $\hat{\vec{d}} = \begin{bmatrix} \hat{\vec{d}}_l^T & \hat{\vec{d}}_r^T \end{bmatrix}^T$. The corresponding global displacement vector \vec{d}_g is related to the local vector $\hat{\vec{d}}$ via the typical linear transformation Λ :

$$\hat{\vec{d}} = \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda \end{bmatrix} \vec{d}_g, \quad \Lambda = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.31)$$

The element constraints of Eqs. (2.23),(2.27) transformed in the global system become:

$$\vec{h}_g^A = \mathbf{V}_1 \vec{d}_g - \Lambda^T \left[\sum_{i=1}^n w_i \mathbf{R}_i(\vec{q}_i + \vec{E}_1) - \ell \vec{E}_1 \right] \quad (2.32a)$$

$$\vec{h}_g^B = \vec{\phi} - \mathbf{V}_2 \vec{d}_g - \mathbf{T} \vec{\kappa} \quad (2.32b)$$

where:

$$\mathbf{V}_1 = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} \vec{0} & \vec{0} & \vec{1} & \vec{0} & \vec{0} & \vec{0} \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{0} \in \mathbb{R}^n$$

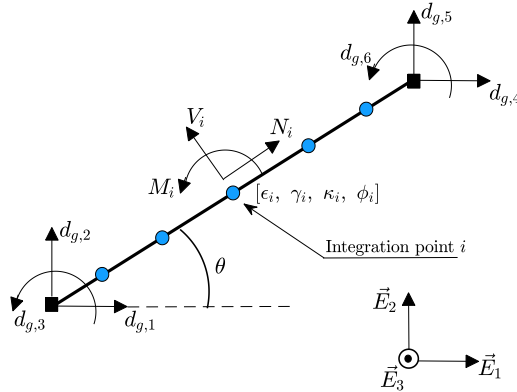


Figure 2.3: Typical element representing one member.

In the equation above the superscript denoting an element is omitted for clarity. The element contribution to the Lagrangian of the assemblage, denoted \mathcal{E}^e , is then:

$$\mathcal{E}^e(\vec{y}_1, \dots, \vec{y}_n, \vec{d}_g, \vec{\lambda}^e) = U^e + (\vec{\lambda}^e)^T \vec{h}_g \quad (2.33)$$

Notice that the external potential function W is not included in Eq. (2.33) as it is added directly via the work done by external forces in the global degrees of freedom and not from element contributions. The element Lagrange multipliers $\vec{\lambda}^e$ are force conjugate measures at the corresponding degrees of freedom and U^e is the sum total of cross-section strain energies. The Lagrangian of the whole system is accordingly given by:

$$\mathcal{L}(\vec{y}, \vec{d}_g, \vec{\lambda}) = U - W + \vec{\lambda}^T \vec{h}_g \quad (2.34)$$

which is a restatement of Eq. (2.30) in the global coordinate system and $\vec{y} = \begin{bmatrix} \vec{y}_1^T & \vec{y}_2^T & \dots & \vec{y}_m^T \end{bmatrix}^T$, with m being the total number of quadrature points in the structure.

2.4.1 Equilibrium

In what follows, it is assumed we are dealing with quantities pertaining to the total assemblage, after all element contributions have been resolved. The first-order necessary optimality condition for the Lagrangian function (see Eq. (2.17)), after the imposition of boundary conditions, yields the following relations:

$$\nabla_{\mathbf{y}_i} \mathcal{L} = \nabla_{\mathbf{y}_i} U + [\nabla_{\mathbf{y}_i} \vec{h}_g] \vec{\lambda} = \vec{0} \quad (2.35a)$$

$$\nabla_{\mathbf{d}_g} \mathcal{L} = -\nabla_{\mathbf{d}_g} W + [\nabla_{\mathbf{d}_g} \vec{h}_g] \vec{\lambda} = \vec{0} \quad (2.35b)$$

$$\nabla_{\lambda} \mathcal{L} = \vec{h}_g = \vec{0} \quad (2.35c)$$

Eq. (2.35a) expresses the equilibrium between external and internal forces acting on the i -th

cross-section for $i = 1, 2, \dots, m$, while Eq. (2.35b) ensures consistency between externally applied loads and Lagrange multipliers. Finally, Eq. (2.35c) requires that all constraints are active when the minimum is attained. Explicit expressions for the first and second derivatives of all quantities involved are given in the Appendix. We should note that derivatives of the strain energy U are computed numerically during the cross-section state determination phase in Sec. (2.4.3).

2.4.2 Hessian matrix

The Hessian matrix of the Lagrangian function contains second order information of the system and is utilized during the solution phase. Eqs. (2.35a)-(2.35c) constitute a set of nonlinear algebraic equations and an iterative scheme is needed to solve the system. In block matrix form, the Hessian is provided as:

$$\mathbf{H} = \begin{bmatrix} \nabla_{yy}^2 \mathcal{L} & \nabla_{yd_g}^2 \mathcal{L} & \nabla_{y\lambda}^2 \mathcal{L} \\ \nabla_{yd_g}^2 \mathcal{L}^T & \nabla_{d_g d_g}^2 \mathcal{L} & \nabla_{d_g \lambda}^2 \mathcal{L} \\ \nabla_{y\lambda}^2 \mathcal{L}^T & \nabla_{d_g \lambda}^2 \mathcal{L}^T & \nabla_{\lambda\lambda}^2 \mathcal{L} \end{bmatrix} = \begin{bmatrix} \nabla_{yy}^2 \mathcal{L} & \mathbf{0} & \nabla_y \vec{h}_g^T \\ \mathbf{0} & \mathbf{0} & \nabla_{d_g} \vec{h}_g^T \\ \nabla_y \vec{h}_g & \nabla_{d_g} \vec{h}_g & \mathbf{0} \end{bmatrix} \quad (2.36)$$

The resulting Hessian is fairly sparse. Moreover, the block matrix $\nabla_{yy}^2 U$ which contains section stiffness information is block-diagonal (see Appendix B) and its symmetry guarantees the full symmetry of \mathbf{H} .

2.4.3 Solution scheme

As mentioned in Appendix B, the nonlinear system of equations in Eq. (2.17) is solved in an incremental-iterative fashion. Linearization around the current iteration k yields:

$$\nabla \mathcal{L}^k + \mathbf{H}^k \delta \vec{z}^k = \vec{0} \quad (2.37)$$

where \vec{z} is the vector of all unknowns:

$$\vec{z} = \begin{bmatrix} \vec{y} \\ \vec{d}_g \\ \vec{\lambda} \end{bmatrix}, \quad \text{with} \quad \vec{y} = \begin{bmatrix} \vec{y}_1 \\ \vdots \\ \vec{y}_m \end{bmatrix}$$

Reference to the current iteration step is omitted subsequently for clarity. Solving Eq. (2.37) directly may be cumbersome in some cases, especially for problems that require denser distribution of integration points (e.g. plasticity), since the Hessian is not banded and may also be badly conditioned. In addition, implementation of continuation schemes in order to attain solutions past critical points is more involved since the state variable vector contains displacements, strains and Lagrange multiplier unknowns. A different approach is hence sought, where the initial linear system is reduced to a smaller one involving only the displacement vector \vec{d}_g . Restating the system in terms of its distinct vector components \vec{y} , \vec{d}_g and $\vec{\lambda}$ gives:

$$\begin{bmatrix} \nabla_y \mathcal{L} \\ \nabla_{d_g} \mathcal{L} \\ \nabla_{\lambda} \mathcal{L} \end{bmatrix} + \begin{bmatrix} \nabla_{yy}^2 \mathcal{L} & \mathbf{0} & \nabla_y \vec{h}_g^T \\ \mathbf{0} & \mathbf{0} & \nabla_{d_g} \vec{h}_g^T \\ \nabla_y \vec{h}_g & \nabla_{d_g} \vec{h}_g & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta \vec{y} \\ \delta \vec{d}_g \\ \delta \vec{\lambda} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \end{bmatrix} \quad (2.38)$$

Utilizing Eqs. (2.35a)-(2.35c) and setting $\delta \vec{\lambda} = \vec{\lambda}^{k+1} - \vec{\lambda}^k$, Eq. (2.38) can be rewritten as:

$$\begin{bmatrix} \nabla_y U \\ -\vec{P} \\ \vec{h}_g \end{bmatrix} + \begin{bmatrix} \nabla_{yy}^2 \mathcal{L} & \mathbf{0} & \nabla_y \vec{h}_g^T \\ \mathbf{0} & \mathbf{0} & \nabla_{d_g} \vec{h}_g^T \\ \nabla_y \vec{h}_g & \nabla_{d_g} \vec{h}_g & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta \vec{y} \\ \delta \vec{d}_g \\ \vec{\lambda}_{k+1} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \end{bmatrix} \quad (2.39)$$

As mentioned previously, the gradient of the strain energy with respect to the strain vector represents the cross-sectional stress resultants. Solving the first equation in the system of

Eq. (2.39) for $\delta\vec{y}$ we get:

$$\delta\vec{y} = -\nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}^{-1} \left[\vec{F}_{sec} + \nabla_{\mathbf{y}} \vec{h}_g^T \vec{\lambda}_{k+1} \right] \quad (2.40)$$

Substituting Eq. (2.40) in the third equation of Eq. (2.39) and solving for $\vec{\lambda}^{k+1}$ yields:

$$\vec{\lambda}^{k+1} = \mathbf{B}^{-1} [\vec{h}_g - \vec{b}] + \mathbf{B}^{-1} [\nabla_{\mathbf{d}_g} \vec{h}_g] \delta\vec{d}_g \quad (2.41)$$

By substituting Eq. (2.41) into the second system equation Eq. (2.35b) we arrive at the familiar static equilibrium form of the system of equations:

$$\mathbf{K} \delta\vec{d}_g = \vec{P} - \vec{F}_{int} \quad (2.42)$$

which can now be solved for the iterative displacement vector $\delta\vec{d}_g$. Vectors \vec{b} , \vec{F}_{int} and matrices \mathbf{B} , \mathbf{K} , are given by the following explicit formulas in the global system:

$$\vec{b} = [\nabla_{\mathbf{y}} \vec{h}_g] [\nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}]^{-1} \vec{F}_{sec} \quad (2.43a)$$

$$\vec{F}_{int} = [\nabla_{\mathbf{d}_g} \vec{h}_g]^T [\mathbf{B}]^{-1} [\vec{h}_g - \vec{b}] \quad (2.43b)$$

$$\mathbf{B} = [\nabla_{\mathbf{y}} \vec{h}_g] [\nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}]^{-1} [\nabla_{\mathbf{y}} \vec{h}_g]^T \quad (2.43c)$$

$$\mathbf{K} = [\nabla_{\mathbf{d}_g} \vec{h}_g]^T [\mathbf{B}]^{-1} [\nabla_{\mathbf{d}_g} \vec{h}_g] \quad (2.43d)$$

where $\vec{b} \in \mathbb{R}^p$, $\vec{F}_{int} \in \mathbb{R}^{3N}$, N is again the number of structural nodes, $\mathbf{B} \in \mathbb{R}^{p \times p}$ with $p = m + 3n_{nel}$, and $\mathbf{K} \in \mathbb{R}^{3N \times 3N}$.

Equivalently, an assembly process can also be implemented by casting Eqs. (2.43a), (2.43c) in local form, where they can be further simplified by being expanded in terms of

element cross-section contributions:

$$\vec{b}^e = \sum_{i=1}^n [\nabla_{\mathbf{y}_i} \vec{h}_g] [\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 \mathcal{L}]^{-1} \vec{F}_{sec}^{(i)} \quad (2.44a)$$

$$\mathbf{B}^e = \sum_{i=1}^n [\nabla_{\mathbf{y}_i} \vec{h}_g] [\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 \mathcal{L}]^{-1} [\nabla_{\mathbf{y}_i} \vec{h}_g]^T \quad (2.44b)$$

where n is the number of element quadrature points, $\vec{b}^e \in \mathbb{R}^{n+3}$ and $\mathbf{B}^e \in \mathbb{R}^{(n+3) \times (n+3)}$. The element stiffness matrix $\mathbf{K}^e \in \mathbb{R}^{6 \times 6}$ and internal force vector $\vec{F}_{int}^e \in \mathbb{R}^6$ are given by the same expressions as in Eqs. (2.43b), (2.43d), but with the gradients now cast in the local element form. For the assembly, the standard FEM routines can be directly employed and the resulting global stiffness operator retains all properties typically associated with it in the context of classical finite element analysis, i.e. it is a symmetric positive definite matrix, it is well-conditioned and, importantly, it is sparse and banded. Hence, in this case, the global internal force vector and the global stiffness matrix are given by the standard assembly process, designated here via operator Λ :

$$\vec{F}_{int} = \bigwedge_{e=1}^{n_{el}} \vec{F}_{int}^e \quad (2.45)$$

$$\mathbf{K} = \bigwedge_{e=1}^{n_{el}} \mathbf{K}^e \quad (2.46)$$

with

$$\vec{F}_{int}^e = [\nabla_{\mathbf{d}_g} \vec{h}_g^e]^T [\mathbf{B}^e]^{-1} [\vec{h}_g^e - \vec{b}^e] \quad (2.47)$$

$$\mathbf{K}^e = [\nabla_{\mathbf{d}_g} \vec{h}_g^e]^T [\mathbf{B}^e]^{-1} [\nabla_{\mathbf{d}_g} \vec{h}_g^e] \quad (2.48)$$

where $\vec{h}_g^e \in \mathbb{R}^{n+3}$ denotes the *element* constraint vector given by Eq. (2.28) and $\nabla_{\mathbf{d}_g} \vec{h}_g^e \in \mathbb{R}^{(n+3) \times 6}$ its gradient with respect to the global displacement degrees of freedom (DOFs)

associated with it, is given in Appendix A. It is clear that with this formulation the inversion of large global matrices for the computation of \mathbf{K} and $\vec{\mathbf{F}}_{int}$ is avoided and, instead, only inversion of the local elements flexibility matrices \mathbf{B}^e is required. Given that for highly nonlinear problems we typically have $n \in \mathbb{N}([5, 10])$ for satisfactory accuracy, this results in an element flexibility matrix of dimension $\dim(\mathbf{B}^e) \leq 13$, thus accelerating the analysis considerably.

Having written the system of equations in the form of Eq. (2.42), implementation of arc-length type schemes is now straightforward. In the present work, we adopt the algorithm proposed by Crisfield [13] whereby the additional equation supplemented to the system is:

$$\Delta \vec{\mathbf{d}}_g^T \Delta \vec{\mathbf{d}}_g = \Delta S^2 \quad (2.49)$$

with ΔS being the user-specified arc-length parameter. The incremental displacement vector $\Delta \vec{\mathbf{d}}_g$ is updated in each iteration as follows:

$$\Delta \vec{\mathbf{d}}_g^k = \Delta \vec{\mathbf{d}}_g^{k-1} + \delta \vec{\mathbf{d}}_g^k \quad (2.50)$$

where k denotes the current iteration and $\delta \vec{\mathbf{d}}_g^k$ is the vector of iterative displacements.

Solution updating

After the determination, at an arbitrary iteration k within step j , of the iterative displacement vector $\delta \vec{\mathbf{d}}_g^k$ from Eq. (2.42), the strain and Lagrange multiplier vectors in Eqs. (2.40),(2.41) have then to be updated. The detailed steps of the updating procedure are as follows:

$$\text{Step } j, \text{ iteration } k: \left\{ \vec{\mathbf{y}}_j^0, \vec{\mathbf{d}}_{g,j}^0, \vec{\boldsymbol{\lambda}}_j^0, \vec{\mathbf{P}}_j, \Delta \vec{\mathbf{y}}_j^{k-1}, \Delta \vec{\mathbf{d}}_{g,j}^{k-1} \right\}$$

- (1):** Get section stiffnesses $\nabla_{\mathbf{y}\mathbf{y}}^2 \mathcal{L}^k$ from Eq. (B.3) in Appendix B and section forces $\vec{\mathbf{F}}_{sec}^k$ from section integration, Eq. (2.55).

- (2): Evaluate \vec{b}^k and \mathbf{B}^k from Eqs. (2.43a), (2.43c) respectively (or, alternatively, $\vec{b}^{e\ k}$, $\mathbf{B}^{e\ k}$ from Eqs. (2.44a), (2.44b)).
- (3): Evaluate \mathbf{K}^k from Eq. (2.43d), (or, alternatively, from Eqs. (2.46), (2.48)).
- (4): Solve Eq. (2.42) for $\delta\vec{d}_g^k$.
- (5): Update incremental displacement vector: $\Delta\vec{d}_{g,j}^k \leftarrow \Delta\vec{d}_{g,j}^{k-1} + \delta\vec{d}_g^k$
- (6): Update displacement vector: $\vec{d}_{g,j}^k \leftarrow \vec{d}_{g,j}^0 + \Delta\vec{d}_g^k$
- (7): Update Langrange multiplier $\vec{\lambda}_j^k$ from Eq. (2.41).
- (8): Evaluate iterative strain vector $\delta\vec{y}^k$ from Eq. (2.40).
- (9): Update incremental strain vector: $\Delta\vec{y}_j^k \leftarrow \Delta\vec{y}_j^{k-1} + \delta\vec{y}^k$
- (10): Update total strain vector: $\vec{y}_j^k \leftarrow \vec{y}_j^0 + \Delta\vec{y}_j^k$
- (11): Evaluate \vec{F}_{int}^k from Eq. (2.43b) (or, alternatively, from Eqs. (2.45), (2.47)).
- (12): Check $\|\vec{P}_j - \vec{F}_{int}^k\| \leq \text{tol} \cdot \|\vec{P}_j - \vec{F}_{int}^0\|$.
- (13): If **FALSE**, set $k \leftarrow k + 1$ and go to (1). If **TRUE**, set $j \leftarrow j + 1$, $\vec{y}_j^0 \leftarrow \vec{y}_{j-1}^k$, $\vec{d}_{g,j}^0 \leftarrow \vec{d}_{g,j-1}^k$, $\vec{\lambda}_j^0 \leftarrow \vec{\lambda}_{j-1}^k$.

2.4.4 Cross-section state determination

During the updating scheme described in the previous section, the section stress resultants \vec{F}_{sec} , along with the matrix $\nabla_{yy}^2 \mathcal{L}$, are required. The latter, as seen subsequently, corresponds to the generalized section stiffnesses. In the general case where inelastic behavior is considered, the stress-strain constitutive law has to be integrated at the cross-section level and the material properties have to be updated accordingly. In this work, each cross-section is discretized in n_l number of layers and the stress update is performed independently for each layer. This is equivalent to the composite midpoint rule applied along the height of

the cross-section. The shear coefficient k_s is introduced as a correction factor for the simplifying aforementioned assumption, in accordance with Cowper [14].

Below we present the kinematic and constitutive relations for a cross-section object. The (multiaxial) constitutive law at the fiber level is treated in the next chapter. The update procedure is regarded as strain-driven, in the sense that the known initial state is updated given an increment in the centerline strains at a particular quadrature point. First, we define the stress and strain vectors associated with a particular fiber:

$$\vec{\sigma}_f = \begin{bmatrix} \sigma_{11} & \sigma_{12} \end{bmatrix}^T, \quad \vec{\epsilon}_f = \begin{bmatrix} \epsilon_{11} & \gamma_{12} \end{bmatrix}^T \quad (2.51)$$

In line with the plane section hypothesis, the fiber strain vector can be determined from the centerline strain vector as follows:

$$\begin{aligned} \epsilon_{11} &= \varepsilon - X_2 \kappa \\ \gamma_{12} &= \varphi(X_2) \gamma \end{aligned} \iff \vec{\epsilon}_f = \mathbf{N}_s \vec{q}, \quad \mathbf{N}_s = \begin{bmatrix} 1 & 0 & -X_2 \\ 0 & \varphi(X_2) & 0 \end{bmatrix} \quad (2.52)$$

where $\varphi(X_2)$ is an as of now unspecified function that defines a shear strain distribution along the height of the cross-section.

Thus, given $\left\{ \vec{q}, \vec{q}^{pl} \right\}$, $\Delta \vec{q}$, along with a set of internal state variables (e.g. accumulated plastic strain), we can evaluate the incremental strains at the midpoint of a layer at distance X_2 from the neutral axis as follows:

$$\vec{\epsilon}_f = \mathbf{N}_s \vec{q} \quad (2.53a)$$

$$\Delta \vec{\epsilon}_f = \mathbf{N}_s \Delta \vec{q} \quad (2.53b)$$

where it is reminded that \vec{q} is the vector containing the neutral axis generalized strains and $\vec{\epsilon}_f$ is the strain vector associated with the fiber midpoint. With Eqs. (2.53a), (2.53b) and the initial state known, we can perform the stress update which will yield the updated

stress vector $\vec{\sigma}_f$ and the (consistent) elastoplastic modulus for the fiber. The conjugate stress resultants associated with the assumed kinematic assumptions are derived from the element virtual work equation:

$$\vec{P}_l^T \delta \vec{d}_l + \vec{P}_r^T \delta \vec{d}_r = \int_V \vec{\sigma}_f^T \delta \vec{\epsilon}_f dV = \int_0^L \left[\int_A \mathbf{N}_s^T \vec{\sigma}_f dA \right]^T \vec{q} dX_1 \quad (2.54)$$

where we define the conjugate section stress resultants associated with \vec{q} as follows:

$$\vec{F}_{sec}^{(i)} = \int_A \mathbf{N}_s^T \vec{\sigma}_f dA \quad (2.55)$$

Application of the composite midpoint rule on Eq. (2.55) yields:

$$\vec{F}_{sec}^{(i)} \approx \begin{bmatrix} \sum_{j=1}^{n_l} \sigma_{11,j} \Delta A_j \\ \sum_{j=1}^{n_l} \sigma_{12,j} \Delta A_j \\ \sum_{j=1}^{n_l} X_{2,j} \sigma_{11,j} \Delta A_j \end{bmatrix} \quad (2.56)$$

where ΔA_j the area of layer j .

Let $\mathbf{C} = \partial \vec{\sigma}_f / \partial \vec{\epsilon}_f$ designate the tangent modulus at a fiber. During elastic steps, $\mathbf{C} \equiv \mathbf{C}^{el} = \text{diag}[E, G]$. In the next chapter we will see that during plastic steps the diagonal components of \mathbf{C} are, generally, also non-zero. The tangent section stiffness is derived as follows:

$$\mathbf{k}_{sec}^{(i)} = \frac{\partial \vec{F}_{sec}^{(i)}}{\partial \vec{q}} = \int_A \mathbf{N}_s^T \frac{\partial \vec{\sigma}_f}{\partial \vec{q}} dA = \int_A \mathbf{N}_s^T \frac{\partial \vec{\sigma}_f}{\partial \vec{\epsilon}_f} \frac{\partial \vec{\epsilon}_f}{\partial \vec{q}} dA = \int_A \mathbf{N}_s^T \mathbf{C} \mathbf{N}_s dA \quad (2.57)$$

Application of the midpoint rule on Eq. (2.57) yields the following component form for

$\mathbf{k}_s^{(i)}$:

$$\mathbf{k}_{sec}^{(i)} = \begin{bmatrix} C_{11}^i & \varphi^i C_{12}^i & -X_2^i C_{11}^i \\ \varphi^i C_{21}^i & (\varphi^i)^2 C_{22}^i & -x_2^i \varphi^i C_{21}^i \\ -X_2^i C_{11}^i & -x_2^i \varphi^i C_{12}^i & (X_2^i)^2 C_{11}^i \end{bmatrix} \quad (2.58)$$

where C_{ij} are the components of the elastoplastic consistent tangent modulus of the fiber.

The section stiffness is then given by the following sum:

$$\mathbf{k}_{sec} = \sum_{i=1}^{n_f} \mathbf{k}_{sec}^{(i)} \Delta A^i \quad (2.59)$$

Given an increment in nodal displacements at the global/structural level, the assembly process has to loop over all elements in order to determine the element stiffness and internal nodal forces. This procedure, illustrated in Fig. 2.4, involves analysis at three different levels: the global/element, the section state determination, and the fiber stress update. Having briefly outlined the basic theory for the hybrid element as it pertains to the first two levels mentioned above, next we introduce the elastoplastic stress update formulation, which pertains to the third level of the solution process and results in a fast return mapping algorithm.

The generalized section stiffness, $\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 \mathcal{L}$, is given by the second derivatives of the Lagrangian with respect to strain vector $\vec{\mathbf{y}}_i = \begin{bmatrix} \vec{\mathbf{q}}_i^T & \phi_i \end{bmatrix}^T$:

$$\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 \mathcal{L} = \nabla_{\mathbf{y}_i \mathbf{y}_i}^2 U + \nabla_{\mathbf{y}_i} ([\nabla_{\mathbf{y}_i} \vec{\mathbf{h}}_g] \vec{\lambda}) \quad (2.60)$$

where:

$$\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 U = w_i \begin{bmatrix} \mathbf{k}_{sec}^{(i)} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$$

For the explicit expression of the second term in the right-hand side of Eq. (2.60), see Ap-

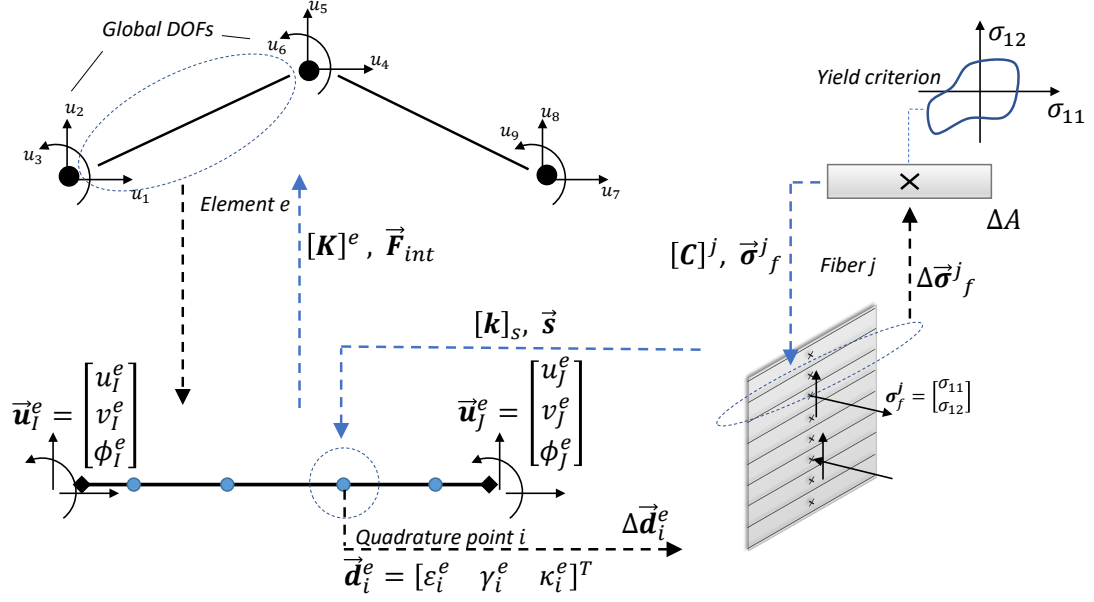


Figure 2.4: Solutions process stages. Quantities \mathbf{K}^e and \vec{F}_{int} stand for stiffness matrix and internal nodal force vector for element e respectively.

pendix B. It generally includes parameters associated with section rotations and Lagrange multipliers, and is greatly simplified if small displacement assumptions are used.

CHAPTER 3

RESULTS

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CHAPTER 4

DISCUSSION

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CHAPTER 5

CONCLUSION

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Appendices

APPENDIX A

EXPERIMENTAL EQUIPMENT

Explicit expressions of the formulation are provided in this Appendix. We note that operations involving the symbol ∇ map scalars to vectors and vectors to matrices. For example, if $f : \mathbb{R}^p \rightarrow \mathbb{R}$ and $\vec{v} : \mathbb{R}^p \rightarrow \mathbb{R}^q$, then $\nabla f : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $\nabla \vec{v} : \mathbb{R}^p \rightarrow \mathbb{R}^{p \times q}$.

The Lagrangian function is used here, as presented in Eq. (2.30):

$$\mathcal{L}(\vec{y}, \vec{d}_g, \vec{\lambda}) = U - W + \vec{\lambda}^T \vec{h}_g \quad (\text{A.1})$$

where it is also reminded that $\vec{y} = \begin{bmatrix} \vec{y}_1^T & \vec{y}_2^T & \dots & \vec{y}_m^T \end{bmatrix}^T$, $\vec{y}_i = \begin{bmatrix} \vec{q}_i & \phi_i \end{bmatrix}^T$ and $\vec{q}_i = \begin{bmatrix} \epsilon_i & \gamma_i & \kappa_i \end{bmatrix}^T$. The number m designates the total number of integration points in the structure.

A.0.1 Gradient of strain energy

The gradient of U is expressed as:

$$\nabla U = \begin{bmatrix} \nabla_{\mathbf{y}} U \\ \nabla_{\mathbf{d}_g} U \\ \nabla_{\lambda} U \end{bmatrix}$$

The gradient with respect to the strain vector \vec{y} gives the stress resultants at all cross-

sections and has the following form for a particular cross-section i :

$$\nabla_{\mathbf{y}_i} U = \begin{bmatrix} w_i N_i \\ w_i V_i \\ w_i M_i \\ 0 \end{bmatrix} = w_i \begin{bmatrix} \vec{\mathbf{F}}_{sec}^{(i)} \\ 0 \end{bmatrix} \quad (\text{A.2})$$

it is usually computed by numerical integration of Eqs. (2.55) over the cross-section height.

In the case of elastic analysis, the above vector can be computed explicitly for each cross-section, since then $\vec{\mathbf{F}}_{sec}^{(i)} = \begin{bmatrix} EA\epsilon_i & k_s GA\gamma_i & EI\kappa_i \end{bmatrix}^T$.

The gradients of U with respect to both $\vec{\mathbf{d}}_g$ and $\vec{\boldsymbol{\lambda}}$ provide zero vectors:

$$\nabla_{\mathbf{d}_g} U = \vec{\mathbf{0}} \in \mathbb{R}^{3N} \quad \text{and} \quad \nabla_{\boldsymbol{\lambda}} U = \vec{\mathbf{0}} \in \mathbb{R}^s$$

where $s = m + 3n_{el}$, n_{el} being the number of elements. Finally, we then get:

$$\nabla U = \begin{bmatrix} \nabla_{\mathbf{y}} U \\ \vec{\mathbf{0}} \\ \vec{\mathbf{0}} \end{bmatrix} \quad (\text{A.3})$$

A.0.2 Gradient of constraints

We restate the constraints equations from Eqs. (2.32a)-(2.32b):

$$\vec{\mathbf{h}}_g = \begin{bmatrix} \vec{\mathbf{h}}_g^A \\ \vec{\mathbf{h}}_g^B \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \vec{\mathbf{d}}_g - \boldsymbol{\Lambda}^T \left[\sum_{i=1}^n w_i \mathbf{R}_i (\vec{\mathbf{q}}_i + \vec{\mathbf{E}}_1) - \ell \vec{\mathbf{E}}_1 \right] \\ \vec{\boldsymbol{\phi}} - \mathbf{V}_2 \vec{\mathbf{d}}_g - \mathbf{T} \vec{\boldsymbol{\kappa}} \end{bmatrix}$$

For notational simplicity, we assume here only one element in the derivations that follow.

When more elements are used, their corresponding constraint vectors are stacked in vector form as $\vec{\mathbf{h}} = \begin{bmatrix} \vec{\mathbf{h}}_g^{1T} & \vec{\mathbf{h}}_g^{2T} & \dots & \vec{\mathbf{h}}_g^{n_{el}T} \end{bmatrix}^T$. The gradient with respect to the strain vector

\vec{y}_i at a particular cross-section is given by:

$$\nabla_{\mathbf{y}_i} \vec{h}_g^A = \left[\nabla_{\mathbf{q}_i} \vec{h}_g^A \quad \frac{d\vec{h}_g^A}{d\phi_i} \right] = -w_i \mathbf{\Lambda}^T \begin{bmatrix} \mathbf{R}_i & \mathbf{R}_i \mathbf{X} [\vec{q}_i + \vec{E}_1] \end{bmatrix}$$

where:

$$\frac{d\mathbf{R}_i}{d\phi_i} = \mathbf{R}_i \mathbf{X} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The gradient of \vec{h}_g^B with respect to strain vector \vec{y}_i , is:

$$\nabla_{\mathbf{y}_i} \vec{h}_g^B = \left[\nabla_{\mathbf{q}_i} \vec{h}_g^B \quad \frac{d\vec{h}_g^B}{d\phi_i} \right] = \begin{bmatrix} \vec{0} & \vec{0} & -\mathbf{T} \hat{\mathbf{E}}_i & \hat{\mathbf{E}}_i \end{bmatrix}$$

where $\hat{\mathbf{E}}_i \in \mathbb{R}^n$ is a unit vector with the i -th component equal to one. Collectively, we have:

$$\nabla_{\mathbf{y}_i} \vec{h}_g = \begin{bmatrix} -w_i \mathbf{\Lambda}^T \mathbf{R}_i & -w_i \mathbf{\Lambda}^T \mathbf{R}_i \mathbf{X} [\vec{q}_i + \vec{E}_1] \\ [\vec{0} & \vec{0} & -\mathbf{T} \hat{\mathbf{E}}_i] & \hat{\mathbf{E}}_i \end{bmatrix} \quad (\text{A.4})$$

The gradient with respect to the displacement vector is derived in a straightforward manner by simple derivations:

$$\nabla_{\mathbf{d}_g} \vec{h}_g = \begin{bmatrix} \mathbf{V}_1 \\ -\mathbf{V}_2 \end{bmatrix} \quad (\text{A.5})$$

$$\mathbf{V}_1 = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \end{bmatrix} \quad , \quad \mathbf{V}_2 = \begin{bmatrix} \vec{\mathbf{0}} & \vec{\mathbf{0}} & \vec{\mathbf{1}} & \vec{\mathbf{0}} & \vec{\mathbf{0}} & \vec{\mathbf{0}} \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \vec{\mathbf{0}}, \vec{\mathbf{1}} \in \mathbb{R}^n, \quad \vec{\mathbf{0}} = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T, \quad \vec{\mathbf{1}} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$$

Finally, the gradient with respect to the vector $\vec{\lambda}$ gives a zero matrix:

$$\nabla_{\lambda} \vec{\mathbf{h}}_g = \mathbf{0} \in \mathbb{R}^{(n+3) \times (n+3)} \quad (\text{A.6})$$

The total constraints gradient is then expressed in block matrix form as:

$$\nabla \vec{\mathbf{h}}_g = \begin{bmatrix} \nabla_{\mathbf{y}_1} \vec{\mathbf{h}}_g & \nabla_{\mathbf{y}_2} \vec{\mathbf{h}}_g & \dots & \nabla_{\mathbf{y}_n} \vec{\mathbf{h}}_g & \nabla_{\mathbf{d}_g} \vec{\mathbf{h}}_g & \nabla_{\lambda} \vec{\mathbf{h}}_g \end{bmatrix}$$

Note that in the case of small displacements, only the expression $\nabla_{\mathbf{y}_i} \vec{\mathbf{h}}_g^A$ has to be modified, so that the strain-displacement relations reduce to the classical linear Timoshenko theory:

$$\nabla_{\mathbf{y}_i} \vec{\mathbf{h}}_g^A = -w_i \mathbf{\Lambda}^T \begin{bmatrix} 1 & -\phi_i & 0 & -\gamma_i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

with the vector of constraints $\vec{\mathbf{h}}_g^A$ being:

$$\vec{\mathbf{h}}_g^A = \mathbf{V}_1 \vec{\mathbf{d}}_g - \mathbf{\Lambda}^T \left[\sum_{i=1}^n w_i [\hat{\mathbf{R}}_i(\vec{\mathbf{q}}_i + \vec{\mathbf{E}}_1) - \vec{\mathbf{g}}_i] - \ell \vec{\mathbf{E}}_1 \right] \quad (\text{A.7})$$

where:

$$\hat{\mathbf{R}}_i = \begin{bmatrix} 1 & -\phi_i & 0 \\ \phi_i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{\mathbf{g}}_i = \begin{bmatrix} 0 \\ \epsilon_i \phi_i \\ 0 \end{bmatrix}$$

The vector $\vec{\mathbf{g}}_i$ is added in order to eliminate the second order term $\epsilon_i \phi_i$ arising from the imposition of the small displacement assumption $\sin \phi_i \approx \phi_i$, $\cos \phi_i \approx 1$ in Eq. (2.32a).

A.0.3 Gradient of the potential energy

The potential energy due to external loads $\vec{\mathbf{P}}$ is:

$$W = \sum_{j=1}^N \vec{\mathbf{P}}_j^T \vec{\mathbf{d}}_{g,j} = \vec{\mathbf{d}}_g^T \vec{\mathbf{P}}$$

The gradient of the potential energy is, again, straightforward in its derivation as it only depends on the displacement vector:

$$\nabla W = \begin{bmatrix} \nabla_{\mathbf{y}} W \\ \nabla_{\mathbf{d}_g} W \\ \nabla_{\lambda} W \end{bmatrix}$$

with $\nabla_{\mathbf{y}} W = \vec{\mathbf{0}} \in \mathbb{R}^m$, $\nabla_{\mathbf{d}_g} W = \vec{\mathbf{P}}$ and $\nabla_{\lambda} W = \vec{\mathbf{0}} \in \mathbb{R}^{n+3}$ for one element. Thus, we have:

$$\nabla W = \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{P}} \\ \vec{\mathbf{0}} \end{bmatrix} \quad (\text{A.8})$$

A.0.4 Gradient of the Lagrangian

We substitute in Eqs. (2.35) the explicit expressions derived here in the Appendix:

$$\nabla_{\mathbf{y}_i} \mathcal{L} = w_i \begin{bmatrix} \vec{\mathbf{F}}_{sec}^{(i)} \\ 0 \end{bmatrix} + \begin{bmatrix} -w_i \boldsymbol{\Lambda}^T \mathbf{R}_i & -w_i \boldsymbol{\Lambda}^T \mathbf{R}_i \mathbf{X} [\vec{\mathbf{q}}_i + \vec{\mathbf{E}}_1] \\ [\vec{\mathbf{0}} \quad \vec{\mathbf{0}} \quad -\mathbf{T} \hat{\mathbf{E}}_i] & \hat{\mathbf{E}}_i \end{bmatrix}^T \vec{\boldsymbol{\lambda}} \quad (\text{A.9})$$

$$\nabla_{\mathbf{d}_g} \mathcal{L} = - \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{P}} \\ \vec{\mathbf{0}} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_1 \\ -\mathbf{V}_2 \end{bmatrix}^T \vec{\boldsymbol{\lambda}} \quad (\text{A.10})$$

$$\nabla_{\boldsymbol{\lambda}} \mathcal{L} = \vec{\mathbf{h}}_g \quad (\text{A.11})$$

APPENDIX B

HESSIAN OF THE LAGRANGIAN

The Hessian form in Eq. (2.36), is restated here:

$$\mathbf{H} = \begin{bmatrix} \nabla_{yy}^2 \mathcal{L} & \mathbf{0} & \nabla_y \vec{h}_g^T \\ \mathbf{0} & \mathbf{0} & \nabla_{dg} \vec{h}_g^T \\ \nabla_y \vec{h}_g & \nabla_{dg} \vec{h}_g & \mathbf{0} \end{bmatrix}$$

The gradients of \vec{h}_g have been derived in the previous section. Thus, for the complete specification of the second order information we need only to determine the second derivative matrix with respect to the strain vector \vec{y} . Since different integration points are independent of one another, then it is clear that $\nabla_{y_i y_j}^2 \mathcal{L} = \mathbf{0}$ for $i \neq j$. Therefore, the matrix $\nabla_{yy}^2 \mathcal{L}$ has a block diagonal form:

$$\nabla_{yy}^2 \mathcal{L} = \begin{bmatrix} \nabla_{y_i y_i}^2 \mathcal{L} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \nabla_{y_i y_i}^2 \mathcal{L} & \cdots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \nabla_{y_m y_m}^2 \mathcal{L} \end{bmatrix}$$

and making use of Eq. (2.60) gives:

$$\nabla_{y_i y_i}^2 \mathcal{L} = \nabla_{y_i y_i}^2 U + \nabla_{y_i} ([\nabla_{y_i} \vec{h}_g]^T \vec{\lambda}) \quad (\text{B.1})$$

As already seen, the second derivative matrix of the strain energy yields the generalized

section stiffness of cross-section i :

$$\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 U = w_i \begin{bmatrix} \mathbf{k}_{sec}^{(i)} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$$

where $\mathbf{k}_{sec}^{(i)}$ is given by Eq. (2.58) after numerical integration of the cross-section. We only now need to derive the explicit form for the second term of the right hand side in Eq. (B.1). To this end, we separate the Lagrange multiplier vector into two parts, which correspond to vectors $\vec{\mathbf{h}}_h^A$ and $\vec{\mathbf{h}}_h^B$ respectively:

$$\vec{\boldsymbol{\lambda}} = \begin{bmatrix} \vec{\boldsymbol{\lambda}}^A \\ \vec{\boldsymbol{\lambda}}^B \end{bmatrix}$$

Eq. (A.9) can be now restated as:

$$\nabla_{\mathbf{y}_i} \mathcal{L} = w_i \begin{bmatrix} \vec{\mathbf{F}}_{sec}^{(i)} \\ 0 \end{bmatrix} - w_i \begin{bmatrix} \mathbf{R}_i^T \boldsymbol{\Lambda} \\ [\vec{\mathbf{q}}_i + \vec{\mathbf{E}}_1]^T \mathbf{X}^T \mathbf{R}_i \boldsymbol{\Lambda} \end{bmatrix} \vec{\boldsymbol{\lambda}}^A + \begin{bmatrix} \vec{\mathbf{0}}^T \\ \vec{\mathbf{0}}^T \\ -\hat{\mathbf{E}}_i^T \mathbf{T}^T \\ \hat{\mathbf{E}}_i^T \end{bmatrix} \vec{\boldsymbol{\lambda}}^B \quad (\text{B.2})$$

Noticing that the gradient of $\vec{\mathbf{h}}_g^B$ yields components independent of strains, we conclude that the second derivatives of \mathcal{L} will include only the part $\vec{\boldsymbol{\lambda}}^A$, as:

$$\nabla_{\mathbf{y}_i \mathbf{y}_i}^2 \mathcal{L} = \nabla_{\mathbf{y}_i \mathbf{y}_i}^2 U + \mathbf{Z}_i \quad (\text{B.3})$$

where:

$$\mathbf{Z}_i = \begin{bmatrix} \mathbf{0}_{(3 \times 3)} & \vec{\mathbf{t}}_i \\ \vec{\mathbf{t}}_i^T & \vec{\mathbf{t}}_i^T \mathbf{X} [\vec{\mathbf{q}}_i + \vec{\mathbf{E}}_1] \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{t}}_i = -w_i \mathbf{X}^T \mathbf{R}_i^T \boldsymbol{\Lambda} \vec{\boldsymbol{\lambda}}^A$$

and the second derivative matrix of the Lagrangian is now fully determined.

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