Numerical Methods HW 7 Number 3

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Show that if $A = a_{ik}$ is Hermitian, then for every diagonal element a_{ii}, there exists an eigenvalue $\lambda(A)$ of A such that

$$|\lambda(A) - a_{ii}| \leq \sqrt{\sum_{k \neq i} |a_{ij}|^2} \tag{1}$$

PROOF

Let λ be an eigenvalue of A with the eigenvector v. Then v satisfies the eigenvalue equation

$$\sum_{j=1}^{N} a_{ij} v_j = \lambda v_i \tag{2}$$

Suppose that v_k is the element of v having the largest absolute value. Then equation 2 for $i = k$ becomes

$$\sum_{j=1}^{N} a_{kj} v_j = \lambda v_k \tag{3}$$

Next, consider the equation $|\lambda(A) - a_{kk}||v_k|$, using equation 3 we can write

$$|\lambda(A) - a_{kk}||v_k| = |\sum_{j=1}^{N} a_{kj} v_j - a_{kk} v_k| = |\sum_{k \neq i} a_{ij} v_j| = \sqrt{|\sum_{k \neq i} a_{ij} v_j|^2} \leq \sqrt{\sum_{k \neq i} |a_{ij}|^2 |v_j|^2} \tag{4}$$

Where the last inequality comes from the Pythagorean Theorem. Since $v_j \leq v_k$ we can continue

$$|\lambda(A) - a_{kk}||v_k| \leq \sqrt{\sum_{k \neq i} |a_{ij}|^2 |v_j|^2} \leq |v_k| \sqrt{\sum_{k \neq i} |a_{ij}|^2} \tag{5}$$

Dividing the first and last expression in equation 5 by v_k we get the desired expression

$$|\lambda(A) - a_{kk}| \leq \sqrt{\sum_{k \neq i} |a_{ij}|^2} \tag{6}$$

Because A is Hermitian there are N linearly independent eigenvectors v, so equation 6 can be written for each eigenvector λ corresponding to that eigenvector v.

Note

This proof is a modification on the proof of Gerschgorin's Disk Theorem found in [1, p. 297].

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[1] Friedberg, S. H.,Insel, A. J., Spence, L. E., *Linear Algebra.* Pearson Education, Inc., New Jersey, 4th Edition, 2003