Term Structure Models - Assignment

Group 5

Contents

1 Ex 01 - Plot real and nominal yields

We wish to investigate market expectations on future inflation in the US. We obtain real and nominal yields from respectively US TIPS and nominal bonds provided by the Federal Reserve. We extract the end of month yield data from the time period Jan-2005 until Aug-2022 for maturities 2, 3, 5, 7, and 10 years. In order to get a feeling of the data, we plot the nominal and real yields in the figures 1 and 2 below.

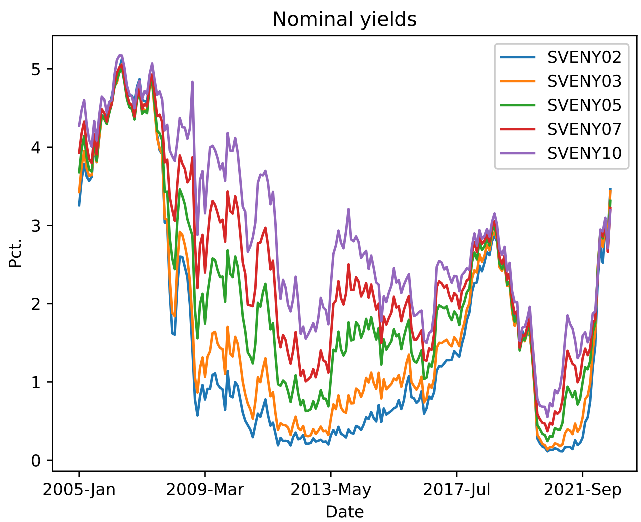

Figure 1: Nominal yields

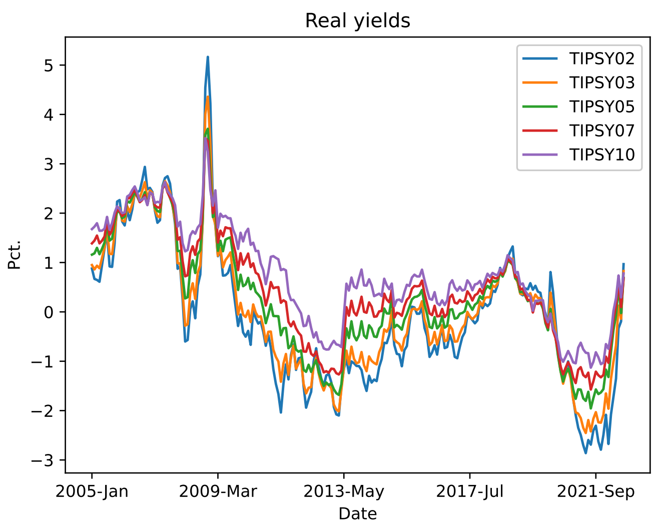

Figure 2: Real yields

2 Ex 02 - Principal Component Analysis (PCA) of yields

First the features[1] are standardized using the map

$$x \mapsto \frac{x - \bar{\mu}_x}{\bar{\sigma}_x},$$

where $\bar{\mu}_x$ and $\bar{\sigma}_x$ denotes the empirical mean and standard deviation of the feature x, receptively. This is done to avoid any dimension being disproportionally inflated compared to others. As such the data is prepared to for the Principal Component Analysis (PCA).

The amount of variation captured by including 1-5 components are shown in table 1.

Components	**EVar_Real**	**EVar_Nom**	**Cum_EVar_Real**	**Cum_EVar_Nom**
1	96.05	93.09	96.05	93.09
2	3.72	6.75	99.77	99.84
3	0.22	0.16	99.99	100.00
4	0.01	0.00	100.00	100.00
5	0.00	0.00	100.00	100.00

Table 1: Amount of variation captured by number of principal components

The table suggests that the first three principal components capture almost the entire variation of both the nominal and real yield curves. In relation to this the plots in figure 3 shows the first three factor loadings for the maturities 1, 2, 5, 7, and 10 years for real and nominal yields, respectively.

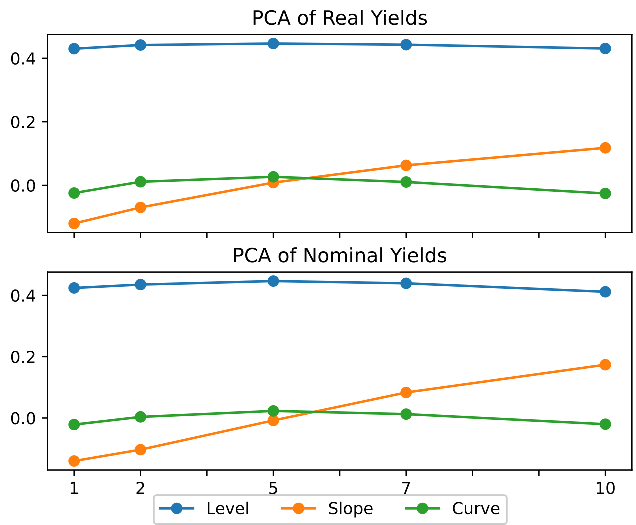

Figure 3: First 3 factor loadings for different maturities of real and nominal yields

[1]Here the set of *features* refers to the used yields.

The interpretation of the of the factor loadings are the same as usual. That is, the first describes the *level*, the second describes the *slope*, and third describes the *curvature* of the yield curve.

3 Ex 03 - Calculate and plot Break Even Inflation (BEI) for 2, 5, 10 years of maturity

To calculate the break even inflation (BEI) rates for maturity-matched nominal and real yields, we use the following formula

$$\pi^{\mathrm{BEI}}(t;\tau) = y^{\mathrm{N}}(t;\tau) - y^{\mathrm{R}}(t;\tau). \tag{3.1}$$

Applying this to formula to the maturities 2, 5, and 10 years produce the time series shown in figure 4

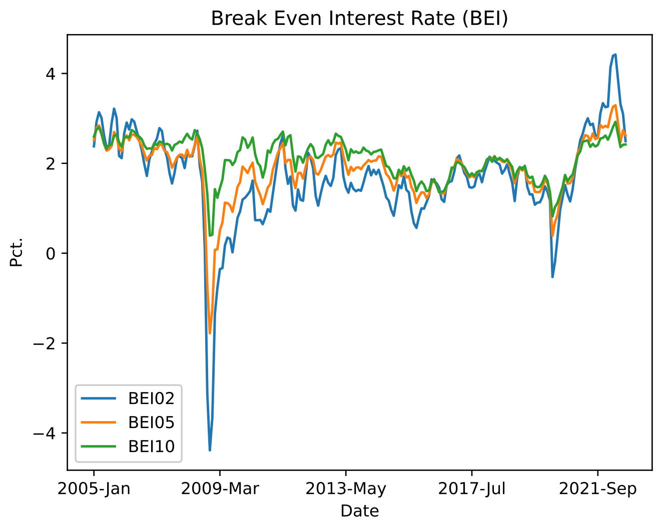

Figure 4: Break Even Inflation (BEI) rates

4 Ex 04 - Derive $A(\tau)$ and $B(\tau)$

We now consider the two-factor AFNS model whit Q-dynamics of the latent state varialbes given by

$$dX_t = K^Q\left[\theta^Q - X_t\right]dt + \Sigma dW_t^Q, \tag{4.1}$$

where

$$\theta^Q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad K^Q = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_L & 0 \\ 0 & \sigma_S \end{bmatrix}.$$

Furthermore, the short rate is defined as

$$r_t = \rho_0 Ex + \rho_1^\top X_t \quad \text{where} \quad \rho_0 = 0,\ \rho_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^\top. \tag{4.2}$$

Finally adopting the level and slope interpretation of the state variables such that $X_t = \begin{pmatrix} L_t & S_t \end{pmatrix}$ we have that $r_t = L_t + S_t$.

We now want to derive the functions $A(\tau)$ and $B(\tau)$ such that zero-coupon yields are computed as

$$y_t(\tau) = -\frac{A(\tau)}{\tau} - \frac{B(\tau)^\top}{\tau} X_t. \tag{4.3}$$

In the affine multifactor setup we consider a process of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where the drift and volatility processes are defined as

$$\mu(x) = K_0 + K_1 x, \quad (\sigma(x)\sigma(x)^\top)_{ij} = H_{0ij} + H_{1ij}x.$$

Relating the notation to our AFNS two-factor model we have

$$K_0 = K^Q\theta^Q, \quad K_1 = -K^Q, \quad H_{0ij} = \Sigma\Sigma^\top, \quad H_{1ij} = 0.$$

Taking outset in the ODE's from Laplace transformation with discounting these can be rewritten as

$$\begin{aligned}
\alpha'(\tau) &= -\rho_0 + K_0^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top H_0 \beta(\tau) = (K^Q\theta^Q)^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top \Sigma\Sigma^\top \beta(\tau) \\
\beta'(\tau) &= -\rho_1 + K_1^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^T H_1 \beta(\tau) = -\begin{pmatrix} 1 & 1 \end{pmatrix}^\top - (K^Q)^\top \beta(\tau),
\end{aligned}$$

with terminal conditions $\alpha(0) = 0$ and $\beta(0) = 0$. The second equation can be rewritten as

$$\beta'(\tau) + K^Q\beta(\tau) = -\begin{pmatrix} 1 & 1 \end{pmatrix}^\top. \tag{4.4}$$

In the first step of solving the ODE's we make the following observation by the use of the product rule

$$\begin{aligned}
\frac{\partial}{\partial\tau}\left[e^{(K^Q)^\top\tau}\beta(\tau)\right] &= (K^Q)^\top e^{(K^Q)^\top\tau}\beta(\tau) + e^{(K^Q)^\top\tau}\beta'(\tau) \\
&= (K^Q)^\top e^{(K^Q)^\top\tau}\beta(\tau) + e^{(K^Q)^\top\tau}\left(-\begin{pmatrix} 1 & 1 \end{pmatrix}^\top - K^Q\beta(\tau)\right). \\
&= -e^{(K^Q)^\top\tau}\begin{pmatrix} 1 & 1 \end{pmatrix}^\top,
\end{aligned}$$

where the expression of $\beta'(\tau)$ is inserted from the ODE system in the second equality. Then integrating both sides yields

$$\int_0^\tau \frac{\partial}{\partial\tau}\left[e^{(K^Q)^\top u}\beta(u)\right] du = -\int_0^\tau e^{(K^Q)^\top u}\begin{pmatrix} 1 & 1 \end{pmatrix}^\top du.$$

Furthermore using the boundary condition $\beta(0) = 0$ and isolate beta we obtain

$$\beta(\tau) = -e^{-(K^Q)^\top\tau}\int_0^\tau e^{(K^Q)^\top u}\begin{pmatrix} 1 & 1 \end{pmatrix}^\top du.$$

We note that the square matrix $K^Q \in \mathbb{R}^{2\times 2}$ is a diagonal matrix and so is $K^Q\tau$, $\tau \in \mathbb{R}_+$. Thus, we know that

$$\exp\left(K^Q\tau\right) = \begin{bmatrix} e^{0\cdot\tau} & 0 \\ 0 & e^{\lambda\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\lambda\tau} \end{bmatrix}, \tag{4.5}$$

and therefor

$$\begin{aligned}
\beta(\tau) &= -\begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda\tau} \end{bmatrix} \int_0^\tau \begin{bmatrix} 1 & 0 \\ 0 & e^{\lambda u} \end{bmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}^\top du \\
&= -\begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda\tau} \end{bmatrix} \int_0^\tau \begin{pmatrix} 1 & e^{\lambda u} \end{pmatrix}^\top du \\
&= -\begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda\tau} \end{bmatrix} \left[\begin{pmatrix} u & \frac{e^{\lambda u}}{\lambda} \end{pmatrix}^\top\right]_0^\tau \\
&= -\begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda\tau} \end{bmatrix} \left[\begin{pmatrix} \tau & \frac{e^{\lambda-1}}{\lambda} \end{pmatrix}^\top\right] \\
&= \begin{pmatrix} -\tau & -\frac{1}{\lambda} + \frac{e^{-\lambda\tau}}{\lambda} \end{pmatrix}^\top.
\end{aligned}$$

Then integrating the first equation of the ODE system and plugging in the arrive solution from above we obtain

$$\begin{aligned}
\alpha(\tau) &= \frac{1}{2}\int_0^\tau \beta(u)^\top \Sigma\Sigma^\top \beta(u) du \\
&= \frac{1}{2}\int_0^\tau \begin{pmatrix} -u & -\frac{1}{\lambda} + \frac{e^{-\lambda u}}{\lambda} \end{pmatrix} \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} \begin{pmatrix} -u & -\frac{1}{\lambda} + \frac{e^{-\lambda u}}{\lambda} \end{pmatrix}^\top du \\
&= \frac{1}{2}\int_0^\tau \sigma_{11}^2 u^2 + \sigma_{22}^2 \left(-\frac{1}{\lambda} + \frac{e^{-\lambda u}}{\lambda}\right)^2 du \\
&= \frac{1}{2}\int_0^\tau \sigma_{11}^2 u^2 + \frac{\sigma_{22}^2}{\lambda^2}\left(1 + e^{-2\lambda u} - 2e^{-\lambda u}\right) du \\
&= \frac{1}{2}\left[\frac{\sigma_{11}^2 u^3}{3} + \frac{\sigma_{22}^2}{\lambda^2}\left(u - \frac{e^{-2\lambda u}}{2\lambda} + \frac{2e^{-\lambda u}}{\lambda}\right)\right]_0^\tau \\
&= \frac{1}{2}\left[\frac{\sigma_{11}^2 \tau^3}{3} + \frac{\sigma_{22}^2}{\lambda^2}\left(\tau - \frac{e^{-2\lambda\tau} - 1}{2\lambda} + \frac{2e^{-\lambda\tau} - 2}{\lambda}\right)\right] \\
&= \frac{\sigma_{11}^2 \tau^3}{6} + \frac{\sigma_{22}^2}{2\lambda^2}\left(\tau - \frac{e^{-2\lambda\tau} - 1}{2\lambda} + \frac{2e^{-\lambda\tau} - 2}{\lambda}\right)
\end{aligned}$$

Concluding we have derived the ZCB yield formula on the form

$$y_t(\tau) = -\frac{\alpha(\tau)}{\tau} - \frac{\beta(\tau)}{\tau}.$$

5 Ex 05 - Calculate Zero Coupon Bonds (ZCB) yields for maturities 1, 5, and 10 years

For the parameter setting $\lambda = 0.5$, $\sigma_L = 0.005$, $\sigma_S = 0.01$, and $X_t = (0.02, -0.02)$ we calculate the zero coupon yields for the maturities 1, 5 and 10 years using the functions $\alpha(\tau)$ and $\beta(\tau)$ from above. The result is shown in table 2.

	1Y	5Y	10Y
ZCB	0.9958	0.9396	0.8567
Yield	0.0042	0.0125	0.0155

Table 2: Yields for ZCB with maturities 1, 5, and 10 years.

6 Ex 06 - Dynamics of latent state variables, X_t, under $\mathbb{P}$-measure

In a Gaussian setup we can relate the Q and P measure by considering the essentially affine market price of risk specified as:

$$\Lambda_t = \lambda_1 + \lambda_2 X_t,$$

with $\lambda_1 \in \mathbb{R}^4$ and $\lambda_2 \in \mathbb{R}^{4\times 4}$. Girsanov's theorem then states

$$dW_t^Q = dW_t^P + \Lambda_t = dW_t^P + \lambda_1 + \lambda_2 X_t.$$

Inserting this in the Q-dynamics of X_t we find

$$\begin{aligned} dX_t &= K^Q \left[\theta^Q - X_t\right] dt + \Sigma\left[\lambda_1 + \lambda_2 X_t\right] dt + \Sigma dW_t^P \\ &= \left(K^Q \left[\theta^Q - X_t\right] + \Sigma\left[\lambda_1 + \lambda_2 X_t\right]\right) dt + \Sigma dW_t^P, \end{aligned}$$

from which we apply a separation of variables argument (collecting terms depending on X_t and not) to obtain the equations:

$$\begin{cases} K^Q\theta^Q + \Sigma\lambda_1 = K^P\theta^P, \\ K^Q - \Sigma\lambda_2 = K^P. \end{cases} \tag{6.1}$$

By these relations we can in fact write up the dynamics of X_t under the P measure:

$$dX_t = K^P \left[\theta^P - X_t\right] dt + \Sigma dW_t^P.$$

Note also that for an appropriate Λ_t we can specify K^P and θ^P independently of K^Q and θ^Q. This is possible as λ_1 and λ_2 can be adjusted as needed since they are free variables.

7 Ex 07 - Derive solution of X under $\mathbb{P}$-measure

In order to derive the solution of X under the $\mathbb{P}$-measure we define

$$g(t, X_t) := e^{K^P t}\left(X_t - \theta^P\right) \tag{7.1}$$

and notice $g \in C^{1,2}$. By suppressing the arguments and apply itô:

$$\begin{aligned} dg &= K^P g dt + e^{K^P t} dX_t \\ &= K^P e^{K^P t}\left(X_t - \theta^P\right) dt + e^{K^P t}\left(K^P\left(\theta^P - X_t\right) dt + \Sigma dW_t^P\right) \\ &= K^P e^{K^P t}\left\{\left(X_t - \theta^P\right) + \left(\theta^P - X_t\right)\right\} dt + e^{K^P t}\Sigma dW_t^P \\ &= e^{K^P t}\Sigma dW_t^P, \end{aligned}$$

with the use of the dynamics of the state variable under P from above. The itô integral (over a finite time increment) then becomes

$$g_{t+\Delta t} = g_t + \Sigma \int_t^{t+\Delta t} e^{K^P u} dW_u^P \quad , \quad \Delta t \in \mathbb{R}_+.$$

Inserting g and rewriting gives us the solution

$$\begin{aligned} e^{K^P(t+\Delta t)} \left(X_{t+\Delta t} - \theta^P \right) &= e^{K^P t} \left(X_t - \theta^P \right) + \Sigma \int_t^{t+\Delta t} e^{K^P u} dW_u^P \\ \Leftrightarrow X_{t+\Delta t} &= \theta^P + e^{-K^P \Delta t} \left(X_t - \theta^P \right) + \Sigma \int_0^{\Delta t} e^{-K^P u} dW_u^P. \end{aligned}$$

8 Ex 08 - Conditional time t mean and variance of X_T under $\mathbb{P}$

Note the shift in time index:

$$X_T = \theta^P + e^{-K^P(T-t)}(X_t - \theta^P) + \Sigma \int_0^{T-t} e^{-K^P u} dW_u^P$$

Then it follows by standard moment calculations that

$$\mathbb{E}_t \left[X_T \right] = \left(I - e^{-K^P(T-t)} \right) \theta^P + e^{-K^P(T-t)} X_t \tag{8.1}$$

$$\mathbb{V}_t \left[X_T \right] = \int_0^{T-t} e^{-K^P u} \Sigma \Sigma^\top e^{-\left(K^P\right)^\top u} du, \tag{8.2}$$

where the conditionality makes the state variable well-known (and can be considered constant) at any previous time-point.

9 Ex 09 - Prediciton and update step in the Kalman filter

As the yields are affine in the latent state variables our measurement equation is of the form

$$y_t = A + BX_t + \varepsilon_t \quad , \quad \varepsilon \overset{\text{iid.}}{\sim} \mathcal{N}\left(0 \; ; \; H\right), \; H = \operatorname{diag}\left(\sigma^{err}\right)^2.$$

In our case $A \in \mathbb{R}^{10}$ has entries of the form $-\frac{A^i(\tau)}{\tau}$ where the first 5 entries are related to the nominal observations and the latter 5 are the real ones. Likewise $B \in \mathbb{R}^{10\times 4}$ is a block matrix of the form

$$B = \begin{bmatrix} \left(\frac{B_L^N(\tau)}{\tau}, \frac{B_S^N(\tau)}{\tau} \right) & \mathbf{0} \\ \mathbf{0} & \left(\frac{B_L^R(\tau)}{\tau}, \frac{B_S^R(\tau)}{\tau} \right) \end{bmatrix} \quad , \quad \dim(\mathbf{0}) = (5 \times 2).$$

The transition equation is defined as

$$X_t = C_t + F_t X_{t-1} + \xi_t, \quad \xi_t \sim \mathcal{N}\left(0 \; ; \; Q_t\right). \tag{9.1}$$

Matching this with equation 8.1, we get

$$C_t = \left(I - e^{-K^P(T-t))}\right)\theta^P,$$
$$F_t = e^{-K^P(T-t)}.$$

We can now write up the prediction and update step.
Prediction step:

$$X_{t|t-1} = \mathbb{E}\left[C_t + F_t X_{t-1} + \xi_t | y_{t-1}, ...y_1\right] = F_t X_{t-1|t-1} + C_t \tag{9.2}$$
$$P_{t|t-1} = \mathbb{V}\left[C_t + F_t X_{t-1} + \xi_t | y_{t-1}, ...y_1\right] = F_t P_{t-|t-1} F_t^\top + Q_t. \tag{9.3}$$

Here we let Q_t be the the conditional variance as in equation 8.2, i.e.

$$Q_t = \int_0^{T-t} e^{-K^P s}\Sigma\Sigma^\top e^{-\left(K^P\right)^\top s} ds. \tag{9.4}$$

To initiate the algorithm a *suitable* choice of X_0 and P_0 is used. In our case, we use the unconditional mean for X_0 and unconditional variance for P_0. As stated section 5 of the note "Analytical second moments affine models" these are

$$X_0 = \mathbb{E}^P[X_t] = \theta^P \quad , \quad P_0 = \mathbb{V}^P[X_t] = Q\bar{V}(\infty)Q^\top.$$

Update step:
Following the slides from Week 3, lecture 1, we have

$$\nu_t = y_t - \mathbb{E}\left[y_t | y_{t-1}, ..., y_1\right] = y_t - A - BX_{t|t-1}$$
$$S_t = H + BP_{t|t-1}B_t^\top$$
$$K_t = P_{t|t-1}B_t^\top S_t^{-1}.$$

As such the last equations for the update step are

$$X_{t|t} = X_{t|t-1} + K_t\nu_t, \tag{9.5}$$
$$P_{t|t} = \left(I - K_t B\right) P_{t|t-1}. \tag{9.6}$$

10 Ex 10 - Defining the Gaussian log-likelihood function

It is derived from the lectures that the Gaussian log-likelihood function is defined as

$$I(y_1, ..., y_T\,; \Theta) = -\frac{NT}{2}\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\left(\log(|S_t|) + \nu_t^\intercal S_t^{-1}\nu_t\right)$$

where S_t and ν_t are given above, and N and T denotes the number of different maturities and number of observations, respectively.

11 Ex 11 - Estimation of model using maximum-likelihood and Kalman filter

An implementation of the Kalman filter is contained in the attached code.

12 Ex 12 - Parameter estimates and plot of filtered state variables

The parameters of the algorithm are as follows:

$$\widehat{K}^P = \begin{pmatrix} 2.27105 & 0.73518 & -1.02171 & -0.07144 \\ 0.59052 & 0.06742 & -2.06254 & -0.10657 \\ 0.85207 & 0.47327 & 0.39878 & 0.04017 \\ 2.21805 & -0.11998 & -3.97592 & 0.22800 \end{pmatrix}, \quad \widehat{\theta}^P = \begin{pmatrix} 0.03694 \\ -0.01585 \\ 0.01168 \\ -0.01354 \end{pmatrix},$$

$$\widehat{\Sigma} = \begin{pmatrix} -0.00011 & 0 & 0 & 0 \\ 0 & 0.00431 & 0 & 0 \\ 0 & 0 & 0.01173 & 0 \\ 0 & 0 & 0 & 0.00184 \end{pmatrix}, \quad \widehat{\lambda} = \begin{pmatrix} \hat{\lambda}^N \\ \hat{\lambda}^R \end{pmatrix} = \begin{pmatrix} 0.17011 \\ 0.21477 \end{pmatrix}, \quad \widehat{\sigma}^{err} = 0.00083.$$

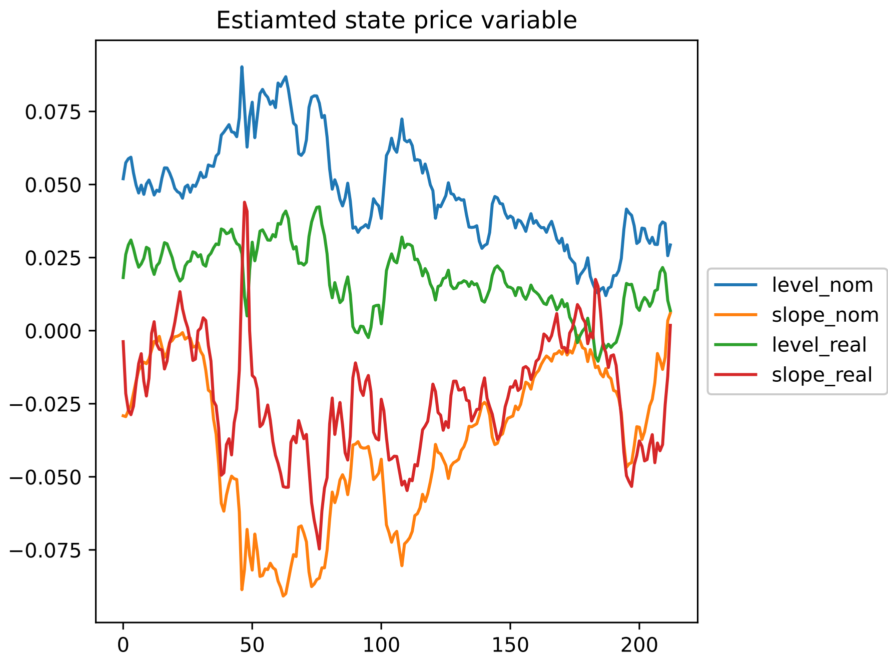

Figure 5: Estimated stateprice dynamics

13 Ex 13 - Root-mean-square-error (RMSE) of yield curves

Letting $\tau = (2, 3, 5, 7, 10, 2, 3, 5, 7, 10)^\top$, where the first five elements represent the nominal yields, and the last the real yields. We compute the RMSE measured in basis points to be

$$\text{RMSE}(\tau) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(y_{t_i}^{fit}(\tau) - y_{t_i}^{obs}(\tau)\right)^2} = \begin{pmatrix} 6.51 \\ 4.63 \\ 7.13 \\ 4.32 \\ 7.61 \\ 10.56 \\ 4.38 \\ 9.62 \\ 5.09 \\ 7.13 \end{pmatrix}.$$

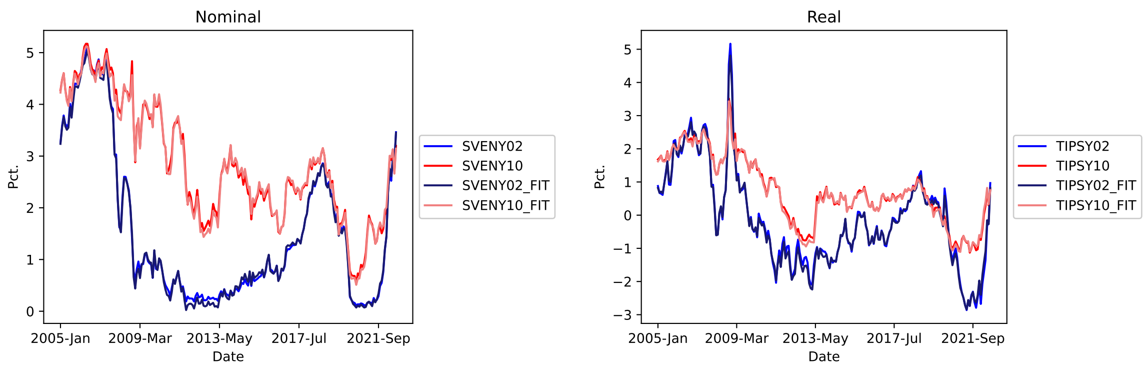

Figure 6: Estimated yields

14 Ex 14 - ODEs specifying the implied inflation expectations, π_t

In order to obtain the ODE's that solves the expected rate of inflation

$$\pi(t, T) = -\frac{1}{T-t}\log \mathbb{E}_t^P\left[e^{-\int_t^T (r_s^N - r_s^R)ds}\right],$$

we recognize it as the discounted Laplace transform with certain parameter specification such that

$$f(t, X_t) = \mathbb{E}_t^P\left[e^{-\int_t^T R(X_s)ds + w^\top X}\right] = \mathbb{E}_t^P\left[e^{-\int_t^T (r_s^N - r_s^R)ds}\right],$$

where $R(x) = \rho_0 + \rho_1^\top X$ is the short rate with parameters

$$w = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^\top, \quad \rho_0 = 0, \quad \rho_1 = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}^\top.$$

Since the dynamics is affine under $\mathbb{P}$ it implies the solution must be exponentially affine, and we can set up the ODE's

$$\alpha'(\tau) = (K^P\theta^P)^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top \Sigma\Sigma^\top \beta(\tau),$$
$$\beta'(\tau) = -\begin{pmatrix}1 & 1 & -1 & -1\end{pmatrix}^\top - (K^P)^\top \beta(\tau).$$

These are computed using Runge-Kutta in the following question.

15 Ex 15 - Compute and plot 2, 5 and 10 year model implied inflation expectation

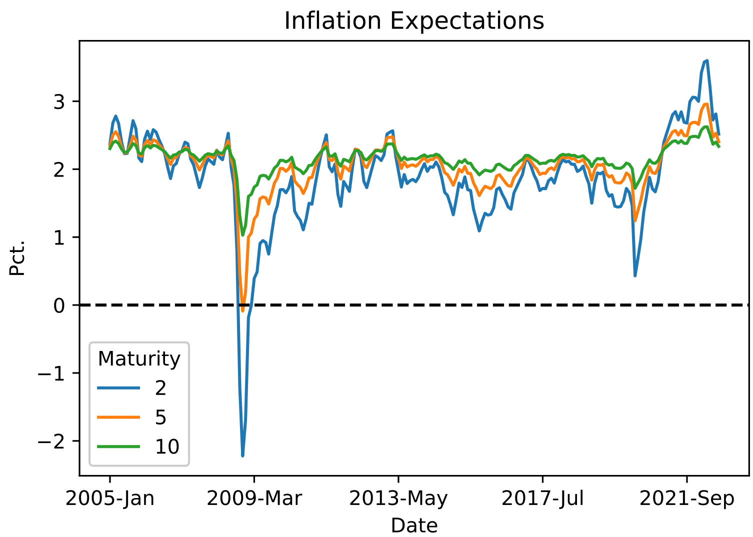

Figure 7: Estimated inflation expectations

16 Ex 16 - Compute and plot 2, 5 and 10 year model implied inflation risk premium

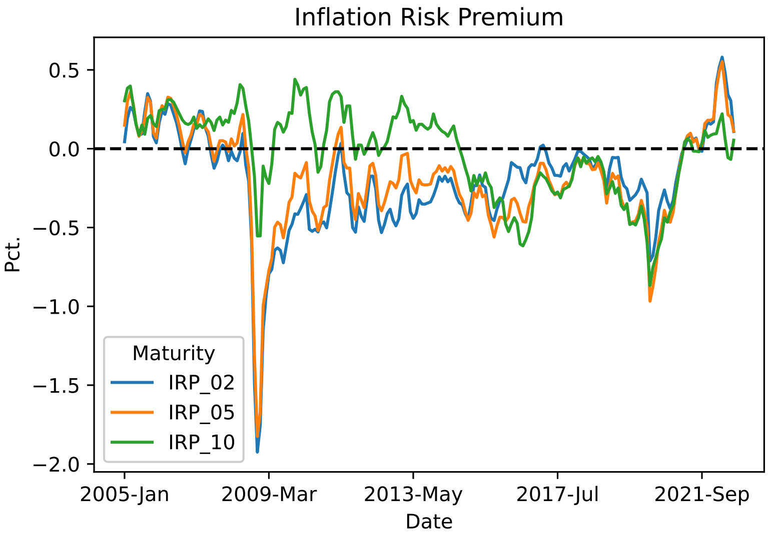

Figure 8: Estimated inflation risk premium

17 Ex 17 - Other relevant factors for decomposing BEI rates

Difference in liquidity across the different maturities might be a relevant factor to consider when decomposing the BEI rates. The liquidity in the bonds may be very different across the different maturities. All things being equal, an investor will demand compensation for the liquidity risk originating from the less liquid bonds. The liquidity premium is not directly related to inflation and cannot be captured in the inflation risk premium or the expected inflation.