Phase Efficiency Calculation

Alvaro Gonzalez, NAOJ, March 11th 2016

1. Taper Efficiency expressed as Amplitude and Phase Efficiency

The taper efficiency describes the coupling of the receiver optics radiation pattern E to the pattern on the secondary created by a plane wave, e$^{-j\mathbf{kr}}$. **Bold font** is used for vectors. The expression for taper efficiency is just a correlation between those two waves, E and e$^{-j\mathbf{kr}}$.

$$\eta_t = \frac{\left|\iint_{AP} E e^{j\vec{k}\cdot\vec{r}} dS\right|^2}{\iint_{AP} |E|^2 dS \cdot \iint_{AP} \left|e^{-j\vec{k}\cdot\vec{r}}\right|^2 dS} = \frac{\left|\iint_{AP} E e^{j\vec{k}\cdot\vec{r}} dS\right|^2}{S_{AP} \cdot \iint_{AP} |E|^2 dS}$$

Measured pattern E is a complex quantity and therefore, E=|E|exp(jϕ). The integrand in the numerator can be expressed as |E|exp(jψ)= |E|cos(ψ)+j|E|sin(ψ), with ψ=ϕ+**kr**. Therefore, the full integral can be divided in two, one for the real part and the other for the imaginary part, both under the $||^2$, which can also be expressed as

$$\eta_t = \frac{\left(\iint_{AP} |E| \cos\psi \, dS\right)^2 + \left(\iint_{AP} |E| \sin\psi \, dS\right)^2}{S_{AP} \cdot \iint_{AP} |E|^2 dS}$$

And this can be divided in two efficiencies, amplitude and phase:

$$\eta_t = \frac{\left(\iint_{AP} |E| dS\right)^2}{S_{AP} \cdot \iint_{AP} |E|^2 dS} \cdot \frac{\left(\iint_{AP} |E| \cos\psi \, dS\right)^2 + \left(\iint_{AP} |E| \sin\psi \, dS\right)^2}{\left(\iint_{AP} |E| dS\right)^2}$$

The phase efficiency is therefore:

$$\eta_{ph} = \frac{\left(\iint_{AP} |E| \cos\psi \, dS\right)^2 + \left(\iint_{AP} |E| \sin\psi \, dS\right)^2}{\left(\iint_{AP} |E| dS\right)^2}$$

At this point, it is important to discuss about phase ψ. As shown before, ψ=ϕ+**kr**, where the optimum illumination of the secondary was supposed to be a plane wave exp(-jkr). If we assumed a wave exp(+jkr), then the phase should be ψ=ϕ-**kr**. The sign will be different if we consider the plane wave in reception or in transmission in the antenna. The sign of the measured phase ϕ will also change depending if the measurement is LSB or USB in the measurement setup used to characterize the receiver optics. Notice that the sign of phase ψ will not change the result of phase efficiency since cos ψ=cos(-ψ) and even though sin(-ψ)=-sin ψ, it does not matter since it is squared outside of the integral. In short, we have to be consistent about the difference in sign between the measured pattern ϕ and ±**kr.** If we decide on a sign for our analysis, let's say – as in this report, **we have to use the pattern ± ϕ which minimizes the phase ψ, defined as ϕ+kr.**

The phase of the plane wave **kr** can be expressed as

$$\vec{k} \cdot \vec{r} = \mathrm{k}(\Delta x \ \sin Az \cos El + \Delta y \ \sin El + \Delta z \ \cos Az \cos El)$$

In the current ALMA efficiency calculator, some approximations are used since the range of Az, El used for analysis is usually small (R. Hills recommended Az=±15deg, El=±15deg in his memo). In this case, sinx=x and cosx=1-x^2/2. If only powers up to 2 in Az, El are used, the product **kr** becomes:

$$\vec{k} \cdot \vec{r} = \mathrm{k}\left(\Delta x \ Az + \Delta y \ El + \Delta z \ \left(1 - \frac{Az^2 + El^2}{2}\right)\right)$$

At this point, we need to use a property of the phase efficiency: constant phases in the phase pattern or in the plane wave pattern do not change the aperture efficiency. This is easily proved. Phase ψ can be divided into two phases Ψ(Az,El), which is function of Az, El, and ψ_0, which is constant. In other words, $\psi = \Psi(Az,El) + \psi_0$. In the numerator of the efficiency calculation:

$$\left(\iint_{AP} |E| \cos(\Psi + \psi_0) \, dS\right)^2 + \left(\iint_{AP} |E| \sin(\Psi + \psi_0) \, dS\right)^2 =$$

$$= \left(\cos\psi_0 \iint_{AP} |E| \sin \Psi \, dS + \sin \psi_0 \iint_{AP} |E| \cos \Psi \, dS\right)^2$$

$$+ \left(\cos\psi_0 \iint_{AP} |E| \cos \Psi \, dS - \sin \psi_0 \iint_{AP} |E| \sin \Psi \, dS\right)^2 =$$

$$= \left(\iint_{AP} |E| \sin \Psi \, dS\right)^2 + \left(\iint_{AP} |E| \cos \Psi \, dS\right)^2$$

Therefore, constant phases do not contribute to the phase efficiency. This is an important results which tells us that we do not need to care about normalizing the phase of the phase pattern or about the constant phase term k Δz in the approximation of **kr**. Using this result, we can obtain the final expression used in the current efficiency calculator:

$$\vec{k} \cdot \vec{r} = \mathrm{k}\left(\Delta x \ Az + \Delta y \ El - \Delta z \ \frac{Az^2 + El^2}{2}\right)$$

2. *Implementation of the phase efficiency calculation*

The current code is:

```
// initial guess for delta_x, delta_y, delta_z
p[1] = 0.0;
p[2] = 0.0;
p[3] = fitPhaseScan -> getZDistance(); // the probe Z distance in mm

// convert to rad/deg or rad/deg^2 prior to fitting:
float k = fitPhaseScan -> getKWaveNumber(); // rad/m
p[1] *= 0.001 * k * (2 * M_PI / 360.0);
p[2] *= 0.001 * k * (2 * M_PI / 360.0);
p[3] *= -0.001 * k * pow(2 * M_PI / 360.0, 2.0);
```

```
// convert measured phase to radians:
phaseRad = phiArray_m[i] * M_PI / 180.0;

// it uses the following equations for the phase fit and phase fit error:
phaseFit = p[1] * Az + p[2] * El + p[3] * (Az^2 + El^2) / 2.0;
phaseErr = phaseRad - phaseFit;
// eta_phase is then calculated from phaseErr.
```

There are a couple of tricky things in these code lines:

```
p[1] *= 0.001 * k * (2 * M_PI / 360.0);
p[2] *= 0.001 * k * (2 * M_PI / 360.0);
p[3] *= -0.001 * k * pow(2 * M_PI / 360.0, 2.0);
```

1. Az and El are expressed in degrees and the conversion to radians is included in p[1], p[2], p[3]. Notice that p[3] is multiplied by the conversion 2pi/360 twice.
2. The – sign for the p[3] term in phaseFit is included in the conversion of p[3]
3. K is included in the value of p[i]
4. Changes in these lines must be properly reproduced in the other part of the code which calculates the phase centers from the values of p[i]

In phaseErr, the difference between phaseRad – phaseFit is taken. This is assuming a plane wave with phase +jkr. This will impact on what sideband must be inverted. **We have to choose the sign of the phase of LSB/USB, phaseRad, so it minimizes phaseErr.**

- With respect to my suggestion to use the exact equations, several changes must be implemented:

1. The conversion factor `(2 * M_PI / 360.0)` for Az, El cannot be included in p[i]. It has to be applied directly to Az, El.
2. The – sign in p[3] must be removed

```
// convert to rad/deg or rad/deg^2 prior to fitting:
float k = fitPhaseScan -> getKWaveNumber(); // rad/m
p[1] *= 0.001 * k
p[2] *= 0.001 * k
p[3] *= 0.001 * k

Az *= M_PI / 180.0
El *= M_PI / 180.0

// convert measured phase to radians:
phaseRad = phiArray_m[i] * M_PI / 180.0;

phaseFit = p[1]*sin(Az)*cos(El) + p[2]*sin(El) + p[3]*cos(Az)*cos(El);
phaseErr = phaseRad - phaseFit;
// eta_phase is then calculated from phaseErr.
```

As explained before, `phaseErr = phaseRad - phaseFit;`
Could be replaced by `phaseErr = phaseRad + phaseFit;`
Then, the sideband we need to phase invert changes. We just need to be consistent. Personally, I have the feeling that the + sign will be the consistent solution… but need to try with real data to be sure about this!