Lecture 7: Computational Geometry

CS 491 CAP

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Friday, October 13th, 2017

Credit for many of the slides on solving geometry problems goes to the Stanford CS 97SI course lecture on computational geometry (http://web.stanford.edu/class/cs97si/09-computational-geometry.pdf)

Table of Contents

Table of Contents

sin, cos, & tan

- $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$
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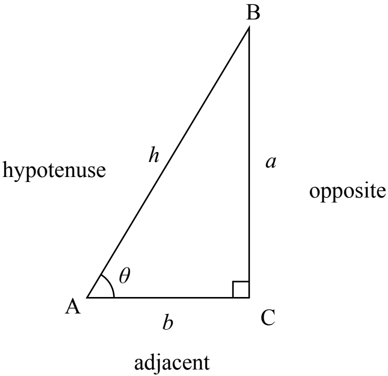

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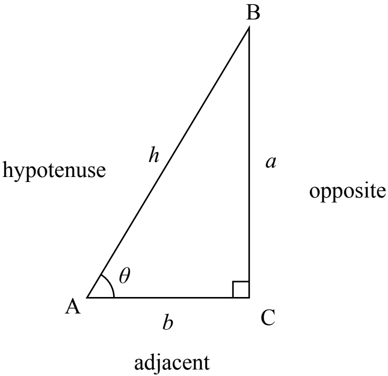

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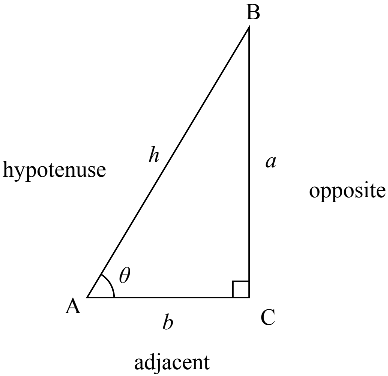

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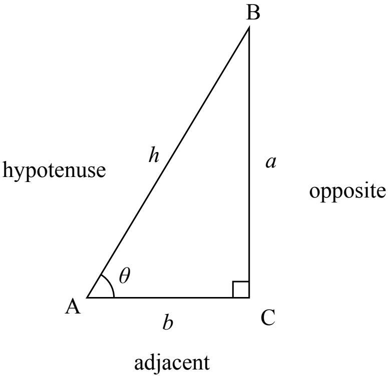
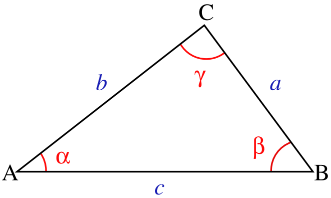


Law of sines & law of cosines



- Law of sines: $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$
- Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$
  - Put another way: $\cos(\alpha) = \frac{-a^2+b^2+c^2}{2bc}$
  - Similar for b and c

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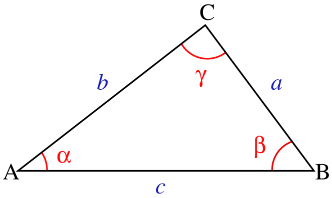

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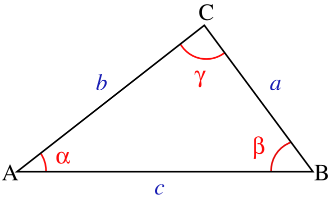

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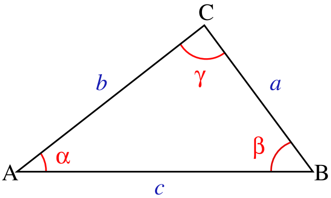

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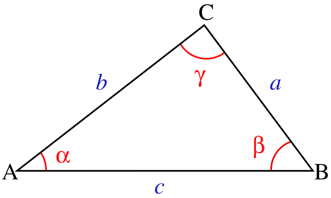

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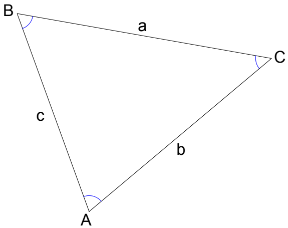

- Area of triangle $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$
- s = semiperimeter $= \frac{a+b+c}{2}$
- But there is an easier way to find the area of a triangle...

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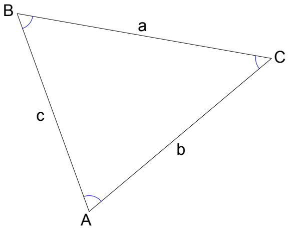

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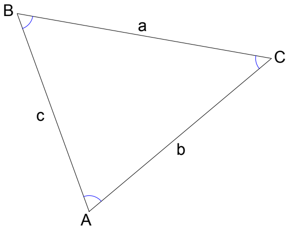

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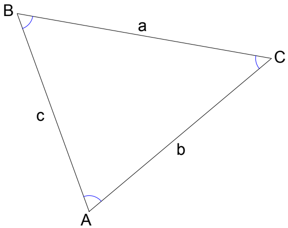

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Table of Contents

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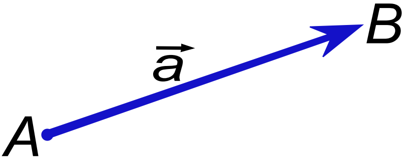
- Magnitude

- Alternatively, magnitude & angle from positive x-axis



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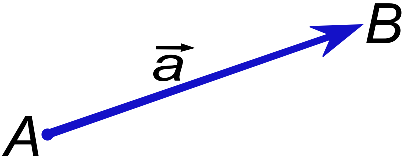

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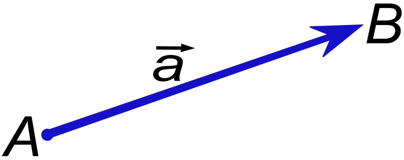

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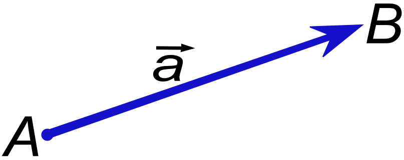

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- Basis for large number of geometry problems
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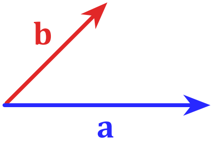
- Place beginning of one vector at end of other
- New vector from beginning of first vector to end of last



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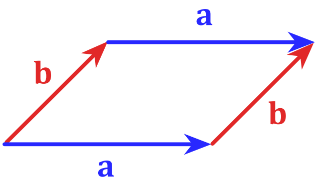

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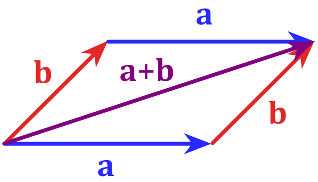
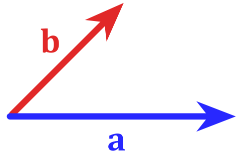


Vector Dot Product

- $\mathbf{A} \cdot \mathbf{B} = x_1 \times x_2 + y_1 \times y_2$
- Why is this important?
  - $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta$
  - Can use to find $\cos \theta$



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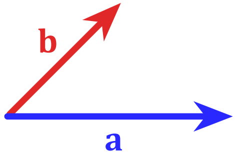

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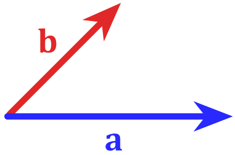

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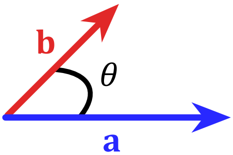

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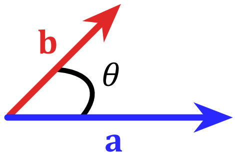

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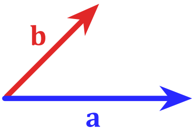

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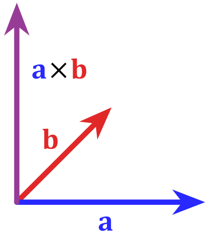

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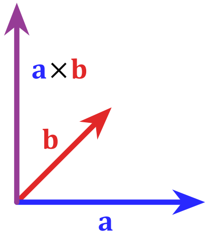

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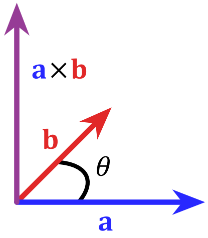

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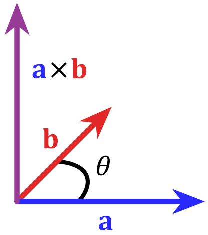

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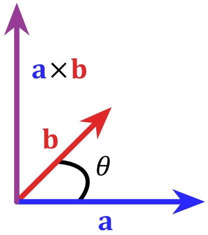

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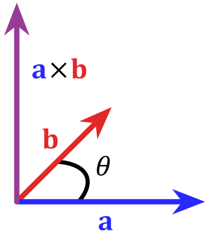

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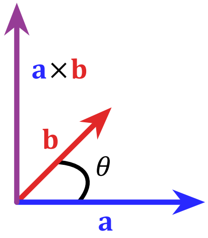

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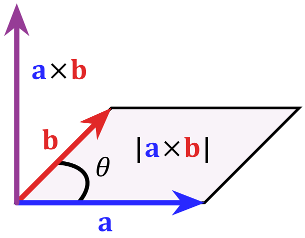

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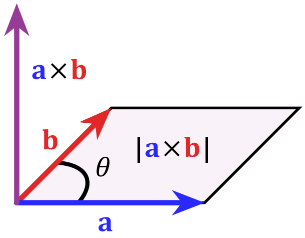

Table of Contents

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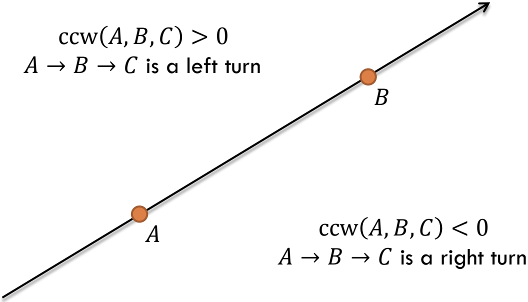
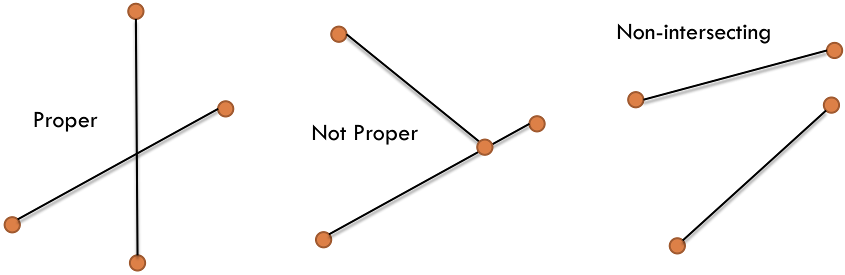
Cross Product

- Define $\text{ccw}(A, B, C) = (B - A) \times (C - A)$



Segment-Segment Intersection Test

- Given two segments AB and CD
- Want to determine if they intersect properly: two segments meet at a single point that are strictly inside both segments



Segment-Segment Intersection Test

- Assume that the segments intersect
  - From A's point of view, looking straight to B, C and D must lie on different sides
  - Holds true for the other segment as well
- The intersection exists and is proper if:
  - $\mathrm{ccw}(A, B, C) \times \mathrm{ccw}(A, B, D) < 0$
  - AND $\mathrm{ccw}(C, D, A) \times \mathrm{ccw}(C, D, B) < 0$

Segment-Segment Intersection Test

- Determining non-proper intersections
  - We need more special cases to consider!
  - e.g. If $\text{ccw}(A, B, C)$, $\text{ccw}(A, B, D)$, $\text{ccw}(C, D, A)$, $\text{ccw}(C, D, B)$ are all zeros, then two segments are collinear
  - Very careful implementation is required

Table of Contents

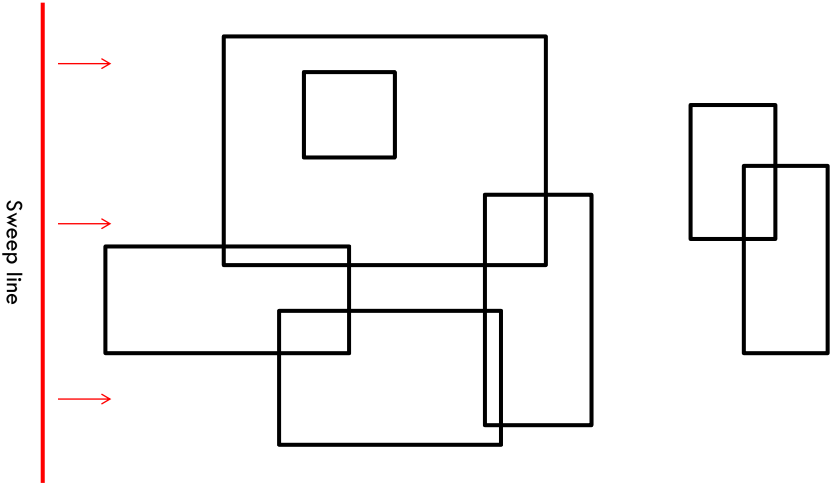
Sweep Line Algorithm

- A problem solving strategy for geometry problems
- The main idea is to maintain a line (with some auxiliary data structure) that sweeps through the entire plane and solve the problem locally
- We can't simulate a continuous process, (e.g. sweeping a line) so we define *events* that causes certain changes in our data structure
  - And process the events in the order of occurrence
- We'll cover one sweep line algorithm

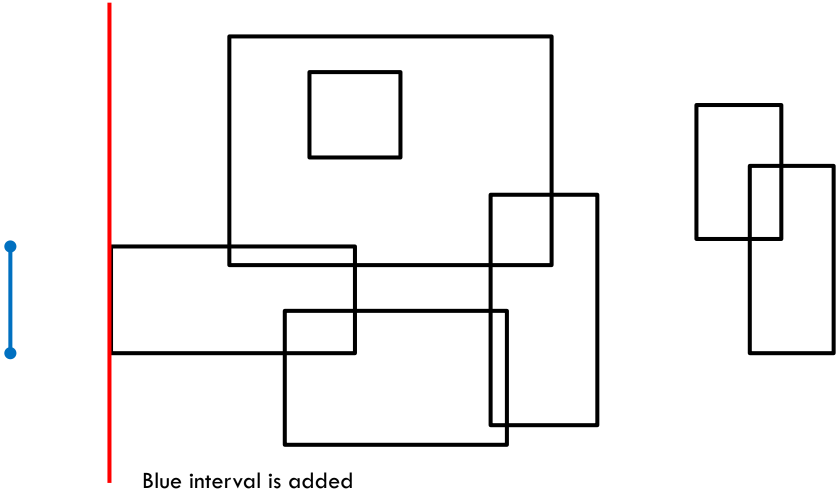
Sweep Line Algorithm

- Problem: Given n axis-aligned rectangles, find the area of the union of them
- We will sweep the plane from left to right
- Events: left and right edges of the rectangles
- The main idea is to maintain the set of “active” rectangles in order
  - It suffices to store the y-coordinates of the rectangles

Example

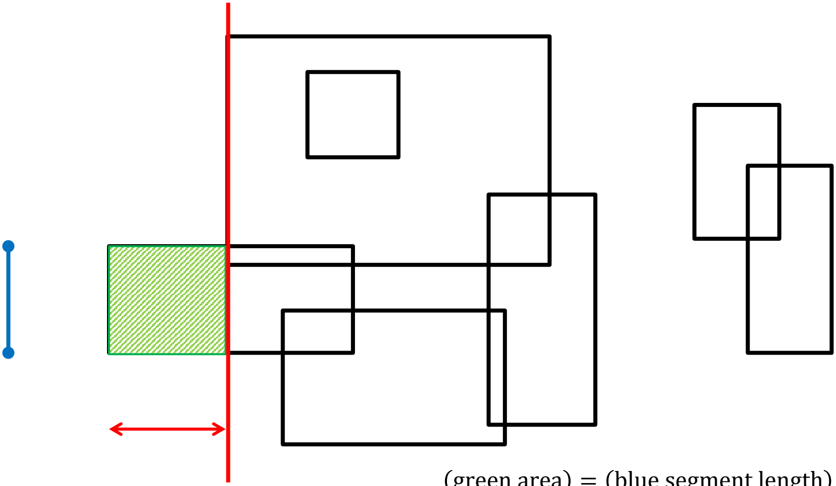


Example



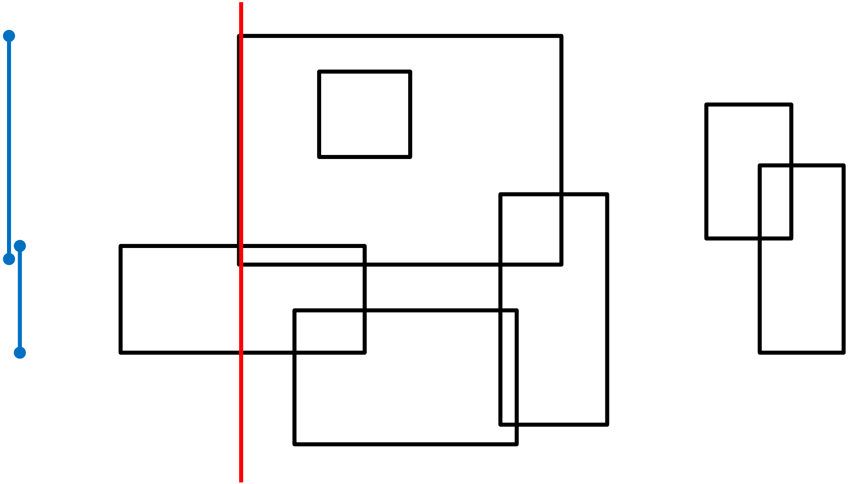
Blue interval is added
to the data structure

Example



(green area) = (blue segment length)
× (distance that the sweep line traveled)

Example



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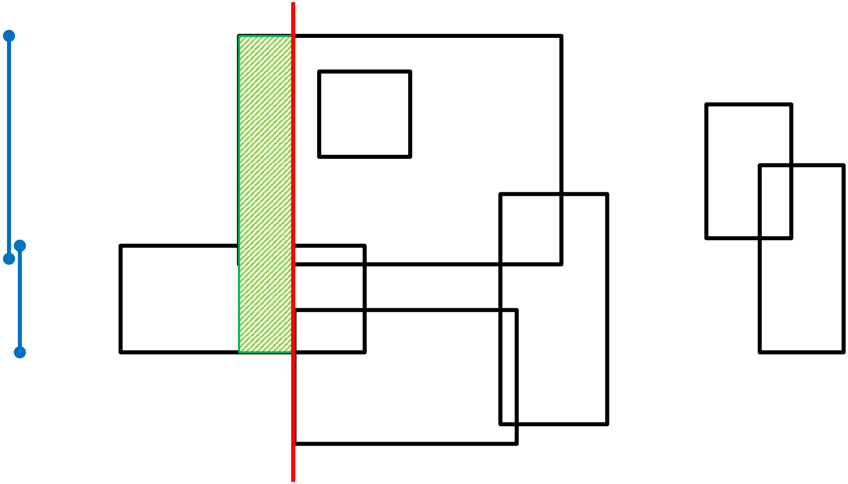

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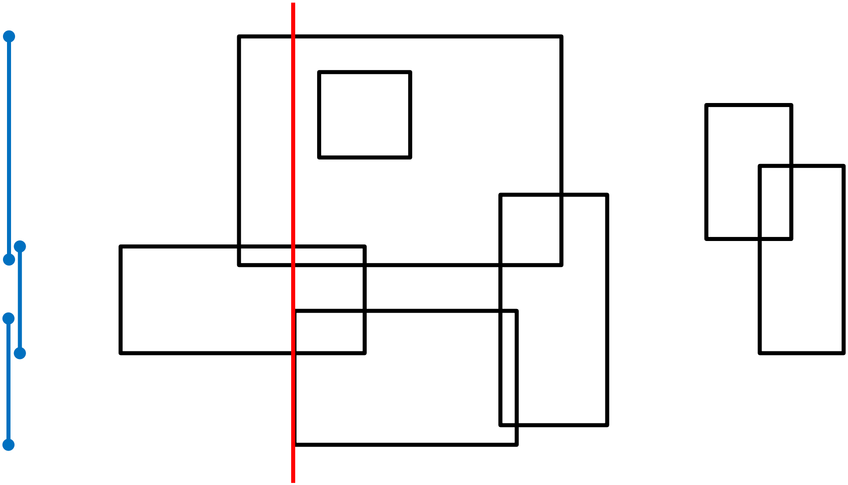

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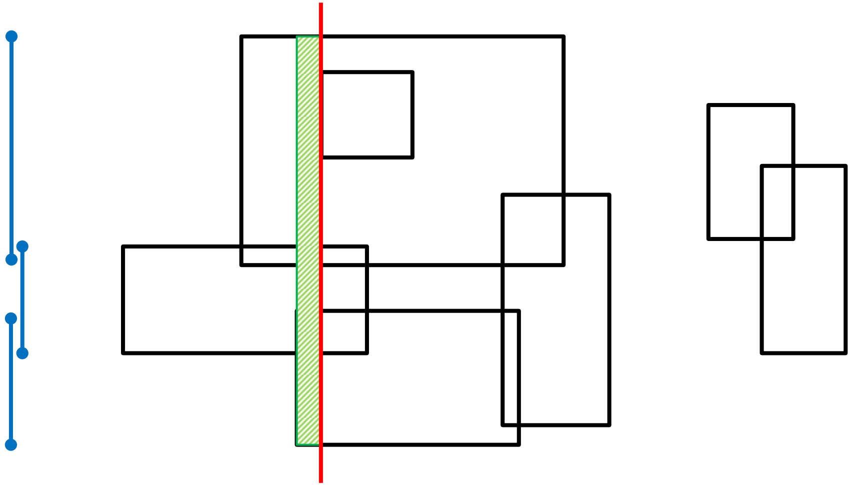

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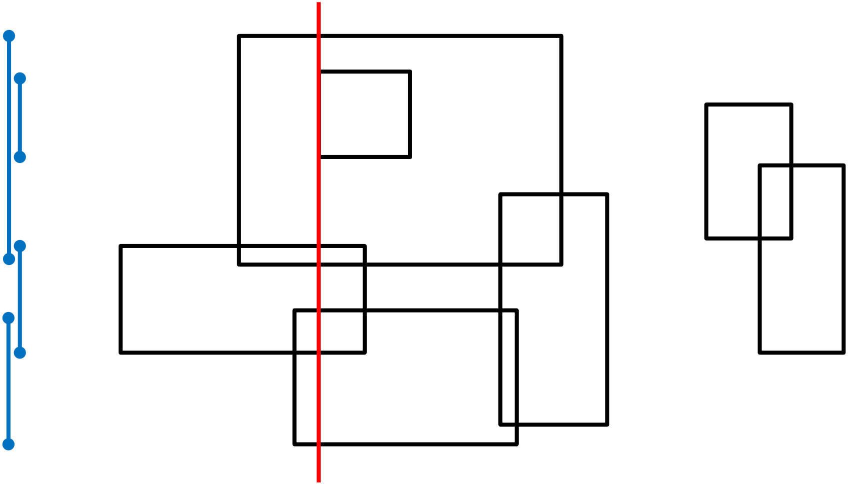

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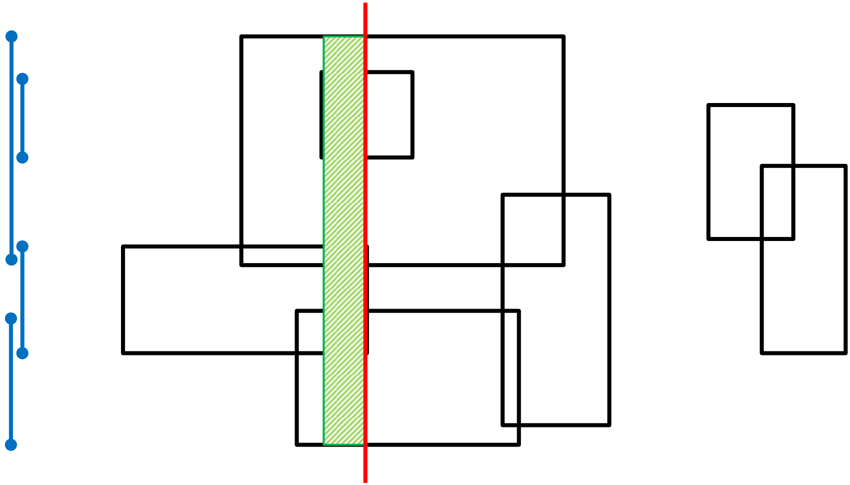

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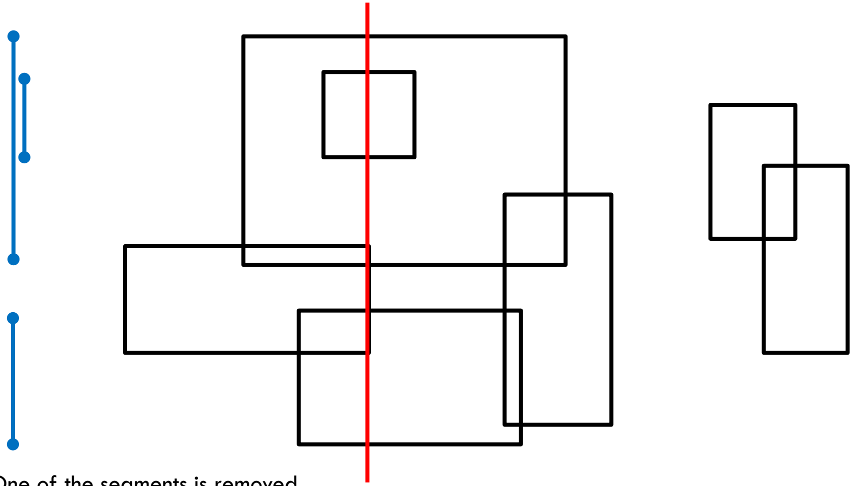
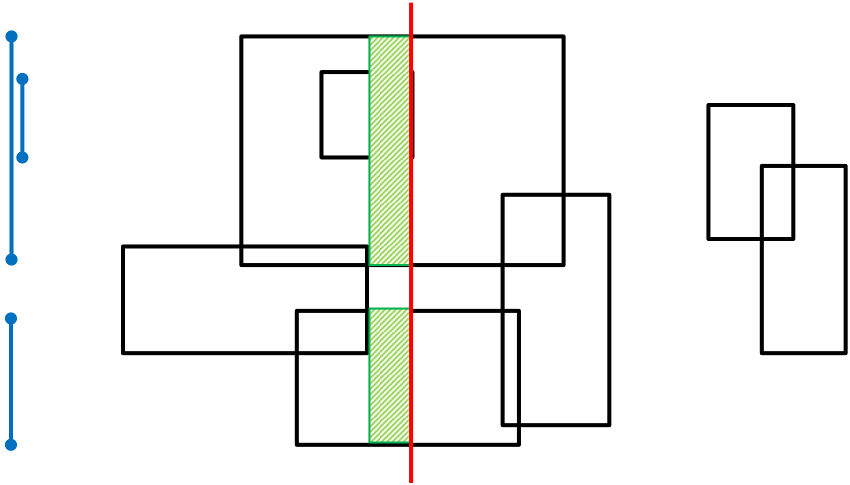


One of the segments is removed

Example



Pseudopseudocode

- If the sweep line hits the left edge of a rectangle
  - Insert it to the data structure
- Right edge?
  - Remove it
- Move to the next event, and add the area(s) of the green rectangle(s)
  - Finding the length of the union of the blue segments is the hardest step
  - There is an easy $O(n)$ method for this step

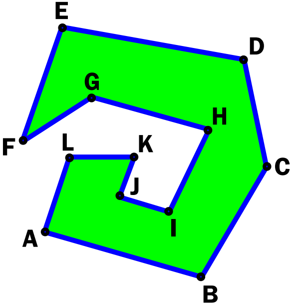
Notes on Sweep Line Algorithms

- Sweep line algorithm is a generic concept
  - Come up with the right set of events and data structures for each problem
- Exercise problems
  - Finding the perimeter of the union of rectangles
  - Finding all k intersections of n line segments in $O\big((n+k)\log n\big)$ time

Table of Contents

Triangulation!

1. Start at any point (let's call it A)
2. Go through all other points, summing areas of triangles made by vectors from A (using cross product)



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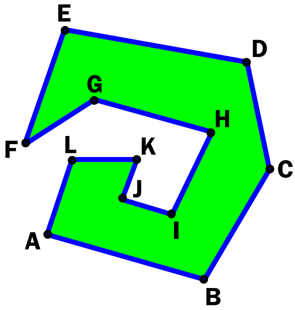

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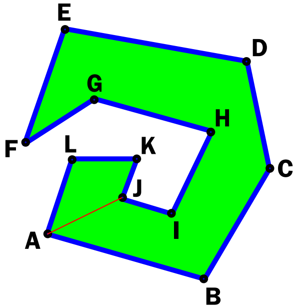

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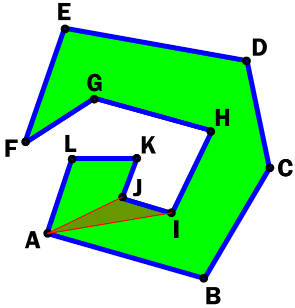

Why does this work?

- For convex polygons, obvious
- For concave polygons, when we turn inwards, area is negative
- Sum of the negative and positive triangles equals area of polygon

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Table of Contents

Note on Binary Search

- Usually, binary search is used to find an item of interest in a sorted array
- There is a nice application of binary search, often used in geometry problems
  - Example: finding the largest circle that fits into a given polygon
    - Don't try to find a closed form solution or anything like that!
    - Instead, binary search on the answer

Table of Contents

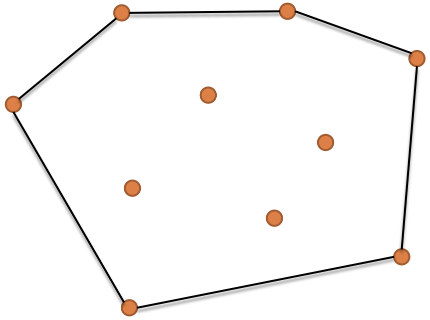
Ternary Search

- Another useful method in many geometry problems
- Finds the minimum point of a "convex" function f
  - Not exactly convex, but let's use this word anyway
- Initialize the search interval $[s, e]$
- Until $e - s$ becomes small:
  - $m_1 = s + (e - s)/3, m_2 = e - (e - s)/3$
  - If $f(m_1) \leq f(m_2)$, then set e to m_2
  - Otherwise, set s to m_1

Table of Contents

Convex Hull Problem

- Given n points on the plane, find the smallest convex polygon that contains all the given points
  - For simplicity, assume that no three points are collinear



Simple $O(n^3)$ algorithm

- AB is an edge of the convex hull iff $\mathrm{ccw}(A, B, C)$ have the same sign for all other given points C
  - This gives us a simple algorithm

- For each A and B:
  - If $\mathrm{ccw}(A, B, C) > 0$ for all $C \neq A, B$:
    - Record the edge $A \rightarrow B$
- Walk along the recorded edges to recover the convex hull

Faster Algorithm: Graham Scan

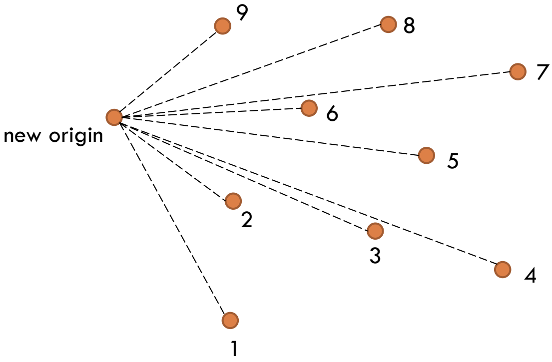
- We know that the leftmost given point has to be in the convex hull
  - We assume that there is a unique leftmost point
- Make the leftmost point the origin
  - So that all other points have positive x coordinates
- Sort the points in increasing order of y/x
  - Increasing order of angle, whatever you like to call it
- Incrementally construct the convex hull using a stack

Incremental Construction

- We maintain a *convex chain* of the given points
- For each i, we do the following:
  - Append point i to the current chain
  - If the new point causes a concave corner, remove the bad vertex from the chain that causes it
  - Repeat until the new chain becomes convex

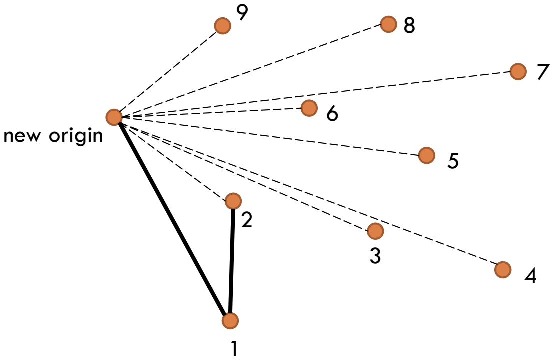
Example

- Points are numbered in increasing order of y/x



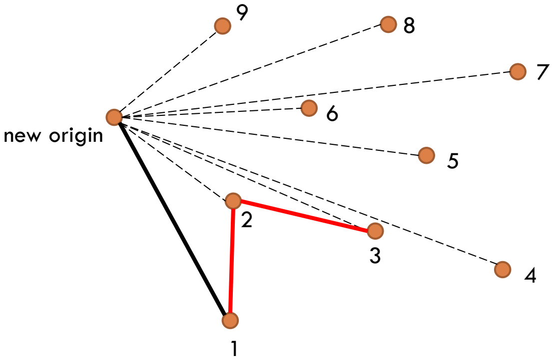
Example

- Add the first two points in the chain



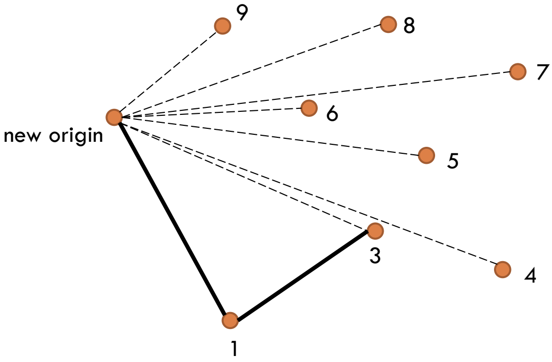
Example

- Adding point 3 causes a concave corner 1-2-3
  - Remove 2



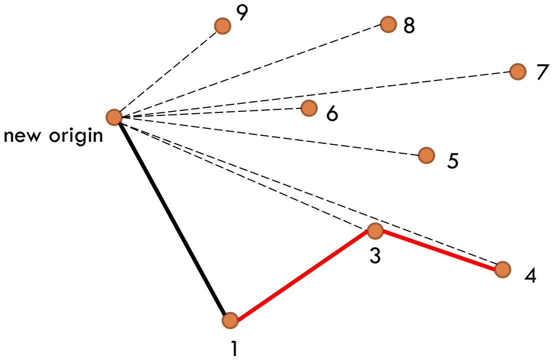
Example

- That's better…



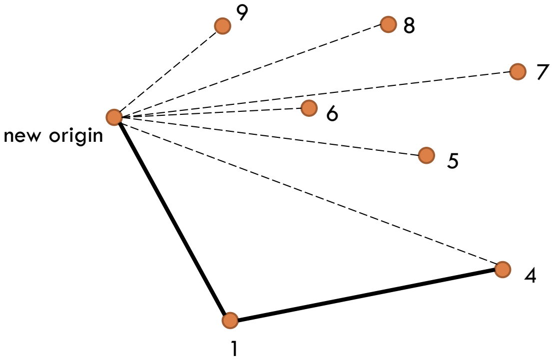
Example

- Adding 4 to the chain causes a problem
  - Remove 3



Example

- Continue adding points…



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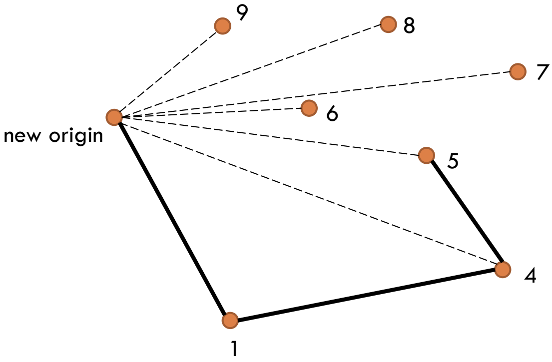

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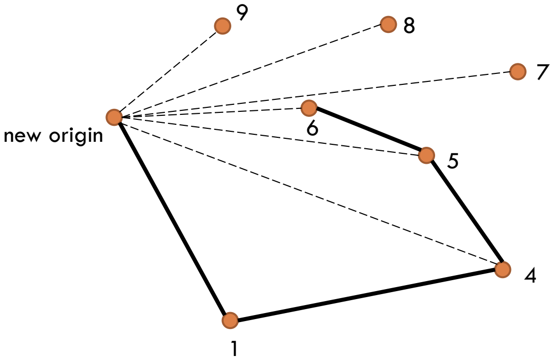
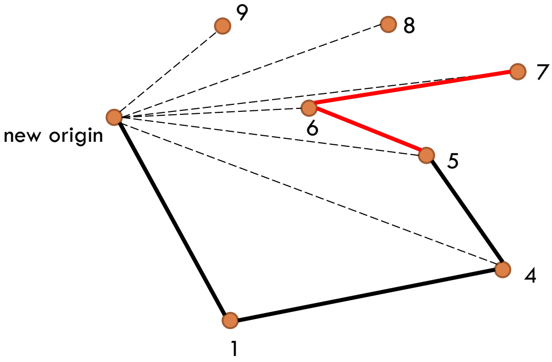


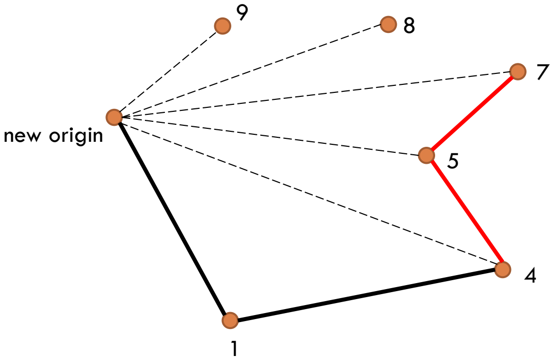
Example

- Bad corner!



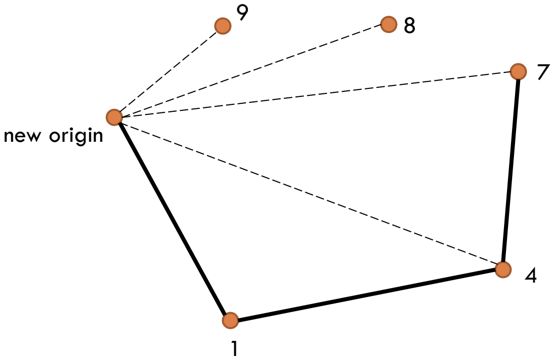
Example

- Bad corner again!



Example

- Continue adding points…



Example

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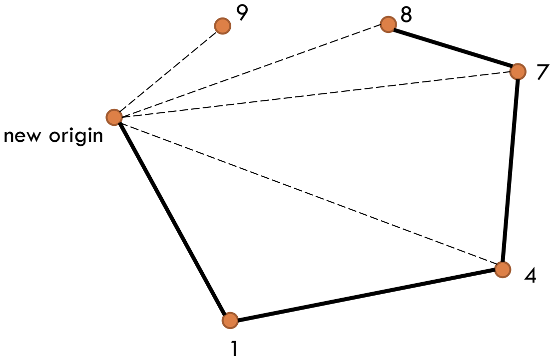

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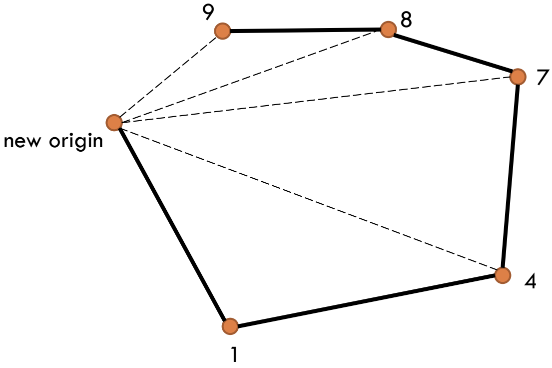
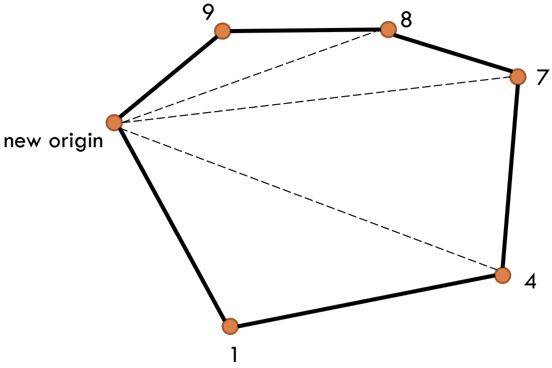


Example

- Done!



Pseudocode

- Set the leftmost point $(0,0)$, and sort the rest of the points in increasing order of y/x
- Initialize stack S
- For $i = 1 \dots n$:
  - Let A be the second topmost element of S, B be the topmost element of S, C be the ith point
  - If $\mathrm{ccw}(A, B, C) < 0$, pop S and go back
  - Push C to S
- Points in S form the convex hull

Graham Scan Complexity

- Sorting of points takes $O(n \log n)$ time
- Construction of the hull requires only traversal of all points once, with removal from the stack at most once, so $O(n)$ time
- Total runtime complexity: $O(n \log n)$

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Table of Contents

There are more geometry concepts to know!

- Circle concepts – chord length, inscribed and circumscribed polygons, circles and intersecting tangent/secant lines
- Sphere concepts – great circle distance
- Polygons – checking if point is inside polygon (extension of concepts covered today)
- For formulas, algorithms, and implementations, see Chapter 7 of Competitive Programming by Steven Halim

Thankfully, most of these concepts do not appear in our regionals

- However, you should know these for World Finals!

Table of Contents

Some geometry resources

- TopCoder Algorithm Tutorials - Basic Geometry Concepts
- TopCoder Algorithm Tutorials - Line Intersection and its Applications
- TopCoder Algorithm Tutorials - Practice TopCoder geometry problems
- Ahmed-Aly list of geometry problems
- Stanford Introduction to ICPC Course
  - Geometry Lecture Slides
- Stanford ICPC Notebook (contains geometry algorithm implementations in C++)
- Chapter 7 of Competitive Programming by Steven Halim