Agenda for Friday, April 19, 2024

Functions	**2**

Reminders

- Office hours Fri 2PM in Locy 203
- Exam practice problems available on Crowdmark
- ! Exam 1 on April 23 in discussion

Warm up

Let $A = \{a, b\}$ and $B = \{-1, 1\}$. List the elements of the Cartesian product $A \times B$. Are they the same as the elements of $B \times A$?

The elements of $A \times B$ are $(a, 1), (a, -1), (b, 1), (b, -1)$. The elements of $B \times A$ are different - the elements are in the other order.

Functions

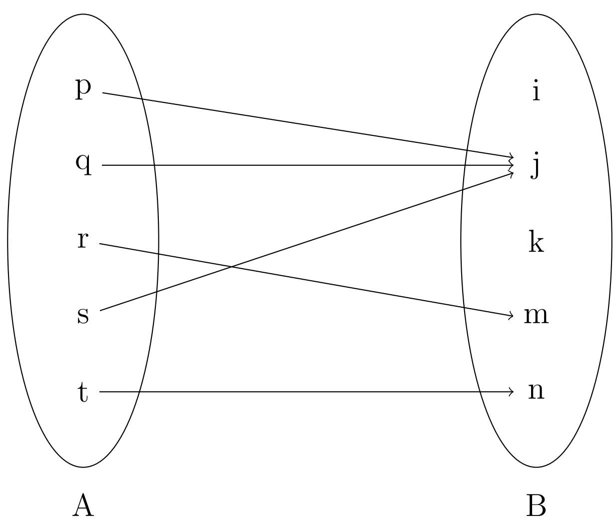
Let's discuss - what's your current definition of a function? How do you determine if two functions are equal?

Intuitively, the notion of a function is some sort of "assignment rule" where we take in some input, and assign to it some output. The rule we must obey is that a given input can only be assigned to one output. We now make precise the definition of a function

Definition. Let A and B be sets. A function (also called a map) f from A to B, denoted $f : A \to B$, is a subset $f \subseteq A \times B$ such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. The set A is the domain of f, the set B is codomain of f.

Remark. A function always consists of this data: the domain A, the codomain B, and the subset $f \subseteq A \times B$.

Remark. In this definition, a function is a *set*. However it's common shorthand notation to write $f(a) = b$ to mean the same thing as $(a, b) \in f$.



Example 1. In the function diagram above, notice these features of a function.

- Every element of A is mapped somewhere, but not every element of B receives a map. A function must mapa every element of the domain to something.
- Multiple elements in A can map to the same element in B. (i.e. several arrows come together). It is not possible to have multiple elements of B receive maps from the same element in A (i.e. arrows never split)

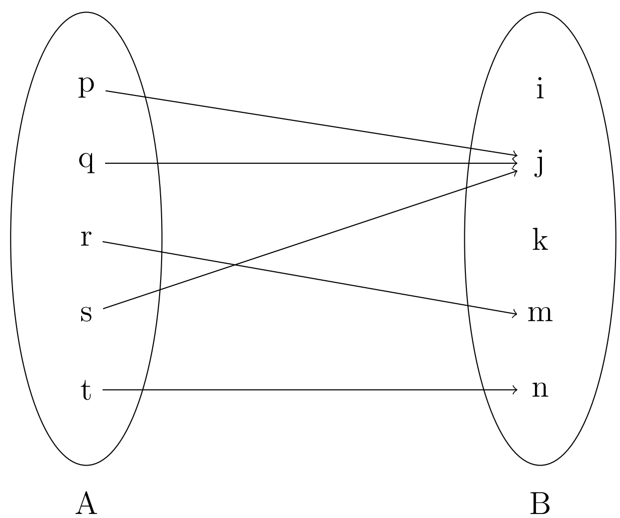
Definition. Let $f : A \to B$ be a function.
The image of a subset $X \subseteq A$, denoted $f(X)$, is

$$\{b \in B \mid \exists a \in X \text{ such that } (a, b) \in F\}$$

The preimage of a subset $Y \subseteq B$, denoted $f^{-1}(Y)$, is

$$\{a \in A \mid \exists b \in Y \text{ such that } (a, b) \in f\}$$

The range of f is the set $f(A)$. Note that the range is a subset of B.



Example 2. Using the function diagram above, what is the image of $\{p, q, r\}$? What is the image of $\{p, q, r, s\}$? What is the preimage of $\{j, m\}$? What is the preimage of $\{j, k, m\}$?

(a) image of $\{p, q, r\}$ is $\{j, m\}$

(b) image of $\{p, q, r, s\}$ is $\{j, m\}$ [Notice that s didn't change the answer from (a). Why?]

(c) preimage of $\{j, n\}$ is $\{p, q, s, t\}$

(d) preimage of $\{j, k, n\}$ is $\{p, q, s, t\}$ [Notice that k didn't change the answer from (c). Why?]

Definition. Two functions $f, g : A \to B$ are equal, denoted $f = g$, if $f = g$ as subsets of $A \times B$. In other words, $(x, f(x)) = (x, g(x))$ for all $x \in A$.

Example 3. Is it a function?

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

yes

(b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x}$

no, not every point in the domain is defined

(c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$

yes

(d) $f : \mathbb{Z} \to \mathbb{N}$ given by $f(x) = 2x$

no, some of the images land outside the codomain.

(e) Let $P = \{$ all living people $\}$ and define $f : P \to P$ such that $f(x) =$ sister of x

no, this is ill-defined, which is how mathematicians describe functions whose outputs are ambiguous

(f) Is the set $f = \{(x^2, x) \mid x \in \mathbb{R}\}$ a function from $\mathbb{R} \to \mathbb{R}$?

no, it's ill-defined

(g) Is the set $f = \{(x^3, x) \mid x \in \mathbb{R}\}$ a function from $\mathbb{R} \to \mathbb{R}$?

yes, every $x^3 \in \mathbb{R}$ is assigned to a single x.

Example 4. Let $f : A \to B$ be a function. Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$. Which of the following are true?

(a) $f^{-1}(f(X_1)) = X_1$ To be continued...

(b) $f(f^{-1}(Y_1)) = Y_1$

(c) $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$

(d) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$