Name: ANSWER KEY

1. For what interval(s) is $f(x) = xe^x$ concave up?

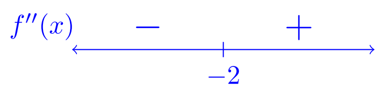
   We seek intervals where $f'' > 0$

   $f'(x) = xe^x + e^x$

   $f''(x) = xe^x + 2e^x = e^x(x+2)$

   $f''(x) = 0$ when $e^x = 0$ (which is never since $e^x = 0$ has no solutions) and when $x + 2 = 0$ (at $x = -2$)

   Now we draw a number line and choose points from each interval to find the sign of $f''(x)$ in each interval.

   In conclusion, $f(x)$ is concave up on $(-2, \infty)$ because $f'' > 0$ there.

2. For what interval(s) is $g(x) = \dfrac{1}{x^2+1}$ decreasing?

   We seek intervals where $g'(x) < 0$

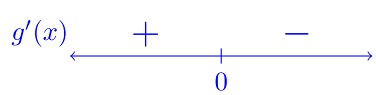
   $g(x) = (x^2+1)^{-1}$

   $g'(x) = -(x^2+1)^{-1}(2x) = -\dfrac{2x}{(x^2+1)^2}$

   $g'(x) = 0$ when $2x = 0$, (at $x = 0$)

   $g'(x)$ is undefined when $(x^2+1)^2 = 0$, (no solutions)

   Now we draw a number line and choose points from each interval to find the sign of $f''(x)$ in each interval.

   We conclude $g(x)$ is decreasing on $(0, \infty)$ because $g'(x) < 0$ on that interval.

3. Write the equation of the tangent line to $f(x) = \tan(e^x)$ at $x = \ln(\frac{\pi}{4})$.

   Slope: $f'(x) = e^x \sec^2(e^x) \quad \Longrightarrow \quad f'\left(\ln\frac{\pi}{4}\right) = e^{\ln\frac{\pi}{4}} \sec^2(e^{\ln\frac{\pi}{4}}) = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

   Point: $f\left(\ln\frac{\pi}{4}\right) = 1$

   Equation of line: $y - 1 = \frac{\pi}{2}(x - \ln\frac{\pi}{4})$

4. The position of a particle undergoing simple harmonic motion is given by $s(t) = A\cos(\omega t + \delta)$ where A, ω, and δ are constants.

   (a) What is the velocity of the particle at time t?

   Velocity is the derivative of the position function, so $v(t) = s'(t) = -\omega A\sin(\omega t + \delta)$.

   (b) When is the velocity of the particle 0?

   $-\omega A\sin(\omega t + \delta) = 0 \quad \Longrightarrow \quad \sin(\omega t + \delta) = 0 \quad \Longrightarrow \quad \omega t + \delta = k\pi \quad \Longrightarrow \quad t = \frac{k\pi - \delta}{\omega}$, for integer k

5. Air is being bumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.

   (a) What do the derivatives $\frac{dV}{dr}$ and $\frac{dV}{dt}$ represent?

   $\frac{dV}{dr}$ is how fast volume changes as radius increases.
   $\frac{dV}{dt}$ is how fast volume changes as time increases.

   (b) Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

   $$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

6. Suppose f is differentiable for all real numbers and a is a real number.

   (a) Let $F(x) = f(x^a)$. Find $F'(x)$.

   $$F'(x) = f'(x^a)(ax^{a-1})$$

   (b) Let $G(x) = [f(x)]^a$. Find $G'(x)$.

   $$G'(x) = a[f(x)]^{a-1}f'(x)$$

7. **Discovery: Extended Chain Rule**

$$\text{Let } y = e^{(5x-1)^2}.$$

(a) The function y given above is the composition of three functions. Suppose $y = f(g(h(x)))$. Can you identify the outer function, the middle function, and the inner function? (Note: there may be more than one right answer.)

Outer function: $f(w) =$ e^w

Middle function: $g(z) =$ z^2

Inner function: $h(x) =$ $5x - 1$

(b) Differentiate $y = e^{(5x-1)^2}$ by first expanding $(5x-1)^2$ and then using the chain rule.

$$y = e^{25x^2-10x+1} \implies y' = (50x-10)e^{25x^2-10x+1}$$

(c) Differentiate $f(w)$, $g(z)$, and $h(x)$. (i.e. the outer, middle, and inner functions from part (a).)

Outer function: $f'(w) =$ e^w

Middle function: $g'(z) =$ $2z$

Inner function: $h'(x) =$ 5

(d) Use your answers from parts (b) and (c) to guess the "extended chain rule". In other words, what is the formula for $[f(g(h(x)))]'$? $f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$