Math 300-0 Final project - Construction of $\mathbb{R}$

We used the Peano axioms to construct $\mathbb{N}$. We used equivalence classes of unreduced fractions to form $\mathbb{Q}$. But what about $\mathbb{R}$? This final project will demonstrate how to construct $\mathbb{R}$, starting with just the natural numbers.

Required materials

The reference for this project is *The Real Numbers and Real Analysis* by Ethan D. Bloch. The book is available in PDF format for free through Springer Online Access. You will have to use your Northwestern credentials to gain access to the book.

Day 1: Dedekind cuts

To do before May 20

- Read Bloch, section 1.6, until Lemma 1.6.7 and its proof. Note that $X - A$ in this book means "set minus," which we have denoted $X \setminus A$ in this course.
- Think about these questions. If you are not able to answer them, consider reviewing the text again.
  - What is a Dedekind cut? Can you give a concrete example?
  - What is a rational cut? Why do we make a distinction?
  - How do you prove a Dedekind cut? What will a proof template look like?
- Write any remaining questions you have in the shared Overleaf document.

To do after May 20

Write solutions to the following exercises.

Problem 1. Exercise 1.6.2(1)

Problem 2. Exercise 1.6.1

Day 2: Dedekind cut arithmetic

To do before May 24

- Read Bloch pp 37-39, Lemma 1.6.8 and its proof.
- Think about these questions. If you are not able to answer them, consider reviewing the text again.
  - Can you sketch some pictures to help you understand the proof?
  - What are the geometric meanings of the three conditions of a Dedekind cut?
- Write any remaining questions you have in the shared Overleaf document.

To do after May 24

Write solutions to the following exercises.

Problem 3. Prove Lemma 1.6.8(3)

Day 3: Constructing real numbers

To do before May 28

- Read Bloch Section 1.7, pp 41-49.
- Think about these questions. If you are not able to answer them, consider reviewing the text again.
  - How is $\mathbb{R}$ constructed? Does the construction refer to any real numbers during the process?
  - Why is multiplication defined differently from addition and negation? What is the missing component from Lemma 1.6.8 that requires this more complicated treatment?
  - What is the property that distinguishes the real numbers from the rational numbers?
- Write any remaining questions you have in the shared Overleaf document.

To do after May 28

Write solutions to the following exercises.

Problem 4. Let $r \in \mathbb{Q}$. Prove that $D_{-r} = -D_r$ and $D_{r^{-1}} = [D_r]^{-1}$ using only Definitions 1.6.4, 1.7.3, and 1.7.5.

Problem 5. Prove Theorem 1.7.6 (7). For this exercise, you may only use parts (1)-(4), parts (10)-(12), part (14) of the theorem, and anything prior to the theorem.

Problem 6. Prove that $\mathbb{Q}$ does not have the least upper bound property, i.e. prove that there exists a subset $X \subseteq \mathbb{Q}$ that does not have a least upper bound in $\mathbb{Q}$.

Final Project submission

Submit your typed exercises on Canvas (not Crowdmark!) by **5pm on June 3**. Remember, collaboration and using outside resources is encouraged, but your write-up must be your own work.