Calculus II Assignment 11

Solutions.

1. Solve the second-order linear differential equations

$$y'' = y \quad \& \quad y'' = -y$$

using a series method.

The game here is to assume our differential equation has a solution $f(x)$ which has a power series expansion:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$ with some positive radius of convergence.

Note. There is no a priori reason for this to be the case; that is, there is no reason to believe that our equations can be solved by something with a power series expansion. But let's play along and see what happens...

So say $f(x) = \sum_{n=0}^{\infty} c_n x^n$ solves the equation $y'' = y$

i.e. $\quad f''(x) = f(x) \quad$ as functions of x.

Putting in the power series for $f(x)$ on both sides, this gives:

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n = f''(x) = f(x) = \sum_{n=0}^{\infty} c_n x^n$$

(differentiate power series term-by-term)

Now two power series are the same only when the coefficient of x^n coincide for all n. So comparing the coefficient of x^n on both sides above, we obtain the relationship:

$$(n+2)(n+1)c_{n+2} = c_n \quad \text{for all } n \geq 0$$

$$\iff \quad c_{n+2} = \frac{c_n}{(n+2)(n+1)} \quad \text{for all } n \geq 0.$$

Since this applies for all $n \geq 0$, we could iteratively apply this relation and relate c_n with c_{n-2},

c_{n-2} with c_{n-4}

$\vdots$

all the way until we cannot drop the subscript by 2, so either c_1 or c_0 depending on whether n is even or odd

That is:

$$c_n = \frac{c_{n-2}}{n(n-1)} = \frac{c_{n-4}}{n(n-1)(n-2)(n-3)} = \cdots = \begin{cases} \dfrac{c_1}{n!} & \text{if } n \text{ is odd} \\ \dfrac{c_0}{n!} & \text{if } n \text{ is even.} \end{cases}$$

Thus all the coefficients c_n of the power series are related to either c_0 or c_1. Let's use this relationship to simplify the expression for $f(x)$:

(separate to even and odd indices)

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} c_{2m} x^{2m} + \sum_{m=0}^{\infty} c_{2m+1} x^{2m+1}$$

$$= \sum_{m=0}^{\infty} \frac{c_0}{(2m)!} x^{2m} + \sum_{m=0}^{\infty} \frac{c_1}{(2m+1)!} x^{2m+1}$$

$$= c_0 \cdot \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$$

$$= c_0 \cosh(x) + c_1 \sinh(x)$$

where the functions $\cosh(x)$ and $\sinh(x)$ are those defined by

$$\cosh(x) := \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}, \quad \sinh(x) := \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$$

A direct computation shows that these functions are related to the exponential function by:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \& \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

This allows us to express $f(x)$ in terms of e^x & e^{-x},

$$f(x) = c_0 \cosh(x) + c_1 \sinh(x)$$

$$= c_0\left(\frac{e^x + e^{-x}}{2}\right) + c_1\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{c_0 + c_1}{2}\right) e^x + \left(\frac{c_0 - c_1}{2}\right) e^{-x}$$

$$=: b_0 e^x + b_1 e^{-x} \quad \text{where} \quad b_0 := \frac{c_0 + c_1}{2}, \quad b_1 := \frac{c_0 - c_1}{2}.$$

Summary: The general solution to $y'' = y$ is given by

$$f(x) = c_0 \cosh(x) + c_1 \sinh(x)$$

$$= b_0 e^x + b_1 e^{-x}$$

for arbitrary constants c_0, c_1 or b_0, b_1

Let's do the same thing with the equation $y''=-y$:

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n = y'' = -y = \sum_{n=0}^{\infty} -c_n x^n.$$

So comparing the coefficients of x^n on both sides, we get the relation:

$$c_n = (-1)\cdot\frac{c_{n-2}}{n(n-1)} = (-1)^2\frac{c_{n-4}}{n(n-1)(n-2)(n-3)} = \cdots = \begin{cases} (-1)^m \dfrac{c_1}{(2m+1)!} & \text{if } n=2m+1 \text{ odd} \\ (-1)^m \dfrac{c_0}{(2m)!} & \text{if } n=2m \text{ even.} \end{cases}$$

Thus: $$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} c_{2m} x^{2m} + \sum_{m=0}^{\infty} c_{2m+1} x^{2m+1}$$

$$= c_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

recognize the series!

$$= c_0 \cos(x) + c_1 \sin(x)$$

Summary. The general solution to $y''=-y$ is given by

$$f(x) = c_0 \cos(x) + c_1 \sin(x)$$

for arbitrary constants c_0 and c_1.

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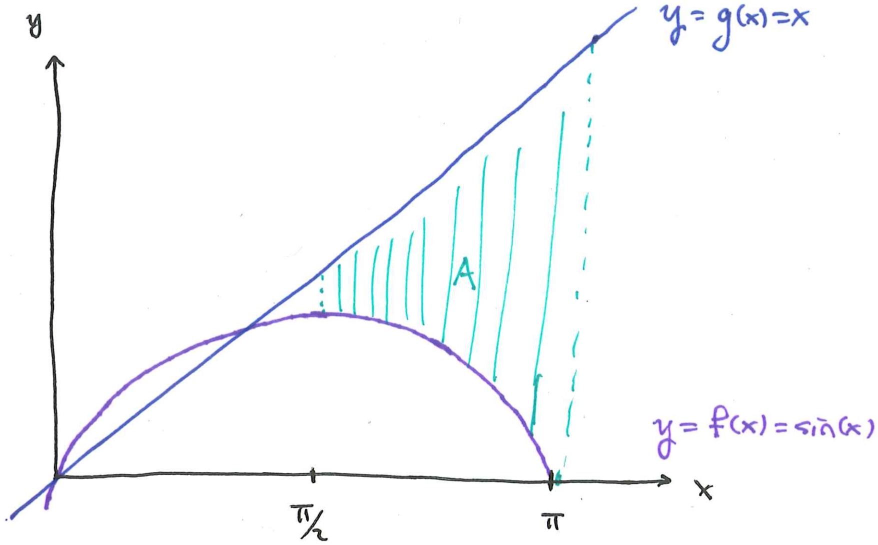
2. Time to find areas between curves!

(i)



upper curve

lower curve.

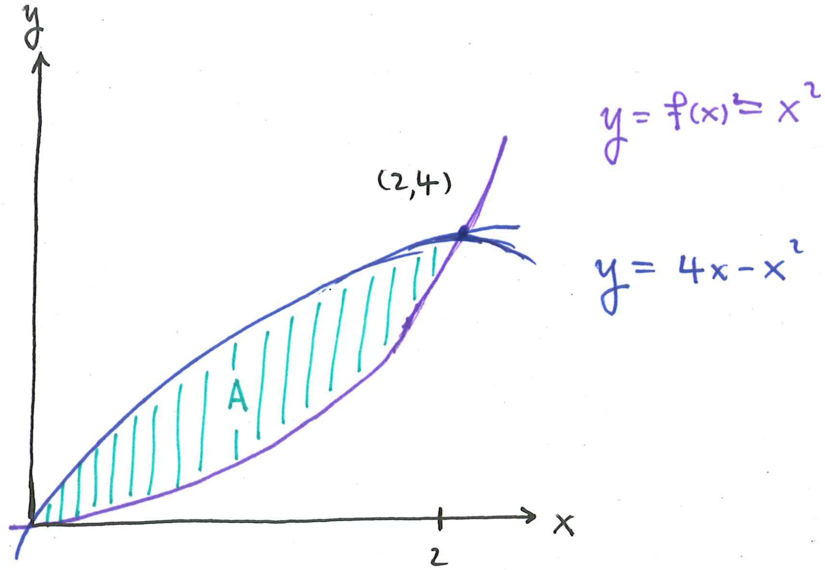
$$A = \int_{\pi/2}^{\pi} x - \sin(x)\, dx$$

$$= \frac{1}{2}x^2 + \cos(x) \Big]_{\pi/2}^{\pi}$$

$$= \left(\frac{1}{2}\pi^2 - 1\right) - \left(\frac{1}{8}\pi^2 + 0\right)$$

$$\boxed{A = \frac{3}{8}\pi^2 - 1.}$$

(ii)



$$A = \int_0^2 (4x - x^2) - x^2 \; dx$$

$$= \int_0^2 4x - 2x^2 \, dx$$

$$= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 = 8 - \frac{16}{3} = \frac{8}{3}.$$

$$\Rightarrow \boxed{A = \frac{8}{3}}$$

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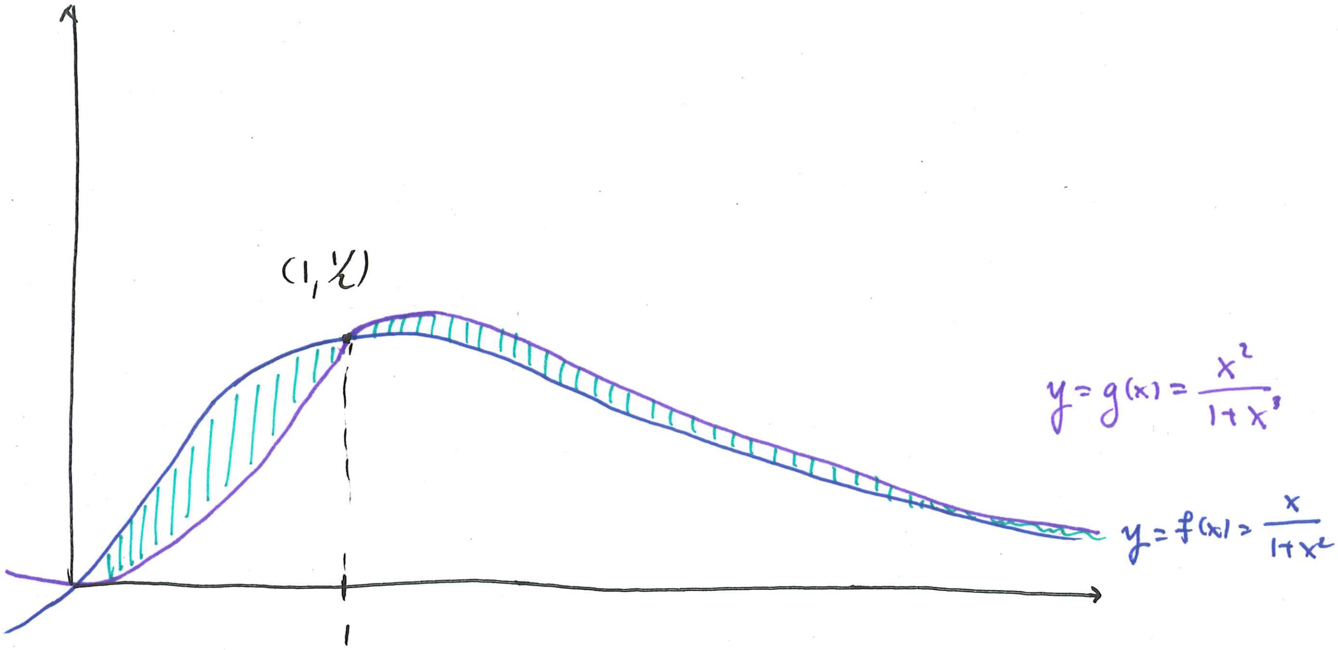
(iii)



$$A = \int_0^\infty \left| \frac{x}{1+x^2} - \frac{x^2}{1+x^3} \right| dx = \int_0^1 \frac{x}{1+x^2} - \frac{x^2}{1+x^3} + \int_1^\infty \frac{x^2}{1+x^3} - \frac{x}{1+x^2}$$

$$\left(\begin{matrix} v = 1+x^3 \\ u = 1+x^2 \end{matrix} \right) \quad = \left(\frac{1}{2}\int_1^2 \frac{du}{u} - \frac{1}{3}\int_1^2 \frac{dv}{v} \right) + \lim_{t\to\infty} \left(\frac{1}{3}\int_2^{1+t^3} \frac{dv}{v} - \frac{1}{2}\int_2^{1+t^2} \frac{du}{u} \right)$$

$$= \left(\frac{1}{2}\log(2) - \frac{1}{3}\log(2) \right) +$$

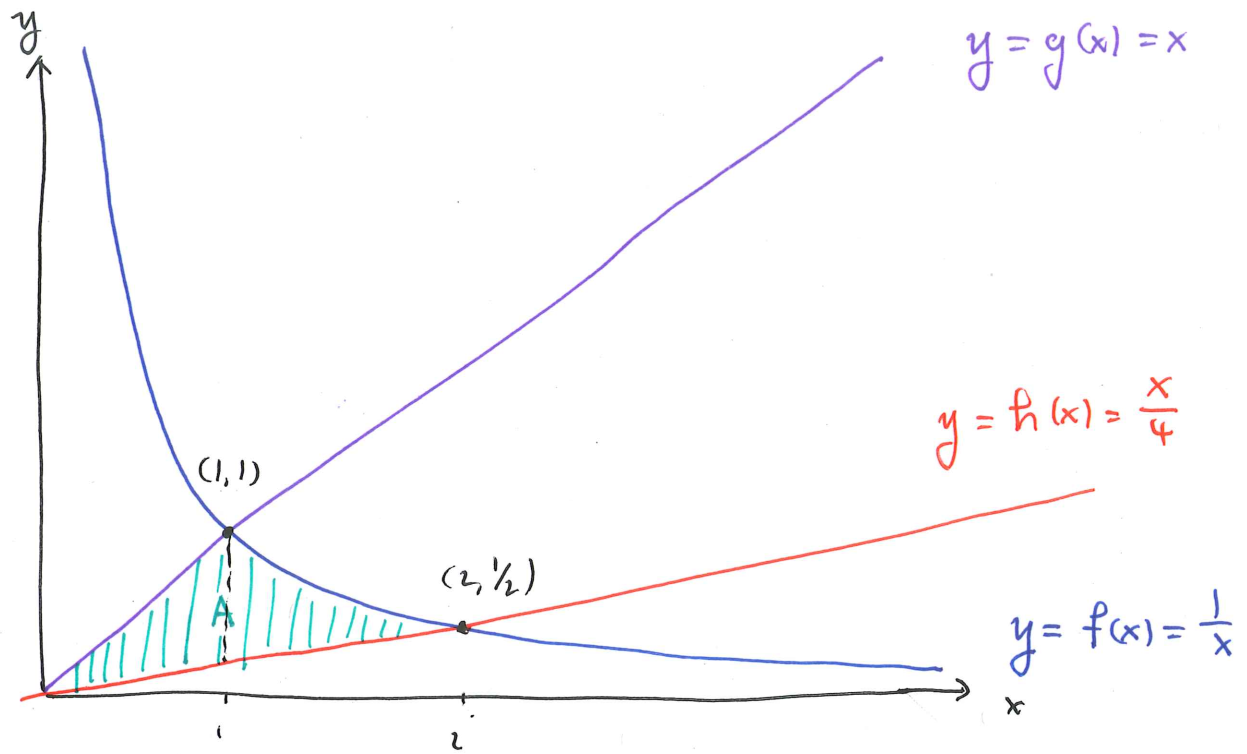
$$+ \lim_{t\to\infty} \frac{1}{3}\log(1+t^3) - \frac{1}{3}\log(2) - \frac{1}{2}\log(1+t^2) + \frac{1}{2}\log(2)$$

$$= \frac{1}{3}\log(2) + \lim_{t\to\infty} \log\left(\frac{(1+t^3)^{1/3}}{(1+t^2)^{1/2}} \right)$$

$$= \frac{1}{3}\log(2) + \cancel{\lim}\log\left(\lim_{t\to\infty} \frac{(1+t^3)^{1/3}}{(1+t^2)^{1/2}} \right)$$

$$= \frac{1}{3}\log(2) + \log(1) \implies \boxed{A = \frac{1}{3}\log(2)}$$

(iv)



$$A = \int_0^1 x - \frac{x}{4}\, dx + \int_1^2 \frac{1}{x} - \frac{x}{4}\, dx$$

$$= \frac{1}{2}x^2\Big]_0^1 + \log(x)\Big]_1^2 - \frac{1}{8}x^2\Big]_0^2$$

$$= \frac{1}{2} + \log(2) - \frac{1}{2}$$

$$\Rightarrow \boxed{A = \log(2)}$$

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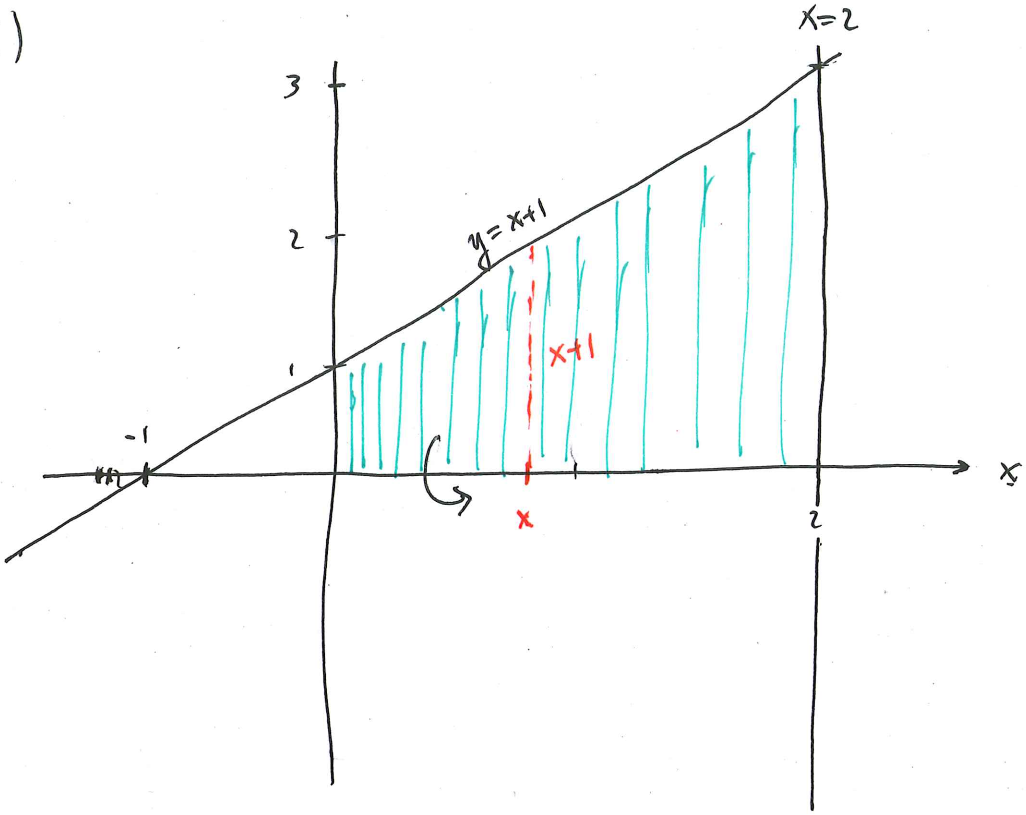
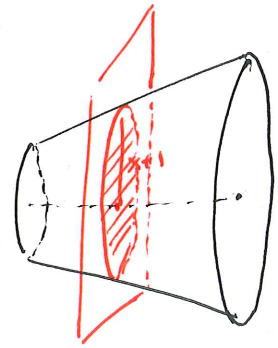
3. Now for some solids of revolution.

(i)



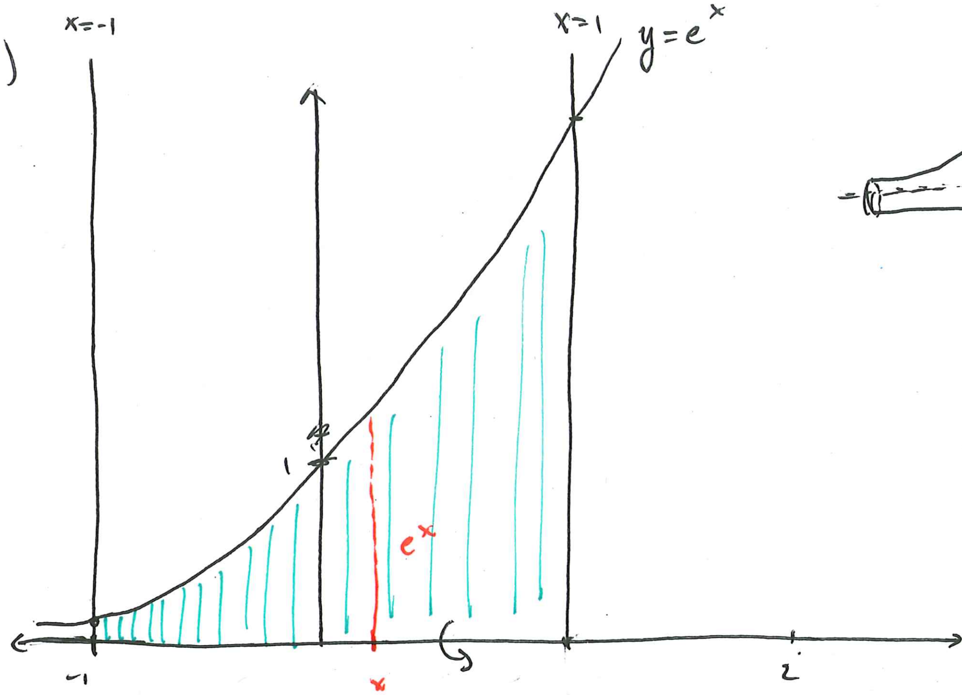


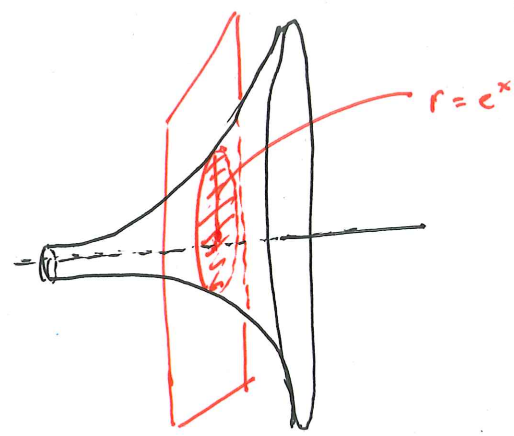
$$V = \int_0^2 \pi (x+1)^2 \, dx = \pi \int_0^2 x^2 + 2x + 1 \, dx$$

$$= \pi \left[\frac{1}{3}x^3 + x^2 + x \right]_0^2 = \pi \left(\frac{8}{3} + 4 + 2 \right)$$

$$\Rightarrow \boxed{V = \frac{26}{3}\pi.}$$

(ii)





$$V = \int_{-1}^{1} \pi e^{2x} \, dx$$

$$= \frac{\pi}{2}\left[e^{2x}\right]_{-1}^{1} = \frac{\pi}{2}\left(e^{2} - e^{-2}\right)$$

$$\Rightarrow \boxed{V = \left(\frac{e^{2} - e^{-2}}{2}\right)\pi = \pi \sinh(2).}$$

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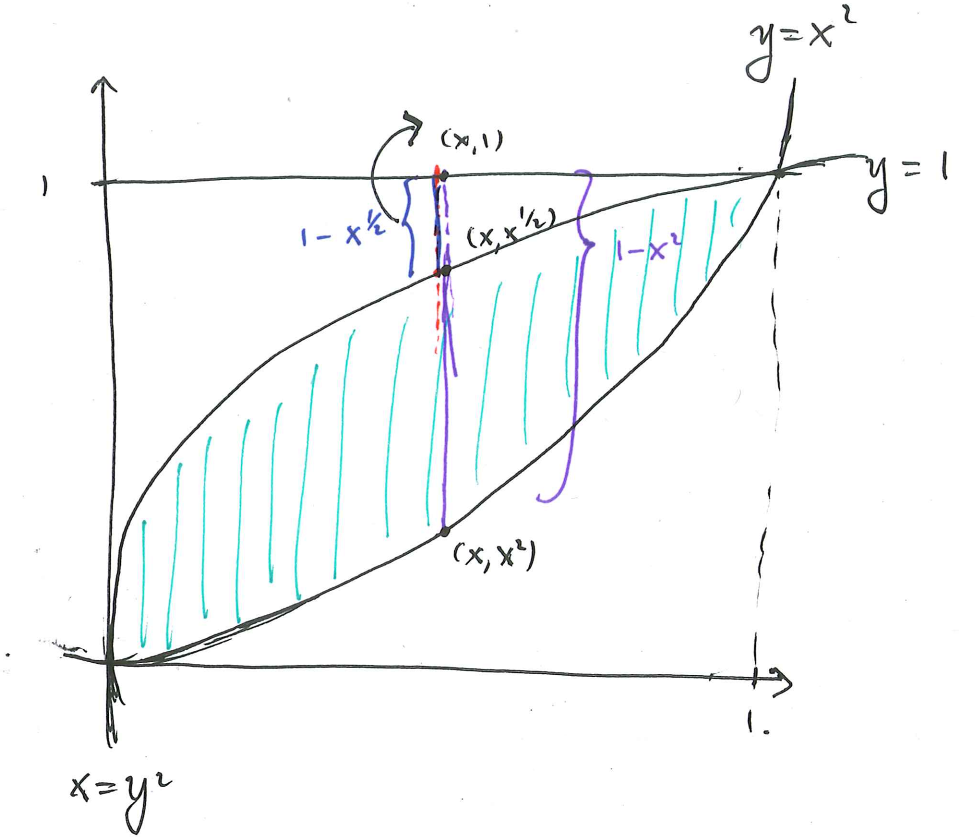
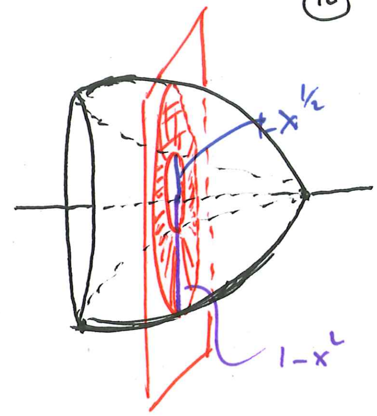
(iii)





$$V = \int_0^1 \pi \cdot \left((1-x^2)^2 - (1-x^{1/2})^2 \right) dx$$

$$= \pi \int_0^1 (1-2x^2+x^4) - (1-2x^{1/2}+x)\ dx$$

$$= \pi \int_0^1 x^4 - 2x^2 - x + 2x^{1/2}\ dx$$

$$= \pi \left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + 2 \cdot \frac{2}{3} \right) = \pi \left(\frac{11}{30} \right)$$

$$\Longrightarrow \boxed{V = \frac{11}{30}\pi}$$

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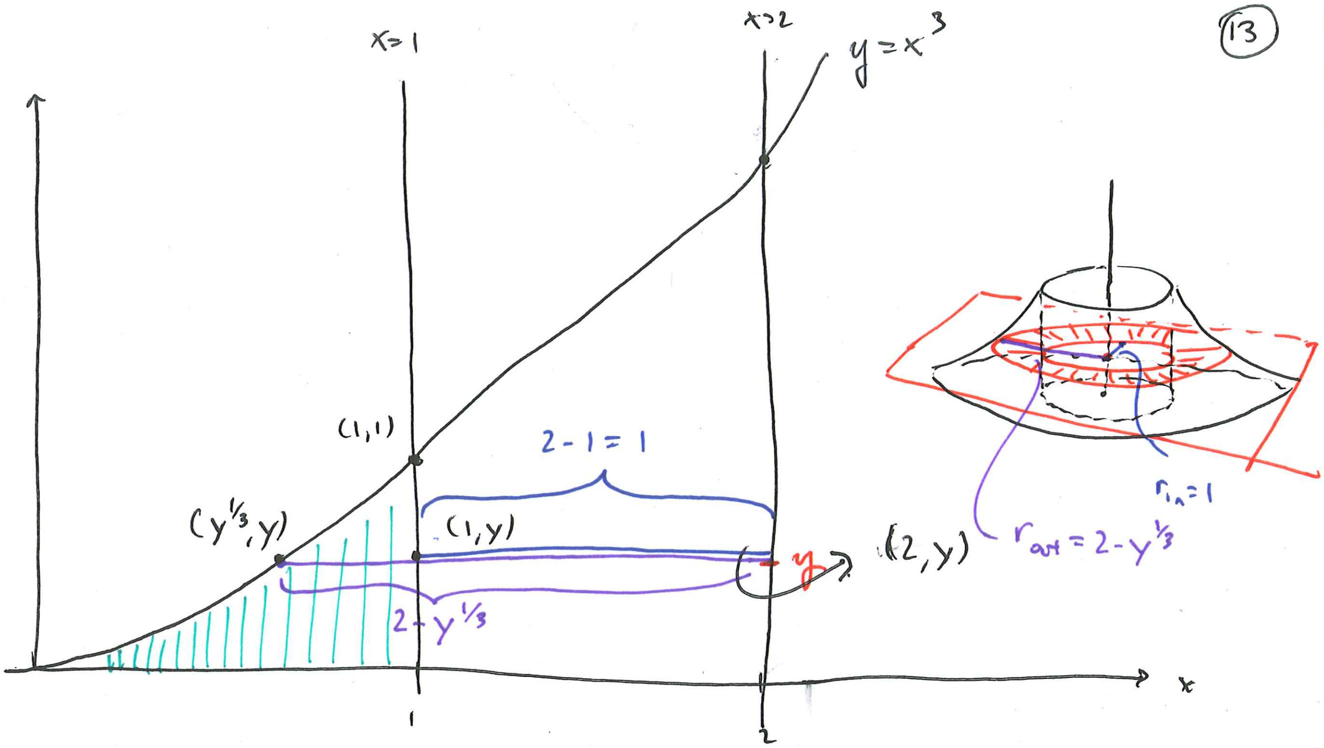
(iv)



$$V = \int_0^1 \pi \left((2-y^{1/3})^2 - 1^2 \right) dy$$

$$= \pi \int_0^1 y^{2/3} - 4y^{1/3} + 3 \, dy$$

$$= \pi \left[\frac{3}{5} - 4 \cdot \frac{3}{4} + 3 \right]$$

$$= \frac{3}{5} \pi$$

$$\Rightarrow \boxed{V = \frac{3}{5}\pi}$$

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