Spectral Sequences

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Throughout this article, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are abelian categories. We use $C(\mathcal{A})$ to denote the category of chain complexes of objects in $\mathcal{A}$, and $K(\mathcal{A})$ to denote the homotopy category of chain complexes. The localizing class of quasi-isomorphisms in $\mathcal{A}$ is usually denoted S, and $D(\mathcal{A}) \coloneqq K(\mathcal{A})[S^{-1}]$ is therefore the derived category of $\mathcal{A}$. If that did not make sense, first skim through Dragan Miličić's notes "Lectures on Derived Categories." [1]

1 Motivation

Let $X^{\bullet,\bullet}$ be a double complex of objects in $\mathcal{A}$, and let $d_>\colon X^{p,q} \to X^{p+1,q}$ and $d_\wedge\colon X^{p,q} \to X^{p,q+1}$ denote the differentials of each row and each column respectively. Suppose that for each $n \geq 0$ we know the cohomology of the horizontal complex $X^{\bullet,n}$. Let

$$X^n \coloneqq \bigoplus_{p+q=n} X^{p,q}, \quad d^n(x^{p,q}) \coloneqq d_>^{p,q}(x^{p,q}) + (-1)^p d_\wedge^{p,q}(x^{p,q}),$$

so that $(X^\bullet, d)$ form a complex, called the **total complex**. Is there some way we can use our knowledge of the cohomologies of each row to compute the cohomology of the total complex?

Spectral sequences provide the machinery to do so. The idea is simple. Naively, we hope that $H^n(X^\bullet)$ is given by $\bigoplus_{p+q=n} H^q(H^p(X^{\bullet,\bullet}))$. In words, first compute the cohomology of the horizontal complex, and then compute the cohomology of the resulting complex of cohomology groups, and then hope the result is the cohomology of the total complex. Unfortunately, in general, it is not. But it provides a good approximation, and spectral sequences give us "higher-order corrections" to this first approximation.

A spectral sequence consists of **pages**, indexed by $r \in \mathbb{Z}_{\geq 0}$ of double complexes. The r-th page is computed using the $(r-1)$-th page, and gives a better approximation to $H^\bullet(X^\bullet)$ than the $(r-1)$-th page. Eventually, for sufficiently large r, the entries on the r-th page that represent the cohomology groups $H^\bullet(X^\bullet)$ **converge**, i.e. stop changing, and we can then read them off directly.

[1] Available at http://www.math.utah.edu/~milicic/Eprints/dercat.pdf.

2 Filtered Complexes

Given a double complex, its total complex has the structure of a filtered complex. The spectral sequence of a double complex is just the spectral sequence of the total complex. Hence we first look at spectral sequences of filtered complexes.

Definition 2.1. An **(descending) filtered chain complex** in $C(\mathcal{A})$ is a (descending) filtered object in $C(\mathcal{A})$, i.e. a (descending) chain

$$\cdots \hookrightarrow F^{p+1}X^{\bullet} \hookrightarrow F^{p}X^{\bullet} \hookrightarrow F^{p-1}X^{\bullet} \hookrightarrow \cdots \hookrightarrow X^{\bullet}$$

where the differentials of $X^{\bullet}$ must respect the filtering, so that $d(F^pX^n) \subset F^pX^{n+1}$. The **associated graded complex** is given by

$$G^pX^{\bullet} = F^pX^{\bullet}/F^{p+1}X^{\bullet}.$$

Definition 2.2. Let $F^{\bullet}X^{\bullet}$ be a filtered chain complex. For $p, q, r \in \mathbb{Z}$, define the following.

1. The module of (p, q)**-cochains** is G^pX^{p+q}.
2. The module of r**-almost** (p, q)**-cocycles** is
$$\begin{aligned} Z_r^{p,q} &:= \{x \in G^pX^{p+q} : dx = 0 \bmod F^{p+r}X^{\bullet}\} \\ &= \{x \in F^pX^{p+q} : dx \in F^{p+r}X^{p+q+1}\}/F^{p+1}X^{p+q}. \end{aligned}$$
3. The module of (p, q)**-cocycles** is
$$Z_\infty^{p,q} := \{x \in F^pX^{p+q} : dx = 0\}/F^{p+1}X^{p+q} = Z(G^pX^{p+q}).$$
4. The module of r**-almost** (p, q)**-coboundaries** is
$$B_r^{p,q} := d(F^{p-r+1}X^{p+q-1}).$$
5. The module of (p, q)**-coboundaries** is
$$B_\infty^{p,q} := d(F^pX^{p+q-1}).$$

As one might suspect, the r-th page of the spectral sequence consists of the cohomologies $Z_r^{p,q}/B_r^{p,q}$. The differential d of $X^{\bullet}$ restrict to the objects on each page, but the indexing is different. The next two propositions set us up for the definition of the spectral sequence.

Proposition 2.3. *The differential d of $X^{\bullet}$ restricts to*

$$d_r \colon Z_r^{p,q} \to Z_r^{p+r,q-r+1}$$

on r-almost cocycles. It still forms a complex: $(d_r)^2 = 0$.

Proof. Let $x \in Z_r^{p,q}$. Then x represents an element of F^pX^{p+q}. By the definition of Z_r^{p+q}, the differential dx lives in $F^{p+r}X^{p+q+1}$. But $d(dx) = 0$, so

$$dx \in Z_\infty^{p+r,q-r+1} \subset Z_r^{p+r,q-r+1}.$$

Clearly $(d_r)^2 = 0$, since $d^2 = 0$. ☐

Proposition 2.4. $Z_{r+1}^{p,q} = \ker(Z_r^{p,q} \xrightarrow{d_r} Z_r^{p+r,q-r+1})$.

Proof. By 2.2, the definition of $Z_r^{p+r,q-r+1}$,

$$x \in \ker(d_r) \iff dx \in F^{p+r+1}X^{p+q+1},$$

since $F^{p+r+1}X^{p+q+1}$ is quotiented out in the definition of $Z_r^{p+r,q-r+1}$. Similarly, by the definition of $Z_{r+1}^{p,q}$,

$$x \in Z_{r+1}^{p,q} \iff dx \in F^{p+r+1}X^{p+q+1}.$$

Hence $x \in \ker(d_r)$ if and only if $x \in Z_{r+1}^{p,q}$. ☐

We are ready to define the spectral sequence of a filtered chain complex, and, in the next section, a spectral sequence in general!

Definition 2.5. Let $F^\bullet X^\bullet$ be a filtered chain complex in $C(\mathcal{A})$. For $p, q, r \in \mathbb{Z}$, the r**-almost** (p,q)**-cohomology** of the filtered complex is the quotient

$$\begin{aligned} E_r^{p,q} &:= Z_r^{p,q}/B_r^{p,q} \\ &= \frac{\{x \in F^pX^{p+q} : dx \in F^{p+r}X^{p+q+1}\}}{d(F^{p-r+1}X^{p+q-1}) \oplus F^{p+1}X^{p+q}}. \end{aligned}$$

The modules $E_r^{p,q}$ along with the differentials $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ form the **spectral sequence of** $F^\bullet X^\bullet$.

3 Spectral Sequences

Definition 3.1. A **(cohomology) spectral sequence** in $\mathcal{A}$ is given by the following data:

1. for each $r \in \mathbb{Z}_{\geq 0}$, a family $(E_r^{p,q})$ of objects in $\mathcal{A}$, for all $p, q \in \mathbb{Z}$, called the r**-th page** of the spectral sequence;
2. morphisms $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $(d_r)^2 = 0$, called **differentials**; and
3. isomorphisms $\alpha_r^{p,q} \colon \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-1}) \to E_{r+1}^{p,q}$.

For the spectral sequence of a filtered chain complex, the isomorphisms $\alpha_r^{p,q}$ arise from 2.4, which says that $Z_r^{p,q} = \ker(d_r^{p,q})$. Clearly $B_r^{p,q} = \operatorname{im}(d_r^{p-r,q+r-1})$, so the identification follows from the construction 2.5 of $E_r^{p,q}$ as $Z_r^{p,q}/B_r^{p,q}$.

To see why the spectral sequence of a filtered chain complex helps us compute cohomology, let's see what the first two pages of the spectral sequence look like.

Proposition 3.2. *We have*

$$
\begin{aligned}
E_0^{p,q} &= G^p X^{p+q} = F^p X^{p+q}/F^{p+1}X^{p+q},\\
E_1^{p,q} &= H^{p+q}(G^p X^\bullet).
\end{aligned}
$$

Proof. Since the differential preserves the filtering, $d(F^p X^{p+q}) \subset F^p X^{p+q+1}$, so

$$
\begin{aligned}
E_0^{p,q} = Z_0^{p,q}/B_0^{p,q} &= \frac{\{x \in F^p X^{p+q} : dx \in F^p X^{p+q+1}\}}{d(F^{p+1}X^{p+q-1}) \oplus F^{p+1}X^{p+q}}\\
&= \frac{F^p X^{p+q}}{F^{p+1}X^{p+q}} = G^p X^{p+q}.
\end{aligned}
$$

Similarly, recalling that $G^p X^\bullet = F^p X^\bullet / F^{p+1} X^\bullet$,

$$
\begin{aligned}
E_1^{p,q} = Z_1^{p,q}/B_1^{p,q} &= \frac{\{x \in G^p X^{p+q} : dx = 0 \bmod F^{p+1}X^{p+q+1}\}}{d(F^p X^{p+q-1})}\\
&= \frac{\{x \in G^p X^{p+q} : dx = 0 \in G^p X^{p+q+1}\}}{d(F^p X^{p+q-1})} = H^{p+q}(G^p X^\bullet),
\end{aligned}
$$

where it suffices to quotient out by $d(F^p X^{p+q-1})$ instead of $d(G^p X^{p+q-1})$ since $d(F^{p+1}X^{p+q-1}) \subset F^{p+1}X^{p+q}$ is quotiented out in $G^p X^{p+q}$ anyway. □

Suppose now that we have a filtered chain complex $F^\bullet X^\bullet$, and we know how to compute the cohomology $H^{p+q}(G^p X^\bullet)$ of its graded pieces. Then we know exactly what the first page of the spectral sequence is. But given a page of the spectral sequence, its construction tells us exactly how to compute the next page. So in principle, starting with the zeroth or first page, we can compute all the pages of the spectral sequence. The idea now is that eventually, the pages of the sequence stabilize, and in particular, they stabilize to the graded pieces $G^p(H^{p+q}X^\bullet)$ of the cohomology of $X^\bullet$.

Definition 3.3. Let $\{E_r^{p,q}\}$ be a spectral sequence such that for each $p, q \in \mathbb{Z}$, there exists $R(p,q) \in \mathbb{Z}$ such that

$$E_r^{p,q} \cong E_{R(p,q)}^{p,q} \quad \forall r \in \mathbb{Z}_{\geq R(p,q)},$$

i.e. after the $R(p,q)$-th page, the (p,q)-th entry stops changing. Then the spectral sequence is said to **converge** to the bigraded object

$$E_\infty \coloneqq \{E_{R(p,q)}^{p,q}\},$$

which is then called the **limit** of the spectral sequence. If $\{E_\infty^{p,q}\}$ can be written as the associated graded complex $\{G^p H^{p+q}\}$ of some graded object $H^\bullet$, then we say the spectral sequence **converges to** $H^\bullet$, and denote this convergence by

$$E_r^{p,q} \Rightarrow H^\bullet.$$

In practice, we often refer to spectral sequences by their first non-trivial page, which is usually $E_2^{p,q}$.

There are many different conditions we can impose on a spectral sequence to make it converge. A fairly general one is that for all $n, r \in \mathbb{Z}$, the number of non-zero terms $E_r^{k,n-k}$ is finite; sequences with this condition are **bounded**. In particular, sequences $\{E_r^{p,q}\}$ with only non-zero terms when $p, q \geq 0$ are bounded. We are mostly concerned with this case only.

Definition 3.4. A filtration $F^\bullet X^\bullet$ on a chain complex $X^\bullet$ is **bounded** if for all $n \in \mathbb{Z}$ there exists $P(n) \in \mathbb{Z}$ such that $F^p X^n = 0$ for all $p \geq P(n)$.

Definition 3.5. Let $F^\bullet X^\bullet$ be a filtered chain complex. For $p \in \mathbb{Z}$, let

$$F^p H^\bullet(X^\bullet) := \operatorname{im}(H^\bullet(F^p X^\bullet) \to H^n(X^\bullet)).$$

The $F^p H^\bullet(X^\bullet)$ define a filtration on $H^\bullet(X^\bullet)$.

Proposition 3.6. *Given a bounded filtered complex $F^\bullet X^\bullet$, the spectral sequence $\{E_r^{p,q}\}$ of $F^\bullet X^\bullet$ has a limit: it converges to the cohomology $H^\bullet(X^\bullet)$, i.e.*

$$E_r^{p,q} \Rightarrow H^\bullet(X^\bullet),$$

with the filtration on $H^\bullet(X^\bullet)$ defined in 3.5.

Proof. By 2.5, the definition of $E_r^{p,q}$, for each $p, q \in \mathbb{Z}$ there exists an $R(p,q) \in \mathbb{Z}$ such that the $R(p,q)$-almost (p,q)-cocycles and coboundaries are actually (p,q)-cocycles and coboundaries. Hence for $r \geq R(p,q)$, we have

$$E_r^{p,q} = Z_\infty^{p,q} / B_\infty^{p,q} = G^p H^{p+q}(X^\bullet). \qquad \square$$

4 Double Complexes

Now that we can use spectral sequences to compute the cohomology of a filtered complex from the cohomology of its graded pieces, we can apply this technology to compute the cohomology of the total complex of a double complex. The first step is to filter the total complex.

Definition 4.1. Let $X^{\bullet,\bullet}$ be a double complex of objects in $\mathcal{A}$, with differentials $d_>\colon X^{p,q} \to X^{p+1,q}$ and $d_\wedge\colon X^{p,q} \to X^{p,q+1}$. The **horizontal filtration** and **vertical filtration** on the total complex $X^\bullet$ are respectively given by

$$F_>^p(X^n) := \bigoplus_{i+j=n,\, i\geq p} X^{i,j}, \qquad F_\wedge^q(X^n) := \bigoplus_{i+j=n,\, j\geq q} X^{i,j}.$$

Definition 4.2. The **horizontal spectral sequence of $X^{\bullet,\bullet}$**, which we denote $\{{}_>\mathrm{E}_r^{p,q}\}$, is the spectral sequence of $F_>^\bullet(X^\bullet)$. Likewise, the **vertical spectral sequence of $X^{\bullet,\bullet}$**, which we denote $\{{}_\wedge\mathrm{E}_r^{p,q}\}$ is the spectral sequence of $F_\wedge^\bullet(X^\bullet)$.

If we begin with a double complex $X^{\bullet,\bullet}$ where the only non-zero terms are $X^{p,q}$ for $p, q \geq 0$, then the resulting filtrations of the total complex are clearly bounded. Hence the resulting spectral sequences have limits: they both converge to the cohomology of the total complex. We can also examine their first few pages.

Definition 4.3. Let $H^{p,q}_{>}(X^{\bullet,\bullet}) \coloneqq H^p(X^{\bullet,q})$ denote the cohomology at $X^{p,q}$ taken with respect to the horizontal differential. Likewise, let $H^{p,q}_{\wedge}(X^{\bullet,\bullet}) \coloneqq H^q(X^{p,\bullet})$ denote the cohomology at $X^{p,q}$ taken with respect to the vertical differential.

Proposition 4.4. *Let $\{{}_{>}\mathrm{E}^{p,q}_r\}$ be the horizontal spectral sequence of a double complex $X^{\bullet,\bullet}$. Then*

$$\begin{aligned} {}_{>}\mathrm{E}^{p,q}_0 &= X^{p,q}, \\ {}_{>}\mathrm{E}^{p,q}_1 &= H^{p,q}_{\wedge}(X^{\bullet,\bullet}), \\ {}_{>}\mathrm{E}^{p,q}_2 &= H^p(H^{\bullet,q}_{\wedge}(X^{\bullet,\bullet})). \end{aligned}$$

Proof. By 3.2, we have

$${}_{>}\mathrm{E}^{p,q}_0 = G^p X^{p+q} = \bigoplus_{i+j=p+q,\, i\geq p} X^{i,j} \Big/ \bigoplus_{i+j=p+q,\, i\geq p+1} X^{i,j} = X^{p,q},$$

so that $G^p X^{\bullet} = X^{p,\bullet}$. Hence

$${}_{>}\mathrm{E}^{p,q}_1 = H^{p+q}(G^p X^{\bullet}) = H^{p+q}(X^{p,\bullet}) = H^{p,q}_{\wedge}(X^{\bullet,\bullet}).$$

It follows that the representatives of $[x] \in {}_{>}\mathrm{E}^{p,q}_1$ are given by $x \in X^{p,q}$ such that $d_{\wedge} x = 0$. Hence $d_1 \colon {}_{>}\mathrm{E}^{p,q}_1 \to {}_{>}\mathrm{E}^{p+1,q}_1$, which is $d_{>} \pm d_{\wedge}$ restricted to $E^{p,q}_1$, acts on these representatives just as $d_{>}$. By 4.2, the construction of ${}_{>}\mathrm{E}^{p,q}_2$,

$${}_{>}\mathrm{E}^{p,q}_2 = \ker(d^{p,q}_1)/\operatorname{im}(d^{p-1,q}_1) = H^p(H^{\bullet,q}_{\wedge}(X^{\bullet,\bullet})). \qquad \square$$

Most spectral sequences that arise naturally converge very quickly, after one or two pages, so this proposition is very powerful. Instead of looking at those spectral sequences, though, we introduce a more general spectral sequence first.

5 The Grothendieck Spectral Sequence

Let's apply spectral sequences to a fairly natural question arising from the theory of derived functors. Let

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be two left exact (additive) functors, and suppose that $\mathcal{A}$ has a right-adapted subcategory $L_{\mathcal{A}}$ for F, and $\mathcal{B}$ has a right-adapted subcategory $L_{\mathcal{B}}$ for G, so that the derived functors

$$D(\mathcal{A}) \xrightarrow{DF} D(\mathcal{B}) \xrightarrow{DG} D(\mathcal{C})$$

exist. What is the relation between $R(G \circ F)$ and $RG \circ RF$?

Theorem 5.1. *Let $L_{\mathcal{A}}$ be a left-adapted subcategory for F and $L_{\mathcal{B}}$ be a left-adapted subcategory for G, and suppose that $F(L_{\mathcal{A}}) \subset L_{\mathcal{B}}$. Then for any object $X^{\bullet} \in D^{+}(\mathcal{A})$, there exists a spectral sequence*

$$E^{p,q}_2 = (R^p G \circ R^q F)(X^{\bullet}) \Rightarrow R^n(G \circ F)(X^{\bullet})$$

functorial in $X^{\bullet}$.

Before we prove the existence of this spectral sequence, we need some machinery. Without loss of generality, suppose the left-adapted subcategories $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$ are actually the collection of injective objects in $\mathcal{A}$ and $\mathcal{B}$, so that their existence implies $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. The key to constructing the desired spectral sequence is the construction of a Cartan-Eilenberg resolution.

Definition 5.2. A **Cartan-Eilenberg resolution** of a complex $X^\bullet \in C^+(\mathcal{A})$ is a double complex $I^{\bullet,\bullet}$ and a morphism $\varepsilon\colon X^\bullet \to I^{\bullet,0}$ such that

1. $I^{\bullet,q} = 0$ for $q < 0$,
2. the complex $I^{p,\bullet}$ is an injective resolution of X^p,
3. the complex $\ker(I^{p,\bullet} \xrightarrow{d_>} I^{p+1,\bullet})$ is an injective resolution of $\ker(d^p_{X^\bullet})$,
4. the complex $\operatorname{im}(I^{p,\bullet} \xrightarrow{d_>} I^{p+1,\bullet})$ is an injective resolution of $\operatorname{im}(d^p_{X^\bullet})$, and
5. the complex $H^{p,\bullet}_>(I^{\bullet,\bullet})$ is an injective resolution of $H^p(X^\bullet)$.

Proposition 5.3. *Let $\mathcal{A}$ have enough injectives and $X^\bullet \in C^+(\mathcal{A})$. Then there exists a Cartan-Eilenberg resolution of $X^\bullet$.*

Proof. This is a brief sketch of the construction; the details are not important to us. Suppose $X^p = 0$ for $p < n$; set $I^{p,\bullet} = 0$ for $p < n$. Now for every $p \geq n$, take the short exact sequences

$$0 \to \ker(d^p) \to X^p \to \operatorname{im}(d^p)) \to 0,$$

$$0 \to \operatorname{im}(d^p) \to \ker(d^{p+1}) \to H^{p+1}(X^\bullet) \to 0,$$

and perform the following steps:

1. choose injective resolutions

   $$\ker(d^n) \to Z^{n,\bullet}, \quad X^n \to I^{n,\bullet}, \quad \operatorname{im}(d^n) \to B^{n+1,\bullet}$$

   such that $0 \to Z^{n,\bullet} \to I^{n,\bullet} \to B^{n+1,\bullet} \to 0$ is a short exact sequence of complexes, and

2. choose injective resolutions

   $$\operatorname{im}(d^n) \to B^{n+1,\bullet}, \quad \ker(d^{n+1}) \to Z^{n+1,\bullet}, \quad H^{n+1}(X^\bullet) \to H^{n+1,\bullet}$$

   such that $0 \to B^{n+1,\bullet} \to Z^{n+1,\bullet} \to H^{n+1,\bullet} \to 0$ is a short exact sequence of complexes.

Having done this for all $p \geq n$, take the maps

$$d^p_>\colon I^{p,\bullet} \to B^{p+1,\bullet} \to Z^{p+1,\bullet} \to I^{p+1,\bullet}$$

to form the Cartan-Eilenberg resolution $I^{\bullet,\bullet}$. $\square$

The existence of a Cartan-Eilenberg resolution enables us to compute derived functors easily, using the spectral sequence of the double complex given by the resolution.

Proposition 5.4. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be left exact, and $X^\bullet \in C^+(\mathcal{A})$. Suppose $\mathcal{A}$ has enough injectives, and let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $X^\bullet$. The horizontal and vertical spectral sequences of the double complex $F(I^{\bullet,\bullet})$ are*

$$
\begin{aligned}
{}_{>}\mathrm{E}_1^{p,q} &= R^q F(X^p) \Rightarrow RF(X^\bullet), \\
{}_{\wedge}\mathrm{E}_2^{p,q} &= R^p F(H^q(X^\bullet)) \Rightarrow RF(X^\bullet).
\end{aligned}
$$

Proof. Recall that the q-th derived functor R^qF acting on an object $X \in \mathcal{A}$ is computed by taking an injective resolution $X \to I^\bullet$, and then taking the cohomology $H^q(F(I^\bullet)) = R^qF(X)$. Hence, by 4.4,

$$
{}_{>}\mathrm{E}_1^{p,q} = H_{\wedge}^{p,q}(F(I^{\bullet,\bullet})) = H^q(F(I^{p,\bullet})) = R^qF(X^p),
$$

since $I^{p,\bullet}$ is an injective resolution of X^p.

Recall that in a Cartan-Eilenberg resolution, $H_{>}^{p,\bullet}(I^{\bullet,\bullet})$ is an injective resolution of $H^p(X^\bullet)$, so $H^q(F(H_{>}^{p,\bullet}(I^{\bullet,\bullet}))) = R^qF(H^p(X^\bullet))$. Hence we can flip the double complex in 4.4 to get

$$
{}_{\wedge}\mathrm{E}_2^{p,q} = H^p(H_{>}^{\bullet,q}(F(I^{\bullet,\bullet}))) = H^p(F(H_{>}^{\bullet,q}(I^{\bullet,\bullet}))) = R^pF(H^q(X^\bullet)),
$$

where the second equality follows because the differentials split.

Finally, note that $\mathrm{tot}(I^{\bullet,\bullet})$ is an injective resolution of $X^\bullet$. So by 3.6, both spectral sequences converge to

$$
H^\bullet(\mathrm{tot}(F(I^{\bullet,\bullet}))) = H^\bullet(F(\mathrm{tot}(I^{\bullet,\bullet}))) = RF(X^\bullet)). \qquad \square
$$

Now that we have this machinery, we can easily construct the Grothendieck spectral sequence $R^pG(R^qF(X^\bullet)) \Rightarrow R(G \circ F)(X^\bullet)$.

Proof. Without loss of generality, suppose the left-adapted subcategories consist of injective objects. Let $X^\bullet \to I^\bullet$ be an injective resolution, and then let $F(I^\bullet) \to I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution, which exists by 5.3. The vertical spectral sequence of the double complex $G(I^{\bullet,\bullet})$, by 5.4 is

$$
\begin{aligned}
{}_{\wedge}\mathrm{E}_2^{p,q} &= R^pG(H^q(F(I^\bullet))) = R^pG(R^qF(X^\bullet)) \\
&\Rightarrow H^\bullet(G(F(I^\bullet))) = R(G \circ F)(X^\bullet).
\end{aligned} \qquad \square
$$

6 Examples of Spectral Sequences

The Grothendieck spectral sequence can be applied to various functors and their compositions. In this section we do so, and obtain a few well-known spectral sequences.

Definition 6.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, and $f : X \to Y$ a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf. Recall the **direct image functor**

$$f_* \colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(\mathcal{O}_Y), \quad \mathcal{F} \mapsto (V \mapsto \mathcal{F}(f^{-1}(V)))$$

between the category $\operatorname{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules and $\operatorname{Mod}(\mathcal{O}_Y)$ of $\mathcal{O}_Y$-modules. Let $R = \mathcal{O}_X(X)$. Recall that the subcategory of injective $\mathcal{O}_X$-modules is an adapted subcategory of the global sections functor $\Gamma_X \colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(R)$. Hence we have a right derived functor

$$R\Gamma_X \colon D^+(\operatorname{Mod}(\mathcal{O}_X)) \to D^+(\operatorname{Mod}(R)),$$

which, by definition, is **sheaf cohomology**. But since $\Gamma_Y \circ f_* = \Gamma_X$, there is a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^\bullet(X, \mathcal{F}),$$

known as the **Leray spectral sequence**.

Definition 6.2. Let G be a group, and A a left G-module. Recall the **invariants functor**

$$(-)^G \colon \operatorname{Mod}(G) \to \operatorname{Mod}(G), \quad A \mapsto A^G := \{a \in A : ga = a,\ g \in G\},$$

and that its derived functors compute group cohomology. If $N \triangleleft G$ a normal subgroup, then $(-)^G = ((-)^N)^{G/N}$. Hence there is a spectral sequence

$$E_2^{p,q} = H^p(G/H, H^q(H, A)) \Rightarrow H^\bullet(G, A),$$

known as the **Hochschild-Serre spectral sequence**.

Definition 6.3. Let X be a complex analytic space. Recall that the splitting $TX = T^{1,0}X \oplus T^{0,1}X$ gives the **Dolbeault double complex** $(\Omega^{\bullet,\bullet})$, with

$$\partial \colon \Omega^{p,q} \to \Omega^{p+1,q}, \quad \bar{\partial} \colon \Omega^{p,q} \to \Omega^{p,q+1}$$

as differentials. Recall also that the **holomorphic de Rham complex** $\Omega^\bullet$ is the complex of holomorphic differential forms with ∂ as the differential; it arises as the total complex of $\Omega^{\bullet,\bullet}$. Finally, recall that by the Dolbeault theorem, the cohomology of the Dolbeault complex, i.e. the horizontal cohomology $H_>^{p,q}(\Omega^{\bullet,\bullet})$, is actually the sheaf cohomology $H^q(X, \Omega^p)$. Apply the spectral sequence of a double complex to the Dolbeault complex to get the spectral sequence

$$_\wedge\mathrm{E}_1^{p,q} = H^q(X, \Omega^p) \Rightarrow H^\bullet(\Omega^\bullet),$$

known as the **Frölicher spectral sequence**.

In fact, for X a compact Kähler manifold, it is a fact that the Frölicher spectral sequence degenerates at the first page, i.e. $_\wedge\mathrm{E}_1^{p,q} = {_\wedge\mathrm{E}_\infty^{p,q}}$. From this fact follows the Hodge decomposition

$$\bigoplus_{p+q=n} H^p(X, \Omega^q) = H^n(X, \mathbb{C}).$$

7 Exact Sequence of Low-Degree Terms

Proposition 7.1. *Let $E_2^{p,q} \Rightarrow H^\bullet$ be a spectral sequence whose terms are nontrivial only for $p, q \geq 0$. Then*

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$$

is exact.

Proof. Since $E_2^{p,q}$ converges to $H^\bullet$, we have $H^1 = E_\infty^{1,0} \oplus E_\infty^{0,1}$. But

$$E_3^{1,0} = \frac{\ker(d_2^{1,0} \colon E_2^{1,0} \to E_2^{3,-1} = 0)}{\operatorname{im}(d_2^{-1,1} \colon E_2^{-1,1} = 0 \to E_2^{1,0})} = E_2^{1,0},$$

and similarly $E_2^{1,0} = E_3^{1,0} = \cdots = E_\infty^{1,0}$. Hence $E_2^{1,0} \to H^1$ is an injection. By a similar argument, $E_3^{0,1} = E_\infty^{0,1}$. Since $E_\infty^{0,1} \to E_2^{0,1}$ is an injection,

$$\ker(H^1 \to E_3^{0,1} = E_\infty^{0,1} \to E_2^{0,1}) = \ker(H^1 \to E_\infty^{0,1}) = E_\infty^{1,0}.$$

Hence the sequence is exact at H^1. The image of H^1 in $E_2^{0,1}$, which contains $E_\infty^{0,1}$ but not $E_\infty^{1,0}$, is precisely $E_\infty^{0,1} = E_3^{0,1}$. But

$$E_3^{0,1} = \frac{\ker(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0})}{\operatorname{im}(d_2^{-2,2} \colon E_2^{-2,2} = 0 \to E_2^{0,1})} = \ker(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0}),$$

so the sequence is exact at $E_2^{0,1}$. Finally, by the same argument showing that $E_2^{1,0} \to H^1$ is an injection, $E_3^{2,0} \to H^2$ is an injection. Now because

$$E_3^{2,0} = \frac{\ker(d_2^{2,0} \colon E_2^{2,0} \to E_2^{4,-1} = 0)}{\operatorname{im}(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0})} = \frac{E_2^{2,0}}{\operatorname{im}(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0})},$$

it follows that

$$\ker(E_2^{2,0} \to E_3^{2,0} \to H^2) = \ker(E_2^{2,0} \to E_3^{2,0}) = \operatorname{im}(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0}).$$

Hence the sequence is exact at $E_2^{2,0}$. □

Example 7.2. From the Grothendieck spectral sequence

$$E_2^{p,q} = R^p G(R^q F(X^\bullet)) \Rightarrow R(G \circ F)(X^\bullet),$$

there is an exact sequence

$$\begin{aligned} 0 &\to R^1 G(F(X^\bullet)) \to R^1(G \circ F)(X^\bullet) \to G(R^1 F(X^\bullet)) \\ &\to R^2 G(F(X^\bullet)) \to R^2(G \circ F)(X^\bullet). \end{aligned}$$

Example 7.3. From the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^\bullet(X, \mathcal{F}),$$

there is an exact sequence

$$0 \to H^1(Y, f_* \mathcal{F}) \to H^1(X, \mathcal{F}) \to \Gamma(Y, R^1 f_* \mathcal{F}) \to H^2(Y, f_* \mathcal{F}) \to H^2(X, \mathcal{F}).$$

Example 7.4. From the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G/H, H^q(H,A)) \Rightarrow H^\bullet(G,A),$$

there is an exact sequence

$$\begin{aligned} 0 &\to H^1(G/H, A^H) \to H^1(G,A) \to H^1(H,A)^{G/H} \\ &\to H^2(G/H, A^H) \to H^2(G,A) \to H^2(H,A)^{G/H}. \end{aligned}$$

8 Exact Couples

We have been constructing spectral sequences by picking various functors to compose, and then sticking them into the Grothendieck spectral sequence. But there are other spectral sequences! For example, they also commonly arise from objects known as exact couples.

Definition 8.1. An **exact couple** in an abelian category $\mathcal{A}$ consists of two objects A and E and three morphisms forming a cyclic exact sequence

$$\cdots \xrightarrow{j} E \xrightarrow{k} A \xrightarrow{\varphi} A \xrightarrow{j} E \xrightarrow{k} A \xrightarrow{\varphi} \cdots.$$

Usually, exact couples are written as triangles; we do not do that. Given an exact couple (A, E, φ, j, k), we can construct its **derived couple** by doing the following.

- Let $d = jk$. Note that $d^2 = jkjk = 0$, since $kj = 0$ by exactness.
- Define $A' = \varphi(A) \subset A$ and $\varphi' = \varphi|_{A'}$ and $E' = \ker d / \operatorname{im} d$.
- Define $j' \colon A' \to E'$ by $j'(\varphi(a)) = [j(a)] \in E'$. This is well-defined since $dj = jkj = 0$, so $j(a) \in \ker d$, and if $\varphi(a) = \varphi(b)$, then $a - b \in \ker \varphi = \operatorname{im} k$, so $j(a) - j(b) \in \operatorname{im}(jk) = \operatorname{im} d$.
- Define $k' \colon E' \to A'$ by $k'[e] = ke$. This lies in $A' = \operatorname{im} \varphi = \ker j$ because $jke = de = 0$. This is well-defined since if $[e] = 0$, then $e \in \operatorname{im} d \subset \operatorname{im} j = \ker k$.

The map d is called the **differential**.

Proposition 8.2. *The derived couple of an exact couple is an exact couple.*

Proof. Unenlightening diagram-chasing. □

Definition 8.3. The **spectral sequence of the exact couple** (A, E, φ, j, k) is given by the sequence $\{(E_r, d_r)\}_{r=0}^\infty$ given by setting $E_0 = E$ and iterating the derived couple construction. For example, $(A_1, E_1, \varphi_1, j_1, k_1)$ is the derived couple of (A, E, φ, j, k), and $(A_2, E_2, \varphi_2, j_2, k_2)$ is the derived couple of $(A_1, E_1, \varphi_1, j_1, k_1)$, and so on.

This definition may seem odd, since we have not put a grading on A or E yet, but our original definition of a spectral sequence involved bigraded pages. Typically, A_0 and E_0 are $\mathbb{Z}$-bigraded objects, and

$$\varphi_0 \colon A_0^{p,q} \to A_0^{p+1,q-1}, \quad j_0 \colon A^{p,q} \to E^{p,q}, \quad k^0 \colon E^{p,q} \to A^{p+1,q}.$$

Example 8.4. Let $F^\bullet X^\bullet$ be a filtered complex, and

$$A_0^{p,q} = F^{p+q}X^\bullet, \quad E_0^{p,q} = F^{p+q}X^\bullet / F^{p+q+1}X^\bullet.$$

The short exact sequence

$$0 \to F^{p+1}X^\bullet \to F^p X^\bullet \to F^p X^\bullet / F^{p+1}X^\bullet \to 0$$

gives rise to a long exact sequence

$$\begin{aligned} \cdots \xrightarrow{k} H^{p+q}(F^{p+1}X^\bullet) \xrightarrow{\varphi} H^{p+q}(F^p X^\bullet) \xrightarrow{j} H^{p+q}(F^p X^\bullet, F^{p+1}X^\bullet) \\ \xrightarrow{k} H^{p+q+1}(F^{p+1}X^\bullet) \xrightarrow{\varphi} H^{p+q+1}(F^p X^\bullet) \xrightarrow{j} \cdots . \end{aligned}$$

Let $A_1^{p,q} = H^{p+q}(F^p X^\bullet)$ and $E_1^{p,q} = H^{p+q}(F^p X^\bullet, F^{p+1}X^\bullet)$, and iterate. The resulting spectral sequence is precisely the spectral sequence of a filtered complex defined in 2.5.

Note that sometimes constructing spectral sequences from exact couples provides more data than constructing them from complexes. For example, if we look at the r-th derived couple, there is an exact sequence

$$\cdots \xrightarrow{k^r} A_r^{p,q} \xrightarrow{\varphi^r} A_r^{p+1,q-1} \xrightarrow{j^r} E_r^{p+r,q-r} \xrightarrow{k^r} E_r^{p+r+1,q-r} \xrightarrow{\varphi^r} \cdots .$$

This data is not present in our previous constructions of spectral sequences, because we lacked the objects A_r. In fact, the low-degree exact sequence in 7.1 arises naturally for exact couples, by looking at the exact sequence above for $r = 2$.

9 More Examples of Spectral Sequences

We turn our attention to some more examples of spectral sequences, this time of homology instead of cohomology. The same definitions as for cohomology apply, except dualized. These spectral sequences all arise from exact couples.

Definition 9.1. Let C be a torsion-free chain complex with coefficients in $\mathbb{Z}$. For p prime, the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

can be tensored with C to give the exact sequence of chain complexes

$$0 \to C \xrightarrow{p} C \to C \otimes \mathbb{Z}/p\mathbb{Z} \to 0.$$

The long exact sequence of homology induces an exact couple

$$\cdots \xrightarrow{k^1} H_\bullet(C) \xrightarrow{\varphi^1} H_\bullet(C) \xrightarrow{j^1} H_\bullet(C \otimes \mathbb{Z}/p\,\mathbb{Z}) \xrightarrow{k^1} H_\bullet(C) \xrightarrow{\varphi^1} \cdots$$

whose resulting spectral sequence is known as the **mod p Bockstein spectral sequence**. Here $\varphi^1 = p$, and $j^1 = \operatorname{mod} p$, and $k^1 = \partial$, the boundary homomorphism. We can put a $\mathbb{Z}$-bigrading $E^1_{p,q} \coloneqq H_{p+q}(C \otimes \mathbb{Z}/p\,\mathbb{Z})$ on the pages but really the spectral sequence is singly graded. The r-th derived couple looks like

$$\cdots \xrightarrow{\partial} p^{r-1}H_\bullet(C) \xrightarrow{p} p^{r-1}H_\bullet(C) \xrightarrow{\text{mod } p \circ p^{-r+1}} E^r_\bullet \xrightarrow{\partial} \cdots,$$

Expanding a portion of the long exact sequence arising from the r-th derived couple in the mod p Bockstein spectral sequence, we get the short exact sequence

$$0 \to (p^{r-1}H_n(C)) \otimes \mathbb{Z}/p\,\mathbb{Z} \to E^r_{n,0} \to \operatorname{Tor}(p^{r-1}H_n(C), \mathbb{Z}/p\,\mathbb{Z}) \to 0.$$

Note that for $r = 1$ this is precisely the **universal coefficient theorem** for homology. Usually the Bockstein spectral sequence is used to extract mod p data like so, instead of being used to compute homology.

Definition 9.2. Recall that a continuous map $p\colon E \to B$ of topological spaces has the **homotopy lifting property** with respect to another topological space X if for every specified homotopy $f\colon X \times [0,1] \to B$ and initial lift $\tilde{f}_0\colon X \to E$, there exists a lifted homotopy $\tilde{f}\colon X \times [0,1] \to E$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = \tilde{f}_0$. Such a map p is a **Serre fibration** if it has the homotopy lifting property for CW complexes.

Give B its skeletal filtration $F_pB \coloneqq B^p$, and define $F_pE \coloneqq= p^{-1}(F_pB)$. Note that $F_{p-1}E \subset F_pE$ is a **cofibration**, and therefore, for any abelian group π, gives a long exact sequence

$$\cdots \to H_n(F_{p-1}E;\pi) \to H_n(F_pE;\pi) \to H_n(F_pE, F_{p-1}E;\pi) \to \cdots.$$

Hence define an exact couple by

$$A^1_{p,q} \coloneqq H_{p+q}(F_pE;\pi), \quad E^1_{p,q} \coloneqq H_{p+q}(F_pE, F_{p-1}E;\pi).$$

The spectral sequence resulting from this exact couple is the **Serre spectral sequence**.

The Serre spectral sequence allows us to compute the homology of the total space E using the homology of the base B and the fibers $F_b = p^{-1}(b)$, which are all homotopy equivalent if B is path-connected (by the homotopy lifting property). The relevant result is that the sequence is given by

$$E^2_{p,q} = H_p(B; H_q(F;\pi)) \Rightarrow H_\bullet(E;\pi).$$

Example 9.3. Fix a prime p, and let $\mathcal{A}$ be the **mod p Steenrod algebra**, i.e. the algebra of stable cohomology operations for mod p, generated by the **Steenrod squares**

$$\operatorname{Sq}^i\colon H^n(X;\mathbb{Z}/2\,\mathbb{Z}) \to H^{n+i}(X;\mathbb{Z}/2\,\mathbb{Z})$$

for $p = 2$ and the **Steenrod reduced p-th powers**

$$P^i\colon H^n(X;\mathbb{Z}/p\,\mathbb{Z}) \to H^{n+2i(p-1)}(X;\mathbb{Z}/p\,\mathbb{Z})$$

otherwise.