THEOREM OF THE BASE

RAYMOND CHENG, LENA JI, MATT LARSON, AND NOAH OLANDER

ABSTRACT. We explain a proof of the Theorem of the Base: the Neron–Severi group of a proper variety is a finitely generated abelian group. We discuss, quite generally, the Picard functor and its torsion and identity components. We study representability and finiteness properties of the Picard functor, both absolutely and in families. Along the way, we streamline the original proof by using alterations, and we discuss some examples of peculiar Picard schemes.

CONTENTS

INTRODUCTION

The Theorem of the Base is a fundamental finiteness result on the Picard group of a proper variety X. A line bundle $\mathscr{L}$ on X is *algebraically trivial* if it is possible to deform $\mathscr{L}$ to $\mathscr{O}_X$; it is *numerically trivial* if it has degree 0 on every curve. See Definition 2.6. The *Néron–Severi group* $\mathrm{NS}(X)$ of X is the Picard group of X modulo algebraic triviality. We have the following result, see Proposition 4.3 and Theorem 7.4:

Theorem $\mathbf{B_{ase}}$. *Let X be a proper scheme over a field. Then*

(i) $\mathrm{NS}(X)$ *is finitely generated, and*
(ii) *its torsion subgroup consists of numerically trivial classes.*

In fact, we prove a stronger version of Theorem $\mathrm{B_{ase}}$ that gives a uniform bound on the rank and size of the torsion subgroup in families over a Noetherian base; see Theorem 7.7.

The proof of Theorem $\mathrm{B_{ase}}$ for a smooth projective variety X over $\mathbf{C}$ is simple: The first Chern class gives an injection $\mathrm{NS}(X) \hookrightarrow H^2(X,\mathbf{Z})$, so the finite generation of $\mathrm{NS}(X)$ follows from the finite generation of the topological cohomology group $H^2(X,\mathbf{Z})$. The Lefschetz

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$(1,1)$-Theorem and the Hard Lefschetz Theorem imply that the Hodge Conjecture holds for curves on X. Therefore, by Poincaré duality, any line bundle $\mathscr{L}$ that is numerically trivial has torsion first Chern class, which implies (ii).

Severi first proved Theorem $\mathrm{B}_{\mathrm{ase}}$ using transcendental methods in [Sev06, Sev34], extending work of Picard in [Pic05]. Néron proved (i) in arbitrary characteristic and gave the first algebraic proof by reinterpreting the question using rational points of an abelian variety related to the relative Jacobian of a curve fibration, see [N52]. Lang–Néron later simplified this proof when they proved the Lang–Néron Theorem in [LN59]. The proof of (ii) in arbitrary characteristic was first done by Matsusaka in [Mat57].

The modern approach, developed by Kleiman in [Kle66] and [SGA71, Exposé XIII], proves the finiteness of the rank of $\mathrm{NS}(X)$ and the torsion subgroup of $\mathrm{NS}(X)$ separately. The proof that the rank of $\mathrm{NS}(X)$ is finite is similar to the simple proof over the complex numbers, except that it uses a Weil cohomology theory, such as étale cohomology, to work in arbitrary characteristic. In order for the Weil cohomology theory to have the desired finiteness properties, we need to reduce to the case of smooth projective varieties. We use the existence of regular alterations [dJ96] to do so. Alternatively, one could reduce to the case of surfaces, and then use that resolution of singularities for surfaces is known in arbitrary characteristic, see [SGA71, Exp. XIII, Section 5].

There is no Weil cohomology theory with integral coefficients in positive characteristic, so this approach does not show the finiteness of the torsion subgroup. In order to show the finiteness of the torsion subgroup, we reduce to the case of projective varieties using Proposition 1.4, a theorem of Raynaud. We then show that all numerically trivial line bundles are parametrized by a single finite type Quot scheme, from which we deduce the finiteness of the torsion subgroup and (ii).

The Theorem of the Base is used throughout algebraic geometry, and is required for the formulation of many fundamental results. It is frequently useful to study *numerical* properties of line bundles, i.e., properties of line bundles that depend only the image of the line bundle in $\mathrm{NS}(X)\otimes\mathbf{Q}$. Many properties of line bundles, such as whether they are big [Laz04, Theorem 2.2.26] or ample [Kle66], can be shown to be numerical properties. Furthermore, the locus in $\mathrm{NS}(X)\otimes\mathbf{Q}$ of line bundles having a given numerical property often has nice topological properties. For example, the cone of ample line bundles is open in $\mathrm{NS}(X)\otimes\mathbf{Q}$ [Kle66], which allows one to deform the polarization of a variety. The formulation of the openness of the ample cone requires the Theorem of the Base: otherwise, when $\dim(X)>2$, there is no natural topology on $\mathrm{NS}(X)\otimes\mathbf{Q}$.

The Theorem of the Base is used very frequently in birational geometry. The *Picard number* $\rho(X)$, defined as the rank of $\mathrm{NS}(X)$, is a basic measure of complexity of a variety, and it is frequently used to show termination of algorithms. For example, the minimal model program for surfaces X consists of repeatedly contracting curves on X. After each contraction, one shows that the Picard number drops. As $\rho(X)<\infty$, this implies that eventually our variety will have no more curves that can be contracted. Many of the deepest results in birational geometry rely on the study of the ample cone and its closure in $\mathrm{NS}(X)\otimes\mathbf{Q}$.

Our chapter is organized as follows. In §§1 and 2, we discuss some fundamental results on the Picard functor and its components. In §§3 and 4, we prove the finiteness of the torsion subgroup of $\mathrm{NS}(X)$ and (ii) using projective geometry. Finiteness of the Picard number and hence finite generation of $\mathrm{NS}(X)$ is proved in §§5–7. We close in §8 with some examples of Picard schemes.

Conventions. Throughout, k is a field. A *variety* is a separated integral scheme of finite type over a field k. Unadorned fibre products are taken over k. We use the Stacks Project [Sta21] as the main technical reference. Results therein are referred to via their four character alphanumeric tags.

1. The Picard Functor

In this section, we recall the definition of the Picard functor in the Stacks Project, and summarize some of its main properties. We will use these definitions in the remainder of the document. For a more detailed treatment, see [FGI+05, Part 5] and [BLR90, Chapter 8].

Let $f : X \to S$ be a morphism of schemes. The Picard functor, restricting the general definition given in Situation 0D25,

$$\mathrm{Pic}_{X/S} \colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to \mathrm{Sets}$$

is the fppf sheafification of the functor sending a scheme T over S to the group $\mathrm{Pic}(X_T)$ of isomorphism classes of invertible $\mathcal{O}_{X_T}$-modules; here, $f_T : X_T \to T$ is the base change of f along $T \to S$. The basic representability result is the following:

Theorem 1.1. *Let $f : X \to S$ be a morphism of schemes. If*

(i) *f is proper, flat, of finite presentation, and*
(ii) *the formation of $f_*\mathcal{O}_X$ commutes with all base changes,*

then $\mathrm{Pic}_{X/S}$ *is an algebraic space. The morphism* $\mathrm{Pic}_{X/S} \to S$ *is quasi-separated and locally of finite presentation.*

Proof. In the case where $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over S, this is Proposition 0D2C and Lemma 0DNI. In general, see [Art69, Theorem 7.3]. ∎

Let $f : X \to S$ and $g \colon Y \to S$ be morphisms of schemes, and let $h \colon Y \to X$ be a morphism over S. Then there exists a morphism of group functors

$$h^* \colon \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{Y/S}$$

obtained as the fppf sheafification of the natural pullback map of invertible modules. The basic finiteness result for pullbacks is the following:

Theorem 1.2. *Let $f : X \to S$ and $g \colon Y \to S$ be morphisms of schemes, and let $h \colon Y \to X$ be a morphism over S. Assume that*

(i) *S is integral and Noetherian,*
(ii) *f and g are proper, and*
(iii) *h is surjective.*

Then there exists a nonempty open subscheme S° of S such that

$$h^*|_{S^\circ} \colon \mathrm{Pic}_{X/S}|_{S^\circ} \to \mathrm{Pic}_{Y/S}|_{S^\circ}$$

is representable by a quasi-affine morphism of finite presentation.

Proof. See [SGA73, Exposé XII, Théorème 1.1]. ∎

For simplicity, we will mostly restrict ourselves to the situation where our base scheme S is the spectrum of a field k. In this case, since separated group algebraic spaces locally of finite type over fields are actually schemes, the basic representability result reads:

Proposition 1.3. *Let X be proper scheme over k. Then $\mathrm{Pic}_{X/k}$ is a separated scheme locally of finite type over k.*

Proof. Theorem 1.1 already shows that $\mathrm{Pic}_{X/k}$ is a quasi-separated algebraic space locally of finite type over k. By Lemma 08BH, $\mathrm{Pic}_{X/k}$ is actually separated, so by Lemma 0B8F, it is a scheme. See also the discussion of Tag 06E9. ∎

Similarly, Theorem 1.2 implies:

Proposition 1.4. *Let $h\colon Y \to X$ be a surjective morphism of schemes which are proper over k. Then $h^*\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{Y/k}$ is a finite type quasi-affine morphism of schemes.* ∎

Finally, although the value of the Picard functor on general schemes T may be rather subtle, its geometric points are as expected. This follows from the following very general consideration:

Lemma 1.5. *Let U be an object of a site $\mathscr{C}$. Assume there exists a cofinal system of coverings of U of the form $\{V \to U\}$ such that each $V \to U$ admits a section. Then $\mathscr{F}(U) = \mathscr{F}^\#(U)$ for any presheaf $\mathscr{F}$ on $\mathscr{C}$.*

Proof. Here, $\mathscr{F}^\# := \mathscr{F}^{++}$ is the sheafification of $\mathscr{F}$, and $\mathscr{F}^+$ is

$$\mathscr{F}^+(U) := \mathrm{colim}_{\mathscr{U}}\, H^0(\mathscr{U}, \mathscr{F})$$

where the colimit is taken over all coverings $\mathscr{U}$ of U, see Section 00W1. Thus the result will follow if we can show that $\mathscr{F}(U) \to H^0(\mathscr{U}, \mathscr{F})$ is bijective for every covering $\mathscr{U} = \{p\colon V \to U\}$ in our cofinal system where each p admits a section σ. For injectivity, simply observe that

$$\mathscr{F}(U) \to H^0(\mathscr{U}, \mathscr{F}) \subset \mathscr{F}(V) \xrightarrow{\sigma^*} \mathscr{F}(U)$$

is the identity. For surjectivity, let $s \in H^0(\mathscr{U}, \mathscr{F})$ and set $t := \sigma^*(s)$. We claim $p^*(t) = s$. Indeed, writing $\mathrm{pr}_1, \mathrm{pr}_2\colon V \times_U V \to V$ for the projections, we have $\mathrm{pr}_1^*(s) = \mathrm{pr}_2^*(s)$ as s is a section on $\mathscr{U}$. Pulling this identity back along $(\mathrm{id}_V, \sigma \circ p) : V \to V \times_U V$ yields

$$\begin{aligned} s = \mathrm{id}_V^*(s) &= (\mathrm{id}_V, \sigma \circ p)^* \mathrm{pr}_1^*(s) \\ &= (\mathrm{id}_V, \sigma \circ p)^* \mathrm{pr}_2^*(s) = p^*\sigma^*(s) = p^*(t). \end{aligned}$$ ∎

Lemma 1.6. *Let X be a proper scheme over an algebraically closed field k. Then* $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$.

Proof. The Hilbert Nullstellensatz, Theorem 00FV, implies that any fppf covering $T \to \mathrm{Spec}(k)$ admits a section. Therefore Lemma 1.5 applies with $U = \mathrm{Spec}(k)$ and $\mathscr{F}$ the fppf presheaf $T \mapsto \mathrm{Pic}(X_T)$ defining the Picard functor. ∎

2. Components of the Picard Functor

The Picard functor as a whole is almost never of finite type as it generally has countably infinitely many connected components. Thus to make sense of finiteness properties for the Picard functor, it is helpful to consider the subgroup functors $\mathrm{Pic}^0_{X/k}$ and $\mathrm{Pic}^\tau_{X/k}$ giving the connected component of the identity and all torsion components, respectively.

Definition 2.1. Let G be a group scheme over k. The *identity component* G^0 of G is the connected component of the identity. The subscheme of *torsion components* G^τ of G is

$$G^\tau := \bigcup\nolimits_{n>0} (g \mapsto g^n)^{-1} G^0$$

the union of the preimage of G_0 under all n-th power maps with $n > 0$.

Lemma 2.2. *Let G be a group scheme locally of finite type over k. Then*

(i) *the formation of G^0 and G^τ commutes with extending k;*
(ii) *G^0 is an open and closed group subscheme of finite type;*
(iii) *G^0 is geometrically irreducible;*
(iv) *G^τ is an open group subscheme; and*
(v) *if G^τ is quasi-compact, then it is closed and of finite type.*

Proof. For the statements about G^0, see Proposition 0B7R. Now (i) for G^τ and (iv) follow from the corresponding properties of G^0. For (v), if G^τ is quasi-compact, then there exists $N > 0$ such that

$$G^\tau = \bigcup\nolimits_{n=1}^{N} (g \mapsto g^n)^{-1} G^0.$$

Thus G^τ is closed and of finite type since the same is true for G^0. ■

The following is the observation that a finite type morphism between two group schemes must respect torsion components in a strong way.

Lemma 2.3. *Let $f : H \to G$ be a finite type morphism of group schemes over k, which are locally of finite type over k. Then $f^{-1}(G^\tau) = H^\tau$.*

Proof. Since $f(H^0) \subseteq G^0$, we have

$$f(H^\tau) = \bigcup_{n>0} f\big((h \mapsto h^n)^{-1}(H^0)\big) = \bigcup_{n>0} (g \mapsto g^n)^{-1}(f(H^0)) \subseteq G^\tau$$

so that $f^{-1}(G^\tau) \supseteq H^\tau$. Conversely, let $f(h) \in G^\tau$. Replace h by h^n for $n > 0$ to assume $f(h) \in G^0$. By Lemma 2.2(ii), G^0 is of finite type; since f is of finite type, the same goes for $f^{-1}(G^0)$ as the composition of finite type morphisms is of finite type, see Lemma 01T3. In particular, it has finitely many components by Lemma 0BA8. So there exist $n > m > 0$ such that h^n and h^m are in the same component of $f^{-1}(G^0)$. Then $h^{n-m} \in H^0$, showing that $h \in H^\tau$. ■

Applying this to the inclusion of a finite type subgroup scheme shows:

Corollary 2.4. *Let G be a group scheme locally of finite type over k. If $H \subseteq G$ is a subgroup scheme of finite type over k, then $H \subseteq G^\tau$.* ■

Thus if G^τ is quasi-compact, then it is the largest subgroup scheme of G of finite type over k.

Now we apply the above notions to obtain subgroup schemes

$$\mathrm{Pic}^0_{X/k} \subseteq \mathrm{Pic}^\tau_{X/k} \subseteq \mathrm{Pic}_{X/k}.$$

These components make sense by the basic finiteness in the representability result Proposition 1.3. Already, we can show that formation of the torsion component commutes with certain pullbacks.

Lemma 2.5. *Let $f : Y \to X$ be a surjective morphism of proper schemes over k. Then $f^{*,-1}(\mathrm{Pic}^\tau_{Y/k}) = \mathrm{Pic}^\tau_{X/k}$.*

Proof. The finiteness of the representability result of Proposition 1.4 allows us to apply Lemma 2.3. ∎

The structure sheaf $\mathscr{O}_X$ is the unit in the group of invertible $\mathscr{O}_X$-modules. So we may attempt to characterize the points of $\mathrm{Pic}^0_{X/k}$ and $\mathrm{Pic}^\tau_{X/k}$ by relating invertible modules with $\mathscr{O}_X$.

Definition 2.6. Let X be a proper scheme over k. An invertible $\mathscr{O}_X$-module $\mathscr{L}$ is called

- (i) *numerically trivial* if $\deg(\mathscr{L}|_C) := \chi(C, \mathscr{L}|_C) - \chi(C, \mathscr{O}_C) = 0$ for every closed integral curve C in X;
- (ii) *algebraically trivial* if there exists a connected scheme T of finite type over k, an invertible $\mathscr{O}_{X\times T}$-module $\mathscr{M}$, and geometric points t_0 and t_1 of T such that $\mathscr{M}|_{X\times t_0} \cong \mathscr{O}_X$ and $\mathscr{M}|_{X\times t_1} \cong \mathscr{L}$; and
- (iii) *τ-trivial* if $\mathscr{L}^{\otimes n}$ is algebraically trivial for some integer $n \neq 0$.

These notions characterize points of components of $\mathrm{Pic}_{X/k}$:

Lemma 2.7. *Let X be a proper scheme over k. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module corresponding to a point $[\mathscr{L}] \in \mathrm{Pic}_{X/k}(k)$. Then*

- (i) *$\mathscr{L}$ is algebraically trivial if and only if $[\mathscr{L}] \in \mathrm{Pic}^0_{X/k}(k)$; and*
- (ii) *$\mathscr{L}$ is τ-trivial if and only if $[\mathscr{L}] \in \mathrm{Pic}^\tau_{X/k}(k)$.*

Proof. If $\mathscr{L}$ is algebraically trivial, then the witnessing data $(T, \mathscr{M}, t_0, t_1)$ is a morphism $[\mathscr{M}] : T \to \mathrm{Pic}_{X/k}$ such that $t_0 \mapsto [\mathscr{O}_X]$ and $t_1 \mapsto [\mathscr{L}]$. Since T is connected, $[\mathscr{L}]$ is in the connected component of the identity.

Suppose $[\mathscr{L}] \in \mathrm{Pic}^0_{X/k}(k)$. Let $f : T' \to \mathrm{Pic}^0_{X/k}$ be a fppf cover with an invertible sheaf $\mathscr{M}'$ on $X_{T'}$ representing the inclusion $\mathrm{Pic}^0_{X/k} \subseteq \mathrm{Pic}_{X/k}$. For $i = 0, 1$, let $T'_i \subseteq T'$ be irreducible components with geometric points t_i such that $f(t_0) = [\mathscr{O}_X]$ and $f(t_1) = [\mathscr{L}]$. Since $\mathrm{Pic}^0_{X/k}$ is irreducible and fppf morphisms are open, see Lemmas 2.2(iii) and 01UA, $f(T'_0) \cap f(T'_1)$ is a nonempty open subset of $\mathrm{Pic}^0_{X/k}$. Let s be a geometric point therein and let s_i be geometric points in the T'_i lying over, so that by Lemma 1.6, $\mathscr{M}'|_{X\times s_0} \cong \mathscr{M}'|_{X\times s_1}$. Up to replacing T' by a further fppf covering, we may assume that the images of s_i in T'_i are closed points with the same residue field. Then Lemma 0B7M allows us to glue the T'_i together along these closed points to obtain a scheme T. Furthermore, the sheaves $\mathscr{M}'|_{X\times T'_i}$ glue to an invertible $\mathscr{O}_{X\times T}$-module $\mathscr{M}$. Then $(T, \mathscr{M}, t_0, t_1)$ witness algebraic triviality of $\mathscr{L}$. ∎

It is not *a priori* clear that numerical triviality characterizes points of some component of $\mathrm{Pic}_{X/k}$, but we will see in Proposition 4.3 that it is actually equivalent to τ-triviality. This will use some finiteness properties of $\mathrm{Pic}^{\tau}_{X/k}$. In any case, we can already prove that numerical triviality is implied by both algebraic and τ-triviality.

Lemma 2.8. *Let X be a proper scheme over k and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. If $\mathscr{L}$ is either algebraically trivial or τ-trivial, then $\mathscr{L}$ is numerically trivial.*

Proof. If $\mathscr{L}$ is algebraically trivial, then there is a connected scheme T over k, an invertible $\mathscr{O}_{X\times T}$-module $\mathscr{M}$, and geometric points t_0 and t_1 of T such that $\mathscr{M}|_{X\times t_0}\cong \mathscr{O}_X$ and $\mathscr{M}|_{X\times t_1}\cong \mathscr{L}$. Let C be any closed integral curve in X. Since Euler characteristics are locally constant in flat proper families, as in Lemma 0B9T, we have

$$\begin{aligned}\deg(\mathscr{L}|_C) &= \chi(C,\mathscr{L}|_C)-\chi(C,\mathscr{O}_C)\\ &= \chi(C\times t_1,\mathscr{M}|_{C\times t_1})-\chi(C,\mathscr{O}_C)\\ &= \chi(C\times t_0,\mathscr{M}|_{C\times t_0})-\chi(C,\mathscr{O}_C)\\ &= \chi(C,\mathscr{O}_C)-\chi(C,\mathscr{O}_C)=0,\end{aligned}$$

so $\mathscr{L}$ is numerically trivial.

If $\mathscr{L}$ is τ-trivial, let n be a positive integer such that $\mathscr{L}^{\otimes n}$ is algebraically trivial. Then we have just proven that $\mathscr{L}^{\otimes n}$ is then numerically trivial. But if C is now any integral closed curve in X, additivity of degrees as from Lemma 0AYX gives

$$\deg(\mathscr{L}|_C)=\frac{1}{n}\deg(\mathscr{L}^{\otimes n}|_C)=0$$

so $\mathscr{L}$ itself is numerically trivial. ■

The various notions of triviality behave well under pullback.

Lemma 2.9. *Let $f : Y \to X$ be a morphism of proper schemes over k. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(i) *If $\mathscr{L}$ is numerically trivial, then $f^*\mathscr{L}$ is numerically trivial.*
(ii) *If $\mathscr{L}$ is algebraically trivial, then $f^*\mathscr{L}$ is algebraically trivial.*
(iii) *If $\mathscr{L}$ is τ-trivial, then $f^*\mathscr{L}$ is τ-trivial.*

Suppose furthermore that f is surjective.

(iv) *If $f^*\mathscr{L}$ is numerically trivial, then $\mathscr{L}$ is numerically trivial.*
(v) *If $f^*\mathscr{L}$ is τ-trivial, then $\mathscr{L}$ is τ-trivial.*

Proof. For (i), if D is any integral curve in Y, its image $C := f(D)$ is an integral curve in X. Thus by compatibility of degrees on curves with pullbacks, Lemma 0AYZ,

$$\deg(f^*\mathscr{L}|_D)=\deg(D\to C)\deg(\mathscr{L}|_C)=0.$$

For (ii), let $(T,\mathscr{M},t_0,t_1)$ be the data witnessing algebraic triviality of $\mathscr{L}$ on X. Let $f_T : Y\times T\to X\times T$ be the base change of f to T, and let $\mathscr{M}'=f_T^*\mathscr{M}$. Then $(T,\mathscr{M}',t_0,t_1)$ witness algebraic triviality of $f^*\mathscr{L}$. Now (iii) follows directly from (ii).

Assume $f : Y\to X$ is surjective. To see (iv), let C be an integral closed curve in X. The closure of a height 1 generic point of height in $f^{-1}(C)$ yields an integral closed curve D in Y with image C. Thus, again,

$$\deg(\mathscr{L}|_C)=\deg(D\to C)^{-1}\deg(f^*\mathscr{L}|_D)=0.$$

Finally, (v) follows from Lemma 2.7(ii) and Lemma 2.5. ■

3. Castelnuovo–Mumford Regularity

When X is projective over k, finiteness of $\mathrm{Pic}^{\tau}_{X/k}$ comes by exhibiting all its points as a quotient of a fixed finite locally free $\mathcal{O}_X$-module: that is, $\mathrm{Pic}^{\tau}_{X/k}$ will be realized as an open subscheme of a Quot scheme. In this section, we use projective techniques to study the cohomology of numerically trivial invertible modules, the main result being Proposition 3.5. This is formulated with the following notion; see Definition 08A3.

Definition 3.1. A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_k$ is said to be *m-regular* if

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0 \quad \text{for } 1 \le i \le n.$$

Let Λ be a set of coherent sheaves on $\mathbf{P}^n_k$. The *Castelnuovo–Mumford regularity* of Λ is the smallest integer m, if it exists, such that each $\mathcal{F} \in \Lambda$ is m-regular.

Regularity has many consequences for the cohomology of sheaves, see Section 089X and [Mum66, Lecture 14], for example. Most important for us, however, is that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated, see Lemma 08A8. Moreover, the definition is designed to be robust under passing to hyperplane sections, see Lemma 08A5. Conversely, and crucially for inductive arguments, we now show that regularity upon passing to a divisor yields vanishing of cohomology:

Lemma 3.2. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Let $\iota \colon H \hookrightarrow \mathbf{P}^n_k$ be an effective Cartier divisor of degree d. If*

(i) *H avoids all associated points of $\mathcal{F}$, and*
(ii) *$\iota_*\mathcal{F}|_H$ is b-regular,*

then $H^i(\mathbf{P}^n_k, \mathcal{F}(\nu)) = 0$ for all $i \ge 2$ and $\nu \ge b-d$.

Proof. Let $\sigma \in H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d))$ be a section defining H. Multiplication by σ is injective on $\mathcal{F}$ as H avoids all its associated points. Thus twisting the ideal sheaf sequence for H by $\mathcal{F}(\nu)$ gives sequences

$$0 \to \mathcal{F}(\nu - d) \xrightarrow{\sigma} \mathcal{F}(\nu) \to \iota_*\mathcal{F}(\nu)|_H \to 0 \quad \text{for all } \nu \in \mathbf{Z}.$$

Now $\iota_*\mathcal{F}|_H$ is ν-regular for all $\nu \ge b$, see Lemma 08A6. Therefore, the long exact sequence in cohomology gives

$$H^i(\mathbf{P}^n_k, \mathcal{F}(\nu-d)) \cong H^i(\mathbf{P}^n_k, \mathcal{F}(\nu)) \quad \text{for all } i \ge 2 \text{ and } \nu \ge b.$$

Serre Vanishing, Lemma 01YS, shows that these vanish for large ν, so

$$H^i(\mathbf{P}^n_k, \mathcal{F}(\nu-d)) = 0 \quad \text{for all } i \ge 2 \text{ and } \nu \ge b. \qquad ■$$

With the language of regularity, the goal of this section is to show that, for each closed subscheme $j \colon X \hookrightarrow \mathbf{P}^n_k$, the set

$$\Lambda(j) := \{ j_*\mathcal{L} \mid \mathcal{L} \text{ a numerically trivial } \mathcal{O}_X\text{-module} \}$$

has finite Castelnuovo–Mumford regularity which can be bounded by an integer depending only on the Hilbert polynomial of X. We deduce this from the following Induction Principle. Compare with [Mum66, pg. 101–103].

Proposition 3.3 (Induction Principle)**.** *Let Λ be a set of coherent sheaves on $\mathbf{P}^n_k$. Assume that there exists*

(i) *a positive integer s such that* $\dim \mathrm{Supp}(\mathscr{F}) = s$*,*

(ii) *a positive number a such that*

$$\dim_k H^0(\mathbf{P}^n_k, \mathscr{F}(j)) \leq a\binom{\nu+s}{s} \quad \text{for every } \nu \in \mathbf{Z}, \text{ and}$$

(iii) *a polynomial $P(t)$ which is the Hilbert polynomial of $\mathscr{F}$*

for every $\mathscr{F} \in \Lambda$. Also assume that there exists

(iv) *an integer b such that for every $\mathscr{F} \in \Lambda$, $\iota_*\mathscr{F}|_H$ is b-regular for some hyperplane $\iota\colon H \hookrightarrow \mathbf{P}^n_k$ not containing any associated points of $\mathscr{F}$.*

Then there exists an integer m depending only on a, b, and $P(t)$ such that every sheaf in Λ is m-regular.

Proof. Consider any $\mathscr{F} \in \Lambda$. Choose a hyperplane $\iota\colon H \hookrightarrow \mathbf{P}^n_k$ for $\mathscr{F}$ as in (iv). Twisting the ideal sheaf sequence of H by $\mathscr{F}(\nu)$ gives sequences

$$0 \to \mathscr{F}(\nu-1) \to \mathscr{F}(\nu) \to \iota_*\mathscr{F}(\nu)|_H \to 0 \quad \text{for every } \nu \in \mathbf{Z}.$$

Note multiplication by an equation of H is injective since it avoids all associated points of $\mathscr{F}$. Apply Lemma 3.2 with $d = 1$ to get

$$H^i(\mathbf{P}^n_k, \mathscr{F}(\nu-1)) = 0 \quad \text{for all } i \geq 2 \text{ and } \nu \geq b.$$

So when $\nu \geq b$, the long exact sequence in cohomology reduces to

$$\begin{aligned} 0 \longrightarrow H^0(\mathbf{P}^n_k, \mathscr{F}(\nu-1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathscr{F}(\nu)) &\xrightarrow{\rho_\nu} H^0(\mathbf{P}^n_k, \iota_*\mathscr{F}(\nu)|_H) \\ \longrightarrow H^1(\mathbf{P}^n_k, \mathscr{F}(\nu-1)) \longrightarrow H^1(\mathbf{P}^n_k, \mathscr{F}(\nu)) &\longrightarrow 0. \end{aligned}$$

Either ρ_ν is surjective, or else

$$\dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(\nu)) < \dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(\nu-1)).$$

But observe: if ρ_ν is surjective, then $\rho_{\nu+1}$ is surjective. Indeed, consider the commutative square

$$\begin{tikzcd} H^0(\mathbf{P}^n_k, \mathscr{F}(\nu)) \otimes_k H^0(\mathbf{P}^n_k, \mathscr{O}_{\mathbf{P}^n_k}(1)) \arrow[r] \arrow[d, "\rho_\nu\otimes\mathrm{id}"'] & H^0(\mathbf{P}^n_k, \mathscr{F}(\nu+1)) \arrow[d, "\rho_{\nu+1}"] \\ H^0(\mathbf{P}^n_k, \iota_*\mathscr{F}(\nu)|_H) \otimes_k H^0(\mathbf{P}^n_k, \mathscr{O}_{\mathbf{P}^n_k}(1)) \arrow[r] & H^0(\mathbf{P}^n_k, \iota_*\mathscr{F}(\nu+1)|_H) \end{tikzcd}$$

where the horizontal arrows are given by multiplication of sections. By assumption, the map on the left given by ρ_ν is a surjection; since $\iota_*\mathscr{F}|_H$ is ν-regular, Lemma 08A7 implies the map on the bottom is a surjection. Commutativity of the square then implies $\rho_{\nu+1}$ is a surjection.

Thus the sequence $\{\dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(\nu)) \mid \nu \geq b\}$ strictly decreases until it reaches 0. From this we see that $\mathscr{F}$ is m-regular for

$$m = b + \dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(b)).$$

It remains to see that the latter quantity is bounded in terms of a and $P(t)$. By the vanishing from Lemma 3.2,

$$\chi(\mathscr{F}(b)) = \dim_k H^0(\mathbf{P}^n_k, \mathscr{F}(b)) - \dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(b)).$$

By (iii), $\chi(\mathscr{F}(b)) = P(b)$. Applying (ii) and rearranging then yields

$$\dim_k H^1(\mathbf{P}^n_k, \mathscr{F}(b)) \leq a\binom{b+s}{s} - P(b).$$

Thus $\mathscr{F}$ is m-regular for

$$m := b + a\binom{b+s}{s} - P(b)$$

and this depends only on a, b, and $P(t)$, as claimed. ■

The following gives the uniform bound on twists of global sections of numerically trivial invertible modules. Compare with [FGI+05, Lemma 9.6.5]

Lemma 3.4. *Assume the base field k is infinite. Let $X \hookrightarrow \mathbf{P}^n_k$ be a closed subscheme. For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a positive number $a(\mathscr{F})$ such that*

$$\dim_k H^0(X, \mathscr{F} \otimes \mathscr{L}(\nu)) \leq a(\mathscr{F})\binom{\nu+s}{s} \quad \textit{where } s := \dim \operatorname{Supp}(\mathscr{F}),$$

for every numerically trivial $\mathscr{O}_X$-module $\mathscr{L}$ and $\nu \in \mathbf{Z}$. Moreover, $a(\mathscr{O}_X)$ can be chosen to depend only on the degree of X.

Proof. We proceed via dévissage, in the form of Lemma 01YM. There are three conditions to check. The first is that for every short exact sequence of coherent $\mathscr{O}_X$-modules

$$0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0$$

in which $a(\mathscr{F}_1)$ and $a(\mathscr{F}_2)$ exist, then $a(\mathscr{F})$ also exists. The cohomology sequence shows we may take $a(\mathscr{F}) := a(\mathscr{F}_1) + a(\mathscr{F}_2)$.

The second condition is that if $\mathscr{F}$ is a $\mathscr{O}_X$-module and $a(\mathscr{F}^{\oplus r})$ exists for some $r \geq 1$, then $a(\mathscr{F})$ also exists. Additivity shows we may take $a(\mathscr{F}) := a(\mathscr{F}^{\oplus r})/r$.

The third and final condition concerns passing to closed subschemes $Z \hookrightarrow X$. Let $Y \hookrightarrow X$ be another closed subscheme not containing Z. Let $\mathscr{I} \subseteq \mathscr{O}_Z$ be the ideal sheaf of $Y \cap Z$ in Z. Then we must show there exists a quasi-coherent subsheaf of ideals $\mathscr{J} \subseteq \mathscr{I}$ for which $a(\mathscr{J})$ exists. Since the restriction $\mathscr{L}|_Z$ is a numerically trivial invertible module on Z, we may replace X by Z and $\mathscr{L}$ by its restriction. Next, every closed subscheme $Y \hookrightarrow X$ is contained in some hypersurface section; hence we may find $\mathscr{J} \subseteq \mathscr{I}$ of the form $\mathscr{O}_X(-d)$ for some nonnegative integer d. So to complete the dévissage, it suffices to construct $a(\mathscr{O}_X(-d))$. In fact, if $d > 0$, then multiplication by a section of $\mathscr{O}_X(d)$ yields an injection

$$H^0(X, \mathscr{L}(-d)) \hookrightarrow H^0(X, \mathscr{L})$$

so we may take $a(\mathscr{O}_X(-d)) = a(\mathscr{O}_X)$.

Thus we are reduced to the special case of the Lemma in which $\mathscr{F} = \mathscr{O}_X$. Proceed by induction on the dimension s of X. If $s = 0$, take $a(\mathscr{O}_X) = \dim_k H^0(X, \mathscr{O}_X)$; note this is the degree of X. Assume $s > 1$ and the statement holds for all closed subschemes of $\mathbf{P}^n_k$ of dimension $s - 1$. Since k is infinite, we may find a hyperplane section $\iota \colon H \hookrightarrow X$

not containing any associated points of X, see Lemma 08A0. This gives a short exact sequence

$$0 \to \mathscr{L}(\nu-1) \to \mathscr{L}(\nu) \to \iota_*\mathscr{L}(\nu)|_H \to 0.$$

Taking global sections and applying induction on $\mathscr{L}(\nu)|_H$ gives inequalities, for all $\nu \in \mathbf{Z}$,

$$\begin{aligned}\dim_k H^0(X, \mathscr{L}(\nu)) &\leq \dim_k H^0(X, \mathscr{L}(\nu-1)) + \dim_k H^0(H, \mathscr{L}(\nu)|_H) \\ &\leq \dim_k H^0(X, \mathscr{L}(\nu-1)) + a(\mathscr{O}_{X\cap H})\binom{\nu+s-1}{s-1}.\end{aligned}$$

Now note that $\mathscr{L}(-1)$ has negative degree on all integral curves in X, so $H^0(X, \mathscr{L}(-1)) = 0$: indeed, if there were a nonzero section, restricting it to a curve C not contained in its zero locus would yield the contradiction $\deg(\mathscr{L}(-1)|_C) > 0$, see Lemma 0B40. Therefore we may iterate the above inequality to obtain

$$\dim_k H^0(X, \mathscr{L}(\nu)) \leq a(\mathscr{O}_{X\cap H}) \sum\nolimits_{\mu=0}^{\nu} \binom{\mu+s-1}{s-1} = a(\mathscr{O}_{X\cap H})\binom{\nu+s}{s},$$

so we may take $a(\mathscr{O}_X) = a(\mathscr{O}_{X\cap H})$. Since the degree of X and $X \cap H$ are the same, this depends only on the degree of X. ■

We now come to the crucial boundedness result: any numerically trivial invertible sheaf has the same Hilbert polynomial as the structure sheaf. The following argument largely follows [FGI+05, Lemma 9.6.6] and works for any projective scheme. It requires a careful induction on dimension that simultaneously computes the Hilbert polynomial and proves boundedness of Castelnuovo–Mumford regularity. A much easier argument in the smooth case can be made via Hirzebruch–Riemann–Roch together with Lemma 5.2. One could also reduce to the smooth case using the existence of regular alterations and Proposition 1.4.

Proposition 3.5. *Assume the base field k is infinite. Let $j\colon X \hookrightarrow \mathbf{P}^n_k$ be a projective scheme. Then every member of the set*

$$\Lambda(j) := \{j_*\mathscr{L} \mid \mathscr{L} \text{ a numerically trivial } \mathscr{O}_X\text{-module}\}$$

has the same Hilbert polynomial $P(t)$, namely, that of $\mathscr{O}_X$ with respect to the embedding j. The Castelnuovo–Mumford regularity of $\Lambda(j)$ is bounded by an integer m depending only on $P(t)$.

Proof. Proceed by induction on $s := \dim(X)$. When $s = 0$, the set in question consists only of the structure sheaf so the conclusion follows. So assume $s \geq 1$. First we show that every member of $\Lambda(j)$ has the same Hilbert polynomial. For that, it suffices to show that if $\mathscr{L}$ is any numerically trivial $\mathscr{O}_X$-module, then the Hilbert polynomial of $\mathscr{L}^{\otimes q}$ is independent of $q \in \mathbf{Z}$, as we may take $q = 0$. Choose d large such that $\mathscr{L}(d)$ is very ample. Now choose effective divisors H and D determined by short exact sequences

$$\begin{aligned}0 \to \mathscr{O}_X(-d) \to \mathscr{O}_X \to \mathscr{O}_H \to 0,\\ 0 \to \mathscr{L}^\vee(-d) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.\end{aligned}$$

Twisting both sequences by $\mathscr{L}^{\otimes q}(d+\nu)$ yields sequences

$$\begin{aligned}0 \to \mathscr{L}^{\otimes q}(\nu) \to \mathscr{L}^{\otimes q}(d+\nu) \to \iota_{H,*}\mathscr{L}^{\otimes q}(d+\nu)|_H \to 0,\\ 0 \to \mathscr{L}^{\otimes q-1}(\nu) \to \mathscr{L}^{\otimes q}(d+\nu) \to \iota_{D,*}\mathscr{L}^{\otimes q}(d+\nu)|_D \to 0.\end{aligned}$$

Taking Euler characteristics and subtracting yields, for all $q, \nu \in \mathbf{Z}$,

$$\chi(\mathscr{L}^{\otimes q}(\nu)) - \chi(\mathscr{L}^{\otimes q-1}(\nu)) = \chi(\mathscr{L}^{\otimes q}(d+\nu)|_D) - \chi(\mathscr{L}^{\otimes q}(d+\nu)|_H).$$

Since the restriction of $\mathscr{L}^{\otimes q}$ to any closed subscheme remains numerically trivial, induction applies to show that the right hand side is a polynomial depending only on ν. Therefore, as a function of q and ν,

$$\chi(\mathscr{L}^{\otimes q}(\nu)) = \varphi_1(\nu)q + \varphi_0(\nu)$$

for some polynomials φ_1 and φ_0.

We now show $\varphi_1 = 0$. If not, choose ν_0 sufficiently large such that

(i) $\varphi_1(\nu)$ is the same sign for all $\nu \geq \nu_0$, and
(ii) $\mathscr{L}^{\otimes q}|_H$ is b-regular for $b := \nu_0 + d$ and all $q \in \mathbf{Z}$,

where the induction hypothesis is used for the second condition. Then Lemma 3.2 applies to show

$$\chi(\mathscr{L}^{\otimes q}(\nu)) = \dim_k H^0(X, \mathscr{L}^{\otimes q}(\nu)) - \dim_k H^1(X, \mathscr{L}^{\otimes q}(\nu))$$

for all $q \in \mathbf{Z}$ and $\nu \geq \nu_0$. Thus we see that

$$\dim_k H^0(X, \mathscr{L}^{\otimes q}(\nu)) \leq \chi(\mathscr{L}^{\otimes q}(\nu)) = \varphi_1(\nu)q + \varphi_0(\nu).$$

Thus taking $q \to \pm\infty$, depending on whether $\varphi(\nu_0)$ is positive or negative, shows that $\dim_k H^0(X, \mathscr{L}^{\otimes q}(\nu)) \to \infty$. But this contradicts Lemma 3.4, which uniformly bounds this dimension independently of q. Therefore $\varphi_1 = 0$ and the Hilbert polynomials of $\mathscr{L}^{\otimes q}$ are independent of q. Hence all members of $\Lambda(j)$ have the same Hilbert polynomial.

Now to show $\Lambda(j)$ has bounded Castelnuovo–Mumford regularity depending only on $P(t)$, we apply the Induction Principle, Proposition 3.3. We verify the hypotheses:

(i) Every member of $\Lambda(j)$ is supported on all of the s-dimensional scheme X;
(ii) The quantity $a := a(\mathscr{O}_X)$ from Lemma 3.4 bounds global sections;
(iii) We have just proven that every member of $\Lambda(j)$ has the same Hilbert polynomial $P(t)$;
(iv) Since k is infinite, we may choose a hyperplane $\iota\colon H \hookrightarrow \mathbf{P}^n_k$ that avoids the associated points of X, see Lemma 08A0; then induction gives an b depending only on the Hilbert polynomial of $H \cap X$ such that the $\iota_*\mathscr{L}|_H$ are b-regular.

Thus Proposition 3.3 applies to give an integer m, depending only on a, b, and $P(t)$ such that all members of $\Lambda(j)$ are m-regular. It remains to see that both a and b depend only on $P(t)$. In fact, the second statement of Lemma 3.4 shows that $a = a(\mathscr{O}_X)$ depends only on the degree of the embedding $j\colon X \hookrightarrow \mathbf{P}^n_k$; this is but the leading coefficient of $P(t)$. As for b, observe from the short exact sequence

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_{X\cap H} \to 0$$

that the Hilbert polynomial of $X \cap H$ is $P(t) - P(t-1)$, and hence depends only on $P(t)$. Thus m depends only on $P(t)$. ∎

4. Boundedness

In this section, we give the finiteness result of $\mathrm{Pic}^{\tau}_{X/k}$.

Definition 4.1. Let X be a proper scheme over k. We say that a set Λ of invertible $\mathscr{O}_X$-modules is *bounded* if there exists

– a scheme T of finite type over k, and
– an invertible $\mathcal{O}_{X\times T}$-module $\mathcal{M}$,

such that for every $\mathcal{L}\in\Lambda$, there exists $t\in T$ such that $\mathcal{L}\cong\mathcal{M}|_{X\times t}$.

Lemma 4.2. *Let X be a projective scheme over an infinite field k. Then the set Λ of numerically trivial invertible $\mathcal{O}_X$-modules is bounded.*

Proof. Fix a very ample invertible module $\mathcal{O}_X(1)$. Then, by Proposition 3.5, all numerically trivial invertible $\mathcal{O}_X$-modules have the same Hilbert polynomial $P(t)$ and Castelnuovo–Mumford regularity bounded by an integer m depending only on $P(t)$. So for every $\mathcal{L}\in\Lambda$, $\mathcal{L}\otimes\mathcal{O}_X(m)$ is globally generated by Lemma 08A8 and its space of global sections has dimension a fixed integer M. Thus every such $\mathcal{L}$ is a quotient of

$$\mathcal{F}:=\mathcal{O}_X(-m)^{\oplus M}\cong H^0(X,\mathcal{L}\otimes\mathcal{O}_X(m))\otimes_k\mathcal{O}_X(-m).$$

Therefore Λ is parameterized by the open subscheme T of $\mathrm{Quot}^P_{\mathcal{F}/X/k}$ parameterizing locally free quotients. The latter is of finite type over k by Lemma 0DPC hence T is of finite type over k. ■

The following characterizes the points of $\mathrm{Pic}^\tau_{X/k}$ numerically. See [SGA71, Exp. XIII, Théorème 4.6] for more.

Proposition 4.3. *Let X be a proper scheme over an infinite field k, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the following are equivalent.*

(i) *the set $\{\mathcal{L}^{\otimes m}\mid m\in\mathbf{Z}\}$ is bounded;*
(ii) *$\mathcal{L}$ is τ-trivial; and*
(iii) *$\mathcal{L}$ is numerically trivial.*

Then (i) $\Rightarrow$ (ii) $\iff$ (iii). *If X is, furthermore, projective over k, then all three statements are equivalent.*

Proof. To see (i) $\Rightarrow$ (ii), assume $\{\mathcal{L}^{\otimes m}\mid m\in\mathbf{Z}\}$ is bounded. Then there exists a scheme T of finite type over k, a line bundle $\mathcal{M}$ on $X\times T$, and, for each $m\in\mathbf{Z}$, a geometric point t_m of T such that $\mathcal{L}^{\otimes m}\cong\mathcal{M}|_{X\times t_m}$. But T has only finitely many connected components, so there exists $m,n\in\mathbf{Z}$ such that t_m and t_n lie in the same connected component, and so $\mathcal{L}^{\otimes m-n}$ is algebraically trivial.

Now (ii) $\Rightarrow$ (iii) is Lemma 2.8. For the converse (ii) $\Leftarrow$ (iii), Chow's Lemma 0200 and Lemma 2.9 together show that it is enough to consider the projective case. But when X is, furthermore, projective, Lemma 4.2 gives (iii) $\Rightarrow$ (i) and all three statements are then equivalent. ■

Theorem 4.4 (Finiteness of $\mathrm{Pic}^\tau_{X/k}$)**.** *Let X be a proper scheme over a field k. Then $\mathrm{Pic}^\tau_{X/k}$ is a quasi-compact closed subscheme of $\mathrm{Pic}_{X/k}$.*

Proof. Using Chow's Lemma 0200, functoriality of the torsion component from Lemma 2.5, and Proposition 1.4 it suffices to consider the case X is projective and k is algebraically closed. Since $\mathrm{Pic}_{X/k}$ is locally of finite type over k from Proposition 1.3, it suffices to show that $\mathrm{Pic}^\tau_{X/k}$ is quasi-compact, from which everything else follows from Lemma 2.3. Now Proposition 4.3 shows that τ-triviality and numerical triviality are the same, and so Lemma 4.2 gives a scheme T of finite type over k and a morphism $[\mathcal{M}]\colon T\to\mathrm{Pic}_{X/k}$ whose image contains all geometric points corresponding to τ-trivial invertible $\mathscr{O}_X$-modules. In other words, by Lemma 2.7(ii) and Lemma 1.6,

$$[\mathscr{M}](T) \supseteq \mathrm{Pic}^\tau_{X/k}.$$

Being finite type over k, T is a Noetherian topological space, and so is any subspace. Thus $\mathrm{Pic}^\tau_{X/k}$ is Noetherian and hence quasi-compact. ■

5. Finiteness of Cycles Modulo Numerical Equivalence

In this section, we show that the group of cycles modulo numerical equivalence is of finite rank using a cycle class map into a Weil cohomology theory, after which finiteness comes from finiteness of Weil cohomology theories generally.

Throughout this section, we take X to be a smooth projective variety of dimension d over an algebraically closed field k. Then Chapters 0AZ6 and 0FFG on intersection theory and Weil cohomology theories apply.

Definition 5.1. Let $0 \leq i \leq d$ and let $\alpha \in \mathrm{CH}^i(X)$.

(i) We say that α is *numerically trivial* if $\deg(\alpha \cdot \beta) = 0$ for every $\beta \in \mathrm{CH}^{d-i}(X)$. Here, $\deg\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(\mathrm{Spec}(k)) = \mathbf{Z}$ is the degree map and $\alpha \cdot \beta$ is the intersection product of Section 0BoG.

(ii) We say that α is *H^*-trivial* it lies in the kernel of the cycle class map $\gamma\colon \mathrm{CH}^i(X) \to H^{2i}(X)(i)$ associated with H^*.

Write $\mathrm{CH}^i(X)_{\mathrm{num}}$ and $\mathrm{CH}^i(X)_{H^*}$ for the subgroups of cycles which are numerically trivial and H^*-trivial, respectively, and let

$$\mathrm{Num}^i(X) := \mathrm{CH}^i(X)/\mathrm{CH}^i(X)_{\mathrm{num}}$$

be the *numerical group of codimension i cycles*.

The goal of this section is to bound the rank of the numerical group in terms of Betti numbers of a Weil cohomology theory. The relevance of this to our situation is given by the first Chern class homomorphism

$$c_1\colon \mathrm{Pic}_{X/k}(k) \to \mathrm{CH}^1(X)$$

see Section 02SI. Since X is smooth, Lemma 0BE9 shows that c_1 is an isomorphism. The following shows that the two notions of numerical triviality in Definitions 2.6 and 5.1 are compatible under c_1.

Lemma 5.2. *An invertible $\mathscr{O}_X$-module $\mathscr{L}$ is numerically trivial if and only if $c_1(\mathscr{L})$ is numerically trivial. Thus c_1 induces an isomorphism of abelian groups* $\mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^\tau_{X/k}(k) \cong \mathrm{Num}^1(X)$.

Proof. The invertible module $\mathscr{L}$ is numerically trivial if and only if for every integral curve C in X, $\deg(\mathscr{L}|_C) = 0$. By Lemma 0BEY, this degree is the numerical intersection number on the left of

$$(\mathscr{L} \cdot C) = \deg(c_1(\mathscr{L}) \cdot [C]).$$

By Lemma 0BFI, the numerical intersection number is compatible with the Chow-theoretic intersection number on the right, the vanishing of which is equivalent to numerical triviality of $c_1(\mathscr{L})$. ■

We now proceed to bound the ranks of the numerical groups.

Lemma 5.3. $\mathrm{CH}^i(X)_{H^*} \subseteq \mathrm{CH}^i(X)_{\mathrm{num}} \subseteq \mathrm{CH}^i(X)$.

Proof. We need to show that any codimension i cycle α that is H^*-trivial is numerically trivial. So consider any $\beta \in \mathrm{CH}^{d-i}(X)$. By the cycle class axioms (C)(c) of Section oFHA, and Lemma oFHR,

$$\deg(\alpha \cdot \beta) = \int_X \gamma(\alpha) \cup \gamma(\beta) = 0. \qquad \blacksquare$$

In particular, this means that the numerical group is naturally a quotient of $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)_{H^*}$. We use this observation to bound the rank over $\mathbf{Z}$ of $\mathrm{Num}^i(X)$ in terms terms of the dimension over F of $H^{2i}(X)$ by exploiting the compatibility between the intersection pairing on cycles and the perfect pairing on H^*. The situation is abstracted into the following technical result:

Lemma 5.4. *Suppose given*

- *a field F containing a ring R,*
- *finite dimensional vector spaces V_1 and V_2 over F,*
- *R-submodules $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$, and*
- *a F-bilinear map $\langle -,- \rangle \colon V_1 \times V_2 \to F$.*

Let $\bar{A}_1 := \langle A_1, - \rangle \subseteq \mathrm{Hom}_F(V_2, F)$. If

(i) *the restriction of $\langle -,- \rangle$ to $A_1 \times A_2$ takes values in R, and*
(ii) *the F-span of A_2 is V_2,*

then $\mathrm{rank}_R(\bar{A}_1) \leq \dim_F(V_2)$.

Proof. Consider the R-module $U := \{\varphi \in \mathrm{Hom}_F(V_2, F) \mid \varphi(A_2) \subseteq R\}$. Condition (i) means that $\bar{A}_1 \subseteq U$. Thus it suffices to show that U is an R-module of rank at most $d = \dim_F(V_2)$. Use condition (ii) to choose a R-submodule $M \subseteq A_2$ free of rank d and such that $M \otimes_R F = V_2$. Then

$$\mathrm{Hom}_F(V_2, F) = \mathrm{Hom}_F(M \otimes_R F, F) \cong \mathrm{Hom}_R(M, F),$$

and since $M \subseteq A_2$, U is mapped under this isomorphism to a submodule of $R^d \cong \mathrm{Hom}_R(M, R) \subseteq \mathrm{Hom}_R(M, F)$. Thus U, and hence A_1, has rank over R at most d. $\blacksquare$

Proposition 5.5. *Let X be a smooth projective variety of dimension d over an algebraically closed field k. Then for every Weil cohomology theory H^* over k and every $0 \leq i \leq d$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Num}^i(X)) \leq \dim_F(H^{2i}(X)).$$

Proof. We apply Lemma 5.4. Set $R := \mathbf{Z}$ and F the characteristic 0 coefficient field of H^*. Set

$$A_1 := \mathrm{CH}^i(X)/\mathrm{CH}^i(X)_{H^*} \subset H^{2i}(X)(i) =: V_1$$

where the inclusion is given by the cycle class map. Set

$$A_2 := \mathrm{CH}^{d-i}(X)/\mathrm{CH}^{d-i}(X)_{H^*}$$

and V_2 the F-span of the image of A_2 inside of $H^{2(d-i)}(X)(d-i)$ under the cycle class map. Finally, let $\langle -,- \rangle \colon V_1 \times V_2 \to F$ be the restriction of the pairing on H^*. Condition

5.4(i) now follows from from compatibility of intersection products with $\langle -, - \rangle$, and 5.4(ii) is true by construction.

It remains to observe that, thanks again to the compatibility between intersection products and the pairing on H^*, the image $\bar{A}_1$ of

$$\begin{aligned}\mathrm{CH}^i(X)/\mathrm{CH}^i(X)_{H^*} \subseteq H^{2i}(X)(i) &\xrightarrow{\cong} \mathrm{Hom}_F(H^{2(d-i)}(X)(d-i), F) \\ &\twoheadrightarrow \mathrm{Hom}_F(\mathrm{CH}^{d-i}(X)/\mathrm{CH}^{d-i}(X)_{H^*}, F) = \mathrm{Hom}_F(V_2, F)\end{aligned}$$

is exactly $\mathrm{Num}^i(X) = \mathrm{CH}^i(X)/\mathrm{CH}^i(X)_{\mathrm{num}}$: indeed, an element in the kernel above is represented by a cycle of $\mathrm{CH}^i(X)$ which intersects every cycle of $\mathrm{CH}^{d-i}(X)$ trivially. Thus the Lemma gives the first inequality in

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Num}^i(X)) \leq \dim_F(V_2) \leq \dim_F(H^{2(d-i)}(X)) = \dim_F(H^{2i}(X))$$

and where the last equality comes from Poincaré duality for H^*. ∎

6. Alterations in Families

In this section, we formulate a version of de Jong's Alteration Theorem [dJ96] in families. This will be used to study the behavior of the Picard rank in families in the following section.

Lemma 6.1. *Let X be a proper scheme over a perfect field k. Then there exists a smooth projective scheme X' over k and a surjective morphism $X' \to X$ over k.*

Proof. Let $\{V_i\}_{i \in I}$ be the reduced irreducible components of X. For each i, apply the de Jong's Alterations Theorem, [dJ96, Theorem 4.1], to choose a smooth projective alteration $V_i' \to V_i$. Set $X' := \coprod_{i \in I} V_i'$ and $X' \to X$ as the composition $X' \to \coprod_{i \in I} V_i \to X$. ∎

Lemma 6.2. *Let A be a Noetherian domain with fraction field K, then the perfect closure K^{perf} is a filtered union of rings B, each of which contains a subring A_f with $0 \neq f \in A$ such that $A_f \subset B$ is finite free and a universal homeomorphism.*

Proof. We may assume $\mathbf{F}_p \subset A$. We will show every finitely generated A-algebra $C = A[x_1, \ldots, x_n]$ with $A \subset C \subset K^{\mathrm{perf}}$ is contained in some B as in the statement of the lemma. Since $K \subset K^{\mathrm{perf}}$ is purely inseparable, there is a p-th power q such that $x_i^q \in K$ for all i. Thus we can find $0 \neq f \in A$ such that $x_i^q \in A_f$ for all i; thus each x_i is integral over A_f. Then $A_f[x_1, \ldots, x_n]$ is a finitely-generated A_f module so there is a further localization $(A_f[x_1, \ldots, x_n])_{f'} = A_{ff'}[x_1, \ldots, x_n]$, $0 \neq f' \in A$, which is free over $A_{ff'}$. Take this as our B. Then $A_{ff'} \subset B$ is finite free and a universal homeomorphism by Lemma 0CNF. ∎

Lemma 6.3. *Let S be an integral scheme. Let $X \to S$ be a proper morphism of schemes. Then there exists a nonempty open set $U \subset S$, a finite locally free universal homeomorphism $S' \to U$, a smooth projective morphism $X' \to S'$, and a surjective morphism $X' \to X \times_S S'$.*

Proof. Let K be the function field of S. Then by Lemma 6.1, there exists a smooth projective scheme Y over the perfect closure K^{perf} of K together with a surjective morphism $Y \to X \times_S \mathrm{Spec}(K^{\mathrm{perf}})$. Lemma 6.2 now gives an open $U \subset S$, a finite free universal homeomorphism $S' \to U$, and a projective scheme morphism $X' \to S'$ with a morphism $X' \to X \times_S S'$ such that the diagram

$$\begin{tikzcd}
X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{\mathrm{perf}}) \arrow[r] & X \times_S S' \\
Y \arrow[u] \arrow[r] \arrow[d] & X' \arrow[u] \arrow[d] \\
\mathrm{Spec}(K^{\mathrm{perf}}) \arrow[r] & S'
\end{tikzcd}$$

is commutative and the bottom square is Cartesian. By possibly further changing S', we may arrange for $X' \to S'$ to be smooth, see Lemma 0CoC, and for $X' \to X \times_S S'$ to be surjective, see Lemma 07RR. ■

7. Theorem of the Base

Let X be a proper scheme over an algebraically closed field k. The *Néron–Severi group* of X is the abelian group

$$\mathrm{NS}(X) := \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^0_{X/k}(k)$$

parameterizing the components of the Picard scheme; its rank $\rho(X)$ is the *Picard rank* of X. The goal of this section is to show in Theorem 7.4 that $\mathrm{NS}(X)$ is a finitely generated abelian group. We then study its behavior in families in Lemmas 7.5 and 7.6. Everything is put together in Theorem 7.7.

Two general observations before we begin. Given a normal subgroup-scheme H of a group scheme G locally of finite type over an algebraically closed field k, Lemma 1.5 shows $(G/H)(k) = G(k)/H(k)$. Second, $Y_{\mathrm{red}}(k) = Y(k)$ for any scheme Y over any field k, since field valued points are insensitive to nonreduced structure.

To formulate our results, we find it helpful to consider the following slight enrichment of the Néron–Severi group:

Definition 7.1. *Néron–Severi scheme* of a proper scheme X over k is the commutative group scheme

$$\mathrm{NS}_{X/k} := \mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k,\mathrm{red}}.$$

This is a scheme of dimension 0 which is, by Proposition 1.3, locally of finite type over k. When k is algebraically closed, we recover the Néron–Severi group as the group of k-points:

$$\mathrm{NS}_{X/k}(k) = \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^0_{X/k}(k) = \mathrm{NS}(X).$$

The constructions of Definition 2.1 give subgroup schemes of $\mathrm{NS}_{X/k}$ closely related to the corresponding subgroup schemes of $\mathrm{Pic}_{X/k}$:

$$\mathrm{NS}^0_{X/k} \cong \mathrm{Pic}^0_{X/k}/\mathrm{Pic}^0_{X/k,\mathrm{red}} \quad \text{and} \quad \mathrm{NS}^\tau_{X/k} \cong \mathrm{Pic}^\tau_{X/k}/\mathrm{Pic}^0_{X/k,\mathrm{red}}.$$

The torsion subgroup and the torsion-free quotient of the Néron–Severi group can now be realized as groups of points as follows:

Lemma 7.2. *Let X be a proper scheme over an algebraically closed field k. Then the torsion subgroup of* $\mathrm{NS}(X)$ *is the finite group*

$$\mathrm{NS}(X)_{\mathrm{tors}} = \mathrm{Pic}^{\tau}_{X/k}(k)/\mathrm{Pic}^{0}_{X/k}(k) = \mathrm{NS}^{\tau}_{X/k}(k).$$

The torsion-free quotient is

$$\mathrm{NS}(X)_{\mathrm{tf}} := \mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tors}} = \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^{\tau}_{X/k}(k).$$

Proof. The class $[\mathscr{L}]$ lies in $\mathrm{NS}(X)_{\mathrm{tors}}$ if and only if

$$n[\mathscr{L}] = [\mathscr{L}^{\otimes n}] = [\mathscr{O}_X] \quad \text{for some integer } n \neq 0.$$

As $\mathrm{NS}(X)$ is the component group of $\mathrm{Pic}_{X/k}$, this means $\mathscr{L}^{\otimes n} \in \mathrm{Pic}^{0}_{X/k}$, so $\mathscr{L} \in \mathrm{Pic}^{\tau}_{X/k}$. Hence the identification of $\mathrm{NS}(X)_{\mathrm{tors}}$. That this is finite then follows from Theorem 4.4: $\mathrm{NS}^{\tau}_{X/k}$ is a zero-dimensional scheme of finite type over k, so it has only finitely many k-points.

As for $\mathrm{NS}(X)_{\mathrm{tf}}$, consider the short exact sequence of group schemes

$$0 \to \mathrm{NS}^{\tau}_{X/k} \to \mathrm{NS}_{X/k} \to \mathrm{NS}_{X/k}/\mathrm{NS}^{\tau}_{X/k} \to 0.$$

Then the preceding discussion shows $\mathrm{NS}_{X/k}/\mathrm{NS}^{\tau}_{X/k} \cong \mathrm{Pic}_{X/k}/\mathrm{Pic}^{\tau}_{X/k}$ and taking points identifies the torsion-free quotient. ■

Theorem 7.3 (Theorem of the Base, Smooth Projective Case)**.** *Let X be a smooth projective scheme over an algebraically closed field k. Then*

(i) $\mathrm{NS}(X)$ *is a finitely generated abelian group, and*
(ii) $\rho(X) \leq \dim_F(H^2(X))$ *for any Weil cohomology theory H^* over k.*

Proof. We have $X = \coprod_i X_i$ with X_i the connected components of X, and $\mathrm{Pic}_{X/k} = \prod_i \mathrm{Pic}_{X_i/k}$, so $NS(X) = \bigoplus_i NS(X_i)$. Since also $H^2(X) = \bigoplus_i H^2(X_i)$, it suffices to prove the Theorem for each X_i. Thus we may assume X is a smooth projective variety. The short exact sequence

$$0 \to \mathrm{NS}(X)_{\mathrm{tors}} \to \mathrm{NS}(X) \to \mathrm{NS}(X)_{\mathrm{tf}} \to 0$$

implies it suffices to show that $\mathrm{NS}(X)_{\mathrm{tors}}$ is finite and that $\mathrm{NS}(X)_{\mathrm{tf}}$ is of finite rank. Lemma 7.2 already gives finiteness of torsion; it moreover identifies the quotient as the numerical group of invertible modules, giving the first isomorphism in

$$\mathrm{NS}(X)_{\mathrm{tf}} \cong \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^{\tau}_{X/k}(k) \cong \mathrm{Num}^1(X).$$

Since X is smooth projective, the first Chern class identifies the numerical group of invertible modules with the numerical group $\mathrm{Num}^1(X)$ of divisors on X, as in Lemma 5.2. Then Proposition 5.5 shows that the rank of this is at most the dimension of $H^2(X)$ for any Weil cohomology theory H^* over k. ■

We can now deduce finite generation of $\mathrm{NS}(X)$ for any proper scheme over k by reducing to the smooth projective case above:

Theorem 7.4 (Theorem of the Base, Proper Case)**.** *Let X be a proper scheme over an algebraically closed field k. Then* $\mathrm{NS}(X)$ *is a finitely generated abelian group.*

Proof. As before, it suffices to show that $\mathrm{NS}(X)$ has finite torsion and that its torsion-free quotient is of finite rank. Finiteness of torsion is again handled by Lemma 7.2. As for the torsion-free quotient, observe that if $f : Y \to X$ is a surjective morphism, then Lemma 2.9 says that an $\mathscr{O}_X$-module $\mathscr{L}$ is τ-trivial if and only if $f^*\mathscr{L}$ is τ-trivial; thus pullback induces an injective homomorphism $f^* \colon \mathrm{NS}(X)_{\mathrm{tf}} \to \mathrm{NS}(Y)_{\mathrm{tf}}$. If the latter is of finite rank, then so is the former. Thus to prove that X has finite Picard rank, we may replace X by schemes surjecting onto it. But by Lemma 6.1, there is a surjective morphism $Y \to X$ with Y a smooth projective scheme over k. We conclude by Theorem 7.3. ■

We consider how the Néron–Severi group varies in families. More precisely, consider a proper morphism $f : X \to S$. For each point $s \in S$, let $\kappa(s)$ be its residue field and let

$$\bar{s} \colon \operatorname{Spec}(\overline{\kappa(s)}) \to S$$

be a geometric point lying above s. Let $X_{\bar{s}} := X \times_S \bar{s}$ be the geometric fibre of f over s. Then the Theorem of the Base allows us to define numerical functions

$$\begin{aligned} \mathrm{tors}_{X/S} \colon S &\to \mathbf{Z} & \qquad \rho_{X/S} \colon S &\to \mathbf{Z} \\ s &\mapsto \#\mathrm{NS}(X_{\bar{s}})_{\mathrm{tors}}, & s &\mapsto \rho(X_{\bar{s}}), \end{aligned}$$

giving the torsion size and Picard rank of geometric fibres.

In order to study the function $\mathrm{tors}_{X/S}$, we will need to make use of the subfunctor $\mathrm{Pic}^{\tau}_{X/S} \subset \mathrm{Pic}_{X/S}$ consisting of sections $\xi \in \mathrm{Pic}_{X/S}(T)$ such that $\xi|_{\bar{t}} \in \mathrm{Pic}^{\tau}_{X_{\bar{t}}/\bar{t}}(\bar{t})$ for every geometric point $\bar{t}$ of T. Note that for a geometric point $\bar{s}$ of S, we have $(\mathrm{Pic}^{\tau}_{X/S})_{\bar{s}} = \mathrm{Pic}_{X_{\bar{s}}/\bar{s}}$.

Lemma 7.5 (Bounded Torsion)**.** *Let $f : X \to S$ be a proper morphism. If S is Noetherian, then the function* $\mathrm{tors}_{X/S}$ *is bounded.*

Proof. By Noetherian induction, it suffices to find a non-empty open of S on which $\rho_{X/S}$ is bounded. Since $\mathrm{tors}_{X/S}$ concerns geometric fibres, we may replace S by its reduction and assume S is reduced. We may then replace S by an irreducible component since a non-empty open of an irreducible component contains a non-empty open of S. Thus we may assume S is integral.

For a point $s \in S$, consider the base change map $\varphi(s) : (f_*\mathscr{O}_X)_s \otimes_{\mathscr{O}_{S,s}} \kappa(s) \to H^0(X_s, \mathscr{O}_{X_s})$. When s is the generic point of S this map is an isomorphism by Lemma 02KH, since $\operatorname{Spec}(\kappa(s)) \to S$ is flat. Therefore, there exists a nonempty open of S consisting of points $s \in S$ such that $\varphi(s)$ is an isomorphism, and after replacing S with this open, the formation of $f_*\mathscr{O}_X$ commutes with arbitrary base change by [Gro63, §7.7]. Therefore by Theorem 1.1, we may assume $\mathrm{Pic}_{X/S}$ is representable by an algebraic space. Then [SGA71, Exposé XIII, Théorème 4.7] shows that $\mathrm{Pic}^{\tau}_{X/S}$ is an algebraic space of finite type over S. Since the generic fibre of $\mathrm{Pic}^{\tau}_{X/S} \to S$ is a scheme being a finite type group algebraic space over a field, by Lemma 07SR we may assume after replacing S by a nonempty open that $\mathrm{Pic}^{\tau}_{X/S}$ is a scheme. For every $s \in S$,

$$\mathrm{tors}_{X/S}(s) = \#(\text{connected components of } (\mathrm{Pic}^{\tau}_{X/S})_{\bar{s}}).$$

There is a nonempty open subset of S on which the number of connected components of fibres is constant, see Lemma 055H; replacing S by such an open completes the proof. ■

To bound Picard ranks in families, we assume that there exists a Weil cohomology theory that varies in families. That is, suppose we have the following situation: fix a coefficient field F; for every scheme S and every geometric point

$$\bar{s}\colon \operatorname{Spec}(\overline{\kappa(s)}) \to S,$$

we have a Weil cohomology theory $H^*_{\bar{s}}$ over $\overline{\kappa(s)}$ with coefficients in F. The important hypothesis is: for every smooth projective morphism $X \to S$, the function $s \mapsto \dim_F(H^2_{\bar{s}}(X_{\bar{s}}))$ is constructible on S.

Such a theory exists: for example, one may take ℓ-adic étale cohomology for suitable ℓ. The required constructibility result is then the Théorème de Finitude et Spécialisations of [SGA73, Exposé XVI Corollaire 2.2]. See also Proposition 0GLI.

Lemma 7.6 (Bounded Ranks)**.** *Assume that there exists a Weil cohomology theory that varies in families as above. Let $f : X \to S$ be a proper morphism of schemes. If S is Noetherian, then $\rho_{X/S}$ is bounded.*

Proof. By Noetherian induction, it suffices to find a non-empty open of S on which $\rho_{X/S}$ is bounded. Since $\rho_{X/S}$ concerns geometric fibres, we may replace S by its reduction to assume S is reduced. We may then replace S by an irreducible component since a non-empty open of an irreducible component contains a non-empty open of S. Thus we assume S is integral.

By the Alterations Theorem in Families, Lemma 6.3, after replacing S by a non-empty open we may assume there exist: a finite surjective morphism $S' \to S$, a smooth projective morphism $X' \to S'$, and a surjective morphism $X' \to X \times_S S'$. Since $S' \to S$ is surjective and $X \to S$ and $X \times_S S' \to S'$ have the same geometric fibres, in order to show $\rho_{X/S}$ is bounded it suffices to show $\rho_{X\times_S S'/S'}$ is bounded. Thus we may replace S by S' and assume there exists a surjective morphism $X' \to X$ with X' smooth and projective over S. But then as observed in the proof of the Theorem of the Base, Lemma 2.9 implies that $\rho_{X/S} \leq \rho_{X'/S}$, so we may replace X with X'.

Thus we are in the situation where $X \to S$ is a smooth projective morphism with S an integral Noetherian scheme. The Theorem of the Base in the projective case, Theorem 7.3, implies that for every $s \in S$, $\rho_{X/S}(s) \leq \dim_F(H^2_{\bar{s}}(X_{\bar{s}}))$. Our hypothesis about the Weil cohomology theory in families is that the latter function is constructible on S, and so since S is Noetherian, it is actually uniformly bounded on S. Therefore $\rho_{X/S}$ is also bounded on S. ■

Putting the results of this section together yields the following result, originally due to [SGA71, Exposé XIII, Théorème 5.1]:

Theorem 7.7 (Boundedness of Néron–Severi in Families)**.** *Let $X \to S$ be a proper morphism. If S is Noetherian, then $\mathrm{NS}(X_{\bar{s}})$ is a finitely generated abelian group for every geometric point $\bar{s}$ of S. Moreover, the order of torsion and ranks of these groups are bounded over S.*

Proof. This is Theorem 7.4 together with Lemmas 7.5 and 7.6. ■

8. Examples of Picard Schemes

We close this chapter by giving three examples of Picard schemes.

Example 8.1 (Picard schemes of quotients)**.** The Picard schemes of quotients by finite commutative group schemes G often contain the Cartier dual $G^\vee$, see [Jen78]. To explain, let S be a scheme and G a finite locally free commutative group scheme over S. The presheaf of abelian groups on the category Sch/S of schemes over S given by

$$G^\vee \colon T \mapsto \mathrm{Hom}_T(G_T, (\mathbf{G}_m)_T),$$

the right side being homomorphisms in the category of group schemes over T, is representable by a finite locally free commutative group scheme over S called the *Cartier dual of* G. The functor $G \mapsto G^\vee$ defines a duality on the category of finite locally free commutative group schemes over S in that $G \mapsto (G^\vee)^\vee$ is isomorphic to the identity functor.

The Cartier dual often appears in the Picard scheme of a quotient:

Lemma 8.2. *Let S be a scheme, G a finite locally free commutative group scheme over S, $\pi \colon Y \to X$ be a G-torsor with X and Y algebraic spaces over S. Assume $(Y \to S)_*\mathcal{O}_Y = \mathcal{O}_S$ holds universally. Then there is an exact sequence*

$$0 \to G^\vee \to \mathrm{Pic}_{X/S} \xrightarrow{\pi^*} \mathrm{Pic}_{Y/S}$$

of sheaves on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Proof. It suffices to see that for every $T \to S$, there is an exact sequence

$$0 \to G^\vee(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}(Y_T).$$

By descent theory, the kernel of $\mathrm{Pic}(X_T) \to \mathrm{Pic}(Y_T)$ can be identified with the set of G_T-equivariant structures on $\mathcal{O}_{Y_T}$, see Lemma 043U. By definition, this is an isomorphism $\mathcal{O}_{G_T \times_T X_T} \to \mathcal{O}_{G_T \times_T X_T}$ satisfying two conditions. Such an isomorphism is given by a section of

$$H^0(G_T \times_T X_T, \mathcal{O}_{G_T \times_T X_T})^\times = H^0(G_T, \mathcal{O}_{G_T})^\times$$

by the assumption on the pushforward of the structure sheaf. This is also the same thing as a morphism $G_T \to (\mathbf{G}_m)_T$ of schemes over T. The two conditions in the definition of an equivariant structure say exactly that this morphism is a homomorphism of group schemes. Thus the kernel is canonically identified with $G^\vee(T)$. ∎

Now for a particular example. Take $S := \mathrm{Spec}(k)$ for k an algebraically closed field of characteristic $p > 0$, and let G be the constant group scheme $\mathbf{Z}/p$. Then $G^\vee = \boldsymbol{\mu}_p$. Fix $d \geq 2$. Then there exists a smooth complete intersection Y of dimension d in some projective space that carries a free G action, see [Ser58, Proposition 15]. Then the fppf quotient sheaf $X := Y/G$ is a scheme and the quotient map $Y \to X$ is a G-torsor in the fppf topology, see Lemma 07S7. It follows from Lemma 0BBM that X is separated. Since $Y \to X$ is étale and surjective and Y is smooth over k, X is smooth over k by Lemma 02K5. Since $Y \to X$ surjective and Y is proper and irreducible, so is X, see Lemma 03GN. Thus Y is a smooth proper variety over k. Since the group scheme $\boldsymbol{\mu}_p$ is connected, the exact sequence of Lemma 8.2 gives an exact sequence

$$0 \to \boldsymbol{\mu}_p \to \mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{Y/k}.$$

On the one hand, $H^1(Y, \mathcal{O}_Y) = 0$ since Y is a complete intersection of dimension $d \geq 2$. On the other hand, the same group is $\mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{O}_Y)$ and hence is the tangent space to the

Picard scheme at the identity, see Lemma 08VW. Thus $\mathrm{Pic}^0_{Y/k} = 0$ and so $\mathrm{Pic}^0_{X/k} = \boldsymbol{\mu}_p$. In particular, the Picard scheme of the smooth proper variety X is nonreduced. ∎

Example 8.3 (Pointless conic)**.** We will give an example of a section of the Picard functor that cannot be represented by a line bundle. Let

$$X := \mathrm{Proj}(\mathbf{R}[x,y,z]/(x^2+y^2+z^2)) \subset \mathbf{P}^2_{\mathbf{R}}.$$

Then X is a smooth curve over $\mathbf{R}$ such that $X(\mathbf{R}) = \emptyset$ and $X_{\mathbf{C}} \cong \mathbf{P}^1_{\mathbf{C}}$. By Lemma 0D27, there is an exact sequence

$$0 \to \mathrm{Pic}(\mathrm{Spec}(\mathbf{R})) \to \mathrm{Pic}(X) \to \mathrm{Pic}_{X/\mathbf{R}}(\mathbf{R}).$$

We show that $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/\mathbf{R}}(\mathbf{R})$ is not surjective.

First, observe that $\mathrm{Pic}_{X/\mathbf{R}}(\mathbf{R}) \to \mathrm{Pic}_{X/\mathbf{R}}(\mathbf{C})$ is an isomorphism. It's injective since $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$ is a covering for the fppf, even étale topology. For surjectivity, note that by Lemmas 0D28 and 0BXJ,

$$\mathrm{Pic}_{X/\mathbf{R}}(\mathbf{C}) = \mathrm{Pic}_{X_{\mathbf{C}}/\mathbf{C}}(\mathbf{C}) = \mathrm{Pic}(\mathbf{P}^1_{\mathbf{C}}) = \mathbf{Z} \cdot \mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(1) =: \mathbf{Z} \cdot \mathcal{O}(1).$$

The class of $\mathcal{O}(1)$ descends along the étale covering $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$. Indeed, its two pullbacks to $X_{\mathbf{C}\otimes_{\mathbf{R}}\mathbf{C}} \cong \mathbf{P}^1_{\mathbf{C}} \coprod \mathbf{P}^1_{\mathbf{C}}$ must give $\mathcal{O}(1)$ on each copy of $\mathbf{P}^1_{\mathbf{C}}$, since for an isomorphism $f : \mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ of schemes, not necessarily over $\mathbf{C}$, $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$: the pullback must generate the Picard group, and among the two possibilities, only one of them has a nonzero global section.

Thus to show $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/\mathbf{R}}(\mathbf{R})$ is not surjective, we just have to show $\mathcal{O}(1)$ is not the pullback of a line bundle on X. If it were, then X would have a degree 1 line bundle $\mathcal{L}$ with a nonzero section, as

$$H^0(X, \mathcal{L}) \otimes_{\mathbf{R}} \mathbf{C} \cong H^0(\mathbf{P}^1_{\mathbf{C}}, \mathcal{O}(1)) \neq 0.$$

But then the zero locus of any nonzero section of $\mathcal{L}$ is an $\mathbf{R}$-rational point, contradicting the fact that $X(\mathbf{R}) = \emptyset$. ∎

Example 8.4 (Nodal curves)**.** Let X be a nodal curve over an algebraically closed field k, see Section 0C47. Let $p_1, \dots, p_n \in X$ be the nodes of X and let $\nu\colon X^\nu \to X$ be its normalization. We will show that there is a short exact sequence of group schemes

$$0 \to \mathbf{G}_m^n \to \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X^\nu/k} \to 0.$$

This shows that if X is not smooth, then $\mathrm{Pic}^0_{X/k}$ is not proper, since it contains $\mathbf{G}_m^n$ as a closed subgroup scheme with $n \geq 1$.

The normalization morphism gives a short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\nu^\#} \nu_*(\mathcal{O}_{X^\nu}) \to \bigoplus_{i=1}^n \mathcal{O}_{p_i} \to 0$$

of coherent sheaves on X. Indeed, the cokernel of consists of skyscraper sheaves supported on the p_i since there are exactly two points in X^ν lying above each p_i, see Lemma 0CBW.

For any flat morphism $Y \to X$, pullback yields a short exact sequence

$$0 \to \mathcal{O}_Y \to \nu_{Y,*}(\mathcal{O}_{X^\nu \times_X Y}) \to \bigoplus_{i=1}^n \mathcal{O}_{Y_{p_i}} \to 0$$

which remains valid upon replacing Y by any object in the small étale site $Y_{\text{ét}}$ of Y, since such objects are flat over Y. Thus upon taking units we obtain a short exact sequence

$$0 \to \mathbf{G}_m \to \nu_{Y,*}\mathbf{G}_{m,X^\nu \times_X Y} \to \bigoplus\nolimits_{i=1}^n (Y_{p_i} \to Y)_*\mathbf{G}_{m,Y_{p_i}} \to 0$$

of sheaves on $Y_{\text{ét}}$. Since $\nu_Y : X^\nu \times_X Y \to Y$ and $Y_{p_i} \to Y$ are finite, the long exact sequence of cohomology gives

$$\begin{aligned} 0 \to H^0(Y, \mathscr{O}_Y)^\times \to H^0(X^\nu \times_X Y, \mathscr{O}_{X^\nu \times_X Y})^\times \to \bigoplus_{i=1}^n H^0(Y_{p_i}, \mathscr{O}_{Y_{p_i}})^\times \\ \to \operatorname{Pic}(Y) \to \operatorname{Pic}(X^\nu \times_X Y) \to \bigoplus\nolimits_{i=1}^n \operatorname{Pic}(Y_{p_i}) \to \cdots. \end{aligned}$$

Now take $Y = X_T$ for T an object in $(\mathrm{Sch}/k)_{\mathrm{fppf}}$. Since $H^0(X, \mathscr{O}_X) = H^0(X^\nu, \mathscr{O}_{X^\nu}) = k$,

$$\begin{aligned} 0 \to H^0(T, \mathscr{O}_T)^\times = H^0(T, \mathscr{O}_T)^\times \xrightarrow{0} \bigoplus\nolimits_{i=1}^n H^0(T, \mathscr{O}_T)^\times \\ \to \operatorname{Pic}(X_T) \to \operatorname{Pic}((X^\nu)_T) \to \bigoplus\nolimits_{i=1}^n \operatorname{Pic}(T) \to \cdots. \end{aligned}$$

Furthermore, elements of $\operatorname{Pic}(T)$ vanish Zariski locally on T, and therefore elements of $\operatorname{Pic}((X^\nu)_T)$ Zariski locally on T come from $\operatorname{Pic}(X_T)$. Since we obtain $\mathrm{Pic}_{X/k}$ by sheafifying the rule $T \mapsto \operatorname{Pic}(X_T)$ and similarly for X^ν, we see that we have obtained a short exact sequence

$$0 \to \mathbf{G}_m^n \to \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X^\nu/k} \to 0,$$

as promised.

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Department of Mathematics, Columbia University, New York, NY 10027, USA
Email address: `rcheng@math.columbia.edu`
Email address: `nolander@math.columbia.edu`
URL: `https://math.columbia.edu/~rcheng/`
URL: `https://math.columbia.edu/~nolander/`

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: `lji@math.princeton.edu`
URL: `https://web.math.princeton.edu/~lji/`

Department of Mathematics, Stanford University, Stanford, CA 94305, USA
Email address: `mwlarson@stanford.edu`
URL: `https://stanford.edu/~mwlarson/`