Affine Root Systems

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Contents

The elements of S are affine-linear functions on a f.d.v.s. real Eucildean space E and satisfy axioms analogous to those of finite root system. The set S of affine roots are infinite, and the Weyl group $W(S)$ is an affine Weyl group, that is to say an infinite group of displacements of E generated by reflections.

One way to construct ARS is from FRS. Let R be a (not necessarily reduced) FRS in V, and let $<,>$ be a scalar product on V which is invariant under the action of the Weyl group $W(R)$. For each $\alpha \in \mathbb{R}, k \in \mathbb{Z}$ be the affine-linear fct on V defined by $a_{\alpha,k} :=< \alpha, x > +k$, then we get a ARS $S(R)$ defined by

$$S(R) := \{a_{\alpha,k} : \text{ where k is odd, if } \frac{1}{2}\alpha \notin R; k \in \mathbb{Z}, \text{ o.w } \}$$

For an ARS S, we define positive roots and negative roots in the usual way, by choosing Weyl chamber C for S. If S is irreducible, C is a rectilinear l-simplex($l =$ dim E).

1 Notation and Terminology

Let E be an affine space over a field K: that is to say E is a set on which a K v.s. V acts faithfully and transtively. (Call the elements of V translations of E).

Definition 1.1. Let E, E' affine spaces $/K$, V, V' its vector space of translation. A map $f : E \to E'$ is an affine linear if there exists a $K-linear$ map $Df : V \to V'$, called derivative of f s.t.

$$f(x+v) = f(x) + Df(v) \tag{1}$$

for all $x \in E$ and $v \in V$. In particular, a function $f : E \to K$ is affine linear iff there exists a linear form $Df : V \to K$ s.t. the above equation holds.
Let F denote the $K-v.s$ of all affine linear fcts $f : E \to K$, then $D : F \to V^*$, whose kernel is the line F^0 in F consisting of the constant fcts.

Now let $K = \mathbb{R}$, then we have a bilinear form $\langle u, v\rangle$,(what?) then (2.1) becomes

$$f(x+v) = f(x) + \langle Df, v\rangle$$

and Df is the gradient of f.

Definition 1.2. Let $\langle f, g\rangle$ be a positive semi-def bilinear form on the spaces F as follows:

$$\langle f, g\rangle := \langle Df, Dg\rangle \tag{2}$$

$f \in F$ is isotropic iff f is a constant fct. For $0 \neq v \in V$, we define $f^\vee := \frac{2f}{\langle f,f\rangle}$ and $H_f := \{x \in E : f(x) = 0\}$ where H_f is an affine hyperplane in E. The reflection is the affine lin isometry $w_f : E \to E$ given by

$$w_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee \tag{3}$$

w_f acts on $F : w_f(g) = g \circ w_f^{-1} = g \circ w_f$. Explicitly, for any $g \in F$

$$w_f(g) = g - f^\vee(x)Df = x - f(x)Df^\vee \tag{4}$$

Definition 1.3. For $0 \neq u \in V$, let $w_u : V \to V$ be the reflection in the hyperplane orthogonal to u, so that $w_u := v - \langle u, v\rangle u^\vee$

Proposition 1.4. *For any non-constant fct $f \in F$ we have $D_{w_f} = w_{Df}$.*

Proof. Let $v \in V$ and $x \in E$, then

$$\begin{aligned}
(D_{w_f})(v) &= w_f(x+v) - w_f(x) \\
&= (x + v - f(x+v)Df^\vee) - (x - f(x)Df^\vee) \\
&= v - (f(x+v) - f(x))Df^\vee \\
&= v - \langle Df, v\rangle Df^\vee = w_{Df}(v)
\end{aligned}$$

□

Finally, let $w : E \to E$ be an affine linear isometry. Then its derivative Dw is a linear isometry of V, i.e. we have $\rangle(Dw)u,(Dw)v\langle$ for all $u, v \in V$. The map w acts by transposition on $F : w(f) = f \circ w^{-1}$ and we have

Proposition 1.5. *D(w(f))=(Dw)(Df)*

Proof. For if $v \in V$ and $x \in E$, then

$$\begin{aligned}
\langle D(wf), v\rangle &= (wf)(x+v) - (wf)(x) \\
&= f(w^{-1}(x+v)) - f(w^{-1}(x)) \\
&= f(w^{-1}(x) + (Dw)^{-1}(v)) - f(w^{-1}(x) \\
&= \langle Df, (Dw)^{-1}v\rangle \\
&= \langle Dw(Df), v\rangle
\end{aligned}$$

□

2 Defn of ARS

Let E be a real Euclidean space of dim l, and let V be its space of translations. We give E the usual topology, defined by the mrtric $\|x-y\|$ si that is locally compact. Let F denote the v.s. of affine linear fcts on E.

Definition 2.1. An affine root system on E is defined to be a subset S of F satisfying the following axioms:
(AR 1) S spans F, and the elements of S are nonisotropic, i.e. there are non-constant functions.
(AR 2) $w_\alpha S = S$
(AR 3) $\langle \alpha, \beta^\vee\rangle \in \mathbb{Z}$ for all $a, b \in S$
(AR 4) $W(S)$(as a discrete group) acts properly on E
The elements of S are called affine roots. Let $W(S) := w_a : a \in S$ be the Weyl group of S

Remark 2.2. In finite case, we deduce that if $a, \lambda a$ are proportional affine roots, then by $(AR3)$. Since $< a, \lambda a^\vee >=< a, \frac{2\lambda a}{<\lambda a, \lambda a>} >= \frac{2}{\lambda} \in \mathbb{Z}$,
and $< \lambda a, a^\vee >=< \lambda a, \frac{2a}{<a,a>} >= 2\lambda \in \mathbb{Z}$. Thus, $\lambda = \pm\frac{1}{2}, \pm 1, \pm 2$.

Definition 2.3. We said root a is indivisible if $a \in S$ but $\frac{1}{2}a \notin S$. We say S is reduced if each $a \in S$ is indivisible(i.e. if the only roots proportional to a are $\pm a$)

Here we have a good example to understand the def of ARS.

Definition 2.4. Let R be a FRS in a $\mathbb{R}$ f.d.v.s V. For each $\alpha \in R$ and $k \in \mathbb{Z}$, define affine linear functional on V

$$a_{\alpha,k}(x) =< \alpha, x > +k \tag{5}$$

where $<,>$ is a pos-def bilinear form on V invariant under the Weyl group of R. Let $S(R) := \{a_{\alpha,k}\}$.

Proposition 2.5. *$S(R) := \{a_{\alpha,k}\}$ is a reduced ARS on V and $n \in \mathbb{Z}$ if $\frac{1}{2}\alpha \notin R$; $n \in 2\,\mathbb{Z}+1$ if $\frac{1}{2}\alpha \in R$*

Proof. To prove this we need to verify four axioms and show it is reduced, we look at (AR 2), let $a = a_{\alpha,k}, b = a_{\beta,l} \in S(R)$, since $< \alpha^\vee, \beta >=< a^\vee, b >$,

$$\begin{aligned} w_a(b)(x) &= b(x)- < a^\vee, b > a(x) && (6)\\ &=< \beta, x > +n- < a^\vee, b > (< \alpha, x > +m) && (7)\\ &=< \beta+ < a^\vee, b > \alpha, x > +n- < a^\vee, b > m && (8)\\ &=< \beta+ < \alpha^\vee, \beta > \alpha, x > +n- < a^\vee, b > m && (9)\end{aligned}$$

hence $w_a(b) = a_{\gamma,k}$, where $\gamma = \beta+ < \alpha^\vee, \beta > \alpha$ and $k = n- < a^\vee, b > m$. □

3 Direct sums and reducibility

Let E_i be f.d. real Euclidean spaces, let V_i be the space of translations of E_i, F_i the space of affine llinear fuctions on E_i. Let E, V be the profuct of E_i, V_i respectively. Thus, E is the translation of V, the action of V on E is defined by

$$(x_1, ..., x_r) + (v_1, ..., v_r) = (x_1 + v_1, ..., x_r + v_r)$$

and the bilinear form on V is defined as

$$< (u_1, ..., u_r), (v_1, ..., v_r) >:= \sum < u_i, v_i >$$

Now let F be the space of affine linear functions on E, let p_i be the projection of E onto E_i. Then the map $\pi_i : F_i \to F$ defined by $\pi_i(f_i) := f_i \circ p_i$ are injective isometries, i.e. it preserves the scalar product (1.2). The space $pi_i(F_i)$ generate F, are mutually orthogonal, and contain the line F^0 of constant functions.

Let S_i be an ARS on E_i and $S_i' = \pi_i(S_i) \subset F$. let

$$\cup_{i=1}^{r} S_i' \tag{10}$$

Proposition 3.1. *S is an ARS on E. We call root system S the direct sum of the S_i, we denote it by $\sqcup_{i=1}^{r} S_i'$*

Proposition 3.2.

$$S_i', S_j' \textit{are orthogonal if } i \neq j \tag{11}$$

Conversely, let E be a f.d. real Eucliden space, V its space of translations, and S an affine root systems on E. Let S_i' be substets of S satisfy $(2.6), (2.7)$. From $(\delta 1)$, it follows that the S_i' are pairwise disjoint, then partition of S

4 Chambers and Bases

Let $\mathfrak{h} := \{H_a : a \in S\}$ a set of affine hyperplanes in the E, $w(H_a) = H_{wa}$, for all $w \in W(S)$ and $a \in S$.

Proposition 4.1. *$\mathfrak{h}$ is locally finite.*

Proof. $E - \cup_{a \in S} H_a$ is open in E since affine roots vanishes on $\mathfrak{h}$ locally finite. Therefore, all connected compoenents of it are open bacause E iis locally connected. These components are called the chambers(some other papers used alcoves, however, they have same literally meaning) of $W(S)$. □

Proposition 4.2. *The Weyl group $W(S)$ acts faithfully and transtively on the set of chambers.*

To simplify statements of results, sps S is irreducible. Then each chamber is an open rectilinear $l-$simplex. If S is irreducible, the chamber are orthogonal products of simplexes.

Choose a chamber C from now on, $x_i, i = 0, ..., l$ be the vertices of C, then $C = \{x = \sum_{i=0}^{l} \lambda_i x_i \in E : \sum_{i=0}^{l} \lambda_i = 1, \lambda_i > 0\}$

Now let B ne the set of indivisible affine roots $a \in S$ which s.t. H_a is a wall of C, and $a(x) > 0$ for all $x \in C$. Then B consists of $l + 1$ roots, one for each wall of C. Clearly B is a basis of F.

Proposition 4.3. *The Weyl group $W(S)$ is generated by the reflections w_a for $a \in B$*

Proposition 4.4. *Let $b \in S$ be indivisible. Then $b = wa$ for some $w \in W(S)$ and some $a \in B$*

Proof. The hyperplane H_b is a wall of some chamber C' on which b is positive. Since the Weyl group $W(S)$ acts faithfully and transtively on the set of chambers, $C' = wC$ for some $w \in W(S)$. Hence $w^{-1}b$ is a positive on C, and $H_{w^{-1}b} = w^{-1}H_b$ is a wall of C, so that $w^{-1}b \in B$ □

Proposition 4.5. *Let L be the lattice in F generated by B, which is a free abelian group of rank $l+1$, then $L = L(S)$*

Proposition 4.6. *Each affine root $a \in S$ is a linear combo of $a_0, a_1, ..., a_l$ with rational integer coefficients which are all ≥ 0 if a is positive, and all ≤ 0 if a is negative.*

Definition 4.7. We call such B a basis of the ARS S

Example 4.8. Let R be a finite irreducible root system, $\alpha_1, ...\alpha_l$ a basis of R, and let ϕ be the highest root of R relative to this basis. Then the affine roots $a_0 = 1-\phi, a_i = \alpha_i (i = 1, ..., l)$ form a basis if the ARS $S(R)$.

5 Classification of irreducible reduced ARS

Let S be an irred reduced ARS, C chamber of S, $x_0, ..., x_l$ vertices of C, and $B = a_0, ..., a_l$ the corresponding basis. For each i, let F_i be the subspace of F vanish at x_i, $S_i = S \cap F_i$. On F_i the bilinear form $< f, g >$ is positive definite. Let W_i be the subgroup of $W(S)$ which fixes x_i.

Proposition 5.1. (1)*S_i is a FRS in F_i, and is reduced if S is reduced.*
(2) *$B - a_i$ is a basis of S_i*
(3) *W_i is the Weyl group of S_i*

Now assume S is reduced. We construct a Dynkin diagram for S according to the usual prescription: the nodes of the diagram for S $a_0, a_1, ..., a_l$ belongs to the basis B, and the bonds and arrows follow the same rules for FRS. One thing to notice here is that for rank 1 ARS S, we have to allow bonds which have infinitely multiplicity

If R is of type X, where X is one of the symbols $A_n, B_n, C_n, D_n, BC_n, E_6, E_7, E_8, F_4, G_2$ we say that $S(R)$ (resp. $S(R)^\vee$) is of type X (resp. $X^\vee$)

Theorem 5.2. *Every irreducible reduced ARS is similar to $S(R)$ or $S(R)^\vee$.*

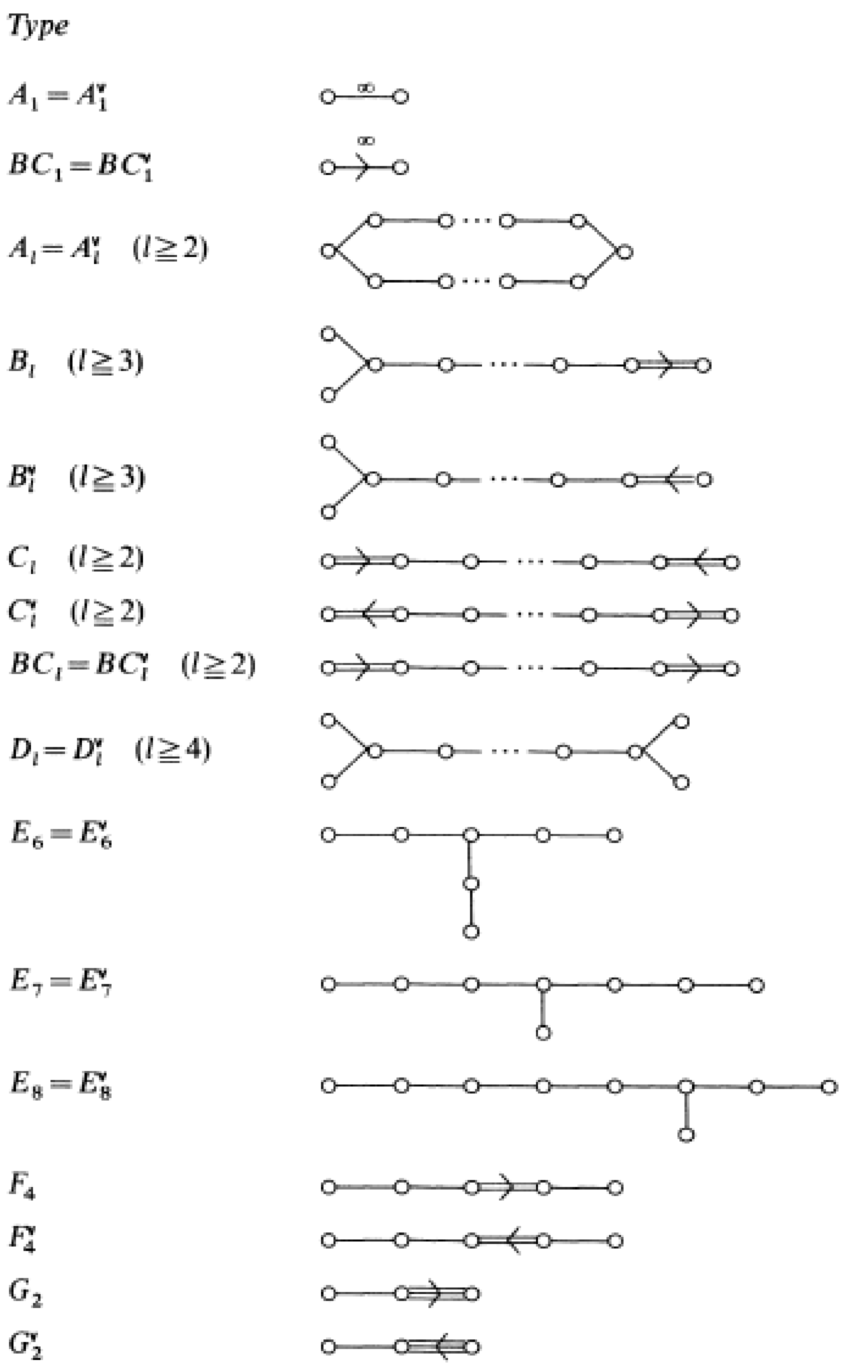

Figure 1: Dynkin Diagrams