CALCULUS II ASSIGNMENT 8 SOLUTIONS

1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{7^n}$,

(ii) $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{6m+2}$,

(iii) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k}{\sqrt{k^3+3}}$,

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(v) $\displaystyle\sum_{r=1}^{\infty} \left(1 + \frac{1}{r}\right)^{r^2}$,

(vi) $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{\log(m)}$.

Solutions. For (i), for the most part, the terms basically only involve powers with an n and so this suggests that either the ratio or the root test are applicable here. Indeed, let's apply both tests: applying the Ratio Test, we need to compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)/7^{n+1}}{n/7^n}\right| = \lim_{n\to\infty} \frac{1}{7} \cdot \frac{n+1}{n} = \frac{1}{7}$$

which is less than 1. Hence the Ratio Test implies that this series is absolutely convergent. Similarly, applying the Root Test instead, we compute:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^{1/n}}{7} = \frac{1}{7}$$

which is again less than 1. Thus the Root Test also allows us to conclude, in this case, that the series is absolutely convergent.

For (ii), first notice that this is an alternating series. Moreover, the denominators get bigger as you go along the series, so absolute values of the terms are decreasing. Finally, the limit of the terms is also 0. Hence the Alternating Series Test applies to show that this series is convergent. To check whether or not this series is absolutely convergent, we need to study the series

$$\sum_{m=1}^{\infty} \left|\frac{(-1)^m}{6m+2}\right| = \sum_{m=1}^{\infty} \frac{1}{6m+2}.$$

Squinting a bit, pretending the 2 is not in the denominator, this looks like the harmonic series. This suggests that this series is divergent. Indeed, let's do a comparison: we have $8m \geq 6m+2$ for all $m \geq 1$, and thus, upon taking reciprocals,

$$\frac{1}{8m} \leq \frac{1}{6m+2} \qquad \text{for all } m \geq 1.$$

Therefore, since the series $\sum \frac{1}{8m}$ is a scalar multiple of the harmonic series, this series is divergent. Alternatively, we can apply the Limit Comparison Test with the harmonic series: this means that we need to compute

$$\lim_{m\to\infty} \frac{1/m}{1/(6m+2)} = \lim_{m\to\infty} \frac{6m+2}{m} = 6.$$

As this limit is positive, the divergence of the harmonic series implies that our series here diverges, too. Hence our series is conditionally convergent.

For (iii), observe again that this is an alternating series whose terms are decreasing in absolute value and are tending to 0, so the Alternating Series Test applies to give convergence. To see whether or not this is absolutely convergence, we need to stare at the series

$$\sum_{k=1}^{\infty} \left| (-1)^k \frac{k}{\sqrt{k^3+3}} \right| = \sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3+3}}$$

and decide whether or not this is convergent. Again, squint and pretend that the "+3" is not in the denominator so that the terms of the series are roughly $\frac{k}{k^{3/2}} = \frac{1}{k^{1/2}}$. Thus this series looks like a p-series with $p = 1/2$, so we expect this to diverge. Indeed, apply the Limit Comparison Test with $\sum \frac{1}{k^{1/2}}$, which is to say, compute:

$$\lim_{k \to \infty} \frac{k}{\sqrt{k^3+3}} \cdot \frac{k^{1/2}}{1} = \lim_{k \to \infty} \frac{k^{3/2}}{\sqrt{k^3+3}} = \lim_{k \to \infty} \frac{1}{\sqrt{1+3/k^{3/2}}} = 1.$$

This is positive, and hence since $\sum \frac{1}{k^{1/2}}$ diverges, so does our series here. Thus our series is conditionally convergent.

For (iv), there are factorials and powers in the series, suggesting that we should apply the Ratio Test to check for convergence. This means we compute

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} \right|$$

$$= \lim_{n \to \infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n \to \infty} (n+1) \cdot \frac{1}{n+1} \left(\frac{n}{n+1} \right)^n = \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n.$$

To compute the final limit, we might proceed as follows:

$$\lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \to \infty} \exp \left(n \log \left(\frac{n}{n+1} \right) \right) = \exp \left(\lim_{n \to \infty} n \log \left(\frac{n}{n+1} \right) \right)$$

by continuity of the exponential function. Now to compute the limit in the argument, write

$$\lim_{n \to \infty} n \log \left(\frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{\log(n/(n+1))}{1/n}$$

and apply l'Hôpital's Rule; note that the calculation of the derivative of the function on top is simplified by noting $\log(n/(n+1)) = \log(n) - \log(n+1)$:

$$\lim_{n \to \infty} n \log \left(\frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{-1/n^2} = \lim_{n \to \infty} \frac{n^2}{n+1} - n = \lim_{n \to \infty} \frac{n^2 - (n^2+n)}{n+1} = \lim_{n \to \infty} \frac{-n}{n+1} = -1.$$

Tracing back, the upshot of this calculation is that

$$\lim_{n \to \infty} \left| \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} \right| = \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n = \exp \left(\lim_{n \to \infty} n \log(n/(n+1)) \right) = e^{-1}.$$

Thus the ratio of successive terms is $e^{-1} < 1$ and hence the Ratio Test allows us to conclude that the series is absolutely convergent.

For (v), the terms are just all powers, so apply the Root Test:

$$\lim_{r \to \infty} \sqrt[r]{(1 + 1/r)^{r^2}} = \lim_{r \to \infty} (1 + 1/r)^r = e,$$

where the final calculation might be made as in the limit computation above. Since $e = 2.71\ldots > 1$, the Root Test implies that this series is divergent.

For (vi), again note that the series is alternating, the absolute values of the terms decreases, and the terms limit to 0. Therefore the Alternating Series Test implies this series converges. To check absolute convergence, note that $\log(m) \leq m$, at least for $m \geq 2$—the logarithm function is an *extremely* slow growing function! Thus $\frac{1}{m} \leq \frac{1}{\log(m)}$ for $m \geq 2$, and so by comparison with the harmonic series $\sum \frac{1}{m}$, we see that the series $\sum \frac{1}{\log(m)}$ diverges. So our series is conditionally convergent.

Incidentally, to show the inequality $\log(m) \leq m$, we can proceed as follows: consider the function $f(x) := x - \log(x)$. Then $f(2) = 2 - \log(2) \approx 1.69\ldots > 0$. Moreover, $f'(x) = 1 - 1/x$ and this function is positive for all $x > 1$, which means f grows for all $x > 1$. Since $f(2) > 0$, this implies that $f(x) > 0$ for all $x \geq 2$; in particular, this means that $f(m) = m - \log(m) \geq 0$, which gives the inequality we were after. ∎

2. Test the series for convergence or divergence. Use any method available to you.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k^3} + \frac{1}{4^k}\right)$,

(iii) $\displaystyle\sum_{\ell=1}^{\infty} \ell \sin(1/\ell)$,

(iv) $\displaystyle\sum_{m=1}^{\infty} \frac{(m!)^m}{m^{4m}}$,

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$,

(vi) $\displaystyle\sum_{p=1}^{\infty} (\sqrt[p]{2} - 1)$.

Solutions. For (i), there are lots of powers and factorials, suggesting that we should test convergence using the Ratio Test. So compute:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+3)!} \cdot \frac{(n+2)!}{2^n n!} = \lim_{n\to\infty} 2\frac{n+1}{n+3} = 2$$

which is greater than 1. Therefore, the Ratio Test implies that the series in (i) is divergent.

For (ii), notice that the terms of series are sums of the terms of $\sum \frac{1}{k^3}$ and $\sum \frac{1}{4^k}$. But these are convergent series, the first being a p series with $p > 1$, and the second being a geometric series with $|r| = 1/4 < 1$. Since sums of convergent series are convergent, the series in (ii) is also convergent.

For (iii), well, as always, the first thing we should do when looking at a series is test whether or not the terms go to zero:

$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{-\cos(1/x)/x^2}{-1/x^2} = \lim_{x\to\infty} \cos(1/x) = \cos(0) = 1$$

where I have applied l'Hôpital's Rule in the middle to compute the limit. Thus the terms of this series do not tend to 0, so by the Divergence Test, the series diverges.

For (iv), there are an awful lot of powers of m, so the Root Test seems like a good choice here. Computing roots of the terms, we get:

$$\lim_{m\to\infty} \sqrt[m]{|a_m|} = \lim_{m\to\infty} \frac{m!}{m^4}.$$

Let's think about the sequence $m!/m^4$. Well, this is a number with 4 things in the denominator and m things in the numerator. Moreover, the growing number of things in the numerator are also getting bigger with m. Thus it would appear that the numerator grows (much) faster than the denominator and hence this sequence should diverge. To make this precise, split off 4 of the largest numbers in the numerator and group them with the denominator:

$$\frac{m!}{m^4} = \frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \frac{m-3}{m} \cdot (m-4)!.$$

Now for $m \geq 4$,

$$\frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \frac{m-3}{m} = \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\left(1 - \frac{3}{m}\right)$$

$$\geq \left(1 - \frac{1}{4}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{3}{4}\right)$$

$$= \frac{3}{32}.$$

Therefore, for $m \geq 4$,

$$\frac{m!}{m^4} \geq \frac{3}{32}(m-4)!$$

and since the right hand side of the inequality tends to ∞ as $m \to \infty$, so does the left hand side. Consequently, the Root Test concludes that the series in (iv) diverges.

For (v), the series looks an awful lot like the Harmonic Series, so apply the Limit Comparison Test with the Harmonic Series:

$$\lim_{n \to \infty} \frac{1/n}{1/n^{1+1/n}} = \lim_{n \to \infty} n^{1/n} = 1,$$

and hence, as the Harmonic Series diverges, so does the series in (v). Note that this limit is computed by writing $n^{1/n} = \exp(\log(n)/n)$ and then computing the limit of the argument $\log(n)/n$ via l'Hôpital's rule.

For (vi), well maybe it's not so easy to figure out what to compare this series to. One way to approach this is to momentarily replace $\sqrt[p]{2}$ with $\sqrt[p]{e} = e^{1/p}$. Then imagine computing $e^{1/p} - 1$ by Taylor series expanding the exponential term:

$$e^{1/p} - 1 = \frac{1}{p} + \frac{1}{2} \cdot \frac{1}{p^2} + \frac{1}{3!} \cdot \frac{1}{p^3} + \cdots.$$

The lowest order term is $1/p$, so this suggests that the terms $\sqrt[p]{2} - 1$ grow at least like $1/p$. Hence this suggests a comparison of our series with the Harmonic Series. So we can now try to apply the Limit Comparison Test with the Harmonic Series:

$$\lim_{p \to \infty} \frac{\sqrt[p]{2} - 1}{1/p} = \lim_{p \to \infty} \frac{-2^{1/p}/p^2}{1/p^2} = \lim_{p \to \infty} 2^{1/p} = 1.$$

This is nonzero, so as the Harmonic Series diverges, we see that our series in (vi) also diverges. Wow, the Limit Comparison Test sure is useful, eh?   ∎

3. Find the radius of convergence and interval of convergence of the series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$,

(iii) $\displaystyle\sum_{m=2}^{\infty} \frac{b^m}{\log(m)} (x-a)^m$ for $b > 0$ fixed,

(ii) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$,

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

Solutions. For (i), the limit we need to compute in the Ratio Test is

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2n+1} \cdot \frac{2n-1}{x^n} \right| = \lim_{n\to\infty} \frac{2n+1}{2n-1}|x| = |x|.$$

Therefore, we see that whenever $|x| < 1$, then the series in (i) converges. To determine the interval of convergence, we need to determine whether or not the series converges when $|x| = 1$, that is, when $x = 1$ or $x = -1$. At these endpoints, our series are

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

respectively. The former diverges by comparison with the Harmonic Series; the latter is an alternating series with terms decreasing in absolute value and tending to 0, so the Alternating Series Test applies to show that this series is convergent. In conclusion, our series has a radius of convergence $R = 1$ and interval of convergence $I = [-1, 1)$.

For (ii), the limit we would compute in the Ratio Test is

$$\lim_{k\to\infty} \left| \frac{x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^{2k+1}} \right| = \lim_{k\to\infty} \frac{|x|^2}{(2k+3)(2k+2)} = 0$$

since we think of x as a fixed number and the denominator tends to ∞ as $k \to \infty$. Therefore this power series has radius of convergence $R = \infty$ and interval of convergence $I = \mathbf{R}$ the entire real line.

For (iii), apply the Ratio Test and compute the limit

$$\lim_{m\to\infty} \left| \frac{b^{m+1}}{\log(m+1)} \frac{\log(m)}{b^m} (x-a) \right| = \lim_{m\to\infty} \frac{\log(m)}{\log(m+1)} b|x-a| = b|x-a|.$$

Therefore our series would converge whenever $b|x-a| < 1$, which is to say that $|x-a| < 1/b$. Now to determine whether or not the series converges when $|x-a| = 1/b$, that is, when $x = a + 1/b$ and $x = a - 1/b$. When $x = a + 1/b$, the series becomes

$$\sum_{m=2}^{\infty} \frac{b^m}{\log(m)} \frac{1}{b^m} = \sum_{m=2}^{\infty} \frac{1}{\log(m)}$$

which diverges by comparison with the Harmonic Series. When $x = a - 1/b$, the series becomes

$$\sum_{m=2}^{\infty} \frac{b^m}{\log(m)} \frac{(-1)^m}{b^m} = \sum_{m=2}^{\infty} \frac{(-1)^m}{\log(m)}$$

which is an alternating series with terms decreasing in absolute value and tending to zero. Thus this series is convergent by the Alternating Series Test. In conclusion, the radius of convergence of this series is $R = 1/b$ and the interval of convergence of is $[a - \frac{1}{b}, a + \frac{1}{b})$. ■

4. The Bessel functions of order 0 and 1 are, respectively,

$$J_0(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(n!)^2 2^{2n}} \quad \text{and} \quad J_1(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

Find the radii and intervals of convergence of these functions.

Solution. There are plenty of powers and factorials in both functions, so we should apply the Ratio Test to find radii and intervals of convergence. To do these simultaneously, let i be either 0 or 1. Then applying the Ratio Test to J_i means that we need to compute the limit

$$\lim_{n\to\infty} \left| \frac{x^{2n+2+i}}{(n+1)!(n+1+i)!2^{2n+2+i}} \frac{n!(n+i)!2^{2n+i}}{x^{2n+i}} \right| = \lim_{n\to\infty} \frac{1}{4(n+1)(n+1+i)} |x|^2 = 0$$

for all x. Therefore the radius of convergence of both J_i is $R = \infty$ and the interval of convergence is $I = \mathbf{R}$, the entire real line. ∎