Calibrations

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These notes are primarily based off of Lotay's *An Invitation to Geometry and Topology via* G_2 notes and Joyce's book *Riemannian Holonomy Groups and Calibrated Geometry.*

1 Minimal Submanifolds

Despite what the name seems to suggest, minimal submanifolds are not "minimal" in any sense of the word; they are stationary. By this we mean that their volume does not change under small variations. But how do we make this precise? It turns out that for several applications (some of which we may see) we need to rigorously define what it means for an *immersed* submanifold to be minimal (the case of embedded submanifolds is quite easy). So, we fix an orientable Riemannian manifold (M, g) as well as an immersed submanifold $i : N \to M$ and proceed as follows.

A **variation of** $i : N \to M$ **with compact support** is a smooth function:

$$F : N \times (-\epsilon, \epsilon) \to M$$

such that $F(x, 0) = i(x)$ for all $x \in N$, each $F(-, t) : N \to M$ is an immersion and there exists an open subset $S \subseteq N$ with $\bar{S}$ compact satisfying

$$F(x, t) = i(x) \text{ for all } x \in N \setminus \bar{S} \text{ and all } t \in (-\epsilon, \epsilon).$$

Now, if $\partial/\partial t$ denotes the vector field on $N \times (-\epsilon, \epsilon)$ given by sending each (x, t_0) to $\partial/\partial t|_{t=t_0}$ then we can also define the **variation vector field**:

$$X := F_* \left(\frac{\partial}{\partial t} \right) \in \Gamma(i^* TM).$$

This will be useful later.

An immersed submanifold $i : N \to M$ of a Riemannian manifold (M, g) is called **minimal** when

$$\left. \frac{d}{dt} \right|_{t=0} \mathrm{Vol}(F(S, t)) = 0$$

for all variations F with compact support S (depending on F), where by $\mathrm{Vol}(F(S,t))$, we mean the volume of S with respect to the metric $F(-,t)^*g$ on N. As one can probably tell, this definition seems nigh impossible to verify given an immersion $i : N \to M$. However, we will simplify it as follows.

For the next little while we will assume that N is compact (the general scenario is very similar). One should be warned that the following computation is quite hairy. Afterwards, we will derive a cleaner characterization of what it means to be minimal. We then fix $p \in N$ and choose normal coordinates $(U, x^1, \ldots, x^n)$ on N at p with respect to the metric $i^*g = F(-,0)^*g$. Hence if we denote

$$e_i(q,t) := F_{*,(q,t)}\left(\frac{\partial}{\partial x^i}\right) \quad \text{and} \quad g_{i,j}(q,t) := g(e_i(q,t), e_j(q,t))$$

then we obtain $g_{i,j}(p,0) = \delta_{i,j}$. Furthermore, if we denote by vol_N the metric volume form on (N, i^*g) then we can compute:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(F(N,t)) &= \frac{d}{dt}\Big|_{t=0} \int_N \sqrt{\det(g_{i,j})}\,\mathrm{vol}_N \\
&\qquad\qquad \text{by the change of variables formula} \\
&= \int_N \frac{d}{dt}\Big|_{t=0} \sqrt{\det(g_{i,j})}\,\mathrm{vol}_N \\
&\qquad\qquad \text{since } \mathrm{vol}_N \text{ doesn't depend on } t \\
&= \int_N \frac{\sum_i \frac{\partial}{\partial t} g_{i,i}(q,t)\big|_{t=0}}{2\sqrt{\det(g_{i,j}(q,0))}}\,\mathrm{vol}_N \quad (\text{write the denominator as } 2f(q)) \\
&= \int_N \frac{1}{2f(q)} \sum_i \frac{\partial}{\partial t}\Big|_{t=0} g(e_i, e_i)\,\mathrm{vol}_N \\
&= \int_N \frac{1}{f(q)} \sum_i g(\nabla_X e_i, e_i)\,\mathrm{vol}_N \\
&\qquad \text{where } \nabla \text{ is the Levi-Civita connection for } i^*g \\
&= \int_N \frac{1}{f(q)} \sum_i g(\nabla_{e_i} X, e_i)\,\mathrm{vol}_N \quad \text{since } [X, e_i] = F_*\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right] = 0.
\end{aligned}$$

The second last equality above makes sense since $\nabla_X e_i$ at a point only depends on X at a point and e_i on a neighbourhood (which is fine since F is locally an embedding). Also, it is worth remarking now that $f(p) = 1$ since we are in normal coordinates centered at p. From here we decompose $i^*TM = TN \oplus \nu$ where ν is the normal bundle to N in M and write $X = X^T + X^\perp$. Since the direct sum is orthogonal we have $\nabla_{e_i} g(X^\perp, e_i) = 0$ and so:

$$\begin{aligned}
\sum_i g(\nabla_{e_i} X, e_i) &= \sum_i g(\nabla_{e_i} X^T, e_i) - \sum_i g(X^\perp, \nabla_{e_i} e_i) \\
&= \sum_i g(\nabla_{e_i} X^T, e_i) - g(X^\perp, H)
\end{aligned}$$

where:
$$H := \sum_i (\nabla_{e_i} e_i)^{\perp}.$$
is what is called the **mean curvature vector**. Now we can rewrite our integral as:
$$\int_N \frac{1}{f(q)} \sum_i g(\nabla_{e_i} X^T, e_i) \operatorname{vol}_N - \int_N \frac{1}{f(q)} g(X^{\perp}, H) \operatorname{vol}_N .$$
Now, for some reason in Lotay's notes he no longer has the $f(q)$ term. In fact, he dropped it way back when we first differentiated $\sqrt{\det(g_{i,j})}$ citing the fact that we were in normal coordinates so $g_{i,j}(p,0) = \delta_{i,j}$. However, $\sqrt{\det(g_{i,j})}$ really does depend on $q \in U$ and is not just fixed at p. The key thing to notice is that Lotay was abusing notation by saying that he was differentiating the volume functional at $t = 0$. What we really do is compute the first variation of the volume form itself at the point $(p, 0)$. This gives:
$$\begin{aligned} \frac{d}{dt}\Big|_{(p,0)} \operatorname{Vol}(F(N,t)) &= \left(\sum_i g(\nabla_{e_i} X^T, e_i) - g(X^{\perp}, H) \right) \operatorname{vol}_N \\ &= (\operatorname{div}_N(X^T) - g(X^{\perp}, H)) \operatorname{vol}_N \end{aligned}$$
and notice that this expression is in fact independent of our choice of coordinates! So, by applying the divergence theorem we actually have the **first variational formula**:
$$\frac{d}{dt}\Big|_{t=0} \operatorname{Vol}(F(N,t)) = - \int_N g(X^{\perp}, H) \operatorname{vol}_N .$$
Now, if we want $i : N \to M$ to be a critical point of this functional then the above must hold for all mean curvature vectors X. One can then show with some work that this happens if and only if $H = 0$. As we will see, this is precisely the characterization of minimal we will state below.

Here we no longer assume that (M, g) is compact. Let $i : N \to M$ be an immersed submanifold and $\nu \to N$ the normal bundle of N in M (so $i^*TM = TN \oplus \nu$). The **second fundamental form** of $i : N \to M$ is a section $B \in \Gamma((S^2T^*N) \otimes \nu)$ satisfying: whenever $X, Y \in \mathfrak{X}(M)$ satisfy $X|_N, Y|_N \in \Gamma(TN)$ then
$$B(X|_N, Y|_N) = \pi_{\nu}((\nabla_X Y)|_N)$$
where ∇ is the Levi-Civita connection on (M, g). This is indeed symmetric in $X|_N, Y|_N$ since $\nabla_X Y = \nabla_Y X + [X, Y]$ and, since $X|_N, Y|_N \in \Gamma(TN)$, $[X, Y]|_N \in \Gamma(TN)$ and so $\pi_{\nu}([X, Y]|_N) = 0$. We then define the **mean curvature vector** $\kappa \in \Gamma(\nu)$ to be the trace of B using g, that is we raise an index of B using g and then take the trace.

Now, if one works in normal coordinates about some point p then κ_p is precisely the H from before. So indeed $\kappa = 0$ if and only if $i : N \to M$ is a minimal submanifold. If one wants to see this in more detail I highly recommand Peter

Li's "Lecture Notes on Geometric Analysis".

2 Calibrations

Looking at our expression for H from before, we can see that the equation $\kappa = 0$ is a second order non-linear PDE in $i : N \to M$. This is because each e_i is already using one derivative of i and we have $\nabla_{e_i} e_i$ in the expression. Furthermore, the equation $\kappa = 0$ is apparently "elliptic module diffeomorphisms" according to Joyce. The basic idea is that if a PDE is invariant under some diffeomorphisms (in some sense) then the principal symbol, which is a linear map, cannot possibly be injective (and hence cannot be invertible). However, by quotienting out the domain and codomain of the principal symbol by some subspaces related to the diffeomorphisms we can occasionally make the principal symbol invertible and apply elliptic theory. Despite this, $\kappa = 0$ is still incredibly difficult to solve. Furthermore, if $i : N \to M$ is a solution then it is merely a critical point of the volume functional, whereas often we want actual minima. Calibrations will fix this problem.

A **calibration** on (M, g) is a closed k-form η (for any k) satisfying $\eta(e_1, ..., e_k) \leq 1$ for all unit tangent vectors $e_1, ..., e_k \in T_pM$ and all $p \in M$. Notice that this is equivalent to $\eta|_V = \alpha_V \operatorname{vol}_V$ with $\alpha_V \leq 1$ for any oriented tangent k-plane V in T_pM (and any $p \in M$). A k-dimensional submanifold $i : N \to M$ is called **calibrated** with respect to η if there exists an orientation on N so that $\operatorname{vol}_{T_xN} = (i^*\eta)_x$ for all $x \in N$. We also call such submanifolds **η-submanifolds**.

As a first example, let $x^1, ..., x^n$ be the standard coordinates on $\mathbb{R}^n$ and consider:

$$\eta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1,...,i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}, \quad a_{i_1,...,i_k} \in \mathbb{R}.$$

Then $\eta : \Lambda^k \mathbb{R}^n \to \mathbb{R}$ is linear and hence bounded since $\Lambda^k \mathbb{R}^n$ and $\mathbb{R}$ are finite dimensional. Since $|\vec{v} \wedge \vec{w}| \leq |\vec{v}||\vec{w}|$ it then follows that the collection $e_1 \wedge \cdots \wedge e_k$ with each $e_i \in \mathbb{R}^n$ a unit vector is bounded and so we can rescale η so that it is a calibration (we already have $d\eta = 0$). Furthermore, η can be scaled in such a way so that $\eta(e_1, ..., e_k) = 1$ for some (not necessarily unique) $e_1, ..., e_k$ unit vectors in $\mathbb{R}^n$.

Theorem:
Let N be an η-submanifold of (M, g). Then N is homologically volume minimizing and, in particular, a minimal submanifold.
Proof: Again we assume N is compact for simplicity. If N' is homologous to N then there exists a compact manifold K with boundary $\partial K = (-N) \sqcup N'$ and so by Stoke's theorem:

$$0 = \int_K d\eta = \int_{N'} \eta - \int_N \eta.$$

Thus:

$$\mathrm{Vol}(N) = \int_N \eta = \int_{N'} \eta \leq \mathrm{Vol}(N') \text{ since } \eta \text{ is a calibration.} \quad \square$$

Notice how the above theorem in conjunction with Poincaré's lemma and Stoke's theorem imply that there are no compact calibrated submanifolds of $\mathbb{R}^n$.

We now provide another example. Let (X, J, ω) be a Kähler manifold and consider:

$$\frac{\omega^k}{k!} \in \Omega^{2k}(X).$$

This form is indeed closed since $d(\omega^k/k!) = kd\omega \wedge \omega^{k-1} = 0$. Our goal now is to show that $\omega^k/k!$ is a calibration and find its calibrated submanifolds. In order to do so, we will assume the following (easy to prove) lemma.

Lemma:
Let η be a calibration and suppose that $*\eta$ is closed. Then $*\eta$ is a calibration.

The next step is to prove the so-called "Wirtinger's inequality". This says that if we define

$$\omega = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j \quad \text{on } \mathbb{C}^n$$

then $\frac{\omega^k}{k!}(e_1, ..., e_{2k}) \leq 1$ for any unit vectors $e_1, ..., e_{2k} \in \mathbb{C}^n$ with equality if and only if $\mathrm{Span}_{\mathbb{R}}\{e_1, ..., e_{2k}\}$ is a complex k-plane. Maybe I'll write down the proof of this later ...

From here (and the fact that $\omega^k/k!$ is closed) it follows immediately that $\omega^k/k!$ is a calibration (on X, J, ω). Furthermore, the calibrated submanifolds are precisely the complex submanifolds. Using our previous theorem, this implies that all compact complex submanifolds of a Kähler manifold are homologically volume minimizing.

3 Special Lagrangian Submanifolds

A **Calabi-Yau** m-fold is a compact Kähler manifold (X, J, g) with $\dim_{\mathbb{C}}(X) = m$ and holonomy in $\mathrm{SU}(m)$ (or at least this is one of the possible inequivalent definitions). The holonomy being in $\mathrm{SU}(m)$ yields the existence of a holomorphic nowhere-vanishing m form $\theta \in \Omega^{m,0}(X)$. For us, we will fix such a form and consider it as part of the definition of a Calabi-Yau manifold. That is, for us a Calabi-Yau manifold is a quadruple (X, J, g, θ) where (X, J, g) is a compact Kähler manifold and $\theta \in \Omega^{m,0}(X)$ is nowhere vanishing and satisfies:

$$\frac{\omega^m}{m!} = (-1)^{m(m-1)/2}(i/2)^m \theta \wedge \bar{\theta}$$

where ω is the Kähler form $\omega(v, w) = g(Jv, w)$. Given a Calabi-Yau manifold (X, J, g, θ) using our definition we have the following:

1. as θ is a nowhere vanishing $(m, 0)$-form, the canonical bundle K_X is trivial and so $c_1(X) = 0$;
2. g is Ricci flat;
3. $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$;
4. if $\mathrm{Hol}(g) = \mathrm{SU}(m)$ (and $m \geq 3$) then (X, J) is projective.

Here we will see how the soon-to-be-defined *special Lagrangian submanifolds* of a Calabi-Yau manifold are related to the symplectic geometry. In particular, a **Lagrangian submanifold** of a symplectic manifold (M, ω) is one on which the symplectic form ω vanishes. Keep this in mind when reading the following lemma.

Lemma: Define $\theta = dz^1 \wedge \cdots \wedge dz^n$ and $\omega = \frac{i}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$ on $\mathbb{C}^n$. Then $|\theta(e_1, ..., e_n)| \leq 1$ for all unit vectors $e_1, ..., e_n \in \mathbb{C}^n$ with equality if and only if $\omega|_P = 0$ where $P = \mathrm{Span}_{\mathbb{R}}\{e_1, ..., e_n\}$ (in this case we call P a *Lagrangian plane*).
Proof:
Let $e_1, ..., e_n$ denote the standard basis for $\mathbb{R}^n$, P be any real n-plane in $\mathbb{C}^n$ and find $A \in \mathrm{GL}(\mathbb{C}^n)$ so that $f_1 := Ae_1, ..., f_n := Ae_n$ form on orthonormal basis for P. Then:

$$
\begin{aligned}
|\theta(f_1, ..., f_n)|^2 = |\det(A)|^2 &= |f_1 \wedge Jf_1 \wedge \cdots \wedge f_n \wedge Jf_n| \\
&\leq |f_1||Jf_1| \cdots |f_n||Jf_n| = 1
\end{aligned}
$$

with equality if and only if $f_1, Jf_1, ..., f_n, Jf_n$ is orthonormal. But $\omega(u, v) = g(Ju, v)$ so $\omega|_P = 0$ if and only if $JP = P^{\perp}$ and we are done. $\square$

Theorem:
Let (X, J, g, θ) be a Calabi-Yau m-fold. Then for any $\varphi \in \mathbb{R}$, $\mathrm{Re}(e^{-i\varphi}\theta)$ is a calibration.
Proof:
Since $d(dz^1 \wedge \cdots \wedge dz^n) = 0$ this follows directly from our lemma. $\square$

The calibrated submanifolds of $\mathrm{Re}(\theta)$ are called **special Lagrangian submanifolds** and, if $\varphi \neq 0$, then the calibrated submanifolds of $\mathrm{Re}(e^{-i\varphi}\theta)$ are called **special Lagrangian submanifolds with phase** $e^{i\varphi}$.

Theorem:
Let $(X, J, g, \omega, \theta)$ be a Calabi-Yau manifold and $\sigma : X \to X$ an involution satisfying $\sigma^*\omega = -\omega$ and $\sigma^*\theta = \bar{\theta}$. Then the collection of fixed points of σ is a special Lagrangian submanifold of X.

Finally we conclude this talk with a brief discussion of the SYZ conjecture. A disclaimer for this paragraph is that I actually don't understand any of this, but here we go anyway. In string theory one can compute something called the super-conformal field theory (SCFT) associated to a Calabi-Yau 3-fold (and apparently space-time is supposed to locally look something like $\mathbb{R}^{(1,3)} \times X$ for some Calabi-Yau 3-fold X due to "supersymmetry"). From this SCFT, it is allegedly possible to recover $H^{1,1}(X)$ and $H^{2,1}(X)$. However, in this process, one has to make a choice of sign corresponding to some sort of $U(1)$-action and so it was conjectured that there would exist a "mirror" Calabi-Yau 3-fold $\hat{X}$ whose SCFT would yield $H^{1,1}(\hat{X}) \cong H^{2,1}(X)$ and $H^{2,1}(\hat{X}) \cong H^{1,1}(X)$. The SYZ conjecture describes some sort of relationship between X and its mirror; a very very basic (and incorrect) version is stated below.

Very Basic SYZ Conjecture: Let $X, \hat{X}$ be mirror Calabi-Yau 3-folds. Then there exists a compact 3-manifold B and surjective continuous maps $f : X \to B$, $\hat{f} : \hat{X} \to B$ as well as a closed subset $\Delta \subseteq B$ with $B \setminus \Delta$ dense so that the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ for $b \in B \setminus \Delta$ are non-singular special Lagrangian 3-tori which are dual in some sense. Furthermore, for $b \in \Delta$, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are singular special Lagrangian 3-folds.

References:
Joyce's book "Riemannian Holonomy Groups and Calibrated Geometry", Lotay's course notes "An Invitation to Geometry and Topology via G_2" and Peter Li's "Lecture Notes on Geometric Analysis".