CALCULUS II ASSIGNMENT 10 SOLUTIONS

1. Find the general solution to the following differential equations:

(i) $y' = 3x^2 y^2$,

(ii) $\dfrac{dy}{dx} - y = e^x$,

(iii) $\dfrac{dz}{dt} + e^{t+z} = 0$,

(iv) $xy' + y = \sqrt{x}$,

(v) $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}}$,

(vi) $t^2\dfrac{dz}{dt} + 3tz = \sqrt{1+t^2}$.

Solutions. The equation in (i) is a separable equation, so upon writing $y' = {}^{dy}/_{dx}$ and isolating variables, we have

$$-\frac{1}{y} = \int \frac{dy}{y^2} = \int 3x^2\,dx = x^3 + C$$

for an arbitrary constant C. Taking negative reciprocals, we have

$$y = \frac{-1}{x^3 + C}$$

as the solution to (i).

The equation in (ii) is a linear differential equation with $P(x) = -1$. Thus the integrating factor for this equation is $I(x) = e^{-\int dx} = e^{-x}$. Hence

$$\frac{d}{dx}\left(e^{-x}y\right) = 1.$$

Integrating both sides, this gives

$$e^{-x}y = \int dx = x + C$$

so, upon solving for y, we get

$$y = e^x(x + C)$$

for any constant C, as the general solution to the equation in (ii).

Equation (iii) is a separable equation. Rearranging and isolating the variables, we have

$$e^{-z} = -\int e^{-z}\,dz = \int e^t\,dt = e^t + C.$$

Taking logarithms on both sides, we see that the general solution of (iii) is

$$z = -\log(e^t + C)$$

for arbitrary constant C.

Equation (iv) is a linear differential equation: upon dividing through by x, we will have $P(x) = 1/x$ and $Q(x) = 1/\sqrt{x}$. Thus the integrating factor for this equation is

$$I(x) = e^{\int \frac{dx}{x}} = e^{\log(x)} = x$$

and thus, upon multiplying through the equation and integrating, we get

$$xy = \int \frac{d}{dx}(xy) = \int \sqrt{x} = \frac{2}{3}x^{3/2} + C.$$

1

Therefore the general solution to (iv) is

$$y = \frac{2}{3}x^{1/2} + \frac{C}{x}.$$

Equation (v) is separable: rearranging and integrating yields

$$\int \frac{\theta}{\sec\theta}\, d\theta = \int te^{-t^2}\, dt.$$

On the left hand side,

$$\int \frac{\theta}{\sec\theta}\, d\theta = \int \theta\cos\theta\, d\theta = \theta\sin\theta - \int \sin\theta\, d\theta = \theta\sin\theta + \cos\theta.$$

On the right hand side,

$$\int te^{-t^2}\, dt = \frac{-1}{2}\int e^u\, du = -\frac{1}{2}e^{-t^2}.$$

Putting things together gives the equation

$$\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C.$$

This gives an implicit equation for θ.

Finally, (vi) is another linear differential equation, as we can see upon dividing through by t^2. Here, $P(t) = \frac{3}{t}$ and $Q(t) = t^{-2}\sqrt{1+t^2}$. Then

$$I(t) = e^{3\int \frac{dt}{t}} = e^{3\log(t)} = e^{\log(t^3)} = t^3.$$

So multiplying through by t^3 gives the equation

$$\frac{d}{dt}\left(t^3 z\right) = t\sqrt{1+t^2}.$$

Integrating both sides and dividing by t^3,

$$z = \frac{1}{t^3}\int t\sqrt{1+t^2}\, dt = \frac{1}{2t^3}\int \sqrt{u}\, du = \frac{1}{3t^3}\left(t^{3/2} + C\right)$$

for C an arbitrary constant. $\blacksquare$

2. Solve the following initial value problems:

(i) $\dfrac{dP}{dt} = \sqrt{Pt}$ with $P(1) = 2$,

(iii) $\dfrac{dL}{dt} = kL^2 \log(t)$ with $L(1) = -1$,

(ii) $xy' + y = x\log(x)$ with $y(1) = 0$,

(iv) $xy' = y + x^2 \sin(x)$ with $y(\pi) = 0$.

Solution. The equation in (i) is separable, so

$$2P^{1/2} = \int \frac{dP}{\sqrt{P}} = \int \sqrt{t}\,dt = \frac{2}{3}t^{3/2} + C.$$

Thus $P = (t^{3/2}/3 + C)^2$. Imposing the initial condition at $t = 1$:

$$2 = P(1) = (C + 1/3)^2 \quad\text{so}\quad C = \pm\sqrt{2} - \frac{1}{3}.$$

In fact, either choice of sign in front of $\sqrt{2}$ would be valid. Thus we actually get two solutions to our initial value problem, given by

$$P_{\pm}(t) := \left(t^{3/2}/3 \pm \sqrt{2} - 1/3\right)^2.$$

The equation in (ii) is a linear differential equation. Either by computing the integrating factor or by inspection, we have

$$\frac{d}{dx}(xy) = x\log(x)$$

so integrating both sides and dividing by x gives

$$y = \frac{1}{x}\int x\log(x)\,dx = \frac{1}{x}\left(\frac{1}{2}x^2\log(x) - \frac{1}{4}x^2 + C\right) = \frac{1}{2}x\log(x) - \frac{1}{4}x + \frac{C}{x}.$$

Putting in $x = 1$, we solve for C as:

$$0 = y(1) = -\frac{1}{4} + C \quad\text{so}\quad C = \frac{1}{4}.$$

So our solution is

$$y = \frac{1}{2}x\log(x) - \frac{1}{4}x + \frac{1}{4x}.$$

The equation in (iii) is separable:

$$-\frac{1}{L} = \int \frac{dL}{L^2} = \int k\log(t)\,dt = k\left(t\log(t) - t\right) + C.$$

Inputting the initial condition $L(1) = -1$ in the above gives

$$1 = -\frac{1}{L(1)} = -k + C \quad\text{so}\quad C = k + 1.$$

Therefore

$$L = \frac{-1}{k(t\log(t) - t) + 1 + k}.$$

The equation (iv) is linear with $P(x) = -1/x$ and $Q(x) = x\sin(x)$. The integrating factor is $e^{-\int dx/x} = e^{-\log(x)} = 1/x$. Multiplying through and simplifying, we get

$$\frac{d}{dx}(y/x) = \sin(x).$$

So integrating through and multiplying by x gives

$$y = x \int \sin(x)\, dx = -x\cos(x) + Cx.$$

Inputting the initial condition, we have

$$0 = y(\pi) = \pi + C\pi \quad \text{so} \quad C = -1.$$

Therefore $y = -x\cos(x) - x$ is the solution. ∎

3. Sometimes, equations that are not obviously of a form that you know how to solve can be massaged into one by a clever change of variables. For instance, consider the equation

$$y' = x + y.$$

This is not separable, but it can be transformed into a separable equation using a change of variables, as follows:

   (i) Consider the substitution $u = x + y$. Express dx in terms of du and dy.
  (ii) Make the substitution $u = x + y$ in $y' = x + y$ and rearrange so that the equation becomes separable. Don't forget to change the differential in $y' = dy/dx$!
 (iii) Solve the differential equation and express the final solution in terms of x.
 (iv) Double check that your proposed solution satisfies the original differential equation.

Solution. For (i), apply d to the equation $u = x + y$ and rearrange:

$$du = dx + dy \quad \text{so} \quad dx = du - dy.$$

Now substituting $u = x + y$:

$$\frac{dy}{du - dy} = u \quad \text{so} \quad dy = u\,du - u\,dy.$$

Adding $u\,dy$ on both sides,

$$(1 + u)\,dy = u\,du \quad \text{so} \quad dy = \frac{u}{1 + u}\,du.$$

Integrating both sides then gives

$$y = \int \frac{u}{1 + u}\,du = \int 1 - \frac{1}{1 + u}\,du = u - \log(1 + u) + C.$$

Now to solve this equation in terms of x, substitute $u = x + y$ back in to get

$$y = x + y - \log(1 + x + y) + C \quad \text{so} \quad \log(1 + x + y) = x + C.$$

Exponentiating both sides gives

$$1 + x + y = e^{x+C} + Ae^x \quad \text{so} \quad y = Ae^x - x - 1$$

where A is some constant.

Let's check that this solves the equation $y' = x + y$:

$$y' = Ae^x - 1 = x + (Ae^x - x - 1) = x + y$$

as we hoped for! ∎

4. In a similar vein to **3.**, consider the equation

$$\frac{dx}{dt} = \frac{t}{2}x + \frac{1}{2x}e^{t^2/2}.$$

This is not a linear equation, a fact that you should explain. However, consider the substitution $y(t) := x(t)^2$. Show that y satisfies the linear differential equation

$$\frac{dy}{dt} = ty + e^{t^2/2}$$

and use this to solve the original equation.

Solution. Well, let's differentiate the relationship $y = x^2$:

$$\frac{dy}{dt} = \frac{d}{dt}x^2 = 2x\frac{dx}{dt} = 2x\left(\frac{t}{2}x + \frac{1}{2x}e^{t^2/2}\right) = tx^2 + e^{t^2/2} = ty + e^{t^2/2},$$

and so y satisfies the equation as claimed. Now this is a linear equation with integrating factor $e^{-\int t\,dt} = e^{-t^2/2}$. Rearranging and multiplying this through, we get

$$\frac{d}{dt}\left(e^{-t^2/2}y\right) = 1$$

so integrating and solving for y, we get

$$y = e^{t^2/2}(t + C)$$

for an arbitrary constant C. Using the relationship $y = x^2$, this means that $x = \pm e^{t^2/4}(t + C)^{1/2}$ is a solution to the original equation. ∎

5. A pretty neat equation I just learned about is the Gompertz function. This type of equation is good for modelling things that start off and end off slow, but may change rapidly in the interim. For instance, one application is apparently in modelling the growth of a tumor.

In any case, consider the equation

$$\frac{dV}{dt} = a(\log(b) - \log(V))V$$

where a and b are positive constants.

  (i) Find the general solution to this equation.

 (ii) Try to come up with a situation—one which is not covered in Wikipedia's list!—that this equation could serve as a useful model.

Proof. To solve the Gompertz equation, observe that the equation is separable, with

$$\int \frac{dV}{a(\log(b) - \log(V))V} = \int dt = t + C.$$

To integrate the left hand side, consider the substitution $u = \log(b) - \log(V)$, so that $du = -dV/V$, giving

$$\int \frac{dV}{a(\log(b) - \log(V))V} = -\frac{1}{a}\int \frac{du}{u} = -\frac{1}{a}\log(\log(b) - \log(V)).$$

Solving for V, we get

$$V = be^{-Ae^{-at}}$$

for some arbitrary constant A. ∎

6. In class, we discussed a very simple model for population growth of a system. Of course, it was too simple and unrealistic: there are physical constraints to the population of a system. A slightly better model is given by the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where k and M are positive constants. In modelling populations, k is a growth rate of the system, and M is the *carrying capacity* of the system: the maximum number of individuals a system can support with its resources.

(i) Solve the logistic equation by recognizing it as a separable equation.

(ii) Solve the logistic equation, again, by making the substitution $z = 1/P$ to obtain the linear differential equation

$$z' + kz = \frac{k}{M}.$$

(iii) Revisit the model that we discussed in class with the logistic model: let

$$P(t) := \text{population of the Earth in year } 2019 + t \text{ in billions}$$

so that the current population is $P(0) = 7.7$. Assume that the annual rate of population growth remains stead at 1.07%, or $k = 0.0107$. One estimate for the carrying capacity of the Earth is $M = 10$. Using these parameters, use your answers above to write down a function that models the population of the Earth for the next century. Sketch a graph representing this.

Solution. First, let's solve the logistic equation via separation of variables: dividing the stuff on the right over to the left gives

$$\int \frac{dP}{kP(1 - P/M)} = \int dt = t + C.$$

To integrate the quantity on the right, use the method of partial fractions: write

$$\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$$

for some numbers A and B. Clearing denominators, this gives a system of equations

$$\begin{cases} (B - A/M) = 0, \\ A = 1 \end{cases} \quad \text{so} \quad \begin{cases} A = 1, \\ B = 1/M. \end{cases}$$

Therefore

$$t + C = \frac{1}{k}\int \frac{dP}{P(1 - P/M)} = \frac{1}{k}\int \frac{dP}{P} + \frac{1}{k}\int \frac{dP}{M - P} = \frac{1}{k}\left(\log(P) - \log(M - P)\right) = \frac{1}{k}\log\left(\frac{P}{M - P}\right).$$

Solving for P, we have

$$Ae^{kt} = \frac{P}{M - P} = \frac{M}{M - P} - 1 \quad \text{so} \quad P = M\left(1 - \frac{1}{1 + Ae^{kt}}\right)$$

for some constant A.

Now let's solve the logistic differential equation as a linear differential equation: making the substitution $P = 1/z$, the equation becomes

$$-\frac{1}{z^2}\frac{dz}{dt} = \frac{d}{dt}z^{-1} = \frac{dP}{dt} = kP(1 - P/M) = \frac{k}{z}\left(1 - \frac{1}{Mz}\right).$$

Multiplying through by $-z^2$ and rearranging gives

$$z' + kz = \frac{k}{M}$$

as desired. Now this is a linear differential equation with integrating factor $I(t) = e^{\int k\,dt} = e^{kt}$. Thus

$$z = e^{-kt}\int \frac{k}{M}e^{kt}\,dt = \frac{1}{M} + Ce^{-kt}.$$

Putting $z = 1/P$ back, this gives

$$P = \frac{1}{Ce^{-kt} + 1/M}.$$

As different as this answer looks from the first, they are actually the same with $C = A^{-1}/M$.

Let's use this expression to model population again. Here, our parameters are $k = 0.0107$ and $M = 10$. Now, use the initial condition to solve for the constant C above:

$$7.7 = P(0) = \frac{1}{C + 1/10} \quad \text{so} \quad C = \frac{1}{7.7} - \frac{1}{10} = \frac{23}{770}.$$

So our model for the population of the Earth is given by

$$P(t) = \frac{1}{23e^{-0.0107t}/770 + 1/10}.$$

A graph of this function between $t = 0$ and $t = 100$ looks something like the following:
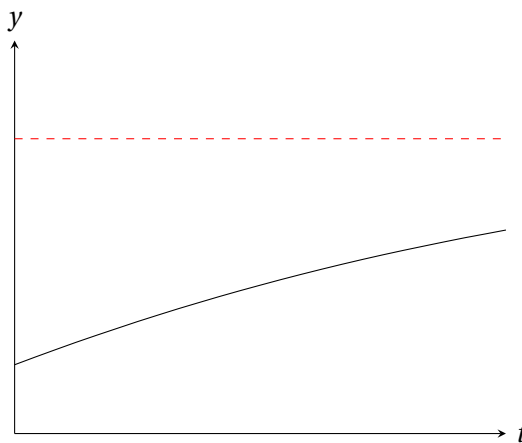
Notice that the curve approaches the red dotted line, positioned at $y = 10$, slower and slower as time goes on, indicating that growth is inhibited by some external factors. In this mode, by 2120 the population is predicted to be approximately 9.07 billion people, rather than the much larger number discussed in class. ∎