HIGHER CYCLE OPERATIONS AND A UNIPOTENT TORELLI THEOREM FOR GRAPHS

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ABSTRACT. Let Γ be a connected bridgeless graph and fix a vertex v of Γ. We introduce a bilinear map $\int : \mathbf{Z}\pi_1(\Gamma, v) \times T\mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Q}$, called the higher cycle pairing, between the group algebra of the fundamental group of (Γ, v) and the tensor algebra on the first homology of Γ. This pairing generalizes the standard inner product on $\mathrm{H}_1(\Gamma, \mathbf{Z})$ and is a combinatorial analogue of Chen's iterated integrals on Riemann surfaces. We use this pairing on the two-step unipotent quotient of the group algebra to prove a Unipotent Torelli Theorem for pointed bridgeless connected graphs: the pair $(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int)$ is a complete invariant for (Γ, v).

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0. INTRODUCTION

Whitney showed in [Whi33] that a finite connected bridgeless graph can be reconstructed with only the knowledge of its set of edges and set of cycles, at least up to two types of ambiguities: first, subgraphs that are dangling to the rest by a single vertex may be misplaced; and, second, components that are joined at two vertices might be joined in a twisted fashion. In modern parlance, Whitney's Theorem says that a graph is determined up to 2-isomorphism by its matroid.

Classical algebraic topological constructions offer a useful reformulation. Let Γ be a finite connected graph without bridges, possibly with loops and multiple edges. Let $\mathrm{C}_1(\Gamma, \mathbf{Z})$ be the free abelian group generated on the edge set of Γ. With a choice of orientation on the edges of Γ, there is a well-defined subgroup $\mathrm{H}_1(\Gamma, \mathbf{Z}) \subseteq \mathrm{C}_1(\Gamma, \mathbf{Z})$, variously referred to as the cycle space of Γ or the first simplicial homology group of Γ, generated by sums of oriented edges making cyclic subgraphs of Γ. Then Whitney's 2-Isomorphism Theorem stated above says that the inclusion map $\mathrm{H}_1(\Gamma, \mathbf{Z}) \hookrightarrow \mathrm{C}_1(\Gamma, \mathbf{Z})$ together with the distinguished generators of $\mathrm{C}_1(\Gamma, \mathbf{Z})$ determine Γ up to 2-isomorphism.

From the point of view of topology, the group $\mathrm{H}_1(\Gamma, \mathbf{Z})$ is the one that is intrinsic to Γ, so one might hope to find a formulation of Whitney's Theorem which only refers to structures defined on $\mathrm{H}_1(\Gamma, \mathbf{Z})$. Many authors [Ger82, Art06, DSSV09, CV10, SW10] approaching the problem from many perspectives have found the following answer: let $(\cdot, \cdot)$ be the natural inner product on $\mathrm{H}_1(\Gamma, \mathbf{Z})$ which takes two cycles in Γ and

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returns their signed intersection number. Then the data of the group $\mathrm{H}_1(\Gamma, \mathbf{Z})$ together with the inner product $(\cdot,\cdot)$ determines Γ up to 2-isomorphism. We, following [Art06, CV10] for instance, refer to this result as a *Torelli Theorem* for graphs, in analogy to the Torelli Theorem for algebraic curves, as the data of the pair $(\mathrm{H}_1(\Gamma, \mathbf{Z}), (\cdot,\cdot))$ is equivalent to the data of the Jacobian of a graph, as defined in [BdlHN97, BF11].

The results discussed so far use homological or abelian invariants to recover a graph up to specified ambiguities. One may ask if these statements may be refined using homotopical or non-abelian invariants. For instance, consider replacing $\mathrm{H}_1(\Gamma, \mathbf{Z})$ with the fundamental group $\pi_1(\Gamma, v)$ of Γ based at some vertex v. Then, correspondingly, $\mathrm{C}_1(\Gamma, \mathbf{Z})$ is replaced by the free group $F\Gamma$ generated on the edge set of Γ, and the inclusion $\mathrm{H}_1(\Gamma, \mathbf{Z}) \hookrightarrow \mathrm{C}_1(\Gamma, \mathbf{Z})$ is replaced by the group homomorphism $\mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma$ which sends a loop in Γ to the word spelled by the oriented edges in it. Covering space theory, for instance, reconstructs Γ from word.

Of course, the map $\mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma$ is rather complicated and contains lots of redundant information. To pare down the situation, linearize by taking group algebras. In fact, going even further, let $J := \ker(\mathbf{Z}\pi_1(\Gamma, v) \to \mathbf{Z})$ and $I := \ker(\mathbf{Z}F\Gamma \to \mathbf{Z})$ be the augmentation ideals of the groups in question and consider the map $\mathsf{word}_\ell\colon \mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1} \to \mathbf{Z}F\Gamma/I^{\ell+1}$ induced on quotients for each $\ell \geq 1$. A classical calculation shows that word_1 recovers the embedding $\mathrm{H}_1(\Gamma, \mathbf{Z}) \hookrightarrow \mathrm{C}_1(\Gamma, \mathbf{Z})$. Thus this sequence of quotients interpolates between homology and the entire fundamental group.

The case $\ell = 2$ is the natural place to start, when the groups involved are, in some sense, almost abelian. The question is then the following: to what extent does the data of $\mathsf{word}_2\colon \mathbf{Z}\pi_1(\Gamma, v)/J^3 \to \mathbf{Z}F\Gamma/I^3$ determine the graph Γ? Somewhat remarkably, it turns out that this tiny bit of extra non-abelian structure is enough to reconstruct Γ completely—in fact, one can even recover the vertex v! More precisely, the second named author established in [Kat14] the following non-abelian analogue of Whitney's 2-Isomorphism Theorem: the homomorphism word_2 together with the distinguished generators of $\mathbf{Z}F\Gamma/I^3$ determine the pointed graph (Γ, v) up to isomorphism.

This paper completes the circle of ideas sketched above with a non-abelian counterpart to the Torelli Theorem for graphs. To formulate the result, we introduce the *higher cycle pairing*: this is a bilinear pairing

$$\int\colon \mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1} \times T_\ell \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Q}$$

between quotients $\mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1}$ of the group algebra of the fundamental group and truncations of the tensor algebra $T_\ell \mathrm{H}_1(\Gamma, \mathbf{Z}) := \bigoplus_{k=0}^{\ell} \mathrm{H}_1(\Gamma, \mathbf{Z})^{\otimes k}$ on homology, and is defined for every $\ell \geq 1$, including $\ell = \infty$. Note that $T_\ell \mathrm{H}_1(\Gamma, \mathbf{Z})$ is the associated graded algebra of $\mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1}$ with respect to the filtration induced by J, recovered as the nilradical of the group algebra. For this reason, the higher cycle pairing can be viewed as a structure associated solely with $\mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1}$. Our main result is the following Torelli counterpart to the non-abelian Whitney Theorem:

Unipotent Torelli Theorem. — *Let (Γ, v) be a connected bridgeless pointed graph. Then the pair $(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int)$ completely determines (Γ, v).*

This is proved in §5. See (5.1) for a precise statement. We refer to this as the *Unipotent Torelli Theorem* for graphs as the pair $(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int)$ is a combinatorial analogue of the lowest weight component of the mixed Hodge structure on the unipotent fundamental group of an algebraic curve.

Beyond this paper, the higher cycle pairing and Unipotent Torelli Theorem lead to a variety of questions and applications. Our proof of the Theorem actually shows that, for a pair of pointed graphs (Γ, v) and (Γ', v'),

$$\mathrm{Isom}\big((\Gamma, v), (\Gamma', v')\big) = \mathrm{Isom}\big((\mathbf{Z}\pi_1(\Gamma, v)/J^3, \textstyle\int), (\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \textstyle\int)\big)$$

where the left hand side is the set of isomorphisms between pointed graphs and the right hand side is the set of isomorphisms between cycle pairing algebras. Because of this, pointed graphs might be called *anabelian* combinatorial objects, being determined by their fundamental groups. Hence the Unipotent Torelli Theorem might be viewed as a result in *anabelian combinatorics*, in analogy with Grothendieck's anabelian program in algebraic geometry [Gro97]. Beyond pointed graphs, what other combinatorial objects might be anabelian? Regular matroids that are not necessarily graphic appear to be a place to start. Indeed, Su and Wagner show in [SW10] that the Torelli Theorem as in (4.1) holds more generally for regular matroids. Is there a unipotent enrichment of general regular matroids which carries an analogue of a fundamental group and higher cycle pairing, with which a generalization of the Unipotent Torelli Theorem can be obtained?

All of our constructions can be applied to metric graphs and tropical curves, and we expect many of our results to have analogues there. For instance, an analogue of the Unipotent Torelli Theorem can be formulated and a proof might be obtained by using approximation techniques of [BF11]. The higher cycle pairing is also closely related to Jacobians and Abel–Jacobi theory of graphs and tropical curves [BdlHN97, MZ08, BF11]; see especially the survey of [BJ16]. In particular, the higher cycle pairing and the constructions in §6 suggest combinatorial and tropical analogues of unipotent Albanese manifolds [Hai87d, Kim09], intermediate Jacobians and refined Abel–Jacobi theory [Hai02, §10–11].

Outside of combinatorics, we have three applications in mind. First, in joint work [KL] between the second named author and Daniel Litt, it is shown that the higher cycle pairing mediates among the various theories of iterated p-adic integration on curves, such as those developed by Vologodsky [Vol03] and Berkovich [Ber07], generalizing the manner in which the tropical Abel–Jacobi map transforms between Berkovich–Coleman and abelian integration [KRZB16, §3.5.2]. This is related to work of Besser and Zerbes [BZ17], and has applications to the non-abelian Chabauty methods of Kim [Kim05, Kim12]. See [KRZB18, §6.5] for a related discussion. Second, the higher cycle pairing arises in the asymptotics of period integrals for variations of Hodge structures associated with truncated fundamental group algebras on a semistable family of curves, analogous the cohomological case as explained in [Gri70, Proposition 13.3] and [Sch73, Theorem 6.6']. Finally, the Unipotent Torelli Theorem provides a crucial step toward a unipotent Torelli-type theorem for stable curves, refining that of [CV11], as an object closely related to the cycle pairing algebra appears in the weight zero part of the mixed Hodge structure on the truncated fundamental group of such a curve.

Many of the constructions in this paper are inspired by Hodge theory. The higher cycle pairing is a combinatorial analogue of Chen's theory of iterated integrals [Che77]. Hain used this theory in [Hai87a, Hai87b], generalizing work of Morgan [Mor78] and others, to put a mixed Hodge structure on truncations of the fundamental group algebra of complex algebraic varieties. See also [Hai87c] and [PS08, Chapter 9]. The pair $(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int)$ is a combinatorial shadow of this mixed Hodge structure. Later, combined with work [Car80, Car87] of Carlson on extensions of mixed Hodge structures, Hain [Hai87c] and Pulte [Pul88] used the mixed Hodge structure on $\mathbf{Z}\pi_1(X, x)/J^3$ to prove a pointed Torelli Theorem for pointed complex algebraic curves (X, x).

Outline. We conclude the Introduction with a few words on the organization of the article. But before that, a reading note: for the reader who wants a beeline to the proof of the main result, we suggest skipping directly to (3.3) for a quick summary of the higher cycle pairing on a graph, after which §5 may be read, with reference to preceding sections as necessary.

In §1, we review basic facts about group algebras and collect some useful computations.

In §2, we construct the higher cycle pairing and establish its basic properties. In particular, we formulate and prove in (2.13) a duality result for the higher cycle pairing.

In §3, we collect general graph-theoretic discussions and constructions. In particular, we summarize the construction of the higher cycle pairing for graphs in (3.3), discuss the context around our main result more precisely, and discuss the crucial relation of concyclicity amongst edges in (3.9).

In §4, we present a proof of the Torelli Theorem, essentially following [CV10]. Importantly, we extract a technical Lemma (4.5), which will feature again in the proof of the Unipotent Torelli Theorem.

In §5, we formulate and prove the Unipotent Torelli Theorem (5.1). See (5.2) for a detailed overview.

In §6, we end by developing an extension theory, inspired by the work [Car80] of Carlson, associated with nondegenerate orthographized groups. Combined with the Unipotent Torelli Theorem, we use this in (6.13) to put coordinates on the set of isomorphism classes of pointed graphs in which the underlying graphs are 2-isomorphic. This is done via a combinatorial analogue of harmonic volumes (6.10).

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1. Group Algebras

This Section is a brief on group algebras. Throughout, G is a group and R is a commutative ring.

(**1.1**) The *group algebra* RG of G with coefficients in R is the associative unital R-algebra with underlying R-module

$$RG = \bigoplus_{g\in G} Re_g$$

and multiplication determined on basis elements as $r_1 e_{g_1} \cdot r_2 e_{g_2} := r_1 r_2 e_{g_1 g_2}$ for all $r_1, r_2 \in R$ and $g_1, g_2 \in G$. The unit of RG is the element e_1, where $1 \in G$ the identity of G. We write g for e_g and confuse 1 with the unit of RG, G, and the ring R.

(**1.2**) **Hopf Structure.** The group algebra RG is naturally a Hopf algebra: the coproduct $\Delta\colon RG \to RG\otimes RG$ is $g \mapsto g\otimes g$; the augmentation homomorphism $\epsilon\colon RG \to R$ is $g \mapsto 1$; and the antipode $\mathsf{S}\colon RG \to RG$ is $g \mapsto g^{-1}$. Dualizing the Hopf structure on RG yields a Hopf structure on $RG^* := \mathrm{Hom}_R(RG, R)$.

(**1.3**) Augmentation Ideal The kernel $J := \ker(\epsilon\colon RG \to R)$ of the augmentation homomorphism is called the *augmentation ideal*. An element of J is of the form $\sum_{g\in G\setminus\{1\}} a_g g$ with $\sum_{g\in G\setminus\{1\}} a_g = 0$. So

$$\sum_{g\in G\setminus\{1\}} a_g g = \sum_{g\in G\setminus\{1\}} a_g(g-1) + \sum_{g\in G\setminus\{1\}} a_g = \sum_{g\in G\setminus\{1\}} a_g(g-1),$$

showing that J is a free R-module with basis $\{(g-1) \mid g \in G\setminus\{1\}\}$. Since both R and J are free R-modules, the defining exact sequence $0 \to J \to RG \xrightarrow{\epsilon} R \to 0$ is split and we have $RG \cong R \oplus J$ as R-modules.

(**1.4**) **Abelianization.** Consider the R-module J/J^2. Let $g_1, g_2 \in G$ and consider the class of $(g_1g_2 - 1)$ in J/J^2. Moving to RG/J^2 for a moment, we can calculate

$$(g_1g_2-1) = \big(((g_1-1)+1)((g_2-1)+1)-1\big) = (g_1-1)(g_2-1)+(g_1-1)+(g_2-1) = (g_1-1)+(g_2-1)$$

as $(g_1-1)(g_2-1) \in J^2$. Writing G^{ab} for the abelianization of G, this calculation essentially shows that the map $R\otimes_{\mathbf{Z}} G^{\mathrm{ab}} \to J/J^2 \colon g \mapsto (g-1)$ is an isomorphism of R-modules.

(**1.5**) We perform two computations in RG/J^3 which we shall use below. Given $g_1, g_2 \in G$,

$$0 = (g_1-1)(g_1^{-1}-1)(g_2-1) = -\big((g_1-1)+(g_1^{-1}-1)\big)(g_2-1)$$

in RG/J^3, so rearranging gives the identity

$$(g_1^{-1}-1)(g_2-1) = -(g_1-1)(g_2-1) \in RG/J^3.$$

Similarly, for any $g \in G$, we have $-(g-1)^2 = (g-1)(g^{-1}-1) = -(g-1)-(g^{-1}-1)$ thereby giving the identity

$$(g^{-1}-1) = (g-1)(g-1)-(g-1) \in RG/J^3.$$

(**1.6**) **Filtration and Grading.** In general, RG is has a descending filtration $RG = J^0 \supseteq J^1 \supseteq J^2 \supseteq \cdots$ by powers of the augmentation ideal. The associated graded algebra with respect to this filtration is

$$\mathrm{gr}_J(RG) := \bigoplus_{i=0}^{\infty} J^i/J^{i+1}.$$

Identifying J/J^2 with $R\otimes_{\mathbf{Z}} G^{\mathrm{ab}}$ as above, there is always a natural map

$$(R\otimes_{\mathbf{Z}} G^{\mathrm{ab}})^{\otimes i} \to J^i/J^{i+1} \colon (g_1-1)\otimes\cdots\otimes(g_i-1) \mapsto (g_1-1)\cdots(g_i-1)$$

for each $i \geq 0$ and where $g_1, \dots, g_i \in G^{\mathrm{ab}}$. In the special case that G is a free group on a set of generators E, this map is an isomorphism and we get that the associated graded algebra of RG

$$\mathrm{gr}_J(RG) \cong \bigoplus_{i=0}^{\infty} (R \otimes_{\mathbf{Z}} G^{\mathrm{ab}})^{\otimes i} =: T(G^{\mathrm{ab}})$$

is isomorphic to the tensor algebra of the free R-module on the set E.

2. Higher Cycle Pairing

In this Section, we introduce the higher cycle pairing. This is a bilinear pairing between the group algebra of a free group and the tensor algebra on the abelianization of the free group. In a sense made precise by (2.5), the higher cycle pairing is a non-abelian refinement of the standard inner product on the tensor algebra itself; in fact, (2.8) shows that the usual inner product can be recovered from the higher cycle pairing upon passing to associated graded algebras. After defining the higher cycle pairing in (2.1, 2.2), we establish some basic properties in (2.3–2.9). We then generalize the construction to a wider class of groups, referred to as *orthographized groups*, in (2.10–2.12). Finally, we prove in (2.13) that the higher cycle pairing establishes a duality between the group algebra of a nondegenerate orthographized group and the tensor algebra associated with its abelianization.

Our constructions are inspired by Chen's theory of iterated line integrals [Che77] and their applications to the construction of a mixed Hodge structure on the fundamental group of an algebraic variety [Hai87c]. After completing this article, we became aware of [BKP09] in which apparently related definitions were made. It may be of interest to compare the constructions.

(**2.1**) Throughout this Section, let F be a free group equipped with a set E of generators. Let $C := F^{\mathrm{ab}}$ be the abelianization of F and write $[\cdot]\colon F \to C$ for the quotient map. Fix a bilinear pairing $(\cdot,\cdot)\colon C \times C \to \mathbf{R}$. Let $TC := \bigoplus_{k \geq 0} C^{\otimes k}$ be the tensor algebra over $\mathbf{Z}$ of C. The *higher cycle pairing* is a bilinear map

$$\int \colon \mathbf{Z}F \times TC \to \mathbf{R}$$

defined inductively as follows: For $1 \in \mathbf{Z} \subset TC$ and any $\alpha \in F$, set

$$\int_\alpha 1 := \epsilon(1),$$

where $\epsilon\colon \mathbf{Z}F \to \mathbf{Z}$ is the augmentation map (1.2). For $k \geq 1$, any $\omega_1 \otimes \cdots \otimes \omega_k \in C^{\otimes k}$, and $e \in E$, set

(**2.1.1**)
$$\int_e \omega_1 \cdots \omega_k := \frac{1}{k!}(\omega_1, [e]) \cdots (\omega_k, [e]).$$

For $\alpha' \in F$ arbitrary, write $\alpha' = \alpha e$ for some $\alpha \in F$ and $e \in E$ and set

(**2.1.2**)
$$\int_{\alpha e} \omega_1 \cdots \omega_k := \sum_{i=0}^{k} \left(\int_\alpha \omega_1 \cdots \omega_i \right) \left(\int_e \omega_{i+1} \cdots \omega_k \right),$$

with the convention that the integral with empty integrand is 1.

(**2.2**) **Lemma. —** *For any $\alpha \in F$, $e \in E$ and $\omega_1, \dots, \omega_k \in C$,*

$$\int_{\alpha e e^{-1}} \omega_1 \cdots \omega_k = \int_\alpha \omega_1 \cdots \omega_k.$$

In particular, the higher cycle pairing is well-defined.

Proof. Applying (2.1.1) twice and rearranging, the left hand quantity becomes

$$\int_{\alpha e e^{-1}} \omega_1 \cdots \omega_k = \sum_{j=0}^{k} \left(\int_\alpha \omega_1 \cdots \omega_j \right) \sum_{i=j}^{k} \left(\int_e \omega_{j+1} \cdots \omega_i \right) \left(\int_{e^{-1}} \omega_{i+1} \cdots \omega_k \right).$$

The inner summand vanishes for $j \neq k$:

$$\sum_{i=j}^{k}\left(\int_e \omega_{j+1}\cdots\omega_i\right)\left(\int_{e^{-1}} \omega_{i+1}\cdots\omega_k\right) = \sum_{i=j}^{k} \frac{(\omega_{j+1},[e])\cdots(\omega_i,[e])(\omega_{i+1},[e^{-1}])\cdots(\omega_k,[e^{-1}])}{(i-j)!(k-i)!}$$
$$= \frac{\Omega}{(k-j)!}\sum_{i=0}^{k-j}(-1)^{k-i}\binom{k-j}{i} = 0,$$

where $\Omega := (\omega_{j+1},[e])\cdots(\omega_k,[e])$ and where we have used $[e^{-1}] = -[e] \in C$. ■

We give a combinatorial formula for the cycle pairing. This is a discrete analogue of the parameterization of geometric iterated integrals over the time ordered simplex [Hai87c, Definition 1.1]. For any positive integer n, write $[n] := \{1,\ldots,n\}$. For k and r positive integers, let

$$\Delta(k,r) := \left\{ f\colon [k] \to [r] \;\middle|\; f(1) \le \cdots \le f(k) \right\}$$

be the set of all weakly increasing functions from $[k]$ to $[r]$. Equivalently, setting $n_i := \# f^{-1}(i)$ for $i = 1,\ldots,r$, an element $f \in \Delta(k,r)$ may be represented as the sequence $(n_1,\ldots,n_r)$. Note $n_1 + \cdots + n_r = k$. With this notation, write $f! := n_1!n_2!\cdots n_r!$.

(**2.3**) **Lemma.** — *Let $\alpha := e_1\cdots e_r \in F$ and $\omega_1,\ldots,\omega_k \in C$. Then*

$$\int_\alpha \omega_1\cdots\omega_k = \sum_{f\in\Delta(k,r)} \frac{1}{f!}(\omega_1,[e_{f(1)}])\cdots(\omega_k,[e_{f(k)}]).$$

Proof. Induct on the length r of α, with $r = 1$ being the definition (2.1.2). When $r > 1$,

$$\int_{e_1\cdots e_r} \omega_1\cdots\omega_k = \sum_{i=0}^{k}\left(\sum_{g\in\Delta(i,r-1)} \frac{(\omega_1,[e_{g(1)}])\cdots(\omega_i,[e_{g(i)}])}{g!}\right)\frac{(\omega_{i+1},[e_r])\cdots(\omega_k,[e_r])}{(k-i)!}$$
$$= \sum_{i=0}^{k} \sum_{\substack{f\in\Delta(k,r)\\ f(i+1)=\cdots=f(k)=r}} \frac{1}{f!}(\omega_1,[e_{f(1)}])\cdots(\omega_k,[e_{f(k)}])$$
$$= \sum_{f\in\Delta(k,r)} \frac{1}{f!}(\omega_1,[e_{f(1)}])\cdots(\omega_k,[e_{f(k)}]),$$

completing the induction. ■

The next result gives the tensor algebra TC, viewed as the set of iterated integrals, the structure of a Hopf algebra. See also (2.13). For positive integers k and ℓ, define the set $\mathrm{Sh}(k,\ell)$ of (k,ℓ)-*shuffles* to be the following subset of the symmetric group $S_{k+\ell}$ on $k+\ell$ symbols:

$$\mathrm{Sh}(k,\ell) := \left\{ \sigma \in S_{k+\ell} \;\middle|\; \sigma^{-1}(1) \le \cdots \le \sigma^{-1}(k) \text{ and } \sigma^{-1}(k+1) \le \cdots \le \sigma^{-1}(k+\ell) \right\}.$$

(**2.4**) **Proposition.** — *Let $\alpha \in F$ and $\omega_1,\ldots,\omega_k \in C$. Then we have the following formulae:*

– *(Product) For any $\omega_{k+1},\ldots,\omega_{k+\ell} \in C$,*

$$\int_\alpha \omega_1\cdots\omega_k \int_\alpha \omega_{k+1}\cdots\omega_{k+\ell} = \sum_{\sigma\in\mathrm{Sh}(k,\ell)} \int_\alpha \omega_{\sigma(1)}\cdots\omega_{\sigma(k+\ell)}.$$

– *(Coproduct) For any $\beta \in F$,*

$$\int_{\alpha\beta} \omega_1\cdots\omega_k = \sum_{i=0}^{k} \int_\alpha \omega_1\cdots\omega_i \int_\beta \omega_{i+1}\cdots\omega_k.$$

– *(Antipode)*

$$\int_{\alpha^{-1}} \omega_1\cdots\omega_k = (-1)^k \int_\alpha \omega_k\cdots\omega_1.$$

Proof. See, for example, [Hai87c, 2.9,2.11,2.12]. ■

The next three formulae are obtained by iterating those proven above. These formulae, especially the Symmetrization Formula (2.5) and the Conjugation Formula (2.7), highlight the non-abelian nature of the higher cycle pairing.

(**2.5**) **Symmetrization Formula. —** *Let $\alpha \in F$ and $\omega_1, \dots, \omega_k \in C$. Then*

$$\sum_{\sigma \in S_k} \int_\alpha \omega_{\sigma(1)}\omega_{\sigma(2)} \cdots \omega_{\sigma(k)} = \int_\alpha \omega_1 \int_\alpha \omega_2 \cdots \int_\alpha \omega_k.$$

(**2.6**) **Iterated Coproduct Formula. —** *Let $\alpha_1, \dots, \alpha_r \in F$ and $\omega_1, \dots, \omega_k \in C$. Then*

$$\int_{\alpha_1 \cdots \alpha_r} \omega_1 \cdots \omega_k = \sum_{\substack{g \in \Delta(k,r) \\ g=(n_1,\dots,n_r)}} \Big(\int_{\alpha_1} \omega_1 \cdots \omega_{n_1}\Big)\Big(\int_{\alpha_2} \omega_{n_1+1} \cdots \omega_{n_1+n_2}\Big) \cdots \Big(\int_{\alpha_r} \omega_{n_1+\dots+n_{r-1}+1} \cdots \omega_k.\Big)$$

(**2.7**) **Conjugation Formula. —** *Let $\alpha, \beta \in F$ and $\omega_1, \omega_2 \in C$. Then*

$$\int_{\beta\alpha\beta^{-1}} \omega_1\omega_2 = \int_\alpha \omega_1\omega_2 + \Big(\int_\beta \omega_1 \int_\alpha \omega_2 - \int_\alpha \omega_1 \int_\beta \omega_2\Big).$$

Proof. We prove the Conjugation Formula. It suffices to compute the image of $\beta\alpha\beta^{-1}$ in $\mathbf{Z}F/I^3$. Since $(\alpha-1)(\beta^{-1}-1) = -(\alpha-1)(\beta-1)$ in $\mathbf{Z}F/I^3$ as in (1.5), we have

$$\beta\alpha\beta^{-1} - 1 = (\alpha-1) + \big((\beta-1)(\alpha-1) - (\alpha-1)(\beta-1)\big).$$

The formula now follows directly from (2.8.1). ∎

For positive integers k and r, denote by

$$\Delta^+(r-1,k-1) := \big\{ f \in \Delta(r-1,k-1) \;\big|\; f \text{ strictly increasing} \big\}.$$

Note that $\Delta^+(0,k-1)$ has the empty function $f\colon \varnothing \to [k-1]$ as its unique element. By convention, for $f \in \Delta^+(r-1,k-1)$, set $f(0) := 0$ and $f(r) := k$.

(**2.8**) **Nilpotence Property. —** *Let $\alpha_1, \dots, \alpha_r \in F$ and $\omega_1, \dots, \omega_k \in C$. Then*

$$\int_{(\alpha_1-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_k = \sum_{f \in \Delta^+(r-1,k-1)} \prod_{i=1}^r \int_{\alpha_i} \omega_{f(i-1)+1} \cdots \omega_{f(i)}.$$

In particular,

(**2.8.1**)
$$\int_{(\alpha_1-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_k = \begin{cases} 0 & r > k, \\ (\omega_1,\alpha_1)\cdots(\omega_k,\alpha_k) & r = k. \end{cases}$$

Proof. Applying (2.6) to the left hand side and using inclusion-exclusion, we have

$$\int_{(\alpha_1-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_k = \sum_{g \in \Delta(k,r) \text{ surjective}} \Big(\int_{\alpha_1} \omega_1 \cdots \omega_{\#g^{-1}([1])}\Big) \cdots \Big(\int_{\alpha_r} \omega_{\#g^{-1}([r-1])+1} \cdots \omega_k\Big)$$

Now note that $g \in \Delta(k,r)$ is surjective if and only if the function $f\colon [r-1] \to [k-1]$ given by $f(i) := \#g^{-1}([i])$ is strictly increasing, i.e. $f \in \Delta^+(r-1,k-1)$. In fact, the map $g \mapsto f$ induces a bijection between the subset of surjective elements of $\Delta(k,r)$ and $\Delta^+(r-1,k-1)$. Reindexing the sum above gives the result. ∎

(**2.9**) **Corollary. —** *Let $I := \ker(\mathbf{Z}F \to \mathbf{Z})$ be the augmentation ideal. For any $\alpha \in I^{\ell+1}$, any $k \leq \ell$, and any $\omega_1, \dots, \omega_k \in C$,*

$$\int_\alpha \omega_1 \cdots \omega_k = 0.$$

Thus the higher cycle pairing descends to a map

$$\int \colon \mathbf{Z}F/I^{\ell+1} \times T_\ell C \to \mathbf{R}$$

where $T_\ell C := \bigoplus_{k=0}^{\ell} C^{\otimes k}$ is the ℓ^{th} truncation of the tensor algebra.

Proof. Any $\alpha \in I^{\ell+1}$ can be written in the form

$$\alpha = \sum_{r>\ell} \sum_{i_1,\dots,i_r} c_{i_1\dots i_r}(e_{i_1}-1)\cdots(e_{i_r}-1)$$

where all but finitely many of the $c_{i_1\dots i_r} \in \mathbf{Z}$ are 0. It follows from (2.8.1) that $\int_\alpha$ is identically zero on $C^{\otimes k}$ for all $k \leq \ell$ since $\Delta^+(k,r) = \varnothing$ for all $r > \ell$. ∎

In the next paragraphs, we extend the higher cycle pairing to any group which can be coordinatized via some homomorphism into some free group. Since subgroups of a free group are themselves free, this may seem to be an empty extension; indeed, in principle, the theory developed thus far is enough to deal with our applications. However, the advantage of the language below is that many free groups in nature—fundamental groups of finite graphs, for example—are not equipped with a canonical set of generators, but do admit a natural embedding into some other free group which does.

(**2.10**) **Orthographized Groups.** An *orthographization* of a group G consists of the data of

- a free group F equipped with a set E of generators;
- a homomorphism $\mathsf{word}\colon G \to F$; and
- a bilinear pairing $(\cdot,\cdot)\colon C \times C \to \mathbf{R}$, where $C := F^{\mathrm{ab}}$.

A group G together with an orthographization is called an *orthographized group*. An orthographized group is often denoted simply by $\mathsf{word}\colon G \to F$, leaving the set E of generators and pairing $(\cdot,\cdot)$ implicit.

A *morphism* between two orthographized groups

$$(\varphi,\Phi)\colon \big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big) \to \big(E', \mathsf{word}'\colon G' \to F', (\cdot,\cdot)'\big)$$

consists of the data of

- a set map $\Phi\colon E \to E'$ between the generators of F and F'; and
- a group homomorphism $\varphi\colon G \to G'$

such that, denoting again by $\Phi\colon F \to F'$ the induced homomorphism on free groups, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ {\scriptstyle \mathsf{word}}\downarrow & & \downarrow{\scriptstyle \mathsf{word}'} \\ F & \xrightarrow{\Phi} & F' \end{array}$$

commutes, and $([x_1],[x_2]) = ([\Phi(x_1)],[\Phi(x_2)])'$ for all $x_1, x_2 \in F$.

(**2.11**) **Cycle Pairing Algebra.** Let $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ be an orthographized group. Then the group algebra functor induces a Hopf algebra homomorphism

$$\mathsf{word}\colon \mathbf{Z}G \to \mathbf{Z}F.$$

In particular, word sends the augmentation ideal $J := \ker(\mathbf{Z}G \to \mathbf{Z})$ of $\mathbf{Z}G$ into the augmentation ideal $I := \ker(\mathbf{Z}F \to \mathbf{Z})$ of $\mathbf{Z}F$. Let $\mathrm{gr}\,\mathsf{word}\colon \mathrm{gr}_J \mathbf{Z}G \to \mathrm{gr}_I \mathbf{Z}F$ be the induced map on associated graded algebras; note that, since F is free, (1.6) gives $\mathrm{gr}_J(\mathbf{Z}F) \cong TC$. Now pullback the construction of (2.1)

$$\int \colon \mathbf{Z}G \times \mathrm{gr}_J(\mathbf{Z}G) \xrightarrow{\mathsf{word}\times\mathrm{gr}\,\mathsf{word}} \mathbf{Z}F \times TC \xrightarrow{\int} \mathbf{R}$$

to obtain a higher cycle pairing between the group algebra of G and its associated graded algebra. Similarly, for each $\ell \geq 0$, (2.8) gives a higher cycle pairing

$$\int \colon \mathbf{Z}G/J^{\ell+1} \times \bigoplus_{k=0}^{\ell} J^k/J^{k+1} \to \mathbf{Z}F/I^{\ell+1} \times T_\ell C \to \mathbf{R}$$

between the ℓ^{th} truncation of the group algebra and the first ℓ pieces of its associated graded.

We refer to pairs $(\mathbf{Z}G, \int)$ constructed in this fashion as *cycle pairing algebras*. Let $(\mathbf{Z}G, \int)$ and $(\mathbf{Z}G', \int)$ be two cycle pairing algebras. A *morphism*

$$\varphi\colon \big(\mathbf{Z}G, \textstyle\int\big) \to \big(\mathbf{Z}G', \textstyle\int\big)$$

between cycle pairing algebras consists of a $\mathbf{Z}$-algebra morphism $\varphi\colon \mathbf{Z}G \to \mathbf{Z}G'$ which preserve the higher cycle pairing in that

$$\int_{\varphi(g)} \mathrm{gr}_J(\varphi)(\omega) = \int_g \omega$$

for all $g \in \mathbf{Z}G$ and $\omega \in \mathrm{gr}_J(\mathbf{Z}G)$, where we write $\mathrm{gr}_J(\varphi)\colon \mathrm{gr}_J(\mathbf{Z}G) \to \mathrm{gr}_{J'}(\mathbf{Z}G')$ for the induced morphism on the associated graded algebras. Similarly, pairs $(\mathbf{Z}G/J^{\ell+1}, \int)$ are referred to as as ℓ^{th} *truncated cycle pairing algebras*. We write $\mathsf{word}_\ell\colon \mathbf{Z}G/J^{\ell+1} \to \mathbf{Z}F/I^{\ell+1}$ for the morphism to the corresponding truncated group algebra. A *morphism* of ℓ^{th} truncated cycle pairing algebras is defined similarly.

(**2.12**) **Nondegeneracy.** An orthographized group $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ is called *nondegenerate* if

- the homomorphism $\mathsf{word}\colon G \to F$ is injective; and
- the restriction $(\cdot,\cdot)\colon G^{\mathrm{ab}} \times G^{\mathrm{ab}} \to \mathbf{R}$ of the form to G^{ab} is nondegenerate.

Upon identifying G with a subgroup of F via word, the Nielsen–Schreier Theorem [Sti93, §2.2.4] implies that G is abstractly a free group. Thus, by (1.6), $\mathrm{gr}_J(\mathbf{Z}G) \cong T(G^{\mathrm{ab}})$, so the construction of (2.11) shows that higher cycle pairing is naturally a bilinear map

$$\int\colon \mathbf{Z}G \times T(G^{\mathrm{ab}}) \to \mathbf{R}.$$

We now show that cycle pairing algebras associated with nondegenerate orthographized groups have a duality property. Compare with Chen's de Rham Theorem [Che77]. See also [Hai87c, Theorem 4.1]. In the statement of the Duality Theorem, the Hopf algebra structure $\mathbf{R}G/J^{\ell+1}$ refers to that discussed in (1.2); the Hopf structure on $T_\ell(G^{\mathrm{ab}})$ comes from identifying this with the space of pairings of length at most ℓ and using the formulae in (2.4).

(**2.13**) **Duality Theorem.** — *Let* $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ *be a nondegenerate orthographized group* G*. For each* $\ell \geq 0$*, the maps*

$$\begin{aligned} T_\ell(G^{\mathrm{ab}}) \otimes_{\mathbf{Z}} \mathbf{R} &\to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}G/J^{\ell+1}, \mathbf{R}), & \mathbf{R}G/J^{\ell+1} &\to \mathrm{Hom}_{\mathbf{Z}}(T_\ell(G^{\mathrm{ab}}), \mathbf{R}), \\ \omega &\mapsto \int \omega, & \gamma &\mapsto \int_\gamma, \end{aligned} \tag{2.13.1}$$

are isomorphisms of Hopf algebras.

Proof. The Nilpotence Property, specifically, the first case of (2.8.1), says that both maps in (2.13.1) are maps of filtered $\mathbf{R}$-algebras. Moreover, the second case of (2.8.1) implies the map induced by, say, the first map in (2.13.1) on the degree i component of the associated graded algebras is

$$(G^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{R})^{\otimes i} \to (G^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{R})^{*,\otimes i}, \qquad \omega_1 \otimes \cdots \otimes \omega_i \mapsto (\omega_1, \cdot) \cdots (\omega_i, \cdot).$$

Since $(\cdot,\cdot)$ is nondegenerate, the maps on associated graded algebras are isomorphisms, from which we conclude that the original maps (2.13.1) are isomorphisms of algebras. Finally, that the Hopf algebra structures are identified under these maps comes from a direct comparison between the formulae in (2.4) with those for the dual Hopf structure on group algebras as described in (1.2). ∎

3. Graphs

This Section is a summary of graph theoretic constructions that will be used in the remainder. The essential paragraphs in this Section are the following: in (3.2), we define the orthographized group associated with a pointed graph, thereby constructing a higher cycle pairing for graphs; note that we also summarize the construction in (3.3), so it may be useful to skip directly there; in (3.8) and (3.9), we discuss the important relation of concyclicity of edges of a graph; finally, we collect most of our notational conventions in (3.4).

Throughout, a *graph* is a finite connected graph, possibly with loops and multiple edges. For topological constructions, we think of graphs as finite one-dimensional Δ-complexes. Given a graph Γ, we write Γ_0 and Γ_1 for the set of vertices and edges of Γ, respectively. We will often assume our graphs are *bridgeless*, that is, there does not exist an edge whose deletion disconnects the graph. For the purposes of constructing algebraic invariants, all graphs are equipped with some orientation; the particular choice of orientation will not be important in what we do, so we will adjust orientations as convenient. Given an edge e, write e^+ and e^- for the head and tail of e in the given orientation.

(**3.1**) **(Co)homology of Graphs.** Here we set our notations and conventions for the (co)homology theory of graphs. Let R be a ring. For $i = 0, 1$, let $\mathrm{C}_i(\Gamma, R)$ be the free R-module generated on the set Γ_i. Let $\partial\colon \mathrm{C}_1(\Gamma, R) \to \mathrm{C}_0(\Gamma, R)$ be the simplicial boundary map defined by $e \mapsto e^+ - e^-$. The homology groups of Γ with coefficients in R are $\mathrm{H}_0(\Gamma, R) := \mathrm{C}_0(\Gamma, R)/\partial\mathrm{C}_1(\Gamma, R)$ and $\mathrm{H}_1(\Gamma, R) := \ker(\partial\colon \mathrm{C}_1(\Gamma, R) \to \mathrm{C}_0(\Gamma, R))$. The elements of $\mathrm{H}_1(\Gamma, R)$ are referred to as *cycles* of Γ.

The free R-modules $\mathrm{C}_i(\Gamma, R)$ come with canonical nondegenerate bilinear forms, defined by

$$(\cdot,\cdot)\colon \mathrm{C}_i(\Gamma, R) \times \mathrm{C}_i(\Gamma, R) \to R \qquad (x, y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Since $\mathrm{H}_1(\Gamma, R) \subseteq \mathrm{C}_1(\Gamma, R)$, the first homology group inherits a bilinear form, also denoted by $(\cdot,\cdot)$. When R is a subring of $\mathbf{R}$, this is a restriction of the standard Euclidean pairing and is positive-definite.

Dualizing the constructions above, we have cochain groups $\mathrm{C}^i(\Gamma, R) := \mathrm{Hom}_R(\mathrm{C}_i(\Gamma, R), R)$, $i = 0, 1$, and a coboundary map $\delta\colon \mathrm{C}^0(\Gamma, R) \to \mathrm{C}^1(\Gamma, R)$ given by $\alpha \mapsto (e \mapsto \alpha(e^+) - \alpha(e^-))$. Hence we obtain cohomology groups $\mathrm{H}^0(\Gamma, R) := \ker(\delta\colon \mathrm{C}^0(\Gamma, R) \to \mathrm{C}^1(\Gamma, R))$ and $\mathrm{H}^1(\Gamma, R) := \mathrm{C}^1(\Gamma, R)/\delta\mathrm{C}^0(\Gamma, R)$. Write

$$g := g(\Gamma) := \mathrm{rank}(\mathrm{H}_1(\Gamma, \mathbf{Z})) = \mathrm{rank}(\mathrm{H}^1(\Gamma, \mathbf{Z})).$$

(**3.2**) **Orthographized Group of a Pointed Graph.** Let Γ be a graph and let $v \in \Gamma_0$ be a fixed vertex. The fundamental group $\pi_1(\Gamma, v)$ of (Γ, v) has a canonical orthographization: Let $\mathrm{Bouq}(\Gamma)$ be the bouquet of $\#\Gamma_1$ circles obtained from Γ by identifying all vertices of Γ to a single vertex $\star$. The fundamental group

$$F\Gamma := \pi_1(\mathrm{Bouq}(\Gamma), \star) \cong \big(\text{free group on } \Gamma_1\big)$$

of $\mathrm{Bouq}(\Gamma)$ is canonically the free group generated by the edge set Γ_1 of Γ; in particular, its abelianization

$$F\Gamma^{\mathrm{ab}} \cong \mathrm{H}_1(\mathrm{Bouq}(\Gamma), \mathbf{Z}) \cong \mathrm{C}_1(\Gamma, \mathbf{Z})$$

is canonically the group of simplicial 1-chains in Γ. Let $\mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma$ be the map on fundamental groups induced by the quotient map $\Gamma \to \mathrm{Bouq}(\Gamma)$. The triple

$$\big(\Gamma_1, \mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma, (\cdot,\cdot)\colon \mathrm{C}_1(\Gamma, \mathbf{Z}) \times \mathrm{C}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}\big),$$

with the pairing the standard inner product from (3.1), is the orthographized group associated with the pointed graph (Γ, v).

In fact, $\mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma$ is nondegenerate in the sense of (2.12). To see that word is injective, choose a spanning tree $T \subset \Gamma$ and consider the graph Γ/T obtained by contracting T. Then Γ/T is a bouquet of $g(\Gamma)$ circles. Let $\star$ also denote the unique vertex of Γ/T. There is a natural map $(\mathrm{Bouq}(\Gamma), \star) \to (\Gamma/T, \star)$ of pointed graphs given by contracting the edges of $\mathrm{Bouq}(\Gamma)$ in T. The map induced on fundamental groups

$$\rho\colon F\Gamma \cong \big(\text{free group on } \Gamma_1\big) \to \big(\text{free group on } \Gamma_1 \setminus T_1\big) \cong \pi_1(\Gamma/T, \star)$$

is the natural quotient map. On the other hand, the composition $\pi_1(\Gamma, v) \xrightarrow{\mathsf{word}} F\Gamma \xrightarrow{\rho} \pi_1(\Gamma/T, \star)$ is the isomorphism induced by the homotopy equivalence from (Γ, v) to $(\Gamma/T, \star)$, so word is injective. Nondegeneracy of $(\cdot,\cdot)$ follows from injectivity of word and positive-definiteness of $(\cdot,\cdot)$ on $\mathrm{C}_1(\Gamma, \mathbf{Z})$.

The constructions of (2.11) give a cycle pairing algebra $(\mathbf{Z}\pi_1(\Gamma, v), \int)$ associated with the pointed graph (Γ, v). Since $\pi_1(\Gamma, v)^{\mathrm{ab}} \cong \mathrm{H}_1(\Gamma, \mathbf{Z})$ and $(\cdot, \cdot)$ is integer valued, the higher cycle pairing is a bilinear map

$$\int : \mathbf{Z}\pi_1(\Gamma, v) \times T\mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Q}$$

between the group algebra of the fundamental group of (Γ, v) and the tensor algebra on the first homology group of Γ. Taking quotients by powers of the augmentation ideal $J := \ker(\mathbf{Z}\pi_1(\Gamma, v) \to \mathbf{Z})$, yield truncated cycle pairing algebras

$$\int : \mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1} \times T_\ell \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Q}$$

for every $\ell \geq 0$.

(**3.3**) *Summary of the Higher Cycle Pairing.* We summarize and explicate the construction (3.2) of the higher cycle pairing for a graph Γ. To pair a pure tensor $\omega_1 \otimes \cdots \otimes \omega_\ell \in \mathrm{H}_1(\Gamma, \mathbf{Z})^{\otimes \ell}$ with a path α in Γ, first write α as a sequence $e_1, \ldots, e_r$ of oriented edges e_i of Γ with $e_i^+ = e_{i+1}^-$ for $i = 1, \ldots, r-1$; in the notation above, this means $\mathsf{word}(\alpha) = e_1 \cdots e_r$. Next, compute inductively: When $r = 1$, set

$$\int_\gamma \omega_1 \cdots \omega_\ell = \int_{e_1} \omega_1 \cdots \omega_\ell = \frac{1}{\ell!}(\omega_1, [e_1]) \cdots (\omega_r, [e_1]),$$

where $[e_1] \in \mathrm{C}_1(\Gamma, \mathbf{Z})$ is the class of the oriented edge e_1 in $\mathrm{C}_1(\Gamma, \mathbf{Z})$, and $(\cdot, \cdot) \colon \mathrm{C}_1(\Gamma, \mathbf{Z}) \times \mathrm{C}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}$ is the usual pairing on $\mathrm{C}_1(\Gamma, \mathbf{Z})$, as in (3.1). When $r > 1$, set

$$\int_\gamma \omega_1 \cdots \omega_\ell = \sum_{i=0}^{r} \Big(\int_{e_1 \cdots e_{r-1}} \omega_1 \cdots \omega_i \Big) \Big(\int_{e_r} \omega_{i+1} \cdots \omega_\ell \Big).$$

Restricting α to be loops of Γ based at v and extending by bilinearity gives the entire pairing

$$\int : \mathbf{Z}\pi_1(\Gamma, v) \times T\mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Q}.$$

(**3.4**) *Notation.* Given a pointed graph (Γ, v), write

$$\mathsf{word} \colon \pi_1(\Gamma, v) \to F\Gamma \quad \text{and} \quad \mathsf{word} \colon \mathbf{Z}\pi_1(\Gamma, v) \to \mathbf{Z}F\Gamma$$

for the homomorphism of the orthographized group associated with (Γ, v) in (3.2) and also the induced map on group algebras. Let $I := \ker(\mathbf{Z}F\Gamma \to \mathbf{Z})$ and $J := \ker(\mathbf{Z}\pi_1(\Gamma, v) \to \mathbf{Z})$ be the augmentation ideals. Since word respects augmentation ideals, for each $\ell \geq 0$, we have a $\mathbf{Z}$-algebra map

$$\mathsf{word}_\ell \colon \mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1} \to \mathbf{Z}F\Gamma/I^{\ell+1}.$$

Write $(\cdot, \cdot) \colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \times \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}$ and $\langle \cdot, \cdot \rangle \colon \mathrm{H}^1(\Gamma, \mathbf{Z}) \times \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}$ for the inner product on homology, as in (3.1), and the canonical duality pairing, respectively.

Let $[\cdot] \colon \pi_1(\Gamma, v) \to \mathrm{H}_1(\Gamma, \mathbf{Z})$ be the homology class map; this is a group homomorphism. For each $\ell \geq 1$, there is also $\mathbf{Z}$-linear homology class map

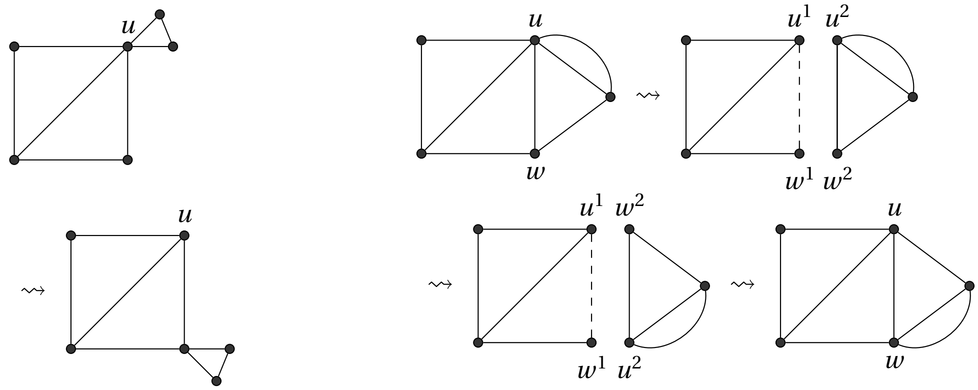
$$[\cdot] \colon \mathbf{Z}\pi_1(\Gamma, v)/J^{\ell+1} \to \mathbf{Z}\pi_1(\Gamma, v)/J^2 \cong \mathbf{Z} \oplus \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma, \mathbf{Z})$$

where the first map is the natural quotient map, the second is the canonical splitting as in (1.3), and the third the projection onto the H_1 component.

We will use obvious modifications of the above notation for other pointed graphs that arise.

The next few paragraphs set up some context for the results of later sections. Specifically, we recall in (3.5) the relation of 2-isomorphism between graphs, allowing us to formulate Whitney's Theorem in (3.6); this result is the primary combinatorial input into the result of §4. Next, (3.7) gives a non-abelian refinement of Whitney's result and, in turn, is the combinatorial input into §5.

(**3.5**) **2-Isomorphic Graphs.** Two graphs Γ and Γ' are said to be *2-isomorphic* if Γ can be transformed into Γ' by a finite sequence of either *vertex cleavings* or *Whitney twists*, collectively referred to as *2-moves*. Briefly, a vertex cleaving of Γ takes a subgraph attached at a disconnecting vertex, breaks the subgraph



(a) Vertex cleaving about the vertex u.

(b) Whitney twisting around $\{u, w\}$.

off at that vertex, and then reattaches this subgraph possibly in a different way; see Figure 1a. A Whitney twists takes a subgraph of Γ which is connected to the rest at two vertices, breaks this off at the vertices, and then reattaches this subgraph with the role of the original vertices flipped; see Figure 1b. See [Oxl11, Section 5.3] for a more precise description. We will also revisit this notion in (3.11).

These notions give a beautiful formulation of

(**3.6**) **Whitney's 2-Isomorphism Theorem. —** [Whi33] *Bridgeless graphs* Γ *and* Γ' *are 2-isomorphic if and only if there exists a bijection* $\Phi\colon \Gamma_1 \to \Gamma'_1$ *such that the induced isometry* $\mathrm{C}_1(\Phi)\colon \mathrm{C}_1(\Gamma,\mathbf{Q}) \to \mathrm{C}_1(\Gamma',\mathbf{Q})$ *restricts to an isomorphism* $\phi\colon \mathrm{H}_1(\Gamma,\mathbf{Q}) \to \mathrm{H}_1(\Gamma',\mathbf{Q})$ *making*

(**3.6.1**)
$$\begin{array}{ccc} \mathrm{C}_1(\Gamma,\mathbf{Q}) & \xrightarrow{\mathrm{C}_1(\Phi)} & \mathrm{C}_1(\Gamma',\mathbf{Q}) \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i'} \\ \mathrm{H}_1(\Gamma,\mathbf{Q}) & \xrightarrow{\phi} & \mathrm{H}_1(\Gamma',\mathbf{Q}) \end{array}$$

commute.

Whitney's Theorem (3.6) may be interpreted as saying that a connected bridgeless graph Γ is determined by the embedding $\mathbf{Q}\pi_1(\Gamma, v)/J^2 \hookrightarrow \mathbf{Q}F\Gamma/I^2$, v any vertex, up to specified ambiguity. If we retain a smidgen of non-abelian information from the fundamental group, we may eliminate this ambiguity and recover the entire graph—and the base vertex too!

(**3.7**) **Non-abelian Whitney Theorem. —** [Kat14] *Let* (Γ, v) *and* (Γ', v') *be bridgeless pointed graphs. Suppose* $\Phi\colon \Gamma_1 \to \Gamma'_1$ *is a bijection such that the induced isomorphism* $\mathbf{Q}F\Phi\colon \mathbf{Q}F\Gamma/I^3 \to \mathbf{Q}F\Gamma'/I'^3$ *restricts to an isomorphism* $\varphi\colon \mathbf{Q}\pi_1(\Gamma, v)/J^3 \to \mathbf{Q}\pi_1(\Gamma', v')/J'^3$*, so that we have a commutative diagram*

(**3.7.1**)
$$\begin{array}{ccc} \mathbf{Q}F\Gamma/I^3 & \xrightarrow{\mathbf{Q}F\Phi} & \mathbf{Q}F\Gamma'/I'^3 \\ {\scriptstyle \mathrm{word}_2}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{word}_2} \\ \mathbf{Q}\pi_1(\Gamma, v)/J^3 & \xrightarrow{\varphi} & \mathbf{Q}\pi_1(\Gamma', v')/J'^3. \end{array}$$

Then Φ *is an isomorphism* $(\Gamma, v) \cong (\Gamma', v')$ *of pointed graphs.*

We now discuss the equivalence relation of concyclicity amongst edges of a graph. From the perspective of matroid theory, concyclicity classes of edges in a graph are maximal sets of parallel elements in the associated cographic matroid. We will instead adopt a definition more explicitly related to the geometry of the graph in (3.8), following [CV10, Definition 2.3.1]; note, however, that Caporaso–Viviani refer to concyclicity classes as "C1-sets", and also work in a slightly more general setting. The notion of concyclicity together with Whitney's Theorem allows us to make sense of a 2-isomorphism of graphs as an honest map between actual sets; this is discussed in (3.11–3.13).

(**3.8**) **Concyclicity.** Two edges e and e' of a bridgeless graph Γ are said to be *concyclic* if for any cyclic subgraph Δ, $e \in \Delta_1$ if and only if $e' \in \Delta_1$. This is an equivalence relation on Γ_1 and its equivalence classes are called *concyclicity classes*. Given $e \in \Gamma_1$, write $C_e \subseteq \Gamma_1$ for the concyclicity class containing e. Let

$$\Gamma^\circ := \left\{ C \subseteq \Gamma_1 \;\middle|\; C \text{ is a concyclicity class of } \Gamma \right\}$$

be the set of concyclicity classes of Γ.

There are many characterizations of concyclicity:

(**3.9**) **Lemma.** — *Let e_1 and e_2 be edges of a bridgeless graph Γ. Let $e_1^*, e_2^* \in \mathrm{C}^1(\Gamma, \mathbf{Z})$ be the corresponding functionals. Then the following are equivalent:*

(i) *e_1 is concyclic with e_2;*
(ii) *$e_1^*|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} = \pm e_2^*|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} \in \mathrm{H}^1(\Gamma, \mathbf{Z})$; and*
(iii) *there exists a set $C \subseteq \Gamma_1$ containing e_1 and e_2 such that the contraction $\Gamma(C)$ of all the edges outside of C is a cyclic graph and the deletion $\Gamma \setminus C$ of edges in C is bridgeless.*

Proof. The equivalence of (i) and (ii) comes from the fact that the functionals e_1^* and e_2^* are signed characteristic functions for whether or not e_1 or e_2 is contained in some cycle of Γ. The equivalence between (iii) and the first two is established in [CV10, Lemma 2.3.2]. See also (3.15). ■

(**3.10**) **Cyclic Orientations.** A *cyclic orientation* of Γ is one which for any concyclicity class $C \subseteq \Gamma_1$ and any pair of $e, f \in C$,

$$e^*|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} = f^*|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} \in \mathrm{H}^1(\Gamma, \mathbf{Z}).$$

Cyclic orientations exist and can be constructed as follows: begin with any orientation on Γ; for each concyclicity class $C \subseteq \Gamma_1$, fix some $e \in C$; for every $f \in C \setminus \{e\}$ flip the orientation of f as necessary so that $e^* = f^*$ in $\mathrm{H}^1(\Gamma, \mathbf{Z})$. Alternatively, [CV10, Corollary 2.4.5] shows that strongly connected orientations on Γ are cyclic orientations.

In the case that Γ is endowed with a cyclic orientation, we can write $C_e^* \in \mathrm{H}^1(\Gamma, \mathbf{Z})$, $e \in \Gamma_1$, for the common functional associated with the edges in C_e. For convenience, we will often tacitly assume our graphs are equipped with a cyclic orientation; for our purposes, this is essentially a notational convenience.

(**3.11**) **2-Isomorphisms of Graphs.** A map $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ between concyclicity classes of Γ and Γ' is said to be *represented* by a map $\Phi \colon \Gamma_1 \to \Gamma'_1$ between edge sets if, for every $e \in \Gamma_1$, $\Phi(e) \in C_{\Phi^\circ(C_e)}$. For example, if Φ° is a bijection satisfying $\#\Phi^\circ(C) = \#C$ for all $C \in \Gamma^\circ$, it is clear that a representing map exists.

A *2-isomorphism* between bridgeless graphs Γ and Γ' is a bijection $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ such that

(i) $\#\Phi^\circ(C) = \#C$ for every $C \in \Gamma^\circ$; and
(ii) there exists a bijection $\Phi \colon \Gamma_1 \to \Gamma'_1$ representing Φ° such that $\mathrm{C}_1(\Phi) \colon \mathrm{C}_1(\Gamma, \mathbf{Z}) \to \mathrm{C}_1(\Gamma', \mathbf{Z})$ preserves homology, i.e. $\mathrm{C}_1(\Phi)(\mathrm{H}_1(\Gamma, \mathbf{Z})) = \mathrm{H}_1(\Gamma', \mathbf{Z})$.

We show in (3.13) that bridgeless graphs Γ and Γ' are 2-isomorphic in the sense of (3.5) if and only if there exists a 2-isomorphism between them. See also [CV10, Proposition 2.3.9].

Given a 2-isomorphism Φ° as above, let $\Phi \colon \Gamma_1 \to \Gamma'_1$ be any bijection as in (ii). Then $\mathrm{C}_1(\Phi)$ induces an isometry $\phi \colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$; this isometry depends only on Φ° and not the choice of Φ.

(**3.12**) **Lemma.** — *Let $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ be a 2-isomorphism between bridgeless graphs Γ and Γ'.*

(i) *If $\Psi \colon \Gamma_1 \to \Gamma'_1$ is any bijection representing Φ°, then $\mathrm{C}_1(\Psi)$ preserves homology.*
(ii) *The isometry $\phi \colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$ induced by a representing map of Φ° is independent of the choice of representing map.*

Proof. Fix a bijection $\Phi \colon \Gamma_1 \to \Gamma'_1$ representing Φ° and for which $\mathrm{C}_1(\Phi)$ preserves homology as in (3.11)(ii). If $\Psi \colon \Gamma_1 \to \Gamma'_1$ is any other bijection representing Φ°, then there is a permutation $\pi \colon \Gamma_1 \to \Gamma_1$ which preserves concyclicity amongst edges and for which $\Psi = \Phi \circ \pi$. Since π preserves concyclicity, $\mathrm{C}_1(\pi)$ permutes the edges in the homology class of any cyclic subgraph of Γ and thus acts as identity on $\mathrm{H}_1(\Gamma, \mathbf{Z})$. Therefore

$$\mathrm{C}_1(\Psi)|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} = \mathrm{C}_1(\Phi \circ \pi)|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} = \mathrm{C}_1(\Phi)|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} \circ \mathrm{C}_1(\pi)|_{\mathrm{H}_1(\Gamma,\mathbf{Z})} = \mathrm{C}_1(\Phi)|_{\mathrm{H}_1(\Gamma,\mathbf{Z})}.$$

This establishes both parts of the Lemma. ■

(**3.13**) **Lemma.** — *Two bridgeless graphs Γ and Γ' are 2-isomorphic if and only if there exists a 2-isomorphism $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$.*

Proof. If Γ and Γ' are 2-isomorphic, Whitney's Theorem (3.6) gives a bijection of edges $\Phi \colon \Gamma_1 \to \Gamma'_1$. Dualize the diagram (3.6.1) to obtain a commutative diagram

(**3.13.1**)
$$\begin{array}{ccc} \mathrm{C}^1(\Gamma,\mathbf{Z}) & \xrightarrow{\mathrm{C}^1(\Phi)} & \mathrm{C}^1(\Gamma',\mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\Gamma,\mathbf{Z}) & \xrightarrow{\varphi} & \mathrm{H}^1(\Gamma',\mathbf{Z}) \end{array}$$

in which the horizontal arrows are isometries. Now commutativity of (3.13.1) combined with (3.9)(ii) show that Φ preserves concyclicity among edges, and morever induces a bijection $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ between concyclicity classes. To verify that Φ° satisfies (3.11)(i), note that injectivity of Φ immediately gives $\#C \leq \#\Phi^\circ(C)$. But Φ is actually bijective, so

$$\#\Gamma'_1 = \sum_{C' \in \Gamma'^\circ} \#C' = \sum_{C \in \Gamma^\circ} \#\Phi^\circ(C) \geq \sum_{C \in \Gamma^\circ} \#C = \#\Gamma_1 = \#\Gamma'_1.$$

Therefore equality holds all the way through and $\#C = \#\Phi^\circ(C)$.

Conversely, assume that $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ is a 2-isomorphism. For each concyclicity class C, (3.11)(i) allows us to pick a bijection $\Phi_C \colon C \to \Phi^\circ(C)$. Let $\Phi \colon \Gamma_1 \to \Gamma'_1$ be the bijection $e \mapsto \Phi_{C_e}(e)$. Then Φ is a bijection representing Φ°. The induced isomorphism $\mathrm{C}_1(\Phi) \colon \mathrm{C}_1(\Gamma,\mathbf{Z}) \to \mathrm{C}_1(\Gamma',\mathbf{Z})$ makes (3.6.1) commute thanks to (3.11)(ii). Whitney's Theorem (3.6) now implies Γ and Γ' are 2-isomorphic. ■

The final paragraphs discuss contractions of graphs. We set notation and discuss maps between algebraic invariants induced by contractions in (3.14). We end with a computation (3.15) of a particular type of contraction.

(**3.14**) **Contractions.** Let Γ be a graph and let $e \in \Gamma_1$. Recall that the contraction Γ/e of e in Γ is the quotient of Γ in which e is identified to a single point. The vertex and edge sets of Γ/e are

$$(\Gamma/e)_0 = \Gamma_0 \setminus \{e^+, e^-\} \cup \{e\} \quad \text{and} \quad (\Gamma/e)_1 = \Gamma_1 \setminus \{e\},$$

and the only incidences that change are that those edges incident with either e^+ or e^- in Γ are now incident with e in Γ/e. For $E \subseteq \Gamma_1$, write Γ/E for the contraction of Γ by all edges in E, and

$$\Gamma(E) := \Gamma/(\Gamma_1 \setminus E)$$

for the contraction of all edges outside of E. Let $\rho_E \colon \Gamma \to \Gamma(E)$ be the quotient map. This induces, by functoriality, maps

$$\begin{aligned} &\mathrm{C}_1(\rho_E) \colon \mathrm{C}_1(\Gamma,\mathbf{Z}) \to \mathrm{C}_1(\Gamma(E),\mathbf{Z}), && \mathrm{H}_1(\rho_E) \colon \mathrm{H}_1(\Gamma,\mathbf{Z}) \to \mathrm{H}_1(\Gamma(E),\mathbf{Z}), \\ &F\rho_E \colon F\Gamma \to F(\Gamma(E)), && \pi_1(\rho_E) \colon \pi_1(\Gamma, v) \to \pi_1(\Gamma(E), v(E)), \end{aligned}$$

where we have written $v(E) \in \Gamma(E)_0$ for the image of the vertex $v \in \Gamma_0$. These maps can be made explicit: The maps $\mathrm{C}_1(\rho_E)$ and $F\rho_E$ are the natural quotient maps

$$\begin{aligned} \mathrm{C}_1(\rho_E) \colon \mathrm{C}_1(\Gamma,\mathbf{Z}) = \textstyle\bigoplus_{e \in \Gamma_1} \mathbf{Z}e \to \bigoplus_{e \in E} \mathbf{Z}e \cong \mathrm{C}_1(\Gamma(E),\mathbf{Z}), \\ F\rho_E \colon F\Gamma = \big(\text{free group on } e \in \Gamma_1\big) \to \big(\text{free group on } e \in E\big) \cong F\Gamma(E), \end{aligned}$$

while $\mathrm{H}_1(\rho_E)$ and $\pi_1(\rho_E)$ are induced from the embeddings $\mathrm{H}_1(\Gamma,\mathbf{Z}) \hookrightarrow \mathrm{C}_1(\Gamma,\mathbf{Z})$ and $\pi_1(\Gamma, v) \hookrightarrow F\Gamma$.

Dually, a contraction $\rho_E\colon \Gamma \to \Gamma(E)$ defines subspaces

$$\mathrm{C}^1(\rho_E)\colon \mathrm{C}^1(\Gamma(E),\mathbf{Z}) \hookrightarrow \mathrm{C}^1(\Gamma,\mathbf{Z}) \quad\text{and}\quad \mathrm{H}^1(\rho_E)\colon \mathrm{H}^1(\Gamma(E),\mathbf{Z}) \hookrightarrow \mathrm{H}^1(\Gamma,\mathbf{Z}).$$

The lattice $\mathrm{H}^1(\Gamma(E),\mathbf{Z})$ is the image of $\mathrm{C}^1(\Gamma(E),\mathbf{Z})$ under the restriction map $\mathrm{C}^1(\Gamma,\mathbf{Z}) \to \mathrm{H}^1(\Gamma,\mathbf{Z})$ and

$$\mathrm{C}^1(\Gamma(E),\mathbf{Z}) \cong \bigoplus\nolimits_{e\in E} \mathbf{Z}e^* \hookrightarrow \bigoplus\nolimits_{e\in\Gamma_1} \mathbf{Z}e^* = \mathrm{C}^1(\Gamma,\mathbf{Z}).$$

This last computation might be seen as a mild extension of (3.9)(iii). We will have use for it in (5.5).

(**3.15**) **Lemma. —** *Let Γ be a bridgeless graph and let $C, D \in \Gamma^\circ$ be distinct. Then*

(i) $\Gamma(C)$ *is a cyclic graph; and*
(ii) $\Gamma(C \cup D)$ *is pair of cyclic graphs joined at a vertex.*

Proof. For (i), recall from (3.14) that the sublattice $\mathrm{H}^1(\Gamma(C),\mathbf{Z}) \hookrightarrow \mathrm{H}^1(\Gamma,\mathbf{Z})$ is the image of $\bigoplus_{e\in C}\mathbf{Z}e^* \hookrightarrow \mathrm{C}^1(\Gamma,\mathbf{Z})$ under the restriction map. But (3.9)(ii) tells us that each of the restrictions $e^*|_{\mathrm{H}_1(\Gamma,\mathbf{Z})}$, e ranging over C, lie in a line, so $g(\Gamma(C)) = \operatorname{rank}\big(\mathrm{H}^1(\Gamma(C),\mathbf{Z})\big) = 1$. Since Γ is bridgeless, each of the functionals e^* have nonzero image in $\mathrm{H}^1(\Gamma,\mathbf{Z})$, so $\Gamma(C)$ contains no leaves. Hence $\Gamma(C)$ is a cyclic graph.

For (ii), again consider $\mathrm{H}^1(\Gamma(C\cup D),\mathbf{Z})$. Since C and D are distinct, C^* and D^* are linearly independent in $\mathrm{H}^1(\Gamma,\mathbf{Z})$. Indeed, by definition, there exists a cycle $\gamma \in \mathrm{H}_1(\Gamma,\mathbf{Z})$ such that $C^*(\gamma) \neq 0$ and $D^*(\gamma) = 0$ and vice versa. Therefore, the same arguments as in (i) show that $g(\Gamma(C\cup D)) = 2$ and that there are no leaves in $\Gamma(C\cup D)$. Since $\Gamma(C\cup D)$ is connected and $C \cap D = \varnothing$, it must be obtained by joining the cyclic graphs $\Gamma(C)$ and $\Gamma(D)$ at some vertex. ■

4. Torelli Theorem for Graphs

In this Section, we present an essentially self-contained proof of the Torelli theorem for graphs, as in [CV10, SW10]. See also [Ger82, Art06, DSSV09]. Throughout, Γ and Γ' will denote graphs. For simplicity, all our graphs are assumed to be bridgeless and equipped with a cyclic orientation (3.10).

(**4.1**) **Torelli Theorem. —** *Let Γ and Γ' be bridgeless graphs. Suppose there exists an isometry*

$$\phi\colon \mathrm{H}_1(\Gamma,\mathbf{Z}) \to \mathrm{H}_1(\Gamma',\mathbf{Z}).$$

Then there exists a unique 2-isomorphism $\Phi^\circ\colon \Gamma^\circ \to \Gamma'^\circ$ inducing ϕ.

Note the converse of (4.1) is an immediate consequence of Whitney's Theorem (3.6).

The proof of (4.1) is completed in (4.9). The point is that a representing function Φ can be constructed from the homology map ϕ because each of the linear functionals on C_1 associated with a concyclicity class (3.10) gives rise to a canonical hyperplane in H_1 which can be accessed directly through the inner product $(\cdot,\cdot)$; see (4.4) and the discussion around it.

(**4.2**) **Delaunay Decompositions.** Concyclicity class functionals can be identified through the Delaunay decomposition associated with the lattice $\mathrm{H}_1(\Gamma,\mathbf{Z})$. In general, the Delaunay decomposition associated with a lattice Λ is a Λ-periodic polyhedral decomposition $\mathrm{Del}(\Lambda)$ of the vector space $\Lambda_{\mathbf{R}} := \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. Cells are constructed in a nearest neighbour fashion: for each $\alpha \in \Lambda_{\mathbf{R}}$, let

$$D(\alpha) := \operatorname{conv}\big\{\lambda \in \Lambda \,\big|\, \|\lambda-\alpha\| \le \|\lambda'-\alpha\| \text{ for all } \lambda' \in \Lambda\big\}$$

be the convex hull in $\Lambda_{\mathbf{R}}$ of the lattice points closest to α. Then $\mathrm{Del}(\Lambda)$ is the set of all such polytopes as α ranges through $\Lambda_{\mathbf{R}}$. By construction, $\mathrm{Del}(\Lambda)$ is Λ-periodic and there are only finitely many distinct polytopes contained in a fundamental domain of $\Lambda_{\mathbf{R}}/\Lambda$.

(**4.3**) As an important example, consider the standard Euclidean lattice $\Lambda := \mathbf{Z}^n$. By periodicity, $\mathrm{Del}(\mathbf{Z}^n)$ is determined by its top-dimensional polytopes in the fundamental domain $[0,1]^n$ of $\mathbf{R}^n/\mathbf{Z}^n$; a direct computation shows that there is a unique top-dimensional polytope here given by

$$\mathrm{Del}\big(\tfrac{1}{2},\dots,\tfrac{1}{2}\big) = \operatorname{conv}\big\{\mathbf{Z}^n \cap [0,1]^n\big\} = [0,1]^n$$

the entire unit cube. Hence $\mathrm{Del}(\mathbf{Z}^n)$ is the decomposition of $\mathbf{R}^n$ into unit cubes with integral vertices.

An important special feature $\mathrm{Del}(\mathbf{Z}^n)$ is that it arises from a hyperplane arrangement: Let

$$\mathscr{H}_n := \bigcup_{i=1}^{n} \bigcup_{m\in\mathbf{Z}} \{x \in \mathbf{R}^n \mid (e_i, x) = m\} \tag{4.3.1}$$

be the union of all integer translates of the coordinate hyperplanes in $\mathbf{R}^n$. Then the top-dimensional cells of $\mathrm{Del}(\mathbf{Z}^n)$ are precisely the closures of connected components of $\mathbf{R}^n \setminus \mathscr{H}_n$.

Now consider the situation of a graph Γ and the Delaunay decomposition $\mathrm{Del}(\Gamma) := \mathrm{Del}(\mathrm{H}_1(\Gamma,\mathbf{Z}))$ associated with the simplicial homology lattice. Remarkably, $\mathrm{Del}(\Gamma)$ is closely related to $\mathrm{Del}(\mathbf{Z}^n)$ as described in (4.3). Consider the tautological embedding

$$\mathrm{H}_1(\Gamma,\mathbf{Z}) \hookrightarrow \mathrm{C}_1(\Gamma,\mathbf{Z}) \cong \mathbf{Z}^{\#\Gamma_1}.$$

The target is a Euclidean lattice. Mumford observed that, through this embedding of lattices, the best possible situation occurs:

(**4.4**) **Proposition. —**[OS79, Proposition 5.5] *For any graph* Γ, $\mathrm{Del}(\Gamma)$ *coincides with the decomposition*

$$\mathrm{Del}(\mathrm{C}_1(\Gamma,\mathbf{Z})) \cap \mathrm{H}_1(\Gamma,\mathbf{R}) = \{D \cap \mathrm{H}_1(\Gamma,\mathbf{R}) \mid D \in \mathrm{Del}(\mathrm{C}_1(\Gamma,\mathbf{Z})), \mathrm{relint}(D) \cap \mathrm{H}_1(\Gamma,\mathbf{R}) \neq \varnothing\}$$

obtained by intersecting the Delaunay decomposition of $\mathrm{C}_1(\Gamma,\mathbf{R})$ *with the subspace* $\mathrm{H}_1(\Gamma,\mathbf{R})$.

This follows from the fact that $\mathrm{H}_1(\Gamma,\mathbf{Z})$ is a *totally unimodular sublattice* of $\mathrm{C}_1(\Gamma,\mathbf{Z})$.

Combining the discussion of (4.3) and Mumford's observation (4.4) shows that $\mathrm{Del}(\Gamma)$ is also induced by a hyperplane arrangement. Specifically, let

$$\mathscr{H}(\Gamma) := \bigcup_{C\in\Gamma^\circ} \bigcup_{m\in\mathbf{Z}} \{x \in \mathrm{H}_1(\Gamma,\mathbf{R}) \mid \langle C^*, x\rangle = m\}.$$

The top-dimensional cells of $\mathrm{Del}(\Gamma)$ are obtained as closures of connected components of $\mathrm{H}_1(\Gamma,\mathbf{R}) \setminus \mathscr{H}(\Gamma)$. Consequently, the subset of $\mathrm{H}^1(\Gamma,\mathbf{Z})$ corresponding to concyclicity classes of edges can be recovered from $\mathrm{Del}(\Gamma)$ as the set of linear functionals defining the hyperplanes in $\mathscr{H}(\Gamma)$.

(**4.5**) **Lemma. —** *Let* $\phi\colon \mathrm{H}_1(\Gamma,\mathbf{Z}) \to \mathrm{H}_1(\Gamma',\mathbf{Z})$ *be an isometry. Let* $\psi\colon \mathrm{H}^1(\Gamma,\mathbf{Z}) \to \mathrm{H}^1(\Gamma',\mathbf{Z})$ *be the dual of* ϕ^{-1}. *Then for every* $C \in \Gamma^\circ$, *there exists a unique* $C' \in \Gamma'^\circ$ *such that* $\psi(C^*) = \pm C'^*$.

Proof. Delaunay decompositions are preserved under isometries, so $\phi(\mathrm{Del}(\Gamma)) = \mathrm{Del}(\Gamma')$. In particular, every hyperplane $\{\langle C^*, \cdot\rangle = 0\} \subset \mathrm{H}_1(\Gamma,\mathbf{R})$, $C \in \Gamma^\circ$, defining $\mathrm{Del}(\Gamma)$ at the origin is mapped isomorphically to a unique hyperplane $\{\langle C'^*, \cdot\rangle = 0\} \subset \mathrm{H}_1(\Gamma',\mathbf{R})$, $C' \in \Gamma'^\circ$, cutting out $\mathrm{Del}(\Gamma')$. Thus $\psi(C^*)$ is a multiple of C'^*. Since C^* and C'^* take value 1 on homology classes corresponding to cyclic subgraphs containing edges from C and C', respectively, each are primitive in their respective lattices. Hence, as isometries preserve primitive elements, $\psi(C^*) = \pm C'^*$. ■

An element $\alpha \in \mathrm{H}_1(\Gamma,\mathbf{Z})$ is called a *simple cycle* if $([e], \alpha) \in \{-1, 0, +1\}$ for every $e \in \Gamma_1$.

(**4.6**) **Corollary. —** *Let* $\phi\colon \mathrm{H}_1(\Gamma,\mathbf{Z}) \to \mathrm{H}_1(\Gamma',\mathbf{Z})$ *be an isometry. Let* α *be a simple cycle in* $\mathrm{H}_1(\Gamma,\mathbf{Z})$. *Then* $\phi(\alpha)$ *is a simple cycle in* $\mathrm{H}_1(\Gamma',\mathbf{Z})$.

Proof. Let $\psi^{-1}\colon \mathrm{H}^1(\Gamma',\mathbf{Z}) \to \mathrm{H}^1(\Gamma,\mathbf{Z})$ be the dual of ϕ. By (4.5), for every $C' \in \Gamma'^\circ$, there exists some $C \in \Gamma^\circ$ such that $\psi^{-1}(C'^*) = \pm C^*$. Therefore

$$\langle C'^*, \phi(\alpha)\rangle = \langle \psi^{-1}(C'^*), \alpha\rangle = \pm\langle C^*, \alpha\rangle \in \{-1, 0, +1\}$$

so $\phi(\alpha)$ is a simple cycle in $\mathrm{H}_1(\Gamma',\mathbf{Z})$. ■

The next Lemma says that any concyclicity class of Γ can be realized as the intersection of exactly two cyclic subgraphs of Γ. Compare with [CV10, Lemma 3.3.1].

(**4.7**) **Lemma. —** *Let* Γ *be a bridgeless graph and let* $C \in \Gamma^\circ$. *Then there exists a pair of cyclic subgraphs* Δ^1 *and* Δ^2 *of* Γ *such that* $\Delta^1_1 \cap \Delta^2_1 = C$.

Proof. Proceed by induction on the cardinality of C. Consider the case when C consists of a single edge. The deletion $\Gamma \setminus C$ is connected since Γ is bridgeless. Moreover, by (3.9)(iii), $\Gamma \setminus C$ is bridgeless. Therefore, Menger's Theorem [Bol98, p.75] gives two edge disjoint paths P^1 and P^2 in $\Gamma \setminus C$ between the endpoints of the edge in C. Then $\Delta^i := P^i \cup C$ is a cyclic subgraph of Γ containing C for $i = 1, 2$, and they satisfy $\Delta_1^1 \cap \Delta_1^2 = C$ by construction.

Now let $n > 1$ and assume the result holds for any concyclicity class of cardinality $n-1$ in any graph. Let $C \in \Gamma^\circ$ be a concyclicity class of size n and choose an edge $e \in C$. Let $\rho\colon \Gamma \to \Gamma/e$ be the contraction of the edge e. Then Γ/e is a connected and also bridgeless: contractions and deletions commute, so a bridge in Γ/e would yield a bridge in Γ. Moreover, $C' := \rho(C \setminus e) \subseteq (\Gamma/e)_1$ is a concyclicity class in the contracted graph. Indeed, using (3.9)(iii), it suffices to see that $(\Gamma/e)(C')$ is cyclic and $(\Gamma/e) \setminus C'$ is bridgeless. Since contractions commute, $(\Gamma/e)(C') = \Gamma(C)/e$. But $\Gamma(C)$ is cyclic since C is a concyclicity class in Γ, so $(\Gamma/e)(C')$ is also cyclic. As for $(\Gamma/e) \setminus C'$, observe that it can be obtained from $\Gamma \setminus C$ by identifying the vertices of e. But glueing vertices together will only decrease the number of bridges, so since $\Gamma \setminus C$ is bridgeless, C being a concyclicity class, $(\Gamma/e) \setminus C'$ is also bridgeless. Thus C' is a concyclicity class of Γ/e.

Therefore the inductive hypothesis applies to Γ/e and C' to yield cyclic subgraphs Δ'^1 and Δ'^2 of Γ/e such that $\Delta_1'^1 \cap \Delta_1'^2 = C'$. Let Δ^i be the subgraph of Γ with edges $\rho^{-1}(\Delta_1'^i) \cup \{e\}$ for $i = 1, 2$. We claim that the Δ^i are cyclic subgraphs of Γ. To see this, note that since $\Delta^i/e = \Delta'^i$ is cyclic, the image of e in the contraction must be a vertex contained in Δ^i. Then either e joins vertices of two adjacent edges of Δ'^i or else e joins a vertex of Δ'^i with a vertex outside of Δ'^i. However, in the latter case, $\Delta^i \setminus \{e\}$ is a cyclic subgraph of Γ containing $C \setminus \{e\}$, contradicting the assumption that C is a concyclicity class. Thus the Δ^i are cyclic subgraphs of Γ containing the entire concyclicity class C and $\Delta_1^1 \cap \Delta_1^2 = C$. ■

(**4.8**) **Lemma. —** *Let $\phi\colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$ be an isometry. Let $\psi\colon \mathrm{H}^1(\Gamma, \mathbf{Z}) \to \mathrm{H}^1(\Gamma', \mathbf{Z})$ be the dual of ϕ^{-1}. Let $C \in \Gamma^\circ$ and $C' \in \Gamma'^\circ$ be corresponding concyclicity class as in* (4.5). *Then $\#C = \#C'$.*

Proof. By (4.7), there exists cycles $\alpha_1, \alpha_2 \in \mathrm{H}_1(G, \mathbf{Z})$ represented by cyclic subgraphs with $\mathrm{supp}(\alpha_1) \cap \mathrm{supp}(\alpha_2) = C$. Set $\alpha_1' := \phi(\alpha_1)$ and $\alpha_2' := \phi(\alpha_2)$. Since ϕ is an isometry,

$$\#C = \#\big(\mathrm{supp}(\alpha_1) \cap \mathrm{supp}(\alpha_2)\big) = |\langle \alpha_1, \alpha_2 \rangle| = |\langle \alpha_1', \alpha_2' \rangle| = \#\big(\mathrm{supp}(\alpha_1') \cap \mathrm{supp}(\alpha_2')\big),$$

where we have used (4.6) in the final equality to conclude that α_1' and α_2' intersect without multiplicity. Thus it suffices to show $\mathrm{supp}(\alpha_1') \cap \mathrm{supp}(\alpha_2') = C'$. Let $D' \in \Gamma'^\circ \setminus \{C'\}$ and apply (4.5) to obtain a concyclicity class $D \in \Gamma^\circ$ such that $\psi^{-1}(D'^*) = \pm D^*$. By the uniqueness statement of (4.5), $D \neq C$. Now,

$$\langle D'^*, \alpha_1' \rangle \langle D'^*, \alpha_2' \rangle = \langle D^*, \alpha_1 \rangle \langle D^*, \alpha_2 \rangle = \#(D \cap \mathrm{supp}(\alpha_1)) \#(D \cap \mathrm{supp}(\alpha_2)).$$

As D is a concyclicity class, (3.9)(i) shows that whenever $D \cap \mathrm{supp}(\alpha_i) \neq \varnothing$, then $D \subseteq \mathrm{supp}(\alpha_i)$. Since D is not contained in $\mathrm{supp}(\alpha_1) \cap \mathrm{supp}(\alpha_2)$, either $D \cap \mathrm{supp}(\alpha_1)$ or $D \cap \mathrm{supp}(\alpha_2)$ is empty. Thus, since $D \neq C$, the above quantity is 0. By the same argument, this time for D', α_1' and α_2' we deduce that D' is not contained in either $\mathrm{supp}(\alpha_1')$ or $\mathrm{supp}(\alpha_2')$, hence D' is not in their intersection. As this is true for all $D' \in \Gamma'^\circ \setminus \{C'\}$, the result follows. ■

(**4.9**) *Proof of* (4.1). Lemma (4.5) gives a bijection $\Phi^\circ\colon \Gamma^\circ \to \Gamma'^\circ$ which induces ϕ on homology. By (4.8), the map Φ° satisfies $\#\Phi^\circ(C) = \#C$ for every $C \in \Gamma^\circ$ and thus is a 2-isomorphism. Uniqueness of Φ° comes from the fact that distinct concyclicity classes give rise to distinct elements of $\mathrm{H}^1(\Gamma, \mathbf{Z})$. ■

5. Unipotent Torelli Theorem for Graphs

In this Section, we use the constructions of §2 and (3.2) to formulate and prove what we call a Unipotent Torelli Theorem. In the following, all graphs are assumed to be connected, bridgeless, and equipped with a cyclic orientation. We will freely use the notation and terminology of §3, especially (3.3), (3.4), (3.9) and, especially later, (3.14).

(**5.1**) **Unipotent Torelli Theorem.** — *Let* (Γ, v) *and* (Γ', v') *be connected bridgeless pointed graphs. Suppose that there exists an isomorphism of* $\mathbf{Z}$*-algebras*

$$\varphi\colon \mathbf{Z}\pi_1(\Gamma, v)/J^3 \to \mathbf{Z}\pi_1(\Gamma', v')/J'^3$$

which preserves the higher cycle pairing. Then there exists a unique isomorphism $\Phi\colon (\Gamma, v) \to (\Gamma', v')$ *of pointed graphs such that the induced map on truncated fundamental group algebras is* φ.

(**5.2**) *Overview of Proof.* The idea of the proof is the following: An isomorphism φ as in the statement yields an isometry $\mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$ upon passing to associated graded algebras. The Torelli Theorem (4.1) then gives a 2-isomorphism $\Phi^\circ\colon \Gamma^\circ \to \Gamma'^\circ$. Therefore, the desired isomorphism be a bijection $\Phi\colon \Gamma_1 \to \Gamma'_1$ representing Φ°, in the sense of (3.11).

Unfortunately, there are generally many bijections $\Gamma_1 \to \Gamma'_1$ which represent a given 2-isomorphism; for instance, if Γ and Γ' were cyclic graphs, any bijection between their edge sets would induce the same map on concyclicity classes. The point is to use the higher cycle pairing to distinguish a canonical bijection representing the 2-isomorphism Φ°. In the case of cyclic graphs, this amounts to saying that the higher cycle pairing allows us to recover the cyclic ordering amongst the edges; this calculation is explained in (5.3). In fact, this computation is key, and the distinguished bijection in the general case is constructed via contraction to the cyclic case. See (5.5).

Once we have constructed a candidate bijection $\Phi\colon \Gamma_1 \to \Gamma'_1$, the problem is to show that this induces a morphism of graphs. The non-abelian Whitney Theorem (3.7) reduces this combinatorial problem to an algebraic statement: we only need to show that the map induced by Φ on group algebras induces a commutative diagram (3.7.1).

So, we should ask: what is it that $\mathsf{word}_2\colon \mathbf{Z}\pi_1(\Gamma, v)/J^3 \to \mathbf{Z}F\Gamma/I^3$ really sees? Due to the truncation by I^3, the word_2 image of the class of a loop remembers the edges present and also the relative ordering between *pairs* of edges. Now the bijection Φ was constructed so that it preserves the relative ordering amongst edges in a single concyclicity class, so it remains to check Φ preserves the ordering between distinct concyclicity classes. In fact, since we are working modulo I^3, it suffices to show Φ preserves the relative ordering or edges between *pairs* of concyclicity classes. The crucial case is now a graph obtained by joining two cyclic graphs at a single vertex; we study this case in (5.6–5.8) and reduce the general case to this via a contraction argument (5.9). Finally, all these ingredients are put together in (5.10) to complete the proof of (5.1).

We now perform the basic calculation that gets the proof of (5.1) off the ground: the cyclic ordering of the edges of a cyclic graph is completely determined by its cycle pairing algebra. In contrast, the homology lattice can only see the number of edges in a cyclic subgraph.

(**5.3**) **Lemma.** — *Let* Δ *be a cyclic graph and let* $v \in \Delta_0$. *Let* Δ' *be any connected bridgeless graph and* $v' \in \Delta'_0$. *Suppose that there exists an isomorphism* $\varphi\colon \big(\mathbf{Z}\pi_1(\Delta, v)/J^3, \textstyle\int\big) \to \big(\mathbf{Z}\pi_1(\Delta', v')/J'^3, \textstyle\int\big)$. *Then there exists a unique isomorphism* $\Phi\colon (\Delta, v) \to (\Delta', v')$ *such that*

$$\begin{tikzcd} \mathbf{Z}F\Delta/I^3 \arrow[r, "\mathbf{Z}F\Phi"] & \mathbf{Z}F\Delta'/I'^3 \\ \mathbf{Z}\pi_1(\Delta, v)/J^3 \arrow[u, "\mathsf{word}_2"] \arrow[r, "\varphi"] & \mathbf{Z}\pi_1(\Delta', v')/J'^3 \arrow[u, "\mathsf{word}_2"'] \end{tikzcd}$$

commutes.

Proof. Let $\phi\colon \mathrm{H}_1(\Delta, \mathbf{Z}) \to \mathrm{H}_1(\Delta', \mathbf{Z})$ be the isometry obtained from φ by taking the degree one component of the isomorphism induced on associated graded algebras. Thus $\mathrm{H}_1(\Delta', \mathbf{Z})$ is a rank one lattice with shortest vector of length $d := \#\Delta_1$, and so Δ' is a cyclic graph of size d.

Let $\gamma \in \pi_1(\Delta, v)$ and $\gamma' \in \pi_1(\Delta', v')$ be the generators compatible with the cyclic orientations (3.10). Since φ is a $\mathbf{Z}$-algebra isomorphism which induces an isometry in degree one,

$$\varphi(\gamma - 1) = (\gamma' - 1) + a(\gamma' - 1)(\gamma' - 1)$$

for some $a \in \mathbf{Z}$. But φ preserves the entire cycle pairing, so

$$\int_{(\gamma-1)} [\gamma][\gamma] = \int_{\varphi(\gamma-1)} [\gamma'][\gamma'] = ad + \int_{(\gamma'-1)} [\gamma'][\gamma'].$$

However, the outermost terms are the same because both are pairings between the generator of the fundamental group and the generator of $\mathrm{H}_1 \otimes \mathrm{H}_1$. Thus $a = 0$ and we conclude $\varphi(\gamma-1) = (\gamma'-1)$.

Now enumerate the edges of Δ in the unique way so that $\mathsf{word}(\gamma) = e_1 \cdots e_d \in F\Delta$, and similarly for Δ'. Observe that this enumeration of Δ_1 can be uniquely recovered from the second term of

$$\mathsf{word}_2(\gamma-1) = \sum_{i=1}^{d} (e_i - 1) + \sum_{1 \leq i < j \leq d} (e_i - 1)(e_j - 1) \in \mathbf{Z}F\Delta / I^3,$$

and similarly for Δ'. Since $\varphi(\gamma-1) = (\gamma'-1)$, the bijection $\Phi \colon \Delta_1 \to \Delta'_1$ given by $e_i \mapsto e'_i$, for $i = 1, \ldots, d$, is the unique bijection on edge sets which is compatible with φ in that it fits into a commutative square as in the statement. But Φ is also an isomorphism of pointed graphs in this case, as required. ∎

We bootstrap this calculation to produce from an isomorphism of cycle pairing algebras of graphs a unique bijection on the edge sets of the underlying graphs. This is done one concyclicity class at a time, and for a given concyclicity class, we reduce to the cyclic case via contractions.

Toward this, we set some notation. Given an isomorphism $\varphi \colon \big(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int\big) \to \big(\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \int\big)$ let $\phi \colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$ be the isometry induced by φ by passing to associated graded algebras and taking degree one components. Write $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ for the 2-isomorphism, in the sense of (3.11), given by the Torelli Theorem (4.1). Given $E \subseteq \Gamma_1$ which is a union $E = C_1 \cup \cdots \cup C_s$ of concyclicity classes $C_i \in \Gamma^\circ$, call $E' := \Phi^\circ(C_1) \cup \cdots \cup \Phi^\circ(C_s) \subseteq \Gamma'_1$ the set of edges *corresponding* to E under φ.

In the next statement, recall the notation for contractions from (3.14).

(**5.4**) **Lemma.** — *Let $\varphi \colon \big(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \int\big) \to \big(\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \int\big)$ be an isomorphism. Let $E \subseteq \Gamma_1$ be a subset of edges which is a union of concyclicity classes of Γ_1 and let $E' \subseteq \Gamma'_1$ be the corresponding set of edges under φ. Then there exists a unique isomorphism of $\mathbf{Z}$-algebras*

$$\varphi_E \colon \mathbf{Z}\pi_1(\Gamma(E), v(E))/J_E^3 \to \mathbf{Z}\pi_1(\Gamma'(E), v'(E))/J'^3_{E'},$$

which preserves the higher cycle pairing and which fits into the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}\pi_1(\Gamma, v)/J^3 \arrow[r, "\varphi"] \arrow[d, "\rho_E"'] & \mathbf{Z}\pi_1(\Gamma', v')/J'^3 \arrow[d, "\rho_{E'}"] \\
\mathbf{Z}\pi_1(\Gamma(E), v(E))/J_E^3 \arrow[r, "\varphi_E"] & \mathbf{Z}\pi_1(\Gamma'(E'), v'(E'))/J'^3_{E'}
\end{tikzcd} \tag{5.4.1}$$

where ρ_E and $\rho_{E'}$ are the contraction mappings.

Proof. Given the solid diagram

$$\begin{tikzcd}
& T_2\mathrm{H}^1(\Gamma, \mathbf{Q}) \arrow[rr, "\psi"] \arrow[dd] & & T_2\mathrm{H}^1(\Gamma', \mathbf{Q}) \arrow[dd] \\
\mathbf{Z}\pi_1(\Gamma, v)/J^3 \arrow[ur, hook, "\int"] \arrow[rr, "\varphi"] \arrow[dd, "\rho_E"] & & \mathbf{Z}\pi_1(\Gamma', v')/J'^3 \arrow[ur, hook, "\int"] \arrow[dd, "\rho_{E'}"] & \\
& T_2\mathrm{H}^1(\Gamma(E), \mathbf{Q}) \arrow[rr, dashed, "\psi_E"] & & T_2\mathrm{H}^1(\Gamma'(E'), \mathbf{Q}) \\
\mathbf{Z}\pi_1(\Gamma(E), v(E))/J_E^3 \arrow[ur, hook, "\int"] \arrow[rr, dashed, "\varphi_E"] & & \mathbf{Z}\pi_1(\Gamma'(E'), v'(E'))/J'^3_{E'} \arrow[ur, hook, "\int"] &
\end{tikzcd}$$

our task is to construct the dashed arrows; here, ψ is the isometry on cohomology induced by φ, and the vertical maps are induced by the contraction maps. Note that the top face commutes since φ preserves the higher cycle pairing, and left and right faces commute by compatibilities amongst contraction maps (3.14).

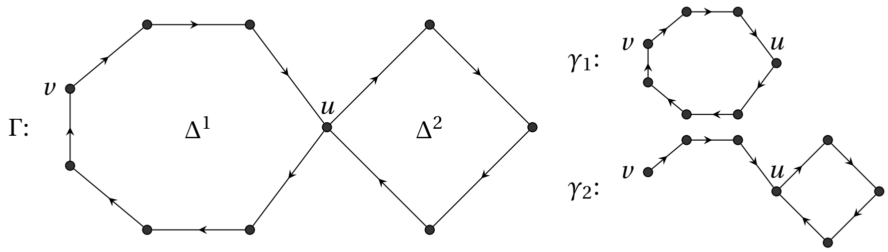


FIGURE 2. An example of a vertex join of two cyclic graphs at some vertex and its minimal generators. In this example, $\mathsf{path}(\gamma_1) = \varnothing$ so $\mathsf{eye}(\gamma_1) = \gamma_1$, and $\mathsf{path}(\gamma_2)$ is a nonempty path in the subgraph Δ^1 of length $c_2 = 3$.

The existence of $\psi_E \colon T_2\mathrm{H}^1(\Gamma(E), \mathbf{Z}) \to T_2\mathrm{H}^1(\Gamma'(E'), \mathbf{Z})$ is easy: the contraction map, say for Γ, is induced by the projection

$$\mathrm{H}^1(\Gamma, \mathbf{Z}) \cong \bigoplus_{C \in \Gamma^\circ} \mathbf{Z}C^* \to \bigoplus_{C \in \Gamma(E)^\circ} \mathbf{Z}C^* \cong \mathrm{H}^1(\Gamma(E), \mathbf{Z})$$

where we note that $\Gamma(E)^\circ = \{C_1, \dots, C_s\} \subset \Gamma^\circ$, C_i the concyclicity classes such that $E = C_1 \cup \cdots \cup C_s$. Since ψ sends $\Gamma(E)^\circ$ to $\Gamma'(E')^\circ$ by definition, it descends to the quotient to give ψ_E. Also, by (4.8), ψ_E is an isometry.

Since the vertical arrows are surjective and the hooked arrows are injective, a diagram chase shows that φ also descends to the quotient to give φ_E, completing the commuting cube. Then the bottom face of the cube also commutes, and this shows that φ_E is an isomorphism preserving the higher cycle pairing. ■

Contractions allow us to use the cyclic case (5.3) to produce from an isomorphism of cycle pairing algebras a distinguished bijection of edge sets. Recall from (3.15)(i) that the graphs $\Gamma(C)$ obtained by contracting all edges away from a given concyclicity class C is cyclic. With all this, the next statement proves itself.

(**5.5**) **Proposition.** — *Let $\varphi \colon \big(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \textstyle\int\big) \to \big(\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \textstyle\int\big)$ be an isomorphism. Then there exists a unique bijection $\Phi \colon \Gamma_1 \to \Gamma'_1$ satisfying the following:*

(i) *The bijection Φ represents the 2-isomorphism $\Phi^\circ \colon \Gamma^\circ \to \Gamma'^\circ$ induced by φ.*

(ii) *For each $C \in \Gamma^\circ$, let $C' := \Phi^\circ(C)$ and let*

$$\varphi_C \colon \big(\mathbf{Z}\pi_1(\Gamma(C), v(C))/J_C^3, \textstyle\int\big) \to \big(\mathbf{Z}\pi_1(\Gamma'(C'), v'(C'))/J'^3_{C'}, \textstyle\int\big)$$

be the isomorphism induced by φ from (5.4). Let $\Phi_C \colon C \to C'$ be the unique bijection inducing φ_C given by (5.3). Then $\Phi|_C = \Phi_C$.

The map Φ constructed in (5.5) must be the bijection we are looking for. To confirm this, it remains to show Φ respects the relative order of edges across concyclicity classes. This brings us to our second fundamental calculation: the case of two cyclic graphs joined at a single vertex.

(**5.6**) **Vertex Joins and their Minimal Generators.** We set some notation and make some preliminary observations before proceeding with the main calculation. Let Δ^1 and Δ^2 be cyclic graphs and let Γ be their *vertex join* at a vertex $u \in \Gamma_0$, that is, Γ is the graph with two cyclic subgraphs, isomorphic to Δ^1 and Δ^2, and where these subgraphs are connected to one another at the single vertex u. By way of notation, write $\Gamma = \Delta^1 \vee_u \Delta^2$. Figure 2 displays a typical example.

Let $v \in \Gamma_0$ be any vertex. The fundamental group $\pi_1(\Gamma, v)$ is abstractly isomorphic to a free group on two generators. We distinguish a pair of generators as follows: let γ_i be the loop based at v which first takes the unique shortest path from v to u which is compatible with the cyclic orientation, then loops around Δ^i compatible with the cyclic orientation of Δ^i, and then returns to v from u along the path it started on. Then $[\gamma_i] = [\Delta^i]$ by construction and hence form generators for the fundamental group. We refer to γ_1 and γ_2 as the *minimal generators* for $\pi_1(\Gamma, v)$. See the right side of Figure 2 for an illustration.

(**5.7**) **Lemma.** — *Let* $\Gamma := \Delta^1 \vee_u \Delta^2$ *be the vertex join of two cyclic graphs and let* $v \in \Gamma_0$ *be any vertex. Let* γ_1 *and* γ_2 *be the minimal generators of* $\pi_1(\Gamma, v)$. *Then, for* $i = 1, 2$,

$$\int_{(\gamma_i - 1)} [\Delta^{3-i}][\Delta^i] = c_i \#\Delta_1^i$$

for some $c_i \in \mathbf{Z}$ *satisfying* $0 \le c_i < \#\Delta_1^{3-i}$. *Moreover,* $c_i = 0$ *if and only if* v *lies in the subgraph* Δ^i.

Proof. By construction of the γ_i, there is a unique factorization

$$\mathsf{word}(\gamma_i) = \mathsf{path}(\gamma_i)\mathsf{eye}(\gamma_i)\mathsf{path}(\gamma_i)^{-1} \in F\Gamma$$

where $\mathsf{path}(\gamma_i)$ is supported in Δ^{3-i}, $\mathsf{eye}(\gamma_i)$ is supported in Δ^i, and $[\gamma_i] = [\mathsf{eye}(\gamma_i)] = [\Delta^i]$ in $\mathrm{C}_1(\Gamma, \mathbf{Z})$. See Figure 2. Then, by the Conjugation Formula (2.7), we compute

$$\begin{aligned}\int_{(\gamma_i - 1)} [\Delta^{3-i}][\Delta^i] &= \int_{(\mathsf{eye}(\gamma_i)-1)} [\Delta^{3-i}][\Delta^i] + \Big(([\mathsf{path}(\gamma_i)], [\Delta^{3-i}])([\Delta^i], [\Delta^i]) - ([\Delta^i], [\Delta^{3-i}])([\mathsf{path}(\gamma_i)], [\Delta^i])\Big) \\ &= \int_{(\mathsf{eye}(\gamma_i)-1)} [\Delta^{3-i}][\Delta^i] + ([\mathsf{path}(\gamma_i)], [\Delta^{3-i}])\#\Delta_1^i\end{aligned}$$

Since the subgraphs Δ^1 and Δ^2 are edge disjoint and $\mathsf{eye}(\gamma_i)$ consists only of edges in Δ^i, the first term vanishes by construction of the higher cycle pairing. Thus, setting $c_i := ([\mathsf{path}(\gamma_i)], [\Delta^{3-i}])$,

$$\int_{(\gamma_i - 1)} [\Delta^{3-i}][\Delta^i] = c_i \#\Delta_1^i.$$

To obtain the bounds on c_i, recall from the construction of γ_i that $\mathsf{path}(\gamma_i)$ is the word associated with the shortest path between v and u and which follows the cyclic orientation of Γ. The shortest path from v to u is necessarily a proper subgraph of Δ^{3-i}, so $c_i < \#\Delta_1^{3-i}$. That $\mathsf{path}(\gamma_i)$ coincides with the orientation shows that $c_i \ge 0$.

For the statement about vertices, note that $c_i = 0$ if and only if $\mathsf{path}(\gamma_i) = \varnothing$. But this happens if and only if the shortest path from v to u in Γ compatible with the cyclic orientation is contained within Δ^i. Since u lies in Δ^i, the shortest path between v and u lies in Δ^i if and only if v is already in Δ^i. ■

(**5.8**) **Lemma.** — *Let* $\Gamma := \Delta^1 \vee_u \Delta^2$ *be the vertex join of two cyclic graphs and let* $v \in \Gamma_0$. *Let* Γ' *be any connected bridgeless graph and* $v' \in \Gamma'_0$. *Assume there exists an isomorphism*

$$\varphi\colon \big(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \textstyle\int\big) \to \big(\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \textstyle\int\big)$$

and let $\Phi\colon \Gamma_1 \to \Gamma'_1$ *be the bijection associated with* φ *by* (5.5). *Then* Φ *is an isomorphism of pointed graphs* $(\Gamma, v) \cong (\Gamma', v')$ *and* φ *is induced by* $\mathbf{Z}F\Phi$.

Proof. As above, φ induces an isometry $\phi\colon \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathrm{H}_1(\Gamma', \mathbf{Z})$ from which we deduce that Γ' must also be a vertex join $\Delta'^1 \vee_{u'} \Delta'^2$ of two cyclic graphs. The edge sets of these cyclic subgraphs form the concyclicity classes of Γ', so by property (5.5)(ii) of Φ, we may arrange the labels so that $\Phi(\Delta^i) = \Delta'^i$ for $i = 1, 2$. In fact, by (5.3), Φ induces an isomorphism between the subgraphs Δ^i of Γ and Δ'^i of Γ', so it remains to show that Φ sends the vertices u and v of Γ to u' and v' of Γ'. To that end, we may assume without loss of generality that v lies in the subgraph Δ^1. Set $d_i := \#\Delta_1^i = \#\Delta_1'^i$ for $i = 1, 2$. Let $\gamma_1, \gamma_2 \in \pi_1(\Gamma, v)$ and $\gamma'_1, \gamma'_2 \in \pi_1(\Gamma', v')$ be the minimal generators, as described in (5.6). We now proceed through a series of Claims.

Claim A. — *There are some* $a_i \in \mathbf{Z}$ *such that*

$$\varphi(\gamma_i - 1) = (\gamma'_i - 1) + a_i\big((\gamma'_1 - 1)(\gamma'_2 - 1) - (\gamma'_2 - 1)(\gamma'_1 - 1)\big)$$

for $i = 1, 2$.

Using only the fact that φ is an algebra isomorphism inducing the isometry ϕ in degree one, we have that, for $i = 1, 2$,

$$\varphi(\gamma_i - 1) = (\gamma'_i - 1) + \sum_{\mu,\nu=1}^{2} a_i^{\mu\nu}(\gamma'_\mu - 1)(\gamma'_\nu - 1)$$

for some $a_i^{\mu\nu} \in \mathbf{Z}$. But contracting, say, Δ^2 yields an isomorphism φ_{Δ^2} between the cycle pairing algebras of Δ^1 and Δ'^1 via (5.4). The contraction map ρ_{Δ^2} is the identity on γ_1 and kills γ_2, so using the commuting diagram (5.4.1), we obtain equations

$$\begin{aligned}\varphi(\gamma_1 - 1) = \varphi_{\Delta^2}(\rho_{\Delta^2}(\gamma_1 - 1)) = \rho_{\Delta'^2}(\varphi(\gamma_1 - 1)) = (\gamma'_1 - 1) + a_1^{11}(\gamma'_1 - 1)(\gamma'_1 - 1),\\ 0 = \varphi_{\Delta^2}(\rho_{\Delta^2}(\gamma_2 - 1)) = \rho_{\Delta'^2}(\varphi(\gamma_2 - 1)) = a_2^{11}(\gamma'_1 - 1)(\gamma'_1 - 1).\end{aligned}$$

Comparing coefficients shows $a_1^{11} = a_2^{11} = 0$. Contracting Δ^1 instead of Δ^2 shows $a_1^{22} = a_2^{11} = 0$.

It remains to show $a_i^{12} = -a_i^{21}$. Since φ preserves the higher cycle pairing, we have

$$\int_{(\gamma_i - 1)} [\Delta^1][\Delta^2] + [\Delta^2][\Delta^1] = \int_{\varphi(\gamma'_i - 1)} [\Delta'^1][\Delta'^2] + [\Delta'^2][\Delta'^1]$$

for $i = 1, 2$. On the one hand, the Symmetrization Formula (2.5) allows us to compute the left hand side as

$$\int_{(\gamma_i - 1)} [\Delta^1][\Delta^2] + [\Delta^2][\Delta^1] = \big([\gamma_i], [\Delta^1]\big)\big([\gamma_i], [\Delta^2]\big) = 0$$

since $[\Delta^1]$ and $[\Delta^2]$ are orthogonal classes in $\mathrm{H}_1(\Gamma, \mathbf{Z})$. On the other hand, expanding $\varphi(\gamma'_i - 1)$ and using orthogonality between $[\Delta'^1]$ and $[\Delta'^2]$ in $\mathrm{H}_1(\Gamma, \mathbf{Z})$ and (2.8.1), we compute

$$0 = \int_{\varphi(\gamma'_i - 1)} [\Delta'^1][\Delta'^2] + [\Delta'^2][\Delta'^1] = (a_i^{12} + a_i^{21}) d_1 d_2.$$

Therefore $a_i^{12} = -a_i^{21}$. Setting $a_i := a_i^{12}$ gives the Claim. ◇

Claim B. — $a_1 = 0$ *so that* $\varphi(\gamma_1 - 1) = (\gamma'_1 - 1)$. *Moreover,* v' *lies in* Δ'^1.

Since φ preserves the higher cycle pairing,

$$\int_{(\gamma_1 - 1)} [\Delta^2][\Delta^1] = \int_{\varphi(\gamma_1 - 1)} [\Delta'^2][\Delta'^1] = \int_{(\gamma'_1 - 1)} [\Delta'^2][\Delta'^1] - a_1 d_1 d_2$$

Since v lies in Δ^1, the left hand side vanishes by (5.7). Thus, rearranging and using (5.7) again,

$$a_1 d_1 d_2 = \int_{(\gamma'_1 - 1)} [\Delta'^2][\Delta'^1] = c'_1 d_1,$$

hence $c'_1 = a_1 d_2$. But $0 \le c'_1 < d_2$, so $a_1 = c'_1 = 0$. The final statement of (5.7) shows v' lies in Δ'^1. ◇

Claim C. — $a_2 = 0$ *so that* $\varphi(\gamma_2 - 1) = (\gamma'_2 - 1)$. *Moreover,* $c_2 = c'_2$.

Computing as in the proof of Claim B, we obtain

$$c_2 d_2 = \int_{(\gamma_2 - 1)} [\Delta^1][\Delta^2] = \int_{\varphi(\gamma_2 - 1)} [\Delta'^1][\Delta'^2] = \int_{(\gamma'_2 - 1)} [\Delta'^1][\Delta'^2] + a_2 d_1 d_2 = (c'_2 + a_2 d_1) d_2$$

for some $0 \le c_2, c'_2 < d_1$ as in (5.7). Since $a_2 \in \mathbf{Z}$, this implies $a_2 = 0$ and $c_2 = c'_2$, as required. ◇

Finally, we show that Φ respects the vertices u and v. To see that Φ sends v to v', note that, by Claim A, v' lies in Δ'^1, so it remains unchanged upon contraction of Δ'^2, that is, in the notation of (3.14), $v'(\Delta'^2) = v'$ as vertices of Δ'^1. Thus (5.4) together with (5.3) show that Φ induces an isomorphism $(\Delta^1, v) \cong (\Delta'^1, v')$ as pointed graphs, which implies Φ sends v to v'.

To see that Φ sends u to u', note that the calculation of (5.7) shows that c is the number of edges on the shortest path from v to u following the underlying orientation, and likewise for c'. But Claim C shows that $c = c'$, so since Φ sends v to v' and Φ is an isomorphism on Δ^1, we see that Φ sends u, viewed as a vertex of Δ^1, to u', as a vertex of Δ'^1. A contraction argument like the one above shows that Φ is compatible with u and u' viewed as vertices of Δ^2 and Δ'^2, respectively. ■

We now show that (5.8), which expresses compatibility of Φ from (5.5) between pairs of concyclicity classes, can be packaged together to deduce compatibility amongst all concyclicity classes.

(**5.9**) **Lemma. —** *Let* (Γ, v) *be a pointed graph. For every pair* $C, D \in \Gamma^\circ$*, let*

$$\rho_{C,D} \colon \mathbf{Z}\pi_1(\Gamma, v)/J^3 \to \mathbf{Z}\pi_1(\Gamma(C \cup D), v(C \cup D))/J^3_{C,D}$$

be the contraction map. Then the map

$$\rho \colon \mathbf{Z}\pi_1(\Gamma, v)/J^3 \xrightarrow{\oplus_{C,D}\rho_{C,D}} \bigoplus_{\{C,D\}\subset\Gamma^\circ} \mathbf{Z}\pi_1(\Gamma(C \cup D), v(C \cup D))/J^3_{C,D}$$

is injective, where the sum is taken over all two element subsets of Γ°*.*

Proof. Since all the $\mathbf{Z}$-modules in question are torsion free, it suffices to check injectivity after extending scalars to $\mathbf{Q}$. Consider $x \in \mathbf{Q}\pi_1(\Gamma, v)/J^3$ in the kernel of ρ. From the construction of the contraction maps (3.14), we have

$$\int_x [C][D] = \int_{\rho_{C,D}(x)} [C][D] = 0$$

for all pairs $C, D \in \Gamma^\circ$. Thus $\int_x$ is zero as a functional on $T_2\mathrm{H}_1(\Gamma, \mathbf{Z})$. But the Duality Theorem (2.13) shows that the map

$$\mathbf{Q}\pi_1(\Gamma, v)/J^3 \to \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma, \mathbf{Z}), \mathbf{Q}) \qquad x \mapsto \int_x$$

is an isomorphism of vector spaces, so $x = 0 \in \mathbf{Q}\pi_1(\Gamma, v)/J^3$. Therefore ρ is injective. ∎

(**5.10**) *Proof of the Unipotent Torelli Theorem.* Suppose we have an isomorphism

$$\varphi \colon \big(\mathbf{Z}\pi_1(\Gamma, v)/J^3, \textstyle\int\big) \to \big(\mathbf{Z}\pi_1(\Gamma', v')/J'^3, \textstyle\int\big).$$

By (5.5), there is a canonically associated bijection $\Phi \colon \Gamma_1 \to \Gamma'_1$. We show that Φ is an isomorphism of pointed graphs.

For each pair $C, D \in \Gamma^\circ$, let $C' := \Phi(C)$ and $D' := \Phi(D)$. By (5.5)(i), C' and D' are concyclicity classes of Γ'. With the notation of (3.14), set

$$G_{C,D} := \pi_1(\Gamma(C \cup D), v(C \cup D)) \quad \text{and} \quad G'_{C',D'} := \pi_1(\Gamma'(C' \cup D'), v'(C' \cup D')).$$

By (5.4), φ induces isomorphisms

$$\varphi_{C,D} \colon \big(\mathbf{Z}G_{C,D}/J^3_{C,D}, \textstyle\int\big) \to \big(\mathbf{Z}G'_{C',D'}/J'^3_{C',D'}, \textstyle\int\big).$$

As is shown in (3.15), each of $\Gamma(C \cup D)$ and $\Gamma'(C' \cup D')$ are vertex joins of two cyclic graphs, so (5.8) applies. Therefore the canonical bijection $\Phi_{C,D} \colon \Gamma(C \cup D)_1 \to \Gamma'(C' \cup D')_1$ associated with $\varphi_{C,D}$ by (5.5) is an isomorphism of pointed graphs $(\Gamma(C \cup D), v(C \cup D)) \cong (\Gamma'(C' \cup D'), v'(C' \cup D'))$. In particular, for each pair of $C, D \in \Gamma^\circ$, writing

$$F_{C,D} := F\Gamma(C \cup D) \quad \text{and} \quad F'_{C',D'} := F\Gamma'(C' \cup D'),$$

we have commutative diagrams

(**5.10.1**)
$$\begin{tikzcd}
\mathbf{Z}F_{C,D}/I^3_{C,D} \arrow[r, "\mathbf{Z}F\Phi_{C,D}"] & \mathbf{Z}F_{C',D'}/I'^3_{C',D'} \\
\mathbf{Z}G_{C,D}/J^3_{C,D} \arrow[u, "\mathsf{word}_2"] \arrow[r, "\varphi_{C,D}"] & \mathbf{Z}G'_{C',D'}/J'^3_{C',D'}. \arrow[u, "\mathsf{word}_2"']
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
& \bigoplus_{C,D} \mathbf{Z}F_{C,D}/I^3_{C,D} \arrow[rr, "\bigoplus_{C,D} \mathbf{Z}F\Phi_{C,D}"] & & \bigoplus_{C',D'} \mathbf{Z}F'_{C',D'}/I'^3_{C',D'} \\
\mathbf{Z}F\Gamma/I^3 \arrow[ur, hook] \arrow[rr, "\mathbf{Z}F\Phi"] & & \mathbf{Z}F\Gamma'/I'^3 \arrow[ur, hook] & \\
& \bigoplus_{C,D} \mathbf{Z}G_{C,D}/J^3_{C,D} \arrow[uu, "\bigoplus_{C,D}\mathsf{word}_2"'] \arrow[rr, "\bigoplus_{C,D}\varphi_{C,D}"] & & \bigoplus_{C',D'} \mathbf{Z}G'_{C',D'}/J'^3_{C',D'} \arrow[uu, "\bigoplus_{C',D'}\mathsf{word}_2"'] \\
\mathbf{Z}\pi_1(\Gamma,v)/J^3 \arrow[uu, "\mathsf{word}_2"] \arrow[ur, hook] \arrow[rr, "\varphi"] & & \mathbf{Z}\pi_1(\Gamma',v')/J'^3 \arrow[uu, "\mathsf{word}_2"'] \arrow[ur, hook] &
\end{tikzcd}$$

the sums over C, D ranging over Γ° and those over C', D' ranging over Γ'°. Now

- the left and right faces commute by construction of the contraction maps (3.14);
- the top face commutes by the compatibility condition (5.5)(ii) between Φ and $\Phi_{C,D}$;
- the bottom face commutes by construction (5.4); and
- the back face is an amalgamation of the commuting squares (5.10.1) and thus commutes.

Since the hooked arrows are indeed injections (5.9), the front face is also commutative. The non-abelian Whitney Theorem (3.7) now applies to show Φ is an isomorphism of pointed graphs $(\Gamma, v) \cong (\Gamma', v')$. ∎

6. Extension Theory

In this final Section, we study the integral structure of the cycle pairing algebra associated with a graph in some more detail. Our main result here coordinatizes the set of pointed graphs 2-isomorphic to some fixed pointed graph; these coordinates are valued in a torsion abelian group. See (6.12) and (6.13). This system of coordinates is produced as extension data associated with the cycle pairing algebra, and can be made explicit in terms of the higher cycle pairing.

To set the stage, the next statement shows that the integral structure of $\mathbf{Z}\pi_1(\Gamma, v)/J^3$ carries nontrivial information with regards to the isomorphism class of (Γ, v) within its 2-isomorphism class. In particular, it would not be enough in (5.1) to have an isomorphism of cycle pairing algebras over fields. Given this, the remainder of the Section aims to extract this information via homological algebra. See also (6.14) for an interpretation.

(**6.1**) **Proposition. —** *Let Γ and Γ' be connected bridgeless graphs and suppose there exists an isometry $\phi\colon \mathrm{H}_1(\Gamma,\mathbf{Z}) \to \mathrm{H}_1(\Gamma',\mathbf{Z})$. Then, for any vertices $v \in \Gamma_0$ and $v' \in \Gamma'_0$, there exists a unique isomorphism*

$$\varphi\colon \big(\mathbf{Q}\pi_1(\Gamma, v)/J^3, \textstyle\int\big) \to \big(\mathbf{Q}\pi_1(\Gamma', v')/J'^3, \textstyle\int\big)$$

which induces ϕ upon passing to associated graded algebras.

Proof. The dual of ϕ^{-1} induces an isomorphism

$$\psi\colon \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma,\mathbf{Z}),\mathbf{Q}) \to \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma',\mathbf{Z}),\mathbf{Q}).$$

One easily verifies that ψ preserves the Hopf algebra structure from (2.4) on both sides. Composing ψ with the isomorphisms from the Duality Theorem (2.13) gives an isomorphism of Hopf algebras

$$\varphi\colon \mathbf{Q}\pi_1(\Gamma, v)/J^3 \cong \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma,\mathbf{Z}),\mathbf{Q}) \xrightarrow{\psi} \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma',\mathbf{Z}),\mathbf{Q}) \cong \mathbf{Q}\pi_1(\Gamma', v')/J'^3.$$

This map is characterized by the equation

$$\textbf{(6.1.1)} \qquad \int_\gamma \phi^{-1}(\omega') = \int_{\varphi(\gamma)} \omega'$$

for all $\gamma \in \mathbf{Q}\pi_1(\Gamma, v)/J^3$ and all $\omega' \in T_2\mathrm{H}_1(\Gamma',\mathbf{Q})$.

It remains to show that φ induces ϕ upon taking the degree one component of the associated graded algebra. Let $\omega' \in \mathrm{H}_1(\Gamma', \mathbf{Q})$ and let $\gamma \in J^2/J^3$. Since $\phi^{-1}(\omega') \in \mathrm{H}_1(\Gamma, \mathbf{Q})$, (2.8) implies that

$$\int_\gamma \phi^{-1}(\omega') = 0.$$

Thus pairing against $\phi^{-1}(\omega')$ depends only on the homology class $[\gamma]$ of γ. But, again by (2.8), the higher cycle pairing reduces to the pairing $(\cdot, \cdot)$ between homology elements, so

$$\int_\gamma \phi^{-1}(\omega') = ([\gamma], \phi^{-1}(\omega')) = (\phi([\gamma]), \omega') = \int_{\phi([\gamma])} \omega'$$

for all $\omega' \in \mathrm{H}_1(\Gamma', \mathbf{Q})$ and $\gamma \in J/J^3$. Comparing this with (6.1.1), it follows that $\varphi(\gamma) + J^2 = \phi([\gamma])$ and thus φ induces ϕ on J/J^2.

To see that φ preserves the higher cycle pairing, for all $\gamma \in \mathbf{Q}\pi_1(\Gamma, v)/J^3$ and $\omega \in T_2\mathrm{H}_1(\Gamma, \mathbf{Q})$,

$$\int_{\varphi(\gamma)} \phi(\omega) = \int_\gamma \phi^{-1}(\phi(\omega)) = \int_\gamma \omega,$$

where the first equality is by the defining equation (6.1.1).

Finally, for the uniqueness statement, let

$$\varphi' \colon \left(\mathbf{Q}\pi_1(\Gamma, v)/J^3, \textstyle\int\right) \to \left(\mathbf{Q}\pi_1(\Gamma', v')/J'^3, \textstyle\int\right)$$

be another isomorphism which induces ϕ on the first graded quotient. Then

$$\int_{\varphi(\gamma)-\varphi'(\gamma)} \phi(\omega) = \int_{\varphi(\gamma)} \phi(\omega) - \int_{\varphi'(\gamma)} \phi(\omega) = \int_\gamma \omega - \int_\gamma \omega = 0,$$

for every $\gamma \in \mathbf{Q}\pi_1(\Gamma, v)/J^3$ and every $\omega \in T_2\mathrm{H}_1(\Gamma, \mathbf{Q})$. This means that the functional

$$\int_{\varphi(\cdot)-\varphi'(\cdot)} \in \mathrm{Hom}_{\mathbf{Z}}(T_2\mathrm{H}_1(\Gamma', \mathbf{Z}), \mathbf{Q})$$

is the zero functional. Using the Duality Theorem (2.13), we conclude that $\varphi = \varphi'$. ■

In the subsequent paragraphs, we develop an extension theory for nondegenerate orthographized groups. Our aim is to construct an interesting extension class for J/J^3 in some group $\mathrm{Ext}^1_{\mathsf{C}}(J/J^2, J^2/J^3)$, J being the augmentation ideal of our orthographized group word: $G \to F$. Note that since G is free, each of J/J^3, J/J^2, and J^2/J^3 are free $\mathbf{Z}$-modules, so $\mathrm{Ext}^1_{\mathbf{Z}}(J/J^2, J^2/J^3) = 0$. Thus, in order to get an interesting extension theory, we take into account the structure map word.

Paragraphs (6.2–6.6) begin with general homological constructions. We specialize to the setting of nondegenerate orthographized groups starting in (6.8).

(**6.2**) Fix a commutative ring R. Let C be the category such that objects in C are R-module homomorphisms $A \to B$ and morphisms from $w\colon A \to B$ to $w'\colon A' \to B'$ in C are pairs of R-module homomorphisms $A \to A'$ and $B \to B'$ such that the square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle w'} \\ B & \longrightarrow & B' \end{array}$$

commutes. Thus, for two objects $w\colon A \to B$ and $w'\colon A' \to B'$ in C,

(**6.2.1**)
$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(A \to B, A' \to B') &= \left\{ (f, g) \in \mathrm{Hom}_R(A, A') \oplus \mathrm{Hom}_R(B, B') \;\middle|\; w' \circ f = g \circ w \right\} \\ &= \ker\left(\mathrm{Hom}_R(A, A') \oplus \mathrm{Hom}_R(B, B') \xrightarrow{+\circ(w'_*, -w^*)} \mathrm{Hom}_R(A, B') \right) \end{aligned}$$

where $w'_*\colon \mathrm{Hom}_R(A, A') \to \mathrm{Hom}(A, B')$ and $w^*\colon \mathrm{Hom}_R(B, B') \to \mathrm{Hom}_R(A, B')$ are the maps induced by w' and w on Hom_R.

A direct calculation shows that C is abelian; alternatively, C can be recognized as the category of sheaves of R-modules on the two point space with a unique closed point, and that C is abelian (and has enough injectives) follows from general considerations [Wei94, Example 2.3.12]. In any case, explicitly, a short exact sequence in C is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0
\end{tikzcd}$$

where each row is an exact sequence of R-modules. From this, it is clear that the functors

$$\begin{aligned} \mathrm{cod}\colon \mathsf{C} &\to \mathsf{Mod}_R, \\ (A \to B) &\mapsto B. \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{dom}\colon \mathsf{C} &\to \mathsf{Mod}_R, \\ (A \to B) &\mapsto A, \end{aligned}$$

are exact.

Consider the functors

$$\begin{aligned} (0 \to \cdot)\colon \mathsf{Mod}_R &\to \mathsf{C}, \\ B &\mapsto (0 \to B), \end{aligned} \qquad \begin{aligned} (\cdot \to \cdot)\colon \mathsf{Mod}_R &\to \mathsf{C}, \\ C &\mapsto (C \xrightarrow{\mathrm{id}_C} C), \end{aligned} \qquad \begin{aligned} (\cdot \to 0)\colon \mathsf{Mod}_R &\to \mathsf{C}, \\ A &\mapsto (A \to 0). \end{aligned}$$

All of these functors are exact. Collecting everything together, we have

$$\begin{tikzcd}
\mathsf{C} \arrow[d, bend right=40, "\mathrm{cod}"'] \arrow[d, bend left=40, "\mathrm{dom}"] \\
\mathsf{Mod}_R \arrow[u, bend left=80, "(0 \to \cdot)"] \arrow[u, "(\cdot \to \cdot)"] \arrow[u, bend right=80, "(\cdot \to 0)"']
\end{tikzcd}$$

which also expresses the adjunction relations stated in the Lemma below.

(**6.3**) **Lemma. —** *With the notation above, we have the following adjunction relations:*

(i) $(0 \to \cdot)$ *is left adjoint to* cod*;*
(ii) cod *is left adjoint to* $(\cdot \to \cdot)$*;*
(iii) $(\cdot \to \cdot)$ *is left adjoint to* dom*;*
(iv) dom *is left adjoint to* $(\cdot \to 0)$.

Proof. A morphism from $(0 \to B)$ to $(A' \to B')$ in C is precisely a morphism $B \to B'$ of R-modules, hence (i). Similarly, a morphism from $(A \to B)$ to $(A' \to 0)$ in C is just a morphism $A \to A'$ of R-modules, hence (iv). For the middle two, consider commutative squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "w"'] & C' \arrow[d, "\mathrm{id}_{C'}"] \\
B \arrow[r, "g"] & C'.
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C \arrow[r, "f'"] \arrow[d, "\mathrm{id}_C"'] & A' \arrow[d, "w'"] \\
C \arrow[r, "g'"] & B'
\end{tikzcd}$$

Then $f = g \circ w$ and $g' = w' \circ f'$, showing (ii) and (iii), respectively. ■

(**6.4**) **Lemma. —** *Let I be an injective R-module.*

(i) *The objects* $\mathrm{id}_I\colon I \to I$ *and* $I \to 0$ *are injective objects of* C.
(ii) *If $I \to J$ is an injective object in C, then I and J are injective R-modules.*
(iii) *The category C has enough injectives.*

Dually, let P be a projective R-module.

(i') *The objects* $\mathrm{id}_P\colon P \to P$ *and* $0 \to P$ *are projective objects of* C.
(ii') *If $P \to Q$ is a projective object in C, then P and Q are projective R-modules.*

(iii') *The category* C *has enough projectives.*

Proof. Items (i), (ii), and their duals follow from (6.3) by standard arguments involving adjoint pairs of functors; see, for example, [Wei94, Proposition 2.3.10].

For (iii), let $A \to B$ be any object in C. Let $A \to I$ and $B \to J$ be injective homomorphisms of R-modules with I and J injective modules. By (i) and the fact that a product of injective objects is injective, $(I \to 0) \times (J \to J) = I \times J \to J$ is an injective object in C and we have an injective map $(A \to B) \to (I \times J \to J)$ in C. Dualizing this argument gives (iii'). ■

Lemma (6.4) allows us to compute Ext groups in C as derived functors of Hom. We now make one such computation.

(**6.5**) **Lemma. —** *Let* $w\colon A \to B$ *and* $w'\colon A' \to B'$ *be objects in* C. *Assume that*

(i) $w\colon A \to B$ *is injective; and*

(ii) $\mathrm{Ext}^1_R(A, A') = \mathrm{Ext}^1_R(B, B') = 0$.

Then

$$\mathrm{Ext}^1_{\mathsf{C}}(A \xrightarrow{w} B, A' \xrightarrow{w'} B') = \frac{\mathrm{Hom}_R(A, B')}{w'_* \mathrm{Hom}_R(A, A') + w^* \mathrm{Hom}_R(B, B')}.$$

Proof. By (6.4)(iii), we can pick an injective resolution

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "w'"] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & B' \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots \end{tikzcd}$$

of $A' \to B'$ in C. Consider the first quadrant double complex

$$\begin{tikzcd} 0 & 0 & 0 & \\ \mathrm{Hom}_R(A, J^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(A, J^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(A, J^2) \arrow[r] \arrow[u] & \cdots \\ \mathrm{Hom}_R(A, I^0) \oplus \mathrm{Hom}_R(B, J^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(A, I^1) \oplus \mathrm{Hom}_R(B, J^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(A, I^2) \oplus \mathrm{Hom}_R(B, J^2) \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

and let $\mathrm{Tot}^\bullet$ be the associated total complex. We compute the cohomology of $\mathrm{Tot}^\bullet$ via the two spectral sequences associated with the double complex.

Going first vertically, for each $p \geq 0$, the homology groups of the complex

$$0 \to \mathrm{Hom}_R(A, I^p) \oplus \mathrm{Hom}_R(B, J^p) \to \mathrm{Hom}_R(A, J^p) \to 0$$

are $\mathrm{H}^0 = \mathrm{Hom}_{\mathsf{C}}(A \to B, I^p \to J^p)$ by (6.2.1), and $\mathrm{H}^1 = 0$ since $A \to B$ is injective by assumption (i) and $\mathrm{Hom}_R(\cdot, J^p)$ is exact by (6.4)(ii), so that $\mathrm{Hom}_R(B, J^p) \to \mathrm{Hom}_R(A, J^p)$ is surjective. Thus the E_1 page for the upwards spectral sequence is simply the following complex in row 0:

$$\mathrm{Hom}_{\mathsf{C}}(A \to B, I^0 \to J^0) \to \mathrm{Hom}_{\mathsf{C}}(A \to B, I^1 \to J^1) \to \mathrm{Hom}_{\mathsf{C}}(A \to B, I^2 \to J^2) \to \cdots.$$

Therefore $\mathrm{H}^i(\mathrm{Tot}^\bullet) = \mathrm{Ext}^i_{\mathsf{C}}(A \to B, A' \to B')$ for all $i \geq 0$.

Computing $\mathrm{H}^*(\mathrm{Tot}^\bullet)$ via the horizontal spectral sequence instead, using the fact (6.4)(ii) that $A' \to I^\bullet$ and $B' \to J^\bullet$ are injective resolutions in Mod_R, gives an E_1 page whose nonzero terms are

$$\begin{tikzcd} \mathrm{Hom}_R(A, B') & \mathrm{Ext}^1_R(A, B') & \mathrm{Ext}^2_R(A, B') & \cdots \\ \mathrm{Hom}_R(A, A') \oplus \mathrm{Hom}_R(B, B') \arrow[u] & \mathrm{Ext}^1_R(A, A') \oplus \mathrm{Ext}^1_R(B, B') \arrow[u] & \mathrm{Ext}^2_R(A, A') \oplus \mathrm{Ext}^2_R(B, B') \arrow[u] & \cdots. \end{tikzcd}$$

Assumption (ii) kills the Ext^1_R groups in the bottom row, from which it follows that $\mathrm{H}^1(\mathrm{Tot}^\bullet)$ is the cokernel of the first map in the first column. This map is $+\circ(w'_*, -w^*)$, so

$$\mathrm{Ext}^1_{\mathsf{C}}(A \to B, A' \to B') = \mathrm{H}^1(\mathrm{Tot}^\bullet) = \frac{\mathrm{Hom}_R(A, B')}{w'_* \mathrm{Hom}_R(A, A') + w^* \mathrm{Hom}_R(B, B')},$$

as desired. ■

As in any abelian category, $\mathrm{Ext}^1_{\mathsf{C}}(A \to B, A' \to B')$ classifies extensions of $A \to B$ by $A' \to B'$ in C. We identify representatives for extension classes in the setting of (6.5).

(**6.6**) **Lemma.** — *Let*

(**6.6.1**)
$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "w'"] & A'' \arrow[r] \arrow[d, "w''"] & A \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B'' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

be a short exact sequence in C. *Assume*

(i) $w \colon A \to B$ *is injective;*
(ii) $\mathrm{Ext}^1_R(A, A') = \mathrm{Ext}^1_R(B, B') = 0$*;*
(iii) $\mathrm{Ext}^1_R(A, A) = \mathrm{Ext}^1_R(B, B) = 0$*; and*
(iv) $\mathrm{Ext}^1_R(A, A'') = \mathrm{Ext}^1_R(B, B'') = 0$.

Then for any pair of sections $\sigma \colon A \to A''$ *and* $\tau \colon B \to B''$,

$$w'' \circ \sigma - \tau \circ w \in \mathrm{Hom}_R(A, B')$$

and is a representative of the extension class $\big[A'' \xrightarrow{w''} B''\big] \in \mathrm{Ext}^1_{\mathsf{C}}(A \to B, A' \to B')$.

Proof. As usual, the extension class $\big[A'' \xrightarrow{w''} B''\big]$ is the image of the identity under the coboundary map

$$\delta \colon \mathrm{Hom}_{\mathsf{C}}(A \to B, A \to B) \to \mathrm{Ext}^1_{\mathsf{C}}(A \to B, A' \to B')$$

appearing in the long exact sequence obtained by applying $\mathrm{Hom}_{\mathsf{C}}(A \to B, \cdot)$ to (6.6.1). The hypotheses (i)–(iv) allow us to run the proof of (6.5) to obtain a diagram

$$\begin{tikzcd}
& & & \mathrm{Hom}_{\mathsf{C}}(A \to B, A \to B) \arrow[d] \arrow[dddll, rounded corners, "\delta"'] & \\
& \mathrm{Hom}_R(A, A') \oplus \mathrm{Hom}_R(B, B') \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_R(A, A'') \oplus \mathrm{Hom}_R(B, B'') \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_R(A, A) \oplus \mathrm{Hom}_R(B, B) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(A, B') \arrow[r] \arrow[d] & \mathrm{Hom}_R(A, B'') \arrow[r] & \mathrm{Hom}_R(A, B) & \\
& \mathrm{Ext}^1_{\mathsf{C}}(A \to B, A' \to B') & & &
\end{tikzcd}$$

in which we have only inlcuded parts relevant to the computation to follow. In particular, the coboundary map δ is the map given by the Snake Lemma connecting the kernel of the last column with the cokernel of the first columm. Now chase the diagram:

– $\mathrm{id}_{A \to B}$ maps to $(\mathrm{id}_A, \mathrm{id}_B) \in \mathrm{Hom}_R(A, A) \oplus \mathrm{Hom}_R(B, B)$;
– choose any $(\sigma, \tau) \in \mathrm{Hom}_R(A, A'') \oplus \mathrm{Hom}_R(B, B'')$ lift of $(\mathrm{id}_A, \mathrm{id}_B)$;
– then (σ, τ) map to $w'' \circ \sigma - \tau \circ w \in \mathrm{Hom}_R(A, B'')$ via the middle vertical map;

– this difference lies in the image of $\mathrm{Hom}_R(A,B')$: we have a solid commutative diagram with dashed arrows sections,

$$\begin{tikzcd} A'' \arrow[r, "q"'] \arrow[d, "w''"'] & A \arrow[l, dashed, bend right, "\sigma"'] \arrow[d, "w"] \\ B'' \arrow[r, "p"] & B \arrow[l, dashed, bend left, "\tau"] \end{tikzcd} \quad \text{so} \quad p\circ(w''\circ\sigma-\tau\circ w) = w\circ(q\circ\sigma)-(p\circ\tau)\circ w = 0.$$

Thus $w''\circ\sigma-\tau\circ w \in \mathrm{Hom}_R(A,B'')$ is a representative for the extension class. ■

(**6.7**) **Corollary. —** *With the notation and assumptions of* (6.6), *suppose further that B'' is equipped with a distingiushed section $\tau\colon B\to B''$. Then there is a canonical section*

$$\tau^\sharp\colon \mathrm{Ext}^1_{\mathsf{C}}(A\to B, A'\to B') = \frac{\mathrm{Hom}_R(A,B')}{w'_*\,\mathrm{Hom}_R(A,A')+w^*\,\mathrm{Hom}_R(B,B')} \to \frac{\mathrm{Hom}_R(A,B')}{w'_*\,\mathrm{Hom}_R(A,A')}$$

to the natural quotient map such that $\tau^\sharp\big([A''\xrightarrow{w''}B'']\big)$ is represented by

$$w''\circ\sigma-\tau\circ w\in\mathrm{Hom}_R(A,B')$$

for any section $\sigma\colon A\to A''$.

Proof. This follows directly the construction in (6.6). ■

We now apply the theory developed in the preceding paragraphs to the context of orthographized groups. Our base ring is now $\mathbf{Z}$ so that objects in C are $\mathbf{Z}$-module homomorphisms $A\to B$.

(**6.8**) Let $\mathsf{word}\colon G\to F$ be an orthographized group and let I and J be the augmentation ideals of $\mathbf{Z}F$ and $\mathbf{Z}G$, respectively. Then there is a commutative diagram of $\mathbf{Z}$-modules

(**6.8.1**)
$$\begin{tikzcd} 0 \arrow[r] & J^2/J^3 \arrow[r] \arrow[d, "\mathsf{word}_2"] & J/J^3 \arrow[r] \arrow[d, "\mathsf{word}_2"] & J/J^2 \arrow[r] \arrow[d, "\mathsf{word}_1"] & 0 \\ 0 \arrow[r] & I^2/I^3 \arrow[r] & I/I^3 \arrow[r] & I/I^2 \arrow[r] & 0 \end{tikzcd}$$

with exact rows. As a short exact sequence in C, the middle term defines an extension class

$$\big[J/J^3\xrightarrow{\mathsf{word}_2}I/I^3\big]\in\mathrm{Ext}^1_{\mathsf{C}}\big(J/J^2\xrightarrow{\mathsf{word}_1}I/I^2, J^2/J^3\xrightarrow{\mathsf{word}_2}I^2/I^3\big).$$

Since F comes with a distinguished set of generators, there is a distinguished section $\tau\colon I/I^2\to I/I^3$. We will often tacitly identify I/I^2 as a submodule of I/I^3 via τ.

Now suppose that $\mathsf{word}\colon G\to F$ is nondegenerate. Then $\mathsf{word}_1\colon J/J^2\to I/I^2$ is injective. Since both G and F are free, all modules in (6.8.1) are free, so $\mathrm{Ext}^1_{\mathbf{Z}}(J/J^2,J^2/J^3)=\mathrm{Ext}^1_{\mathbf{Z}}(I/I^2,I^2/I^3)=0$. Thus the hypotheses of (6.5) are satisfied with $(A\to B)=(J/J^2\to I/I^2)$ and $(A'\to B')=(J^2/J^3\to I^2/I^3)$, so

$$\mathrm{Ext}^1_{\mathsf{C}}\big(J/J^2\xrightarrow{\mathsf{word}_1}I/I^2, J^2/J^3\xrightarrow{\mathsf{word}_2}I^2/I^3\big) = \frac{\mathrm{Hom}_{\mathbf{Z}}(J/J^2,I^2/I^3)}{\mathsf{word}_{2*}\,\mathrm{Hom}_{\mathbf{Z}}(J/J^2,J^2/J^3)+\mathsf{word}_1^*\,\mathrm{Hom}_{\mathbf{Z}}(I/I^2,I^2/I^3)}.$$

In fact, using the canonical section $\tau\colon I/I^2\to I/I^3$, (6.7) gives a section

(**6.8.2**)
$$\tau^\sharp\colon \mathrm{Ext}^1_{\mathsf{C}}\big(J/J^2\xrightarrow{\mathsf{word}_1}I/I^2, J^2/J^3\xrightarrow{\mathsf{word}_2}I^2/I^3\big)\to\frac{\mathrm{Hom}_{\mathbf{Z}}(J/J^2,I^2/I^3)}{\mathsf{word}_{2*}\,\mathrm{Hom}_{\mathbf{Z}}(J/J^2,J^2/J^3)}.$$

In the next Lemma, we map the extension class $\big[J/J^3\xrightarrow{\mathsf{word}_2}I/I^3\big]$ into a group that is slightly more intrinsic to G. This is done with the help of the nondegenerate pairing $(\cdot,\cdot)\colon I/I^2\otimes I/I^2\to\mathbf{Z}$, which we assume for simplicity to be valued in $\mathbf{Z}$. Define

$$\mathscr{J}_2\big(E,\mathsf{word}\colon G\to F,(\cdot,\cdot)\big) := \frac{\mathrm{Hom}_{\mathbf{Z}}(J/J^2,(J^2/J^3)^*)}{\mathrm{Hom}_{\mathbf{Z}}(J/J^2,J^2/J^3)}$$

where $(\cdot)^* := \mathrm{Hom}_{\mathbf{Z}}(\cdot,\mathbf{Z})$ and $\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3)$ is identified as a submodule of $\mathrm{Hom}_{\mathbf{Z}}(J/J^2,(J^2/J^3)^*)$ via the inclusion $J^2/J^3 \cong J/J^2 \otimes J/J^2 \hookrightarrow (J/J^2)^* \otimes (J/J^2)^* \cong (J^2/J^3)^*$ given by the pairing $(\cdot,\cdot)$.

(**6.9**) **Lemma.** — *Let $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ be a nondegenerate orthographized group and let I and J be the augmentation ideals of $\mathbf{Z}F$ and $\mathbf{Z}G$, respectively. Then $(\cdot,\cdot)$ and restriction induce a homomorphism*

$$\textbf{(6.9.1)} \qquad \frac{\mathrm{Hom}_{\mathbf{Z}}(J/J^2, I^2/I^3)}{\mathsf{word}_{2*}\,\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3)} \to \frac{\mathrm{Hom}_{\mathbf{Z}}(J/J^2, (J^2/J^3)^*)}{\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3)} =: \mathscr{J}_2\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big).$$

Proof. First we define two auxiliary maps. Since F is free and $(\cdot,\cdot)\colon I/I^2 \times I/I^2 \to \mathbf{Z}$ is nondegenerate, there is a canonical map $\xi\colon I^2/I^3 \cong I/I^2 \otimes I/I^2 \hookrightarrow (I/I^2)^* \otimes (I/I^2)^* \cong (I^2/I^3)^*$. Let $\rho\colon (I^2/I^3)^* \to (J^2/J^3)^*$ be the dual map to $\mathsf{word}_2\colon J^2/J^3 \to I^2/I^3$. Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3) \arrow[r, "\mathsf{word}_{2*}"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}_{\mathbf{Z}}(J/J^2, I^2/I^3) \arrow[r] \arrow[d, "\xi_*"] & Q_1 \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3) \arrow[r, "\xi_* \circ \mathsf{word}_{2*}"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}_{\mathbf{Z}}(J/J^2, (I^2/I^3)^*) \arrow[r] \arrow[d, "\rho_*"] & Q_2 \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3) \arrow[r, "\rho_* \circ \xi_* \circ \mathsf{word}_{2*}"] & \mathrm{Hom}_{\mathbf{Z}}(J/J^2, (J^2/J^3)^*) \arrow[r] & Q_3 \arrow[r] & 0
\end{tikzcd}$$

in which the Q_i are the quotient modules making each row exact.

Clearly, Q_1 is the left hand side of (6.9.1). We shall identify Q_3 with the right hand side, and then the composition $Q_1 \to Q_2 \to Q_3$ will be the desired map. So we need to show that $\rho_* \circ \xi_* \circ \mathsf{word}_{2*}$ in the bottom left corner coincides with the map induced by the inclusion $J^2/J^3 \hookrightarrow (J^2/J^3)^*$ given by $(\cdot,\cdot)$. But this just means that the diagram

$$\begin{tikzcd}
I^2/I^3 \arrow[r, hook, "\xi"] & (I^2/I^3)^* \arrow[d, "\rho"] \\
J^2/J^3 \arrow[u, "\mathsf{word}_2"] \arrow[r, hook] & (J^2/J^3)^*,
\end{tikzcd}$$

should commute. This is the case since the pairing on J^2/J^3 is induced from that on I^2/I^3 by restriction. ∎

(**6.10**) **Harmonic Volume.** Let $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ be a nondegenerate orthographized group and let I and J be as above. Consider the composition

$$\begin{aligned}
\nu\colon \mathrm{Ext}^1_{\mathsf{C}}\big(J/J^2 \xrightarrow{\mathsf{word}_1} I/I^2, J^2/J^3 \xrightarrow{\mathsf{word}_2} I^2/I^3\big) &\xrightarrow{(6.9.1)\circ\tau^\sharp} \mathscr{J}_2\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big) \\
\big[J/J^3 \xrightarrow{\mathsf{word}_2} I/I^3\big] &\longmapsto \nu\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)
\end{aligned}$$

which takes the extension class of (6.8.1) to the image of $\tau^\sharp\big(\big[J/J^3 \xrightarrow{\mathsf{word}_2} I/I^3\big]\big)$ under the homomorphism of (6.9), where $\tau^\sharp$ is the section from (6.8.2). The element $\nu\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ is called the *harmonic volume* of the orthographized group, in analogy to the notion of harmonic volumes of compact Riemann surfaces [Har83, Pul88]. The harmonic volume can be made explicit via the higher cycle pairing.

(**6.11**) **Lemma.** — *Let $\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ be a nondegenerate orthographized group, $\sigma\colon J/J^2 \to J/J^3$ any section, and $\tau\colon I/I^2 \to I/I^3$ the canonical section associated with the generators E. Then the map*

$$\begin{aligned}
\mu\colon J/J^2 &\to (J/J^2)^* \otimes (J/J^2)^* \cong (J^2/J^3)^* \\
\alpha &\mapsto \Big(\omega_1 \otimes \omega_2 \mapsto \int_{\sigma(\alpha)} \omega_1\omega_2 - \int_{\tau(\mathsf{word}_1(\alpha))} \omega_1\omega_2\Big)
\end{aligned}$$

is a representative for $\nu(\mathsf{word}\colon G \to F, (\cdot,\cdot))$. If $\sigma'\colon J/J^2 \to J/J^3$ is another section with associated function $\mu'\colon J/J^2 \to (J^2/J^3)^*$, then, for any $\alpha \in J/J^2$, there are some $\beta_1^i, \beta_2^i \in J/J^2$ such that

$$(\mu - \mu')(\alpha)(\omega_1 \otimes \omega_2) = \int_{\sigma(\alpha)-\sigma'(\alpha)} \omega_1\omega_2 = \sum_i (\beta_1^i, \omega_1)(\beta_2^i, \omega_2)$$

for all $\omega_1 \otimes \omega_2 \in J^2/J^3$.

Proof. By definition of $\nu\big(E, \mathsf{word}\colon G \to F, (\cdot,\cdot)\big)$ in (6.10), we must compute the image of the extension class $\big[J/J^3 \xrightarrow{\mathsf{word}_2} I/I^3\big]$ under $\tau^\sharp$ of (6.8.2), and then under the map (6.9.1). With our choice of σ and τ, (6.7) shows that

$$\mathsf{word}_2 \circ \sigma - \tau \circ \mathsf{word}_1 \in \mathrm{Hom}_{\mathbf{Z}}(J/J^2, I^2/I^3)$$

is a representative for $\tau^\sharp\big(\big[J/J^3 \xrightarrow{\mathsf{word}_2} I/I^3\big]\big)$. Now (6.9.1) acts by post-composing this representative map with the embedding $I^2/I^3 \hookrightarrow (I^2/I^3)^*$ given by the pairing, followed restriction to J^2/J^3. Thus

$$\mu(\alpha)(\omega_1 \otimes \omega_2) = \big((\mathsf{word}_2 \circ \sigma)(\alpha), \omega_1 \otimes \omega_2\big) - \big((\tau \circ \mathsf{word}_1)(\alpha), \omega_1 \otimes \omega_2\big)$$

for any $\omega_1 \otimes \omega_2 \in J^2/J^3$. By (2.8), the pairing between an element of I^2/I^3 and an element of J^2/J^3 coincides with the higher cycle pairing between them, so

$$\mu(\alpha)(\omega_1 \otimes \omega_2) = \int_{\mathsf{word}_2(\sigma(\alpha))} \omega_1\omega_2 - \int_{\tau(\mathsf{word}_1(\alpha))} \omega_1\omega_2.$$

Since the higher cycle pairing over J/J^3 is constructed in (2.11) as pullback via word of the higher cycle pairing from I/I^3, we may omit the mapping word_2 in the first term. Hence we may write

$$\mu(\alpha)(\omega_1 \otimes \omega_2) = \int_{\sigma(\alpha)} \omega_1\omega_2 - \int_{\tau(\mathsf{word}_1(\alpha))} \omega_1\omega_2$$

If $\sigma'\colon J/J^2 \to J/J^3$ is another section, then $\sigma - \sigma' \in \mathrm{Hom}_{\mathbf{Z}}(J/J^2, J^2/J^3)$. Hence, for any $\alpha \in J/J^2$, there are some $\beta_1^i, \beta_2^i \in J/J^2$ such that

$$\sigma(\alpha) - \sigma'(\alpha) = \sum_i (\beta_1^i - 1)(\beta_2^i - 1).$$

Combining this with (2.8) gives the second statement. ■

(**6.12**) **Harmonic Volume of Pointed Graphs.** Let Γ be a connected, bridgeless graph. Set

$$\mathscr{J}_2(\Gamma) := \mathscr{J}_2\big(\mathsf{word}\colon \pi_1(\Gamma, v) \to F\Gamma, (\cdot,\cdot)\big) = \frac{\mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_1(\Gamma,\mathbf{Z}), \mathrm{H}^1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}^1(\Gamma,\mathbf{Z})\big)}{\mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_1(\Gamma,\mathbf{Z}), \mathrm{H}_1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}_1(\Gamma,\mathbf{Z})\big)}.$$

Recall here that the quotient comes from viewing $\mathrm{H}_1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}_1(\Gamma,\mathbf{Z})$ as a sublattice of $\mathrm{H}^1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}^1(\Gamma,\mathbf{Z})$ via the inclusion $\mathrm{H}_1(\Gamma,\mathbf{Z}) \hookrightarrow \mathrm{H}^1(\Gamma,\mathbf{Z})$ induced by the cycle pairing $(\cdot,\cdot)$. Let $W(\Gamma)$ be the set of isomorphism classes of pairs (Γ', v') where Γ' is a graph 2-isomorphic to Γ and $v' \in \Gamma'_0$ is some vertex. By Whitney's Theorem (3.6), we may identify the integral homology and cohomology groups of all graphs in $W(\Gamma)$ with $\mathrm{H}_1(\Gamma,\mathbf{Z})$ and $\mathrm{H}^1(\Gamma,\mathbf{Z})$. Thus there is a well-defined map

$$\begin{aligned} \nu_\Gamma\colon W(\Gamma) &\to \mathscr{J}_2(\Gamma) \\ (\Gamma', v') &\mapsto \nu\big(\mathsf{word}\colon \pi_1(\Gamma', v') \to F\Gamma', (\cdot,\cdot)\big) \end{aligned}$$

that sends a pointed graph (Γ', v') to the harmonic volume (6.10) of its orthographized group (3.2).

As a consequence of the Unipotent Torelli Theorem (5.1), we have the following:

(**6.13**) **Proposition. —** *Let Γ be a connected bridgeless graph. Then the map ν_Γ is injective.*

Proof. Explicitly, we need to show that if (Γ', v') and (Γ'', v'') are elements of $W(\Gamma)$ with the same harmonic volume, then (Γ', v') and (Γ'', v'') are isomorphic as pointed graphs. By Whitney's Theorem (3.6), we may choose isometries

$$\phi'\colon \mathrm{H}_1(\Gamma',\mathbf{Z}) \to \mathrm{H}_1(\Gamma,\mathbf{Z}) \quad \text{and} \quad \phi''\colon \mathrm{H}_1(\Gamma'',\mathbf{Z}) \to \mathrm{H}_1(\Gamma,\mathbf{Z}).$$

Set $\phi := (\phi'')^{-1} \circ \phi'\colon \mathrm{H}_1(\Gamma',\mathbf{Z}) \to \mathrm{H}_1(\Gamma'',\mathbf{Z})$. Write

$$\nu' := \phi'^* \nu_\Gamma(\Gamma', v') \in \mathscr{J}_2(\Gamma') \quad \text{and} \quad \nu'' := \phi''^* \nu_\Gamma(\Gamma'', v'') \in \mathscr{J}_2(\Gamma'')$$

so that our hypothesis means $\phi^* \nu'' = \nu'$. Pick generators $\gamma'_1,\dots,\gamma'_g \in \pi_1(\Gamma', v')$ so that the homology classes $[\gamma'_1],\dots,[\gamma'_g] \in \mathrm{H}_1(\Gamma',\mathbf{Z})$ form a basis. Then the $\phi([\gamma'_j])$ together form a basis for $\mathrm{H}_1(\Gamma'',\mathbf{Z})$. Choose sections $\sigma'\colon J'/J'^2 \to J'/J'^3$ and $\sigma''\colon J''/J''^2 \to J''/J''^3$ thereby yielding isomorphisms

$$\mathbf{Z}\pi_1(\Gamma',v')/J'^3 \cong T_2\mathrm{H}_1(\Gamma',\mathbf{Z}) \quad \text{and} \quad \mathbf{Z}\pi_1(\Gamma'',v'')/J''^3 \cong T_2\mathrm{H}_1(\Gamma'',\mathbf{Z})$$

as $\mathbf{Z}$-algebras. Under this identification, the $\sigma'([\gamma'_1]),\dots,\sigma'([\gamma'_g])$ and $\sigma''(\phi([\gamma'_1])),\dots,\sigma''(\phi([\gamma'_g]))$ are $\mathbf{Z}$-algebra generators for $\mathbf{Z}\pi_1(\Gamma',v')/J'^3$ and $\mathbf{Z}\pi_1(\Gamma'',v'')/J''^3$, respectively.

By (6.11), σ' and σ'' give rise to functions $\mu'\colon J'/J'^2 \to (J'^2/J'^3)^*$ and $\mu''\colon J''/J''^2 \to (J''^2/J''^3)^*$ which represent ν' and ν'', respectively. Thus our assumption that $\phi^*\nu'' = \nu'$ means $\phi^*\mu''$ is another representative of ν'. So by the second statement of (6.11), we see that, for every $j = 1,\dots,g$, there are some $\beta'^i_{1j}, \beta'^i_{2j} \in J'/J'^2$ such that

(6.13.1) $$\mu'([\gamma'_j]) - \phi^*\mu''(\phi([\gamma'_j])) = \textstyle\sum_i (\beta'^i_{1j},\cdot)(\beta'^i_{2j},\cdot) \in \mathrm{Hom}_{\mathbf{Z}}(J'^2/J'^3,\mathbf{Z}).$$

Consider the following $\mathbf{Z}$-algebra map defined on the generators $\sigma'([\gamma'_j])$:

$$\begin{aligned}\varphi\colon \mathbf{Z}\pi_1(\Gamma',v')/J'^3 &\to \mathbf{Z}\pi_1(\Gamma'',v'')/J''^3\\ \sigma'([\gamma'_j]) &\mapsto \sigma''(\phi([\gamma'_j])) + \textstyle\sum_i \big(\sigma''(\phi(\beta'^i_{1j})) - 1\big)\big(\sigma''(\phi(\beta'^i_{2j})) - 1\big).\end{aligned}$$

Since σ' and σ'' are sections to the homology class maps, φ induces the isometry ϕ on homology, and hence φ is an isomorphism of $\mathbf{Z}$-algebras.

We claim that φ preserves the higher cycle pairing. We use (6.13.1) to obtain a relationship on higher cycle pairings. First, we claim that for any $\alpha', \omega'_1, \omega'_2 \in \mathrm{H}_1(\Gamma',\mathbf{Z})$,

(6.13.2) $$\int_{\mathsf{word}_1(\alpha')} \omega'_1\omega'_2 = \int_{\mathsf{word}_1(\phi(\alpha'))} \phi(\omega'_1)\phi(\omega'_2),$$

where we view $\mathsf{word}_1(\alpha') \in \mathrm{C}_1(\Gamma',\mathbf{Z})$ as an element of $\mathrm{C}_1(\Gamma',\mathbf{Z}) \oplus \mathrm{C}_1(\Gamma',\mathbf{Z}) \otimes \mathrm{C}_1(\Gamma',\mathbf{Z})$ via the canonical inclusion τ', which we suppress in the notation above. Similar remarks go for $\mathrm{C}_1(\Gamma'',\mathbf{Z})$. By the Torelli Theorem (4.1), there exists an isometry $\Phi\colon \mathrm{C}_1(\Gamma',\mathbf{Z}) \to \mathrm{C}_1(\Gamma'',\mathbf{Z})$ that lifts the isometry $\phi\colon \mathrm{H}_1(\Gamma',\mathbf{Z}) \to \mathrm{H}_1(\Gamma'',\mathbf{Z})$. Writing $\mathsf{word}_1(\alpha') = \sum_i c'_i e'_i \in \mathrm{C}_1(\Gamma',\mathbf{Z})$, for some $c'_i \in \mathbf{Z}$ and $e'_i \in \Gamma'_1$, we have

$$\begin{aligned}\int_{\mathsf{word}_1(\phi(\alpha'))} \phi(\omega'_1)\phi(\omega'_2) &= \sum_i c'_i \int_{\Phi(e'_i)} \Phi(\omega'_1)\Phi(\omega'_2)\\ &= \sum_i \frac{c'_i}{2}(\Phi(e'_i),\Phi(\omega'_1))(\Phi(e'_i),\Phi(\omega'_2))\\ &= \sum_i \frac{c'_i}{2}(e'_i,\omega'_1)(e'_i,\omega'_2)\\ &= \sum_i c'_i \int_{e'_i} \omega'_1\omega'_2 = \int_{\mathsf{word}_1(\alpha')} \omega'_1\omega'_2,\end{aligned}$$

where the first equality comes from viewing $\mathrm{H}_1(\Gamma'',\mathbf{Z})$ as a sublattice of $\mathrm{C}_1(\Gamma'',\mathbf{Z})$, that Φ lifts φ, and the construction of the higher cycle pairing (2.11); the second equality comes from the definition of the higher cycle pairing (2.1); the third equality from the isometric nature of Φ; and the remaining equalities due to the analogous reasons as the first three.

Now expand the left hand side of (6.13.1) using the formula of (6.11). The second terms in the expressions for μ' and μ'' cancel thanks to (6.13.2). Thus, for every $j = 1,\dots,g$ and any $\omega'_1,\omega'_2 \in \mathrm{H}_1(\Gamma',\mathbf{Z})$,

$$\int_{\sigma'([\gamma'_j])} \omega'_1\omega'_2 = \int_{\sigma''(\phi([\gamma'_j]))} \phi(\omega'_1)\phi(\omega''_2) + \textstyle\sum_i \big(\sigma''(\phi(\beta'^i_{1j})),\phi(\omega'_1)\big)\big(\sigma''(\phi(\beta'^i_{2j})),\phi(\omega'_2)\big).$$

Comparing this with the definition of φ and applying (2.8), we see that this says that

$$\int_{\sigma'([\gamma'_j])} \omega'_1\omega'_2 = \int_{\varphi(\sigma'([\gamma'_j]))} \phi(\omega'_1)\phi(\omega'_2).$$

Since ϕ is the map on homology induced by φ, this equation precisely says φ preserves the higher cycle pairing over $\sigma'([\gamma'_j])$. Since this is true for all $j = 1, \ldots, g$ and the $\sigma'([\gamma'_1]), \ldots, \sigma'([\gamma'_g])$ are generators for $\mathbf{Z}\pi_1(\Gamma', v')/J'^3$, φ preserves the higher cycle pairing. The Unipotent Torelli Theorem (5.1) then applies to show that (Γ', v') and (Γ'', v'') are isomorphic as pointed graphs. ■

(**6.14**) *Remark.* Proposition (6.13) provides another interpretation of (6.1). In general, the embedding $\mathrm{H}_1(\Gamma, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(\Gamma, \mathbf{Z})$ given by the inner product is not surjective and the cokernel is a torsion group; in fact, the cokernel is the Jacobian of the graph Γ. Likewise, the embedding

$$\mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_1(\Gamma,\mathbf{Z}), \mathrm{H}_1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}_1(\Gamma,\mathbf{Z})\big) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_1(\Gamma,\mathbf{Z}), \mathrm{H}^1(\Gamma,\mathbf{Z}) \otimes \mathrm{H}^1(\Gamma,\mathbf{Z})\big)$$

has torsion cokernel. Hence the harmonic volume of a pointed graph takes values in a torsion abelian group. Thus $\mathscr{J}_2(\Gamma) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$ so different pointed graphs (Γ', v') cannot be distinguished by means of their harmonic volume or higher cycle pairing on the rational group algebra.

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