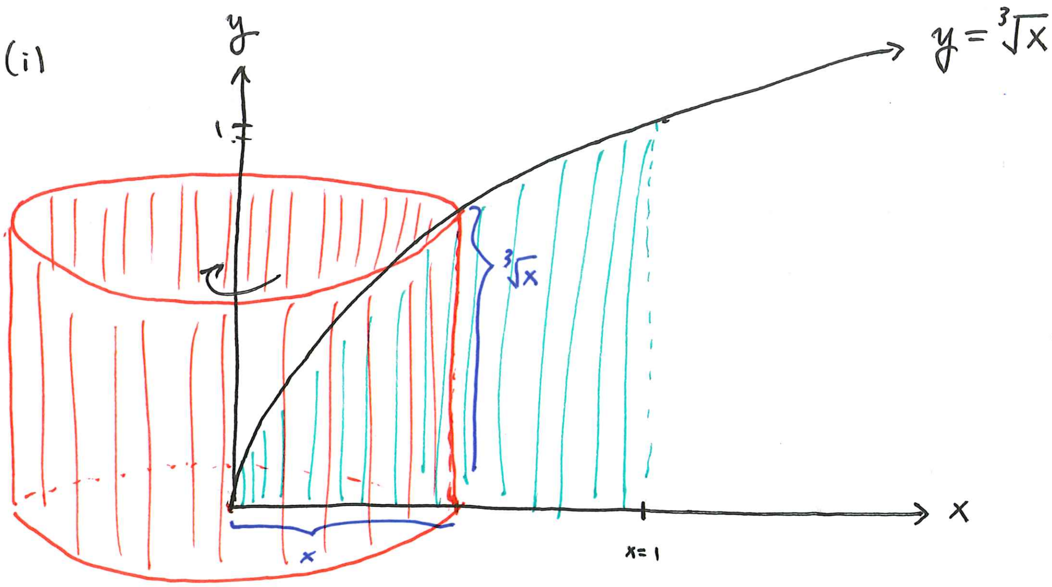
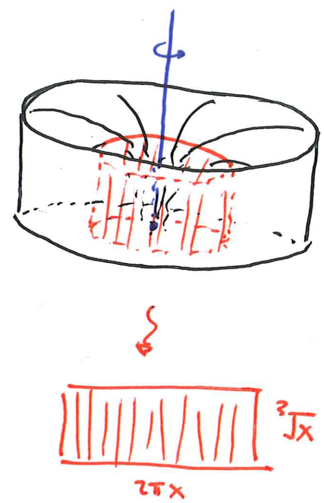
Calculus II Assignment 12

Solutions

1. Time to find some volumes of solids of revolution via cylindrical shells.

(i)



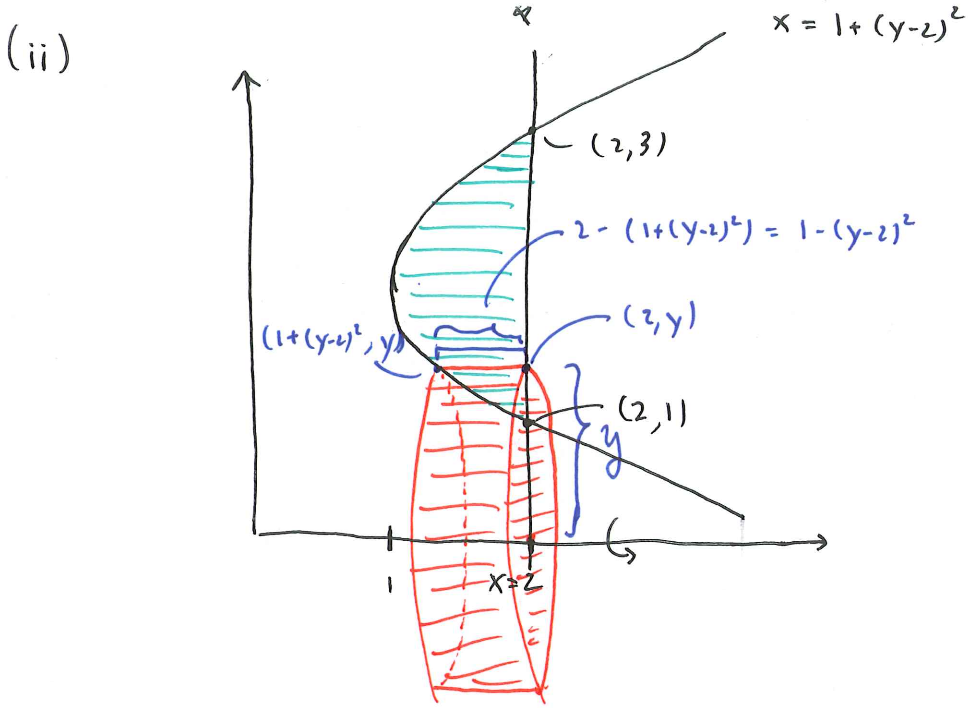


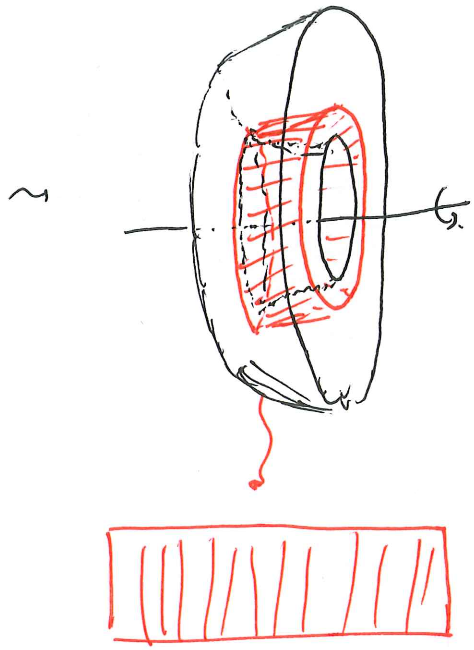
$$V = \int_0^1 2\pi x \cdot \sqrt[3]{x} \; dx$$

$$= 2\pi \int_0^1 x^{4/3} \, dx = 2\pi \cdot \frac{3}{7} x^{7/3} \Big|_0^1$$

$$\Longrightarrow \boxed{V = \frac{6\pi}{7}} \quad \text{———} //$$

(ii)





$$V = \int_1^3 2\pi y \cdot (1-(y-2)^2)\, dy$$

$$= 2\pi \int_1^3 y(4y - y^2 - 3)\, dy$$

$$= 2\pi \left[\frac{4}{3}y^3 - \frac{1}{4}y^4 - \frac{3}{2}y^2 \right]_1^3$$

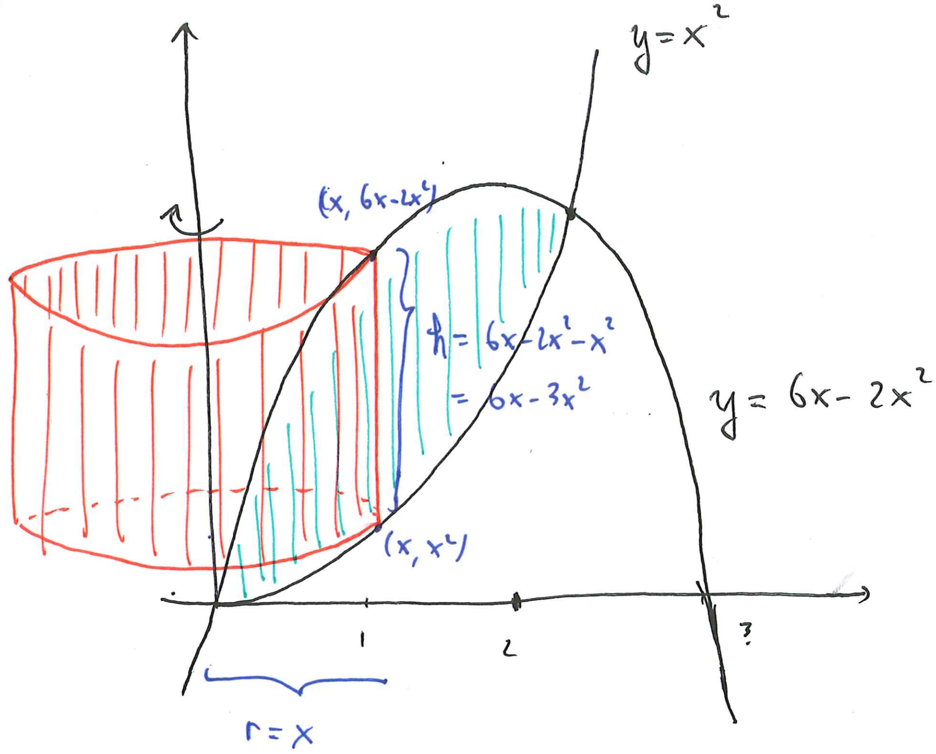
$$= 2\pi \left(36 - \frac{81}{4} - \frac{27}{2} - \frac{4}{3} + \frac{1}{4} + \frac{3}{2} \right)$$

$$= 2\pi \left(36 - 20 - 12 - \frac{4}{3} \right) = 2\pi \cdot \left(4 - \frac{4}{3} \right)$$

$$\Longrightarrow \boxed{V = \frac{16\pi}{3}}$$

//

(iii)



$$\Rightarrow \quad V = \int_0^2 2\pi x \cdot (6x - 3x^2)\, dx$$

$$= 6\pi \int_0^2 2x^2 - x^3 \, dx$$

$$= 6\pi \left[\frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^2$$

$$= 6\pi \left(\frac{16}{3} - \frac{16}{4} \right)$$

$$\Rightarrow \quad \boxed{V = 8\pi}$$

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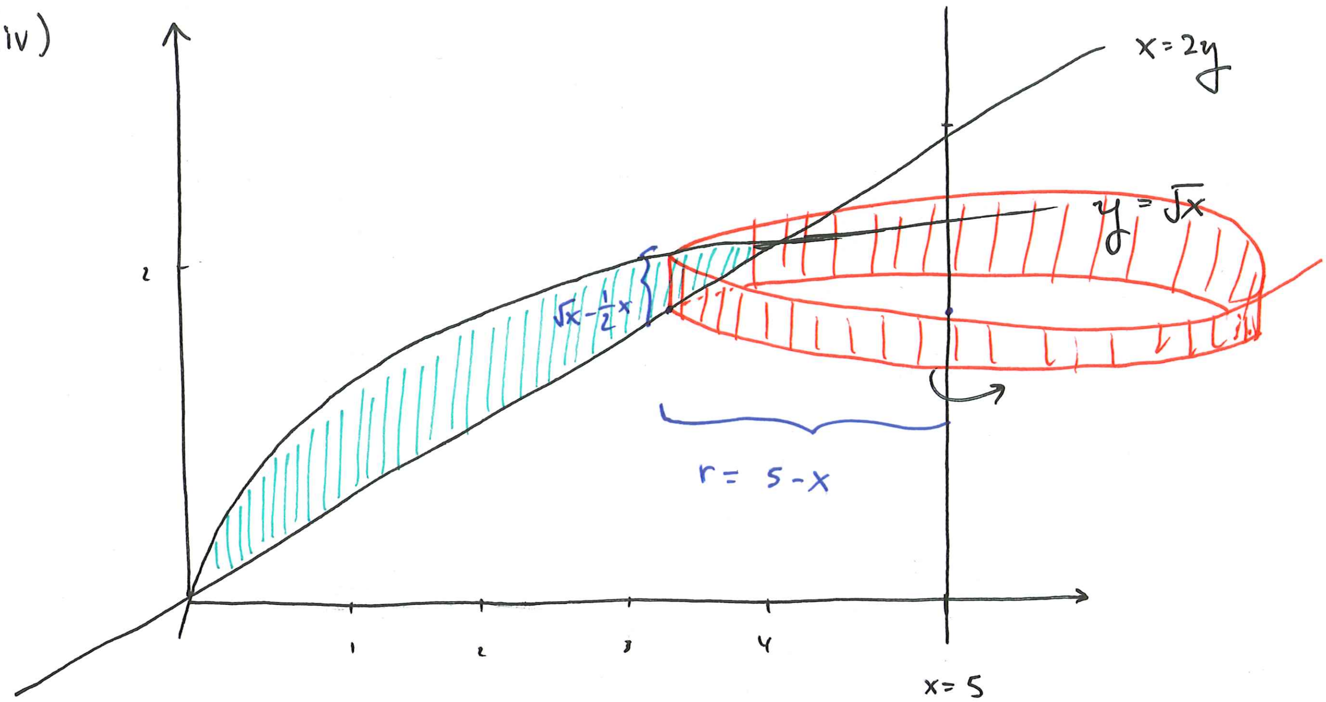
(iv)



$$\Rightarrow \quad V = \int_0^4 2\pi(5-x)\cdot\left(\sqrt{x} - \frac{1}{2}x\right)\,dx$$

$$= 2\pi \int_0^4 5\sqrt{x} - \frac{5}{2}x - x^{3/2} + \frac{1}{2}x^2 \, dx$$

$$= 2\pi \left[\frac{10}{3}x^{3/2} - \frac{5}{4}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{6}x^3 \right]_0^4$$

$$= 2\pi \left[\frac{80}{3} - 5 - \frac{64}{5} + \frac{4}{3} \right]$$

$$= 2\pi \left(28 - 5 - \frac{64}{5} \right) = 2\pi \left(23 - \frac{64}{5} \right)$$

$$\Rightarrow \quad \boxed{V = \frac{102\pi}{5}}$$

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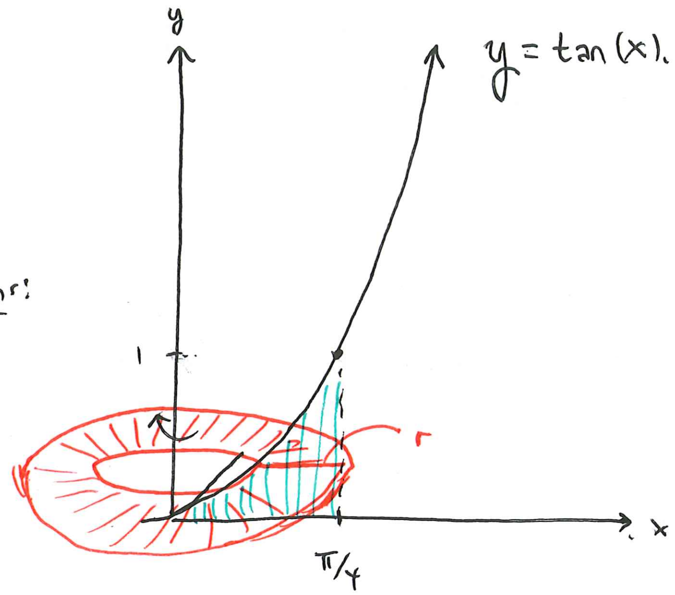
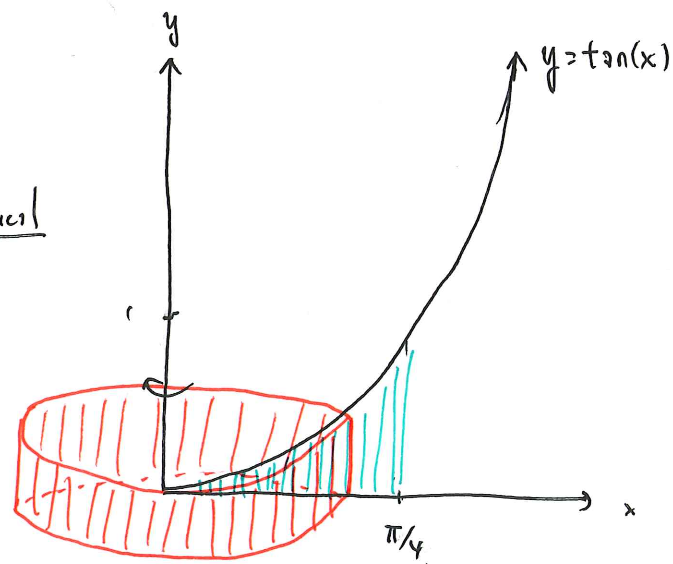
2. Let's use both the cross-sectional and cylindrical methods to compute the following solid of revolution:

Cross-sections:



$$V = \int_0^1 \pi \cdot \left(\left(\frac{\pi}{4}\right)^2 - \arctan(y)^2 \right) dy$$

cylindrical



$$V = \int_0^{\pi/4} 2\pi x \tan(x)\, dx$$

A computer then can tell you that $V \approx 1.16732\ldots$

in both cases.

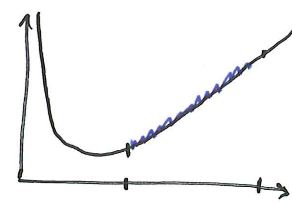
$\rule{2cm}{0.4pt}$ //

3. Time to find the lengths of some curves. This amounts to applying the formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Most of the integrals are impossible...

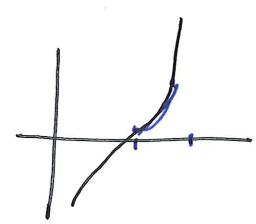
(i) $y = x - \log(x)$ between $1 \leq x \leq 4$.



$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x}, \text{ so:}$$

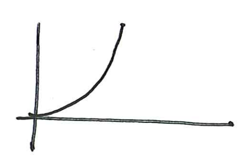
$$L = \int_1^4 \sqrt{1 + (1 - \tfrac{1}{x})^2}\, dx \approx 3.4467\ldots$$

(ii) $y = \dfrac{x^3}{3} - \dfrac{1}{4x}$ between $1 \leq x \leq 2$



$$L = \int_1^2 \sqrt{1 + (x^2 + \tfrac{1}{4x^2})^2} \approx 21.5258\ldots$$

(iii) $y = \log(\sec(x))$ between $0 \leq x \leq \frac{\pi}{4}$

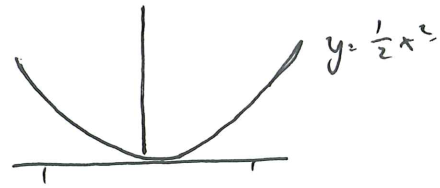


$$\frac{dy}{dx} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x).$$

$$\Rightarrow L = \int_0^{\pi/4} \sqrt{1 + \tan(x)^2}\, dx = \int_0^{\pi/4} \sec(x)\, dx$$

$$= \log(\sec(x) + \tan(x))\Big|_0^{\pi/4} = \log(\sqrt{2} + 1) \approx 0.8814\ldots$$

(iv) $y = \frac{1}{2}x^2$ between $-1 \le x \le 1$.



$$\Rightarrow \frac{dy}{dx} = x$$

$$\Rightarrow L = \int_{-1}^{1} \sqrt{1+x^2}\, dx = 2\int_0^1 \sqrt{1+x^2}\, dx$$

$x = \tan(\theta)$

$dx = \sec(\theta)^2 d\theta$

$$= 2\int_0^{\pi/4} \sec(\theta)^3\, d\theta$$

$$= \Big[\sec(\theta)\tan(\theta) + \log(\sec(\theta) + \tan(\theta)) \Big]_0^{\pi/4}$$

$$= \sqrt{2} + \log(1+\sqrt{2})$$

$$\approx 2.295587\ldots$$

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<u>Aside</u>. The integral above can be done by the following:

$$\int \sec(\theta)^3 d\theta = \int \sec(\theta)(1+\tan(\theta)^2)\, d\theta = \int \sec(\theta)\, d\theta + \int \underline{\sec(\theta)\tan(\theta)^2 d\theta}$$

$$= \log(\sec(\theta)+\tan(\theta)) + \left[\sec(\theta)\tan(\theta) - \int \sec(\theta)^3 d\theta \right]$$

$\underset{\text{integrate}}{\underline{\sec(\theta)\tan(\theta)}} \cdot \underset{\text{diff.}}{\underline{\tan(\theta)}}$

$$\Longrightarrow 2\int \sec(\theta)^3 d\theta = \log(\sec(\theta)+\tan(\theta)) + \sec(\theta)\tan(\theta)$$

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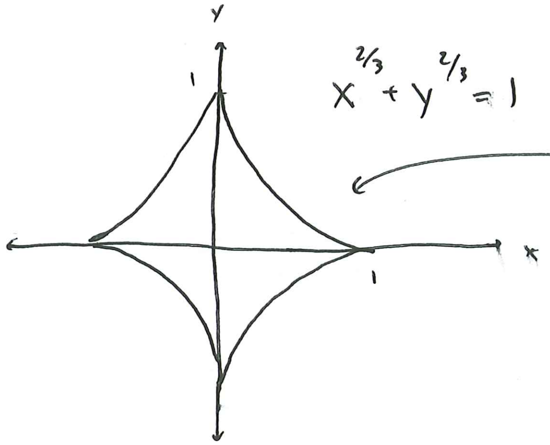
4. Now to find the length of the curve



L

we find the length of this segment and multiply by 4 to get everything — Yay symmetry!

First! $\dfrac{dy}{dx}$ is found by differentiating the relation given:

$$0 = d(1) = d\left(x^{2/3} + y^{2/3}\right) = \frac{2}{3}x^{-1/3}\,dx + \frac{2}{3}y^{-1/3}\,dy$$

$$\Longrightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}.$$

Thus! $$L = \int_0^1 \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}\,dx$$

$y^{2/3} = 1 - x^{2/3}$ from the defining equation

$$= \int_0^1 \sqrt{1 + \frac{1 - x^{2/3}}{x^{2/3}}}\,dx$$

$$= \int_0^1 \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3}\Big|_0^1 = \frac{3}{2}$$

$$\Longrightarrow \boxed{\text{Length}\left(\quad\right) = 4L = 6.}$$ —//