SEMINAR ON COMBINATORIAL RECIPROCITY THEOREMS: LECTURE 1

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This seminar is centred around the combinatorial study of discrete geometric objects such as polytopes, the central questions being enumerative. Before bringing geometry into the fray, we should discuss a little about counting. To begin, let's set some notation: for a positive integer n, let

$$[n] := \{1, \ldots, n\}$$

be the set consisting of the first n positive integers.

(**1**) **Question. —** *How many subsets of $[n]$ are there? More precisely, what is the cardinality of the set*

$$2^{[n]} := \{S \subseteq [n]\}$$

consisting of subsets of $[n]$?

As an example, consider the case $n = 1$:

$$2^{[1]} = \{\varnothing, [1]\}$$

and thus we see that

$$\#2^{[1]} = 2 (= 2^1).$$

Now in the next few paragraphs, I shall give two different approaches to answering this question, each interesting for exposing a different approach to counting.

(**2**) **First Answer: Induction.** This first approach to (1) is via induction, in a form that some might refer to as recursion. The observation is that the set $2^{[n]}$ has two distinguished disjoint subsets:

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \sqcup \{S \subseteq [n] \mid n \in S\}. \tag{2.1}$$

The first member of (2.1) is none other than the set $2^{[n-1]}$; after all, a subset of $[n]$ not containing n is simply a subset of $[n] \setminus \{n\} = [n-1]$. But the second member (2.1) is also basically $2^{[n-1]}$: if everyone contains n, then no one contains n. More precisely, there is a bijection

$$\begin{aligned} \{S \subseteq [n] \mid n \in S\} &\leftrightarrow 2^{[n-1]} \\ S &\mapsto S \setminus \{n\} \\ T \cup \{n\} &\leftarrow\!\shortmid T \end{aligned}$$

In particular, both members on the right hand side of (2.1) have the same size as $2^{[n-1]}$. Taking cardinalities on both sides of (2.1), we arrive at

$$\#2^{[n]} = \#2^{[n-1]} + \#2^{[n-1]} = 2 \cdot \#2^{[n-1]}.$$

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Applying this argument to $2^{[n-1]}$, then $2^{[n-2]}$, and so forth, we see that

$$\#2^{[n]} = 2 \cdot \#2^{[n-1]} = 2^2 \cdot \#2^{[n-2]} = \cdots = 2^{n-1} \cdot \#2^{[1]} = 2^n.$$

(**3**) **Second Answer: Bijection.** The second approach to (1) is perhaps what some might call more combinatorial, as I am going to relate $2^{[n]}$ with a set of objects that is a bit easier to count. Toward this, set some notation: given positive integers a and b, let

$$[a]^{[b]} := \{ f \colon [b] \to [a] \}$$

be the set of functions from the set $[b]$ to the set $[a]$. I now claim that

$$\#[a]^{[b]} = a^b. \tag{3.1}$$

Note there are absolutely no restrictions on what sorts of functions that appear in $[a]^{[b]}$ and this is what makes counting the elements of $[a]^{[b]}$ manageable. The basic idea is that each of the b elements of $[b]$ have a possible places to go in $[a]$, so

$$\begin{aligned} \#[a]^{[b]} &= \#(\text{places for } 1 \in [b]) \times \#(\text{places for } 2 \in [b]) \times \cdots \times \#(\text{places for } b \in [b]) \\ &= a \times a \times \cdots \times a = a^b. \end{aligned}$$

As an aside, this argument can be formalized by saying there is a bijection

$$\begin{aligned} [a]^{[b]} \leftrightarrow \prod_{i=1}^{b} [a] &:= \{ (x_1, x_2, \ldots, x_b) \mid x_i \in a \} \\ f \mapsto (f(1), f(2), &\ldots, f(b)) \end{aligned}$$

and the right hand side has cardinality a^b.

To return to (1), I now claim there is a bijection between $2^{[n]}$ and $[2]^{[n]}$, which sends a subset $S \subseteq [n]$ to its *characteristic function*

$$\begin{aligned} f_S \colon [n] &\to [2] \\ i &\mapsto \begin{cases} 1 & \text{if } i \notin S, \\ 2 & \text{if } i \in S. \end{cases} \end{aligned}$$

The function $2^{[n]} \to [2]^{[n]} \colon S \mapsto f_S$ is easily seen to be a bijection, so

$$\#2^{[n]} = \#[2]^{[n]} = 2^n$$

by (3.1).

(**4**) So far, I have presented two approaches to (1). Each method illuminates a different structural property of the set $2^{[n]}$ and exemplify a different principle in approaching counting problems. Let me try to digest some of these ideas:

– One of the first things to try to do when trying to count a set of things is to try to break the set into smaller pieces consisting of things you might know how to count. In practice, one way of doing so is to try to find recursive structure in the set involved so that you reduce your problem to a smaller problem, until you get to something you can really just write down.
– Another thing to do is to find more convenient descriptions of the objects you wish to count, which is to say, you would like to relate the objects you wish to count with something else that is more amenable to direct investigation.

Of course, these are vague, overly general principles. The only way to really get some meaning out of these is to work out some examples and see them in practice, as in the Examples above.

On that note, let's begin to consider a slightly more interesting counting problem.

(**5**) **Question. —** *For any positive integers n and k, how many k-element subsets does $[n]$ have? More precisely, what is the size of the set*

$$\binom{[n]}{k} := \{ S \subseteq [n] \mid \#S = k \}$$

consisting of k-element subsets of $[n]$?

I will provide two answers to this question, yet again showcasing some different methods to counting.

(**6**) **First Answer: Algebraic Trick.** This first method is based on an observation that goes back to Dirichlet. What is a k-element subset? Well, by definition, it is an unordered collection of k distinct elements of n. This can be encoded, in a sense, algebraically, by considering the product

$$(1+x_1)(1+x_2)\cdots(1+x_n), \tag{6.1}$$

where $x_1, x_2, \ldots, x_n$ are distinct indeterminates. Specifically, consider expanding this product. To do so, you would choose either 1 or x_1 from the first parenthesized term, then choose either 1 or x_2 from the second, and so forth until you have finally chosen either 1 or x_n from the final term. You would then multiply the collection of elements you have chosen and that is one of the terms in the expansion of (6.1). The full expansion of (6.1) involves doing this for all possible choices. Meditating on this, we find that

$$\#\binom{[n]}{k} = \text{number of degree } k \text{ monomials in } (1+x_1)(1+x_2)\cdots(1+x_n). \tag{6.2}$$

This doesn't quite simplify the problem of counting subsets of $[n]$, but the simple—albeit very clever—observation is that if we set each of the variables $x_1, \ldots, x_n$ to a single variable x, then the right hand side of (6.2) amounts to finding the coefficient of x^k:

$$\begin{aligned}
\#\binom{[n]}{k} &= \text{number of degree } k \text{ monomials in } (1+x_1)(1+x_2)\cdots(1+x_n) \\
&= \text{coefficient of } x^k \text{ in } (1+x)(1+x)\cdots(1+x) \\
&= \text{coefficient of } x^k \text{ in } (1+x)^n.
\end{aligned}$$

Now, either by induction or by the Binomial Theorem or by some other means,

$$\text{coefficient of } x^k \text{ in } (1+x)^n = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

and the number on the right is traditionally denoted by $\binom{n}{k}$. Therefore, we see that

$$\#\binom{[n]}{k} = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{6.3}$$

(**7**) **Second Answer: Ordering a Subset.** The answer to (5) given in (6) introduces an powerful algebraic technique in counting. However, it is ultimately unsatisfying to me because, well, I would like to prove the Binomial Theorem using the observation (6.2). So here let me present a more combinatorial approach which shall produce the same answer via some bijection of sets.

As mentioned in (6), a k element subset of $[n]$ is the same as an unordered collection of k distinct elements in $[n]$. Now the point to this second approach is that counting becomes a lot easier when considering ordered sequence of distinct elements. So consider the sets

$$\begin{aligned} \mathfrak{S}_k &:= \{\text{orderings on } k \text{ elements}\}, \\ \mathrm{Seq}(k,n) &:= \{\text{ordered sequences of } k \text{ distinct elements of } [n]\}. \end{aligned}$$

Then there is a bijection

$$\begin{gathered} \mathfrak{S}_k \times \binom{[n]}{k} \leftrightarrow \mathrm{Seq}(k,n) \\ \left(\begin{matrix}\text{ordering on} \\ k \text{ elements}\end{matrix}, \begin{matrix}\text{unordered collection} \\ \text{of } k \text{ elements of } [n]\end{matrix}\right) \mapsto \left(\begin{matrix}\text{ordered sequence of} \\ k \text{ elements of } [n]\end{matrix}\right). \end{gathered} \tag{7.1}$$

By the arguments used to count $[a]^{[b]}$ in (3), one shows that

$$\#\mathfrak{S}_k = k! \quad \text{and} \quad \#\mathrm{Seq}(k,n) = n(n-1)\cdots(n-k+1)$$

so that the bijection (7.1) yields

$$\#\mathfrak{S}_k \times \#\binom{[n]}{k} = k! \cdot \#\binom{[n]}{k} = n(n-1)\cdots(n-k+1) = \#\mathrm{Seq}(k,n).$$

Rearranging gives the equality (6.3).

(**8**) *Aside: Making the Above Argument Rigorous.* The argument sketched in (7) contains the essential ideas to solving (5), and will be completely comprehensible to someone with some training in combinatorics. But let's try to make the argument a bit more precise. For that, let

$$\begin{aligned} \mathfrak{S}_k &:= \{\pi \colon [k] \to [k] \mid \pi \text{ is a bijection}\}, \\ \mathrm{Seq}(n,k) &:= \{(x_1, x_2, \ldots, x_k) \mid x_i \in [n] \text{ distinct}\}. \end{aligned}$$

These definitions make sense of the ideas of orderings on k elements and ordered sequences of k distinct elements of $[n]$. Then the bijection (7.1) is given by the map

$$\begin{gathered} \mathfrak{S}_k \times \binom{[n]}{k} \leftrightarrow \mathrm{Seq}(n,k) \\ \big(\pi, \{i_1 < \cdots < i_k\}\big) \mapsto (i_{\pi(1)}, \ldots, i_{\pi(k)}). \end{gathered}$$

(**9**) **Relating (1) and (5).** Questions (1) and (5) are, of course related: each subset of $[n]$ has a size between 0 and n, so there is a decomposition

$$2^{[n]} = \binom{[n]}{0} \sqcup \binom{[n]}{1} \sqcup \binom{[n]}{2} \sqcup \cdots \sqcup \binom{[n]}{n} \tag{9.1}$$

of the collection of subsets of $[n]$ into subcollections according to the size of the given subset of $[n]$. Taking the sizes of both sides of (9.1) and using our knowledge of the sizes of each set involved gives the following equation between numbers:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}. \tag{9.2}$$

Thus the equality of *sets* in (9.1) is a "lift" of the equaltiy of *numbers* in (9.2). In fancy, modern parlance, one might call the identity (9.1) as a *categorification* of the identity (9.2); maybe more on this in the future.

(**10**) **Polynomial Counting Function.** Let's look at the quantity (6.3) and make the following rather simple observation: the number

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

is the evaluation at n of the following *polynomial*, say, in the variable x of degree k:

$$\frac{x(x-1)\cdots(x-k+1)}{k!}. \tag{10.1}$$

From now on, we abusively view $\binom{n}{k}$ as signifying the polynomial (10.1), which allows us to put things other than positive integers in the top argument. In particular, we can consider the quantity

$$\binom{-n}{k} = \frac{-n(-n-1)\cdots(-n-k+1)}{k!}. \tag{10.2}$$

A question that one with an enumerative bent might ask is: do the numbers (10.2) count something? Posed as such, the answer to this question is: at least half the time, no. The reason is simply that (10.2) may not be a positive integer, and thus has no chance of being a count of anything.

But this is a relatively minor issue: taking out the minus signs in each term of the numerator of (10.2), we have the identity

$$\binom{-n}{k} = \frac{-n(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k\frac{n(n+1)\cdots(n+k-1)}{k!} = (-1)^k\binom{n+k-1}{k}.$$

Thus we have the following sensible

(**11**) **Question. —** *Does the number*

$$(-1)^k\binom{-n}{k} = \binom{n+k-1}{k}$$

count something related to $[n]$*?*

(**12**) To provide an answer to (11), let me define a *multiset* to be a "set with multiplicities": think of a multiset as consisting of the data of a set, and for each element of the set, a positive integer that tells you how many times this particular element shows up. The size of a multiset is then the sum of the multiplicities of each element in the underlying set. Define a *multisubset* of a set S to be a multiset whose underlying set is a subset of S. I now claim that

$$\#\{\text{size } k \text{ multisubsets of } [n]\} = (-1)^k \binom{-n}{k}.$$

Let me give a cheap proof, in the spirit of (6):

$$\begin{aligned}\#\{\text{size } k \text{ multisubsets of } [n]\} &= \text{number of degree } k \text{ monomials in } \prod_{i=1}^{n}(1 + x_i + x_i^2 + \cdots) \\ &= \text{number of degree } k \text{ monomials in } \prod_{i=1}^{n}\frac{1}{1-x_i} \\ &= \text{coefficient of } x^k \text{ in } \left(\frac{1}{1-x}\right)^n\end{aligned}$$

where the second equality is due to the geometric series formula, and the second and third lines are to be made sense of upon taking a formal power series expansion. To find the coefficient of x^k in the last line, we use the following algebraic method: given any formal power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$,

$$[x^k]f(x) := \text{coefficient of } x^k \text{ in } f(x) = a_k = \frac{1}{k!}\frac{d^k}{dx^k}f(x)|_{x=0}.$$

Now, by the chain rule,

$$\begin{aligned}\frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{1}{1-x}\right)^n &= \frac{1}{k!}\frac{d^k}{dx^k}(1-x)^{-n} = (-1)\frac{-n}{k!}\frac{d^{k-1}}{dx^{k-1}}(1-x)^{-n-1} \\ &= \cdots = (-1)^k\frac{-n(-n-1)\cdots(-n-k+1)}{k!}(1-x)^{-n-k}.\end{aligned}$$

Evaluating at $x = 0$, we see that

$$\#\{\text{size } k \text{ multisubsets of } [n]\} = (-1)^k\frac{-n(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k\binom{-n}{k}$$

as claimed.

Putting the answers to (5) and (11) together, we have proven

(**13**) **Theoerm. —** *Let k be a positive integer. Then for any positive integer n,*

$$\#\{\text{size } k \text{ subsets of } [n]\} = \binom{n}{k},$$

$$\#\{\text{size } k \text{ multisubsets of } [n]\} = (-1)^k\binom{-n}{k}.$$

This is our first *Combinatorial Reciprocity Theorem*. I am basically echoing Richard Stanley, but the precise meaning of what a Combinatorial Reciprocity Theorem is might be a bit murky, but you will know it when you see it. I hope, however, that they will not offend you, unlike some other things you just know when you see...