CALCULUS II MIDTERM II PRACTICE SOLUTIONS

These are model solutions for the practice midterm. They are much more detailed than I expect your response to be, but do make sure that your arguments include the core parts of the arguments here.

1. Consider the sequence $\{a_n\}_n$ defined recursively by $a = 1$ and for $n \geq 2$,

$$a_n = \frac{1}{3}(a_{n-1} + 4).$$

(i) Assuming that the sequence $\{a_n\}_n$ is convergent, compute its limit.
(ii) State the Monotone Convergence Theorem.
(iii) Show that $\{a_n\}_n$ is increasing and that $a_n < 2$ for all n by considering the function $f(x) := \frac{1}{3}(x+4)$ and its derivative.
(iv) Deduce that the sequence $\{a_n\}_n$ is convergent.

Solutions. For (i), assuming that $\{a_n\}_n$ is convergent with limit L, we compute:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3}(a_{n-1} + 4) = \frac{1}{3}(L+4).$$

Rearranging gives $2L = 4$, so $L = 2$.

The Monotone Convergence Theorem requested in (ii) is the following: *Every bounded, monotonic sequence is convergent.* Recall that a sequence $\{a_n\}_n$ is *bounded* if there are numbers m and M such that $m \leq a_n \leq M$ for every n; it is said to be *monotonic* if it is either increasing or decreasing.

To show that the sequence in question is increasing, consider the function $f(x) = \frac{1}{3}(x+4)$, the point being that $a_n = f(a_{n-1})$. The graph $y = f(x)$ of f is a line with slope $1/3$. Moreover, this line intersects the line $y = x$ at $x = 2$, and for $x < 2$, the graph of $y = f(x)$ lies over that of $y = x$; in other words, $f(x) > x$ for all $x < 2$. Finally, drawing the horizontal line at $y = 2$, we see that $f(x) < 2$ for all $x < 2$. Since $a_1 = 1$ and $a_n = f(a_{n-1})$, these together show that $a_n < 2$ for all n and that the sequence is increasing.

Finally, for (iv), since we have shown that the sequence $\{a_n\}_n$ is increasing, is bounded from above by 2, and bounded below by 1, it follows from the Monotone Convergence Theorem stated in (ii) that $\{a_n\}_n$ actually converges. ■

2. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\log(n)}}$$

is convergent or divergent. Explain.

Solution. The series is divergent. Since the terms of the series are all positive and decreasing, we can apply the Integral Test, that is, consider the associated improper integral

$$\int_2^\infty \frac{dx}{x\sqrt{\log(x)}} = \lim_{N\to\infty}\int_{\log(2)}^{\log(N)} \frac{du}{\sqrt{u}} = \lim_{N\to\infty} 2\sqrt{\log(N)} - 2\sqrt{\log(2)} = \infty$$

where I have used the substitution $u = \log(x)$ in the first equality. Since the improper integral diverges, the Integral Test implies that the associated series is also divergent. ■

3. Is the series

$$\sum_{n=1}^\infty (-1)^{n-1}\frac{\sqrt{n}}{n+1}$$

convergent or divergent? Justify your answer.

Solution. We claim that this series has terms that are tending to zero and which are decreasing in absolute value. Since this is an alternating series, once we have verified these claims, the Alternating Series Test applies and allows us to conclude that this series is convergent.

To see that the terms limit to zero, we compute:

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = \lim_{n\to\infty}\frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0.$$

To see that the terms of the series decrease in absolute value, one could directly compare the terms and show that

$$\frac{\sqrt{n+1}}{n+2} \le \frac{\sqrt{n}}{n+1}.$$

A slicker way to do this is the following: consider the function $f(x) := \sqrt{x}/(x+1)$. Note that absolute value of the n^{th} term of the series is $f(n)$. We show that $f(x)$ is a decreasing function, at least for $x \ge 1$. The derivative of f is

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}(x+1)} - \frac{\sqrt{x}}{(x+1)^2} = \frac{(x+1)-2x}{2\sqrt{x}(x+1)^2} = \frac{-x+1}{2\sqrt{x}(x+1)^2}.$$

Thus $f'(x) \le 0$ for $x \ge 1$, which means that the function $f(x)$ is decreasing for $x \ge 1$. In particular, $f(n+1) \le f(n)$ for all $n \ge 1$, and hence the terms are decreasing. This shows our two claims above and we are done. ■

4. Does the series

$$\sum_{n=1}^\infty \log\left(\frac{n}{3n+1}\right)$$

converge or diverge? Why?

Solution. The series diverges. Indeed, consider the limit of the terms of the series:

$$\lim_{n\to\infty} \log\left(\frac{n}{3n+1}\right) = \log\left(\lim_{n\to\infty}\frac{n}{3n+1}\right) = \log(1/3) \neq 0.$$

Therefore, by the Divergence Test, the series diverges. ■

5. Determine the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} n^3 x^n.$$

Bonus: Find its sum.

Proof. Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(n+1)^3x^{n+1}}{n^3x^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3|x| = |x|.$$

Therefore we see that the series will converge for all x satisfying $|x| < 1$. To find the interval of convergence, we need to further see whether or not the series will converge when $|x| = 1$, that is, when $x = 1$ or $x = -1$. In these cases, the series takes the form

$$\sum_{n=1}^{\infty} n^3 \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^n n^3$$

respectively. In both cases, the terms of the series do not tend to 0, so both series diverge by the Divergence Test. Hence our series has a radius of convergence $R = 1$ and interval of convergence $I = (-1,1)$.

To find its sum, we squint and see that the series sort of looks like

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

though with some complications coming from the coefficients of n^3. But we can produce those coefficients in the above series by differentiating with respect to x, and then multiplying again:

$$\sum_{n=1}^{\infty} nx^n = x\frac{d}{dx}\sum_{n=1}^{\infty} x^n = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2},$$
$$\sum_{n=1}^{\infty} n^2x^n = x\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = x\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{x(x+1)}{(1-x)^3},$$
$$\sum_{n=1}^{\infty} n^3x^n = x\frac{d}{dx}\sum_{n=1}^{\infty} n^2x^n = x\frac{d}{dx}\frac{x(x+1)}{(1-x)^3} = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

The final line is the sum we are looking for. The equality makes sense on the interval of convergence $(-1,1)$. ■

6. Let p be a number. Consider the series

(i) Find the values of p for which the series

$$S_p := \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots .$$

is convergent.

(ii) Consider the power series

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}.$$

What is the radius of convergence of this power series?

(iii) What is the interval of convergence of the power series in (ii)? Your answer will depend on p.

(iv) Using the power series for $\frac{1}{1-x}$, find a Taylor series for $-\log(1-x)$ centred at 0.

(v) Show that

$$\mathrm{Li}_2(x) = \int \frac{-\log(1-x)}{x}\, dx,$$

perhaps up to a constant of integration.

Solutions. For (i), apply the Integral Test:

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \lim\limits_{N\to\infty} \dfrac{1}{1-p}\left(\dfrac{1}{N^{1-p}} - 1\right) & \text{if } p \neq 1, \\ \lim\limits_{N\to\infty} \log(N) & p = 1, \end{cases} = \begin{cases} 0 & \text{if } p > 1, \\ \infty & \text{if } p \leq 1. \end{cases}$$

Therefore we see that the series converges when $p > 1$ and diverges otherwise.

Apply the Ratio Test for the power series in (ii):

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)^p}{x^n/n^p} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p |x| = |x|.$$

Thus $\mathrm{Li}_p(x)$ converges whenever $|x| < 1$, so the radius of convergence is 1.

To find the interval of convergence requested in (iii), we need to see whether or not the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

converge or diverge. Consider the cases $p > 1$, $1 \leq p > 0$, and $p \leq 0$ separately.

When $p > 1$, then the first series is convergent by (i). Hence the second series is absolutely convergent, hence convergent. The interval of convergence in this case is $I = [-1, 1]$.

When $1 \leq p > 0$, the first series is divergent by (i). The second series is an alternating series with terms tending to 0 and decreasing in absolute value. Therefore the Alternating Series Test implies this series converges. The interval of convergence in this case is $I = [-1, 1)$.

When $p \leq 0$, the terms of both sequence do not tend to 0 as $n \to \infty$. Therefore, by the Divergence Test, both series diverge. The interval of convergence in this case is $I = (-1, 1)$.

For (iv), observe that

$$-\log(1-x) = \int \frac{dx}{1-x} = \int \sum_{n=0}^{\infty} x^n \, dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

for some constant of integration C. To find C, evaluate both sides at $x = 0$: $C = -\log(1-0) = 0$. Therefore

$$-\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Finally, for (v), we compute the integral:

$$\int \frac{-\log(1-x)}{x} \, dx = \int \sum_{n=0}^{\infty} \frac{x^n}{n+1} \, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2}$$

where we shall take the constant of integration to be 0. This is precisely the power series of $\mathrm{Li}_2(x)$, as desired. ■

7. Suppose that you have a function $f(x)$ such that $f(0) = \pi$ and

$$\frac{d}{dx} f(x) = \log(1+x).$$

Use this relationship to derive a Taylor series centred at 0 for $f(x)$. Find the radius and interval of convergence for the resulting power series.

Solution. Apply the Fundamental Theorem of Calculus to transform the differential relationship to an integral one, from which knowledge of the power series of $\log(1+x)$ gives a power series for $f(x)$:

$$f(x) = \int \log(1+x) \, dx = \int \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \, dx = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+1}}{n(n+1)}$$

for some constant of integration C. To find C, evaluate both sides at $x = 0$: $C = f(0) = \pi$. Hence

$$f(x) = \pi + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+1}}{n(n+1)} = \pi + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots.$$

Find the radius of convergence of this series by applying the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+2}}{(n+1)(n+2)} \frac{n(n+1)}{x^{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+2} |x| = |x|.$$

Therefore, this series converges whenever $|x| < 1$, which is to say that the radius of convergence of the series is $R = 1$. ■