q-BIC FORMS

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ABSTRACT. A *q-bic* form is a pairing $V \times V \to \mathbf{k}$ that is linear in the second variable and q-power Frobenius linear in the first; here, V is a vector space over a field $\mathbf{k}$ containing the finite field $\mathbf{F}_{q^2}$. This article develops a geometric theory of q-bic forms in the spirit of that of bilinear forms. I find two filtrations intrinsically attached to a q-bic form, with which I define a series of numerical invariants. These are used to classify, study automorphism group schemes of, and describe specialization relations in the parameter space of q-bic forms.

CONTENTS

INTRODUCTION

Eschewing the semi-linear description of the abstract, a *q-bic form* is a bilinear pairing β between a finite-dimensional vector space V over a field $\mathbf{k}$ and its Frobenius twist $V^{[1]} := \mathbf{k} \otimes_{\mathrm{Fr},\mathbf{k}} V$, where the scalar action is twisted by the q-power Frobenius map $\mathrm{Fr}\colon \mathbf{k} \to \mathbf{k}$. Equivalently, this is a linear map

$$\beta : V^{[1]} \otimes_{\mathbf{k}} V \to \mathbf{k}.$$

This article develops an intrinsic theory of q-bic forms, with an eye towards the geometry of their hypersurfaces of isotropic vectors: the *q-bic hypersurface* described below.

To begin to appreciate the main aspects of the theory, assume from now on that the base field $\mathbf{k}$ contains the finite field $\mathbf{F}_{q^2}$, so that the q-power Frobenius is a nontrivial endomorphism of $\mathbf{k}$. A q-bic form β then pairs two distinct $\mathbf{k}$-vector spaces which, however, are related by the canonical q-linear map $V \to V^{[1]}$ given by $v \mapsto v^{[1]} := 1 \otimes v$. To emphasize: this map is not $\mathbf{k}$-linear and is not even a bijection for imperfect fields. In any case, a basis $V = \langle e_1, \ldots, e_n \rangle$ induces a basis $V^{[1]} = \langle e_1^{[1]}, \ldots, e_n^{[n]} \rangle$ and the q-bic form β may be explicitly understood through its *Gram matrix*:

$$\mathrm{Gram}(\beta; e_1, \ldots, e_n) := \left(\beta(e_i^{[1]}, e_j)\right)_{i,j=1}^{n}.$$

Moreover, if $(e_1', \ldots, e_n') = (e_1, \ldots, e_n) \cdot A^\vee$ is another basis of V, related to the original via an invertible matrix A, then the two Gram matrices are related by *q-twisted conjugation*:

$$\mathrm{Gram}(\beta; e_1', \ldots, e_n') = A^{[1],\vee} \cdot \mathrm{Gram}(\beta; e_1, \ldots, e_n) \cdot A$$

where $A^{[1],\vee}$ is obtained from the transpose of A by raising each matrix entry to the q-th power. In concrete terms, this article studies the invariants, orbits, and stabilizers of q-twisted conjugation.

Returning to the intrinsic situation, contemplate: what are invariants of β? The key observation is that the Frobenius map makes it possible to iteratively take left and right orthogonals, thereby canonically attaching two sequences of vector spaces to β. Left orthogonals give rise to a sequence of subspaces of various Frobenius twists of V, collectively referred to as the $\perp^{[\cdot]}$-*filtration*; for instance, the first space is the left kernel $\mathrm{P}_1'V^{[1]} = V^{\perp} := \ker(\beta^\vee \colon V^{[1]} \to V^\vee)$. Right orthogonals give rise to a finite filtration on V, the $\perp$-*filtration*, which has the form

$$\mathrm{P}_{\bullet}V : \{0\} =: \mathrm{P}_{-1}V \subseteq \mathrm{P}_1V \subseteq \mathrm{P}_3V \subseteq \cdots \subseteq \mathrm{P}_-V \subseteq \mathrm{P}_+V \subseteq \cdots \subseteq \mathrm{P}_4V \subseteq \mathrm{P}_2V \subseteq \mathrm{P}_0V =: V$$

where the odd- and even-indexed pieces form increasing and decreasing filtrations, limiting to subspaces P_-V and P_+V, respectively; this time, the first piece of the filtration is the right kernel $\mathrm{P}_1V = V^{[1],\perp} := \ker(\beta \colon V \to V^{[1],\vee})$. Numerical invariants of β are then obtained by taking dimensions of graded pieces of this filtration: for each $m \geq 1$, set $\epsilon := (-1)^m$, and

$$a := \dim_{\mathbf{k}} \mathrm{P}_+V/\mathrm{P}_-V, \qquad a_m := \dim_{\mathbf{k}} \mathrm{P}_{m-\epsilon-1}V/\mathrm{P}_{m+\epsilon-1}V, \qquad b_m := a_m - a_{m+1}.$$

The first results of this article classify β in terms of its *type* $(a; b_m)_{m\geq 1}$ and properties of its $\perp^{[\cdot]}$-filtration. This is simplest over an algebraically closed field, where β is completely classified by its type. The finitely many isomorphism classes may be described in terms of a normal form for its Gram matrix. In the following statement, write $\mathbf{1}$ for the 1-by-1 identity matrix, $\mathbf{N}_m$ for the m-by-m Jordan block with 0 on the diagonal, and $\oplus$ for the block diagonal sum of matrices:

Theorem A. — *Let (V, β) be a q-bic form of type $(a; b_m)_{m \geq 1}$ over an algebraically closed field $\mathbf{k}$. Then there exists a basis $V = \langle e_1, \ldots, e_n \rangle$ such that*

$$\mathrm{Gram}(\beta; e_1, \ldots, e_n) = \mathbf{1}^{\oplus a} \oplus \Big(\bigoplus\nolimits_{m \geq 1} \mathbf{N}_m^{\oplus b_m} \Big).$$

See 4.2. Over an arbitrary field $\mathbf{k}$, such a splitting exists if and only if the $\perp^{[\cdot]}$-filtration admits a purely inseparable descent to a filtration on V, and if a certain Galois cohomology class for a finite unitary group vanishes: see 4.1 and 2.7.

The basis exhibiting β in the normal form above is never unique, meaning that q-bic forms carry many automorphism. The next results describe the basic structure of the automorphism group *scheme* $\mathbf{Aut}_{(V,\beta)}$ of a q-bic form (V, β). This group is typically of positive-dimension and nonreduced; the following computes its dimension and that of its Lie algebra in terms of the type of β:

Theorem B. — *Let (V, β) be a q-bic form of type $(a; b_m)_{m\geq 1}$ over a field $\mathbf{k}$. Then*

$$\dim_{\mathbf{k}} \mathrm{Lie}\,\mathbf{Aut}_{(V,\beta)} = \dim_{\mathbf{k}} \mathrm{Hom}_{\mathbf{k}}(V, V^{[1],\perp}) = \Big(a + \sum\nolimits_{m\geq 1} m b_m\Big)\Big(\sum\nolimits_{m \geq 1} b_m\Big), \text{ and}$$

$$\dim \mathbf{Aut}_{(V,\beta)} = \sum\nolimits_{k \geq 1} \Big[k(b_{2k-1}^2 + b_{2k}^2) + \Big(a + \sum\nolimits_{m \geq 2k} m b_m\Big) b_{2k-1} + 2k\Big(\sum\nolimits_{m \geq 2k+1} b_m\Big) b_{2k}\Big].$$

The Lie algebra statement is 5.5, where the two vector spaces are in fact canonically identified, and the dimension statement is 5.15. The proof proceeds by identifying the reduced subgroup scheme of $\mathbf{Aut}_{(V,\beta)}$, say when $\mathbf{k}$ is perfect, with the subgroup of automorphisms that additionally preserve a descent of the $\perp^{[\cdot]}$-filtration to V, see 5.13; the latter is shown to be reduced in 5.12 by studying infinitesimal deformations of the identity automorphism.

Consider now the parameter space of q-bic forms on the fixed n-dimensional vector space V:

$$q\text{-}\mathbf{bics}_V := \mathbf{A}(V^{[1]} \otimes_{\mathbf{k}} V)^\vee := \mathrm{Spec}\,\mathrm{Sym}^*(V^{[1]} \otimes_{\mathbf{k}} V).$$

Via the Gram matrix construction, this may be identified with the affine space of n-by-n matrices. Orbits of the natural $\mathbf{GL}_V$ action, corresponding to q-Frobenius twisted conjugation of matrices, are determined by Theorem A as the locally closed subschemes

$$q\text{-}\mathbf{bics}_{V,\mathrm{type}(\beta)} := \left\{ [\beta'] \in q\text{-}\mathbf{bics}_V \;\middle|\; \mathrm{type}(\beta) = \mathrm{type}(\beta') \right\}$$

parameterizing q-bic forms with the same type as a given form β; as usual, this is a smooth and irreducible subscheme, and its codimension is $\dim \mathbf{Aut}_{(V,\beta)}$, which is determined by Theorem B.

Taken together, these subschemes yield a finite stratification of the space q-$\mathbf{bics}_V$. The final results of this article partially characterize the closure relations amongst the strata in terms of inequalities amongst types. For a q-bic form β of type $(a; b_m)_{m\geq 1}$ and each integer $m \geq 1$, write

$$\Psi_m(\beta) := \begin{cases} b_{2k-1} + 2\sum_{\ell=1}^{k-1}(k-\ell)b_{2\ell-1} & \text{if } m = 2k-1, \\ \sum_{\ell=1}^{k-1} \ell\, b_{2\ell} + k\Big(\sum_{\ell\geq 1} b_{2\ell-1} + \sum_{\ell\geq k} b_{2\ell}\Big) & \text{if } m = 2k, \end{cases}$$

and $\Theta_m(\beta) := \sum_{k=1}^{m} b_{2k-1}$. With this notation, the result is as follows:

Theorem C. — *Let β and β' be q-bic forms on a $\mathbf{k}$-vector space V.*

(i) *If the orbit closure of β contains β', then $\Psi_m(\beta) \leq \Psi_m(\beta')$ for all $m \geq 1$.*
(ii) *If $\Psi_m(\beta) \leq \Psi_m(\beta')$ and $\Theta_m(\beta) \leq \Theta_m(\beta')$ for all $m \geq 1$, then the orbit closure of β contains β'.*

See 6.2 and 6.4. Clearly, this result can be sharpened: for instance, the hypothesis in (ii) can be relaxed. See 6.8 for further comments and questions about the moduli of q-bic forms.

Applications and interrelations. — Isotropic lines for a nonzero q-bic form β on V are parameterized by a degree $q+1$ hypersurface X in the projective space $\mathbf{P}V$: such is a *q-bic hypersurface*. For example, if β admits $\mathbf{1}^{\oplus n}$ as a Gram matrix, then X is the Fermat hypersurface of degree $q+1$. Smooth q-bic hypersurfaces have been of recurring interest for many reasons, for example: they were amongst the first varieties on which the Tate conjecture could be verified, see [Tat65]; they are related to the Deligne–Lusztig varieties of [DL76] for finite unitary groups; and they are unirational by [Shi74]. See [Che22, pp.7–11] for further references and discussion.

The theory of q-bic forms brings new deformation and moduli theoretic methods to bear in the study of q-bic hypersurfaces. For one, this endows their Fano schemes of linear spaces with a new moduli interpretation, making them akin to orthogonal Grassmannian: see [Che23b]. In another direction, this exposes new structure in the geometry of rational curves on q-bic hypersurfaces: see [Che23c]. These results suggest an analogy between q-bic hypersurfaces and low degree hypersurfaces, such as quadrics and cubics, which is most striking for threefolds: see [Che23a].

A few words on precedents: Thinking of q-bic hypersurfaces in terms of a bilinear form is related to the study of finite Hermitian varieties, as in [BC66, Seg65]. This perspective is also adopted by [Shi01] in a spirit closer to that of the present article. The classification of q-conjugacy classes of matrices has been considered in varying degrees of generality: for nonsingular matrices, this follows from Lang's Lemma, see [Lan56, Theorem 1], and can be found in [Hef85, Bea90]; for corank 1 matrices, this is [HH16, Proposition 1]; and for general matrices, this is [KKP+22, Theorem 7.1].

Outline. — Basic definitions and constructions are given in §1. The relationship between q-bic forms and Hermitian forms over $\mathbf{F}_{q^2}$ is explained in §2; in particular, 2.7 shows that any nonsingular q-bic form over a separably closed field admits an orthonormal basis. The $\perp$- and $\perp^{[\cdot]}$-filtrations are constructed in §3, and certain basic properties such as symmetry and the meaning of the b_m are established: see 3.6 and 3.8. Classification of q-bic forms by their numerical invariants is proven in §4. Automorphism group schemes of q-bic forms are studied in §5. The article closes in §6 with a basic study of the moduli space of q-bic forms.

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1. Basic notions

This Section begins with the basic definitions, constructions, and properties of q-bic forms. For the schematic constructions later, this material is developed in the setting of finite projective modules over an $\mathbf{F}_{q^2}$-algebra R. To set notation: Write $\mathrm{Fr}\colon R \to R$ for the q-power Frobenius morphism. Given an R-module M, write $M^\vee := \mathrm{Hom}_R(M,R)$ for its dual; the double dual is tacitly identified with M itself. For each integer $i \geq 0$, write $M^{[i]} := R \otimes_{\mathrm{Fr}^i,R} M$ for the *i-th Frobenius twist* of M: the (left) R-module on which the action of R is twisted by the q^i-power Frobenius. The canonical map $M \to M^{[i]}$ given by $m \mapsto m^{[i]} := 1 \otimes m$ is the universal Fr^i-linear map out of M. Given a submodule $N' \subseteq M^{[i]}$, a submodule $N \subseteq M$ such that $N^{[i]} = N'$ is called a *Frobenius descent* of N' to M.

1.1. Definitions. — A *q-bic form* over R is a pair (M,β) consisting of an R-module M and an R-linear map $\beta : M^{[1]} \otimes_R M \to R$. A *morphism* $\varphi : (M_1,\beta_1) \to (M_2,\beta_2)$ between two q-bic forms is a morphism $\varphi : M_1 \to M_2$ of R-modules such that, for every $m \in M_1$ and $m' \in M_1^{[1]}$,

$$\beta_1(m',m) = \beta_2(\varphi^{[1]}(m'),\varphi(m))$$

where $\varphi^{[1]}\colon M_1^{[1]} \to M_2^{[1]}$ is the Frobenius twist of φ. The morphism φ is an *isomorphism* if the underlying module map is an isomorphism.

Adjunction induces two mutually dual R-linear maps which, by an abuse of notation, are denoted

$$\beta : M \to M^{[1],\vee} \quad \text{and} \quad \beta^\vee : M^{[1]} \to M^\vee.$$

The form (M,β) is said to be *nondegenerate* if the map $\beta : M \to M^{[1]}$ is injective, and *nonsingular* if this map is an isomorphism.

1.2. Gram matrices. — Suppose that M is, moreover, a finite free R-module. The *Gram matrix* of a q-bic form β with respect a basis $\varphi : \bigoplus_{i=1}^n R\cdot e_i \xrightarrow{\sim} M$ is the n-by-n matrix

$$\mathrm{Gram}(\beta;e_1,\ldots,e_n) := \big(\beta(\varphi(e_i)^{[1]},\varphi(e_j))\big)_{i,j=1}^n.$$

Equivalently, this is the matrix of the map $\beta : M \to M^{[1],\vee}$ with respect to the bases $\varphi : \bigoplus_{i=1}^n R\cdot e_i \xrightarrow{\sim} M$ and $\varphi^{[1]}\colon \bigoplus_{i=1}^n R\cdot e_i^{[1]} \xrightarrow{\sim} M^{[1]}$.

Given another basis $\varphi'\colon \bigoplus_{i=1}^n R\cdot e_i' \xrightarrow{\sim} M$, view the change of basis isomorphism $A := \varphi^{-1}\circ\varphi'$ as an invertible n-by-n matrix over R. Its Frobenius twist $A^{[1]}$ is the matrix obtained from A by taking q-powers entrywise, and the two Gram matrices are related by

$$\mathrm{Gram}(\beta;e_1',\ldots,e_n') = A^{[1],\vee}\cdot\mathrm{Gram}(\beta;e_1,\ldots,e_n)\cdot A.$$

It is now clear that a q-bic form β on a free module M is nonsingular if and only if its Gram matrix in some—equivalently, any—basis is invertible.

1.3. Standard forms. — Gram matrices provide a convenient way to encode q-bic forms. Given an n-by-n matrix B over R, let $(R^{\oplus n},B)$ denote the unique q-bic form with Gram matrix B in the given basis. Particularly simple, and important, are the following: For each integer $m \geq 1$, let

$$\mathbf{N}_m := \begin{pmatrix} 0 & 1 & \cdots & & \\ & 0 & \cdots & & \\ & & \cdots & & \\ & & \cdots & 0 & 1 \\ & & \cdots & & 0 \end{pmatrix}$$

denote the m-by-m Jordan block with 0 on the diagonal, and write $\mathbf{1}$ for the 1-by-1 identity matrix. Given two matrices B_1 and B_2, write $B_1 \oplus B_2$ for their block diagonal sum. Let $\mathbf{b} := (a; b_1, b_2, \ldots)$ be a sequence of nonnegative integers such that $n = a + \sum_{m \geq 1} m b_m$. The q-bic form

$$\Big(R^{\oplus n}, \mathbf{1}^{\oplus a} \oplus \Big(\bigoplus\nolimits_{m \geq 1} \mathbf{N}_m^{\oplus b_m}\Big)\Big)$$

is the *standard q-bic form of type* $\mathbf{b}$.

1.4. Orthogonals. — To discuss orthogonals, consider generally a map $\beta \colon M_2 \otimes_R M_1 \to R$ of R-modules, viewed as a pairing between M_1 and M_2. Given submodules $N_i \subseteq M_i$ for $i = 1, 2$, write

$$N_1^{\perp} := \ker(M_2 \xrightarrow{\beta} M_1^{\vee} \to N_1^{\vee}) \quad \text{and} \quad N_2^{\perp} := \ker(M_1 \xrightarrow{\beta^{\vee}} M_2^{\vee} \to N_2^{\vee}).$$

These are the *orthogonals*, with respect to β, of N_1 and N_2, respectively. The orthogonals $M_2^{\perp} \subseteq M_1$ and $M_1^{\perp} \subseteq M_2$ are called the *kernels* of β (and $\beta^{\vee}$). It is formal to see that taking orthogonals is an inclusion-reversing operation. Precisely: for submodules N_1 and N_1' of M_1:

(i) if $N_1' \subseteq N_1$, then $N_1'^{\perp} \supseteq N_1^{\perp}$;
(ii) $(N_1 + N_1')^{\perp} = N_1^{\perp} \cap N_1'^{\perp}$; and
(iii) $(N_1 \cap N_1')^{\perp} \supseteq N_1^{\perp} + N_1'^{\perp}$.

With further assumptions, orthogonals behave as expected:

1.5. Lemma. — *Suppose that the image of the map $\beta \colon M_1 \to M_2^{\vee}$ is a local direct summand, and that $N_1', N_1 \subseteq M_1$ are local direct summands. Then there are exact sequences:*

(i) $0 \to M_2^{\perp} \to M_1 \to M_2^{\vee} \to M_1^{\perp,\vee} \to 0$,
(ii) $0 \to N_1 \cap M_2^{\perp} \to N_1 \to M_2^{\vee} \to N_1^{\perp,\vee} \to 0$, *and*
(iii) $0 \to N_1 \cap M_2^{\perp}/N_1' \cap M_2^{\perp} \to N_1/N_1' \to (N_1'^{\perp}/N_1^{\perp})^{\vee} \to 0$ *if additionally* $N_1' \subseteq N_1$.

The operation of taking orthogonals further satisfy:

(iv) $N_1^{\perp,\perp} = M_2^{\perp} + N_1$, *that is, reflexivity, and*
(v) $(N_1 \cap N_1')^{\perp} = N_1^{\perp} + N_1'^{\perp}$ *if additionally* $M_2^{\perp} \subseteq N_1$.

Proof. For (i), it remains to identify $\operatorname{coker}(\beta)$ with $M_1^{\perp,\vee}$ for the kernel-cokernel exact sequence of $\beta \colon M_1 \to M_2^{\vee}$. The hypotheses imply that the cokernel is a local direct summand of $M_2^{\vee}$, so that dualizing the sequence and making the canonical identifications as in **1.1** gives an exact sequence

$$0 \to \operatorname{coker}(\beta)^{\vee} \to M_2 \xrightarrow{\beta^{\vee}} M_1^{\vee} \to M_2^{\perp,\vee} \to 0.$$

This identifies $\operatorname{coker}(\beta)^{\vee}$ as $\ker(\beta^{\vee}) =: M_1^{\perp}$. Reflexivity shows $\operatorname{coker}(\beta) = M_1^{\perp,\vee}$. Sequence (ii) now follows upon restricting β to N_1. Comparing this sequence for nested submodules $N_1' \subseteq N_1$ then gives a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_1' \cap M_2^{\perp} & \longrightarrow & N_1' & \longrightarrow & (M_2/N_1'^{\perp})^{\vee} & \longrightarrow & 0 \\
 & & \cap\!| & & \cap\!| & & \cap\!| & & \\
0 & \longrightarrow & N_1 \cap M_2^{\perp} & \longrightarrow & N_1 & \longrightarrow & (M_2/N_1^{\perp})^{\vee} & \longrightarrow & 0
\end{array}$$

from which (iii) follows upon taking cokernels.

For (iv), note that $N_1^{\perp,\perp} = \ker(\beta \colon M_1 \to M_2^{\vee} \to N_1^{\perp,\vee}) = M_2^{\perp} + \beta^{-1}(\ker(M_2^{\vee} \to N_1^{\perp,\vee}))$, so that the result follows from (ii), which shows that the second term is N_1.

For (v), apply **1.4**(ii) to $N_1^{\perp}$ and $N_1'^{\perp}$, take an orthogonal, apply (iv), and use the assumption $M_2^{\perp} \subseteq N_1$ to obtain the first equality in

$$N_1^{\perp} + N_1'^{\perp} = (N_1 \cap (N_1' + M_2^{\perp}))^{\perp} = (N_1 \cap N_1' + M_2^{\perp})^{\perp} = (N_1 \cap N_1')^{\perp}$$

The second equality comes from **1.6** below, and the final equality comes from **1.4**(ii) and (iv). ∎

1.6. Lemma. — *Let M be an R-module and $K, N, N' \subseteq M$ submodules. If $K \subseteq N$, then*

$$N \cap (N' + K) = N \cap N' + K.$$

Proof. The inclusion "$\supseteq$" always holds and is clear. For "$\subseteq$", let $x \in N \cap (N'+K)$ and view it as an element of N. Being in the intersection means there are $x' \in N'$ and $y \in K$ such that $x = x' + y$. But $x' = x - y \in N$ since $K \subseteq N$, so $x' \in N \cap N'$. This shows $x \in N \cap N' + K$, as required. ∎

1.7. Frobenius twists and orthogonals. — Return to the situation of a q-bic form (M, β) over R. Twisting by Frobenius yields a sequence of associated q-bic forms $(M^{[i]}, \beta^{[i]})$ for each integer $i \geq 0$; the i-th Frobenius twisted form is characterized by the formula

$$\beta^{[i]}(m'^{[i]}, m^{[i]}) = \beta(m', m)^{q^i} \quad \text{for every } m \in M \text{ and } m' \in M^{[1]}.$$

Observe that the module $M^{[i]}$ is involved with the two Frobenius twists of β, namely $\beta^{[i-1]}$ and $\beta^{[i]}$. Accordingly, a submodule $N \subseteq M^{[i]}$ induces two orthogonals:

$$\begin{aligned} N^{\perp^{[i-1]}} &:= \ker\big(\beta^{[i-1]} \colon M^{[i-1]} \to M^{[i],\vee} \to N^{\vee}\big), \text{ and} \\ N^{\perp^{[i]}} &:= \ker\big(\beta^{[i],\vee} \colon M^{[i+1]} \to M^{[i],\vee} \to N^{\vee}\big). \end{aligned}$$

These are the $(i-1)$-th and i-th *Frobenius-twisted orthogonals of N*.

In this setting, suppose further that taking Frobenius twists commutes with taking kernels: for instance, suppose that R is regular so that the Frobenius morphism is flat by [Kun69]; or suppose that the image in $N^{\vee}$ of $\beta^{[i-1]}$ or $\beta^{[i],\vee}$ has vanishing Tor_1^R. Then the Frobenius-twisted orthogonal commutes with taking Frobenius twists in the following sense: for any integer $j \geq 0$,

$$N^{\perp^{[i-1]},[j]} = N^{[j],\perp^{[i+j-1]}} \quad \text{and} \quad N^{\perp^{[i]},[j]} = N^{[j],\perp^{[i+j]}}$$

as submodules of $M^{[i+j-1]}$ and $M^{[i+j+1]}$, respectively.

1.8. Total orthogonals and complements. — The *total orthogonal* of a submodule $N \subseteq M$ is

$$N^{\perp} \cap N^{[2],\perp^{[1]}} = \big\{\, m' \in M^{[1]} \;\big|\; \beta(m', n) = \beta^{[1]}(n^{[2]}, m') = 0 \text{ for all } n \in N \,\big\}.$$

This is a submodule of $M^{[1]}$ and, in general, it need not have a Frobenius descent to M. The *radical* of (M, β) is the total orthogonal of M itself, and may be written as

$$\mathrm{rad}(\beta) = \big\{\, m' \in M^{[1]} \;\big|\; \beta^{[1]}(m'^{[1]}, m^{[1]}) = \beta^{[1]}(m^{[2]}, m') = 0 \text{ for all } m \in M \,\big\}.$$

In particular, $\beta^{[1]}$ passes to the quotient and induces a q-bic form on $M^{[1]}/\mathrm{rad}(\beta)$.

An *orthogonal complement* to a submodule M' is a submodule M'' such that $M = M' \oplus M''$ and $M''^{[1]}$ lies in the total orthogonal of M'. This situation is signified by

$$(M, \beta) = (M', \beta') \perp (M'', \beta'')$$

where β' and β'' denote the restriction of β to M' and M'', respectively; when the underlying modules are not crucial, also write $\beta = \beta' \oplus \beta''$. Orthogonal complements need not exist, and when they do, need not be unique: one exists if and only if the total orthogonal contains a module complement that descends to M; it is unique if and only if the total orthogonal is the complement.

2. HERMITIAN FORMS

A q-bic form (M, β) over R linearizes a biadditive map $M \times M \to R$ which is R-linear in the second variable, but only q-linear in the first. In the case that $R = \mathbf{F}_{q^2}$, such forms are sesquilinear with respect the q-power Frobenius, and the Hermitian property gives a notion of symmetry. In general, the Hermitian condition does not make sense. This Section identifies a distinguished subset of any q-bic form consisting of those elements that satisfy the Hermitian equation. This construction provides an invariant of q-bic forms, sensitive to the arithmetic of its ring of definition.

2.1. Hermitian elements. — An element $m \in M$ is said to be *Hermitian* if

$$\beta(-, m) = \beta^{[1]}(m^{[2]}, -) \text{ as elements of } M^{[1],\vee}$$

or equivalently, if $\beta(n^{[1]}, m) = \beta(m^{[1]}, n)^q$ for all $n \in M$, see 1.7. It is straightforward to check that the set M_{Herm} of Hermitian elements is a vector space over $\mathbf{F}_{q^2}$ and that $\beta(m_1^{[1]}, m_2) \in \mathbf{F}_{q^2}$ for every $m_1, m_2 \in M_{\mathrm{Herm}}$. Therefore the restriction of β to the space of Hermitian elements gives

$$\beta_{\mathrm{Herm}} : M_{\mathrm{Herm}}^{[1]} \otimes_{\mathbf{F}_{q^2}} M_{\mathrm{Herm}} \to \mathbf{F}_{q^2}$$

a Hermitian form for the quadratic extension $\mathbf{F}_{q^2}/\mathbf{F}_q$. This partially justifies the nomenclature: if M is a free module with basis $\langle e_1, \ldots, e_n \rangle$ consisting of Hermitian elements, then the associated Gram matrix is a Hermitian matrix over $\mathbf{F}_{q^2}$ in that it satisfies

$$\mathrm{Gram}(\beta; e_1, \ldots, e_n)^\vee = \mathrm{Gram}(\beta; e_1, \ldots, e_n)^{[1]}.$$

The space M_{Herm} may not be of finite dimension over $\mathbf{F}_{q^2}$. For instance, it contains the preimage in M of the radical of β. It is therefore helpful to note that taking Hermitian vectors is compatible with orthogonal decompositions:

2.2. Lemma. — *An orthogonal decomposition $(M, \beta) = (M', \beta') \perp (M'', \beta'')$ induces an orthogonal decomposition of Hermitian spaces*

$$(M_{\mathrm{Herm}}, \beta_{\mathrm{Herm}}) = (M'_{\mathrm{Herm}}, \beta'_{\mathrm{Herm}}) \perp (M''_{\mathrm{Herm}}, \beta''_{\mathrm{Herm}}).$$

Proof. Let $m \in M_{\mathrm{Herm}}$, and let $m = m' + m''$ be its decomposition with $m' \in M'$ and $m'' \in M''$. Consider m'. Since M' and M'' are orthogonal, for every $n \in M'$,

$$\beta'(n^{[1]}, m') = \beta(n^{[1]}, m' + m'') = \beta((m' + m'')^{[1]}, n)^q = \beta'(m'^{[1]}, n)^q.$$

Therefore $m' \in M'_{\mathrm{Herm}}$. An analogous argument shows $m'' \in M''_{\mathrm{Herm}}$. ∎

In the remainder of this Section, let (V, β) be a q-bic form over a field $\mathbf{k}$ containing $\mathbf{F}_{q^2}$.

2.3. Scheme of Hermitan vectors. — The subgroup V_{Herm} of Hermitian vectors of V may be endowed with the structure of a closed subgroup scheme of the affine space $\mathbf{A}V := \operatorname{Spec} \operatorname{Sym} V^\vee$ on V. Namely, the functor taking a $\mathbf{k}$-algebra R to the group of Hermitian elements of the q-bic form $(V \otimes_{\mathbf{k}} R, \beta \otimes_{\mathbf{k}} R)$ over R obtained by extension of scalars is represented by the closed subscheme

$$\mathbf{A}V_{\mathrm{Herm}} = \mathrm{V}\big(\beta(\ \ , -) - \beta^{[1]}(-^{[2]}, \ \ )\big)$$

cut out by the difference of two morphisms $\mathbf{A}V \to \mathbf{A}V^{[1],\vee}$ which are induced by the linear maps

$$v \mapsto \beta(\ \ , v) \in \mathrm{Hom}_{\mathbf{k}}(V^{[1]}, V^\vee) \quad \text{and} \quad v \mapsto \beta^{[1]}(v^{[2]}, \ \ ) \in \mathrm{Hom}_{\mathbf{k}}(V^{[1]}, V^{\vee,[2]}),$$

where $V^{\vee,[2]}$ is identified as the subspace of q^2-powers in $\mathrm{Sym}^{q^2}(V^\vee) \subset \Gamma(\mathbf{A}V, \mathcal{O}_{\mathbf{A}V})$. Concretely, choose a basis $V = \langle e_1, \ldots, e_n \rangle$, let $B := \mathrm{Gram}(\beta; e_1, \ldots, e_n)$ be the associated Gram matrix, and let

$\mathbf{x}^\vee := (x_1, \ldots, x_n)$ be the associated coordinates for $\mathbf{A}V \cong \mathbf{A}^n$. Then $\mathbf{A}V_{\mathrm{Herm}}$ is the closed subscheme given by the system of equations

$$B\mathbf{x} - B^{[1],\vee}\mathbf{x}^{[2]} = 0.$$

Observe furthermore that, by 2.1, $\mathbf{A}V_{\mathrm{Herm}}$ is a scheme of vector spaces over $\mathbf{F}_{q^2}$, and is equipped with a Hermitian bilinear form

$$\mathbf{A}\beta_{\mathrm{Herm}} \colon \mathbf{A}V_{\mathrm{Herm}}^{[1]} \times_{\mathbf{k}} \mathbf{A}V_{\mathrm{Herm}} \to \mathbf{F}_{q^2,\mathbf{k}}$$

with values in the constant group scheme associated with $\mathbf{F}_{q^2}$.

2.4. Examples. — Schemes of Hermitian vectors for the standard forms of 1.3 are determined with the help of 2.2 by the following two computations:

First, let $(V,\beta) = (\mathbf{k}^{\oplus n}, \mathbf{N}_n)$ be the standard form with just one nilpotent block. Then

$$\mathbf{A}V_{\mathrm{Herm}} = \operatorname{Spec}\mathbf{k}[x_1,\ldots,x_n]/(x_2, x_3 - x_1^{q^2}, \ldots, x_n - x_{n-2}^{q^2}, x_{n-1}^{q^2}).$$

The structure of this scheme depends on the parity of n:

$$\frac{\mathbf{k}[x_1,\ldots,x_n]}{(x_2, x_3 - x_1^{q^2}, \ldots, x_n - x_{n-2}^{q^2}, x_{n-1}^{q^2})} \cong \begin{cases} \mathbf{k}[x_1] & \text{if } n \text{ is odd, and} \\ \mathbf{k}[x_1]/(x_1^{q^n}) & \text{if } n \text{ is even.} \end{cases}$$

Indeed, the first $n-1$ equations imply that the even-indexed variables vanish, and that x_{2k+1} is the q^{2k}-power of x_1. As for the final equation: when n is odd, this is implied by the vanishing of the even-indexed variables, whereas when n is even, this shows that $x_{n-1}^{q^2} = x_1^{q^n} = 0$.

Second, let $V = \langle e \rangle$ be a 1-dimensional vector space and let $\beta(e^{[1]}, e) = \lambda$ for some $\lambda \in \mathbf{k}$. Then

$$\mathbf{A}V_{\mathrm{Herm}} = \operatorname{Spec}\mathbf{k}[x]/(\lambda x - \lambda^q x^{q^2}).$$

This is a form of the constant group scheme on $\mathbf{F}_{q^2}$ that splits with a $(q+1)$-th root of λ. By 2.2, this determines the structure the scheme of Hermitian vectors for all diagonal q-bic forms. This also suggests that the scheme is quite simple for all nonsingular q-bic forms. Indeed:

2.5. Proposition. — *If (V,β) is a nonsingular q-bic form, then $\mathbf{A}V_{\mathrm{Herm}}$ is an étale group scheme of degree q^2n over $\mathbf{k}$, geometrically isomorphic to the constant group scheme associated with $\mathbf{F}_{q^2}^{\oplus n}$.*

Proof. Since β is nonsingular, the equations for $\mathbf{A}V_{\mathrm{Herm}}$ in 2.3 may be written as

$$\mathbf{x}^{[2]} = B^{[1],\vee,-1}B\mathbf{x}.$$

This is a system of n equations in n variables, with Jacobian equations the linear system given by $B^{[1],\vee,-1}B$. Since β is nonsingular, this is of full rank, and so $\mathbf{A}V_{\mathrm{Herm}}$ is étale of degree q^2n over $\mathbf{k}$. Since the group of points $\mathbf{A}V_{\mathrm{Herm}}(\mathbf{k}^{\mathrm{sep}})$ is a vector space over $\mathbf{F}_{q^2}$ by 2.1, $\mathbf{A}V_{\mathrm{Herm}}$ must be a form of the constant group scheme associated with $\mathbf{F}_{q^2}^{\oplus n}$, see [Mil17, 2.16]. ■

This implies that a nonsingular q-bic form is spanned by its Hermitian vectors after a finite separable field extension. This is not true in general: 2.4 shows that a form of type $\mathbf{N}_{2k}$ has no nonzero Hermitian vectors.

2.6. Proposition. — *If (V,β) is a nonsingular q-bic form over a separably closed field $\mathbf{k}$, then the natural map $V_{\mathrm{Herm}} \otimes_{\mathbf{F}_{q^2}} \mathbf{k} \to V$ is an isomorphism.*

Proof. The two $\mathbf{k}$-vector spaces have the same dimension by 2.5, so it suffices to show that the map is injective. If not, there is a linear relation in V of the form

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m \quad \text{with } m \geq 1,\ v_i \in V_{\mathrm{Herm}}, \text{ and } a_i \in \mathbf{k}.$$

Choose such a relation with m minimal. Minimality implies that $v_1, \ldots, v_m$ are linearly independent in V. Since β is nonsingular, there exists $w \in V$ such that $\beta(w^{[1]}, v_i) = 0$ for $1 \leq i \leq m-1$, and $\beta(w^{[1]}, v_m) \neq 0$; up to scaling v_m, this last value may be taken to be 1. Since the v_i are Hermitian,

$$a_m = \beta(w^{[1]}, v_{m+1}) = \beta(v_{m+1}^{[1]}, w)^q = \sum\nolimits_{i=1}^m a_i^{q^2} \beta(v_i^{[1]}, w)^q = \sum\nolimits_{i=1}^m a_i^{q^2} \beta(w^{[1]}, v_i) = a_m^{q^2}$$

so $a_m \in \mathbf{F}_{q^2}$, whence $v_m' := v_{m+1} - a_m v_m$ lies in V_{Herm} by **2.1**. The relation $v_m' = a_1 v_1 + \cdots + a_{m-1} v_{m-1}$ then contradicts the minimality of m, as required. ■

In particular, since Hermitian forms over $\mathbf{F}_{q^2}$ always have an orthonormal basis, this implies that all nonsingular q-bic forms over a separably closed field have a Gram matrix given by the identity:

2.7. Corollary. — *If (V, β) is a nonsingular q-bic form over a separably closed field, then there exists a basis $V = \langle e_1, \ldots, e_n \rangle$ such that* $\mathrm{Gram}(\beta; e_1, \ldots, e_n) = \mathbf{1}^{\oplus n}$. ■

When $\mathbf{k}$ is separably closed, V is spanned by its Hermitian vectors V_{Herm} by **2.6**, and so the restriction map $\mathrm{Aut}(V, \beta) \to \mathrm{Aut}(V_{\mathrm{Herm}}, \beta_{\mathrm{Herm}})$ from the automorphism group of a nonsingular q-bic form to that of its associated Hermitian form is an isomorphism. Combined with **2.7**, Galois descent implies that isomorphism classes of nonsingular q-bic forms of dimension n over a general field $\mathbf{k}$ are in bijection with forms of the standard Hermitian form on $\mathbf{F}_{q^2}^{\oplus n}$ over $\mathbf{k}$. This implies:

2.8. Corollary. — *Two nonsingular q-bic forms are isomorphic over $\mathbf{k}$ if and only if their group schemes of Hermitian vectors equipped with their Hermitian forms are isomorphic over $\mathbf{k}$.* ■

3. Canonical filtrations

Although a q-bic form β pairs distinct modules M and $M^{[1]}$, the canonical q-linear map $M \to M^{[1]}$ makes it possible to iterate left and right orthogonals. This gives rise to two intrinsic filtrations whose interactions ultimately encode the structure of β. After their definition in **3.1** and **3.2**, a symmetry relation is established in **3.6**, and a series of numerical invariants of β are defined in **3.7** and **3.9**.

3.1. $\perp$-filtration. — The map $\beta \colon M \to M^{[1],\vee}$ gives rise to a sequence $\mathrm{P}_\bullet M$ of submodules of M as follows: set $\mathrm{P}_{-1}M := \{0\}$, and for each $i \geq 0$, inductively set

$$\mathrm{P}_i M := \mathrm{P}_{i-1} M^{[1],\perp} := \ker\big(\beta \colon M \to M^{[1],\vee} \to \mathrm{P}_{i-1} M^{[1],\vee}\big).$$

For instance, $\mathrm{P}_0 M = M$ is the entire module and $\mathrm{P}_1 M = M^{[1],\perp}$ is the kernel of β. Induction with the inclusion-reversing property **1.4(i)** implies that:

- the odd-indexed submodules form an increasing filtration $\mathrm{P}_{2k-1}M \subseteq \mathrm{P}_{2k+1}M$,
- the even-indexed submodules form a decreasing filtration $\mathrm{P}_{2k}M \supseteq \mathrm{P}_{2k+2}M$, and
- the odd-indexed submodules are totally isotropic $\mathrm{P}_{2k-1}M \subseteq \mathrm{P}_{2k-1}M^{[1],\perp} =: \mathrm{P}_{2k}M$.

Therefore the sequence $\mathrm{P}_\bullet M$ fits into two interwoven filtrations

$$\{0\} = \mathrm{P}_{-1}M \subseteq \mathrm{P}_1 M \subseteq \mathrm{P}_3 M \subseteq \cdots \subseteq \mathrm{P}_- M \subseteq \mathrm{P}_+ M \subseteq \cdots \subseteq \mathrm{P}_4 M \subseteq \mathrm{P}_2 M \subseteq \mathrm{P}_0 M = M$$

called the $\perp$-*filtration* of (M, β); here, $\mathrm{P}_- M$ and $\mathrm{P}_+ M$ are the limiting submodules for the increasing odd-, and decreasing even-filtrations, respectively.

3.2. $\perp^{[\cdot]}$-filtration. — The map $\beta^\vee \colon M^{[1]} \to M^\vee$ and its Frobenius twists give rise to a sequence of submodules $\mathrm{P}_i' M^{[i]} \subseteq M^{[i]}$ as follows: set $\mathrm{P}_{-1}' M^{[-1]} := \{0\}$, and for each $i \geq 0$, inductively set

$$\mathrm{P}_i' M^{[i]} := \mathrm{P}_{i-1}' M^{[i-1],\perp^{[i-1]}} := \ker\big(\beta^{[i-1],\vee} \colon M^{[i]} \to M^{[i-1],\vee} \to \mathrm{P}_{i-1}' M^{[i-1],\vee}\big),$$

where notation is as in 1.7. For instance, $\mathrm{P}'_0M := \mathrm{P}'_0M^{[0]} = M$ is the module itself and $\mathrm{P}'_1M^{[1]} = M^\perp$ is the kernel of $\beta^\vee$. For each integer j, write

$$\mathrm{P}'_iM^{[i+j]} := (\mathrm{P}'_iM^{[i]})^{[j]}$$

for the submodule of $M^{[i+j]}$ which for $j \geq 0$ is the j-th Frobenius twist of $\mathrm{P}'_iM^{[i]}$, and for $j \leq 0$ is the j-th Frobenius descent, if it exists; in the latter case, say that the i-th piece of the $\perp^{[\cdot]}$-filtration *descends to* $M^{[i+j]}$ *over* R. As with the $\perp$-filtration, the inclusion-reversing property 1.4(i) inductively implies that the modules fit together to yield interwoven filtrations, so that for each $i \geq 0$, there is a filtration of $M^{[i]}$ of the form:

$$\{0\} = \mathrm{P}'_{-1}M^{[i]} \subseteq \mathrm{P}'_1M^{[i]} \subseteq \mathrm{P}'_3M^{[i]} \subseteq \cdots \subseteq \mathrm{P}'_iM^{[i]} \subseteq \cdots \subseteq \mathrm{P}'_2M^{[i]} \subseteq \mathrm{P}'_0M^{[i]} = M^{[i]}.$$

Since the two filtrations are inductively defined via kernels of β and $\beta^\vee$, it is straightforward to check that their formation is compatible with orthogonal decompositions:

3.3. Lemma. — *If $(M,\beta) = (M',\beta') \perp (M'',\beta'')$ is an orthogonal decomposition, then for every i,*

$$\mathrm{P}_iM = \mathrm{P}_iM' \oplus \mathrm{P}_iM'' \quad \textit{and} \quad \mathrm{P}'_iM^{[i]} = \mathrm{P}'_iM'^{[i]} \oplus \mathrm{P}'_iM''^{[i]}. \qquad \blacksquare$$

3.4. Example. — Consider the $\perp$- and $\perp^{[\cdot]}$-filtrations for the standard forms defined in 1.3. Note that the filtrations are trivial when (M,β) is nondegenerate, since they are constructed by taking iterated kernels of β and $\beta^\vee$. So by 3.3, it remains to describe the filtrations when β has a Gram matrix given by $\mathbf{N}_n$ for some basis $\langle e_1,\ldots,e_n\rangle$ of M. A direct computation shows that the $\perp$-filtration has n steps, given for $1 \leq i \leq n$ by

$$\mathrm{P}_iM = \begin{cases} \bigoplus_{\ell=1}^k R\cdot e_{2\ell-1} & \text{if } i = 2k-1, \text{ and} \\ \left(\bigoplus_{\ell=1}^k R\cdot e_{2\ell-1}\right) \oplus \left(\bigoplus_{j=2k+1}^n R\cdot e_j\right) & \text{if } i = 2k. \end{cases}$$

In particular, $\mathrm{P}_-M = \mathrm{P}_+M$ is the span of the odd-indexed basis elements.

The $\perp^{[\cdot]}$-filtration similarly has n steps. Moreover, in this special case, each step of the filtration descends to M, and is given for $1 \leq i \leq n$ by

$$\mathrm{P}'_iM = \begin{cases} \bigoplus_{\ell=1}^k R\cdot e_{n-2\ell+2} & \text{if } i = 2k-1, \text{ and} \\ \left(\bigoplus_{\ell=1}^k R\cdot e_{n-2\ell+2}\right) \oplus \left(\bigoplus_{j=1}^{n-2k} R\cdot e_j\right) & \text{if } i = 2k. \end{cases}$$

For the remainder of the Section, specialize to the situation of a q-bic form (V,β) on a finite-dimensional vector space over a field $\mathbf{k}$.

3.5. Symmetry. — Dimensions of various pieces of the $\perp$- and $\perp^{[\cdot]}$-filtration provide a series of numerical invariants for (V,β); taking just the first pieces gives the familiar

$$\mathrm{rank}(V,\beta) := \mathrm{rank}(\beta\colon V \to V^{[1],\vee}) = \mathrm{rank}(\beta^\vee\colon V^{[1]} \to V^\vee), \text{ and}$$
$$\mathrm{corank}(V,\beta) := \dim_{\mathbf{k}} V - \mathrm{rank}(V,\beta) = \dim_{\mathbf{k}} \mathrm{P}_1V = \dim_{\mathbf{k}} \mathrm{P}'_1V^{[1]}.$$

In particular, the first pieces of the two filtrations have the same dimension. As is also suggested by the examples in 3.4, this dimensional symmetry persists amongst higher pieces. To prove this, first observe that restricting the j-th Frobenius twisted pairing $\beta^{[j]}\colon V^{[j+1]} \otimes V^{[j]} \to \mathbf{k}$ to $\mathrm{P}_{i-1}V^{[j+1]}$ and $\mathrm{P}'_jV^{[j]}$ and using 1.5(ii) twice gives an exact sequence

$$0 \to \mathrm{P}_iV^{[j]} \cap \mathrm{P}'_jV^{[j]} \to \mathrm{P}'_jV^{[j]} \xrightarrow{\beta^{[j]}} \mathrm{P}_{i-1}V^{[j+1],\vee} \to (\mathrm{P}_{i-1}V^{[j+1]} \cap \mathrm{P}'_{j+1}V^{[j+1]})^\vee \to 0.$$

The symmetry statement is the following:

3.6. Proposition. — $\dim_{\Bbbk} \mathrm{P}_i V^{[j]} \cap \mathrm{P}'_j V^{[j]} = \dim_{\Bbbk} \mathrm{P}_j V^{[i]} \cap \mathrm{P}'_i V^{[i]}$ *for all integers* $i, j \geq 0$.

Proof. By symmetry, it suffices to consider $i \geq j \geq 0$. Proceed by induction on $i + j$. The base case is when $i = j = i + j = 0$ so that the result follows from the definitions in 3.1 and 3.2. Now fix the quantity $i + j \geq 1$ and further induct on the difference $\delta := i - j \geq 0$. There are two base cases: If $\delta = 0$, then there is nothing to prove. If $\delta = 1$, considering the sequence in 3.5 with $i = j + 1$ gives

$$\dim_{\Bbbk} \mathrm{P}_{j+1} V^{[j]} \cap \mathrm{P}'_j V^{[j]} - \dim_{\Bbbk} \mathrm{P}'_j V^{[j]} = \dim_{\Bbbk} \mathrm{P}_j V^{[j+1]} \cap \mathrm{P}'_{j+1} V^{[j+1]} - \dim_{\Bbbk} \mathrm{P}_j V^{[j+1]}.$$

Since $j < i + j = 2j + 1$, induction applies to the negative terms to show

$$\dim_{\Bbbk} \mathrm{P}'_j V^{[j]} = \dim_{\Bbbk} \mathrm{P}_0 V^{[j]} \cap \mathrm{P}'_j V^{[j]} = \dim_{\Bbbk} \mathrm{P}_j V \cap \mathrm{P}'_0 V = \dim_{\Bbbk} \mathrm{P}_j V$$

whence $\dim_{\Bbbk} \mathrm{P}_{j+1} V^{[j]} \cap \mathrm{P}'_j V^{[j]} = \dim_{\Bbbk} \mathrm{P}_j V^{[j+1]} \cap \mathrm{P}'_{j+1} V^{[j+1]}$.

Assume $\delta \geq 2$. Taking dimensions in the sequence of 3.5 gives the first equation in:

$$\begin{aligned}
\dim_{\Bbbk} \mathrm{P}_i V^{[j]} \cap \mathrm{P}'_j V^{[j]} &= \dim_{\Bbbk} \mathrm{P}_{i-1} V^{[j+1]} \cap \mathrm{P}'_{j+1} V^{[j+1]} - \dim_{\Bbbk} \mathrm{P}_{i-1} V^{[j+1]} + \dim_{\Bbbk} \mathrm{P}'_j V^{[j]} \\
&= \dim_{\Bbbk} \mathrm{P}_{j+1} V^{[i-1]} \cap \mathrm{P}'_{i-1} V^{[i-1]} - \dim_{\Bbbk} \mathrm{P}'_{i-1} V^{[i-1]} + \dim_{\Bbbk} \mathrm{P}_j V^{[i]} \\
&= \dim_{\Bbbk} \mathrm{P}_j V^{[i]} \cap \mathrm{P}'_i V^{[i]}.
\end{aligned}$$

Since $i - 1 - (j + 1) = \delta - 2$ and $\max\{i - 1, j\} < i + j$, induction gives the equality in the middle. The final equality follows from the sequence in 3.5 upon substituting $i \mapsto j + 1$ and $j \mapsto i - 1$. ■

3.7. Numerical invariants. — A sequence of numerical invariants of (V, β) is now obtained by taking dimensions of graded pieces for the filtrations: For each integer $m \geq 0$, set $\epsilon := (-1)^m$, set

$$a_m(V, \beta) := \dim_{\Bbbk} \mathrm{P}_{m-\epsilon-1} V / \mathrm{P}_{m+\epsilon-1} V = \dim_{\Bbbk} \mathrm{P}'_{m-\epsilon-1} V^{[i]} / \mathrm{P}'_{m+\epsilon-1} V^{[m]},$$

and set $a(V, \beta) := \dim_{\Bbbk} \mathrm{P}_+ V / \mathrm{P}\_V$. Since V is finite-dimensional, $a_m(V, \beta)$ vanish for large m. Slightly more convenient are the successive differences of these dimensions: for each $m \geq 0$, set

$$b_m(V, \beta) := a_m(V, \beta) - a_{m+1}(V, \beta) \quad \text{so that} \quad a_m(V, \beta) = \sum\nolimits_{i \geq m} b_i(V, \beta).$$

In fact, these differences are also dimensions:

3.8. Lemma. — $b_m(V, \beta) = \dim_{\Bbbk} \left(\dfrac{\mathrm{P}_{m-\epsilon-1} V^{[1]} \cap \mathrm{P}'_1 V^{[1]}}{\mathrm{P}_{m+\epsilon-1} V^{[1]} \cap \mathrm{P}'_1 V^{[1]}} \right) = \dim_{\Bbbk} \left(\dfrac{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}'_{m-\epsilon-1} V^{[m]}}{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}'_{m+\epsilon-1} V^{[m]}} \right).$

Proof. This follows directly from the definitions in 3.1 and 3.2 of the filtrations, and taking dimensions of the exact sequence 1.5(iii) applied to the nested sequences of subspaces

$$\mathrm{P}_{m+\epsilon-1} V^{[1]} \subseteq \mathrm{P}_{m-\epsilon-1} V^{[1]} \subseteq V^{[1]} \quad \text{and} \quad \mathrm{P}'_{m+\epsilon-1} V^{[m]} \subseteq \mathrm{P}'_{m-\epsilon-1} V^{[m]} \subseteq V^{[m]}$$

for the maps $\beta^\vee$ and $\beta^{[m]}$, respectively. ■

3.9. Type. — The sequence $(a(V, \beta); b_m(V, \beta))_{m \geq 1}$ is the *type* of (V, β) and is the fundamental invariant: all other intrinsic numerical invariants may be expressed in terms of the type. For instance,

$$\begin{aligned}
\operatorname{corank}(V, \beta) &= a_1(V, \beta) = \sum\nolimits_{m \geq 1} b_m(V, \beta), \text{ and} \\
\dim_{\Bbbk} V &= a(V, \beta) + \sum\nolimits_{m \geq 1} a_m(V, \beta) = a(V, \beta) + \sum\nolimits_{m \geq 1} m b_m(V, \beta).
\end{aligned}$$

Dimensions of each intersection $\mathrm{P}_i V^{[j]} \cap \mathrm{P}'_j V^{[j]}$ may be expressed similarly. Particularly useful is the following direct consequence of 3.8: for every integer $k \geq 1$,

$$\begin{aligned}
\dim_{\Bbbk} \mathrm{P}_{2k-1} V^{[1]} \cap \mathrm{P}'_1 V^{[1]} &= \sum\nolimits_{\ell=1}^{k} b_{2\ell-1}(V, \beta), \text{ and} \\
\dim_{\Bbbk} \mathrm{P}_{2k} V^{[1]} \cap \mathrm{P}'_1 V^{[1]} &= \sum\nolimits_{\ell=1}^{k} b_{2\ell-1}(V, \beta) + \sum\nolimits_{m \geq 2k+1} b_m(V, \beta).
\end{aligned}$$

Examining the description of the filtrations for standard forms given in 3.4 shows that this notion of type matches that given for standard forms in 1.3. In particular, $a(V,\beta)$ is the dimension of the nonsingular summand of a standard form. This is true more generally in the following sense:

3.10. Lemma. — *The restriction of β to P_+V has P_-V as its radical, and the induced q-bic form on $\mathrm{P}_+V/\mathrm{P}_-V$ is nonsingular.*

Proof. It follows from definitions that $\mathrm{P}_\pm V = \mathrm{P}_\mp V^{[1],\perp}$, so the kernels of β restricted to P_+V are

$$\ker(\beta : \mathrm{P}_+V \to \mathrm{P}_+V^{[1],\vee}) = \mathrm{P}_+V \cap \mathrm{P}_+V^{[1],\perp} = \mathrm{P}_-V, \text{ and}$$
$$\ker(\beta^\vee : \mathrm{P}_+V^{[1]} \to \mathrm{P}_+V^\vee) = \mathrm{P}_+V^{[1]} \cap \mathrm{P}_+V^\perp = \mathrm{P}_+V^{[1]} \cap (\mathrm{P}_-V^{[1]} + \mathrm{P}'_1V^{[1]}),$$

where the final equality follows from 1.5(iv). Since the two kernels have the same dimension, the latter must simply be $\mathrm{P}_-V^{[1]}$. Thus β restricted to P_+V has P_-V as its radical, and since this is the entire kernel, the form induced on the quotient is nonsingular. ■

3.11. Descending the $\perp^{[\cdot]}$-filtration. — Since V is finite-dimensional, the filtrations are finite, and all pieces of the $\perp^{[\cdot]}$-filtration are defined on all sufficiently large Frobenius twists of V. Thus

$$\nu(V,\beta) := \min\left\{ i \in \mathbf{Z}_{\geq 0} \,\middle|\, \perp^{[\cdot]}\text{-filtration descends to } (V^{[i]}, \beta^{[i]}) \text{ over } \mathbf{k} \right\}$$

is well-defined. This depends on $\mathbf{k}$: for instance, $\nu(V,\beta) = 0$ for any form over a perfect field. The following gives an *a priori* upper bound $\nu(V,\beta) \leq \nu_0$ depending only on the type of β. The statement is optimal, as can be shown by considering infinitesimals in automorphism groups as in 5.14 and comparing with the examples 5.7, 5.8, and $\mathbf{Aut}_{(\mathbf{k}^{\oplus 5}, \mathbf{N}_2 \oplus \mathbf{N}_3)}$.

3.12. Lemma. — *Assume that β is degenerate and let $\mu := \max\{ m \in \mathbf{Z}_{\geq 1} \mid b_m(V,\beta) \neq 0\}$. Then the $\perp^{[\cdot]}$-filtration canonically descends to $V^{[\nu_0]}$ for*

$$\nu_0 := \begin{cases} \mu - 2 & \text{if } \mu > 1 \text{ and } b_m(V,\beta) = 0 \text{ for all even } m, \\ \mu - 1 & \text{if } \mu \text{ is odd or } \mu \text{ is even and } a(V,\beta) = 0, \text{ and} \\ \mu & \text{otherwise.} \end{cases}$$

Proof. Since μ is the length of the $\perp^{[\cdot]}$-filtration, the last case is clear. When μ is even and $a(V,\beta) = 0$, then $\mathrm{P}'_\mu V^{[\mu]} = \mathrm{P}'_{\mu-1}V^{[\mu]}$, so the last step of the filtration canonically descends to $V^{[\mu-1]}$. It remains to consider the situation when μ is odd. Since $b_m(V,\beta) = 0$ for all $m > \mu$ the formulae of 3.7 imply that $a_\mu(V,\beta) = b_\mu(V,\beta)$, so that by comparing dimensions via 3.8, the natural injection

$$\mathrm{P}_1V^{[\mu]} \cap \mathrm{P}'_\mu V^{[\mu]} / \mathrm{P}_1V^{[\mu]} \cap \mathrm{P}'_{\mu-2}V^{[\mu]} \hookrightarrow \mathrm{P}'_\mu V^{[\mu]} / \mathrm{P}'_{\mu-2}V^{[\mu]}$$

is an isomorphism. This implies that $\mathrm{P}'_\mu V^{[\mu]}$ is spanned by $\mathrm{P}_1V^{[\mu]} \cap \mathrm{P}'_\mu V^{[\mu]}$ and $\mathrm{P}'_{\mu-2}V^{[\mu-2]}$. By the formulae in 3.9, the intersection coincides with $\mathrm{P}_1V^{[\mu]} \cap \mathrm{P}'_{\mu-1}V^{[\mu]}$. Therefore

$$\mathrm{P}'_\mu V^{[\mu-1]} = \mathrm{P}'_{\mu-2}V^{[\mu-1]} + \mathrm{P}_1V^{[\mu-1]} \cap \mathrm{P}'_{\mu-1}V^{[\mu-1]}$$

is a canonical descent of $\mathrm{P}'_\mu V^{[\mu]}$ to $V^{[\mu-1]}$. Furthermore, when $\mu > 1$ and $b_m(V,\beta) = 0$ for all even m, the intersection is all of $\mathrm{P}_1V^{[\mu-1]}$ and so the sum descends further to $V^{[\mu-2]}$. By 1.5(iv),

$$\mathrm{P}'_{\mu-1}V^{[\mu-3]} := \mathrm{P}'_\mu V^{[\mu-2],\perp^{[\mu-3]}} := \ker\left(\beta^{[\mu-3]} : V^{[\mu-3]} \to \mathrm{P}'_\mu V^{[\mu-2],\vee}\right)$$

and so the entire $\perp^{[\cdot]}$-filtration admits a descent to $V^{[\mu-2]}$ in this case. ■

4. Classification

The object of this Section is to prove the following Classification Theorem, which says that, after passing to a suitable Frobenius twist upon which its $\perp^{[\cdot]}$-filtration is defined as in 3.11, a q-bic form over a field is essentially classified by its type, as defined in 3.9:

4.1. Theorem. — *Let (V,β) be a q-bic form over a field $\mathbf{k}$ of type $(a;b_m)_{m\geq 1}$, and let $\nu := \nu(V,\beta)$. Then there exists an orthogonal decomposition*

$$\beta^{[\nu]} = \beta_0 \oplus \Big(\bigoplus\nolimits_{m\geq 1} \beta_m\Big)$$

such that β_0 is nonsingular of dimension a, and β_m has a Gram matrix given by $\mathbf{N}_m^{\oplus b_m}$ for each $m \geq 1$.

Combined with the classification of nonsingular forms in 2.7 and the remarks on ν from 3.11, this gives a normal form for q-bic forms over an algebraically closed field:

4.2. Corollary. — *If $\mathbf{k}$ is algebraically closed, then there exists a basis $V = \langle e_1,\ldots,e_n\rangle$ such that*

$$\mathrm{Gram}(\beta; e_1,\ldots,e_n) = \mathbf{1}^{\oplus a} \oplus \Big(\bigoplus\nolimits_{m\geq 1} \mathbf{N}_m^{\oplus b_m}\Big). \qquad \blacksquare$$

Throughout this Section, let (V,β) denote a q-bic form over a field $\mathbf{k}$. The idea of proof is as follows: let $m \geq 1$ be an integer, and choose subspace of dimension $b := b_m(V,\beta)$ in

$$\begin{cases} \mathrm{P}_1 V^{[\nu]} \cap \mathrm{P}'_{2k-1} V^{[\nu]} & \text{if } m = 2k-1, \text{ or} \\ \mathrm{P}_1 V^{[\nu]} \cap \mathrm{P}'_{2k-2} V^{[\nu]} & \text{if } m = 2k, \end{cases}$$

lifting the quotient from 3.8. The seesaw relation between the two filtrations, as in 4.6, extends this subspace to an mb-dimensional subspace of V on which β has a Gram matrix given by $\mathbf{N}_m^{\oplus b}$, and which admits an orthogonal complement. Since the filtrations are compatible with orthogonal decompositions, inductively continuing this method on the complement yields the Theorem.

4.3. Recognizing a standard form. — An intrinsic formulation of the property that a q-bic form (V,β) has a Gram matrix given by $\mathbf{N}_m^{\oplus b}$ is: there exists a vector space decomposition $V = \bigoplus_{i=1}^m V_i$ such that, for integers $1 \leq i,j, \leq m$, the map

$$\beta_{i,j}\colon V_j \subseteq V \xrightarrow{\beta} V^{[1],\vee} \twoheadrightarrow V_i^{[1],\vee} \text{ is } \begin{cases} \text{an isomorphism} & \text{if } j = i+1, \text{ and} \\ \text{zero} & \text{otherwise.} \end{cases}$$

Indeed, if V has a basis with Gram matrix $\mathbf{N}_m^{\oplus b}$, set V_i to be the span of the vectors corresponding to the i-th column of each $\mathbf{N}_m$ block. Conversely, given the vector space decomposition, begin with a basis of V_1 and use the maps $\beta_{i,i+1}$ to construct a basis of V with the desired Gram matrix.

The following is a more flexible characterization of such forms:

4.4. Lemma. — *Assume that there is a vector space decomposition $V = \bigoplus_{i=1}^m V_i$ such that*

(i) *$V_1 = \mathrm{P}_1 V$ and $V_m^{[1]} = \mathrm{P}'_1 V^{[1]}$,*
(ii) *$\beta_{1,2}\colon V_2 \subseteq V \to V^{[1],\vee} \twoheadrightarrow V_1^{[1],\vee}$ is an isomorphism, and*
(iii) *$\mathrm{im}(\beta\colon V_{i+1} \to V^{[1],\vee}) = \mathrm{im}(\beta^{[1],\vee}\colon V_{i-1}^{[2]} \to V^{[1],\vee})$ for each $1 < i < m$.*

Then β has a Gram matrix given by $\mathbf{N}_m^{\oplus b}$ with $b := \dim_{\mathbf{k}} V_1$.

Proof. The task is to adjust the given decomposition of V so that the maps $\beta_{i,j}$ from 4.3 are isomorphisms when $j = i+1$ and zero otherwise. Post-composing the maps appearing in (iii) with the restriction $V^{[1],\vee} \to V_j^{[1],\vee}$ shows that: for all $1 < i < m$ and $1 \le j \le m$,

$$\operatorname{rank}(\beta_{j,i+1} \colon V_{i+1} \to V_j^{[1],\vee}) = \operatorname{rank}(\beta_{i-1,j} \colon V_j \to V_{i-1}^{[1],\vee}).$$

The maps $\beta \colon V_{i+1} \to V^{[1],\vee}$ and $\beta^\vee \colon V_{i-1}^{[1]} \to V^\vee$ are injections for each $1 < i < m$ by assumption (i), so the equality of ranks combined with (ii) implies that each subspace V_i is b-dimensional, and that the map $\beta_{i,i+1}$ is an isomorphism for all $1 \le i < m$. The rank condition together with (i) implies that

$$\beta_{i,j} = 0 \text{ whenever } j \neq i+1 \text{ and } \begin{cases} \min\{i,j\} \equiv 1 \pmod 2, \\ \max\{i,j\} \equiv m \pmod 2. \end{cases}$$

The even-indexed subspaces will now be adjusted to arrange for the remaining $\beta_{i,j}$ to vanish, via a construction depending on the parity of m.

Case m is odd. It remains to arrange for $\beta_{i,j} = 0$ when both i and j are even. Let $1 \le k < m/2$ and inductively assume that $\beta_{i,j} = 0$ whenever $\min\{i,j\} < 2k$. The task is to modify the subspaces V_i with i even and $2k \le i \le m$ so that vanishing furthermore holds when $\min\{i,j\} = 2k$.

Consider the subspace $V' := \bigoplus_{i=2k}^m V_i$ and let β' be the restriction of β thereon. For $2k \le i \le m$, set $V_i' := V_i$ for odd i, and

$$V_i' := \begin{cases} \mathrm{P}_1 V' & \text{if } i = 2k, \text{ and} \\ \ker(\beta \colon V_{2k+1} \oplus V_i \to V_{2k}'^{[1],\vee}) & \text{otherwise.} \end{cases}$$

Since $\beta_{i,i+1}^\vee$ is an isomorphism and $\beta^\vee$ vanishes on V_m by (i), it follows that $\mathrm{P}_1' V'^{[1]} = V_m^{[1]}$, so

$$\dim_{\mathbf{k}} V_{2k}' = \dim_{\mathbf{k}} \mathrm{P}_1 V' = \dim_{\mathbf{k}} \mathrm{P}_1' V'^{[1]} = \dim_{\mathbf{k}} V_m^{[1]} = b.$$

Observe that the projections $V_i' \to V_i$ are isomorphisms for all i. This is clear for i odd. For $i = 2k$, this is because both V_{2k}' and V_{2k} are b-dimensional and, since $\beta_{j,j+1} \colon V_{j+1} \to V_j^{[1],\vee}$ are isomorphisms,

$$\ker(V_{2k}' \to V_{2k}) = \mathrm{P}_1 V' \cap \Big(\bigoplus\nolimits_{i=2k+1}^m V_i\Big) = \ker\Big(\beta \colon \bigoplus\nolimits_{i=2k+1}^m V_i \to V'^{[1],\vee}\Big) = \{0\}.$$

Since $\beta_{j,2k+1} = 0$ for $2k+1 \le j \le m$, this furthermore implies that the map $V_{2k+1} \to V_{2k}'^{[1],\vee}$ induced by β is an isomorphism. It then follows from the definition of V_i' for i even and different from $2k$, that the projection $V_i' \to V_i$ is an isomorphism.

Therefore the subspaces V_i' yield a new direct sum decomposition

$$V' = V_{2k}' \oplus V_{2k+1}' \oplus \cdots \oplus V_m'.$$

It is then straightforward to check that replacing the V_i with the V_i', for $2k \le i \le m$, that the maps $\beta_{i,j}$, in addition to their previous properties, vanish whenever i and j are both even and $\min\{i,j\} = 2k$. This completes the proof in the case m is odd.

Case m is even. It remains to arrange vanishing of $\beta_{i,j}$ when the smaller index is even, and the larger index odd. This can be done all at once: for each $1 \le k \le m/2$, set $V_{2k-1}' := V_{2k-1}$ and

$$V_{2k}' := \ker\Big(\beta \colon V_{2k} \oplus \Big(\bigoplus\nolimits_{\ell=k+1}^{m/2} V_{2\ell}\Big) \to \bigoplus\nolimits_{\ell=k+1}^{m/2} V_{2\ell-1}^{[1],\vee}\Big).$$

The projection $V_{2k}' \to V_{2k}$ is an isomorphism since β restricts to an isomorphism between the two sums appearing in its definition. Thus this gives a new decomposition $V = \bigoplus_{i=1}^m V_i'$, and it is straightforward to check that the maps

$$\beta_{i,j}' \colon V_j' \subseteq V \xrightarrow{\beta} V^{[1],\vee} \twoheadrightarrow V_i'^{[1],\vee}$$

induced by β on this decomposition satisfy the vanishing conditions of 4.3: First, the subspaces $\bigoplus_{k=1}^{m/2} V_{2k-1}$ and $\bigoplus_{k=1}^{m/2} V_{2k}$ are totally isotropic for β, and so the maps $\beta_{i,j}'$ vanish whenever i and j

are of the same parity. Next, it follows by construction of the V_i' that $\beta_{i,j}'$ vanishes whenever $\min\{i,j\}$ is even and $\max\{i,j\}$ is odd. Thus the restriction of β to V_{2k}' factors as

$$V_{2k}' \subset V_{2k} \oplus \Big(\bigoplus\nolimits_{\ell=k+1}^{m/2} V_{2\ell}\Big) \xrightarrow{\beta} \bigoplus\nolimits_{s=1}^{k} V_{2s-1}^{[1],\vee}.$$

Since $\beta_{2s-1,2\ell} = 0$ for each $1 \leq s < \ell \leq m/2$, the image of V_{2k}' coincides with that of V_{2k}, and is the space $V_{2k-1}^{[1],\vee}$. This gives the remaining conditions on $\beta_{i,j}'$, completing the proof. ■

4.5. Seesawing between filtrations. — Conditions (ii) and (iii) of 4.4 will be arranged by studying the relationship between the $\perp$- and $\perp^{[\cdot]}$-filtrations under β and its dual. The fundamental relationship is the following seesaw relationship between intersections of pieces of the two filtrations:

4.6. Lemma. — *For integers $i, j \geq 0$, there is an equality of subspaces of $V^{[j]}$ given by*

$$(\mathrm{P}_{i+1}V^{[j-1]} \cap \mathrm{P}_{j-1}'V^{[j-1]})^{\perp^{[j-1]}} = \mathrm{P}_i V^{[j]} + \mathrm{P}_j' V^{[j]} = (\mathrm{P}_{i-1}V^{[j+1]} \cap \mathrm{P}_{j+1}'V^{[j+1]})^{\perp^{[j]}}.$$

In particular, there is an equality of subspaces of $V^{\vee,[j]}$ given by

$$\mathrm{im}\big(\beta^{[j-1]} \colon \mathrm{P}_{i+1}V^{[j-1]} \cap \mathrm{P}_{j-1}'V^{[j-1]} \to V^{\vee,[j]}\big) = \mathrm{im}\big(\beta^{[j],\vee} \colon \mathrm{P}_{i-1}V^{[j+1]} \cap \mathrm{P}_{j+1}'V^{[j+1]} \to V^{\vee,[j]}\big).$$

Proof. Since $V^{[j],\perp^{[j-1]}} = \mathrm{P}_1 V^{[j-1]} \subseteq \mathrm{P}_{i+1}V^{[j-1]}$ and $V^{[j],\perp^{[j]}} = \mathrm{P}_1' V^{[j+1]} \subseteq \mathrm{P}_{j+1}' V^{[j+1]}$, 1.5(v) applies and shows that the two orthogonals in the statement are given by the left and right sides of:

$$\mathrm{P}_{i+1}V^{[j-1],\perp^{[j-1]}} + \mathrm{P}_{j-1}'V^{[j-1],\perp^{[j-1]}} = \mathrm{P}_i V^{[j]} + \mathrm{P}_j' V^{[j]} = \mathrm{P}_{i-1}V^{[j+1],\perp^{[j]}} + \mathrm{P}_{j+1}'V^{[j+1],\perp^{[j]}}.$$

The two middle equalities follow from the commutation relationship for Frobenius twists in 1.7 and reflexivity of orthogonals as in 1.5(iv), yielding the first statement. The second statement now follows, since the two images are dual to the quotient of $V^{[j]}$ by the common orthogonal. ■

The following consequence of 4.6 with $i = 2$ is helpful in relating the recognition principle condition 4.4(ii) with the numerical invariants of β via 3.8:

4.7. Corollary. — *Let $m \geq 1$ be an integer and $\epsilon := (-1)^m$. Then $\beta^{[m-1],\vee}$ induces an exact sequence*

$$0 \to \frac{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}_{m-\epsilon-1}' V^{[m]}}{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}_{m+\epsilon-1}' V^{[m]}} \to \left(\frac{\mathrm{P}_{m+\epsilon-2}' V^{[m-1]}}{\mathrm{P}_{m-\epsilon-2}' V^{[m-1]}}\right)^{\vee} \to \left(\frac{\mathrm{P}_2 V^{[m-1]} \cap \mathrm{P}_{m+\epsilon-2}' V^{[m-1]}}{\mathrm{P}_2 V^{[m-1]} \cap \mathrm{P}_{m-\epsilon-2}' V^{[m-1]}}\right)^{\vee} \to 0.$$

Proof. Applying 4.6 with $i = 2$ shows that $\beta^{[j],\vee}$ induces a short exact sequence

$$0 \to \mathrm{P}_1 V^{[j+1]} \cap \mathrm{P}_1' V^{[j+1]} \to \mathrm{P}_1 V^{[j+1]} \cap \mathrm{P}_{j+1}' V^{[j+1]} \to \big(V^{[j]}/(\mathrm{P}_2 V^{[j]} + \mathrm{P}_j' V^{[j]})\big)^{\vee} \to 0.$$

Comparing $j = m-1$ and $j = m-2\epsilon-1$ then gives the isomorphism

$$\frac{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}_{m-\epsilon-1}' V^{[m]}}{\mathrm{P}_1 V^{[m]} \cap \mathrm{P}_{m+\epsilon-1}' V^{[m]}} \cong \left(\frac{\mathrm{P}_2 V^{[m-1]} + \mathrm{P}_{m+\epsilon-2}' V^{[m-1]}}{\mathrm{P}_2 V^{[m-1]} + \mathrm{P}_{m-\epsilon-2}' V^{[m-1]}}\right)^{\vee}$$

which implies the claimed exact sequence. ■

Finally, a semi-linear algebra statement used in the main step of the Classification Theorem to produce an orthogonal complement. The following says that given a q-linear map $V \to W$ between vector spaces, under certain conditions, any subspace of V admits a complement with linearly disjoint image. The hypothesis below is necessary, as can be seen by considering any surjective map.

4.8. Lemma. — *Let V and W be vector spaces over $\mathbf{k}$, and let $f \colon V^{[1]} \to W$ be a linear map. Assume that $\ker(f)$ descends to V. Then any subspace V' of V admits a complement V'' such that*

$$f(V'^{[1]}) \cap f(V''^{[1]}) = \{0\}.$$

Proof. Let $\tilde{f}\colon V \to W$ be the q-linear map obtained by precomposing f with the canonical map $V \to V^{[1]}$. The additive subgroup $K := \ker(\tilde{f})$ is, in fact, a linear subspace of V, and is the Frobenius descent of $\ker(f)$. Choose a complement V_1'' to $V' + K$ in V, and choose a complement V_2'' to $V' \cap K$ in K. Then $V'' := V_1'' \oplus V_2''$ is a complement to V' in V that satisfies the condition: since

$$f(V'^{[1]}) \cap f(V''^{[1]}) = f\big((V'^{[1]} + \ker(f)) \cap V''^{[1]}\big) = f\big(((V' + K) \cap V'')^{[1]}\big)$$

and since the rightmost intersection is contained inside $K^{[1]} = \ker(f)$, the intersection is $\{0\}$. ■

The following explains how to split off orthogonal summands of standard type $\mathbf{N}_m$ once the first m steps of the $\perp^{[\cdot]}$-filtration are visible on V over $\mathbf{k}$:

4.9. Proposition. — *Let $m \geq 1$ be an integer and assume that the first m steps of the $\perp^{[\cdot]}$-filtration descend to V over $\mathbf{k}$. Then there exists an orthogonal decomposition*

$$\beta = \beta' \oplus \beta''$$

where β' has a Gram matrix given by $\mathbf{N}_m^{\oplus b}$ for $b := b_m(V, \beta)$.

Proof. When $m = 1$, let $V' := \mathrm{P}_1 V \cap \mathrm{P}_1' V$ and let V'' be any complementary subspace in V. Restricting β to these subspaces gives subforms β' and β'', respectively, satisfying the properties of the statement in this case. Thus, in the remainder, assume that $m \geq 2$.

Construct subspaces $V_i' \subseteq V$ for $1 \leq i \leq m$ as follows: Let $\epsilon := (-1)^m$. By 3.8 and 4.7, it is possible to choose a b-dimensional subspace $V_1' \subseteq \mathrm{P}_1 V \cap \mathrm{P}_{m-\epsilon-1}' V$ such that the map

$$V_1'^{[1]} \xrightarrow{\sim} \left(\frac{\mathrm{P}_1 V \cap \mathrm{P}_{m-\epsilon-1}' V}{\mathrm{P}_1 V \cap \mathrm{P}_{m+\epsilon-1}' V}\right)^{[1]} \hookrightarrow \left(\frac{\mathrm{P}_{m+\epsilon-2}' V}{\mathrm{P}_{m-\epsilon-2}' V}\right)^{\vee} \hookrightarrow \mathrm{P}_{m+\epsilon-2}' V^{\vee}$$

induced by $\beta^\vee$ is an isomorphism onto its image W_1. Since the kernel of $\beta^\vee\colon V^{[1]} \to \mathrm{P}_{m+\epsilon-2}' V^\vee$ is $\mathrm{P}_{m+\epsilon-1}' V^{[1]}$, which admits a descent to V by hypothesis, 4.8 applies to give a complement V_1'' to V_1' in V whose images under $\beta^\vee$ are linearly disjoint. Extend the image of $V_1''^{[1]}$ under $\beta^\vee$ to a complement W_2 of W_1 so that $\mathrm{P}_{m+\epsilon-2}' V^\vee = W_1 \oplus W_2$. Viewing W_1 as a quotient then determines a b-dimensional subspace $V_2' \subseteq \mathrm{P}_{m+\epsilon-2}' V$, disjoint from $\mathrm{P}_{m-\epsilon-2}' V$, such that $\beta^\vee\colon V_1'^{[1]} \to V_2'^\vee$ is an isomorphism, and

$$V_2'^{\perp} := \ker(\beta^\vee\colon V^{[1]} \to V_2'^\vee) = \ker(\beta^\vee\colon V^{[1]} \to W_1) = V_1''^{[1]}.$$

Starting from V_1' and V_2', apply the second part of 4.6 to successively choose b-dimensional subspaces

$$\begin{cases} V_{2k-1}' \subseteq \mathrm{P}_{2k-1} V \cap \mathrm{P}_{m-\epsilon-2k+1}' V & \text{for } 1 \leq k \leq \lceil m/2 \rceil, \text{ and} \\ V_{2k}' \subseteq \mathrm{P}_{2k-2} V \cap \mathrm{P}_{m+\epsilon-2k}' V & \text{for } 1 \leq k \leq \lfloor m/2 \rfloor, \end{cases}$$

such that $\mathrm{im}(\beta\colon V_{i+1}' \to V^{[1],\vee}) = \mathrm{im}(\beta^{[1],\vee}\colon V_{i-1}'^{[2]} \to V^{[1],\vee})$ for each $1 < i < m$, and such that the V_i' project into the graded piece of the $\perp^{[\cdot]}$-filtration for the index displayed above.

The V_i' are therefore disjoint and together span an mb-dimensional subspace

$$V' := \bigoplus\nolimits_{i=1}^m V_i' \subseteq V.$$

The restriction β' of β satisfies the hypotheses of 4.4 with respect to this decomposition: (ii) follows from the choice of V_1' and V_2', and (iii) follows from the construction of the V_i' via 4.6. Since each V_i' is b-dimensional, this implies that β' has corank b. Since $V_1' \subseteq \mathrm{P}_1 V'$ and $V_m' \subseteq \mathrm{P}_1' V'$ by construction, comparing dimensions then give (i). Therefore β' has a Gram matrix given by $\mathbf{N}_m^{\oplus b}$.

It remains to observe that the total orthogonal of V' descends to a complement V'' in V. Taking orthogonals and applying 1.4(ii) to the direct sum decomposition shows that the total orthogonal is

$$V'^{\perp} \cap V'^{[2],\perp^{[1]}} = \Big(\bigcap\nolimits_{i=1}^m V_i'^{\perp}\Big) \cap \Big(\bigcap\nolimits_{i=1}^m V_i'^{[2],\perp^{[1]}}\Big).$$

By construction, $V'^{\perp}_{i+1} = V'^{[2],\perp^{[1]}}_{i-1}$ for each $1 < i < m$ and $V'^{\perp}_1 = V^{[1]} = V'^{[2],\perp^{[1]}}_m$, so

$$V'^{\perp} \cap V'^{[2],\perp^{[1]}} = V'^{\perp}_2 \cap \Big(\bigcap\nolimits_{i=1}^{m-1} V'^{[2],\perp^{[1]}}_i\Big) = \Big(V''_1 \cap \Big(\bigcap\nolimits_{i=1}^{m-1} V'^{[1],\perp}_i\Big)\Big)^{[1]} =: V''^{[1]}$$

where the middle equality is due to the choice of V'_2. Thus the total orthogonal admits a Frobenius descent to a subspace V'' of V which has codimension at most mb. Since β' does not have a radical, V' and V'' are linearly disjoint, and it follows that they are complementary subspaces of V. Taking β'' to be the restriction of β to V'' then gives the result. ∎

Proof of 4.1. By definition of ν from 3.11, the $\perp^{[\cdot]}$-filtration descends to $V^{[\nu]}$ over $\mathbf{k}$. Since formation of the filtrations is compatible with orthogonal decompositions by 3.3, 4.9 may be successively applied for each integer $m \geq 1$ to produce orthogonal summands β_m with Gram matrices $\mathbf{N}_m^{\oplus b_m}$. Since only finitely many of the invariants b_m are nonzero, this eventually terminates, and the remaining piece β_0 is a nonsingular subform of β lifting that on $\mathrm{P}_+V/\mathrm{P}_-V$ as in 3.10. ∎

5. AUTOMORPHISMS

This Section is concerned with automorphism group schemes of q-bic forms. For nondegenerate forms, this recovers the classical unitary group, see 5.6. In general, these schemes are positive-dimensional, and are often nonreduced, as can be seen the examples 5.7 and 5.8. The main observation in this Section is that all nonreducedness arises from a failure of preserving a descent of the $\perp^{[\cdot]}$-filtration, compare 5.2 and 5.4 with the main result 5.12 and its consequence 5.13. This is used to compute the dimension of the automorphism groups in 5.15.

5.1. Automorphism group schemes. — Let (M,β) be a q-bic form over a $\mathbf{F}_{q^2}$-algebra R. Given an R-algebra S, write $M_S := M \otimes_R S$ and $\beta_S := \beta \otimes \mathrm{id}_S$ for the q-bic form over S obtained by extension of scalars. Consider the group-valued functor

$$\mathbf{Aut}_{(M,\beta)} \colon \mathrm{Alg}_R \to \mathrm{Grps} \qquad S \mapsto \mathrm{Aut}(M_S,\beta_S)$$

that sends R-algebra S to the automorphism group of (M_S,β_S). This is the subfunctor of linear automorphisms $\mathbf{GL}_M$ of M which stabilizes the element $\beta \in \mathrm{Hom}_R(M^{[1]} \otimes_R M, R)$. Since M is always assumed to be finite projective, this is represented by a closed subgroup scheme of $\mathbf{GL}_M$ by [DG70, II.1.2.4 and II.1.2.6], and is referred to as the *automorphism group scheme of* (M,β).

Since the $\perp$- and $\perp^{[\cdot]}$-filtrations from 3.1 and 3.2 are intrinsic to a q-bic form (M,β), they are preserved by points of the automorphism group scheme whenever the constructions are compatible with scalar extension, for instance when each piece is a local direct summand. In other words:

5.2. Lemma. — *Assume that the submodules $\mathrm{P}_i M \subseteq M$ and $\mathrm{P}'_i M^{[i]} \subseteq M^{[i]}$ are local direct summands for every integer $i \geq 0$. Then $\mathbf{Aut}_{(M,\beta)}$ factors through the closed subgroup scheme of $\mathbf{GL}_M$ given by*

$$\mathbf{Aut}_{(M,\mathrm{P}_\cdot M,\mathrm{P}'_\cdot M^{[\cdot]})} \colon S \mapsto \Big\{ g \in \mathrm{GL}(M_S) \;\Big|\; g \cdot \mathrm{P}_i M_S = \mathrm{P}_i M_S \textit{ and } g^{[i]} \cdot \mathrm{P}_i M_S^{[i]} = \mathrm{P}_i M_S^{[i]} \textit{ for each } i \geq 0 \Big\}. \;\blacksquare$$

5.3. — However, $\mathbf{GL}_M$ acts on $M^{[i]}$ through its R-linear Frobenius morphism, which, in coordinates, raises each matrix coefficient to its q^i-th power. Importantly, this means that infinitesimal members of $\mathbf{Aut}_{(M,\beta)}$ might not preserve a descent $\mathrm{P}'_\cdot M$ of the $\perp^{[\cdot]}$-filtration to M. So, in the case that the $\perp^{[\cdot]}$-filtration descends to M over R, let

$$\mathbf{Aut}_{(M,\beta,\mathrm{P}'_\cdot M)} \colon S \mapsto \big\{ g \in \mathrm{Aut}(M_S,\beta_S) \;\big|\; g \cdot \mathrm{P}'_i M_S = \mathrm{P}'_i M_S \text{ for each } i \in \mathbf{Z}_{\geq 0} \big\}$$

be the closed subgroup scheme of $\mathbf{Aut}_{(M,\beta)}$ that additionally preserves $\mathrm{P}'_\cdot M$. Then, at least:

5.4. Lemma. — *Assume that the $\perp^{[\bullet]}$-filtration descends to M and that each piece $\mathrm{P}'_i M$ is a local direct summand. Then, as subschemes of $\mathbf{GL}_M$, the reduced subscheme of $\mathbf{Aut}_{(M,\beta)}$ is contained in $\mathbf{Aut}_{(M,\beta,\mathrm{P}'_\bullet M)}$.*

Proof. It suffices to show that if R is reduced, then every $g \in \mathrm{Aut}(M,\beta)$ preserves $\mathrm{P}'_\bullet M$. Since the i-th Frobenius twist of g preserves $\mathrm{P}'_i M^{[i]}$, the image $N := g \cdot \mathrm{P}'_i M$ fits into a commutative diagram

$$\begin{tikzcd} \mathrm{P}'_i M \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & M/\mathrm{P}'_i M \arrow[d] \\ \mathrm{P}'_i M^{[i]} \arrow[r, "g^{[i]}"] & \mathrm{P}'_i M^{[i]} \arrow[r, "0"] & M^{[i]}/\mathrm{P}'_i M^{[i]} \end{tikzcd}$$

where the vertical maps are induced by the canonical map $M \to M^{[i]}$. Since $\mathrm{P}'_i M$ is a local direct summand of M, all the modules are finite projective; together with reducedness of R, it follows that all the vertical maps are injective. Commutativity of the right square implies that the map $N \to M/\mathrm{P}'_i M$ vanishes, and so $N \subseteq \mathrm{P}'_i M$, implying the result. ■

The remainder of this Section is concerned with q-bic forms (V,β) over a field $\mathbf{k}$, and the dimension of their automorphism group schemes. First, their Lie algebras are as follows:

5.5. Proposition. — *There is a canonical identification of $\mathbf{k}$-vector spaces given by*

$$\mathrm{Lie}\,\mathbf{Aut}_{(V,\beta)} \cong \mathrm{Hom}_{\mathbf{k}}(V, \mathrm{P}_1 V),$$

and, furthermore, when the $\perp^{[\bullet]}$-filtration descends to V,

$$\mathrm{Lie}\,\mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)} \cong \left\{ \varphi \in \mathrm{Hom}_{\mathbf{k}}(V,\mathrm{P}_1 V) \;\middle|\; \varphi(\mathrm{P}'_i V) \subseteq \mathrm{P}_1 V \cap \mathrm{P}'_i V \text{ for each } i \in \mathbf{Z}_{\geq 0} \right\}.$$

Proof. The Lie algebra of $\mathbf{Aut}_{(V,\beta)}$ is the subset of $\mathrm{Aut}(V_D,\beta_D)$, with $D := \mathbf{k}[\epsilon]/(\epsilon^2)$, consisting of automorphisms that restrict to the identity upon setting $\epsilon = 0$. Such elements may be written as $\mathrm{id} + \epsilon\varphi$ for a unique $\mathbf{k}$-linear map $\varphi \colon V \to V$. Since $\epsilon^2 = 0$, that $\mathrm{id}+\epsilon\varphi$ preserves β_D means that, for every $u \in V_D^{[1]}$ and $v \in V_D$,

$$\beta_D(u,v) = \beta_D\big((\mathrm{id}+\epsilon\varphi)^{[1]}(u), (\mathrm{id}+\epsilon\varphi)(v)\big) = \beta_D(u,v) + \epsilon\beta(\bar{u}, \varphi(\bar{v})),$$

where $\bar{u} \in V^{[1]}$ and $\bar{v} \in V$ are the images of u and v upon setting $\epsilon = 0$. Since $\bar{u}$ is arbitrary, this implies that $\varphi(\bar{v}) \in V^{[1],\perp} = \mathrm{P}_1 V$, giving the first identification. This implies the second statement since the Lie algebra of $\mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)}$ is the subspace of $\mathrm{Lie}\,\mathbf{Aut}_{(V,\beta)}$ consisting of those φ which, in addition, preserve the filtration $\mathrm{P}'_\bullet V$. ■

Before continuing, consider a few examples of automorphism groups and schemes of q-bic forms over a field. Some more examples and explicit computations may be found in [Che22, Chapter 3].

5.6. Unitary groups. — When β is nonsingular, **5.5** implies that $\mathbf{Aut}_{(V,\beta)} =: \mathrm{U}(V,\beta)$ is an étale group scheme over $\mathbf{k}$, and might called the *unitary group* of β. By **2.7** and the comments thereafter, its group of points over a separable closure of $\mathbf{k}$ is isomorphic to the classical finite unitary group $\mathrm{U}_n(q)$, as in [CCN+85, §2.1]. Following Steinberg as [Ste16, Lecture 11], the unitary group can also be described as $\mathrm{U}(V,\beta) = \mathbf{GL}_V^F$, the fixed points for the morphism of algebraic groups given by

$$F\colon \mathbf{GL}_V \to \mathbf{GL}_V \qquad g \mapsto \beta^{-1} \circ \mathrm{Fr}^*(g)^{\vee,-1} \circ \beta.$$

5.7. Type $\mathbf{N}_n$. — The automorphism group of a form β with a Gram matrix $\mathbf{N}_n$ can be roughly determined by analyzing the proof of 4.9: First, there is the choice of a basis vector for $V_1 = \mathrm{P}_1 V \cap \mathrm{P}'_{n-\epsilon-1}V$. This now uniquely determines, via 4.6, spaces V_i together with a basis vector whenever i has the same parity as n. Finally, when the parity of i and n are different, the V_i and its basis are only determined up to an additive factor. In particular, this gives $\dim \mathbf{Aut}_{(V,\beta)} = \lceil n/2 \rceil$.

In contrast, 5.5 shows that $\dim_{\mathbf{k}} \mathrm{Lie}\,\mathbf{Aut}_{(V,\beta)} = n$, so these schemes are not reduced for any $n \geq 2$. The schematic structure can be explictly determined in low dimensions; for example, $\mathbf{Aut}_{(V,\beta)}$ may be described for $n \in \{2,3,4\}$ as closed subschemes of $\mathbf{GL}_n$ consisting of matrices

$$\begin{pmatrix} \lambda & \epsilon_1 \\ 0 & \lambda^{-q} \end{pmatrix}, \quad \begin{pmatrix} \lambda & t & \epsilon_1 \\ 0 & \lambda^{-q} & 0 \\ 0 & -\lambda^{q(q-1)}t^q & \lambda^{q^2} \end{pmatrix}, \quad \begin{pmatrix} \lambda & \epsilon_3 & t & \epsilon_1 \\ 0 & \lambda^{-q} & 0 & 0 \\ 0 & \epsilon_2 & \lambda^{q^2} & -\lambda^{-q^2(q-1)}\epsilon_2^q \\ 0 & -\lambda^{-q^2(q+1)}t^q & 0 & \lambda^{-q^3} \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $t \in \mathbf{G}_a$, and $\epsilon_i \in \boldsymbol{\alpha}_{q^i}$, and when $n = 4$, the entries satisfy

$$\epsilon_2^q t^q - \lambda^{q(q^2-q+1)}\epsilon_2 - \lambda^{q^3}\epsilon_3^q = 0.$$

5.8. Type $\mathbf{1}^{\oplus a} \oplus \mathbf{N}_2^{\oplus b}$. — One other situation in which $\mathbf{Aut}_{(V,\beta)}$ admits a reasonably neat description is when β admits a Gram matrix of the form $\mathbf{1}^{\oplus a} \oplus \mathbf{N}_2^{\oplus b}$. Then the automorphism group is a b^2-dimensional closed subgroup scheme of $\mathbf{GL}_{a+2b}$, and is described in [Che22, 1.3.7]; for $a = b = 1$, this is the 1-dimensional subgroup $\mathbf{GL}_3$ given by matrices of the form

$$\begin{pmatrix} \zeta & 0 & -\zeta\epsilon_2^q/\lambda^q \\ \epsilon_2 & \lambda & \epsilon_1 \\ 0 & 0 & \lambda^{-q} \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $\epsilon_1 \in \boldsymbol{\alpha}_q$, $\epsilon_2 \in \boldsymbol{\alpha}_{q^2}$, and $\zeta \in \boldsymbol{\mu}_{q+1}$.

5.9. Filtered automorphisms. — The examples suggest that the nonreducedness of $\mathbf{Aut}_{(V,\beta)}$ arises from its failure to preserve a descent of its $\perp^{[\cdot]}$-filtration to V. In particular, the subgroups in 5.7 and 5.8 that preserve $\mathrm{P}'_{\bullet}V$ are in fact reduced. This turns out to be the general situation and is established below by studying infinitesimal deformations of the identity automorphism $(V,\beta,\mathrm{P}'_{\bullet}V)$. In preparation, establish some notation: write

$$\begin{aligned} \bar{V} &:= V/\mathrm{P}_1 V, & \mathrm{P}_{\bullet}\bar{V} &:= \mathrm{P}_{\bullet}V/\mathrm{P}_1 V, & \mathrm{P}'_{\bullet}\bar{V} &:= \mathrm{P}'_{\bullet}V/\mathrm{P}_1 V \cap \mathrm{P}_{\bullet}V, \\ \bar{V}' &:= V/\mathrm{P}'_1 V, & \mathrm{P}'_{\bullet}\bar{V}' &:= \mathrm{P}'_{\bullet}V/\mathrm{P}_{\bullet}V \cap \mathrm{P}_1 V, & \mathrm{P}'_{\bullet}\bar{V}' &:= \mathrm{P}'_{\bullet}V/\mathrm{P}'_1 V \end{aligned}$$

for the quotients of V by the two kernels of β and for the quotient filtrations induced by $\mathrm{P}_{\bullet}V$ and $\mathrm{P}'_{\bullet}V$. Scalar extensions of these spaces are denoted, as usual, via subscripts. Write $\mathbf{Aut}_{(V,\mathrm{P}_{\bullet}V,\mathrm{P}'_{\bullet}V)}$ for the algebraic group of linear automorphisms of V that preserve the two filtrations $\mathrm{P}_{\bullet}V$ and $\mathrm{P}'_{\bullet}V$, and analogously with V replaced by $\bar{V}$ and $\bar{V}'$. Then the pairing induced by β on the quotients satisfies:

5.10. Lemma. — *The perfect pairing $\bar{\beta} \colon \bar{V}'^{[1]} \otimes_{\mathbf{k}} \bar{V} \to \mathbf{k}$ induces an isomorphism*

$$\bar{\beta}_* \colon (\mathbf{Aut}_{(\bar{V}',\mathrm{P}_{\bullet}\bar{V}',\mathrm{P}'_{\bullet}\bar{V}')})^{[1]} \to \mathbf{Aut}_{(\bar{V},\mathrm{P}_{\bullet}\bar{V},\mathrm{P}'_{\bullet}\bar{V})}$$

of linear algebraic groups over $\mathbf{k}$ *whose value on an* S*-point* $\varphi \colon \bar{V}_S'^{[1]} \to \bar{V}_S'^{[1]}$ *is the composite*

$$\bar{\beta}_S^{-1} \circ \varphi^{\vee,-1} \circ \bar{\beta}_S \colon \bar{V}_S \to \bar{V}_S'^{[1],\vee} \to \bar{V}_S'^{[1],\vee} \to \bar{V}_S.$$

Proof. It remains to observe that $\bar{\beta}_{*,S}(\varphi)$ defined as above preserves the two filtrations on $\bar{V}_S$. This follows from the seesaw relation between the filtrations: 4.6 implies that $\bar{\beta}$ defines an isomorphism

$$\bar{\beta} \colon \mathrm{P}_{i+1}\bar{V} \cap \mathrm{P}_{j-1}\bar{V} \xrightarrow{\sim} \big(\bar{V}'/(\mathrm{P}_i\bar{V}' + \mathrm{P}_j\bar{V}')\big)^{[1],\vee}$$

for all $i, j \geq 0$, compatible with scalar extension. Then since φ preserves the two filtrations on $\bar{V}_S'^{[1]}$, its transpose inverse induces an isomorphism on the right-hand quotient. Transporting this via $\bar{\beta}^{-1}$ shows that $\bar{\beta}_{*,S}(\varphi)$ preserves the two filtrations on $\bar{V}_S$. ■

Let $\mathrm{Art}_{\mathbf{k}}$ be the category of Artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$. Let a *q-small extension* denote a surjection $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ whose kernel is annihilated by the q-power Frobenius. The following construction produces unique lifts, up to Frobenius, along q-small extensions:

5.11. Lemma. — *Let $B \to A$ be a q-small extension. For any $\mathbf{k}$-scheme X such that $X(B) \to X(A)$ is surjective, there exists a canonical map $\phi \colon X(A) \to X^{[1]}(B)$ fitting into a commutative diagram*

$$\begin{tikzcd} X(B) \arrow[r, "\mathrm{Fr}_{X/\mathbf{k}}"] \arrow[d] & X^{[1]}(B) \arrow[d] \\ X(A) \arrow[r, "\mathrm{Fr}_{X/\mathbf{k}}"'] \arrow[ur, "\phi"] & X^{[1]}(A). \end{tikzcd}$$

Proof. Given $x \in X(A)$, let $\phi(x) := y \circ \mathrm{Fr}_{X/\mathbf{k}}$ for any lift $y \in X(B)$. This is well-defined because, if y' is another lift, then the difference $y^{\#} - y'^{\#}$ of the induced maps on structure sheaves factors through $\ker(B \to A)$ and is therefore annihilated by Frobenius. The top triangle now commutes by construction, and the bottom triangle because Frobenius commutes with ring homomorphisms. ■

The following is the main result of this Section:

5.12. Theorem. — *Let (V, β) be a q-bic form over a field $\mathbf{k}$ that admits a descent $\mathrm{P}'_{\bullet}V$ of its $\perp^{[\bullet]}$-filtration to V over $\mathbf{k}$. Then the algebraic group $\mathbf{Aut}_{(V,\beta,\mathrm{P}'_{\bullet}V)}$ is reduced and smooth over $\mathbf{k}$.*

Proof. It suffices to verify the infinitesimal lifting criterion, as in [Stacks, 02HX], at the identity. Explicitly, consider the functor of infinitesimal deformations of the identity automorphism:

$$G \colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Grps} \qquad A \mapsto \left\{ g \in \mathrm{Aut}(V_A, \beta_A, \mathrm{P}'_{\bullet}V_A) \;\middle|\; g \otimes_A A/\mathfrak{m}_A = \mathrm{id}_V \right\}.$$

Then it is enough to show that $G(B) \to G(A)$ is surjective for every q-small extension $B \to A$. Fix such an extension and fix an element $g \in G(A)$. A lift $h \in G(B)$ will be constructed in several steps. The notation of 5.9 will be used throughout the proof.

Step 1. View g as an element of the group $\mathrm{Aut}(V_A, \mathrm{P}_{\bullet}V_A, \mathrm{P}'_{\bullet}V_A)$. In particular, g preserves the submodules P_1V_A and P'_1V_A, so it induces automorphisms on the quotients:

$$\bar{g} \in \mathrm{Aut}(\bar{V}_A, \mathrm{P}_{\bullet}\bar{V}_A, \mathrm{P}'_{\bullet}\bar{V}_A) \quad \text{and} \quad \bar{g}' \in \mathrm{Aut}(\bar{V}'_A, \mathrm{P}_{\bullet}\bar{V}'_A, \mathrm{P}'_{\bullet}\bar{V}'_A).$$

Apply 5.11 to the smooth algebraic group $X = \mathbf{Aut}_{(\bar{V}',\mathrm{P}_{\bullet}\bar{V}',\mathrm{P}'_{\bullet}\bar{V}')}$ of bifiltered linear automorphisms of $\bar{V}'$ to obtain a homomorphism

$$\phi \colon \mathrm{Aut}(\bar{V}'_A, \mathrm{P}_{\bullet}\bar{V}'_A, \mathrm{P}'_{\bullet}\bar{V}'_A) \to \mathrm{Aut}(\bar{V}_B'^{[1]}, \mathrm{P}_{\bullet}\bar{V}_B'^{[1]}, \mathrm{P}'_{\bullet}\bar{V}_B'^{[1]})$$

factoring the relative Frobenius homomorphism of X on B-points. Identify the target of ϕ with the group of bifiltered automorphisms of $\bar{V}_B$ via 5.10. This yields an element

$$\bar{h} := \bar{\beta}_{B,*}(\phi(\bar{g}')) \in \mathrm{Aut}(\bar{V}_B, \mathrm{P}_{\bullet}\bar{V}_B, \mathrm{P}'_{\bullet}\bar{V}_B).$$

By commutativity of the diagram in 5.11, the definition of the isomorphism $\bar{\beta}_*$ from 5.10, and the fact that g preserves β_A, it follows that $\bar{h}$ reduces along $B \to A$ to

$$\bar{\beta}_{A,*}(\bar{g}'^{[1]}) = \bar{g} \in \mathrm{Aut}(\bar{V}_A, \mathrm{P}_{\bullet}\bar{V}_A, \mathrm{P}'\bar{V}_A).$$

Step 2. The quotient map $V \to \bar{V}$ induces a surjection of linear algebraic groups

$$\mathbf{Aut}_{(V,\mathrm{P}_\bullet V,\mathrm{P}'_\bullet V)} \to \mathbf{Aut}_{(\bar{V},\mathrm{P}_\bullet \bar{V},\mathrm{P}'_\bullet \bar{V})}$$

whose kernel is the group of linear maps $V \to \mathrm{P}_1 V$ which preserve the filtrations induced by $\mathrm{P}'_\bullet V$, and restricts to an isomorphism on $\mathrm{P}_1 V \subseteq V$. The surjection is therefore smooth and, in terms of deformation theory, this means that the canonically induced map of sets

$$\mathrm{Aut}(V_B,\mathrm{P}_\bullet V_B,\mathrm{P}'_\bullet V_B) \to \mathrm{Aut}(V_A,\mathrm{P}_\bullet V_A,\mathrm{P}'_\bullet V_A) \times_{\mathrm{Aut}(\bar{V}_A,\mathrm{P}_\bullet \bar{V}_A,\mathrm{P}'_\bullet \bar{V}_A)} \mathrm{Aut}(\bar{V}_B,\mathrm{P}_\bullet \bar{V}_B,\mathrm{P}'_\bullet \bar{V}_B)$$

is surjective. See, for example, [Stacks, 06HJ]. Step 1 produced an element $(g \mapsto \bar{g} \leftarrow\!\shortmid \bar{h})$ of the fibre product on the right; let $h \in \mathrm{Aut}(V_B,\mathrm{P}_\bullet V_B,\mathrm{P}'_\bullet V_B)$ be any lift along this surjection.

Step 3. It remains to see that this element h preserves the q-bic form β_B. Since h is a lift of g, and since reduction maps commute with quotient maps, the image $\bar{h}'$ of h in $\mathrm{Aut}(\bar{V}'_B,\mathrm{P}_\bullet \bar{V}'_B,\mathrm{P}'_\bullet \bar{V}'_B)$ is a lift of the element $\bar{g}'$ from Step 1. Therefore the diagram of **5.11** gives

$$\bar{h}'^{[1]} = \phi(\bar{g}') \in \mathrm{Aut}(\bar{V}_B'^{[1]},\mathrm{P}_\bullet \bar{V}_B'^{[1]},\mathrm{P}'_\bullet \bar{V}_B'^{[1]}).$$

By the definitions of the perfect pairing $\bar{\beta}_B$, of h, and of the isomorphism $\bar{\beta}_{B,*}$ from **5.10**, it follows that, for any $u \in V_B^{[1]}$ and $v \in V_B$,

$$\beta_B(h^{[1]}\cdot u, h\cdot v) = \bar{\beta}_B(\bar{h}'^{[1]}\cdot \bar{u}, \bar{h}\cdot \bar{v}) = \bar{\beta}_B(\phi(\bar{g}')^{-1}\cdot\phi(\bar{g}')\cdot\bar{u},\bar{v}) = \bar{\beta}_B(\bar{u},\bar{v}) = \beta_B(u,v),$$

where $\bar{u} \in \bar{V}_B'^{[1]}$ and $\bar{v} \in \bar{V}_B$ are the images of u and v, respectively. Therefore h preserves β_B. Together with its properties from Step 2, this shows that h lies in $G(B)$ and is a lift of $g \in G(A)$. ■

As a first consequence, this gives a modular interpretation to the reduced subscheme of $\mathbf{Aut}_{(V,\beta)}$:

5.13. Corollary. — *If the $\perp^{[\cdot]}$-filtration descends to V over $\mathbf{k}$, then $\mathbf{Aut}_{(V,\beta),\mathrm{red}} = \mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)}$.*

Proof. That "$\subseteq$" is **5.4**. That "$\supseteq$" follows from **5.12**: since $\mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)}$ is reduced, its closed immersion into $\mathbf{Aut}_{(V,\beta)}$ factors through the reduced subscheme. ■

Second, the following shows that the exponent of nilpotent members in the automorphism group scheme is determined by the number of Frobenius twists required before the $\perp^{[\cdot]}$-filtration is canonically defined; compare with **3.11**:

5.14. Corollary. — *Let ν be such that every piece of the $\perp^{[\cdot]}$-filtration of (V,β) can be defined on $V^{[\nu]}$ in terms of $\beta^{[\nu]}$. Then the quotient of $\mathbf{Aut}_{(V,\beta)}$ by its q^ν-power Frobenius kernel is reduced.*

Proof. Comparing **5.2** and **5.3**, the hypothesis implies that the q^ν-power Frobenius of $\mathbf{Aut}_{(V,\beta)}$ factors through $\mathbf{Aut}_{(V^{[\nu]},\beta^{[\nu]},\mathrm{P}'_\bullet V^{[\nu]})}$. Since the latter is reduced by **5.12**, so is the image of the former. ■

Finally, the main application is to compute dimensions of automorphism groups of q-bic forms in terms of the numerical invariants defined in **3.9**:

5.15. Theorem. — *Let (V,β) be a q-bic form over a field $\mathbf{k}$ of type $(a;b_m)_{m\geq 1}$. Then*

$$\dim \mathbf{Aut}_{(V,\beta)} = \sum\nolimits_{k\geq 1}\Big[k(b_{2k-1}^2 + b_{2k}^2) + \Big(a + \sum\nolimits_{m\geq 2k} m b_m\Big)b_{2k-1} + 2k\Big(\sum\nolimits_{m\geq 2k+1} b_m\Big)b_{2k}\Big].$$

Proof. If β is nondegenerate, then $\dim \mathbf{Aut}_{(V,\beta)} = 0$ follows from the Lie algebra computation **5.5**. Otherwise, replace (V,β) by a sufficiently large Frobenius twist to assume that the $\perp^{[\cdot]}$-filtration descends over $\mathbf{k}$; this is possible by **3.12**, and allowable since the automorphism group scheme of the twist of (V,β) is the twist of the automorphism group scheme (V,β) and are therefore of the same dimension. Since dimension is insensitive to nilpotents, **5.13** gives

$$\dim \mathbf{Aut}_{(V,\beta)} = \dim \mathbf{Aut}_{(V,\beta),\mathrm{red}} = \dim \mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)}.$$

Since $\mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)}$ is smooth by 5.12, its dimension is that of its Lie algebra. With a choice of identification between V and its associated graded for the filtration $\mathrm{P}'_\bullet V$, 5.5 gives

$$\begin{aligned}\operatorname{Lie}\mathbf{Aut}_{(V,\beta,\mathrm{P}'_\bullet V)} \cong &\Big(\bigoplus\nolimits_{\ell\geq 1}\operatorname{Hom}_{\mathbf{k}}(\mathrm{P}'_{2\ell-1}V/\mathrm{P}'_{2\ell-3}V, \mathrm{P}_1V\cap \mathrm{P}'_{2\ell-1}V)\Big)\\ &\oplus\Big(\operatorname{Hom}_{\mathbf{k}}(\mathrm{P}'_+V/\mathrm{P}'_-V, \mathrm{P}_1V\cap\mathrm{P}'_+V)\Big)\oplus\Big(\bigoplus\nolimits_{\ell\geq 1}\operatorname{Hom}_{\mathbf{k}}(\mathrm{P}'_{2\ell-2}V/\mathrm{P}'_{2\ell}V,\mathrm{P}_1V\cap\mathrm{P}'_{2\ell-2}V)\Big)\end{aligned}$$

where P'_-V and P'_+V are, analogous to their unprimed counterparts from 3.1, the limits of the increasing odd and decreasing even pieces of $\mathrm{P}'_\bullet V$, respectively. Express the dimension of each parenthesized term in terms of a and the b_i using the formulae from 3.7 and 3.9, and the symmetry relation 3.6: The easiest is the central summand, with dimension

$$a\dim_{\mathbf{k}}\mathrm{P}_1V\cap\mathrm{P}'_+V = a\sum\nolimits_{k\geq 1} b_{2k-1}.$$

Next, the first parenthesized term has dimension

$$\begin{aligned}\sum\nolimits_{\ell\geq 1} a_{2\ell-1}\dim_{\mathbf{k}}\mathrm{P}_1V\cap\mathrm{P}'_{2\ell-1}V &= \sum\nolimits_{\ell\geq 1}\Big(\sum\nolimits_{m\geq 2\ell-1} b_m\Big)\Big(\sum\nolimits_{k=1}^{\ell} b_{2k-1}\Big)\\ &=\sum\nolimits_{k\geq 1}\Big(\sum\nolimits_{\ell\geq k}\sum\nolimits_{m\geq 2\ell-1} b_m\Big)b_{2k-1} = \sum\nolimits_{k\geq 1}\Big(\sum\nolimits_{\ell\geq k}(\ell-k+1)(b_{2\ell-1}+b_{2\ell})\Big)b_{2k-1}.\end{aligned}$$

Finally, the third parenthesized term has dimension

$$\begin{aligned}\sum\nolimits_{\ell\geq 1} a_{2\ell}\dim_{\mathbf{k}}\mathrm{P}_1V\cap\mathrm{P}'_{2\ell-2}V &= \sum\nolimits_{\ell\geq 1}\Big(\sum\nolimits_{m\geq 2\ell} b_m\Big)\Big(\sum\nolimits_{k=1}^{\ell} b_{2k-1}+\sum\nolimits_{m\geq 2\ell} b_m\Big)\\ &=\sum\nolimits_{k\geq 1}\Big(\sum\nolimits_{\ell\geq k}(\ell-k+1)(b_{2\ell}+b_{2\ell+1})\Big)b_{2k-1}+\sum\nolimits_{\ell\geq 1}\Big(\sum\nolimits_{m\geq 2\ell} b_m\Big)^2.\end{aligned}$$

The final term here may be written as

$$\sum\nolimits_{k\geq 1} k\Big(\Big(b_{2k}+2\sum\nolimits_{m\geq 2k+1} b_m\Big)b_{2k}+\Big(b_{2k+1}+2\sum\nolimits_{m\geq 2k+2} b_m\Big)b_{2k+1}\Big).$$

Adding the expressions together gives the claimed formula. ■

It is interesting and useful to disentangle this formula to see how dimensions of automorphism groups grow under sums of q-bic forms:

5.16. Corollary. — *Let (V,β) and (W,γ) be q-bic forms over $\mathbf{k}$. Then*

$$\dim\mathbf{Aut}_{(V\oplus W,\beta\oplus\gamma)} = \dim\mathbf{Aut}_{(V,\beta)}+\dim\mathbf{Aut}_{(W,\gamma)}+\Big(\sum\nolimits_{k\geq 1} b_{2k-1}\Big)c+\sum\nolimits_{m\geq 1}\Phi_m(\beta)d_m$$

where β and γ are of types $(a;b_m)_{m\geq 1}$ and $(c;d_m)_{m\geq 1}$, respectively, and

$$\Phi_m(\beta) := \begin{cases}\dim_{\mathbf{k}} V + b_{2k-1}+2\sum_{\ell=1}^{k-1}(k-\ell)b_{2\ell-1} & \text{if } m = 2k-1, \text{ and}\\ \sum_{\ell=1}^{k-1}2\ell b_{2\ell}+2k\Big(\sum_{\ell\geq 1}b_{2\ell-1}+\sum_{\ell\geq k}b_{2\ell}\Big) & \text{if } m=2k.\end{cases}$$

Proof. This follows directly from 5.15 together with the formulae of 3.9. ■

6. Moduli

The parameter space for q-bic forms on a fixed n-dimensional vector space V over the field $\mathbf{k}$ is given by the n^2-dimensional affine space

$$q\text{-}\mathbf{bics}_V := \mathbf{A}(V^{[1]}\otimes_{\mathbf{k}} V)^\vee := \operatorname{Spec}\operatorname{Sym}^*(V^{[1]}\otimes_{\mathbf{k}}V).$$

Multiplication in the symmetric algebra induces the universal q-bic form

$$\beta_{\mathrm{univ}}\colon V^{[1]}\otimes_{\mathbf{k}}V\otimes_{\mathbf{k}}\mathcal{O}_{q\text{-}\mathbf{bics}_V}\to\mathcal{O}_{q\text{-}\mathbf{bics}_V}.$$

In particular, q-$\mathbf{bics}_V$ represents the functor $\mathrm{Sch}_{\mathbf{k}}^{\mathrm{opp}}\to\mathrm{Set}$ that sends

$$X\mapsto\big\{\,\beta\colon V^{[1]}\otimes_{\mathbf{k}}V\otimes_{\mathbf{k}}\mathcal{O}_X\to\mathcal{O}_X \text{ a } q\text{-bic form over } \mathcal{O}_X\,\big\}$$

a $\mathbf{k}$-scheme X to the set of q-bic forms on V over $\mathcal{O}_X$. The linear action of $\mathbf{GL}_V$ on $V^{[1]} \otimes_{\mathbf{k}} V$ induces a schematic action of the algebraic group $\mathbf{GL}_V$ on q-$\mathbf{bics}_V$. By the Classification Theorem 4.1, the orbits of this action consist of the finitely many locally closed subschemes

$$q\text{-}\mathbf{bics}_{V,\mathbf{b}} := \{\, [\beta] \in q\text{-}\mathbf{bics}_V \mid \mathrm{type}(\beta) = \mathbf{b} \,\}$$

parameterizing q-bic forms with a given type $\mathbf{b} = (a; b_m)_{m\geq 1}$, where $a + \sum_{m\geq 1} m b_m = n$ as in 3.9. Together, these form the *type stratification* of the affine space q-$\mathbf{bics}_V$. This refines the natural stratification by corank.

Let $\mathbf{Aut}_{(V,\mathbf{b})}$ denote the automorphism group scheme of the standard form of type $\mathbf{b}$, as in 1.3; note that the automorphism group scheme of any other form of type $\mathbf{b}$ over $\mathbf{k}$ will be a form of $\mathbf{Aut}_{(V,\mathbf{b})}$ over $\mathbf{k}$. Since the type strata are orbits under an algebraic group, they enjoy the following standard properties, see, for example, [Mil17, Propositions 1.65, 1.66, and 7.12]:

6.1. Lemma. — *Each stratum q-$\mathbf{bics}_{V,\mathbf{b}}$ is a smooth, irreducible, locally closed subscheme of q-$\mathbf{bics}_V$ of codimension $\dim \mathbf{Aut}_{(V,\mathbf{b})}$, and its closure is a union of type stratum of strictly smaller dimension.* ∎

This induces a partial ordering amongst types of q-bic forms on V: write $\mathbf{b} \geq \mathbf{b}'$ if and only if the closure of q-$\mathbf{bics}_{V,\mathbf{b}}$ contains q-$\mathbf{bics}_{V,\mathbf{b}'}$. Simple observations: First, the maximal and minimal types are those corresponding to $\mathbf{1}^{\oplus n}$ and $\mathbf{N}_1^{\oplus n}$, respectively. Second, by comparing dimensions with 5.15, it follows that the maximal type in the codimension c^2 locus of corank c forms is

$$\begin{cases} \mathbf{1}^{\oplus n-2c} \oplus \mathbf{N}_2^{\oplus c} & \text{if } 0 \leq c \leq n/2, \text{ and} \\ \mathbf{N}_1^{\oplus 2c-n} \oplus \mathbf{N}_2^{\oplus n-c} & \text{if } n/2 \leq c \leq n. \end{cases}$$

Third, by dividing out radicals, it follows that the subposet of types containing $\mathbf{N}_1$ is isomorphic to the poset of q-bic types on a vector space of dimension 1 less.

The goal of this Section is to characterize this partial ordering. As a first step, rephrase this in terms of specialization relations for q-bic forms: $\mathbf{b} \geq \mathbf{b}'$ if and only if for some—equivalently, for any—pair of q-bic forms β and β' on V of types $\mathbf{b}$ and $\mathbf{b}'$, respectively, there exists a discrete valuation ring R over $\mathbf{k}$, a q-bic form (M, γ) over R, and isomorphisms

$$(V_K, \beta_K) \cong (M_K, \gamma_K) \quad \text{and} \quad (V_\kappa, \beta'_\kappa) \cong (M_\kappa, \gamma_\kappa)$$

as q-bic forms over the fraction field K and residue field κ of R, respectively. Denote this situation by $\beta \rightsquigarrow \beta'$ and say that β *specializes to* β'.

The remainder of this Section will be phrased in terms of specialization relations amongst q-bic forms on V, and the goal is to determine necessary and sufficient conditions for a specialization $\beta \rightsquigarrow \beta'$ to exist. A sequence of necessary conditions is obtained by combining the summation formula 5.16 with the fact that boundary strata in the closure have smaller dimension:

6.2. Proposition. — *If there exists a specialization $\beta \rightsquigarrow \beta'$, then $\Phi_m(\beta) \leq \Phi_m(\beta')$ for all $m \geq 1$.*

Proof. By specializing subforms, the given specialization $\beta \rightsquigarrow \beta'$ induces specializations $\beta \oplus \gamma \rightsquigarrow \beta' \oplus \gamma$ for all q-bic forms (W, γ). Thus 6.1 gives the inequalities

$$\dim \mathbf{Aut}_{(V\oplus W, \beta\oplus\gamma)} \leq \dim \mathbf{Aut}_{(V\oplus W, \beta'\oplus\gamma)}.$$

According to 5.16, the dimensions of the automorphism groups grow linearly in the invariants of γ, so comparing coefficients of d_m gives the result. ∎

The following constructs a collection of basic specializations amongst standard q-bic forms:

6.3. Lemma. — *Let $s \geq t \geq 1$ be integers. There exists specializations of q-bic forms:*

$$\mathbf{N}_{2s+1} \rightsquigarrow \mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-1}, \qquad \mathbf{N}_{2s} \rightsquigarrow \mathbf{1} \oplus \mathbf{N}_{2s-1}, \qquad \mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-2} \rightsquigarrow \mathbf{N}_{2s},$$
$$\mathbf{N}_{2s-2t} \oplus \mathbf{N}_{2s+2} \rightsquigarrow \mathbf{N}_{2s-2t+2} \oplus \mathbf{N}_{2s}, \qquad \mathbf{N}_{2s+1} \oplus \mathbf{N}_{2s+2t-1} \rightsquigarrow \mathbf{N}_{2s-1} \oplus \mathbf{N}_{2s+2t+1}.$$

Proof. Let R be a discrete valuation ring over $\mathbf{k}$ with uniformizing parameter π, residue field $\mathbf{k}$, and fraction field K. Let $M := \bigoplus_{i=1}^n R \cdot e_i$ be a free R-module of rank n, and let γ be a q-bic form on M determined by one of the following Gram matrices:

$$\left[\begin{array}{c|cc} \mathbf{N}_{2s-1} & \begin{matrix} 0 \\ \vdots \\ \pi \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right], \quad \left[\begin{array}{c|cc} \mathbf{N}_{2s-2} & \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 \\ \pi \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right], \quad \left[\begin{array}{c|c} \mathbf{N}_{2s-1} & \begin{matrix} 0 \\ \vdots \\ \pi \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right], \quad \left[\begin{array}{c|c|cc} \mathbf{N}_{2s} & & \begin{matrix} 0 \\ \vdots \\ \pi \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & \mathbf{N}_{2s-2t} & \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right], \quad \left[\begin{array}{c|c|cc} \mathbf{N}_{2s-1} & & \begin{matrix} 0 \\ \vdots \\ \pi \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & \mathbf{N}_{2s+2t-1} & \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right].$$

Then the reduction of γ modulo π yields a q-bic form of type

$$\mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-1}, \quad \mathbf{N}_{2s}, \quad \mathbf{1} \oplus \mathbf{N}_{2s-1}, \quad \mathbf{N}_{2s-2t+2} \oplus \mathbf{N}_{2s}, \quad \mathbf{N}_{2s-1} \oplus \mathbf{N}_{2s+2t+1},$$

respectively. To conclude, by 4.1, it suffices to compute the $\perp$-filtration of $\beta := \gamma \otimes_R K$ on $V := M \otimes_R K$; do this by noting that, for each $k \geq 1$,

$$\mathrm{P}_{2k-1}V = \beta^{-1}\big((V/\mathrm{P}_{2k-2}V)^{[1],\vee}\big) \quad \text{and} \quad (V/\mathrm{P}_{2k}V)^\vee = \mathrm{im}\big(\beta^\vee : \mathrm{P}_{2k-1}V^{[1]} \to V^\vee\big).$$

Consider the first three Gram matrices. Their shape shows that $\mathrm{P}_1 V = \langle e_1 \rangle$, and that the maps $\beta : V \to V^{[1],\vee}$ and $\beta^\vee : V^{[1]} \to V^\vee$ satisfy

$$\beta : e_i \mapsto e_{i-1}^{[1],\vee} \quad \text{and} \quad \beta^\vee : e_i^{[1]} \mapsto e_{i+1}^\vee \quad \text{for each } 1 \leq i \leq 2s-2,$$

where $e_0 := 0$. Therefore $\mathrm{P}_{2k-1}V = \bigoplus_{\ell=1}^k K \cdot e_{2\ell-1}$ and $V/\mathrm{P}_{2k}V \cong \bigoplus_{\ell=1}^k K \cdot e_{2\ell}$ for each $1 \leq k \leq s-1$. Now consider each case in turn:

The first Gram matrix satisfies $\beta : e_{2s-1} \mapsto e_{2s-2}^{[1],\vee}$, $\beta^\vee : e_{2s-1}^{[1]} \mapsto \pi e_{2s}^\vee$, and $\beta : e_{2s+1} \mapsto e_{2s}^{[1],\vee}$. Since π is invertible in K, these respectively imply

$$\mathrm{P}_{2s-1}V = \bigoplus\nolimits_{k=1}^s K \cdot e_{2k-1}, \qquad V/\mathrm{P}_{2s}V \cong \bigoplus\nolimits_{k=1}^s K \cdot e_{2k}, \qquad \mathrm{P}_{2s+1}V = \bigoplus\nolimits_{k=1}^{s+1} K \cdot e_{2k-1},$$

and so β is of type $\mathbf{N}_{2s+1}$. A similar computation applies to the second Gram matrix to show that, in that case, β is of type $\mathbf{N}_{2s}$.

The third Gram matrix satisfies $\beta : e_{2s-1} \mapsto e_{2s-2}^{[1],\vee} + \pi e_{2s}^{[1],\vee}$ so neither $e_{2s-2}^{[1],\vee}$ nor $e_{2s}^{[1],\vee}$ lies in the image of β. Therefore the $\perp$-filtration has length $2s-2$ and

$$\mathrm{P}_+V/\mathrm{P}\_V = \mathrm{P}_{2s-2}V/\mathrm{P}_{2s-3}V \cong K \cdot e_{2s-1} \oplus K \cdot e_{2s}$$

meaning that β is of type $\mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-2}$.

Completely analogous computations show that for the fourth and fifth Gram matrices, β is of type $\mathbf{N}_{2s-2t} \oplus \mathbf{N}_{2s+2}$ and $\mathbf{N}_{2s+1} \oplus \mathbf{N}_{2s+2t-1}$, respectively. ■

For each $m \geq 1$ and q-bic form β of type $(a; b_m)_{m \geq 1}$, write $\Theta_m(\beta) := \sum_{k=1}^m b_{2k-1}$. The following gives a partial converse to 6.2. The result is almost enough to completely determine the closure relations in q-$\mathbf{bics}_V$ up to dimension 6: see [Che22, 3.1.2, 3.4.3, and 3.8.2] for dimensions ≤ 4, and Figures 1 and 2 for dimensions 5 and 6, respectively, where $\mathbf{0}$ is written in place of $\mathbf{N}_1$ for emphasis. See 6.8 for additional comments.

6.4. Proposition. — *Let β and β' be q-bic forms of type $(a; b_m)_{m \geq 1}$ and $(a'; b'_m)_{m \geq 1}$, respectively. If $\Phi_m(\beta) \leq \Phi_m(\beta')$ and $\Theta_m(\beta) \leq \Theta_m(\beta')$ for all $m \geq 1$, then there exists a specialization $\beta \rightsquigarrow \beta'$.*

$$
\begin{tikzcd}
\mathbf{1}^5 \arrow[r, squiggly] & \mathbf{1}^3\mathbf{N}_2 \arrow[r, squiggly] & \mathbf{1}\mathbf{N}_4 \arrow[r, squiggly] \arrow[dr, squiggly] & \mathbf{N}_5 \arrow[r, squiggly] & \mathbf{1}^2\mathbf{N}_3 \arrow[r, squiggly] \arrow[dr, squiggly] & \mathbf{0}\mathbf{1}^4 \arrow[dr, squiggly] & \\
 & & & \mathbf{1}\mathbf{N}_2^2 \arrow[rr, squiggly] & & \mathbf{N}_2\mathbf{N}_3 \arrow[r, squiggly] & \mathbf{0}\mathbf{1}^2\mathbf{N}_2
\end{tikzcd}
$$

$\mathbf{b}$	$\mathbf{1}^5$	$\mathbf{1}^3\mathbf{N}_2$	$\mathbf{1}\mathbf{N}_4$	$\mathbf{N}_5$	$\mathbf{1}^2\mathbf{N}_3$	$\mathbf{1}\mathbf{N}_2^2$	$\mathbf{0}\mathbf{1}^4$	$\mathbf{N}_2\mathbf{N}_3$	$\mathbf{0}\mathbf{1}^2\mathbf{N}_2$
$\dim q\text{-}\mathbf{bics}_{V,\mathbf{b}}$	25	24	23	22	21	21	20	20	19

FIGURE 1. Immediate specialization relations amongst 5-dimensional q-bic forms and dimensions of the corresponding strata, up to the first few with nontrivial radical.

$$
\begin{tikzcd}
\mathbf{1}^6 \arrow[r, squiggly] & \mathbf{1}^4\mathbf{N}_2 \arrow[r, squiggly] & \mathbf{1}^2\mathbf{N}_4 \arrow[r, squiggly] \arrow[dr, squiggly] & \mathbf{N}_6 \arrow[r, squiggly] \arrow[dr, squiggly] & \mathbf{1}\mathbf{N}_5 \arrow[r, squiggly] & \mathbf{1}^3\mathbf{N}_3 \arrow[r, squiggly] \arrow[dr, squiggly] & \mathbf{0}\mathbf{1}^5 \arrow[dr, squiggly] & & \\
 & & & \mathbf{1}^2\mathbf{N}_2^2 \arrow[r, squiggly] & \mathbf{N}_2\mathbf{N}_4 \arrow[rr, squiggly] & & \mathbf{1}\mathbf{N}_2\mathbf{N}_3 \arrow[r, squiggly] \arrow[dr, squiggly] \arrow[dd, squiggly] & \mathbf{0}\mathbf{1}^3\mathbf{N}_2 & \\
 & & & & & & & \mathbf{N}_3^2 \arrow[r, squiggly] & \mathbf{0}\mathbf{N}_5 \\
 & & & & & & \mathbf{N}_2^3 \arrow[rr, squiggly] & & \mathbf{0}\mathbf{1}\mathbf{N}_2^2
\end{tikzcd}
$$

FIGURE 2. Immediate specialization relations amongst for 6-dimensional q-bic forms, up to the first few with nontrivial radical.

Proof. For the arguments below, it is convenient to replace Φ_m from 5.16 by

$$
\Psi_m(\beta) := \begin{cases} b_{2k-1} + 2\sum_{\ell=1}^{k-1}(k-\ell)b_{2\ell-1} & \text{if } m = 2k-1, \text{ and} \\ \sum_{\ell=1}^{k-1} \ell\, b_{2\ell} + k\Big(\sum_{\ell\geq 1} b_{2\ell-1} + \sum_{\ell \geq k} b_{2\ell}\Big) & \text{if } m = 2k, \end{cases}
$$

in which redundant constants are removed, so that $\Phi_m(\beta) \leq \Phi_m(\beta')$ if and only if $\Psi_m(\beta) \leq \Psi_m(\beta')$.

The argument proceeds in steps. Each time, 6.3 is applied to a subform of β to produce an intermediate specialization β''. The new form will satisfy $\Psi_m(\beta'') \leq \Psi_m(\beta')$ for all m, together with successively stronger conditions that, in particular, imply $\Theta_m(\beta'') \leq \Theta_m(\beta')$ for all m. Replace β by β'' and continue in this manner until $\Psi_m(\beta) = \Psi_m(\beta')$ for all m. This implies that β and β' are of the same type, at which point the Proposition follows from the Classification Theorem 4.1.

In what follows, abbreviate $\Psi_m(\beta)$, $\Psi_m(\beta')$, and $\Psi_m(\beta'')$ to Ψ_m, Ψ'_m, and Ψ''_m; similarly for Θ_m, etc.

Step 1. *After a sequence of specializations of the form*

$$
\mathbf{N}_{2s+1} \rightsquigarrow \mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-1},
$$

may assume that $b_{2k-1} \leq b'_{2k-1}$ *for every* $k \geq 1$.

The inequalities $\Psi_{2k-1} \leq \Psi'_{2k-1}$ imply that b_{2k-1} must be smaller than b'_{2k-1} the first time they differ. This implies: if s is minimal such that $b_{2s+1} > b'_{2s+1}$, then $\Psi_{2s-1} < \Psi_{2s-1}$ and $\Theta_{2s-1} < \Theta'_{2s-1}$. The former inductively implies that $\Psi_{2k-1} < \Psi'_{2k-1}$ for every $k \geq s$ since

$$
\Psi_{2k+1} - \Psi_{2k-1} = \Theta_{k+1} + \Theta_{k-1} \leq \Theta'_{k+1} + \Theta'_{k-1} = \Psi'_{2k+1} - \Psi'_{2k-1}
$$

upon using the inequalities $\Theta_m \leq \Theta'_m$ for $m = k-1$ and $m = k+1$. So if β'' is obtained from β via a specialization of the form $\mathbf{N}_{2s+1} \rightsquigarrow \mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-1}$, then

$$
\Psi''_m = \begin{cases} \Psi_{2k-1} + 1 & \text{if } m = 2k-1 \geq 2s-1, \\ \Psi_m & \text{otherwise,} \end{cases} \quad \text{and} \quad \Theta''_m = \begin{cases} \Theta_{2s-1} + 1 & \text{if } m = 2s-1, \\ \Theta_m & \text{otherwise,} \end{cases}
$$

and so with the inequalities above, this implies $\Psi''_m \leq \Psi'_m$ and $\Theta''_m \leq \Theta'_m$ for all m.

Step 2. *After a sequence of specializations of the form*

$$\mathbf{N}_{2s} \rightsquigarrow \mathbf{1}^{\oplus 2s-2t+1} \oplus \mathbf{N}_{2t-1} \quad \textit{or} \quad \mathbf{1}^{\oplus 2t-2s-1} \oplus \mathbf{N}_{2s} \rightsquigarrow \mathbf{N}_{2t-1},$$

may assume that $b_{2k-1} = b'_{2k-1}$ *for every* $k \geq 1$.

From now on, assume that $b_{2k-1} \leq b'_{2k-1}$ for all $k \geq 1$, superseding the inequalities $\Theta_m \leq \Theta'_m$. Note this also implies $\Psi_{2k-1} \leq \Psi'_{2k-1}$ for all k, so it will suffice to verify the inequalities $\Psi_{2k} \leq \Psi'_{2k}$.

To begin this Step, suppose first that the inequalities

$$\sum\nolimits_{k \geq m} b_{2k} \leq \sum\nolimits_{k \geq m} b'_{2k} \tag{6.5}$$

are satisfied for every $m \geq 1$. Comparing the formulae from 3.9 for $\dim_{\mathbf{k}} V$ in terms of the invariants of β and β', and successively applying the inequalities (6.5) for increasing m gives the inequality

$$a = a' + \sum\nolimits_{i \geq 1} i(b_i - b'_i) \geq a' + \sum\nolimits_{k \geq 1} (2k-1)(b_{2k-1} - b'_{2k-1}). \tag{6.6}$$

Let t be any index such that $b_{2t-1} < b'_{2t-1}$. Then (6.6) implies $a \geq 2t-1$, meaning β contains a subform of type $\mathbf{1}^{\oplus 2t-1}$. Let β'' be obtained via a specialization $\mathbf{1}^{\oplus 2t-1} \rightsquigarrow \mathbf{N}_{2t-1}$. The choice of t ensures that $b''_{2k-1} \leq b'_{2k-1}$ for all k. Combined with the $m = 1$ case of (6.5), this implies that

$$\Psi''_2 = \sum\nolimits_{i \geq 1} b''_i \leq \sum\nolimits_{i \geq 1} b'_i = \Psi'_2.$$

Step 1 together with the $m = k+1$ case of (6.5) gives, for each $k \geq 1$, inequalities

$$\Psi''_{2k+2} - \Psi''_{2k} = \sum\nolimits_{\ell \geq 1} b''_{2\ell-1} + \sum\nolimits_{\ell \geq k+1} b''_{2\ell} \leq \sum\nolimits_{\ell \geq 1} b'_{2\ell-1} + \sum\nolimits_{\ell \geq k+1} b'_{2\ell} = \Psi'_{2k+2} - \Psi'_{2k}. \tag{6.7}$$

Starting from $\Psi''_2 \leq \Psi'_2$, this inductively implies that $\Psi''_{2k} \leq \Psi'_{2k}$ for all $k \geq 1$.

It remains to consider the situation when at least one of the inequalities in (6.5) fails. Since $b_{2k} = b'_{2k} = 0$ for large k, there is a maximal s such that

$$\sum\nolimits_{k \geq s} b_{2k} > \sum\nolimits_{k \geq s} b'_{2k}.$$

Comparing this with (6.5) where $m = s+1$ gives $b_{2s} > b'_{2s}$. Now consider two cases:

Case 2A. *There exists* $t \leq s$ *such that* $b_{2t-1} < b'_{2t-1}$.

Let β'' be any specialization of β obtained via $\mathbf{N}_{2s} \rightsquigarrow \mathbf{1}^{\oplus 2s-2t+1} \oplus \mathbf{N}_{2t-1}$. Then $\Psi''_{2k} = \Psi_{2k} \leq \Psi'_{2k}$ for all $k \leq s$. Step 1 together with maximality of s implies that (6.7) holds for all $k \geq s$, which then inductively implies that $\Psi''_{2k} \leq \Psi'_{2k}$ also holds for $k > s$.

Case 2B. *Every* k *for which* $b_{2k-1} < b'_{2k-1}$ *satisfies* $k > s$.

Applying the inequality $\Psi'_{2s} - \Psi_{2s} \geq 0$ to the middle term of (6.6) gives the inequality

$$a \geq a' + \sum\nolimits_{k \geq s+1} (2k-2s)(b'_{2k} - b_{2k}) + \sum\nolimits_{k \geq 1} (2k-2s-1)(b'_{2k-1} - b_{2k-1}).$$

The first sum on the right is nonnegative by (6.5) and maximality of s. The assumptions in this case imply that the second sum is nonnegative, and that if t is any index such that $b_{2t-1} < b'_{2t-1}$, then in fact $a \geq 2t - 2s - 1 > 0$. In other words, β contains $\mathbf{1}^{\oplus 2t-2s-1} \oplus \mathbf{N}_{2s}$ as a subform; let β'' be obtained by specializing such a subform to $\mathbf{N}_{2t-1}$. Arguing as above shows that $\Psi''_{2k} \leq \Psi'_{2k}$ for all k.

Step 3. *After a sequence of specializations of the form*

$$\mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-2} \rightsquigarrow \mathbf{N}_{2s}$$

may assume that $\sum_{k \geq 1} k b_{2k} = \sum_{k \geq 1} k b'_{2k}$.

After Step 2, by passing to subforms, it may be assumed that $b_{2k-1} = b'_{2k-1} = 0$ for all k.

Let $s \geq 1$ be maximal such that $\Psi_{2s-2} = \Psi'_{2s-2}$, setting $\Psi_0 = \Psi'_0 = 0$ so that the maximum always exists. If $s = \max\{\, k \mid b_{2k-2} \neq 0 \text{ or } b'_{2k-2} \neq 0 \,\}$, then this is equivalent to $\sum_{k \geq 1} k b_{2k} = \sum_{k\geq 1} k b'_{2k}$, as required. Otherwise, $\sum_{k\geq 1} k b_{2k} < \sum_{k \geq 1} k b'_{2k}$, so combined with the first equation in (6.6),

$$a = a' + 2\sum\nolimits_{k \geq 1} k(b'_{2k} - b_{2k}) \geq 2.$$

Since $\Psi_{2k} \leq \Psi'_{2k}$ for all k with strict inequality for $k \geq s$, it follows that

$$b_{2s-2} = 2\Psi_{2s-2} - \Psi_{2s-4} - \Psi_{2s} > 2\Psi'_{2s-2} - \Psi'_{2s-4} - \Psi'_{2s} = b'_{2s-2}.$$

Thus, in this case, β contains a subform of type $\mathbf{1}^{\oplus 2} \oplus \mathbf{N}_{2s-2}$; let β'' be obtained by specializing it to a form of type $\mathbf{N}_{2s}$. Then $\Psi''_{2k} = \Psi_{2k}$ for $1 \leq k \leq s-1$ and $\Psi''_{2k} = \Psi_{2k} + 1$ for $k \geq s$, so maximality of s, implies $\Psi''_{2k} \leq \Psi'_{2k}$ for all k.

Step 4. *After a sequence of specializations of the form*

$$\mathbf{N}_{2s} \rightsquigarrow \mathbf{N}_2 \oplus \mathbf{N}_{2s-2}$$

may assume that $\sum_{k\geq 1} b_{2k} = \sum_{k \geq 1} b'_{2k}$.

Let $s \geq 1$ be minimal such that $\Psi_{2s} = \Psi'_{2s}$. If $s = 1$, this is equivalent to the desired equality. Otherwise, as in Step 3,

$$b_{2s} = 2\Psi_{2s} - \Psi_{2s-2} - \Psi_{2s+2} > 2\Psi'_{2s} - \Psi'_{2s-2} - \Psi'_{2s+2} = b'_{2s}.$$

Therefore β contains a subform of type $\mathbf{N}_{2s}$; specialize it to $\mathbf{N}_2 \oplus \mathbf{N}_{2s-2}$ to obtain β''. Then $\Psi''_{2k} = \Psi_{2k} + 1$ for $1 \leq k < s$ and $\Psi''_{2k} = \Psi_{2k}$ for $k \geq s$, so the choice of s implies $\Psi''_{2k} \leq \Psi'_{2k}$ for all k.

Step 5. *After a sequence of specializations of the form*

$$\mathbf{N}_{2t} \oplus \mathbf{N}_{2s} \rightsquigarrow \mathbf{N}_{2t+2} \oplus \mathbf{N}_{2s-2}$$

where $s > t$, may assume that $\Psi_m = \Psi'_m$ for every $m \geq 1$.

Consider any pair $s > t$ such that $\Psi_{2t} = \Psi'_{2t}$, $\Psi_{2s} = \Psi'_{2s}$, and $\Psi_{2k} < \Psi'_{2k}$ for each $t < k < s$. Arguing as in Steps 3 and 4, this implies that $b_{2s} > b'_{2s}$ and $b_{2t} > b'_{2t}$, so that β contains $\mathbf{N}_{2t} \oplus \mathbf{N}_{2s}$; specialize this to $\mathbf{N}_{2t+2} \oplus \mathbf{N}_{2s-2}$ to obtain β''. Then $\Psi''_{2k} = \Psi_{2k} + 1$ for $t < k < s$, and $\Psi''_{2k} = \Psi_{2k}$ otherwise. By choice of s and t, this implies that $\Psi''_{2k} \leq \Psi'_{2k}$ for all k. ■

6.8. Remarks. — First, the hypotheses of 6.4 may be relaxed: it is enough that there is some specialization $\beta \rightsquigarrow \beta''$ such that $\Theta_m(\beta'') \leq \Theta_m(\beta')$ for all m. For instance, the specialization $\mathbf{N}_3^{\oplus 2} \rightsquigarrow \mathbf{N}_1 \oplus \mathbf{N}_5$ found at the right end of Figure 2 does not satisfy the inequality Θ_2, but nonetheless exists by 6.3; more generally, any immediate specialization via

$$\mathbf{N}_{2s+1} \oplus \mathbf{N}_{2s+2t-1} \rightsquigarrow \mathbf{N}_{2s-1} \oplus \mathbf{N}_{2s+2t+1},$$

the fifth basic specialization of 6.3, will violate some of the inequalities Θ_m. Note further that the proof of 6.4 did not incorporate these specializations at all.

Second, there are examples of q-bic forms β and β' that satisfy the inequalities $\Phi_m(\beta) \leq \Phi_m(\beta')$, but cannot be specialized to one another via a sequence of basic specializations from 6.3. The first examples appear in dimension 15 and corank 3; for instance, consider

$$\beta := \mathbf{1} \oplus \mathbf{N}_3^{\oplus 2} \oplus \mathbf{N}_8 \quad \text{and} \quad \beta' := \mathbf{N}_1 \oplus \mathbf{N}_7^{\oplus 2}.$$

I do not know whether or not β specializes to β'.

To go further, it would be interesting to study q-bic forms over discrete valuation rings and the fine geometry of the closure of q-$\mathbf{bics}_{V,\mathbf{b}}$. For the former, it may be helpful to reformulate the canonical filtrations in terms of quotients so as to deal with jumps in types upon specialization. For the latter, it would be interesting to know the degrees and singularities of these varieties.

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