The Kobayashi-Hitchi correspondence

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1 Introduction

This article is a presentation of a result in complex geometry that was conjectured by N.Hitchin and S.Kobayashi. The conjecture was in large part inspired by the work of S.K.Donaldson, S.T.Yau and K.Uhlenbeck on moduli spaces of connections over compact complex (in fact Kahler) manifolds. In the case of complex dimension 1, these results end up being a disguised version of a much earlier theorem of Narasimhan and Seshadri characterizing stability of vector bundles in the context of algebraic geometry. The proof of this result is due to the work of Donaldson, Yau, Uhlenbeck, Li and Buchdahl. More about the history of the correspondence can be found in the introductory chapter of [1], which was used as a main reference for this presentation.

2 Tools of Complex geometry

This section will serve as an introduction to complex geometry. Most objects we will be dealing with in later sections are defined here in a way that serves the purpose of the rest of the presentation. For a more detailed exposition of this material, the reader is refered to [2] and [3]. The fundamental object in our story is a complex n-dimensional manifold we denote by X. the old Riemannian metric on the tangent bundle shall be replaced by a Hermitian metric on its complexification; The complex tangent bundle. In local holomorphic coordinates, This can be thought of as a 2-tensor $g = g_{kl} dz^k \otimes d\overline{z}^l$, with the matrix (g_{kl}) being Hermitian positive definite. A Hermitian metric g has an associated $(1, 1)$-form, locally defined as: $\omega_g = \frac{\sqrt{-1}}{2} g_{kl} dz^k \wedge d\overline{z}^l$. This form offers a useful viewpoint of looking at the metric. For one, it automatically provides us with a volume form $\mathrm{vol}_g := \frac{1}{n!}\omega_g$, which in turn makes it straightforward to characterize the complex Hodge $*$ operator: $a \wedge *b = g(a, b)\mathrm{vol}_g$. Furthermore, we now have a Lefschetz operator $L_g : A^*(X) \longrightarrow A^*(X)$ defined by $L_g(\alpha) = \alpha \wedge \omega_g$ together with its formal adjoint Λ_g, which is essentially a contraction by ω_g, defined through the equation: $g(\Lambda_g(a), b) = g(a, L_g(b))$. This operator will play a key role in our future analysis.

In a similar fashion, it is possible to speak of Hermitian metrics h on any complex vector bundle E over X. Similar to the splitting of the exterior derivative $d = \partial + \overline{\partial}$ on forms, it should also be of no surprise to the reader that a covariant derivative d_A on E, with connection 1-form A, which we regard as on operator

on $A^*(E)$, will also split, just like the exterior derivative, into a holomorphic and an anti-holomorphic part: $d_A = \partial_A + \overline{\partial}_A$. In what follows, we will usually refer to A and d_A as connections and we generally think of them as the same object. As usual, compatibility with the metric is formulated in terms of a product rule.
Definition 1 An connection A on E is said to be h-unitary if:

$$dh(s,t) = h(d_A s, t) + h(s, d_A t)$$

In this context, it is very natural to ask what could upgrade a smooth complex vector bundle into a holomorphic one. Intuitively, a holomorphic structure on E is a designation of some smooth sections of E to then call them holomorphic. It is therefore very tempting to define holomorphic sections as those annihilated by $\overline{\partial}_A$. However, this turns out to be rather inconvenient as $\overline{\partial}_A$ does not always square to 0. This motivates the following:
Defintion 2 A holomorphic structure on E is a differential linear operator $\overline{\delta}$: $A^0(E) \longrightarrow A^{0,1}(E)$, squaring to 0, and satisfying the Leibniz rule:

$$\overline{\delta}(fs) = \overline{\partial} f \otimes s + f\overline{\delta}(s)$$

In particular, an integrable connection (one with $\overline{\partial}_A^2 = 0$) defines a holomorphic structure. In fact, all holomorphic structures come in this way: Given a holomorphic structure $(E, h, \overline{\delta})$, There exists a unique h-unitary integrable connection inducing this same holomorphic structure. We call it the <u>Chern connection</u>. This is not so hard to see because ∂_A can always be recovered from $\overline{\partial}_A$ via the identity:

$$\partial h(s,t) = h(\partial_A s, t) + h(s, \overline{\partial}_A t)$$

3 The quest for an ideal connection

Our bag of ingredients now consists a complex manifold (X, g) with some metric on it, and a complex vector bundle E over X. We saw in the previous section how this setup can be further enriched by a holomorphic structure $\overline{\delta}$, a metric h, or a connection A. At this stage, it is very natural to ask for the "best" holomorphic structure (resp. best metric, resp. best connection). Because of the relations between these structures and how they induce each other, we see two ways of approaching this question:

1. Fix a holomorphic structre $\overline{\delta}_0$ on E and look for the best metric. The best connection should then be the corresponding Chern connection.

2. Fix a metric h_0 on E, and look for the best h_0-unitary connection on E. The best holomorphic strucutre should then be the one induce by our connection.

Furthermore, the two approaches are equivalent, thanks to the correspondence between integrable connections and holomorphic structures discussed in the previous section. We will therefore alternate between the two viewpoints in our approach to the problem. From this angle, It is natural to require our best h_0-unitary connection to be integrable so it induces a holomorphic structure. This in fact forces its curvature to be a $(1,1)$-form. The other condition, perhaps less intuitive, is a Hermitian-Einstein condition on the connection, which can be formulated as:

$$K_A := \sqrt{-1}\Lambda_g F_A = \gamma_A \cdot \mathrm{id}_E$$

For some constant γ_A. It only takes a bit of staring at the definition of the scalar curvature K_A above to realize it is only a natural extension of the usual scalar curvature know in Riemannian geometry, which usually defines from the Levi-Civitta connection, and that the equation above is none but the know constant scalar curvature equation, which has been quite well studied in the framework of Kahler manifolds (usually abbreviated as cscK, see [4]). We have already seen that any holomorphic structure induces and integrable h_0-unitary connections and vice-versa. We therefore expect bundles admitting Hermitian-Einstein connections, to only admit some special class of holomorphic structures. This will turn out to be the class of stable holomorphic structures, previously discovered by Mumford (see [5]) in the context of algebraic geometry.

4 Stability

The interest in stability of vector bundles in the context of algebraic geometry stems from the problem of constructing Moduli spaces. A central result in Mumford's geometric invariant theory, which he developed to deal with this problem is the Hilbert-Mumford criterion for stability (see [5]), stating that a vector bundle E over an algebraic variety V is stable, if for any non-zero proper sub-bundle F of E, we have:

$$\frac{\deg(E)}{\mathrm{rank}(E)} > \frac{\deg(F)}{\mathrm{rank}(F)}$$

If the above inequality holds with non-strict inequality sign, we say the bundle is semi-stable. The degree of a vector bundle is a well-known number-valued invariant in the framework of algebraic geometry (see [2]) that satisfies the following key properties:

$$\deg(E_1 \otimes E_2) = \deg(E_1) + \deg(E_2) \tag{1}$$
$$\deg(E) = \deg(\det(E)) \tag{2}$$

Which often fully determine the degree map (up to a multiplicative constant, with no much bearing on our notion of stability). We attempt to mimic this by defining:

$$\deg_g(L) := \int_X c_1(L, A_h) \cdot \omega_g^{n-1}$$

A little analysis of Chern connections shows that in order for this to be independent of h, it suffices to have the Gauduchon condition satisfied $\partial\overline{\partial}\omega_g^{n-1} = 0$. This is not much of a requirement as these exist in every conformal class.

Now we can fix a Gauduchon metric g on X, and use it to define stability (resp, semi-stability) as explained above. In the rest of this section, we briefly explain why a vector bundle admitting a Hermitian-Einstein metric h is semi-stable. The first ingredient of the proof is to notice that for a given vector bundle (E, h_0), an h_0-unitary Hermitian-Einstein connection A (if it exists) must have its Hermitian-Einstein factor λ_A equal to:

$$\gamma(\mathcal{E}) := \frac{2\pi}{(n-1)!\mathrm{vol}_g(X)}\mu_g(\mathcal{E})$$

Where $\mu_g(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ is the slope of the holomorphic bundle $\mathcal{E} := (E, \overline{\partial}_A)$.

Now, a rank s-holomorphic sub-bundle (or more generally a coherent subsheaf) $\mathcal{F}$ of $\mathcal{E}$ comes with an injective morphism $\phi : \mathcal{F} \longrightarrow \mathcal{E}$. This morphism induces an injection: $\det(\phi) : \det(\mathcal{F}) \longrightarrow \Lambda^s\mathcal{E}$ Which in turn yields a non-zero section of the holomorphic bundle $\Lambda^s\mathcal{E} \otimes \det(\mathcal{F})^*$. Furthermore, one can show that holomorphic line bundles always admit a Hermitian-Einstein metric since for line bundles, the stability requirement is void (see next section). Moreover, a known argument due to Kobayashi shows that a holomorphic bundle with negative Hermitian-Eintein constant has no non-trivial holomorphic section (see [1]). Putting all of this together, we deduce that

$$s\gamma(\mathcal{E}) - \gamma(\det(\mathcal{F})) = \gamma(\Lambda^s\mathcal{E} \otimes \det(\mathcal{F})^*) \geq 0$$

Refering back to the formula of γ above, we obtain: $\mu_g(\mathcal{E}) \geq \mu_g(\mathcal{F})$. Which establishes that E is semi-stable, as desired.

5 The Existence of Hermitian-Einstein metrics

The hard direction in the Kobayashi-Hitchin correspondence is establishing that a polystable holomorphic bundle admits a Hermitian-Einstein metric. Since a polystable bundle is a direct sum of stable bundles, it suffices to establish the result for stable bundles. In this section, we will attempt to give a brief demonstration of the key steps in the proof. The problem of finding the desired Hermitian-Einstein metric boils down to solving a non-linear PDE as follows: We start by fixing a Hermitian metric h_0 on our holomorphic stable bundle. Standard linear algebra tells us that any other Hermitian metric h, is related to h_0 through the equation $h(s,t) = h(f(s),t)$ where f is an h_0-selfadjoint operator on E. If the Chern connection induced by h_0 is $d_0 = \partial_0 + \overline{\partial}_0$, the it can be computed that the Chern connection associated with h is $d = d_0 + f^{-1} \circ \partial_0(f)$. Therefore, the corresponding curvature is:

$$K_h = K_0 + \sqrt{-1}\Lambda_g \overline{\partial}(f^{-1} \circ \partial_0 f)$$

Let γ be the Hermitian-Einstein constant of $\mathcal{E}$, which we recall is independent of h. We then see that that our problem boils down to the non-linear PDE (in f):

$$K^0 + \sqrt{-1}\Lambda_g \overline{\partial}(f^{-1} \circ \partial_0 f) = 0$$

Where $K^0 = K_0 - \gamma \mathrm{id}_E$. We approach the PDE above along the classical continuity method: We start by considering the perturbed equation:

$$L_\epsilon(f) := K^0 + \sqrt{-1}\Lambda_g \overline{\partial}(f^{-1} \circ \partial_0 f) + \epsilon \log(f) = 0$$

The goal is to show that $L_0(f) = 0$ has a solution. The approach to this goal is as follows:
Step 1 We first show that for an appropriately chosen initial h_0, the equation $L_1(f) = 0$ has a solution. An initial step towards this is by first establishing that:

$$\mathrm{tr} L_\epsilon(f) = \mathrm{tr} K^0 + \sqrt{-1}\Lambda_g \overline{\partial}\partial(\mathrm{tr}(\log(f))) + \epsilon \mathrm{tr}(\log(f))$$

Step 2 For this fixed h_0, we let J be the set of ϵ in $(0, 1]$ for which the perturbed equation has a solution f_ϵ. Theb we show that J us both open and closed. This is the most technical part of the proof. Accomplishing step 2 means that $J = (0, 1]$
Step 3 Now we study the behaviour of the solutions f_ϵ we constructed in step 2, as ϵ appraoches 0. Here we have two cases:

- The easier case is when the norms $\left\|\log(f_\epsilon)\right\|_{L^2}$ is uniformly bounded in ϵ. In this case, one can show that the f_ϵ converge (after possibly passing to a subsequence) to a solution f_0 of the unperturbed equation, which in turn yields the desired Hermitian-Einstein metric as explained above.
- The more intricate case is when the sequence above is not uniformly bounded. In this case, the idea is to show that this contradicts stability. In other words, one proceeds to construct a destabilizing subsheaf if $\mathcal{E}$.

A priori, it is not so obvious how one could use this data obtained from a differential equation to construct a destabilizing sheaf. In fact, this step makes use of characterization of coherent subsheafs in holomorphic bundle due to Uhlenbeck and Yau. Their idea was that a coherent subsheaf (see [6] for more about sheaves and coherent subsheaves) of a holomorphic vector bundle $\mathcal{E}$ can be realized as the image of an element $\pi \in L_1^2(\mathrm{End}(\mathcal{E}))$ satisfying the following two conditions:

$$\pi^* = \pi = \pi^2 \qquad \text{and} \qquad (\mathrm{id}_{\mathcal{E}} - \pi) \circ \overline{\partial}\pi = 0$$

In our case, the π that will yield a destabilizing sheaf is constructed, using a sequence ϵ_i approaching 0 such that the corresponding sequence of norms $\left\|\log(f_\epsilon)\right\|_{L^2}$ diverges, as:

$$\pi = \mathrm{id}_{\mathcal{E}} - \lim_{\sigma \to 0} \left(\lim_{i \to \infty} \rho(\epsilon_i) \cdot f_{\epsilon_i} \right)^{\sigma}$$

where $\rho(\epsilon)$ is carefully chosen function of ϵ.

6 References

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