PROJECTIVITY OF MODULI OF CURVES

Contents

1. Introduction

In this article, we exposit a proof, due to Kollár in [9], of the fact that the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ admits a coarse moduli space $\overline{M}_g$ which is projective over $\mathbf{Z}$. In particular, this means that the coarse moduli space $\overline{M}_g$, which is *a priori* but an algebraic space, is actually a projective scheme over $\mathbf{Z}$. Together with the work of Deligne–Mumford [3], the moduli space $\overline{M}_g$ is actually an irreducible smooth projective scheme over $\mathbf{Z}$.

Kollár's method differs from other existing proofs of projectivity of $\overline{M}_g$ in two main ways: First, the technique is independent of the powerful, though bulky, methods of Geometric Invariant Theory, on which the proofs of [13, 4, 2] are reliant on. Second, Kollár's criterion does not require one to directly check that a line bundle on the moduli space is ample, in contrast to the approach of Knudsen–Mumford [8, 6, 7]; rather, one only needs to show that some vector bundle on the moduli space is nef. Since nefness is a condition that only depends on curves in the space, concretely, this condition can be checked in moduli situations by considering families over a 1-dimensional base. Thus, the positivity input in this proof amounts to studying positivity of the pushfoward of relative canonical sheaves on families of stable curves over a 1-dimensional base.[1]

August 15, 2017.

[1] **TODO.** Write a legitimate introduction. What is this document about; what does it aim to do; how did it come about. Add some references.

2. Positivity of invertible sheaves

For material on positivity, we largely follow [11, 12].

Definition 2.1. Let X be a proper scheme over a field k. An invertible $\mathcal{O}_X$-module is said to be *numerically effective* or *nef* if $(\mathcal{L} \cdot C) \geq 0$ for every irreducible closed subscheme $C \subset X$ of dimension 1.

Lemma 2.2. *Let $f : Y \to X$ be a morphism of proper schemes over k. If $\mathcal{L}$ is a nef invertible sheaf on X, then $f^*\mathcal{L}$ is a nef invertible sheaf on Y.*

Proof. ...[2] □

Lemma 2.3. *Let $f : Y \to X$ be a surjective proper morphism of schemes over k. Assume X is proper over k. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is nef on X if and only if $f^*\mathcal{L}$ is nef on Y.*

Proof. ...[3] □

Lemma 2.4. *Let X be a proper scheme of dimension 1 over a field k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is nef, then $\deg(\mathcal{L}) \geq 0$.*

Proof. In the case X is irreducible, then the conclusion follows from [14, Tag 0BEY] and the definitions.

In general, let $C_1, \ldots, C_t$ be the irreducible components of X, viewed as subschemes of X with the reduced induced subscheme structure. By [14, Tag 0AYW],

$$\deg(\mathcal{L}) = \sum\nolimits_i m_i \deg(\mathcal{L}|_{C_i})$$

for some positive integers m_i. The irreducible case gives $\deg(\mathcal{L}|_{C_i}) \geq 0$, and thus $\deg(\mathcal{L}) \geq 0$. □

Definition 2.5. Let k be a field. Let X be a proper scheme over k. An invertible sheaf $\mathcal{L}$ on X is said to be *big* if...[4].

3. Tensor sheaves

Linear algebra constructions give many constructions to combine locally free sheaves. These constructions can be unified by viewing each as coming from a representation of a product of general linear groups, and then the composite sheaf can be described by applying the representation to the transition functions of the constituent sheaves.

Definition 3.1. Let k be a field. Let X be a scheme over k. Let $\mathcal{E}_1, \ldots, \mathcal{E}_s$ be locally free sheaves of finite ranks $r_1, \ldots, r_s$ on X. Let $\rho : \prod_t \mathrm{GL}_{r_t} \to \mathrm{GL}_n$ be a homomorphism of group schemes over k. Then $\rho(\mathcal{E}_1, \ldots, \mathcal{E}_s) = \mathcal{E}$ is the locally free sheaf of finite rank n on X constructed as follows: Let $X = \bigcup_i U_i$ be an open cover over which each of $\mathcal{E}_1, \ldots, \mathcal{E}_s$ is trivial and suppose $\mathcal{E}_t$ is given by Čech cocycles $\{(U_j, \varphi_{t,ij})\}$. Then $\mathcal{E}$ is the locally free sheaf given by the Čech cocycle $\{(U_i, \rho(\varphi_{1,ij}, \ldots, \varphi_{s,ij}))\}$.

[2] **TODO.** Prove this!
[3] **TODO.** Prove this!
[4] **TODO.** Fill in the definition and show a few characterizations.

Following the terminology of [5, p.76], we may refer to coherent locally free sheaves of the form $\rho(\mathcal{E}_1, \dots, \mathcal{E}_s)$ as *tensor sheaves*. We generally refer to this procedure as applying *tensor operations* to locally free sheaves.

Remark 3.2. The construction in Definition 3.1 can be described more generally for any homomorphism $\rho : G \to H$ of group schemes and any G-torsor $\mathcal{E}$.

Remark 3.3. The construction in Definition 3.1 and its generalization in Remark 3.2 can be described more abstractly. Let X be a scheme. Let $\rho : G \to H$ be a homomorphism of group schemes and write $\rho^* : H^1(X, G) \to H^1(X, H)$ be the induced map on first cohomology sets. Let $\mathcal{E}$ be a G-torsor over X. By [14, Tag 02FQ], $\mathcal{E}$ corresponds to a class $[\mathcal{E}] \in H^1(X, G)$. Invoking, once again, the correspondence between torsors and classes in H^1, $\rho^*([\mathcal{E}]) \in H^1(X, H)$ gives rise to a H-torsor. This is $\rho(E)$.

Lemma 3.4. *Let k be a field. Let $f : X' \to X$ be a morphism of schemes over k.. Let $\mathcal{E}_1, \dots, \mathcal{E}_s$ be locally free sheaves of finite ranks $r_1, \dots, r_s$ on X. Let $\rho : \prod_t \mathrm{GL}_{r_t} \to \mathrm{GL}_n$ be a homomorphism of group schemes over k. Then $\rho(f^*\mathcal{E}_1, \dots, f^*\mathcal{E}_s)$ and $f^*\rho(\mathcal{E}_1, \dots, \mathcal{E}_s)$ are canonically isomorphic as locally free $\mathcal{O}_{X'}$-modules.*

Proof. This is immediate from the description given in Remark 3.3. □

Certain tensor operations on sheaves preserve nonnegativity properties of locally free sheaves. We single one class of such tensor operations in the next definition.

Definition 3.5. Let k be a field. Let $V_1, \dots, V_s, V$ be finite dimensional vector spaces over k. A homomorphism $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V)$ of group schemes over k is called *nonnegative* if it extends to a homomorphism $\tilde{\rho} : \prod_t \mathrm{End}(V_t) \to \mathrm{End}(V)$ of monoids over k.

Roughly, a nonnegative representation is one where there are "not too many dets in the denominator." The choice of terminology in Definition 3.5 comes about as a generalization of *positive representations* as defined in [5, p.76]. A positive representation, as defined in Hartshorne's paper, is a nonnegative representation.

Lemma 3.6. *Let k be a field. Let $V_1, \dots, V_s$ be finite dimensional vector spaces over k. Then the natural homomorphism $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V_1 \oplus \cdots \oplus V_s)$ is nonnegative.*

Proof. Omitted. □

Lemma 3.7. *Let k be a field. Let $V_1, \dots, V_s$ be finite dimensional vector spaces over k. Then the natural homomorphism $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V_1 \otimes_k \cdots \otimes_k V_s)$ is nonnegative.*

Proof. Omitted. □

Lemma 3.8. *Let k be a field. Let $V_1, \dots, V_s, V$ be finite dimensional vector spaces over k. Let $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V)$ be a homomorphism of group schemes over k Assume ρ is a nonnegative representation. Suppose $V' \subset V$ is an invariant subspace, defining a subrepresentation $\rho' : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V')$. Then ρ' is a nonnegative representation.*

Proof. Let $\rho : G \to \mathrm{GL}(W)$ be a nef representation and let $W' \subset W$ be a subrepresentation. We may easily reduce to the case $G = \mathrm{GL}(V)$. Then, ρ extends to a morphism of monoids $\overline{\rho} : \mathrm{End}(V) \to \mathrm{End}(W)$. The representation ρ also gives a morphism of schemes $\phi : \mathrm{GL}(V) \times W \to W$, where W has been given the obvious scheme structure. Similarly, $\overline{\rho}$ gives rise to an extension $\overline{\phi} : \mathrm{End}(V) \times W \to W$.

The fact that $W' \subset W$ is a subrepresentation is equivalent to the fact that the image of ϕ when restricted to the closed subscheme $\mathrm{GL}(V) \times W'$ is W'. It is sufficient to prove the same statement for the closed subscheme $\mathrm{End}(V) \times W'$ of $\mathrm{End}(V) \times W$. However, by assumption the dense open subscheme $\mathrm{GL}(V) \times W'$ of $\mathrm{End}(V) \times W'$ maps to W', so $\mathrm{End}(V) \times W'$ must as well, as desired. One should check that the resulting map on W' is linear; details omitted.[5] $\square$

Lemma 3.9. *Let k be a field. Let $V_1, \ldots, V_s, V$ be finite dimensional vector spaces over k. Let $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V)$ be a homomorphism of group schemes over k Assume ρ is a nonnegative representation. Suppose $V \to V''$ is a quotient of V such that ρ descends to a representation $\rho'' : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V'')$. Then ρ'' is a nonnegative representation.*

Proof. Omitted.[6] $\square$

Lemma 3.10. *Let k be a field. Let $V_1, \ldots, V_s, V$ be finite dimensional vector spaces over k. Let $\rho : \prod_t \mathrm{GL}(V_t) \to \mathrm{GL}(V)$ be a homomorphism of group schemes over k. Let $\sigma : \prod_t \mathrm{GL}(V_t^\vee) \to \prod_t \mathrm{GL}(V_t)$ be the antihomomorphism of monoids taking an automorphism of $V_t^\vee$ to its dual. Let $\tau : \mathrm{GL}(V) \to \mathrm{GL}(V^\vee)$ denote the antihomomorphism of monoids taking an automorphism of V to its dual. If ρ is a nonnegative representation, then*

$$\tau \circ \rho \circ \sigma : \prod \mathrm{GL}(V_t^\vee) \to \mathrm{GL}(V)$$

is a nonnegative representation.

Proof. ...[7] $\square$

Lemma 3.11. *Let k be a field. Let V be finite dimensional vector spaces over k. Then, for every $r \geq 0$, the natural homomorphism $\rho : \mathrm{GL}(V) \to \mathrm{GL}(\bigwedge^r V)$ is nonnegative.*

Proof. This follows from Lemma 3.6, Lemma 3.7, and Lemma 3.9. $\square$

Lemma 3.12. *Let k be a field. Let V be finite dimensional vector spaces over k. Then, for every $d \geq 0$, the natural homomorphism $\rho : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^d(V))$ is nonnegative.*

Proof. This follows from Lemma 3.6, Lemma 3.7, and Lemma 3.9. $\square$

Lemma 3.13. *Let k be a field. Let X be a scheme over k. Let $\mathcal{E}_1, \ldots, \mathcal{E}_s, \mathcal{E}'_1, \ldots, \mathcal{E}'_s$ be locally free sheaves of finite rank on X. Assume that for each $t = 1, \ldots, s$, $r_t = \mathrm{rank}(\mathcal{E}_t) = \mathrm{rank}(\mathcal{E}'_t)$. For each $t = 1, \ldots, s$, let $f_t : \mathcal{E}_i \to \mathcal{E}'_i$ be a morphism of $\mathcal{O}_X$-modules. Let $\rho : \prod \mathrm{GL}_{r_t} \to \mathrm{GL}_n$ be a representation. If ρ is nef, then the $f_1, \ldots, f_s$ induce a morphism*

$$\rho(f_1, \ldots, f_s) : \rho(\mathcal{E}_1, \ldots, \mathcal{E}_s) \to \rho(\mathcal{E}'_1, \ldots, \mathcal{E}'_s)$$

[5] **TODO.** Edit this to match notation.
[6] **TODO.** Maybe comment that the proof is basically the same as the subrep case.
[7] **TODO.** Draw the required diagram.

of $\mathcal{O}_X$-modules.

Proof. Choose an open cover $X = \bigcup_i U_i$ over which all the locally free sheaves are trivial and fix a trivialization of each sheaf. With respect to this open cover, each morphism f_t can be locally described by a $r_t \times r_t$ matrix. Since ρ is nonnegative, there exists an extension $\tilde{\rho} : \prod \mathrm{Mat}_{r_t} \to \mathrm{Mat}_n$. The matrices $\tilde{\rho}(f_1, \dots, f_t)$ locally define the morphism of $\mathcal{O}_X$-modules in the statement. Details omitted. ☐

Lemma 3.14. *Let k be a field. Let X be a k-scheme. For $i = 1, 2, \dots, n$, let $V_i = \bigoplus_j \mathcal{L}_{ij}$ be a locally free sheaf of rank r_i that splits as a direct sum of line bundles $\mathcal{L}_{ij}$. Let $\rho : \prod_i \mathrm{GL}_{r_i} \to \mathrm{GL}_m$ be a nef representation. Then, $\rho(\mathcal{V}_1, \dots, V_n)$ is a direct sum of line bundles of the form $\bigoplus_{i,j} \bigotimes \mathcal{L}_{ij}^{a_{ij}}$, where $a_{ij} \geq 0$ for each i, j.*[8]

Proof. For each i, we have a diagonal embedding $\psi_i : \prod_{j=1}^{r_i} \mathrm{GL}_1 \to \mathrm{GL}_{r_i}$. Taking the product over i and post-composing with ρ yields a map

$$\psi : \prod_{i=1}^{n} \prod_{j=1}^{r_i} \mathrm{GL}_1 \to \mathrm{GL}_m,$$

which factors through a maximal torus $\prod_{k=1}^{m} \mathrm{GL}_1 \subset \mathrm{GL}_m$, as in the following diagram:

$$\begin{tikzcd}
\prod_{i=1}^{n} \prod_{j=1}^{r_i} \mathrm{GL}_1 \arrow[r, "\psi"] \arrow[dr] & \prod_{i=1}^{n} \mathrm{GL}_{r_i} \arrow[r, "\rho"] & \mathrm{GL}_m \\
& \prod_{k=1}^{m} \mathrm{GL}_1 \arrow[ur] \arrow[d, "\mathrm{pr}_t"] & \\
& \mathrm{GL}_1 &
\end{tikzcd}$$

Now consider the projection on the t^{th} factor $\mathrm{pr}_t : \prod_{k=1}^{m} \mathrm{GL}_1 \to \mathrm{GL}_1$. The composition $\prod_{i=1}^{n} \prod_{j=1}^{r_i} \mathrm{GL}_1 \to \mathrm{GL}_1$ is a character of the form $(z_{ij})_{i,j} \mapsto \prod z_{ij}^{a_{ij,t}}$. On a trivializing cover of X, the z_{ij} correspond to the transition functions of $\mathcal{L}_{ij}$, hence $\prod z_{ij}^{a_{ij,t}}$ correspond to $\otimes \mathcal{L}_{ij}^{a_{ij,t}}$.

Pulling back to $\prod_{k=1}^{m} \mathrm{GL}_m$, we see that $\rho(V_1, \dots, V_n)$ splits as the direct sum of the line bundles $\bigoplus_t \bigotimes_{i,j} \mathcal{L}_{ij}^{a_{ij,t}}$. ☐

4. Nef locally free sheaves

In this section, we define and study basic properties of nef bundles.

Lemma 4.1. *Let X be a proper scheme over a field k. Let $\mathcal{E}$ be a coherent locally free sheaf on X. Then the following are equivalent:*

1. *$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is nef on $\mathbf{P}(\mathcal{E})$; and*
2. *for every k-morphism $f : C \to X$ from a proper k-scheme C of dimension 1, and for every surjection $f^*\mathcal{E} \to \mathcal{L}$ of locally free $\mathcal{O}_C$-modules with $\mathrm{rank}(\mathcal{L}) = 1$, we have $\deg_C(\mathcal{L}) \geq 0$.*

[8]**TODO.** Edit this to be consistent with this section.

Proof. Let $\pi : \mathbf{P}(\mathcal{E}) \to X$ be the projection map from the projective bundle $\mathbf{P}(\mathcal{E})$ down to X. Assume (1) holds and let $f : C \to X$ be a morphism of k-schemes, with 1 dimensional C which is smooth and proper over k, and let $\mathcal{L}$ be an invertible quotient of $f^*\mathcal{E}$ over C. Applying Sym^* to the surjection $f^*\mathcal{E} \to \mathcal{L}$, [14, Tag 01O9] gives a morphism $r : C \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} \cong r^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Since both C and $\mathbf{P}(\mathcal{E})$ are proper over k, [14, Tag 01W6] shows r is proper. Thus by Lemma **??**, $\mathcal{L}$ is nef. Lemma 2.4 now gives $\deg_C(\mathcal{L}) \geq 0$.

Assume (2) and let $g : C \hookrightarrow \mathbf{P}(\mathcal{E})$ be any irreducible closed subscheme of dimension 1. Let $\mathcal{L} = g^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and note that it is nef by Lemma **??**. Set $f = \pi \circ g : C \to X$ to be the closed immersion of C into the projective bundle followed by the projection morphism. By the universal property of relative Proj, the morphism g gives rise to a surjection $f^*\mathcal{E} \to \mathcal{L}$. Now [14, Tag 0BEY] together with our hypothesis gives

$$(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \cdot C) = \deg_C(\mathcal{L}) \geq 0.$$

Therefore $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is nef. □

Definition 4.2. Let k be a field. Let X be a proper scheme over k. A coherent locally free sheaf $\mathcal{E}$ on X is said to be *nef* if the equivalent conditions of Lemma 4.1 are satisfied.

Lemma 4.3. *Let k be a field. Let X be a proper scheme over k. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a surjection of locally free sheaves on X. If $\mathcal{E}$ is nef, then $\mathcal{F}$ is nef.*

Proof. Let $f : C \to X$ be a morphism of schemes over k from a proper k-scheme C of dimension 1 and let $f^*\mathcal{F} \to \mathcal{L}$ be an invertible quotient. Since f^* is right exact, $f^*\varphi$ induces a surjection $f^*\mathcal{E} \to \mathcal{L}$. Since $\mathcal{E}$ is nef, $\deg_C(\mathcal{L}) \geq 0$. This shows $\mathcal{F}$ is nef. □

Lemma 4.4. *Let k be a field. Let X be a proper scheme over k. Let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

be a short exact sequence of coherent locally free sheaves on X. If $\mathcal{E}'$ and $\mathcal{E}''$ are both nef, then $\mathcal{E}$ is nef.

Proof. ...[9] □

Lemma 4.5. *Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. If $\mathcal{E}$ is a nef coherent locally free sheaf on X, then $f^*\mathcal{E}$ is a nef coherent locally free sheaf on Y.*

Proof. Let $g : C \to Y$ be a morphism of schemes over k from a proper k-scheme C of dimension 1 and let $g^*(f^*\mathcal{E}) \to \mathcal{L}$ be an invertible quotient. Since $g^*(f^*\mathcal{E}) = (f \circ g)^*\mathcal{E}$ and $\mathcal{E}$ is nef, $\deg_C(\mathcal{L}) \geq 0$. Thus $f^*\mathcal{E}$ is nef. □

Lemma 4.6. *Let X and Y be schemes. Let $f : Y \to X$ be a flat morphism of schemes. Let $\mathcal{V}$ be a locally free sheaf of finite rank on Y. Then, the set of points $x \in X$ such that the locally free sheaf $\mathcal{V}_x$ is nef on the fiber Y_x is open.*

[9]**TODO.** Prove this!

Proof. Let $\pi : \mathbb{P}_Y(V) \to Y$ be the projection map. Given a point $x \in X$, by Lemma **??** the vector bundle $\mathcal{V}_x$ is nef if and only if $\mathcal{O}_{\mathbb{P}_Y(V)}(1)_x$ is nef. The compositve morphism $f \circ \pi$ is flat, because π is flat. The claim now follows from the fact that nefness is an open condition in flat families.[10] □

Lemma 4.7. *Let k be a field. Let C be a smooth, proper, and connected k-scheme of of genus g. Let $\mathcal{E}$ be a vector bundle on C. Let $\mathcal{L}$ be a line bundle on C of degree at least $2g$. Then, $\mathcal{E} \otimes \mathcal{L}$ is globally generated.*

Proof. By [14, Tag 0B57], it suffices to assume that k is algebraically closed. Let $x \in C$ be a closed point with ideal sheaf $\mathcal{O}(-x)$. We have a short exact sequence

$$0 \to \mathcal{E} \otimes \mathcal{L}(-x) \to \mathcal{E} \otimes \mathcal{L} \to \mathcal{E} \otimes \mathcal{L} \mid_x \to 0 \tag{4.7.1}$$

It suffices to show that $H^1(C, \mathcal{E} \otimes \mathcal{L}(-x)) = 0$. By Serre Duality[11] and since $\mathcal{L}(-x)$ is locally free,

$$H^1(C, \mathcal{E} \otimes \mathcal{L}(-x)) \cong \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{E} \otimes \mathcal{L}(-x), \omega_C)^\vee \cong \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{E}, \omega_C \otimes \mathcal{L}^{-1}(x))^\vee.$$

Consider a morphism $f : \mathcal{E} \to \omega_C \otimes \mathcal{L}^{-1}(x)$ of $\mathcal{O}_C$-modules. Since $\mathcal{E}$ is locally free, the image of f is a torsion-free subsheaf of $\omega_C \otimes \mathcal{L}^{-1}(x)$.[12] □

Lemma 4.8. *Let k be a field of characteristic $p > 0$. Let C be a smooth, proper, and connected k-scheme of dimension 1. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be nef vector bundles of ranks $r_1, \ldots, r_n$ on C. Let $\rho : \prod \mathrm{GL}_{r_i} \to \mathrm{GL}_m$ be a nef representation. Then $\rho(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is a nef vector bundle.*

Proof. Let g be the genus of C, and fix a line bundle $\mathcal{L}$ of degree at least $2g$. By Lemma 4.7, $\mathcal{V}_i \otimes \mathcal{L}$ is generated by global sections for each i, so there exist maps

$$f_i : (\mathcal{L}^{-1})^{\oplus r_i} \to \mathcal{V}_i,$$

which are surjective at a given closed point of $x \in C$, and hence generically surjective.

Applying Lemma 3.4, Lemma 3.14, and Lemma 3.13, we get a generically surjective map

$$\rho(f_1, \ldots, f_n) : \bigoplus_i \mathcal{L}^{-\otimes b_i} \to \rho(\mathcal{V}_1, \ldots, \mathcal{V}_n),$$

where the source is $\rho(\mathcal{L}^{-\otimes r_1}, \ldots, \mathcal{L}^{-\otimes r_n})$. Thus the b_i are positive integers depending on the ranks of the V_i but not the V_i themselves. In particular, there exists a positive integer N, independent of the $\mathcal{V}_i$, such that $\min \deg \mathcal{L}^{-b_i} = -N$. Moreover, $\rho(f_1, \ldots, f_n)$ is surjective at x by the construction in Lemma 3.13, as the image of an r-tuple of invertible matrices is invertible under $\overline{\rho}$.

Suppose $\mathcal{M}$ is an invertible quotient of $\rho(V_1, \ldots, V_n)$. By pre-composing with $\rho(f_1, \ldots, f_n)$, we obtain a nonzero map

$$\bigoplus_i \mathcal{L}^{-\otimes b_i} \to \mathcal{M}.$$

[10]**TODO.** Find a reference for this.

[11]**TODO.** Find a reference in the Stacks Project for this

[12]**TODO.** Is this argument right? *The target of such a nonzero map is a locally free sheaf of negative degree, so the image subsheaf is torsion-free and hence locally free sheaf of negative degree. However, $\mathcal{E}$ has no negative line bundle quotients, so no such map can exist.* I am not sure why the image is locally free? All I can see right now is that the image is locally free away from finitely many points.

Thus, we conclude that

$$\deg(\mathcal{M}) \geq -N. \tag{4.8.1}$$

Finally, suppose that $\rho(V_1, \ldots, V_n)$ has a line bundle quotient $\mathcal{N}$ of degree $d < 0$. Let $F : C \to C$ be the absolute Frobenius map on C. Pulling back the surjection $\rho(V_1, \ldots, V_n) \to \mathcal{N}$ by F^t, we obtain a quotient line bundle of $\rho((F^t)^* V_1, \ldots, (F^t)^* V_n)$ of degree $dp^t < -N$ for sufficiently large t, by Lemma 3.4. However, as $(F^t)^* V_i$ is nef for each i, this contradicts (4.8.1), so hence $\mathcal{N}$ cannot exist. $\square$

Lemma 4.9. *Let k be a field. Let C be a smooth, proper, and connected k-scheme of dimension 1. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be nef vector bundles of ranks $r_1, \ldots, r_n$ on C. Let $\rho : \prod \mathrm{GL}_{r_i} \to \mathrm{GL}_m$ be a nef representation. Then $\rho(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is a nef vector bundle.*

Proof. If the characteristic of k is positive, this is Lemma 4.8. If the characteristic of k is zero, after possibly enlarging k, we may find an integral and finite type $\mathbb{Z}$-algebra A over which C, $\mathcal{V}_1, \ldots, \mathcal{V}_n$, and ρ are all defined, and such that we have a smooth morphism of schemes $\pi : \mathcal{C} \to \mathrm{Spec}(A)$. By Lemma 4.6, we may replace $\mathrm{Spec}(A)$ with an open subset so that each $\mathcal{V}_i$ is nef when restricted to any fiber of π. By Lemma 4.8, $\mathcal{W} = \rho(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is nef when restricted to any fiber of π above a point of $\mathrm{Spec}(A)$ with residue field of positive characteristic. This locus on $\mathrm{Spec}(A)$ is non-empty, so by another application of Lemma 4.6, so $\mathcal{W} = \rho(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is nef when restricted to the generic fiber, which is what was needed. Some additional details may be necessary.[13] $\square$

Lemma 4.10. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{E}_1, \ldots, \mathcal{E}_s$ be nef coherent locally free sheaves on X of ranks $r_1, \ldots, r_s$. Let $\rho : \prod_t \mathrm{GL}_{r_t} \to \mathrm{GL}_n$ be a nonnegative representation. Let $\mathcal{E} = \rho(\mathcal{E}_1, \ldots, \mathcal{E}_n)$. Then $\mathcal{E}$ is nef on X.*

Proof. Immediate from Lemma 4.9. $\square$

Lemma 4.11. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{E}$ be a nef coherent locally free sheaf on X. Then for every $r \geq 0$, $\bigwedge^r(\mathcal{E})$ is nef on X.*

Proof. This is Lemma 4.10 applied to the nef representation $\bigwedge^r$, see Lemma 3.11. $\square$

Lemma 4.12. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{E}$ a nef coherent locally free sheaf on X. Then for every $d \geq 0$, $\mathrm{Sym}^d(\mathcal{E})$ is nef on X.*

Proof. This is Lemma 4.10 applied to the nef representation Sym^d, see Lemma 3.12. $\square$

Lemma 4.13. *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{E}$ be a nef coherent locally free sheaf on X. Then for every $d \geq 0$, $\mathrm{Sym}^d(\mathcal{E}^\vee)^\vee$ is nef on X.*

Proof. This is Lemma 4.10 applied to the nef representation $(-)^\vee \circ \mathrm{Sym}^d \circ (-)^\vee$, see Lemma 3.12 and Lemma **??**. $\square$

[13] **TODO.** Does this even work??

Lemma 4.14. *Let k be a field. Let X be a k-scheme. Let $\mathcal{V}_1, \dots, \mathcal{V}_n$ be nef vector bundles on X of ranks $r_1, \dots, r_n$, respectively. Let $G = \prod_{i=1}^n \mathrm{GL}_{r_i}$. Let $\rho : G \to \mathrm{GL}_m$ be a nef representation. Let $\mathcal{W} = \rho(\mathcal{V}_1, \dots, \mathcal{V}_n)$. Let $\mathcal{V} \subset \mathcal{W}$ be a G-invariant subsheaf of $\mathcal{W}$. Then $\mathcal{V}$ is nef.*

Proof. Let $x \in X$ be any point. The G-invariant subspace $\mathcal{V}_x \subset \mathcal{W}_x$ defines a subrepresentation σ of ρ, and we see that $\mathcal{V} = \sigma(\mathcal{V}_1, \dots, \mathcal{V}_n)$. By Lemma **??**(**??**), σ is a nef representation, so by Lemma **??**, $\mathcal{V}$ is a nef locally free sheaf. □

5. Nef locally free sheaves on spaces

In this section, we generalize the discussion of Section 4 to algebraic spaces.[14]

6. Nakai–Moishezon for schemes

In this section, we prove an ampleness criterion for invertible sheaves on schemes over a field using intersection theory. The next result can be seen as a converse to [14, Tag 0BEV].

Lemma 6.1 (Nakai–Moishezon Criterion)**.** *Let X be a proper scheme over an algebraically closed field k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample on X if and only if for every integral closed subscheme Y of X, $(\mathcal{L}^{\dim(Y)} \cdot Y) > 0$.*

Proof. If $\mathcal{L}$ is ample on X, then the positivity of the intersection number on integral closed subschemes follows from [14, Tag 0BEV].

Conversely, assume that for every integral closed subscheme Y of X, $(\mathcal{L}^{\dim(Y)} \cdot Y) > 0$. We show that $\mathcal{L}$ is ample. Using [14, Tag 0B5V] and [14, Tag 09MS], we reduce to the case X is integral. We proceed by induction on $\dim(X)$. When $\dim(X) = 1$, our assumption says that $\deg(\mathcal{L}) > 0$ and hence $\mathcal{L}$ is ample by [14, Tag 0B5X].

Now suppose $\dim(X) > 1$ and that the theorem is true for all proper schemes of lower dimension. Since X is integral, $\mathcal{L}$ has a regular meromorphic section by [14, Tag 02OZ]. Let $\mathcal{I}_1$ be the sheaf of denominators of $\mathcal{L}$ and set $\mathcal{I}_2 = \mathcal{I}_1 \otimes \mathcal{L}$. Let Y_j be the closed subschemes defined by $\mathcal{I}_j$ with $j = 1, 2$. By [14, Tag 02P0], $\dim(Y_j) < \dim(X)$. Then, for all $m \geq 0$, we the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m} \arrow[r] \arrow[d, equal] & \mathcal{L}^{\otimes m} \arrow[r] & \mathcal{L}^{\otimes m}|_{Y_1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I}_2 \otimes \mathcal{L}^{\otimes (m-1)} \arrow[r] & \mathcal{L}^{\otimes (m-1)} \arrow[r] & \mathcal{L}^{\otimes (m-1)}|_{Y_2} \arrow[r] & 0.
\end{tikzcd}$$

By induction, $\mathcal{L}|_{Y_j}$ is ample on Y_j for $j = 1, 2$. Hence by [14, Tag 0B5U], there is some $m_0 \geq 0$ such that for all $m \geq m_0$, $H^i(Y_j, \mathcal{L}^{\otimes m}|_{Y_j}) = 0$ for all $i > 0$. Thus, taking the long exact sequence in cohomology of the sequences above, for $i \geq 2$,

$$h^i(X, \mathcal{L}^{\otimes m}) = h^i(X, \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m}) = h^i(X, \mathcal{I}_2 \otimes \mathcal{L}^{\otimes (m-1)}) = h^i(X, \mathcal{L}^{\otimes (m-1)})$$

for all $m > m_0$. Hence, for all $m > m_0$,

$$N := \sum_{i=2}^{\dim(X)} (-1)^i \, h^i(X, \mathcal{L}^{\otimes m})$$

[14] **TODO.** Port over results.

is a constant. Now since $\chi(X, \mathcal{L}^{\otimes m})$ has leading coefficient $(\mathcal{L}^{\dim X} \cdot X)$, which is positive by assumption, we see that

$$\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + N \to \infty$$

as $m \to \infty$. Thus, $h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) \to \infty$ as $m \to \infty$; in particular, $h^0(X, \mathcal{L}^{\otimes m}) \to \infty$ as $m \to \infty$. By Lemma [14, Tag 01PS], we may replace $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ to assume $\mathcal{L} = \mathcal{O}_X(D)$ for some effective Cartier divisor D.

We now claim that $\mathcal{L}^{\otimes m} = \mathcal{O}_X(mD)$ is generated by its global sections for $m \gg 0$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0.$$

Since $\mathcal{O}_D(mD)$ is ample by the inductive hypothesis, Serre vanishing [14, Tag 0B5U] implies $H^1(D, \mathcal{O}_D(mD)) = 0$ for $m \gg 0$, hence the maps

$$\rho_m : H^1(X, \mathcal{O}_X((m-1)D)) \longrightarrow H^1(X, \mathcal{O}_X(mD))$$

arising from the long exact sequence on cohomology are surjective for all $m \gg 0$. Since the vector spaces $H^1(X, \mathcal{O}_X(mD))$ are all finite-dimensional, we have that for $m \gg 0$, the sequence

$$h^1(X, \mathcal{O}_X(mD)) \geq h^1(X, \mathcal{O}_X((m+1)D)) \geq \cdots$$

is a nonincreasing sequence of nonnegative integers, and thus stabilizes for $m \gg 0$. The maps ρ_m are therefore bijective for $m \gg 0$, hence

$$H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(D, \mathcal{O}_D(mD)) \tag{6.1.1}$$

is surjective for $m \gg 0$. We can now show $\mathcal{O}_X(mD)$ is generated by global sections. If $x \in X \setminus D$, then a global section in $H^0(X, \mathcal{O}_X(mD))$ defining mD generates $(\mathcal{O}_X(mD))_x$, since $\mathcal{O}_X(mD)$ restricted to $X \setminus D$ is trivial. Otherwise, suppose $x \in D$. By inductive hypothesis, since $\mathcal{O}_D(D)$ is ample, the invertible sheaf $\mathcal{O}_D(mD)$ is generated by global sections for all $m \gg 0$, i.e., the morphism

$$H^0(D, \mathcal{O}_D(mD)) \otimes_k \mathcal{O}_D \longrightarrow \mathcal{O}_D(mD)$$

is surjective. Thus, in the commutative diagram

$$\begin{tikzcd}
H^0(X, \mathcal{O}_X(mD)) \otimes_k \mathcal{O}_X/\mathfrak{m}_x \arrow[r] \arrow[d, two heads] & \mathcal{O}_X(mD) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x \arrow[d, "\simeq"] \\
H^0(D, \mathcal{O}_D(mD)) \otimes_k \mathcal{O}_D/\mathfrak{m}_x \arrow[r, two heads] & \mathcal{O}_D(mD) \otimes_{\mathcal{O}_D} \mathcal{O}_D/\mathfrak{m}_x
\end{tikzcd}$$

the bottom arrow is surjective. By (6.1.1), the left arrow is surjective. The commutativity of the diagram implies the top row is surjective, hence

$$H^0(X, \mathcal{O}_X(mD)) \otimes_k \mathcal{O}_{X,x} \longrightarrow (\mathcal{O}_X(mD))_x$$

is surjective by Nakayama's lemma, i.e., $\mathcal{O}_X(mD)$ is generated by global sections.

We therefore see that $\mathcal{O}_X(mD)$ induces a morphism $f : X \to \mathbf{P}^n_k$ such that $f^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{O}_X(mD)$ for $m \gg 0$. As X is proper, f is proper. We now claim that for all $y \in \mathbf{P}^n_k$, the fiber X_y has finitely many points. If not, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathbf{P}^n_k \\
X_y \arrow[u, hook] \arrow[r] & \mathrm{Spec}(k(y)) \arrow[u, hook] \\
C \arrow[u, hook] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the top square is cartesian, and C is a complete curve in X_y. By commutativity of the diagram, we see that

$$\mathcal{L}|_C = (f^*\mathcal{O}_{\mathbf{P}^n}(1))|_C \simeq \pi^*\mathcal{O}_{\mathrm{Spec}(k(y))} = \mathcal{O}_C,$$

which is a contradiction, since $(\mathcal{L}^{\dim X-1}\cdot C) > 0$ by (2), whereas $\mathcal{O}_C$ has constant Euler characteristic. Thus, f is quasi-finite; since it is proper by the fact that both X and $\mathbf{P}^n_k$ are proper, we moreover have that f is finite by [14, Tag 02OG]. Since the pullback of an ample invertible sheaf via a quasi-finite morphism is ample by [14, Tag 0892] and finite morphisms are quasi-finite by definition, $\mathcal{O}_X(mD)$ is ample for $m \gg 0$. Hence $\mathcal{O}_X(D)$ is ample by [14, Tag 01PS]. $\square$

7. Numerical intersection theory for spaces

For the proof of the Nakai–Moishezon critera for algebraic spaces, we will need to enhance [14, Tag 0DN0] and [14, Tag 0DN3] with generalizations of results from [14, Tag 0BEI] and [14, Tag 0BEL].

To begin, we require a generalization of [14, Tag 0BEK].

Lemma 7.1. *Let k be a field. Let $f : Y \to X$ be a morphism of proper algebraic spaces over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then*

$$\chi(Y,\mathcal{G}) = \sum(-1)^i\chi(X, R^if_*\mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^if_*\mathcal{G}$ are coherent and only a finite number of them are nonzero, see [14, Tag 08AR] and [14, Tag 073G]. By the Leray spectral sequence [14, Tag 0732] there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^qf_*\mathcal{G})$$

converging to $H^{p+q}(Y,\mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum(-1)^{p+q}\dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y,\mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p+1 = n$ (this is the meaning of convergence of the spectral sequence), we conclude. $\square$

This is a generalization of [14, Tag 0BET] to algebraic spaces.

Lemma 7.2. *Let k be a field. Let $f : Y \to X$ be a morphism of proper algebraic spaces over k. Let $Z \subset Y$ be an integral closed subspace of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ *is as in* [14, Tag 0AD6] *or* 0 *if* $\dim(f(Z)) < d$.[15]

8. Nakai–Moishezon for spaces

Lemma 8.1. *Let R be a Noetherian ring. Let X and Y be algebraic spaces over R. Let $f : Y \to X$ be a proper morphism of algebraic spaces over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume f is finite and surjective. Then $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample.*

Proof. Suppose that $\mathcal{L}$ is ample. Then $X \to \mathrm{Spec}(R)$ is representable and hence X is a scheme. But $f : Y \to X$ is finite and hence affine, so, by [14, Tag 03WG], f is representable. Therefore Y is a scheme. Then $f^*\mathcal{L}$ is ample by the schemes case, [14, Tag 0B5V].

Assume that $f^*\mathcal{L}$ is ample. Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which suffices to prove that $\mathcal{L}$ is ample. We are going to apply [14, Tag 07UT]. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be reduced closed subspace with $|Z|$ irreducible. Form the fibre product diagram

$$\begin{tikzcd} W \arrow[r, "i'"'] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

Let $\mathcal{G} = f'_*\mathcal{O}_W$. Since f' is surjective, part (3)(a) of [14, Tag 07UT] holds. For part (3)(b), let $\mathcal{I} \subset \mathcal{O}_Z$ be a nonzero sheaf of ideals. Let $\mathcal{G}' = \mathcal{I}\mathcal{G}$. We have

$$\mathcal{I}\mathcal{G} = f'_*(\mathcal{I}')$$

where $\mathcal{I}' = \mathrm{Im}((f')^*\mathcal{I} \to \mathcal{O}_W)$. This is true because f' is a (representable) affine morphism of algebraic spaces and hence the result can be checked on an étale covering of Z by a scheme in which case the result is [14, Tag 01YP]. Finally, f' is affine, hence $R^p f'_*\mathcal{I}' = 0$ for all $p > 0$ by [14, Tag 073H]. Hence

$$\begin{aligned} H^p(X, \mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(X, i_* f'_*\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \\ &= H^p(X, f_*(i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, i'_*\mathcal{I}' \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

since $f^*\mathcal{L}$ is ample. This verifies (3)(c) of [14, Tag 07UT] as desired. □

[15]**TODO.** Prove this.

Lemma 8.2 (Nakai–Moishezon Criterion for algebraic spaces)**.** *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample on X with respect to k if and only if for every integral closed subspace Y of X, $(\mathcal{L}^{\dim(Y)} \cdot Y) > 0$.*

Proof. Assume that $\mathcal{L}$ is ample on X with respect to k. Then X is a scheme, and $\mathcal{L}$ is ample on X/k in the scheme-theoretic sense by [14, Tag 0D32]. Now [14, Tag 0BEV] implies the positivity of interesction numbers.

Suppose that for every integral closed subspace Y of X, $(\mathcal{L}^{\dim(Y)} \cdot Y) > 0$. We show that $\mathcal{L}$ is ample on X with respect to k. By [14, Tag 09YC], there is a finite surjective map $p : X' \to X$ from a scheme X'. By Lemma 8.1, $\mathcal{L}$ is ample if and only if $p^*\mathcal{L}$ is ample. Thus, by the Nakai–Moishezon criterion for schemes (Lemma 6.1), it suffices to show that for every integral closed subscheme Y' in X', we have $(p^*\mathcal{L}^{\dim(Y')} \cdot Y') > 0$. Let Y be the image of Y' in Y; by Lemma 7.2, we then have

$$((p^*\mathcal{L})^{\dim Y'} \cdot Y') = \deg(p|_{Y'} : Y' \to Y)(\mathcal{L}^{\dim Y'} \cdot Y).$$

Note $\dim Y' = \dim Y$ since $p|_{Y'} : Y' \to Y$ is finite surjective. Since $(\mathcal{L}^{\dim Y} \cdot Y) > 0$ by (2), we therefore see that $((p^*\mathcal{L})^{\dim Y'} \cdot Y') > 0$. □

9. Ampleness Lemma

In this section, we formulate and prove the Ampleness Lemma of [9, Lemmas 3.9 and 3.13].

Lemma 9.1. *Let k be a field. Let X be a scheme over k. Let $\mathcal{E}$ be a locally free sheaf of rank n. Let $\mathbf{P} = \mathbf{P}((\mathcal{E}^\vee)^{\oplus n})$. Let $\pi : \mathbf{P} \to X$ be the structure map. Then there exists a closed subscheme $D \subset \mathbf{P}$ such that the the restriction*

$$\pi : \mathbf{P} \setminus D \to X$$

exhibits $\mathbf{P} \setminus D$ as a PGL_n-torsor over X.

Proof. First we should describe the PGL_n action on $\mathbf{P}$. Let T be a scheme over k and $x : T \to \mathbf{P}$ be a T-point. By [14, Tag 01NK], this corresponds to a morphism $f : T \to X$ of k-schemes together with a surjection

$$\varphi = (\varphi_1, \ldots, \varphi_n) : f^*(\mathcal{E}^\vee)^{\oplus n} \to \mathcal{L},$$

where $\mathcal{L}$ an invertible sheaf on T. Let $g \in \mathrm{PGL}_n(T)$ and pick a representative matrix $(g_{ij}) \in \mathrm{GL}_n(T)$ for g. Consider the morphism

$$g \cdot \varphi = \left(\sum\nolimits_j g_{1j}\varphi_j, \ldots, \sum\nolimits_j g_{nj}\varphi_j \right) : f^*(\mathcal{E}^\vee)^{\oplus n} \to \mathcal{L}$$

of $\mathcal{O}_T$-modules. Since φ is surjective and (g_{ij}) is invertible, $g \cdot \varphi$ is surjective. Therefore $g \cdot \varphi$ defines a T-point of $\mathbf{P}$. This is $g \cdot x$. Note that $g \cdot x$ is independent of the representative (g_{ij}) of g as a different representative of g differs by an element of $\mathbf{G}_m(T)$ and the resulting morphism differs by an automorphism of $\mathcal{L}$, so the resulting T-point of $\mathbf{P}$ will be the same.

Write $\mathbf{P} = \mathbf{P}((\mathcal{E}^\vee)^{\oplus n})$ and let $\pi : \mathbf{P} \to X$ be the structure morphism. Using the projection formula [14, Tag 01E6],

$$\begin{aligned} H^0(\mathbf{P}, \pi^*\mathcal{E}(1)) &\cong H^0(X, \pi_*(\pi^*\mathcal{E} \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_\mathbf{P}(1))) \\ &\cong H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_\mathbf{P}(1)) \\ &\cong H^0(X, \mathcal{E} \otimes (\mathcal{E}^\vee)^{\oplus n}) \\ &\cong H^0(X, \mathrm{End}(\mathcal{E})^{\oplus n}) \end{aligned}$$

has n sections, corresponding to the identity on each summand. We therefore have an evaluation morphism

$$B : \mathcal{O}_\mathbf{P}^{\oplus n} \longrightarrow \pi^*\mathcal{E}(1).$$

Taking determinants, this gives a map

$$\det(B) : \mathcal{O}_\mathbf{P} \longrightarrow \det(\pi^*\mathcal{E})(n).$$

Let $\mathcal{I} \otimes \det(\pi^*\mathcal{E})(n)$ be the image of $\det B$, and let D be the subscheme of $\mathbf{P}$ defined by $\mathcal{I}$. Then, D defines the locus for which $\det B$ vanishes, which is the locus of $\mathbf{P}$ where the corresponding matrices are not invertible. We therefore see that $\mathbf{P} \setminus D$ corresponds to those matrices that are invertible, i.e., are elements of PGL_n. This turns $\mathbf{P} \setminus D$ into a principal PGL_n-bundle over X.

□

Lemma 9.2. *Let X be a scheme over a field k. Let $\mathcal{E}$ be a locally free sheaf on X of rank n. Let $\mathcal{F} = \mathrm{Sym}^d(\mathcal{E})$ for some $d \geq 1$ and let $N = \binom{n+d}{d}$ be the rank. Let $\mathcal{F} \to \mathcal{Q}$ be a locally free quotient of rank q. Then the following hold:*

1. *There exists a closed subscheme $D \subset \mathbf{P}((\mathcal{E}^\vee)^{\oplus n})$ such that*
$$\mathbf{P}((\mathcal{E}^\vee)^{\oplus n}) \setminus D \longrightarrow X$$
is a principal PGL_n-bundle.
2. *There is a PGL_n-action on $\mathbf{G}(q, N)$, and a PGL_n-equivariant map*
$$\mathbf{P}((\mathcal{E}^\vee)^{\oplus n}) \setminus D \longrightarrow \mathbf{G}(q, N).$$
3. *There exists a map of stacks*
$$X \longrightarrow [\mathbf{G}(q, N)/\mathrm{PGL}_n].$$

Proof. We first prove (1). For (2), note that PGL_n acts on $\mathbf{G}(q, N)$ via

$$\mathrm{GL}_n \longrightarrow (\mathrm{GL}_n)^{\times d} \longrightarrow \mathrm{Aut}(\mathrm{Sym}^d(k^{\oplus n})),$$

which descends to their quotients by k^*; note that $\mathrm{Aut}(\mathrm{Sym}^d(k^{\oplus n}))/k^*$ is a subgroup of the automorphism group of the Grassmannian $\mathbf{G}(q, N)$. To show the morphism claimed exists, we fist consider what happens when we apply Sym^d to B, giving a morphism

$$\mathrm{Sym}^d(B) : \mathcal{O}_\mathbf{P}^{\oplus N} \to \pi^*\mathrm{Sym}^d(\mathcal{E})(d) = \pi^*\mathcal{F}(d).$$

Composing this will the given surjection $\mathcal{F} \to \mathcal{Q}$, we obtain a morphism

$$U : \mathcal{O}_\mathbf{P}^{\oplus N} \to \pi^*\mathcal{F}(d) \to \pi^*\mathcal{Q}(d)$$

which is surjective away from D. By the universal property of the Grassmannian, we obtain a morphism $\mathbf{P} \setminus D \to \mathbf{G}(q, N)$. It is G-equivariant by tracing the definitions.

For (3), this follows from general properties of quotient stacks, plus the fact that $(\mathbf{P} \setminus D)/\mathrm{PGL}_n \simeq X$. □

Lemma 9.3. *Let X be a normal projective variety over a field k. Let $\mathcal{E}$ be a nef locally free sheaf on X of rank n. Set $\mathcal{F} = \mathrm{Sym}^d(\mathcal{E})$ for some $d \geq 1$ and set $N = \binom{n+d}{d}$ be the rank. Let $\mathcal{Q}$ be a locally free quotient of $\mathcal{F}$ of rank q. Assume that the map*

$$u : X \to [\mathbf{G}(q, N)/G]$$

constructed in Lemma 9.2 is generically quasi-finite. Then $(\det(\mathcal{Q})^{\dim(Y)}) > 0$.

Proof. Let $\mathbf{P} = \mathbf{P}((\mathcal{E}^\vee)^{\oplus n})$. Let $D \subset \mathbf{P}$ be the closed subscheme given by part (1) of Lemma 9.2. Then there is a fibre product diagram[16]

$$\begin{tikzcd} \mathbf{P} \setminus D \arrow[r] \arrow[d] & \mathbf{G}(q, N) \arrow[d] \\ X \arrow[r, "u"] & {[\mathbf{G}(q, N)/\mathrm{PGL}_m]} \end{tikzcd}$$

Since u is generically quasi-finite, the map $\mathbf{P} \setminus D \to \mathbf{G}(q, N)$ is generically quasi-finite.

As in Lemma 9.2,[17] the morphism $\mathbf{P} \setminus D \to \mathbf{G}(q, N)$ is defined by the morphism of locally free sheaves

$$U : \mathcal{O}_{\mathbf{P}}^{\oplus n} \to \pi^*\mathcal{F}(d) \to \pi^*\mathcal{Q}(d).$$

Here, the map $\mathcal{O}_{\mathbf{P}}^{\oplus N} \to \pi^*\mathcal{F}$ is an isomorphism away from D and thus U is surjective away from D. Taking q^{th} exterior powers, we obtain a morphism

$$\bigwedge^q U : \mathcal{O}_{\mathbf{P}}^{\oplus \binom{N}{q}} \to \pi^* \det(\mathcal{Q})(qd)$$

which is surjective away from D. Thus the image of $\bigwedge^q U$ is of the form $\pi^* \det(\mathcal{Q})(qd) \otimes \mathcal{I}$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbf{P}}$.

Let $g : \mathbf{P}' \to \mathbf{P}$ be the blow up of $\mathbf{P}$ along the sheaf $\mathcal{I}$. Then the rational map from $\mathbf{P}$ to $\mathbf{G}(q, N)$ defined above resolves to a morphism

$$u' : \mathbf{P}' \to \mathbf{G}(q, N)$$

such that

$$g^*\pi^* \det(\mathcal{Q})(qd) \cong u'^*\mathcal{O}_{\mathbf{G}(q,N)}(1) \otimes_{\mathcal{O}_{\mathbf{P}'}} \mathcal{O}_{\mathbf{P}'}(E)$$

where E is the exceptional divisor of $g : \mathbf{P}' \to \mathbf{P}$.

Since u is generically quasi-finite and g is birational, u' is generically quasi-finite. Hence, by ...[18] $u'^*\mathcal{O}_{\mathbf{G}(q,N)}(1)$ is big[19] on $\mathbf{P}'$. Fix an ample invertible sheaf $\mathcal{L}$ on Y. Then by ...[20], there is some $m \geq 0$ such that $\mathcal{O}_{\mathbf{G}(q,N)}(m) \otimes_{\mathcal{O}_{\mathbf{P}'}} g^*\pi^*\mathcal{L}^{-1}$ has a nonzero section. But

$$\mathcal{O}_{\mathbf{G}(q,N)}(m) = \left(g^*\pi^* \det(\mathcal{Q})(qd) \otimes_{\mathcal{O}_{\mathbf{P}'}} \mathcal{O}_{\mathbf{P}'}(-E)\right)^{\otimes m}.$$

Thus we obtain a nonzero morphism of sheaves

$$\mathcal{O}_{\mathbf{P}'} \to \left(g^*\pi^* \det(\mathcal{Q})(qd) \otimes_{\mathcal{O}_{\mathbf{P}'}} \otimes_{\mathcal{O}_{\mathbf{P}'}} \mathcal{O}_{\mathbf{P}'}(-E)\right)^{\otimes m} \otimes_{\mathcal{O}_{\mathbf{P}'}} g^*\pi^*\mathcal{L}^{-1}$$

[16]**TODO.** Why, exactly, is this a fibre product diagram? Also, the u is off.
[17]**TODO.** Maybe factor this statement out?
[18]**TODO.** Prove this or reference this somewhere.
[19]**TODO.** Define big somewhere
[20]**TODO.** Using properties of big. I guess this depends on what the definition is and then what the characterizations are.

for all $m \geq 0$ large. Since E is effective, we can compose this morphism with the natural inclusion $\mathcal{O}_{\mathbf{P}'}(-E) \to \mathcal{O}_{\mathbf{P}'}$ to obtain a nonzero morphism

$$\mathcal{O}_{\mathbf{P}'} \to g^*\big(\pi^* \det(\mathcal{Q})^{\otimes m}(mqd) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^*\mathcal{L}^{-1}\big).$$

Pushing this forward to X and applying the projection formula, we obtain a nonzero morphism

$$\mathcal{O}_X \to \det(\mathcal{Q})^{\otimes m} \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_{\mathbf{P}}(mqd) \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}.$$

Since each term on the right is locally free, this is equivalent to a nonzero morphism

$$\tau : (p_*\mathcal{O}_{\mathbf{P}}(mqd))^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \to \det(\mathcal{Q})^{\otimes m}.$$

As before, there exists an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that the image of τ is the subsheaf $\mathcal{J} \otimes_{\mathcal{O}_X} \det(\mathcal{Q})^{\otimes m}$ of $\det(\mathcal{Q})^{\otimes m}$. Let $s : X' \to X$ be the blow up of X along $\mathcal{J}$. Set $P = \mathbf{P}((\pi_*\mathcal{O}_{\mathbf{P}}(mqd))^\vee \otimes_{\mathcal{O}_X} \mathcal{L})$. Then τ gives rise to a morphism $v : X' \to P$ such that

$$v^*\mathcal{O}_P(1) = s^* \det(\mathcal{Q})^{\otimes m} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-F)$$

for the exceptional divisor F of the blow up $s : X' \to X$. Set $P' = \mathbf{P}((\pi_*\mathcal{O}_{\mathbf{P}}(mqd))^\vee)$. Then there exists a canonical isomorphism[21] $P \to P'$. Let $v' : X' \to P'$ be the composition of v with this isomorphism. Then $\mathcal{L} \otimes_{\mathcal{O}_{X'}} v'^*\mathcal{O}_{P'}(1) = v^*\mathcal{O}_P$, so

$$s^* \det(\mathcal{Q})^{\otimes m} = s^*\mathcal{L} \otimes_{\mathcal{O}_{X'}} v'^*\mathcal{O}_{P'}(1) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(F).$$

We now compute the top self-intersection of $\det(\mathcal{Q})^{\otimes m}$. Since $s : X' \to X$ is birational, by [14, Tag 0BET],

$$((\det(\mathcal{Q})^{\otimes m})^{\dim(X)}) = ((s^* \det(\mathcal{Q})^{\otimes m})^{\dim(X')}).$$

Write $\mathcal{L}' = v'^*\mathcal{O}_{P'}(1) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(F)$. By [14, Tag 0BER] we further have

$$((\det(\mathcal{Q})^{\otimes m})^{\dim(X)}) = (s^*\mathcal{L}^{\dim(X)}) + \sum_{i=1}^{\dim(X)} (s^*\mathcal{L}^{\dim(X)-i}(s^* \det(\mathcal{Q})^{\otimes m})^{i-1}\mathcal{L}').$$

Since $(\pi_*\mathcal{O}_{\mathbf{P}}(mqd))^\vee$[22] is nef by Lemma 4.13, $\mathcal{O}_{P'}(1)$ is nef and hence its v' pullback to X' is nef. As F is effective, $\mathcal{L}'$ is nef. Since $\mathcal{L}$ is ample, $s^*\mathcal{L}$ is nef, so $s^* \det(\mathcal{Q})^{\otimes m}$ is nef. All this shows that the second term in the sum above is nonnegative. Applying [14, Tag 0BET] again,

$$(s^*\mathcal{L}^{\dim(X)}) = (\mathcal{L}^{\dim(X)}) > 0$$

since $\mathcal{L}$ is ample on X. Thus $((\det(\mathcal{Q})^{\otimes m})^{\dim(X)}) > 0$, so by additivity of numerical intersections again, $(\det(\mathcal{Q})^{\dim(X)}) > 0$. □

Lemma 9.4. *Let k be a field. Let X be a proper algebraic space over k. Let $\mathcal{E}$ be a nef locally free sheaf of rank n on X. Set $\mathcal{F} = \mathrm{Sym}^d(\mathcal{E})$ for some $d \geq 1$ and let $N = \binom{n+d}{d}$ be the rank. Let $\mathcal{Q}$ be a locally free quotient of $\mathcal{F}$ of rank q. Assume that the map*

$$u : X \to [\mathbf{G}(q, N)/\mathrm{PGL}_n]$$

constructed in Lemma 9.2 is generically quasi-finite. Then $\det(\mathcal{Q})$ *is ample.*

[21] **TODO.** Find a reference for this.

[22] **TODO.** Explain why this is of the required form.

Proof. By the Nakai–Moishezon Criterion for spaces, Lemma 8.2, we must show that $\det(Q)$ has positive top intersection with every irreducible subspace $Z \subset X$. So let $Z \subset X$ be any irreducible subspace and write $i : Z \to X$ for the inclusion. By [14, Tag 088U] there is a projective scheme Z' and a proper birational morphism $g : Z' \to Z$. Now

$$(\det(Q)^{\dim(Z)} \cdot Z) = (\det(i^*Q)^{\dim(Z)}) = (\det(g^*i^*Q)^{\dim(Z')})$$

so we need to show that the top self-intersection of $\det(g^*i^*\mathcal{Q})$ on Z' is positive. But now $g^*i^*\mathcal{E}$ is nef by Lemma **??**, $g^*i^*\mathcal{F} \cong \mathrm{Sym}^d(g^*i^*\mathcal{E})$, and $g^*i^*\mathcal{Q}$ is a quotient bundle of $g^*i^*\mathcal{F}$ of rank q on Z'. Also, the map $u \circ i \circ g : Z' \to [\mathbf{G}(q,N)/\mathrm{PGL}_n]$ is quasi-finite on the dense open on which g is an isomorphism. Thus Lemma 9.3, $(\det(g^*i^*\mathcal{Q})^{\dim(Z')}) > 0$ and we are done. □

10. Nefness results

In this section, we prove a nefness result, Lemma 10.8, for the pushforward of the relative dualizing sheaf of families of curves.

Situation 10.1. Let k be a field. Let C be a connected, proper smooth k-scheme of dimension 1. Let S be a connected, proper smooth k-scheme of dimension 2. Let $f : S \to C$ be a surjective morphism of k-schemes. Assume that the general fiber of f is smooth (over its residue field) of genus at least 2. Let $m \geq 2$ be an integer.

Lemma 10.2. *Suppose we are in Situation 10.1. Then, $f_*\omega_{S/C}^{\otimes m}$ is an invertible sheaf on C.*

Proof. Immediate from Grauert's Theorem and Riemann-Roch. □

Lemma 10.3. *Let S be a connected, proper smooth k-scheme of dimension 2. Assume that S is a surface of general type. Let $m \geq 2$ be an integer. Let $\mathcal{L}$ be an invertible sheaf on S that is effective. Assume that $\mathcal{L}^2 = 0$. Then, we have $H^1(\omega_X^{\otimes m} \otimes \mathcal{L}) = 0$ if $\mathrm{char}(k) \neq 2$ and $H^1(\omega_X^{\otimes m} \otimes \mathcal{L}) = 0 \leq 1$ if $\mathrm{char}(k) = 2$.*

Proof. Let $D \subset S$ be an effective divisor such that $\mathcal{L} \cong \mathcal{O}_S(D)$. Then we get an exact sequence

$$0 \to \omega_S^{\otimes m} \to \omega_S^{\otimes m} \otimes \mathcal{L} \to (\omega_S^{\otimes m} \otimes \mathcal{L})|_D \to 0.$$

From the long exact sequence for cohomology we get

$$H^1(S, \omega_S^{\otimes m}) \to H^1(S, \omega_S^{\otimes m} \otimes \mathcal{L}) \to H^1(S, (\omega_S^{\otimes m} \otimes \mathcal{L})|_D)$$

By assumption, $D^2 = 0$, so

$$(mK_S + D)|_D = (mK_S + mD)|_D = mK_D,$$

where the last equality follows from the adjunction formula. Then for $m \geq 2$ we get

$$H^1(S, (\omega_S^{\otimes m} \otimes \mathcal{L})|_D) \cong H^1(D, \omega_D^{\otimes m}) \cong H^0(D, \omega_D^{\otimes(1-m)})^\vee = 0$$

by Serre Duality.

Therefore, the result follows from the analogous result of Ekedahl [23]for $H^1(S, \omega_S^{\otimes m})$. □

[23]**TODO.** add ref

Lemma 10.4. *Suppose we are in Situation 10.1. Assume the characteristic of k is equal to $p > 0$. Assume that S is minimal, i.e. it has no (-1)-curves. Assume that the genus of C is at least 2. Then, $f_*\omega_{S/C}^{\otimes m}$ is nef for $m \geq 2$.*

Proof. First, observe that S is of general type. [24]

Suppose there exists a surjection of invertible sheaves

$$f_*(\omega_{S/C}^{\otimes m}) \to \mathcal{M}^{-1},$$

where $\mathcal{M}$ an invertible sheaf and $\deg \mathcal{M} = d < 0$. Let $F_S : S \to S$ and $F_C : C \to C$ denote the absolute Frobenius morphisms By functoriality for the relative dualizing sheaf,

$$F_C^* f_*(\omega_{S/C}^{\otimes m}) \cong f_* F_S^*(\omega_{S/C}^{\otimes m}) = f_*(\omega_{S/C}^{\otimes m})$$

Thus, $f_*(\omega_{S/C}^{\otimes m})$ has the quotient invertible sheaf $F^*\mathcal{M}^{-1}$ with $\deg F^*\mathcal{M}^{-1} = dp$, and we can replace $\mathcal{M}$ with $F^*\mathcal{M}^{-1}$. Hence, we can assume that $\deg \mathcal{M} = d >> 0$, and assume that $\mathcal{M} \cong \omega_C^{\otimes m} \otimes \mathcal{L}$ where $\mathcal{L}$ is very ample.

We therefore have a surjection of sheaves

$$\omega_C^{\otimes m} \otimes \mathcal{L} \otimes f_*(\omega_{S/C}^{m}) \to \mathcal{O}_C \to 0$$

which yields the surjection

$$H^1(C, \omega_C^{\otimes m} \otimes \mathcal{L} \otimes f_*(\omega_{S/C}^{\otimes m})) \to H^1(\mathcal{O}_C) \to 0$$

after applying the long exact sequence for cohomology, as C has dimension 1.

By Serre Duality, we have

$$\dim_k H^1(\mathcal{O}_C) \cong \dim_k H^0(\omega_C) = g.$$

Therefore,

$$\dim_k H^1(C, \omega_C^{\otimes m} \otimes \mathcal{L} \otimes f_*(\omega_{S/C}^{\otimes m})) \geq g$$

Let $\mathcal{F} = f^*(\omega_C^{\otimes m} \otimes \mathcal{L}) \otimes \omega_{S/C}^{\otimes m} \cong \omega_S^{\otimes m} \otimes f^*\mathcal{L}$ by [25]. We have the Leray Spectral Sequence

$$E_2^{p,q} \quad H^p(C, R^q f_*\mathcal{F}) \implies H^{p-q}(S, \mathcal{F}),$$

whose only non-zero entries on the E_2-page are $(p, q) = (0, 0), (0, 1), (1, 0), (1, 1)$, and thus degenerates on the E_2 page.

Hence, we have the short exact sequence

$$0 \to H^1(C, f_*\mathcal{F}) \to H^1(S, \mathcal{F}) \to H^0(C, R^1 f_*\mathcal{F}) \to 0$$

By the projection formula,

$$f_*\mathcal{F} \cong \mathcal{L} \otimes f_*(\omega_{S/C}^{\otimes m}) \otimes \omega_C^m$$

Thus, $\dim_k H^1(C, f_*\mathcal{F}) \geq g$, from which we conclude

$$\dim_k H^1(S, \mathcal{F}) = \dim_k H^1(S, \omega_S^{\otimes m} \otimes f^*\mathcal{L}) \geq g \geq 2.$$

However, the invertible sheaf $f^*\mathcal{L}$ is effective on S because $\mathcal{L}$ is very ample on C, and has self-intersection zero because it is a union of fibers, so this contradicts Lemma 10.3. □

[24]**TODO.** why?
[25]**TODO.** add ref

Lemma 10.5. *Suppose we are in Situation 10.1. Assume the characteristic of k is equal to $p > 0$. Assume that the genus of C is at least 2. Then, $f_*\omega_{S/C}^{\otimes m}$ is nef for $m \geq 2$.*

Proof. We will reduce to the case in which S is minimal, then apply Lemma 10.4.

Suppose that S contains a (-1)-curve C'. Then, if C' is not contained in a fiber of f, then $f|_{C'} : C' \to C$ is a dominant morphism from a curve of genus 0 to a curve of genus 2, which is a contradiction. Thus, all (-1)-curves of S are contained in fibers of f. Thus, we have a k-smooth minimal model S_{min} along with k-morphisms $b : S \to S_{min}$ and $f' : S_{min} \to C$ such that $f = f' \circ b$.

The morphism f' satisfies the hypotheses of Lemma 10.4, so $f'_*(\omega_{S_{min}/C}^{\otimes m})$ is nef. Then

$$\begin{aligned} f_*(\omega_{S/C}^{\otimes m}) &\cong f'_*b_*(\omega_{S/C}^{\otimes m}) \cong f'_*b_*(\omega_{S/S_{min}}^{\otimes m} \otimes b^*\omega_{S_{min}/C}^{\otimes m}) \\ &\cong f'_*(b_*\omega_{S/S_{min}}^{\otimes m} \otimes \omega_{S_{min}/C}^{\otimes m}) \cong f'_*\omega_{S_{min}/C}^{\otimes m} \end{aligned}$$

because $b_*\omega_{S/S_{min}}^{\otimes m} \cong \mathcal{O}_{S_{min}}$.[26] The conclusion follows. ☐

Lemma 10.6. *Suppose we are in Situation 10.1. Assume the characteristic of k is equal to $p > 0$. Then, $f_*\omega_{S/C}^{\otimes m}$ is nef for $m \geq 2$.*

Proof. [27] We will reduce to the case in which the genus of C is at least 2, then apply Lemma 10.4.

Because f is generically smooth, there exists a connected, smooth proper k-scheme C' of dimension 1 and a generically étale k- morphism $f : C' \to C$ branched only at points of C above which the fiber of f is smooth[28]. We then have a fiber diagram

$$\begin{tikzcd} S & S' \arrow[l, "\pi'"'] \arrow[d, "f'"] \\ C \arrow[from=u, "f"'] & C' \arrow[l, "\pi"] \end{tikzcd}$$

and $f' : S' \to C'$ satisfies the hypotheses of Lemma 10.4, so $f'_*(\omega_{S'/C'})$ is nef.

By Cohomology and Base Change and compatibility for relative dualizing sheaves, we have

$$\pi^*(f_*(\omega_{S/C})) \cong f'_*\pi'^*(\omega_{S/C}) \cong f'_*(\omega_{S'/C'}).$$

Now, the locally free sheaf $f_*(\omega_{S/C})$ becomes nef after pullback, and is therefore nef. ☐

Compare the following with [9, Theorem 4.3].

Lemma 10.7. *Suppose we are in situation Lemma 10.1. Then $f_*(\omega_{S/C}^{\otimes m})$ is nef for $m \geq 2$.*

Proof. By Lemma 10.6, we may assume that k has characteristic 0. Spread out. [29] ☐

Compar the following with [9, Theorem 4.6].

[26]**TODO.** this is probably in the stacks project somewhere, find ref
[27]**TODO.** get rid of the beginning of this and cite **??** instead – that needs to be moved up
[28]**TODO.** add ref
[29]**TODO.** add details

Lemma 10.8. *Let k be a field. Let C be a proper connected smooth k-scheme of dimension 1. Let $f : S \to C$ be a family of stable curves, as defined in Lemma* **??**. *Let C_t be a section of f. Then $\omega_{S/C} \cdot C_t \geq 0$.*

Proof. Let $i : C_t \hookrightarrow S$ be the inclusion map, and $g = f \circ i : C_t \to C$ be the corresponding isomorphism. Consider the short exact sequence:

$$0 \to \mathcal{O}_S(-C_t) \to \mathcal{O}_S \to i_*\mathcal{O}_{C_t} \to 0.$$

Tensoring with $\omega_{S/C}^{\otimes m}$, we obtain:

$$0 \to \omega_{S/C}^{\otimes m}(-C_t) \to \omega_{S/C}^{\otimes m} \to i_*\mathcal{O}_{C_t} \otimes \omega_{S/C}^{\otimes m} \to 0.$$

Applying the functor f_* to the sequence, we get a map

$$f_*\omega_{S/C}^{\otimes m} \to f_*(i_*\mathcal{O}_{C_t} \otimes \omega_{S/C}^{\otimes m}) \to R^1f_*(\omega_{S/C}^{\otimes m}(-C_t)).$$

For $m \gg 0$, using cohomology and base change we obtain $R^1f_*(\omega_{S/C}^{\otimes m}(-C_t)) = 0$, so we have a surjection:

$$f_*\omega_{S/C}^{\otimes m} \to f_*(i^*\mathcal{O}_{C_t} \otimes \omega_{S/C}^{\otimes m}) \to 0.$$

Applying the functor g^* to the above sequence, we obtain:

$$g^*f_*\omega_{S/C}^{\otimes m} \to g^*f_*(i^*\mathcal{O}_{C_t} \otimes \omega_{S/C}^{\otimes m}) \to 0.$$

We now claim that the term on the right is $i^*\omega_{S/C}^{\otimes m}$, since

$$\begin{aligned} i^*\omega_{S/C}^{\otimes m} &= g^*g_*(i^*\omega_{S/C}^{\otimes m}) \\ &= g^*f_*i_*(i^*\omega_{S/C}^{\otimes m}) \\ &= g^*f_*(i^*\mathcal{O}_{C_t} \otimes \omega_{S/C}^{\otimes m}). \end{aligned}$$

This way we obtain a contradiction, since the quotient $g^*f_*\omega_{S/C}^{\otimes m} \to i^*\omega_{S/C}^{\otimes m} \to 0$ contradicts nefness. $\square$

We make the following definition, following [9, Definition 4.1(i)].

Definition 10.9. Let k be a field. Let X be an algebraic variety over k. We say X is *semismooth* if all of its closed points are analytically isomorphic to one of the following:

(1) a smooth point;
(2) a double crossing point $\{x_1x_2 = 0\} \subset \mathbf{A}^n$; or
(3) a pinch point $\{x_1^2 - x_2^2x_3 = 0\} \subset \mathbf{A}^n$.

In this case the singular locus is smooth, and we call it the *double divisor* of X.

Situation 10.10. Let k be a field. Let S be a complete Gorenstein reduced k-scheme of dimension 2 that is semismooth. Let C be a complete integral smooth k-scheme of dimension 1. Let $f : S \to C$ be a surjective map onto C, such that the general fiber of f has only nodes as singularities.

Lemma 10.11. *Consider Situation 10.10. Let C' be a complete integral smooth k-scheme of dimension 1, and consider the cartesian square*

$$\begin{tikzcd} S' \arrow[r, "g'"] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ C' \arrow[r, "g"] & C \end{tikzcd}$$

where $g : C' \to C$ is surjective. If for some m, the sheaf $f'_*(\omega_{S'/C'}^{\otimes m})$ is nef, then $f_*(\omega_{S/C}^{\otimes m})$ is nef.

Proof. Suppose $f_*(\omega_{S/C}^{\otimes m})$ is not nef, i.e., there exists a quotient

$$f_*(\omega_{S/C}^{\otimes m}) \longrightarrow \mathcal{L}$$

where $\mathcal{L}$ is an invertible sheaf of negative degree. Pulling back to C', we then obtain a quotient

$$g^* f_*(\omega_{S/C}^{\otimes m}) \longrightarrow g^* \mathcal{L}$$

of $g^* f_*(\omega_{S/C}^{\otimes m})$ that has negative degree on C'. Since f is flat [14, Tag 00R4], we can apply flat base change [14, Tag 02KH] to obtain

$$g^* f_*(\omega_{S/C}^{\otimes m}) \simeq f'_* g'^*(\omega_{S/C}^{\otimes m}) \simeq f'_*(\omega_{S'/C'}^{\otimes m})$$

where the second isomorphism is by the compatibility of the relative dualizing sheaf with pullbacks [14, Tag 0E4P]. We therefore obtain a negative quotient $g^*\mathcal{L}$ of $f'_*(\omega_{S'/C'}^{\otimes m})$, which contradicts the assumption that $f'_*(\omega_{S'/C'}^{\otimes m})$ was nef. $\square$

Compare the following with [9, Theorem 4.3].

Theorem 10.12. *In Situation 10.10, suppose moreover that the generic fiber of f is a stable curve. Then the sheaf $f_*(\omega_{S/C}^{\otimes m})$ is nef for $m \geq 2$.*

Proof. We first claim that we may assume that every double curve in S dominates C. Suppose that a double curve D maps to a point in C. Blowing up D gives a map $b : \widetilde{S} \to S$. Suppose we know the sheaf $(b \circ f)_*(\omega_{\widetilde{S}/C}^{\otimes m})$ is nef. The Grothendieck trace morphism

$$\mathrm{Tr}_b : b_*(\omega_{\widetilde{S}/C}^{\otimes m}) \longrightarrow \omega_{S/C}^{\otimes m}$$

for the blowup is injective, since b is birational, and is an isomorphism away from D [10, Prop. 5.77]. We therefore have a short exact sequence

$$0 \longrightarrow b_*(\omega_{\widetilde{S}/C}^{\otimes m}) \longrightarrow \omega_{S/C}^{\otimes m} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ is supported in D. Pushing this forward Tr_b to the curve C, we then have the left-exact sequence

$$0 \longrightarrow (f \circ b)_*(\omega_{\widetilde{S}/C}^{\otimes m}) \longrightarrow f_*(\omega_{S/C}^{\otimes m}) \longrightarrow f_*\mathcal{Q}.$$

Since D is contained in a fiber over C, the sheaf $f_*\mathcal{Q}$ is a skyscraper sheaf supported at a point on C. We can therefore replace $f_*\mathcal{Q}$ with the subsheaf $\mathcal{Q}'$ making the sequence

$$0 \longrightarrow (f \circ b)_*(\omega_{\widetilde{S}/C}^{\otimes m}) \longrightarrow f_*(\omega_{S/C}^{\otimes m}) \longrightarrow \mathcal{Q}' \longrightarrow 0$$

exact. Since $\mathcal{Q}'$ is a skyscraper sheaf, it has positive degree on C. Thus, assuming $(f \circ b)_*(\omega_{\widetilde{S}/C}^{\otimes m})$ is nef, we have that $f_*(\omega_{S/C}^{\otimes m})$ is also. After repeating finitely many blowups of double curves not dominating C, all remaining double curves will dominate C.

We now claim that we may assume S has no pinch points. If there are pinch points, then blowing them up gives a new surface birational to S that only has double crossing points; the argument of the previous paragraph shows we may reduce to this case, since pinch points are isolated (as in, there cannot be a curve that consists wholely of pinch points).

We now claim that we may assume that every double curve D in S is a section of $f : S \to C$. If it is not, then consider the base change by $g : D \to C$:

$$\begin{tikzcd} S \times_C D \arrow[r, "g'"] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ D \arrow[r, "g"] & C \end{tikzcd}$$

It suffices to show the statement of the Theorem for the morphism f' by Lemma 10.11, and the inverse image of D in $S \times_C D$ is now a section of f' by using the cartesianness of the square:

$$\begin{tikzcd} D \arrow[dr, dashed] \arrow[drr, hook, bend left] \arrow[ddr, equal, bend right] & & \\ & S \times_C D \arrow[r, "g'"] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ & D \arrow[r, "g"] & C \end{tikzcd}$$

Repeating finitely many base changes, we can ensure that every double curve is a section of $f : S \to C$.

We now claim that if $\nu : S' \to S$ is the normalization of S, then the preimage D' of the double curve D still consists of sections. Since the double curve only consists of double crossing points, the normalization (analyticall) locally splits up the double curve into two separate curves on two different irreducible components. These two curves will still be sections.

We have therefore reduced the statement to the situation of Lemma 11.8. $\square$

11. Nefness after twisting by sections

Situation 11.1. Let k be a field. Let C be a connected, proper smooth k-scheme of dimension 1. Let S be a proper smooth k-scheme of dimension 2. Let $f : S \to C$ be a family of stable nodal curves whose fibers have arithmetic genus g. Let $C_1, \ldots, C_n$ be a set of pairwise distinct sections of f. Let $m \geq 2$ be an integer. Let $a_1, \ldots, a_n$ be non-negative integers such that $a_i \leq m$ for each i.

Lemma 11.2. *Suppose we are in Situation 11.1 where $g = 0$ and the sections C_i are pairwise disjoint. Assume that $\sum a_i \leq 2k - 1$. Then $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i)) = 0$.*

Proof. Immediate from the fact that the sheaf $\omega_{S/C}^{\otimes m}(\sum a_i C_i)$ has negative degree on each fiber of f. $\square$

Lemma 11.3. *Suppose we are in Situation 11.1 where $g = 1$ and the sections C_i are pairwise disjoint. Then $f_*(\omega_{S/C}^{\otimes m})$ is a nef locally free sheaf.*

Proof. By Bombieri-Mumford [30], $\omega_{S/C}$ is nef. By Grauert's Theorem, $f_*\omega_{S/C}$ is locally free. Moreover, we can check fiberwise that $f^* f_* \omega_{S/C} \to \omega_{S/C}$ is a surjection, and hence and isomorphism of invertible sheaves. In particular, $f_*\omega_{S/C}$ must be an invertible sheaf of non-negative degree on C, hence nef. Details omitted. $\square$

[30]**TODO.** add reference

Lemma 11.4. *Suppose we are in Situation 11.1 where $g = 0$ and the sections C_i are pairwise disjoint. Then $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ is a nef locally free sheaf.*

Proof. This proof is nearly identical to that in genus 1 and genus at least 2.

We proceed by induction on $\sum a_i$. By the Hodge Index Theorem, we may assume that among $C_1, \ldots, C_n$, C_1 is the only section with positive self-intersection. The base cases where $a_i \leq 2m - 1$ is Lemma 11.2.

Assume the claim is proven for $D_{j-1} = \sum a_i C_i$ where $\sum a_i \geq 2m - 1$; we will prove it for $D_j = D_{j-1} + C_t$. Because $a_1 \leq m$, we may assume $C_1 \neq C_t$. By the adjunction formula, $\omega_{S/C}(C_t)|_{C_t} \cong \mathcal{O}_{C_t}$, so $\omega_{S/C} \cdot C_t = -C_t^2 \geq 0$, as $C_t \neq C_1$.

Consider the exact sequence

$$0 \to \omega_{S/C}^{\otimes m}(D_{j-1}) \to \omega_{S/C}^{\otimes m}(D_j) \to \omega_{S/C}^{\otimes m}(D_j)|_{C_t} \to 0$$

obtained by tensoring the closed subscheme exact sequence for C_t with $\omega_{S/C}^{\otimes m}(D_j)$.

Then, $\omega_{S/C}^{\otimes m}(D_j)|_{C_t} \cong \omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$, because the C_i are pairwise disjoint. Because $a_t + 1 \leq k$ by assumption, this invertible sheaf has non-negative degree.

We now claim that $R^1 f_* \omega_{S/C}^{\otimes m}(D_{j-1})) = 0$. Indeed, by Cohomology and Base Change and the fact that f has relative dimension 1, it suffices to note that $H^1(S_x, \omega_{S_x/\kappa(x)}^{\otimes m}(D_{j-1} \cdot S_x)) = 0$ for each fiber S_x of f, which is clear by Serre Duality and degree considerations.

Thus, applying f_* to (11) expresses $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ as an extension of the positive degree invertible sheaf $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ by the nef locally free sheaf $f_*\omega_{S/C}^{\otimes m}(D_{j-1})$, which is nef by Lemma **??**. Here we have used the fact that C_t is a section, hence $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ is an invertible sheaf on C of the same degree as that of $\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ on C_t. □

Lemma 11.5. *Suppose we are in Situation 11.1 where $g = 1$ and the C_i are pairwise disjoint. Then $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ is a nef locally free sheaf.*

Proof. This proof is nearly identical to that in genus 0 and genus at least 2.

We proceed by induction on $\sum a_i$. The base case where all of the a_i are equal to zero is Lemma 11.3.

Assume the claim is proven for $D_{j-1} = \sum a_i C_i$; we will prove it for $D_j = D_{j-1} + C_t$. By Bombieri-Mumford[31] $\omega_{S/C}$ is nef so in particular, $\omega_{S/C} \cdot C_t \geq 0$.

Consider the exact sequence

$$0 \to \omega_{S/C}^{\otimes m}(D_{j-1}) \to \omega_{S/C}^{\otimes m}(D_j) \to \omega_{S/C}^{\otimes m}(D_j)|_{C_t} \to 0 \tag{11.5.1}$$

obtained by tensoring the closed subscheme exact sequence for C_t with $\omega_{S/C}^{\otimes m}(D_j)$.

Then, $\omega_{S/C}^{\otimes m}(D_j)|_{C_t} \cong \omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$, because the C_i are pairwise disjoint. Because $a_t + 1 \leq k$ by assumption, this invertible sheaf has non-negative degree.

We now claim that $R^1 f_* \omega_{S/C}^{\otimes m}(D_{j-1})) = 0$, unless $D_{j-1} = 0$. Indeed, by Cohomology and Base Change and the fact that f has relative dimension 1, it suffices to note that $H^1(S_x, \omega_{S_x/\kappa(x)}^{\otimes m}(D_{j-1} \cdot S_x)) = 0$ for each fiber S_x of f, which is clear by Serre Duality and degree considerations. On the other hand, when $D_{j-1} = 0$, Cohomology and Base Change shows that $R^1 f_* \omega_{S/C}^{\otimes m}(D_{j-1}))$ is an invertible sheaf.

[31] **TODO.** add reference

Thus, if $D_{j-1} \neq 0$, applying f_* to (11.5.1) expresses $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ as an extension of the positive degree invertible sheaf $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ by the nef locally free sheaf $f_*\omega_{S/C}^{\otimes m}(D_{j-1})$, which is nef by Lemma **??** Here we have used the fact that C_t is a section, hence $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ is an invertible sheaf on C of the same degree as that of $\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ on C_t.

If $D_{j-1} = 0$, the situation is similar; we obtain the exact sequence

$$0 \to f_*(\omega_{S/C}^{\otimes m}) \to f_*((\omega_{S/C}^{\otimes m})(C_t)) \to f_*((\omega_{S/C}^{\otimes m})(C_t))|_{C_t} \to R^1 f_*(\omega_{S/C}^{\otimes m}) \to 0$$

because $R^1 f_*\omega_{S/C}^{\otimes m}(C_t) = 0$. The rightmost map is a surjection of invertible sheaves, hence an isomorphism. Therefore the leftmost map is an isomorphism as well, and we obtain the desired conclusion once again. $\square$

Lemma 11.6. *Suppose we are in Situation 11.1 where $g \geq 2$ and the sections C_i are pairwise disjoint. Then $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ is a nef locally free sheaf.*

Proof. This proof is nearly identical to that in genus 0 and genus 1.

We proceed by induction on $\sum a_i$. The base case where all of the a_i are equal to zero is Lemma [32].

Assume the claim is proven for $D_{j-1} = \sum a_i C_i$; we will prove it for $D_j = D_{j-1} + C_t$. By [33], $\omega_{S/C} \cdot C_t \geq 0$.

Consider the exact sequence

$$0 \to \omega_{S/C}^{\otimes m}(D_{j-1}) \to \omega_{S/C}^{\otimes m}(D_j) \to \omega_{S/C}^{\otimes m}(D_j)|_{C_t} \to 0$$

obtained by tensoring the closed subscheme exact sequence for C_t with $\omega_{S/C}^{\otimes m}(D_j)$.

Then, $\omega_{S/C}^{\otimes m}(D_j)|_{C_t} \cong \omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$, because the C_i are pairwise disjoint. Because $a_t + 1 \leq k$ by assumption, this invertible sheaf has non-negative degree.

We now claim that $R^1 f_*\omega_{S/C}^{\otimes m}(D_{j-1})) = 0$. Indeed, by Cohomology and Base Change and the fact that f has relative dimension 1, it suffices to note that $H^1(S_x, \omega_{S_x/\kappa(x)}^{\otimes m}(D_{j-1} \cdot S_x)) = 0$ for each fiber S_x of f, which is clear by Serre Duality and degree considerations.

Thus, applying f_* to (11) expresses $f_*(\omega_{S/C}^{\otimes m}(\sum a_i C_i))$ as an extension of the positive degree invertible sheaf $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ by the nef locally free sheaf $f_*(\omega_{S/C}^{\otimes m}(D_{j-1})$, which is nef by Lemma **??**. Here we have used the fact that C_t is a section, hence $f_*\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ is an invertible sheaf on C of the same degree as that of $\omega_{S/C}^{\otimes(k-a_t-1)}|_{C_t}$ on C_t. $\square$

Lemma 11.7. *Suppose we are in Situation 11.1. Then $f_*\left(\omega_{S/C}^{\otimes m}\left(\sum a_i C_i\right)\right)$ is nef.*

Proof. Let $b : \widetilde{S}/ \to S$ be a blowup morphism such that the proper transforms C_i' of the C_i are pairwise disjoint, and are sections of $f \circ b$. By Lemma 11.4, Lemma 11.5, and Lemma 11.6, $(f \circ b)_*\omega_{\widetilde{S}/C}^{\otimes m}$ is nef.

The Grothendieck trace morphism

$$\mathrm{Tr}_b : b_*(\omega_{\widetilde{S}/C}^{\otimes m}) \longrightarrow \omega_{S/C}^{\otimes m}$$

[32]**TODO.** insert ref to kollar 4.5

[33]**TODO.** insert ref to our version of kollar 4.6

for the blowup is injective, since b is birational, and is an isomorphism away from D [10, Prop. 5.77]. Tensoring with $\mathcal{O}_S\left(\sum a_iC_i\right)$ and applying the projection formula yields an injection

$$\mathrm{Tr}_b' : b_*\left(\omega_{\widetilde{S}/C}^{\otimes m} \otimes \mathcal{O}_{\widetilde{S}}\left(\sum a_i b^* C_i\right)\right) \longrightarrow \omega_{S/C}^{\otimes m}\left(\sum a_i C_i\right).$$

Therefore, we have an injection

$$\mathrm{Tr}'' : (f \circ b)_*\left(\omega_{\widetilde{S}/C}^{\otimes m} \otimes \mathcal{O}_{\widetilde{S}}\left(\sum a_i C_i'\right)\right) \longrightarrow f_*\left(\omega_{S/C}^{\otimes m}\left(\sum a_i C_i\right)\right)$$

after precomposing with the inclusions of C_i' into b^*C_i and applying f_*.

Because b is a blowup of points on S, Tr'' is generically an isomorphism. We thus conclude that an invertible quotient of $\mathcal{L}$ of $f_*\left(\omega_{S/C}^{\otimes m}\left(\sum a_iC_i\right)\right)$ has a subsheaf that is a quotient of $(f \circ b)_*\left(\omega_{\widetilde{S}/C}^{\otimes m} \otimes \mathcal{O}_{\widetilde{S}}\left(\sum a_iC_i'\right)\right)$, and thus must have non-negative degree by the nefness of $(f \circ b)_*\left(\omega_{\widetilde{S}/C}^{\otimes m} \otimes \mathcal{O}_{\widetilde{S}}\left(\sum a_iC_i'\right)\right)$. This completes the proof. $\square$

Compare with [9, Theorem 4.9].

Lemma 11.8. *Let k be a field. Let C be a proper connected smooth k-scheme of dimension 1. Let S be a semismooth surface. Let $f : S \to C$ be a surjective map whose general fiber is a stable curve. Let D be a double curve on S. Assume D consists of section of $f : S \to C$. Let $g : S' \to S$ be the normalization of S. Let $D' = g^{-1}(D)$. Assume each component of D' is a section. Let $m \geq 2$ any integer. Then $f_*(\omega_{S/C}^{\otimes m})$ is nef.*

Proof. By 11.1[34] we know that $(f \circ g)_*(\omega_{S'/C}^{\otimes m})$ is nef. Observe that $g^*\omega_{S/C} \cong \omega_{S'/C}(D')$. Now we consider the short exact sequence;

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

Tensoring the sequence with $\omega_{S/C}$ gives

$$0 \to g_*(\omega_{S'/C}) \to \omega_{S/C} \to \omega_{S/C}|_D \cong \mathcal{O}_D \to 0.$$

Now we tensor the sequence with $\omega_{S/C}^{\otimes(m-1)}$ and apply the projection formula, obtaining:

$$0 \to g_*\omega_{S'/C}^{\otimes m}((m-1)D') \to \omega_{S/C}^{\otimes m} \to \mathcal{O}_D \to 0.$$

Now we apply the functor f_* to the sequence and use that $R^1(f \circ g)_*\omega_{S'/C}^{\otimes m}((m-1)D') = 0$ for $m \geq 2$, obtaining the following sequence:

$$0 \to (f \circ g)_*\omega_{S'/C}^{\otimes m}((m-1)D') \to f_*\omega_{S/C}^{\otimes m} \to f_*\mathcal{O}_D \to 0.$$

The sheaf $f_*\mathcal{O}_D$ is isomorphic to the sum of copies of $\mathcal{O}_C$, therefore $f_*\omega_{S/C}^{\otimes m}$ is nef for all $k \geq 2$. $\square$

[34]**TODO.** This appears to be the wrong reference?

12. Projectivity of $\overline{\mathcal{M}_g}$

In this section, we show that the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is projective over $\mathrm{Spec}(\mathbf{Z})$ in the sense that it admits a coarse moduli space which is projective over $\mathrm{Spec}(\mathbf{Z})$. This is done by, first, specializing the ampleness criterion of [9, Theorem 2.6] to the case at hand in Lemma 12.6 to show that $\overline{M}_g$ is projective over any field k. This is then used to show $\overline{M}_g$ is projective over $\mathbf{Z}$ in Theorem 12.7.

Lemma 12.1. *The stacks $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_g$ defined in* [14, Tag 0E77] *admit uniform categorical moduli spaces $f : \overline{\mathcal{M}} \to \overline{M}$ and $f_g : \overline{\mathcal{M}}_g \to \overline{M}_g$ such that f and f_g are separated, quasi-compact, and universal homeomorphisms.*

Proof. The stacks of stable curves have finite inertia by [14, Tag 0E7A] and [14, Tag 0DSW]. The existence of f and f_g now follows from by [14, Tag 0DUT]. □

Definition 12.2. The algebraic space $\overline{M}$ from Lemma 12.1 is called the *coarse moduli space of curves*. Similarly, for $g \geq 2$, the algebraic space $\overline{M}_g$ is called the *coarse moduli space of curves of genus g*.

Lemma 12.3. *Let $\overline{\mathcal{M}}_g$ be the stack of stable curves of genus $g \geq 2$ over* $\mathrm{Spec}(\mathbf{Z})$*. Then for each $m \geq 2$, there exists an invertible sheaf λ_m on $\overline{\mathcal{M}}_g$ such that for every family of curves $f : X \to S$ with classifying map $g : S \to \overline{\mathcal{M}_g}$, $g^*\lambda_m = \det(f_*\omega_{X/S})$.*

Proof. ...[35] □

Lemma 12.4. *Let k be a field. Let $g \geq 2$ and let $\overline{M}_g$ be the coarse moduli space of stable curves of genus g. Then there exists a family of stable curves $f : X \to S$ such that S a finite type scheme over k equipped with a finite surjective map $p : S \to \overline{M}_g$ such that for every $s \in S(\overline{k})$, the point $p(s) \in \overline{M}_g(\overline{k})$ is the moduli point corresponding to the stable curve $f_s : X \times_S \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(\kappa(s))$.*[36]

Proof. ...[37] □

Lemma 12.5. *Let k be a field. Let $g \geq 2$ and let $f : X \to S$ be a family of stable curves of genus g such that the classifying map $S \to \overline{\mathcal{M}}_g$ is finite. Then* $\det(f_*\omega_{X/S})$ *is ample on S.*

Proof. We will apply Lemma 9.4 to the multiplication map

$$\mu : \mathrm{Sym}^d(f_*(\omega_{X/S}^{\otimes m})) \to f_*(\omega_{X/S}^{\otimes md}),$$

for some choice of m and d. Specifically, choose positive integers m and d such that the following hold:

(1) $\omega_{X/S}^{\otimes m}$ is f-very ample,
(2) $R^i f_*(\omega_{X/S}^{\otimes m}) = 0$ for all $i > 0$ and the map $f_*(\omega_{X/S}^{\otimes m})_s \to H^0(X_s, \omega_{X/\kappa(s)})$ is an isomorphism for every $s \in S$,
(3) the multiplication map $\mathrm{Sym}^d(f_*\omega_{X/S}^{\otimes m}) \to f_*\omega_{X/S}^{\otimes md}$ is surjective, and

[35] **TODO.** Prove this at some point. Perhaps [1, §13.2] will be useful. Note that I really need for this to be integral.

[36] **TODO.** State this better.

[37] **TODO.** Prove this. Compare with [9, Proposition 2.7].

(4) for every $s \in S$, the fibre $X_s = X \times_S \operatorname{Spec}(\kappa(s))$ embedded via $\omega^{\otimes m}_{X_s/\kappa(s)}$ is defined by equations of degree at most d[38].

Item (1) can be arranged for by the fact [3, Corollary of (1.2)] that $\omega_{X/S}$ is f-ample together with [14, Tag 01VU]; (2) can be arranged, by [14, Tag 0D2M] and quasi-compactness of S, after possibly taking a larger m; (3) can be arranged by the arguments in the proof of [14, Tag 0C6T]; and (4) can be arranged since pluricanonically embedded curves are defined by quadratic equations[39].

Now we set up our application of Lemma 9.4 to the map μ. Note that, by (3), μ is a quotient map. Moreover, by Lemma 10.12, $f_*\omega^{\otimes md}_{X/S}$ is nef. It remains to check that the classifying map

$$u : S \to [\mathbf{G}(q, N)/\mathrm{PGL}_n]$$

is quasi-finite[40], where $n = 2m(g-1)$, $N = \binom{2m(g-1)+d}{d}$ and $q = 2md(g-1)$. It suffices to check that the fibre S_x of u is a finite set for every geometric point $x : \operatorname{Spec}(\overline{k}) \to [\mathbf{G}(q, N)/\mathrm{PGL}_n]$. By [14, Tag 04UV], x corresponds to a diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"'] \arrow[d] & \mathbf{G}(q, N) \arrow[d] \\ \operatorname{Spec}(\overline{k}) \arrow[r] & \operatorname{Spec}(k) \end{tikzcd}$$

where $P \to \operatorname{Spec}(\overline{k})$ is a PGL_n-torsor, $\varphi : P \to \mathbf{G}(q, N)$ is a PGL_n-equivariant morphism of $\operatorname{Spec}(k)$-schemes. The morphism φ to the Grassmannian[41], by construction [14, Tag 089U], corresponds to an isomorphism class of surjections

$$\mu_x : \mathcal{O}_P^{\oplus N} \to \mathcal{Q}$$

where $\mathcal{Q}$ is a locally free $\mathcal{O}_P$-module of rank q. Thus, by construction of the classifying map from Lemma 9.2, the points $s : \operatorname{Spec}(\overline{k}) \to S$ in the fibre S_x are those where the multiplication map

$$\mu_s : \operatorname{Sym}^d(f_*(\omega^{\otimes m}_{X/S}))_s \to f_*(\omega^{\otimes md}_{X/S})_s$$

is isomorphic to μ_x. By assumption (2), μ_s can be identified as the multiplication map

$$\mu_s : \operatorname{Sym}^d(H^0(X_s, \omega^{\otimes m}_{X_s/\overline{k}})) \to H^0(X_s, \omega^{\otimes md}_{X_s/\overline{k}}).$$

Now, up to the action of PGL_n, we may identify

$$\operatorname{Sym}^d(H^0(X_s, \omega^{\otimes m}_{X_s/\overline{k}})) = \operatorname{Sym}^d(H^0(\mathbf{P}^{n-1}_{\overline{k}}, \mathcal{O}_{\mathrm{P}^{n-1}_{\overline{k}}}(1)))$$

via the embedding of X_s by $\omega^{\otimes m}_{X_s/\overline{k}}$. Thus the isomorphism class of μ_s is determined by the subspace of degree d equations defining X_s in $\mathbf{P}_{\overline{k}}$. By assumption (4), each fibre X_s is determined up to isomorphism by such equations. Thus all members of the family $X \times_S S_x \to S_x$ are isomorphic. It remains to remark that $S \to \overline{\mathcal{M}}_g$ is finite and hence, for every stable curve C, there are at most finitely many $s \in S$

[38]**TODO.** Make this hypothesis cleaner.

[39]**TODO.** Check this and find a reference

[40]**TODO.** I want the fibres of this map to be finite; is this really the same as demanding that this is quasi-finite as a morphism of stacks?

[41]**TODO.** Grassmannians in the Stacks project parameterize subs, so need to change q to $N - q$ is most places...

such that X_s is isomorphic to C. Thus S_x is finite, completing the proof that u is quasi-finite. □

Lemma 12.6. *Let k be a field. Let $\overline{\mathcal{M}}_g$ be the stack of stable curves of genus $g \geq 2$ over k and let $\overline{M}_g$ be its coarse moduli space. Then $\overline{M}_g$ is projective over k.*

Proof. We show that the invertible sheaf λ_m on $\overline{\mathcal{M}}_g$ obtained by base change of the invertible sheaf from Lemma 12.3 descends to an ample invertible sheaf $\lambda_m^{\otimes N}$ on $\overline{M}_g$ for N sufficiently divisible[42]. By Lemma 12.4, there exists a family of stable curves $f : X \to S$ such that S is a finite type k-scheme together with a finite surjective map $p : S \to \overline{M}_g$ such that for every $s \in S(\overline{k})$, $p(s) \in \overline{M}_g(\overline{k})$ is the moduli point corresponding to the stable curve $f_s : X \times_S \operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(\kappa(s))$. Now by Lemma 8.1, $\lambda_m^{\otimes N}$ is ample on $\overline{M}_g$ if and only if $p^*(\lambda_m^{\otimes N})$ is ample on S. Let $\lambda_m(f)$ denote the pullback under the classifying map $S \to \overline{\mathcal{M}}_g$ of the invertible sheaf λ_m on the stack $\overline{\mathcal{M}}_g$. Then $p^*(\lambda_m^{\otimes N}) = \lambda_m(f)^{\otimes N}$. Hence by [14, Tag 01PT], it suffices to show $\lambda_m(f)$ is ample on S. But recall from Lemma 12.3 that $\lambda_m(f) = \det(f_* \omega_{X/S})$. Ampleness of $\lambda_m(f)$ on S now follows from Lemma 12.5. □

Theorem 12.7. *Let $\overline{\mathcal{M}_g}$ be the moduli space of stable curves of genus $g \geq 2$ over $\mathbb{Z}$ and let $\overline{M}_g$ be its coarse moduli space. Then $\overline{M}_g$ is projective over $\mathbb{Z}$.*

Proof. We show that the invertible sheaf λ_m, with $m \geq 3$ on $\overline{M}_g$ constructed in Lemma 12.3 is relatively ample on $\overline{M}_g / \operatorname{Spec}(\mathbf{Z})$. We already know from Lemma 12.6 that the base change of λ_m to $\overline{M}_g / \operatorname{Spec}(k)$ is relatively ample therein. In particular, this means that λ_m restricted to each fibre of $\overline{M}_g \to \operatorname{Spec}(\mathbf{Z})$ is relatively ample. Since $\overline{\mathcal{M}}_g \to \operatorname{Spec}(\mathbf{Z})$ is proper by [14, Tag 0E9C], [14, Tag 0DUZ] shows $\overline{M}_g \to \operatorname{Spec}(\mathbf{Z})$ is proper. Hence [14, Tag 0D2N] applies to show λ_m is ampld on all of $\overline{M}_g / \operatorname{Spec}(\mathbf{Z})$, as desired. □

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[42]**TODO.** Make sense of this at some point and find a reference for what this means.

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