GEOMETRY WORKING SEMINAR
PRIMER ON DEFORMATION THEORY

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ABSTRACT. I shall use this talk as an excuse to figure out precisely how or why deformation theory is useful in geometry and the theory of moduli. Thus I want to present some examples of geometric consequences one can extract from deformation theory. I hope to leave this seminar with some example questions for which this theory can tell us something.

CONTENTS

1. INTRODUCTION

These are notes for a talk at the Geometry Working Seminar. I aimed to quickly introduce the basic concepts in deformation theory and work out an important example. I especially wanted to sketch some basic applications of deformation theory. Thus one will find only a smattering of basic deformation theory here, but hopefully enough so that one is motivated to go learn more about the subject, and to figure out how it might apply to some problems to one's interest.

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2. DEFORMATIONS

2.1. **General Deformations.** The term "deformation" is used in conjunction with different mathematical nouns in order to refer to generally different things. Varied notions of deformations, however, are generally related by a similar theme: a deformation of some fixed mathematical object consists of a continuously varying family of similar objects with a fixed member of the family identified as the original one. Consider a simple geometric case: A *deformation* of a smooth variety X consists of a pointed scheme (B, b_0), a flat family $\varphi\colon \mathscr{X} \to B$ and an isomorphism $\psi\colon X \xrightarrow{\sim} \mathscr{X}_{b_0}$. This data can be efficiently expressed in a diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d] & \mathscr{X} \arrow[d, "\varphi"] \\ \operatorname{Spec} k \arrow[r, "b_0"] & B. \end{tikzcd} \tag{2.1.1}$$

Better, we can express this data in terms of a *deformation functor* which takes pointed schemes to flat families with distinguished special fibre.

Deformations of other objects can typically be formulated in a similar manner and expressed in terms of an analogous diagram. That is to say, deformations of many objects involve a flat family of such objects over various base schemes together with an identification of your original object with a distinguished fibre.

2.2. **Deformations and Moduli.** Before continuing on, let me remark on one simple connection between deformation theory and moduli problems. As noted at the end of the previous paragraph, deformations involve flat families of some class of objects. Also recall that moduli problems consist of a moduli functor

$$\begin{aligned} \mathcal{F}\colon (\mathsf{Sch}) &\to (\mathsf{Set}) \\ B &\mapsto \{\mathscr{X} \to B \text{ flat family of objects of interest}\}. \end{aligned}$$

In the ideal case that this functor is representable, we have a universal family of geometric objects from which all other families are uniquely pulled back from.

As a basic example, consider the moduli space $\mathcal{M}_g$ of genus g curves—to be precise, the associated moduli functor is *not* representable in the category of schemes, but rather, in the category of Deligne–Mumford stacks, but for our purposes, this is unimportant. Now fix X a smooth, genus g curve and consider a deformation like (2.1.1). Since $\mathscr{X} \to B$ is a flat family of genus g curves, the universal property of $\mathcal{M}_g$ gives a unique map $B \to \mathcal{M}_g$ for which $\mathscr{X}$ is the pullback of the universal curve $\mathcal{C}_g \to \mathcal{M}_g$. We therefore have a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathscr{X} \arrow[r] \arrow[d] & \mathcal{C}_g \arrow[d] \\ \operatorname{Spec} k \arrow[r] & B \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

where X can now be identified with the fibre in $\mathcal{C}_g$ over the basepoint $b_0 \in B$ in $\mathcal{M}_g$. In other words, deformations of X amount to giving subschemes containing X in the moduli space. In particular, we should be able to study properties of the moduli space around a given point X by studying the deformations of X.

Of course, we could view this picture the other way around. Suppose we have a moduli space of objects containing some particular member X. Then, if our notion of moduli space is reasonable, deformations of X better be subschemes of the moduli space containing X; after all, moduli spaces are supposed to classify all objects like X and then put those points in a manner which expresses continuous variation. In any case, the point of deformations is that we do not need a representable moduli functor in order to make sense of small variations of some fixed object.

2.3. **Infinitesimal Deformations.** A first step in studying the deformation theory of an object is, typically, to study its infinitesimal deformations. As usual, global objects, like arbitrary deformations, are difficult to study and classify, so it is better to linearize the problem and study the local behaviour of deformations first. A very general setup for doing so is to consider deformations of X as above only on spectra of local Artin rings A—recall that an Artin ring is a zero-dimensional noetherian ring. In the setup of (2.1), this means that we wish to consider local Artin rings $(A, \mathfrak{m})$ with residue field $A/\mathfrak{m} = k$, and then consider diagrams of the form

(2.3.1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

where X is isomorphic to the generic fibre of $\mathscr{X}$. Note that $\operatorname{Spec} A$ consists of a single point, so X and $\mathscr{X}$ have the same underlying topological space; the isomorphism statement is about the scheme structure afforded by the structure sheaves.

For the most part, we will be interested in studying deformations over the Artin ring $\mathbf{I} := \operatorname{Spec} k[\epsilon]/(\epsilon^2)$. Deformations over $\mathbf{I}$ are traditionally referred to as *first-order deformations*. Sometimes, deformations over $\mathbf{I}^n := \operatorname{Spec} k[\epsilon]/(\epsilon^n)$ are referred to as *n^{th}-order deformations*; these higher order deformations are generally much more difficult to calculate. Anyway, let us denote by

$$\mathrm{Def}_1(X) := \{\text{Deformations of } X \text{ over } \mathbf{I}\}.$$

The great interest in $\mathrm{Def}_1(X)$ is for its amenability to calculation. Indeed, as we will see, $\mathrm{Def}_1(X)$ typically turns out to be a cohomology group of some sheaf associated to X. This strange regularity in the appearance of groups of cocycles for $\mathrm{Def}_1(X)$ can be vaguely understood as a linearization phenomenon on the level of categories. See [Ane] for an interesting discussion of this.

2.4. **Versal Deformations.** A lingering question when studying infinitesimal deformations is whether or not n^{th}-order deformations can be lifted to $(n+1)^{\text{st}}$-order deformations, or more generally, whether or not deformations over a local Artin ring A can be lifted to one over a A' containing A. One would then like to integrate these infinitesimal deformation and hopefully construct a *versal family* for the deformation problem. For reference, a *versal family* is a deformation $\mathscr{X} \to B$ of X from which every other deformation of X is locally the pullback of.

One reason for studying deformations over arbitrary Artin rings as opposed to only the rings $\mathbf{I}^n$ is that work by Schlessinger [Sch68] gives good conditions for when infinitesimal deformations can be integrated to a formal finite deformation—by a formal deformation, you should think of one over some formal power series ring like $k[[t]]/I$, I some ideal. This is a good step, but in order to proceed, in order to construct a versal family for instance, one needs to worry about convergence issues or about *algebraizing* the formal deformation. These are much more subtle problems.

2.5. Basic Deformation Theory Model. Applying deformation theory is a three step process. Following [HM98, p.89], the basic plan is as follows:

(i) Pose an appropriate deformation theoretic problem;
(ii) Calculate the space of first-order deformations; and
(iii) Construct, if one exists, a versal deformation.

In the coming sections, we will work out a few examples of the first two points. Versal deformations are left for sometime in the future.

3. First-Order Deformations of Affine Varieties

In preparation for the examples to come, we need a technical result characterizing the first-order deformations of an affine variety. As you might expect, affine varieties are wobbly and so if you poke them a little, they should not be fundamentally different from your original variety. Thus we would expect that all first-order deformations of affine things are trivial, in the sense that all the families fitting into (2.3.1) are product families. Another reason you might expect this is because affine varieties do not have any higher cohomology groups. Following [Oss], we will prove the following:

3.1. Affine First-Order Deformations are Trivial. — *If X is a smooth affine variety, then every first-order deformation of X is trivial.*

Proof. Let's unwrap the data of a first-order deformation. In what follows, we will abuse notation a bit and think of the sheaf of rings for X to be a ring itself. There is no harm in doing so as X is affine, so we could very well work with global sections instead. That said, we will ultimately show that the data of a first-order deformation is equivalent to that of

- A scheme $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ over k;
- A fixed map $i^\#\colon \mathscr{O}_{\mathscr{X}} \twoheadrightarrow \mathscr{O}_X$ with kernel $\mathscr{I}$;
- A fixed isomorphism $\pi\colon \mathscr{I} \xrightarrow{\sim} \mathscr{O}_X$.

The data of the scheme $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ together with the ideal sheaf $\mathscr{I}$ satisfying $\mathscr{I}^2$, $(\mathscr{X}, \mathscr{O}_{\mathscr{X}}/\mathscr{I}) \cong (X, \mathscr{O}_X)$ and $\mathscr{I} \cong \mathscr{O}_X$—this isomorphism is as $\mathscr{O}_X$-modules after applying the isomorphism—is sometimes referred to as an *infinitesimal extension of X by $\mathscr{O}_X$*. Anyway, the point is that a first-order deformation of X is really an infinitesimal extension of X with the extra structure of a $k[\epsilon]/(\epsilon^2)$-algebra on the structure sheaf $\mathscr{O}_{\mathscr{X}}$, which is due to the fixed maps $i^\#$ and π. In fact, the only thing we really need to do is to figure out where flatness of $\mathscr{X} \to \mathbf{I}$ is hiding in the above algebraic data.

Let's begin with a first-order deformation $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ of X. By definition, this means that $\mathscr{X}$ is a scheme over $\mathbf{I}$, flat, fitting into a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & \mathscr{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\epsilon]/(\epsilon^2). \end{tikzcd} \tag{3.1.1}$$

By definition, the top arrow in (3.1.1) gives us the morphism $i^\#\colon \mathscr{O}_{\mathscr{X}} \twoheadrightarrow i_*\mathscr{O}_X$; let $\mathscr{I}$ be its kernel. Next, note that saying the square (3.1.1) is Cartesian means we have an isomorphism

$$\mathscr{O}_X \cong \mathscr{O}_{\mathscr{X}} \otimes_{k[\epsilon]/\epsilon^2} k \cong \mathscr{O}_{\mathscr{X}}/\epsilon\mathscr{O}_{\mathscr{X}}, \tag{3.1.2}$$

where k is a $k[\epsilon]/\epsilon^2$-module from the bottom map in (3.1.1) so that ϵ acts by 0 on k. Thus $\mathscr{I} \cong \epsilon\mathscr{O}_{\mathscr{X}}$. Combining this observation with (3.1.2), we have a map

$$\mathscr{O}_X \cong \mathscr{O}_{\mathscr{X}}/\epsilon\mathscr{O}_{\mathscr{X}} \xrightarrow{\times\epsilon} \epsilon\mathscr{O}_{\mathscr{X}} \cong \mathscr{I}.$$

Flatness of $\mathscr{X} \to \mathbf{I}$ together with the Infinitesimal Flatness Lemma (3.2) implies that the multiplication map in the middle is also an isomorphism. Therefore we have an isomorphism $\pi\colon \mathscr{O}_X \to \mathscr{I}$ of $\mathscr{O}_X$-modules.

Conversely, starting with $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ and the maps $i^\#\colon \mathscr{O}_{\mathscr{X}} \twoheadrightarrow \mathscr{O}_X$ and $\pi\colon \mathscr{O}_X \xrightarrow{\sim} \mathscr{I}$, with $\mathscr{I}^2 = 0$, we need to construct the flat $\mathbf{I}$-structure on $\mathscr{O}_{\mathscr{X}}$, which amounts to making sense of the "multiplication by ϵ" endomorphism. Well, there is only one composition of maps giving an endomorphism of $\mathscr{O}_{\mathscr{X}}$:

$$\mathscr{O}_{\mathscr{X}} \xrightarrow{i^\#} \mathscr{O}_X \xrightarrow{\pi} \mathscr{I} \hookrightarrow \mathscr{O}_{\mathscr{X}}.$$

Since $\mathscr{I}^2 = 0$, we can indeed identify the above endomorphism as $\times\epsilon$, thereby giving a map $\mathscr{X} \to \mathbf{I}$. Doing that, we therefore identify $\mathscr{I}$ as the submodule $\epsilon\mathscr{O}_{\mathscr{X}}$. We can finally apply the Infinitesimal Flatness Lemma (3.2) to $\mathscr{O}_{\mathscr{X}}/\mathscr{I} \cong \mathscr{O}_X$, where we recognize the multiplication map in question simply expresses the fact that $\mathscr{I} = \ker(i^\#)$, to conclude that $\mathscr{X} \to \mathbf{I}$ is a flat family and thus a first-order deformation.

First order-deformations are therefore the same as infinitesimal extensions with a little extra data. One can show that infinitesimal extensions of nonsingular affine varieties are always trivial; see [Har77, Exercise II.8.7]. Thus the only first-order deformation of a nonsingular affine variety is the trivial one. ∎

3.2. Infinitesimal Flatness Lemma. — *A $k[\epsilon]/(\epsilon^2)$ module M is flat if and only if the multiplication map $M/\epsilon M \to \epsilon M$ is an isomorphism.*

Proof. This is but a special case of an general characterization of flatness: an A-module M is flat if and only if for every nonzero ideal $I \subseteq A$, the natural map $I \otimes_A M \to IM$ is an isomorphism. Since $k[\epsilon]/(\epsilon^2)$ is a local Artin ring, the only nonzero ideal here is (ϵ). The result follows immediately. ∎

4. Example Calculation

This section is solely devoted to the calculation of the space of first-order deformations of a smooth variety. The calculation follows a method of Artin, as presented in [HM98].

4.1. Deformations of a Smooth Variety. Let X be a smooth variety over k. As remarked before, a *deformation* of X consists of a flat family $\mathscr{X} \to B$ together with a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & B. \end{array}$$

We now calculate the space of first-order deformations of X. For this, choose a affine open cover $\{U_\alpha\}$ of X. By (3.1), all first-order deformations of affine varieties are trivial, so we can find trivializations

$$\varphi_\alpha \colon \mathscr{X}|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbf{I}.$$

We therefore obtain glueing maps

$$\varphi_{\alpha\beta} \colon U_\alpha \times \mathbf{I}|_{U_\beta} \xrightarrow{\sim} U_\beta \times \mathbf{I}|_{U_\alpha}.$$

On the level of sheaves, this gives ring maps

$$\varphi_{\alpha\beta} \colon \mathscr{O}_{U_{\alpha\beta}} \otimes k[\epsilon]/(\epsilon^2) \xrightarrow{\sim} \mathscr{O}_{U_{\alpha\beta}} \otimes k[\epsilon]/(\epsilon^2)$$

which restrict to the identity modulo ϵ. In other words, the $\varphi_{\alpha\beta}$ satisfy

$$\begin{aligned} \varphi_{\alpha\beta}(\epsilon) &= \epsilon, \\ \varphi_{\alpha\beta}(f) &= f + \epsilon D_{\alpha\beta}(f), \end{aligned} \tag{4.1.1}$$

for $f \in \mathscr{O}_{U_{\alpha\beta}}$ and where $D_{\alpha\beta}$ is some k-linear map—this comes from k-linearity of $\varphi_{\alpha\beta}$. Strictly speaking, the first equation in (4.1.1) is not true for all deformations, but it is a normalization we can take for our transition functions. The point is that the $D_{\alpha\beta}$ parameterize our deformation, so we should find the properties for which they satisfy. We do so by using the algebraic properties of $\varphi_{\alpha\beta}$:

$$\begin{array}{ccc} \varphi_{\alpha\beta}(fg) = \varphi_{\alpha\beta}(f)\varphi_{\alpha\beta}(g) & \Longrightarrow & D_{\alpha\beta}(fg) = f \cdot D_{\alpha\beta}(g) + D_{\alpha\beta}(f) \cdot g, \\ \varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} & \Longrightarrow & D_{\alpha\gamma} = D_{\alpha\beta} + D_{\beta\gamma}. \end{array}$$

The first condition says that the $D_{\alpha\beta}$ are k-derivations on the structure sheaf whereas the second condition says that they are additive cocycles. These are exactly the 1-cocycles making up $H^1(X, T_X)$. Therefore

$$\operatorname{Def}_1(X) = H^1(X, T_X). \tag{4.1.2}$$

We give an application of this calculation in (5.1).

5. Applications of Deformation Theory

Let's take a look at some applications of deformation theory. I will first start with some mentioned in [Oss].

5.1. Local Geometry of Moduli Spaces. As mentioned in (2.2), deformation theory amounts to studying the geometry of a moduli space around a fixed point. In particular, infinitesimal deformations are a study of local properties around a point. For instance, the space of first-order deformations of an object X in a moduli space $\mathcal{M}$ is the tangent space $T_X\mathcal{M}$ to $\mathcal{M}$ at X. Indeed, for each first-order deformation $\mathscr{X} \to \mathbf{I}$, we have a diagram

$$\begin{array}{ccccc} X & \longrightarrow & \mathscr{X} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathbf{I} & \longrightarrow & \mathcal{M}. \end{array}$$

But the space of all maps $\mathbf{I} \to \mathcal{M}$ is precisely the Zariski tangent space to $\mathcal{M}$ at X. In particular, this means that we can bound the dimension of a component of $\mathcal{M}$ containing X by $\dim \operatorname{Def}_1(X)$. In fact, there are times when there is a space $\operatorname{Obs}(X)$ of *obstructions* to deformations and which give the inequality

$$\dim \operatorname{Def}_1(X) - \dim \operatorname{Obs}(X) \le \dim T_X\mathcal{M} \le \dim \operatorname{Def}_1(X).$$

In particular, we see that if we are able to show that a given object X has no obstruction theory, that is $\dim \operatorname{Obs}(X) = 0$ always, then we have an equality $\dim \operatorname{Def}_1(X) = \dim T_X\mathcal{M}$.

We can go even further. If all infinitesimal deformations of X lift, then one can actually conclude that $\mathcal{M}$ is actually nonsingular at X. In that case, $\dim \mathcal{M} = \dim T_X\mathcal{M} = \dim \operatorname{Def}_1(X)$. In other words, deformation theory can allow us to compute the dimension of a moduli space.

For instance, we can use the calculation in (4.1) to obtain an estimate on the dimension of the moduli space of curves. Let C be a genus $g \ge 2$ curve. Using (4.1.2), we have that

$$T_C\mathcal{M}_g = \operatorname{Def}_1(C) = H^1(X, T_C).$$

Applying Serre duality, we have that

$$T_C^*\mathcal{M}_g = H^0(X, K_C^{\otimes 2}).$$

This is already interesting: this says that the cotangent space of the moduli space at a point corresponding to a curve C is the space of quadratic differentials on C. Taking dimensions and applying Riemann–Roch, we have that

$$\dim \mathcal{M}_g \le \dim H^0(X, K_C^{\otimes 2}) = 3g - 3.$$

In fact, one can show that smooth curves are smooth points of $\mathcal{M}_g$ and so that the inequality is actually an equality.

5.2. Finiteness Arguments. Here is a nice two step process for showing that a class of objects is finite:

(i) Show that your objects are parameterized by a moduli space of finite-type; then
(ii) Show that every object has no non-trivial first-order deformations.

Consequently, the moduli space must be a discrete set of points and so is a finite set of points by the finite-type assumption. One example application of this sort is the following result:

Arakelov–Parshin Rigidity. — [Par68; Ara71] *Let B be a smooth proper curve over a field k and fix an integer $g \geq 2$. Then there exists only finitely many non-isotrivial families of curves $X \to B$ which are smooth and proper, with each fibre a genus g curve.*

Actually, the two papers referenced together prove a lot more, but the basic method is deformation theoretic.

5.3. Existence of Families. Given a variety X over a field k, there are many reasons one would like to find a family $\mathscr{X} \to B$ with good properties in which X is a member of. The hope is that we can learn something about the geometry of X by deforming it in particular ways. Deformation theory is clearly crucial in constructing such families. One such result is the following.

Winter's Theorem. — *Let X be a proper, geometrically integral curve over a field k with at worst nodal singularities. Then there exists a pointed scheme (B, b) and a scheme $\mathscr{X} \to B$ such that*

- *$k(b) = k$ and $\mathscr{X}_b \cong X$;*
- *B is regular of dimension 1 and can be chosen to be a curve over k; when $\operatorname{char} k > 0$, B can be taken to be $\operatorname{Spec} A$ for A a discrete valuation ring with residue field k;*
- *$\mathscr{X}$ is a regular surface with smooth generic fibre.*

Two main uses of this might be as follows. First, when X is smooth but is defined over a field k of positive characteristic. Then one might want to find a family $\mathscr{X} \to B$ whose generic fibre $\tilde{X}$ is a smooth curve over characteristic 0 and whose special fibre is our curve X. Then we can think of $\tilde{X}$ as a lift of X to characteristic 0; that is, X might be thought of as a specialization of some curve over characteristic 0.

A second use of such families is that we can start with a singular X and find a family $\mathscr{X}$ with smooth generic containing X. We can then think of the variation from the smooth generic fibre to the fibre containing X as a degeneration of something smooth to something singular. A spectacular application of this result, for instance, is in certain steps toward the Brill–Noether Theorem [EH83].

5.4. Existence of Rational Curves on Varieties. One more example of how deformation theoretic techniques might be used to prove geometric results. We will describe the proof of the following result, following the description [Mo1].

Theorem. — [Mor79] *Let X be a smooth complex projective variety such that $-K_X$ is ample. Then X contains a rational curve. In fact, through every point $x \in X$, there is a rational curve D such that*

$$0 < -(D \cdot K_X) \leq \dim X + 1.$$

The proof goes about as follows. Begin with a map $f \colon C \to X$ from a smooth pointed curve $(C, 0)$ with $f(0) = x$. By the deformation theory of maps, the deformation space of the morphism f with fixed image at 0 has dimension at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) - \dim X = -((f_*C)\cdot K_X) - g(C)\cdot \dim X$$

where the equality is from Riemann–Roch. Hence when the right hand side is positive, there must be a non-trivial family of deformations of the map $f\colon C \to X$ keeping 0 fixed. By Mori's *bend-and-break*, one can actually show that the image curve under the morphism breaks into several components, one of which is the desired rational curve through x.

The interesting case is when $-((f_*C)\cdot K_X) - g(C)\cdot \dim X$ is less than zero. In that case, we pass to characteristic p and precompose the map $f\colon C_p \to X_p$ with some power of the Frobenius endomoprhism $F\colon C_p \to C_p$. This has the effect of increasing the intersection number $-((f_*C)\cdot K_X)$ without increasing the genus. Thus at some point, this quantity becomes positive and we can apply the argument before to find a rational curve through the point $x_p \in X_p$. This argument works for every prime number p, so elimination theory allows us to lift this result from positive characteristic back to characteristic 0.

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