CALCULUS II ASSIGNMENT 2 SOLUTIONS

1. Compute the following trigonometric integrals:

(i) $\displaystyle\int \sin(\theta)^2 \cos(\theta)^3 \, d\theta$,

(ii) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx$,

(iii) $\displaystyle\int_0^{\pi/2} \sin(t)^2 \cos(t)^2 \, dt$,

(iv) $\displaystyle\int \tan(y)^2 \, dy$,

(v) $\displaystyle\int \tan(z)^3 \sec(z) \, dz$,

(vi) $\displaystyle\int \sin(8u)\cos(5u) \, du$.

Solutions. For (i), note that the exponent of $\cos(\theta)$ is odd, so we attempt to make the substitution $u = \sin(\theta)$, so that $du = \cos(\theta)\, d\theta$, and apply the Pythagorean identity:

$$\begin{aligned}\int \sin(\theta)^2 \cos(\theta)^3 \, d\theta &= \int \sin(\theta)^2 \cos(\theta)^2 \, (\cos(\theta) d\theta) \\ &= \int u^2 (1-u^2) \, du \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 = \frac{1}{3}\sin(\theta)^3 - \frac{1}{5}\sin(\theta)^5.\end{aligned}$$

For (ii), use the substitution $u = \sqrt{x}$, so that $du = dx/2\sqrt{x}$:

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx = 2\int \sin(u)\, du = -2\cos(\sqrt{x}).$$

For (iii), apply the half-angle identities to each of $\sin(t)^2$ and $\cos(t)^2$ in order to compute:

$$\begin{aligned}\int_0^{\pi/2} \sin(t)^2 \cos(t)^2 \, dt &= \int_0^{\pi/2} \frac{1-\cos(2t)}{2} \cdot \frac{1+\cos(2t)}{2} \, dt \\ &= \frac{1}{4}\int_0^{\pi/2} 1 - \cos(2t)^2 \, dt \\ &= \frac{\pi}{8} - \frac{1}{4}\int_0^{\pi/2} \frac{1+\cos(4t)}{2} \, dt = \frac{\pi}{8} - \frac{\pi}{16} - \frac{1}{32}\sin(4t)\Big|_0^{\pi/2} = \frac{\pi}{16}.\end{aligned}$$

For (iv), use the Pythagorean identity $\tan(y)^2 = \sec(y)^2 - 1$ and the knowledge that $\sec(y)^2$ is the derivative of $\tan(y)$:

$$\int \tan(y)^2 \, dy = \int \sec(y)^2 - 1 \, dy = \tan(y) - y.$$

For (v), make the substitution $u = \sec(z)$ so that $du = \tan(z)\sec(z)\, dz$ and then use the Pythagorean identity:

$$\int \tan(z)^3 \sec(z) \, dz = \int u^2 - 1 \, du = \frac{1}{3}\sec(z)^3 - \sec(z).$$

For (vi), use one of the prosthaphaeresis formulae:

$$\int \sin(8u)\cos(5u)\, du = \frac{1}{2}\int \sin(3u) + \sin(13u)\, du = -\frac{1}{6}\cos(3u) - \frac{1}{26}\cos(13u).$$

■

2. In this Problem, we are going to compute the following relations: for positive integers n and m,

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0.$$

Solutions. First, let's dispense with the cases where $m = n$:

$$\int_{-\pi}^{\pi} \sin(mx)^2\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2mx)}{2} = \frac{x}{2} - \frac{\sin(2mx)}{2m}\Big|_{x=-\pi}^{x=\pi} = \pi,$$

$$\int_{-\pi}^{\pi} \cos(mx)^2\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2mx)}{2} = \frac{x}{2} + \frac{\sin(2mx)}{2m}\Big|_{x=-\pi}^{x=\pi} = \pi.$$

Now suppose that $m \neq n$. Using the various prosthaphaeresis formulae,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x) - \cos((m+n)x)\,dx \\ &= \frac{1}{2(m-n)}\sin((m-n)x) - \frac{1}{2(m+n)}\sin((m+n)x)\Big|_{x=-\pi}^{x=\pi} = 0, \\ \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x) + \cos((m+n)x)\,dx \\ &= \frac{1}{2(m-n)}\sin((m-n)x) + \frac{1}{2(m+n)}\sin((m+n)x)\Big|_{x=-\pi}^{x=\pi} = 0,\end{aligned}$$

where I have used the assumption that $m \neq n$ to do the integral of $\cos((m-n)x)$; notice that in the case $m = n$, $\cos((m-n)x) = \cos(0) = 1$, and this recovers the half angle identities I used in the first case.

For the final integral, notice that for any positive m and n, the function $\sin(mx)\cos(nx)$ is odd. Thus the integral of this function along the symmetric interval $[-\pi, \pi]$ vanishes by the work in the previous assignment! ∎

3. Use the trigonometric substitution $x = 3\sin(\theta)$ to evaluate

$$\int_0^1 x^2\sqrt{9-x^2}\,dx.$$

Be careful about the bounds of integration once you do your substitution: what must θ be when $x = 0$ or $x = 1$?

Solution. The substitution $x = 3\sin(\theta)$ gives rise the differential equality $dx = 3\cos(\theta)\,d\theta$. Moreover, to get the bounds of integration:

- for the lower bound, $0 = x = 3\sin(\theta)$, so $\sin(\theta) = 0$, thus $\theta = 0$; and
- for the upper bound, $1 = x = 3\sin(\theta)$, so $\sin(\theta) = 1/3$, so $\theta = \arcsin(1/3)$.

Thus

$$\begin{aligned}
\int_0^1 x^2\sqrt{9-x^2}\,dx &= \int_0^{\arcsin(1/3)} 9\sin(\theta)^2\sqrt{9(1-\sin(\theta)^2)}\cdot 3\cos(\theta)\,d\theta \\
&= 81\int_0^{\arcsin(1/3)} \sin(\theta)^2\cos(\theta)^2\,d\theta \\
&= \frac{81}{4}\int_0^{\arcsin(1/3)} 1-\cos(2\theta)^2\,d\theta \\
&= \frac{81}{4}\arcsin(1/3) - \frac{81}{8}\int_0^{\arcsin(1/3)} 1-\cos(4\theta)\,d\theta \\
&= \frac{81}{8}\arcsin(1/3) + \frac{81}{32}\sin(4\arcsin(1/3)).
\end{aligned}$$

Ah, this number is much uglier than I hoped for: I number I wanted in the square root in the original integrand. Oh well, not all computations lead to pretty numbers. ∎

4. Sometimes you are going to have to do some manipulations before being able to perform a trigonometric substitution. Here is an example:

(i) Write the polynomial $3-2x-x^2$ in the form $a-(x+b)^2$, for some numbers a and b, by completing the square.

(ii) Do the substitution $u = x+b$ followed by a trigonometric substitution to evaluate the integral

$$\int \sqrt{3-2x-x^2}\,dx.$$

Solution. To complete the square, you can equate the target form $a-(x+b)^2$ with the form $3-2x-x^2$ that we already have. Then a comparison of coefficients will allow you to solve for a and b. Specifically, equating the coefficients on the two sides of

$$3-2x-x^2 = a-(x+b)^2 = (a-b^2)-2bx-x^2$$

yields the equations $2b = 2$ and $a-b^2 = 3$. The first equation immediately implies $b = 1$; inputting this into the second equation then gives $a = 4$. Therefore

$$3-2x-x^2 = 4-(x+1)^2.$$

Now consider the substitution $u = x+1$ in the integral in (ii):

$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4-u^2}\,dx$$

and in this form, we may perform the trigonometric substitution $u = 2\sin(\theta)$ that we know and love:

$$\begin{aligned}\int \sqrt{4-u^2}\,du &= \int \sqrt{4(1-\sin(\theta)^2)}\cdot 2\cos(\theta)\,d\theta \\ &= 4\int \cos(\theta)^2\,d\theta \\ &= 2\theta+\sin(2\theta).\end{aligned}$$

Tracing through our substitutions, we see that $\theta = \arcsin((x+1)/2)$ so

$$\int \sqrt{3-2x-x^2}\,dx = 2\arcsin\Big(\frac{x+1}{2}\Big)+\sin\Big(2\arcsin\Big(\frac{x+1}{2}\Big)\Big)$$

which may be simplified a bit further by applying some double angle identities to the final term. ■

5. Given a circle of radius a, its circumference is $2\pi a$ and its area is πa^2.

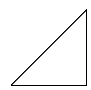
(i) Compute the integral $\int_0^a 2\pi r\, dr$.

(ii) Thinking about polar coordinates, try to explain how the computation in (i) is a way of computing the area of a circle of radius r.

As an analogy, it might be helpful to think about how the integral

$$\int_0^1 x\, dx$$

computes the area of the right triangle



where the vertices are at $(0,0)$, $(0,1)$, and $(1,1)$.

Solution. For (i), compute:

$$\int_0^a 2\pi r\, dr = \pi r^2 \Big|_{r=0}^{r=a} = \pi a^2.$$

One way to interpret this is that we are computing the area of a circle in polar coordinates by drawing a series of concentric circles with radii ranging from 0 to a within the area of the larger circle of radius a. One of these circles have circumference $2\pi r$ and adding all these together will make up the area of the circle. ■