

CALCULUS II ASSIGNMENT 7 SOLUTIONS

1. Use the Integral Test to determine whether the series is convergent or divergent:

$$\begin{array}{ll} \text{(i)} \sum_{n=1}^{\infty} n^2 e^{-n^3}, & \text{(iii)} \sum_{k=1}^{\infty} k e^{-k}, \\ \text{(ii)} \sum_{m=1}^{\infty} \frac{1}{m\sqrt{2}}, & \text{(iv)} \sum_{\ell=1}^{\infty} \frac{2}{5\ell-1}. \end{array}$$

Solutions. For (i), we need to consider the improper integral

$$\begin{aligned} \int_1^{\infty} x^2 e^{-x^3} dx &= \lim_{N \rightarrow \infty} \int_1^N x^2 e^{-x^3} dx \\ &= \lim_{N \rightarrow \infty} \int_1^N \frac{1}{3} e^{-u} du \\ &= \lim_{N \rightarrow \infty} -\frac{1}{3e^N} + \frac{1}{3e} = \frac{1}{3e}, \end{aligned}$$

where I substituted $u = x^3$ in the second equality. This improper integral is convergent, so by the Integral Test, the corresponding series in (i) is also convergent.

For (ii), we need to consider the integral

$$\int_1^{\infty} \frac{dx}{x\sqrt{2}} = \lim_{N \rightarrow \infty} \int_1^N \frac{dx}{x\sqrt{2}} = \lim_{N \rightarrow \infty} \frac{1}{1-\sqrt{2}} \left(\frac{1}{N^{\sqrt{2}-1}} - 1 \right) = \frac{1}{\sqrt{2}-1}$$

where the final inequality comes from the fact that $\sqrt{2}-1 \approx 0.41 > 0$ and so when $N \rightarrow \infty$, the denominator of the first term tends to infinity and hence the fraction goes to zero. This improper integral converges, so by the Integral Test, the corresponding series in (ii) converges.

For (iii), we are concerned with the integral

$$\begin{aligned} \int_1^{\infty} x e^{-x} dx &= \lim_{N \rightarrow \infty} \int_1^N x e^{-x} dx \\ &= \lim_{N \rightarrow \infty} -x e^{-x} \Big|_1^N + \int_1^N e^{-x} dx \\ &= \lim_{N \rightarrow \infty} -\frac{N}{e^N} + \frac{1}{e} - \frac{1}{e^N} + \frac{1}{e} = \frac{2}{e}. \end{aligned}$$

Therefore, this integral converges and the series converges by the Integral Test.

For (iv), consider the integral

$$\int_1^{\infty} \frac{2 dx}{5x-1} = \lim_{N \rightarrow \infty} \int_1^N \frac{2 dx}{5x-1} = \lim_{N \rightarrow \infty} 2 \log(5N-1) - 2 \log(4) = \infty$$

as the argument of the first logarithm tends to infinity. This integral therefore diverges and so does the corresponding series, á la the Integral Test. ■

2. The harmonic series $\sum \frac{1}{n}$ is divergent, but only just so. This is perhaps substantiated by the following exercise: determine the values of p for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log(n)^p}$$

converges.

Solution. Let's try to do an integral test for the series. We are then concerned with the integral

$$\int_2^{\infty} \frac{dx}{x \log(x)^p} = \lim_{N \rightarrow \infty} \int_2^N \frac{dx}{x \log(x)^p} = \lim_{N \rightarrow \infty} \int_{\log(2)}^{\log(N)} \frac{du}{u^p}$$

where I have performed the substitution $u = \log(x)$. To proceed with evaluating this integral, let's split it up into cases depending on what p is:

First, if $p > 1$, then

$$\int_2^{\infty} \frac{dx}{x \log(x)^p} = \lim_{N \rightarrow \infty} \frac{1}{(1-p)u^{p-1}} \Big|_2^N = \lim_{N \rightarrow \infty} \frac{1}{(1-p)u^{p-1}} - \frac{1}{2-2p} = \frac{1}{2p-2}$$

since $p-1 > 0$. Thus, when $p > 1$, this improper integral converges and hence so does the series.

Second, if $p = 1$, then

$$\int_2^{\infty} \frac{dx}{x \log(x)^p} = \lim_{N \rightarrow \infty} \log(u) \Big|_2^N = \lim_{N \rightarrow \infty} \log(N) - \log(2) = \infty$$

and, so, when $p = 1$, this improper integral diverges and thus so does the series.

Finally, if $p < 1$, then

$$\int_2^{\infty} \frac{dx}{x \log(x)^p} = \lim_{N \rightarrow \infty} \frac{u^{1-p}}{1-p} \Big|_2^N = \lim_{N \rightarrow \infty} \frac{N^{1-p}}{1-p} - \frac{2^{1-p}}{1-p} = \infty$$

since $1-p > 0$ this time. Therefore, this improper integral diverges and so does the series.

In summary, we see that the series $\sum \frac{1}{n \log(n)^p}$ converges if and only if $p > 1$, very much like the p -series themselves! Incidentally, the divergence of the series at $p = 1$ might point at how absurdly slow $\log(n)$ grows. ■

3. What does the decimal representation of a real number really mean? Say we write a number as $0.d_1d_2d_3d_4\dots$, where d_i is one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This is really a short hand for the series

$$0.d_1d_2d_3d_4\dots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \dots$$

To make sense of this statement, show that for any choice of $d_1, d_2, d_3, d_4, \dots$, the series on the right is convergent.

Try to generalize this to where the denominators on the right are powers of a number different than 10 and where the digits d_i perhaps range through a different set of values.

Solution. To show that the series $\sum \frac{d_n}{10^n}$ is convergent for any choice of $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we perform a comparison. First, note that for any such choice of d_n , the terms of the series are positive. What is the largest the numerators of the terms of the series can be? Well, the largest digit that is allowed is 9, thus certainly,

$$\frac{d_n}{10^n} \leq \frac{9}{10^n} \text{ for all } n \geq 1.$$

This suggests that we should compare the series $\sum \frac{d_n}{10^n}$ with the series $\sum \frac{9}{10^n}$. This latter series is a geometric series:

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} + \frac{9}{10} \cdot \frac{1}{10} + \frac{9}{10} \cdot \left(\frac{1}{10}\right)^2 + \dots$$

with $a = \frac{9}{10}$ and $r = \frac{1}{10}$. Since $|r| < 1$, this geometric series is convergent. Thus, by Comparison, $\sum \frac{d_n}{10^n}$ is convergent for any allowable choice of d_n .

Incidentally the value of the geometric series $\sum \frac{9}{10^n}$ above is

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9/10}{1 - \frac{1}{10}} = 1$$

and this is expressing the two facts that: 0.999999... is actually another decimal representation for 1; and that decimal numbers that are $0.d_1d_2d_3d_4\dots$ are all smaller than 1.

More generally, we might consider series of the form

$$\sum_{n=1}^{\infty} \frac{d_n}{b^n} = \frac{d_1}{b} + \frac{d_2}{b^2} + \frac{d_3}{b^3} + \frac{d_4}{b^4} + \dots$$

where b is some positive integer and d_n may be any number $0, 1, 2, 3, \dots, b-1$ —these should be the allowable “digits” in base b . Again, any of the above series converge by comparison with the geometric series $\sum \frac{b-1}{b^n}$, which again sums to 1. In the end, a different base b offers simply a slightly different way of writing the same set of numbers. Some fun base systems that are of actual use are when $b = 2$ or $b = 16$, for instance; in fact, they have English names: binary and hexadecimal respectively. The Babylonians also liked base 12, which is part of the reason why some of the common units of measurement in this country like multiples of 12. ■