CALCULUS II ASSIGNMENT 4 SOLUTIONS

1. Evaluate the integrals

(i) $\displaystyle\int \frac{e^{\arctan(y)}}{1+y^2}\,dy$,

(ii) $\displaystyle\int_0^{\pi} t\cos(t)\,dt$,

(iii) $\displaystyle\int_0^1 \big(1+\sqrt{x}\big)^8\,dx$,

(iv) $\displaystyle\int \sqrt{u}e^{\sqrt{u}}\,du$,

(v) $\displaystyle\int \frac{\sec(\theta)^2}{\tan(\theta)^2-16}\,d\theta$,

(vi) $\displaystyle\int \phi\tan(\phi)^2\,d\phi$,

(vii) $\displaystyle\int x\arctan(x)\,dx$,

(viii) $\displaystyle\int \frac{\sin(\psi)\cos(\psi)}{\sin(\psi)^4+\cos(\psi)^4}\,d\psi$.

Solutions. To compute (i), one might recognize that the derivative of $\arctan(y)$ is $1/(1+y^2)$, immediately suggesting a substitution $u=\arctan(y)$. Doing so, the first integral simplifies to

$$\int \frac{e^{\arctan(y)}}{1+y^2}\,dy = \int e^u\,du = e^u = e^{\arctan(y)}.$$

For (ii), observe that $t\cos(t)$ is a product, suggesting that we should use integration by parts; upon differentiating t, the integral simplifies. Thus

$$\int_0^{\pi} t\cos(t)\,dt = t\sin(t)\Big|_{t=0}^{t=\pi} - \int_0^{t}\sin(t)\,dt = 0+\cos(t)\Big|_{t=0}^{t=\pi} = -2.$$

For (iii), one could expand the binomial. Alternatively, in trying to be lazy and also to prevent arithmetic mistakes, one might try to do a substitution $u=1+\sqrt{x}$. Taking the differential, we see that

$$du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2(u-1)} \quad \text{so} \quad dx = 2(u-1)du.$$

Upon making this substitution, the bounds of integration change as:

- For the upper bound, $x=1$ implies $u=1+1=2$; and
- For the lower bound, $x=0$ implies $u=1+0=1$.

Putting everything together, we see that

$$\begin{aligned}\int_0^1 \big(1+\sqrt{x}\big)^8\,dx &= 2\int_1^2 u^8(u-1)\,du \\ &= \frac{2}{10}u^{10}-\frac{2}{9}u^9\Big|_{u=1}^{u=2} = \Big(\frac{1024}{5}-\frac{1024}{9}\Big)-\Big(\frac{1}{5}-\frac{2}{9}\Big) = \frac{4\cdot 1024+1}{45} = \frac{4097}{45}.\end{aligned}$$

For (iv), perhaps a first instinct is to try to integrate by parts to get rid of the $\sqrt{u}$ factor in front of the exponential. But one is left integrating $e^{\sqrt{u}}$, which does not look so easy. So another thing that comes to mind might be to do the substitution $x=\sqrt{u}$ in hopes that it will make things easier. The differential of this substitution is

$$dx = \frac{du}{2\sqrt{u}} = \frac{du}{2x} \quad \text{so} \quad du = 2xdx.$$

Putting this into the integrand then puts it that yells, “integration by parts!!!!”:

$$\begin{aligned}\int \sqrt{u}e^{\sqrt{u}}\,du &= \int 2x^2e^x\,dx\\ &= 2x^2e^x - 4\int xe^x\,dx\\ &= 2x^2e^x - 4xe^x + 4e^x = 2(u-2\sqrt{u}+2)e^{\sqrt{u}}.\end{aligned}$$

For (v), one might notice that the derivative of $\tan(\theta)$ is $\sec(\theta)^2$, so this calls for a substitution $u = \tan(\theta)$. Doing so, one is reduced to an integral of a rational function:

$$\int \frac{\sec(\theta)^2}{\tan(\theta)^2 - 16}\,d\theta = \int \frac{du}{u^2-16} = \int \frac{du}{(u-4)(u+4)}.$$

To integrate the rational function, we use the method of partial fractions; that is, we aim to find numbers A and B so that

$$\frac{1}{(u-4)(u+4)} = \frac{A}{u-4} + \frac{B}{u+4}.$$

Clearing denominators and then collecting like terms, we get equations imposed by

$$1 = A(u+4) + B(u-4) = (A+B)u + (4A-4B).$$

Comparing coefficients of u implies $A = -B$; comparing constant terms then implies that $8A = 1$, so $A = 1/8$, $B = -1/8$. Therefore

$$\int \frac{\sec(\theta)^2}{\tan(\theta)^2-16}\,d\theta = \frac{1}{8}\int\frac{du}{u-4} - \frac{1}{8}\int\frac{du}{u+4} = \frac{1}{8}\big(\log(u-4) - \log(u+4)\big) = \frac{1}{8}\log\Big(\frac{\tan(\theta)-4}{\tan(\theta)+4}\Big).$$

For (vi), I stare at it, and draw some blanks since I don’t seem to know how to directly integrate $\tan(\phi)^2$, nor does there look like any good substitutions. This suggests that I now need to use some trigonometric identities, the first of which comes to mind being the Pythagorean identity

$$\tan(\phi)^2 = \sec(\phi)^2 - 1.$$

Now I know how to integrate ϕ; what about $\phi\sec(\phi)^2$? Well, since this is a product, I again think about integration by parts. This time, I know how to integrate $\sec(\phi)^2$ directly: it is the derivative of $\tan(\phi)$! Thus

$$\int \phi\tan(\phi)^2\,d\phi = \int \phi\sec(\phi)^2\,d\phi - \int\phi\,d\phi = \Big(\phi\tan(\phi) - \int\tan(\phi)\,d\phi\Big) - \frac{1}{2}\phi^2.$$

To do the remaining integral, recall the definition of $\tan(\phi)$:

$$\int\tan(\phi)\,d\phi = \int\frac{\sin(\phi)}{\cos(\phi)}\,d\phi = -\int\frac{du}{u} = -\log(\cos(\phi)),$$

where I performed the substitution $u = \cos(\phi)$. Putting everything together,

$$\int\phi\tan(\phi)^2\,d\phi = \phi\tan(\phi) + \log(\cos(\phi)) - \frac{1}{2}\phi^2.$$

For (vii), I see a product of functions, and I immediately yearn to integrate by parts; I know how to differentiate $\arctan(x)$ but not integrate it, so there is a clear choice of what to try:

$$\begin{aligned}\int x\arctan(x)\,dx &= \frac{x^2}{2}\arctan(x) - \int\frac{x^2}{1+x^2}\,dx\\ &= \frac{x^2}{2}\arctan(x) - \int\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\,dx\\ &= \frac{x^2}{2}\arctan(x) - x + \arctan(x)\end{aligned}$$

where I added and subtracted 1 in the numerator of the rational function to simplify it a bit.

Finally, for (viii), there are a lot of trigonometric functions, and in particular, some in the numerator touching the differential: this calls for a substitution! Set $u = \sin(\psi)$, then the integral simplifies to

$$\int \frac{\sin(\psi)\cos(\psi)}{\sin(\psi)^4 + \cos(\psi)^4}\,d\psi = \int \frac{u}{u^4 + (1-u^2)^2}\,du = \int \frac{u}{2u^4 - 2u^2 + 1}\,du.$$

The denominator of this rational function secretly looks like a quadratic polynomial; indeed, this can be made a reality upon performing the substitution $v = u^2$. Then $dv = 2u\,du$ so

$$\int \frac{u}{2u^4 - 2u^2 + 1}\,du = \frac{1}{2}\int \frac{dv}{2v^2 - 2v + 1}.$$

We might try to factor the denominator by, say, using the quadratic formula to find the roots of the polynomial there; however, doing so, we find that the roots ought to be

$$\frac{2 \pm \sqrt{2^2 - 4\cdot 8}}{2\cdot 2} = \frac{2 \pm \sqrt{-28}}{4}$$

and the square root is not something we can do right now. This means that the denominator is an irreducible quadratic polynomial and so we should compute this integral using a trigonometric substitution. We can do this upon completing the square in the denominator; that is, write

$$2v^2 - 2v + 1 = 2\Big(v^2 - v + \frac{1}{2}\Big) = 2\Big(\Big(v - \frac{1}{2}\Big)^2 + \frac{1}{4}\Big).$$

Putting this into the denominator of our rational function, this suggests that we then need to make the trigonometric substitution

$$v - \frac{1}{2} = \frac{1}{2}\tan(\theta).$$

Putting these together, we get that

$$\begin{aligned}
\int \frac{\sin(\psi)\cos(\psi)}{\sin(\psi)^4 + \cos(\psi)^4}\,d\psi &= \frac{1}{2}\int \frac{dv}{2v^2 - 2v + 1} \\
&= \frac{1}{2}\int \frac{dv}{2((v-1/2)^2 + 1/4)} \\
&= \frac{1}{4}\int \frac{\sec(\theta)^2}{\frac{1}{4}(\tan(\theta)^2 + 1)}\,d\theta \\
&= \int 1\,d\theta = \theta = \arctan(2v - 1) = \arctan(2\sin(\psi)^2 - 1).
\end{aligned}$$

We can simplify the argument of arctan a bit more: using the half-angle identity for squares for sin,

$$\int \frac{\sin(\psi)\cos(\psi)}{\sin(\psi)^4 + \cos(\psi)^4}\,d\psi \arctan(2\sin(\psi)^2 - 1) = \arctan(-\cos(2\psi)) = -\arctan(\cos(2\psi)).$$

And that's a wrap! ■

2. Compute
$$\int_0^{\pi/2} \sin(x)^{2n}\,dx$$
for $n = 0,1,2,3$. Can you guess what the value of the integral might be for arbitrary n?

Solution. It is good for your soul to compute by hand the initial values of this integral. But now, once you have computed a few values, to try to find a pattern you might try to relate the integral of $\sin(x)^{2n}$ to, say, $\sin(x)^{2n-2}$; after all, since you have done some work at this point, might as well try to use it! Thinking about this a bit, you might realize that one way to do this would be integration by parts:
$$\begin{aligned}\int_0^{\pi/2} \sin(x)^{2n}\,dx &= \int_0^{\pi/2} \sin(x)\sin(x)^{2n-1}\,dx \\ &= -\cos(x)\sin(x)^{2n-1}\Big|_{x=0}^{x=\pi/2} + (2n-1)\int_0^{\pi/2} \cos(x)^2\sin(x)^{2n-2}\,dx.\end{aligned}$$
Now the first term on the extreme right is rather special: since $\cos(\pi/2) = 0$ and $\sin(0) = 0$, both evaluations of $\cos(x)\sin(x)^{2n-1}$ must vanish. Thus the integration by parts formula simplifies to
$$\int_0^{\pi/2} \sin(x)^{2n}\,dx = (2n-1)\int_0^{\pi/2} \cos(x)^2\sin(x)^{2n-2}\,dx.$$
Now the right hand side can be expressed purely in terms of sine functions upon invoking Pythagoreas:
$$\int_0^{\pi/2} \cos(x)^2\sin(x)^{2n-2}\,dx = \int_0^{\pi/2} \sin(x)^{2n-2}\,dx - \int_0^{\pi/2} \sin(x)^{2n}\,dx.$$
Therefore
$$\int_0^{\pi/2} \sin(x)^{2n}\,dx = (2n-1)\int_0^{\pi/2} \sin(x)^{2n-2}\,dx - (2n-1)\int_0^{\pi/2} \sin(x)^{2n}\,dx.$$
Now we see that we should combine the integrals of $\sin(x)^{2n}$ on both sides; upon rearranging and dividing by $2n$, we obtain the reduction formula
$$\int_0^{\pi/2} \sin(x)^{2n}\,dx = \frac{2n-1}{2n}\int_0^{\pi/2} \sin(x)^{2n-2}\,dx.$$
Therefore, in general,
$$\int_0^{\pi/2} \sin(x)^{2n}\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\dots\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{(2n-1)(2n-3)\cdots3\cdot1}{2n(2n-2)\cdots4\cdot2}\cdot\frac{\pi}{2}.$$
In particular, the first few integrals are
$$\int_0^{\pi/2} \sin(x)^0\,dx = \frac{\pi}{2}, \qquad \int_0^{\pi/2} \sin(x)^2\,dx = \frac{\pi}{4}, \qquad \int_0^{\pi/2} \sin(x)^4\,dx = \frac{3\pi}{16}, \qquad \int_0^{\pi/2} \sin(x)^6\,dx = \frac{5\pi}{32}.$$
And now you can go on, and on, and on... ■

3. Evaluate the integral

$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$$

by completing the square in the denominator and doing a trigonometric substitution.

Solution. One can directly complete the square and perform a trigonometric substitution to compute this integral. Instead, let me split this integral up to simpler ones to compute, the point being that the numerator almost looks like the polynomial in the denominator:

$$\begin{aligned}\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx &= \int \frac{(x^2-2x+2)+(2x-2)+1}{(x^2-2x+2)^2}\,dx \\ &= \int \frac{dx}{x^2-2x+2} + \int \frac{2x-2}{(x^2-2x+2)^2}\,dx + \int \frac{dx}{(x^2-2x+2)^2}\end{aligned}$$

To compute the first and third integral, we will need to complete the square in the denominator and proceed with a trigonometric substitution. But first, let's do the second integral. The numerator was carefully chosen so that I can perform the substitution $u = x^2-2x+2$, since then $du = (2x-2)\,dx$. Therefore

$$\int \frac{2x-2}{(x^2-2x+2)^2}\,dx = \int \frac{du}{u^2} = \frac{-1}{u} = \frac{-1}{x^2-2x+2}.$$

Okay, now for the first and third integrals. Complete the square:

$$x^2-2x+2 = (x-1)^2+1.$$

Thus we perform the trigonometric substitution $x-1 = \tan(\theta)$. Then the first integral amounts to

$$\int \frac{dx}{x^2-2x+2} = \int \frac{\sec(\theta)^2}{\tan(\theta)^2+1}\,d\theta = \theta = \arctan(x-1).$$

As for the third integral, we simplify, use the definition of $\sec(\theta)$ and then a half-angle identity to get:

$$\begin{aligned}\int \frac{dx}{(x^2-2x+2)^2} &= \int \frac{\sec(\theta)^2}{(\tan(\theta)^2+1)^2}\,d\theta \\ &= \int \frac{d\theta}{\sec(\theta)^2} \\ &= \int \cos(\theta)^2\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = \frac{1}{2}\arctan(x-1) + \frac{1}{4}\sin(2\arctan(x-1)).\end{aligned}$$

Now it's fairly good practice to try to simplify expressions like $\sin(2\arctan(x-1))$. Perusing a list of trigonometric identities, one might stumble upon the double angle formulae, which then gives

$$\sin(2\arctan(x-1)) = \frac{2\tan(\arctan(x-1))}{1+\tan(\arctan(x-1))^2} = \frac{2(x-1)}{x^2-2x+2}.$$

Putting everything together, we get

$$\begin{aligned}\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx &= \frac{3}{2}\arctan(x-1) + \frac{1}{2}\frac{x-1}{x^2-2x+2} - \frac{1}{x^2-2x+1} \\ &= \frac{1}{2}\left(3\arctan(x-1) + \frac{x-3}{x^2-2x+2}\right).\end{aligned}$$

Phew! ■

4. Let's think about the calculations that we performed in Assignment 2, Question **2**. A function of the form

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_N \sin(Nx)$$

is called a *finite Fourier series.* Show that

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx.$$

Moreover, verify that

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = 0$$

for all positive integers m. This gives a way of extracting the coefficients a_i in a function of this form!

Solution. Okay, well, let's evaluate the integral at hand by plugging in the definition of the function $f(x)$ and split the integral up using linearity:

$$\begin{aligned}\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx &= \frac{1}{\pi} \int_{-\pi}^{\pi} \big(a_1 \sin(x) + a_2 \sin(x) + \cdots + a_N \sin(Nx)\big) \sin(mx)\, dx \\ &= \frac{a_1}{\pi} \int_{-\pi}^{\pi} \sin(x) \sin(mx)\, dx + \frac{a_2}{\pi} \int_{-\pi}^{\pi} a_2 \sin(2x) \sin(mx)\, dx + \cdots + \frac{a_N}{\pi} \int_{-\pi}^{\pi} \sin(Nx) \sin(mx)\, dx.\end{aligned}$$

Now by Assignment 2, Question **2**, whenever $n \neq m$,

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) = 0.$$

Therefore, the only term that contributes to the integral of $f(x)\sin(mx)$ is the term corresponding to $a_m \sin(mx)$. In other words,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx = \frac{a_m}{\pi} \int_{-\pi}^{\pi} \sin(mx) \sin(mx)\, dx = \frac{a_m}{\pi}\pi = a_m$$

as claimed. The vanishing

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx =$$

now comes from the exact same calculation, but now using the fact that $\sin(nx)\cos(mx)$ always pairs to 0 when integrated between $-\pi$ and π. ■

5. The calculation in **3.** suggests a way to approximate certain functions. Consider the function

$$f(x) := \begin{cases} -1 & \text{if } -\pi \leq x < 0, \\ 1 & \text{if } 0 \leq x \leq \pi. \end{cases}$$

(i) Calculate

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx$$

for $m = 1, 2, 3, 4, 5, 6, 7, 8$.

(ii) Let

$$f_i(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_{2i-1} \sin((2i-1)x) + a_{2i} \sin(2ix).$$

With the help of a computer, graph the functions f_1, f_2, f_3, and f_4 in the interval $[-\pi, \pi]$. Also draw the graph of f.

Solution. In fact, we can compute the coefficients a_m in general; their behaviour only depends on the parity of m. To compute the integral, split it at 0 so that we can substitute the piecewise definition of f:

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx &= \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin(mx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin(mx)\, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{0} -\sin(mx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin(mx)\, dx \\ &= \frac{1}{m\pi} \cos(mx) \Big|_{x=-\pi}^{x=0} + \frac{1}{m\pi} \cos(mx) \Big|_{x=\pi}^{x=0} \\ &= \frac{1}{m\pi} \big(2\cos(0) - \cos(-m\pi) - \cos(m\pi) \big). \end{aligned}$$

Now $\cos(m\pi) = 1$ if m is even and $\cos(m\pi) = -1$ if m is even. Putting this into the final line, we see that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx = \begin{cases} 0 & \text{if } m \text{ is even}, \\ \dfrac{4}{m\pi} & \text{if } m \text{ is odd}. \end{cases}$$

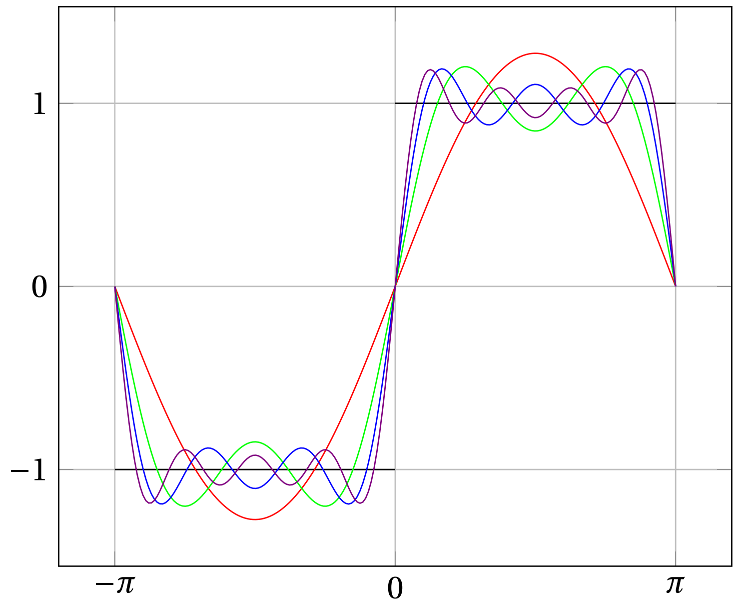
In particular, the first few values of a_m are

$$\frac{4}{\pi}, 0, \frac{4}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{4}{7\pi}, 0.$$

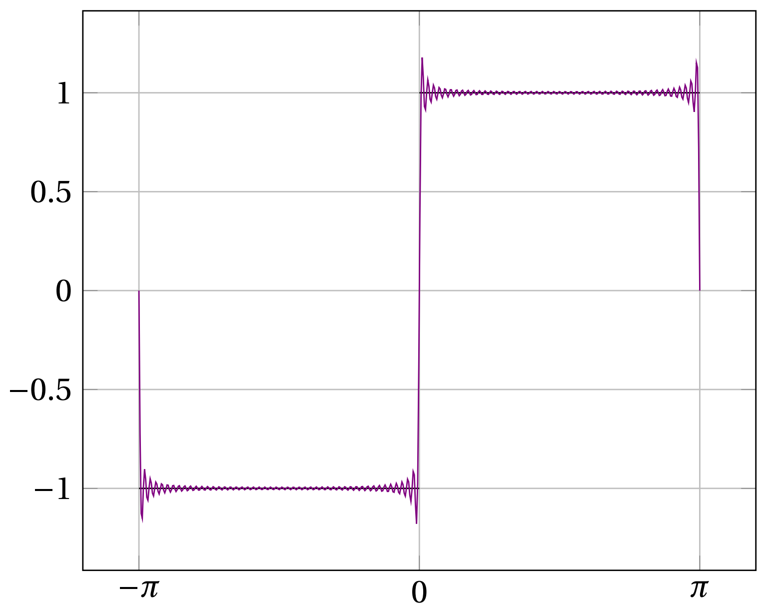
With the computations from (i), we see that the functions f_i defined above take the form

$$\begin{aligned} f_1(x) &= \frac{4}{\pi} \sin(x), \\ f_2(x) &= \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x), \\ f_3(x) &= \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \frac{4}{5\pi} \sin(5x), \\ f_4(x) &= \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \frac{4}{5\pi} \sin(5x) + \frac{4}{7\pi} \sin(7x). \end{aligned}$$

Plotting these functions, we have the following:



In this picture, f_1 is pictured in red, f_2 in green, f_3 in blue, and f_4 in violet. Notice how the more terms we add to f_i, the more it wiggles around the function f, pictured in thicker black. One might imagine that as we add more terms to the f_i, the closer f_i hugs f. Indeed, this is what happens: consider, for example, the following plot of f_{50}:



Perhaps in the limit, f_i, in some sense, becomes f! This is a subject that you might encounter in more depth in courses on Fourier analysis, signal processing, etc. ■