Derived Categories and Spectral Sequences

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1 Derived Categories

Setting $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are categories (abelian unless otherwise mentioned). Objects are given by Ob and morphisms by Mor (or Hom if abelian). Morphisms denoted by lowercase, functors denoted F, G, H.

Localization What can be done for rings can be done for categories. Let $\mathcal{A}$ be a category and S an arbitrary class of morphisms in $\mathcal{A}$. There exists a category $\mathcal{A}[S^{-1}]$ unique up to isomorphism and a functor $Q\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$ such that

1. $Q(s)$ is an isomorphism for every $s \in S$, and
2. for any category $\mathcal{B}$ and functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all s, there exists a unique functor $G\colon \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$.

Construction: literally add in all the backward arrows in S^{-1}.

Localizing class If S is arbitrary, hard to preserve any structure that $\mathcal{A}$ might have, e.g. additive (**Ab**-enriched with finite (bi)products), abelian (additive with (co)kernels and normal monos and epis). A class of morphisms S in $\mathcal{A}$ is a localizing class if:

1. for any $M \in \mathrm{Ob}(\mathcal{A})$, the identity id_M is in S, and S is closed under composition,
2. given the solid lines, the dotted lines must exist:

$$
\begin{tikzcd}
K \arrow[r, dashed, "g\in\mathrm{Mor}\,\mathcal{A}"] \arrow[d, dashed, "t\in S"] & L \arrow[d, "s\in S"] \\
M \arrow[r, "f\in\mathrm{Mor}\,\mathcal{A}"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
K & L \arrow[l, dashed, "g\in\mathrm{Mor}\,\mathcal{A}"'] \\
M \arrow[u, dashed, "t\in S"] & N, \arrow[l, "f\in\mathrm{Mor}\,\mathcal{A}"] \arrow[u, "s\in S"']
\end{tikzcd}
$$

3. if $f, g\colon M \to N$ be two morphisms, $\exists s \in S : s \circ f = s \circ g \iff \exists t \in S : f \circ t = g \circ t$.

Morphisms added by localizing may be weird: path components do not commute. Localizing class forces them to, so now we can represent elements of $\mathcal{A}[S^{-1}]$ by **left/right roofs**:

$$
\begin{tikzcd}
 & L \arrow[dl, "s\in S"', "\sim"] \arrow[dr, "f\in\mathrm{Mor}\,\mathcal{A}"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K & \\
M \arrow[ur, "g\in\mathrm{Mor}\,\mathcal{A}"] & & N \arrow[ul, "t\in S"', "\sim"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L \arrow[dl, "s\in S"', "\sim"] \arrow[dr, "f"] & \\
M & H \arrow[u, "p"] \arrow[d, "q"] & N \\
 & K. \arrow[ul, "t\in S", "\sim"'] \arrow[ur, "g"'] &
\end{tikzcd}
$$

These two roofs are **equivalent** if there exists a common denominator, i.e. an object $H \in \mathrm{Ob}\,\mathcal{A}$ and morphisms $p\colon H \to L$ and $q\colon H \to K$ such that the third diagram commutes and $s \circ p = t \circ q \in S$.

Localization wrt class Preserves additive and abelian categories. Localizing functor Q is exact.

Category of complexes of $\mathcal{A}$-objects Denoted $C(\mathcal{A})$, for an additive category $\mathcal{A}$. Direct sum of complexes makes $C(\mathcal{A})$ an additive category, and $C\colon \mathcal{A} \to C(\mathcal{A})$ is fully faithful.

Homotopy category of complexes of $\mathcal{A}$-objects Denoted $K(\mathcal{A})$, and is $\operatorname{Ob} C(\mathcal{A})$ with classes of (chain) homotopic morphisms as morphisms. Since cohomology is invariant under homotopy, cohomology is well-defined on $K(\mathcal{A})$.

Quasi-isomorphisms in $K(\mathcal{A})$ A morphism $f\colon X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ is a quasi-isomorphism if induced maps $H^p(f)\colon H^p(X^\bullet) \to H^p(Y^\bullet)$ are isomorphisms for all $p \in \mathbb{Z}$. Quasi-isomorphism is invariant under homotopy.

Derived category of $\mathcal{A}$ Let S be the set of quasi-isomorphisms in $K(\mathcal{A})$. Fact: S is a localizing class. Derived category is $D(\mathcal{A}) := K(\mathcal{A})[S^{-1}]$.

Lift of functor Given $F\colon \mathcal{A} \to \mathcal{B}$ an additive (homomorphism on each abelian group) functor, can lift F to $C(F)\colon C(\mathcal{A}) \to C(\mathcal{B})$ by

$$C(F)(X^\bullet)^p = F(X^p), \quad C(F)(f^\bullet)^p = F(f^p) \quad \forall p \in \mathbb{Z}\,.$$

Can even lift to $K(F)$, with $K(G \circ F) = K(G) \circ K(F)$. Fact: $K(F)\colon K(\mathcal{A}) \to K(\mathcal{B})$ is **exact**. But how to lift to $D(\mathcal{A}) \to D(\mathcal{B})$?

Derived functors Use UP of localization! The composition $Q_\mathcal{B} \circ K(F)\colon K(\mathcal{A}) \to D(\mathcal{B})$ factors through $D(\mathcal{A})$ . . . no it doesn't: for a quasi-isomorphism $s\colon X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$, the resulting $K(F)(s)\colon K(F)(X^\bullet) \to K(F)(Y^\bullet)$ may not be a quasi-isomorphism.

Specific case Fact: if $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor (instead of additive one), then condition holds, and $Q_\mathcal{B} \circ K(F)$ factors as $D(F) \circ Q_\mathcal{A}$.

General case Approximate the factorization. A **right derived functor** of $F\colon \mathcal{A} \to \mathcal{B}$ additive is an exact functor $RF\colon D(\mathcal{A}) \to D(\mathcal{B})$ and a (graded) morphism of functors $\epsilon_F\colon Q_\mathcal{B} \circ K(F) \to RF \circ Q_\mathcal{A}$ initial among all such functors, i.e. satisfying the following UP:

1. Let $G\colon D(\mathcal{A}) \to D(\mathcal{B})$ be an exact functor and $\epsilon\colon Q_\mathcal{B} \circ K(F) \to G \circ Q_\mathcal{A}$ a graded morphism of functors. Then there exists a unique graded morphism of functors $\eta\colon RF \to G$ such that ϵ factors as $\eta \circ Q_\mathcal{A} \circ \epsilon_F$.

Similar for left derived functor: $LF\colon D(\mathcal{A}) \to D(\mathcal{B})$ and $\epsilon_F\colon LF \circ Q_\mathcal{A} \to Q_\mathcal{B} \circ K(F)$ with dualized UP.

Composition of derived functors Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be additive functors. Assume the derived functors $RF, RG, R(G \circ F)$ exist, i.e. there exist graded morphisms of functors

$$Q_\mathcal{C} \circ K(G) \xrightarrow{\epsilon_G} RG \circ Q_\mathcal{B}, \quad Q_\mathcal{B} \circ K(F) \xrightarrow{\epsilon_F} RF \circ Q_\mathcal{A},$$

composing with which we get

$$Q_\mathcal{C} \circ K(G) \circ K(F) \xrightarrow{\epsilon_G \circ K(F)} RG \circ Q_\mathcal{B} \circ K(F), \quad RG \circ Q_\mathcal{B} \circ K(F) \xrightarrow{RG \circ \epsilon_F} RG \circ RF \circ Q_\mathcal{A}.$$

The composition $\kappa\colon Q_\mathcal{C} \circ K(G) \circ K(F) \to RG \circ RF \circ Q_\mathcal{A}$ factors through $R(G \circ F) \circ Q_\mathcal{A}$ by the UP of right derived functors: $\kappa = (\eta \circ Q_\mathcal{A}) \circ \epsilon_{G \circ F}$ with $\eta\colon R(G \circ F) \to RG \circ RF$ a graded morphism of functors.

Fact Under mild restrictions (which we'll see later), η is an isomorphism.

2 Spectral Sequences

Goal Explicitly write down this isomorphism in a computable fashion.

Filtered chain complex in $C(\mathcal{A})$ Is a filtered object in $C(\mathcal{A})$, i.e. a (descending) chain

$$\cdots \hookrightarrow F^{p+1}X^\bullet \hookrightarrow F^pX^\bullet \hookrightarrow F^{p-1}X^\bullet \hookrightarrow \cdots \hookrightarrow X^\bullet$$

where the differentials of $X^\bullet$ must respect the filtering, so that $d(F^pX^n) \subset F^pX^{n+1}$. The **associated graded complex** is given by $G^pX^\bullet = F^pX^\bullet/F^{p+1}X^\bullet$.

Objects Let $F^\bullet X^\bullet$ be a filtered chain complex. For $p, q, r \in \mathbb{Z}$, define the following.

1. The module of (p, q)-**cochains** is G^pX^{p+q}.
2. The module of r-**almost** (p, q)-**cocycles** is

$$\begin{aligned} Z_r^{p,q} &:= \{x \in G^pX^{p+q} : dx = 0 \bmod F^{p+r}X^\bullet\} \\ &= \{x \in F^pX^{p+q} : dx \in F^{p+r}X^{p+q+1}\}/F^{p+1}X^{p+q}. \end{aligned}$$

3. The module of (p, q)-**cocycles** is

$$Z_\infty^{p,q} := \{x \in F^pX^{p+q} : dx = 0\}/F^{p+1}X^{p+q} = Z(G^pX^{p+q}).$$

4. The module of r-**almost** (p, q)-**coboundaries** is $B_r^{p,q} := d(F^{p-r+1}X^{p+q-1})$.
5. The module of (p, q)-**coboundaries** is $B_\infty^{p,q} := d(F^pX^{p+q-1})$.

Proposition 2.1. *The differential d of $X^\bullet$ restricts to*

$$d_r \colon Z_r^{p,q} \to Z_r^{p+r,q-r+1}$$

on r-almost cocycles. It still forms a complex: $(d_r)^2 = 0$.

Proof. Let $x \in Z_r^{p,q}$. Then x represents an element of F^pX^{p+q}. By the definition of Z_r^{p+q}, the differential dx lives in $F^{p+r}X^{p+q+1}$. But $d(dx) = 0$, so

$$dx \in Z_\infty^{p+r,q-r+1} \subset Z_r^{p+r,q-r+1}.$$

Clearly $(d_r)^2 = 0$, since $d^2 = 0$. □

Proposition 2.2. $Z_{r+1}^{p,q} = \ker(Z_r^{p,q} \xrightarrow{d_r} Z_r^{p+r,q-r+1})$.

Proof. By the definition of $Z_r^{p+r,q-r+1}$,

$$x \in \ker(d_r) \iff dx \in F^{p+r+1}X^{p+q+1},$$

since $F^{p+r+1}X^{p+q+1}$ is quotiented out in the definition of $Z_r^{p+r,q-r+1}$. Similarly, by the definition of $Z_{r+1}^{p,q}$,

$$x \in Z_{r+1}^{p,q} \iff dx \in F^{p+r+1}X^{p+q+1}.$$

Hence $x \in \ker(d_r)$ if and only if $x \in Z_{r+1}^{p,q}$. □

Definition 2.3. Let $F^\bullet X^\bullet$ be a filtered chain complex in $C(\mathcal{A})$. For $p, q, r \in \mathbb{Z}$, the r-**almost** (p, q)-**cohomology** of the filtered complex is the quotient

$$\begin{aligned} E_r^{p,q} &:= Z_r^{p,q}/B_r^{p,q} \\ &= \frac{\{x \in F^pX^{p+q} : dx \in F^{p+r}X^{p+q+1}\}}{d(F^{p-r+1}X^{p+q-1}) \oplus F^{p+1}X^{p+q}}. \end{aligned}$$

The modules $E_r^{p,q}$ along with the differentials $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ form the **spectral sequence of the filtered complex $F^\bullet X^\bullet$**.

Spectral Sequences A **(cohomology) spectral sequence** in $\mathcal{A}$ is given by the following data:

1. for each $r \in \mathbb{Z}_{\geq 0}$, a family $(E_r^{p,q})$ of objects in $\mathcal{A}$, for all $p, q \in \mathbb{Z}$, called the **r-th page** of the spectral sequence;
2. morphisms $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $(d_r)^2 = 0$, called **differentials**; and
3. isomorphisms $\alpha_r^{p,q} \colon \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-1}) \to E_{r+1}^{p,q}$.

For the spectral sequence of a filtered complex, we showed $Z_{r+1}^{p,q} \cong \ker(d_r^{p,q})$ and $B_{r+1}^{p,q} \cong \operatorname{im}(d_r^{p-r,q+r-1})$ is straightforward; that's where the isomorphism α_r comes from.

Proposition 2.4. *For a filtered complex, we have*

$$E_0^{p,q} = G^p X^{p+q} = F^p X^{p+q}/F^{p+1}X^{p+q},$$
$$E_1^{p,q} = H^{p+q}(G^p X^\bullet).$$

Proof. Since the differential preserves the filtering, $d(F^p X^{p+q}) \subset F^p X^{p+q+1}$, so

$$E_0^{p,q} = Z_0^{p,q}/B_0^{p,q} = \frac{\{x \in F^p X^{p+q} : dx \in F^p X^{p+q+1}\}}{d(F^{p+1}X^{p+q-1}) \oplus F^{p+1}X^{p+q}} = \frac{F^p X^{p+q}}{F^{p+1}X^{p+q}} = G^p X^{p+q}.$$

Similarly, recalling that $G^p X^\bullet = F^p X^\bullet / F^{p+1} X^\bullet$,

$$\begin{aligned} E_1^{p,q} = Z_1^{p,q}/B_1^{p,q} &= \frac{\{x \in G^p X^{p+q} : dx = 0 \bmod F^{p+1}X^{p+q+1}\}}{d(F^p X^{p+q-1})} \\ &= \frac{\{x \in G^p X^{p+q} : dx = 0 \in G^p X^{p+q+1}\}}{d(F^p X^{p+q-1})} = H^{p+q}(G^p X^\bullet), \end{aligned}$$

where it suffices to quotient out by $d(F^p X^{p+q-1})$ instead of $d(G^p X^{p+q-1})$ since $d(F^{p+1}X^{p+q-1}) \subset F^{p+1}X^{p+q}$ is quotiented out in $G^p X^{p+q}$ anyway. □

Summary Suppose now that we have a filtered chain complex $F^\bullet X^\bullet$, and we know how to compute the cohomology $H^{p+q}(G^p X^\bullet)$ of its graded pieces. Then we know exactly what the first page of the spectral sequence is. But given a page of the spectral sequence, its construction tells us exactly how to compute the next page. So in principle, starting with the zeroth or first page, we can compute all the pages of the spectral sequence. The idea now is that eventually, the pages of the sequence stabilize, and in particular, they stabilize to the graded pieces $G^p(H^{p+q}X^\bullet)$ of the cohomology of $X^\bullet$.

Convergence Let $\{E_r^{p,q}\}$ be a spectral sequence such that for each $p, q \in \mathbb{Z}$, there exists $R(p,q) \in \mathbb{Z}$ such that

$$E_r^{p,q} \cong E_{R(p,q)}^{p,q} \quad \forall r \in \mathbb{Z}_{\geq R(p,q)},$$

i.e. after the $R(p,q)$-th page, the (p,q)-th entry stops changing. Then the spectral sequence is said to **converge** to the bigraded object

$$E_\infty := \{E_{R(p,q)}^{p,q}\},$$

which is then called the **limit** of the spectral sequence. If $\{E_\infty^{p,q}\}$ can be written as the associated graded complex $\{G^p H^{p+q}\}$ of some graded object $H^\bullet$, then we say the spectral sequence **converges to** $H^\bullet$, and denote this convergence by

$$E_r^{p,q} \Rightarrow H^\bullet.$$

In practice, we often refer to spectral sequences by their first non-trivial page, which is usually $E_2^{p,q}$.

Cohomology filtration Let $F^\bullet X^\bullet$ be a filtered chain complex. For $p \in \mathbb{Z}$, let

$$F^p H^\bullet(X^\bullet) \coloneqq \operatorname{im}(H^\bullet(F^p X^\bullet) \to H^n(X^\bullet)).$$

The $F^p H^\bullet(X^\bullet)$ define a filtration on $H^\bullet(X^\bullet)$.

Proposition 2.5. *Given a bounded filtered complex $F^\bullet X^\bullet$, the spectral sequence $\{E_r^{p,q}\}$ of $F^\bullet X^\bullet$ has a limit: it converges to the cohomology $H^\bullet(X^\bullet)$, i.e.*

$$E_r^{p,q} \Rightarrow H^\bullet(X^\bullet),$$

with the filtration on $H^\bullet(X^\bullet)$ just defined.

Proof. By the definition of $E_r^{p,q}$, for each $p, q \in \mathbb{Z}$ there exists an $R(p,q) \in \mathbb{Z}$ such that the $R(p,q)$-almost (p,q)-cocycles and coboundaries are actually (p,q)-cocycles and coboundaries. Hence for $r \geq R(p,q)$, we have

$$E_r^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} = G^p H^{p+q}(X^\bullet).$$

□

Spectral sequence of double complex Let $X^{\bullet,\bullet}$ be a double complex of objects in $\mathcal{A}$, with differentials $d_>\colon X^{p,q} \to X^{p+1,q}$ and $d_\wedge\colon X^{p,q} \to X^{p,q+1}$. The **horizontal filtration** and **vertical filtration** on the total complex $X^\bullet$ are respectively given by

$$F_>^p(X^n) \coloneqq \bigoplus_{i+j=n,\, i\geq p} X^{i,j}, \qquad F_\wedge^q(X^n) \coloneqq \bigoplus_{i+j=n,\, j\geq q} X^{i,j}.$$

The **horizontal spectral sequence of** $X^{\bullet,\bullet}$, which we denote $\{{}_>\mathrm{E}_r^{p,q}\}$, is the spectral sequence of $F_>^\bullet(X^\bullet)$. Likewise, the **vertical spectral sequence of** $X^{\bullet,\bullet}$, which we denote $\{{}_\wedge\mathrm{E}_r^{p,q}\}$ is the spectral sequence of $F_\wedge^\bullet(X^\bullet)$.

Proposition 2.6. *Let $\{{}_>\mathrm{E}_r^{p,q}\}$ be the horizontal spectral sequence of a double complex $X^{\bullet,\bullet}$. Then*

$$\begin{aligned} {}_>\mathrm{E}_0^{p,q} &= X^{p,q}, \\ {}_>\mathrm{E}_1^{p,q} &= H_\wedge^{p,q}(X^{\bullet,\bullet}), \\ {}_>\mathrm{E}_2^{p,q} &= H^p(H_\wedge^{\bullet,q}(X^{\bullet,\bullet})). \end{aligned}$$

Proof. Very similar to the low degree pages of the spectral sequence of a filtered complex; exercise. □

3 Grothendieck Spectral Sequence

From now on $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ are additive functors between abelian categories. We assume $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have enough injectives.

Theorem 3.1 (Grothendieck spectral sequence)**.** *Suppose that for every injective object $I \in \mathcal{A}$ the object $F(I) \in \mathcal{B}$ is G-acyclic. Then for any object $X^\bullet \in D^+(\mathcal{A})$, there exists a spectral sequence*

$$E_2^{p,q} = (R^p G \circ R^q F)(X^\bullet) \Rightarrow R^n(G \circ F)(X^\bullet)$$

functorial in $X^\bullet$.

The key to constructing the desired spectral sequence is the construction of a Cartan-Eilenberg resolution.

Definition 3.2. A **Cartan-Eilenberg resolution** of a complex $X^\bullet \in C^+(\mathcal{A})$ is a double complex $I^{\bullet,\bullet}$ and a morphism $\varepsilon\colon X^\bullet \to I^{\bullet,0}$ such that

1. $I^{\bullet,q} = 0$ for $q < 0$,

2. the complex $I^{p,\bullet}$ is an injective resolution of X^p,
3. the complex $\ker(I^{p,\bullet} \xrightarrow{d_>} I^{p+1,\bullet})$ is an injective resolution of $\ker(d^p_{X^\bullet})$,
4. the complex $\operatorname{im}(I^{p,\bullet} \xrightarrow{d_>} I^{p+1,\bullet})$ is an injective resolution of $\operatorname{im}(d^p_{X^\bullet})$, and
5. the complex $H^{p,\bullet}_>(I^{\bullet,\bullet})$ is an injective resolution of $H^p(X^\bullet)$.

Proposition 3.3. *Let $\mathcal{A}$ have enough injectives and $X^\bullet \in C^+(\mathcal{A})$. Then there exists a Cartan-Eilenberg resolution of $X^\bullet$.*

The existence of a Cartan-Eilenberg resolution enables us to compute derived functors easily, using the spectral sequence of the double complex given by the resolution.

Proposition 3.4. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be left exact, and $X^\bullet \in C^+(\mathcal{A})$. Suppose $\mathcal{A}$ has enough injectives, and let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $X^\bullet$. The horizontal and vertical spectral sequences of the double complex $F(I^{\bullet,\bullet})$ are*

$$
\begin{aligned}
{}_>\mathrm{E}_1^{p,q} &= R^qF(X^p) \Rightarrow RF(X^\bullet),\\
{}_\wedge\mathrm{E}_2^{p,q} &= R^pF(H^q(X^\bullet)) \Rightarrow RF(X^\bullet).
\end{aligned}
$$

Construction of Grothendieck SS. Steps:

1. Can pick an **injective resolution** $X^\bullet \to I^\bullet$ of $X^\bullet$, i.e. $I^\bullet$ consists of all injectives, and the map is a quasi-isomorphism.
2. Construct the **Cartan–Eilenberg resolution** $F(I^\bullet) \to J^{\bullet,\bullet}$, i.e. where $H^{p,\bullet}_>(J^{\bullet,\bullet})$ is an injective resolution of $H^p(X^\bullet)$.
3. The vertical spectral sequence of $G(J^{\bullet,\bullet})$ is therefore (by the proposition)

$$
\begin{aligned}
{}_\wedge\mathrm{E}_2^{p,q} &= R^pG(H^q(F(I^\bullet))) = R^pG(R^qF(X^\bullet))\\
&\Rightarrow H^\bullet(G(F(I^\bullet))) = R(G\circ F)(X^\bullet).
\end{aligned}
$$

□

4 Examples

Definition 4.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, and $f : X \to Y$ a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf. Recall the **direct image functor**

$$
f_*\colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(\mathcal{O}_Y), \quad \mathcal{F} \mapsto (V \mapsto \mathcal{F}(f^{-1}(V)))
$$

between the category $\operatorname{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules and $\operatorname{Mod}(\mathcal{O}_Y)$ of $\mathcal{O}_Y$-modules. Let $R = \mathcal{O}_X(X)$. Recall that the subcategory of injective $\mathcal{O}_X$-modules is an adapted subcategory of the global sections functor $\Gamma_X\colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(R)$. Hence we have a right derived functor

$$
R\Gamma_X\colon D^+(\operatorname{Mod}(\mathcal{O}_X)) \to D^+(\operatorname{Mod}(R)),
$$

which, by definition, is **sheaf cohomology**. But since $\Gamma_Y \circ f_* = \Gamma_X$, there is a spectral sequence

$$
E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F}) \Rightarrow H^\bullet(X, \mathcal{F}),
$$

known as the **Leray spectral sequence**.

Definition 4.2. Let X be a complex analytic space. Recall that the splitting $TX = T^{1,0}X \oplus T^{0,1}X$ gives the **Dolbeault double complex** $(\Omega^{\bullet,\bullet})$, with

$$\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}, \quad \bar{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$$

as differentials. Recall also that the **holomorphic de Rham complex** $\Omega^\bullet$ is the complex of holomorphic differential forms with ∂ as the differential; it arises as the total complex of $\Omega^{\bullet,\bullet}$. Finally, recall that by the Dolbeault theorem, the cohomology of the Dolbeault complex, i.e. the horizontal cohomology $H^{p,q}_{>}(\Omega^{\bullet,\bullet})$, is actually the sheaf cohomology $H^q(X, \Omega^p)$. Apply the spectral sequence of a double complex to the Dolbeault complex to get the spectral sequence

$${}_{\wedge}\mathrm{E}_1^{p,q} = H^q(X, \Omega^p) \Rightarrow H^\bullet(\Omega^\bullet),$$

known as the **Frölicher spectral sequence**.

5 Exact Sequence of Low-Degree Terms

Proposition 5.1. *Let $E_2^{p,q} \Rightarrow H^\bullet$ be a spectral sequence whose terms are non-trivial only for $p, q \geq 0$. Then*

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$$

is exact.

Proof. Since $E_2^{p,q}$ converges to $H^\bullet$, we have $H^1 = E_\infty^{1,0} \oplus E_\infty^{0,1}$. But

$$E_3^{1,0} = \frac{\ker(d_2^{1,0}\colon E_2^{1,0} \to E_2^{3,-1} = 0)}{\operatorname{im}(d_2^{-1,1}\colon E_2^{-1,1} = 0 \to E_2^{1,0})} = E_2^{1,0},$$

and similarly $E_2^{1,0} = E_3^{1,0} = \cdots = E_\infty^{1,0}$. Hence $E_2^{1,0} \to H^1$ is an injection. By a similar argument, $E_3^{0,1} = E_\infty^{0,1}$. Since $E_\infty^{0,1} \to E_2^{0,1}$ is an injection,

$$\ker(H^1 \to E_3^{0,1} = E_\infty^{0,1} \to E_2^{0,1}) = \ker(H^1 \to E_\infty^{0,1}) = E_\infty^{1,0}.$$

Hence the sequence is exact at H^1. The image of H^1 in $E_2^{0,1}$, which contains $E_\infty^{0,1}$ but not $E_\infty^{1,0}$, is precisely $E_\infty^{0,1} = E_3^{0,1}$. But

$$E_3^{0,1} = \frac{\ker(d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0})}{\operatorname{im}(d_2^{-2,2}\colon E_2^{-2,2} = 0 \to E_2^{0,1})} = \ker(d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}),$$

so the sequence is exact at $E_2^{0,1}$. Finally, by the same argument showing that $E_2^{1,0} \to H^1$ is an injection, $E_3^{2,0} \to H^2$ is an injection. Now because

$$E_3^{2,0} = \frac{\ker(d_2^{2,0}\colon E_2^{2,0} \to E_2^{4,-1} = 0)}{\operatorname{im}(d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0})} = \frac{E_2^{2,0}}{\operatorname{im}(d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0})},$$

it follows that

$$\ker(E_2^{2,0} \to E_3^{2,0} \to H^2) = \ker(E_2^{2,0} \to E_3^{2,0}) = \operatorname{im}(d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}).$$

Hence the sequence is exact at $E_2^{2,0}$. □

Example 5.2. From the Grothendieck spectral sequence

$$E_2^{p,q} = R^pG(R^qF(X^\bullet)) \Rightarrow R(G \circ F)(X^\bullet),$$

there is an exact sequence

$$\begin{aligned} 0 \to R^1G(F(X^\bullet)) \to R^1(G \circ F)(X^\bullet) \to G(R^1F(X^\bullet)) \\ \to R^2G(F(X^\bullet)) \to R^2(G \circ F)(X^\bullet). \end{aligned}$$