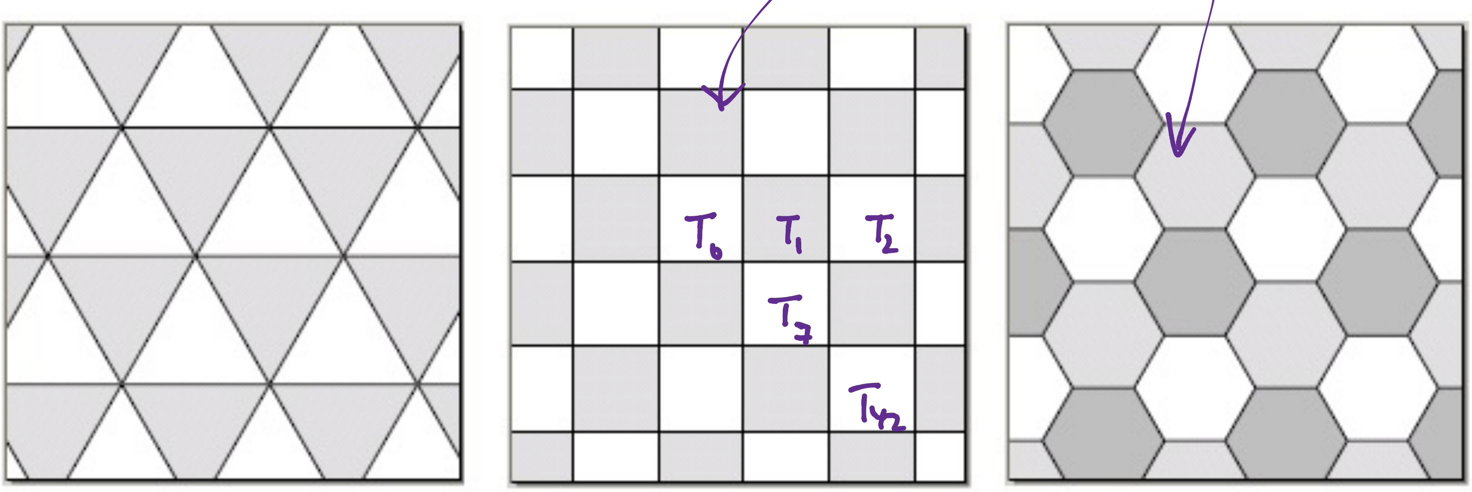
Tilings

Meetings: Fridays 1 PM – 3PM, Mathematics 507

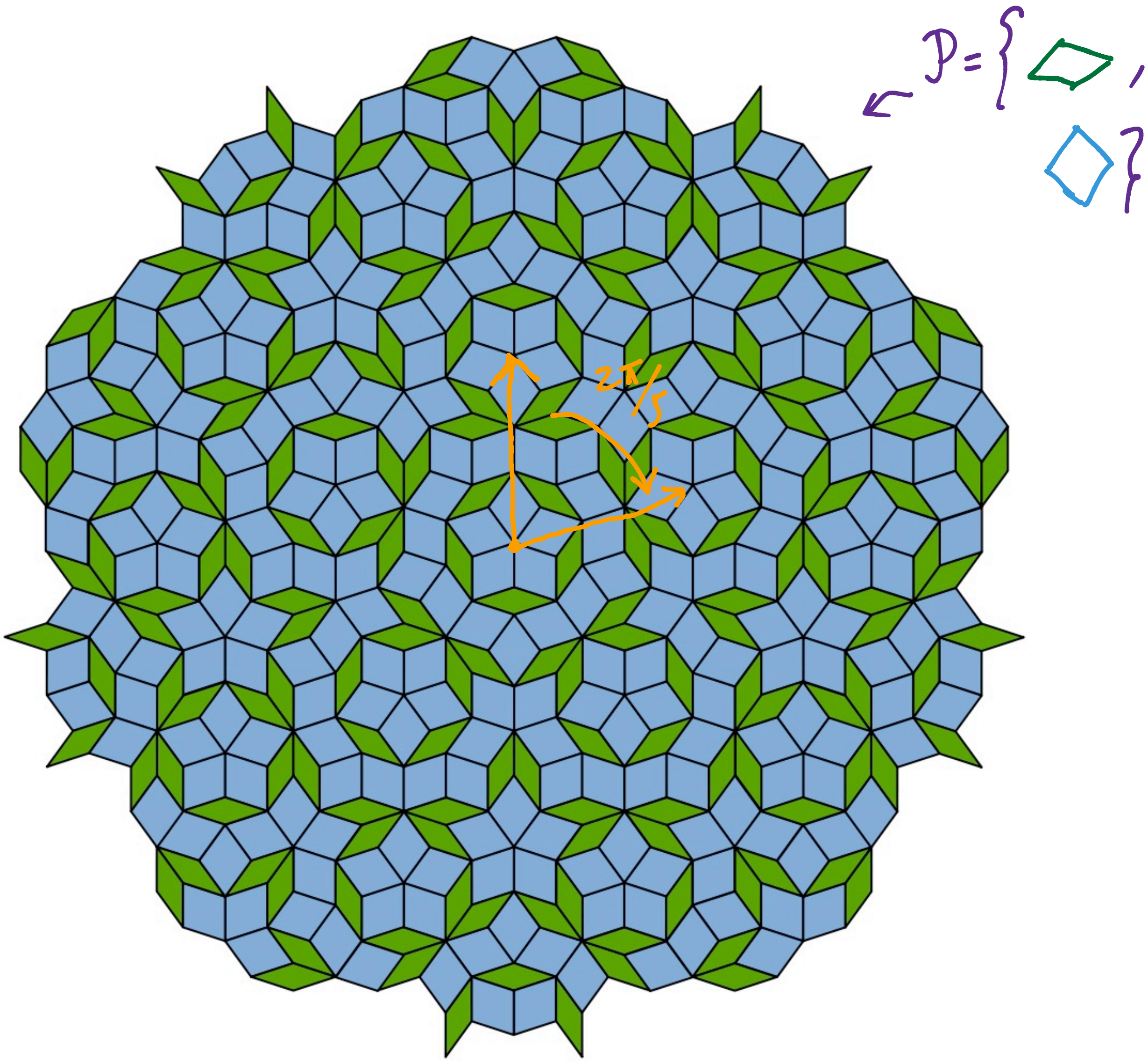
Webpage: math.columbia.edu/~rcheng/S2022

What is a tiling?

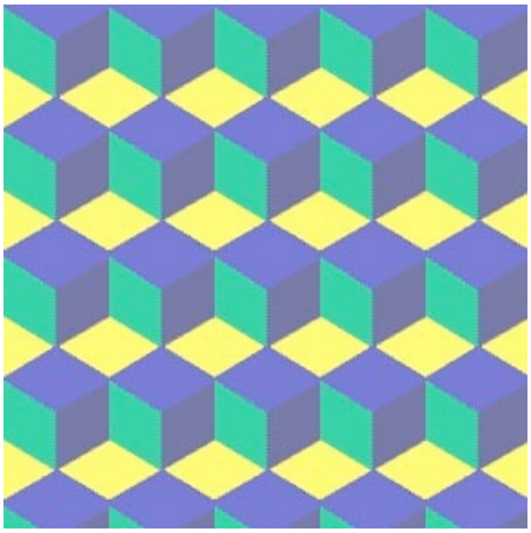
$\mathcal{P} = \{ \square \}$ $\quad\quad$ $\mathcal{P} = \{ \text{⬡} \}$

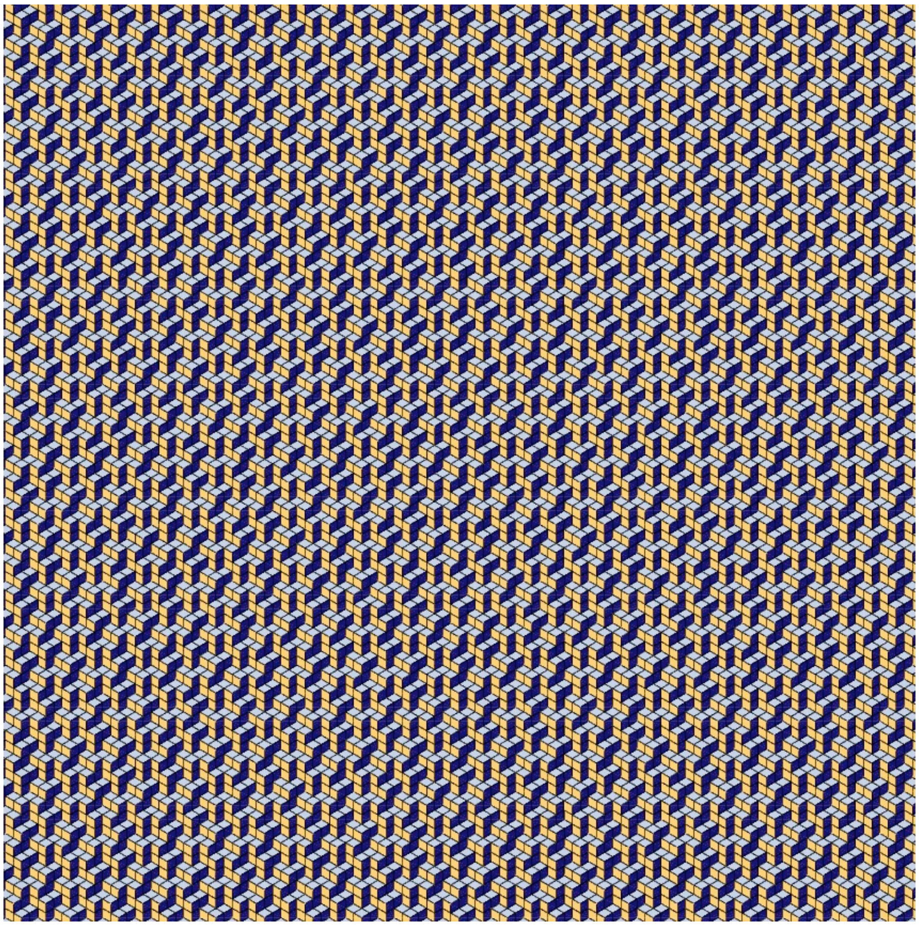




P = { , }
2π/5





Roughly speaking, a tiling amounts to:

"Covering a big shape by small shapes."

More precisely, can define a tiling as follows:

Def$^{\underline{n}}$ A *tiling* of a region $R \subseteq \mathbb{R}^2$ is a collection

$$\mathcal{T} = \{ T_1, T_2, T_3, \dots \}$$

of closed subsets $T_i \subseteq R$ such that :

(i) they cover R : $\bigcup_i T_i = R$

(ii) they do not overlap : $T_i^{\circ} \cap T_j^{\circ} = \emptyset$ if $i \neq j$.

This definition is way too general !

e.g.
- disconnected tiles
- tiles all w/ area zero

$\underline{\text{Def}^{\underline{n}}}$ $\underline{\text{Prototiles}}$ for $\underline{(\mathbb{R}, \mathcal{T})}$ is a set

$$\mathcal{P} = \{P_1, P_2, P_3, \dots\} \text{ w/ } P_i \subseteq \mathbb{R}^2$$

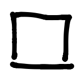
s.t. every $T_i \in \mathcal{T}$ congruent to a $P_j \in \mathcal{P}$

Basic Questions.

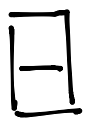
- Does there exist a tiling of a region R with prototiles $\mathcal{P}$?
- If so, how many?
- If not, how to prove?
- Relations amongst tilings?
- Symmetries?

Polynominoes. These are tetris block of general size:

Monomino.

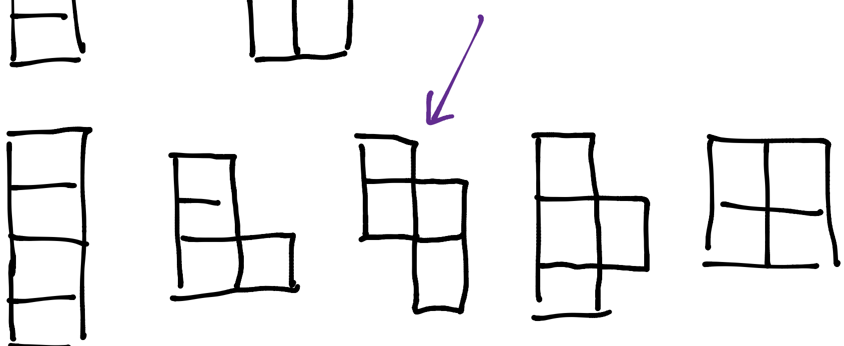


Domino.

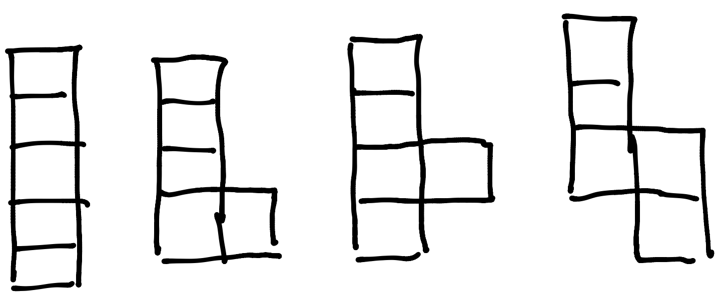


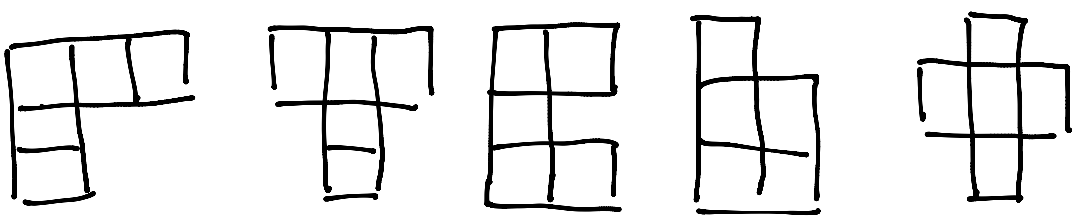
Trinomino.

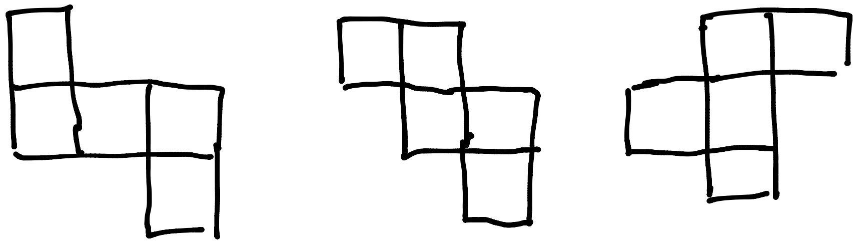
Tetromino.



Pentomino.

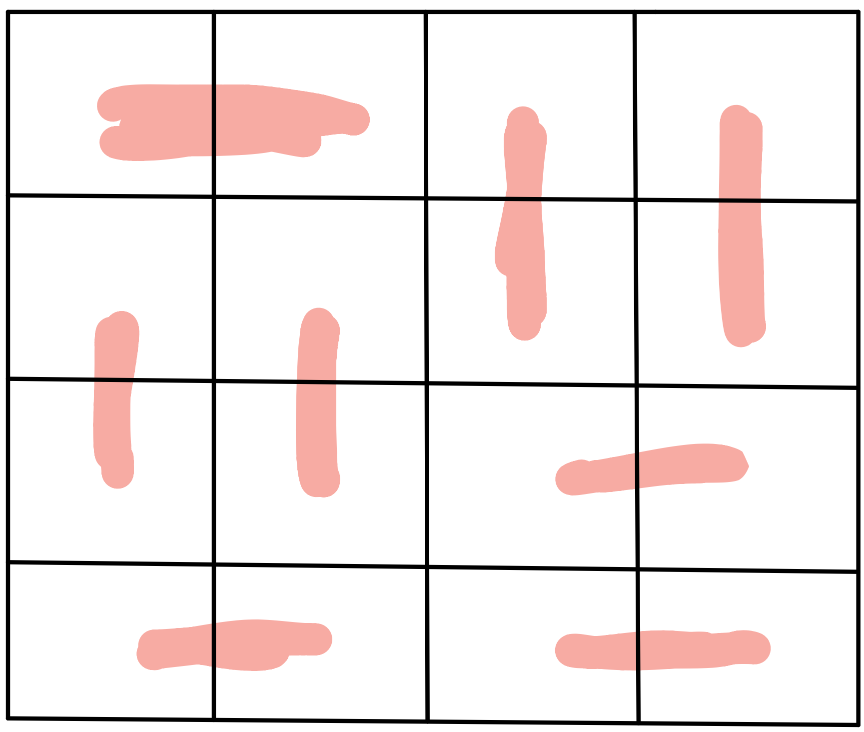






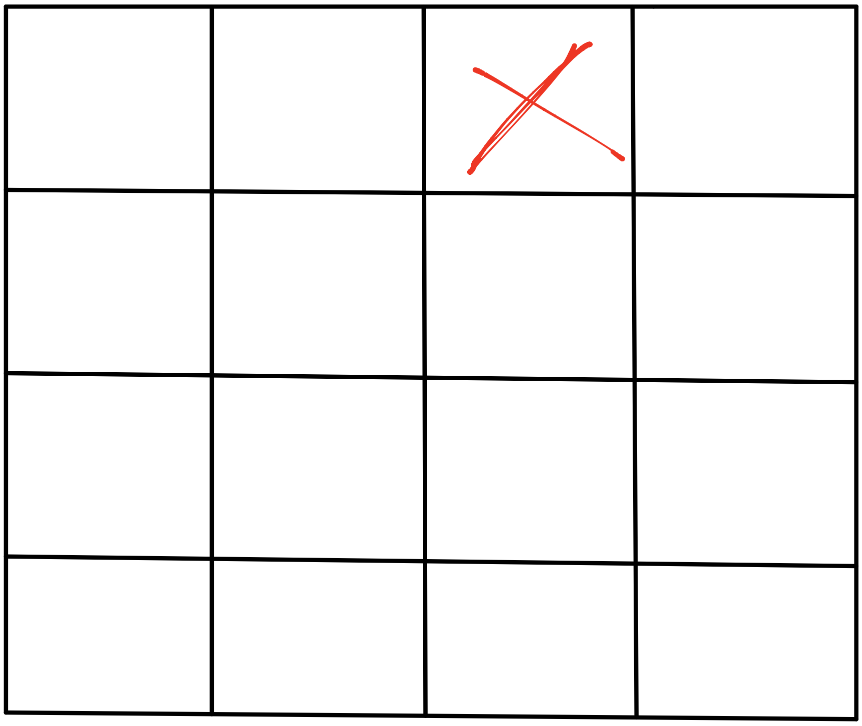
Q. What sorts of regions can we tile with (specific) polyominoes?

Easy. Can we tile a 4x4 box with dominoes?



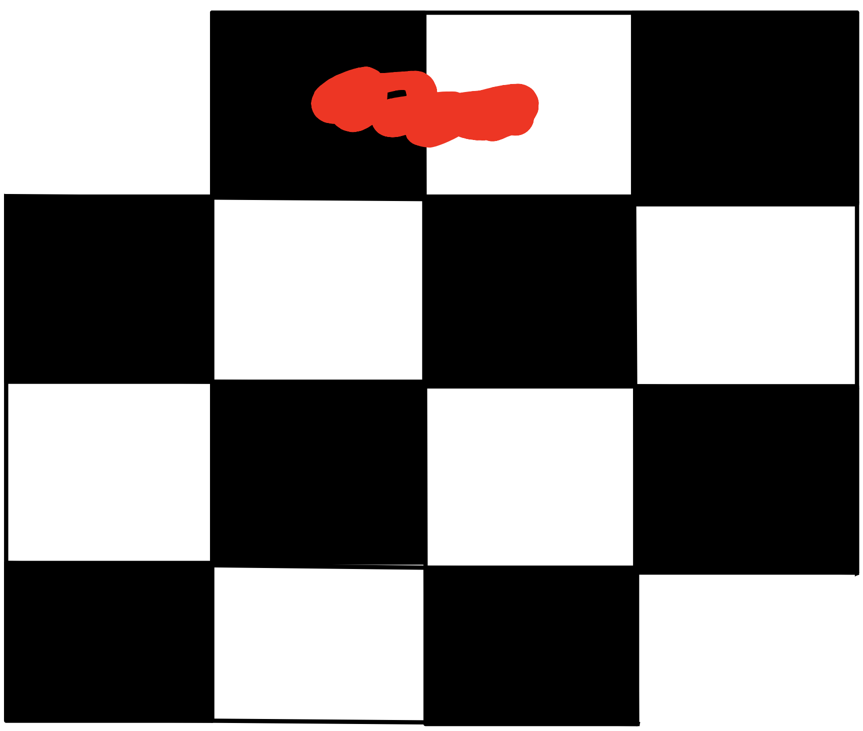
Still Easy. How about upon removing exactly one box?

No! Dominoes fill up an even area!



Trickier. How about after removing opposite corners?

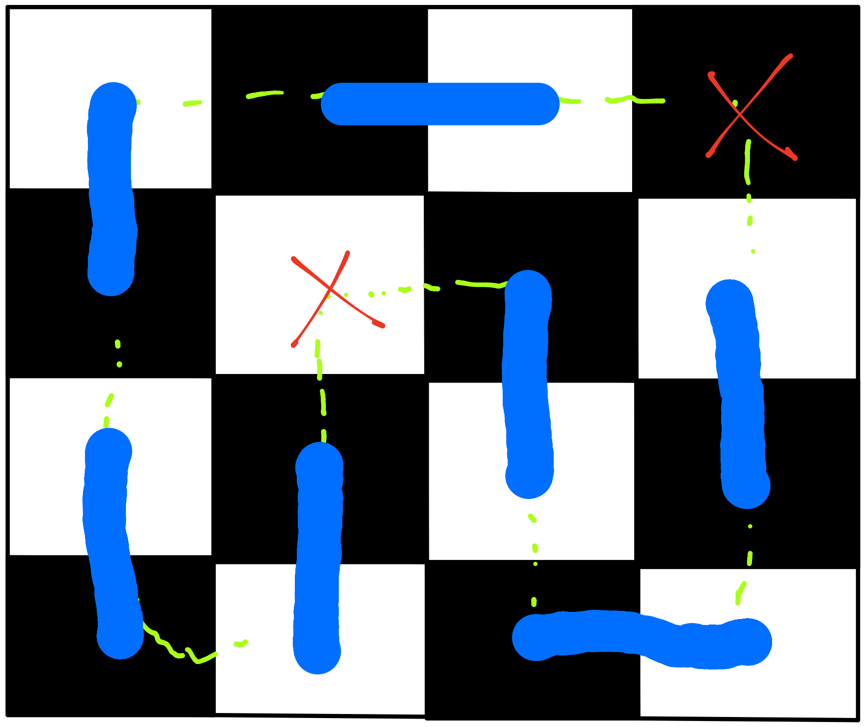
No!



■ = 8

□ = 6

Tricky. How about removing one black and one white square — your choice which!



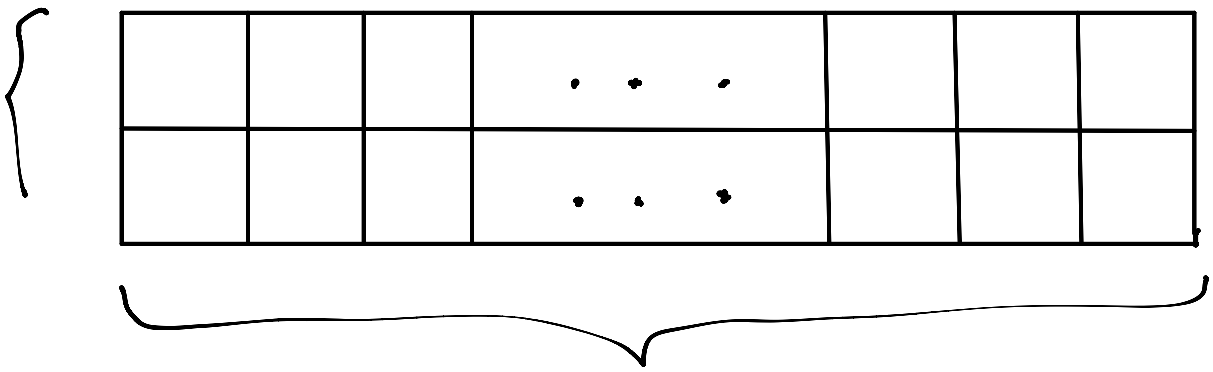
Always Yes!

Generally, not so easy to decide whether there exists a tiling by dominoes, but in a sense, always possible!

- If $\exists$ $\rightsquigarrow$ exhibit tiling
- If $\nexists$ $\rightsquigarrow$ can find a "forbidden configuration"

__Counting__. How many ways to tile a $2 \times k$ rectangle by dominoes?

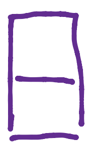
2

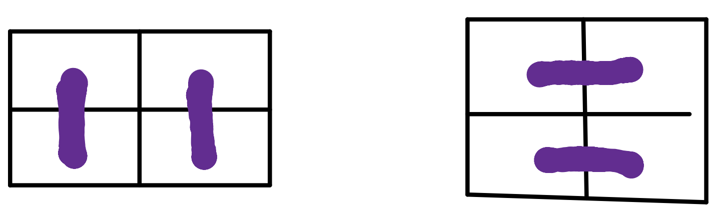


k

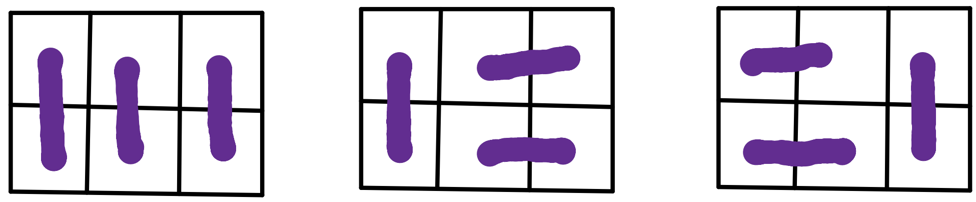
$F_k :=$ # of such tilings.

__$k=0$__ $\emptyset$ $\rightsquigarrow$ $F_0 = 1$

__$k=1$__  $\rightsquigarrow$ $F_1 = 1$

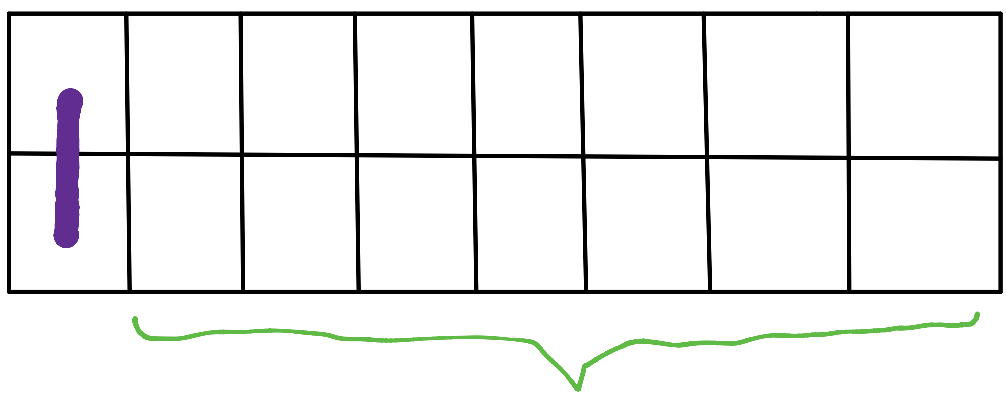
__$k=2$__  $\rightsquigarrow$ $F_2 = 2$

$k = 3$

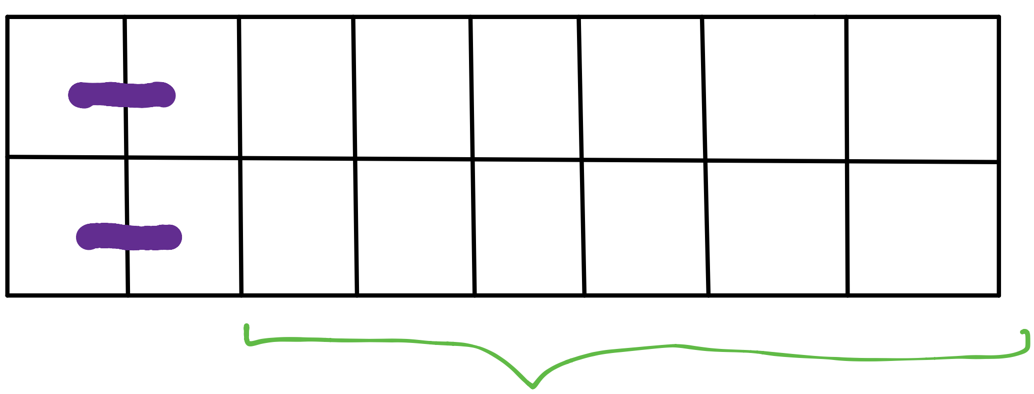


$\rightsquigarrow F_3 = 3$

More generally, let's think about how we might place the leftmost dominoes:



$k-1$



$k-2$

$$\rightsquigarrow \begin{cases} F_k = F_{k-1} + F_{k-2} \end{cases}$$

$F_0 = F_1 = 1$

Conclusion. F_k = # of tilings of $2 \times k$ rectangle by dominoes

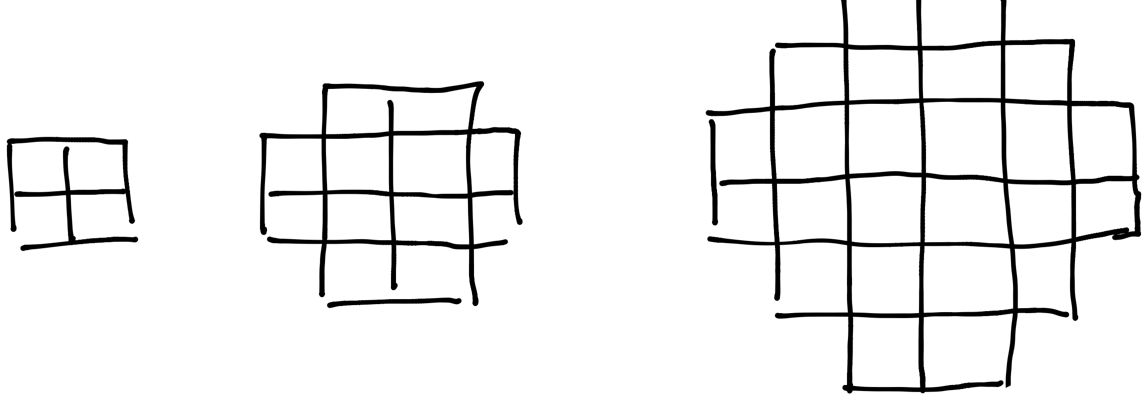
= kth Fibonacci number !

More generally, have the following cool result:

THEOREM. (Fisher - Temperley, Kastelyn) The number of tilings of a $2m \times 2n$ rectangles by dominoes is:

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2\left(\frac{j\pi}{2m+1}\right) + \cos^2\left(\frac{k\pi}{2n+1}\right) \right).$$

Aztec diamonds.



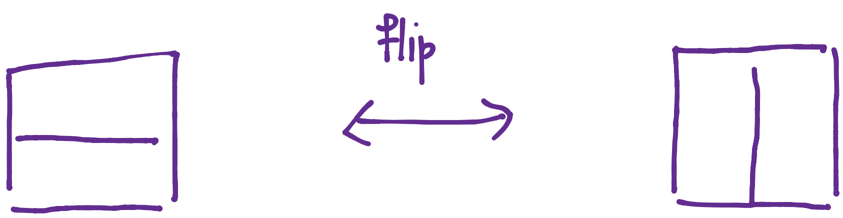
AZ(1) AZ(2) AZ(3)

Q. How many tilings by dominoes for AZ(n)?

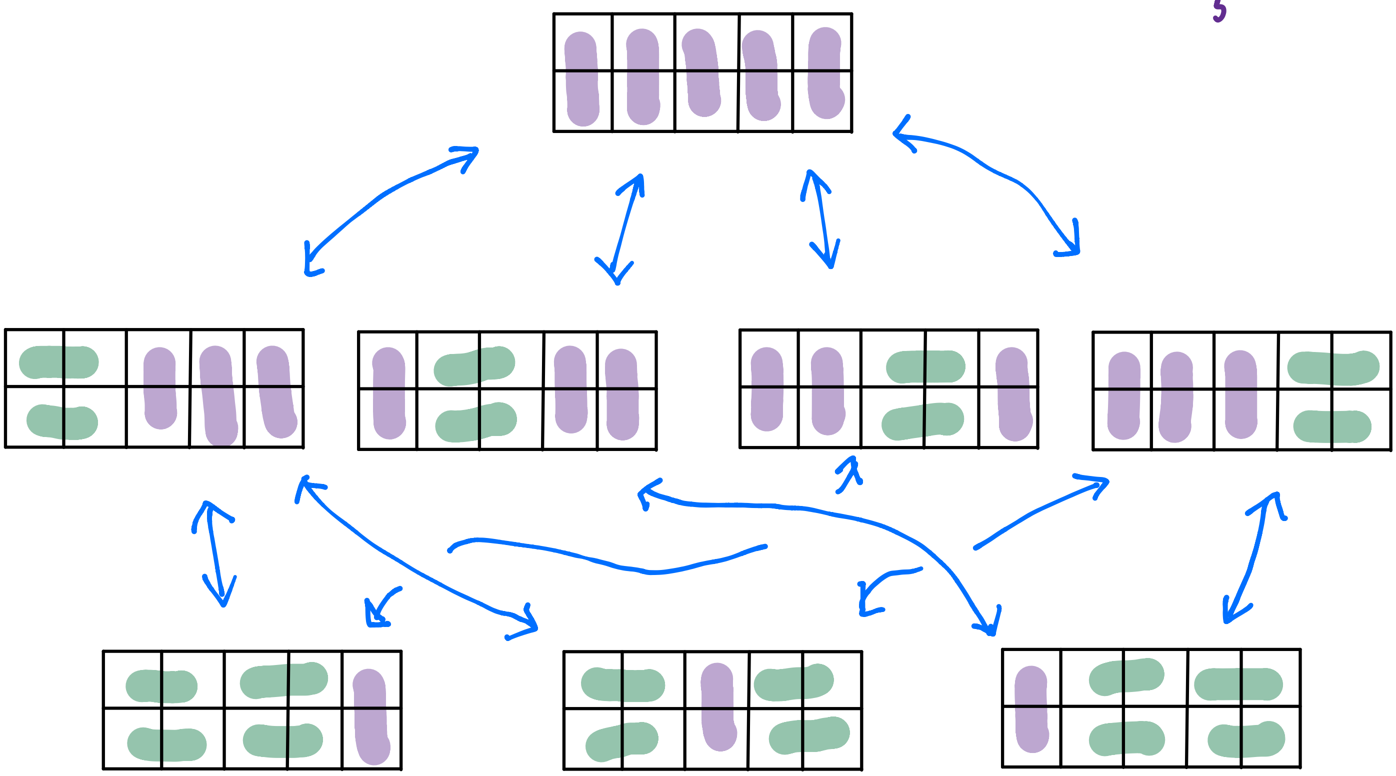
THEOREM. [Elkies-Kuperberg-Larsen-Propp]

$$\#\{\text{domino tilings of } AZ(n)\} = 2^{\frac{n(n+1)}{2}}$$

Relations amongst tilings.

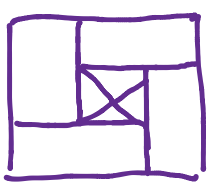


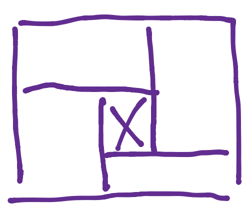
$F_5 = 8$



<u>THEOREM</u>. (Thurston) For any region R with no holes, any pair of domino tilings of R are related by a finite sequence of flips!

Not true if R has holes!





hit's schedule.

02/04 Madeline

02/11 Mia + Andrew

02/18 Sahr + Cassandra

02/25 Joseph + Ashwin

03/04 Ben + Ashely

03/11 Ting + Oliver

03/25 Imanol + Ashton