CALCULUS II ASSIGNMENT 5 SOLUTIONS

1. Explain in your own words what it means for a sequence to converge or diverge. Give two examples of sequences which converge, and two examples of sequences which converge. Try to carefully justify, using the definition from class, why your sequences do what they do.

Solutions. A sequence is convergent when the terms of the sequence eventually stabilize near some limit value. On the other hand, a sequence is divergent when the terms never stabilize around some value.

For an example of a sequence that converges, consider the sequence $a_n := \frac{1}{n}$. Let's show that this sequence converges to 0. By definition, this means that for every $\epsilon > 0$, I need to find some positive integer N such that for every $n > N$, I have $a_n = \frac{1}{n} < \epsilon$. But ϵ is just some tiny positive number, so there is necessarily some gigantic N such that $\frac{1}{N} < \epsilon$; choose any such N. Then for every $n > N$,

$$a_n = \frac{1}{n} < \frac{1}{N} < \epsilon$$

as required. This shows that the limit of the a_n is 0.

For a second example of a convergent sequence, consider $b_n := \frac{1}{2^n}$. Again, let's show that this sequence converges to 0. This means, as above, that given $\epsilon > 0$, I need to produce a positive integer N such that for all $n > N$, $b_n = \frac{1}{2^n} < \epsilon$. Okay, to find the N in this case, apply the logarithm $\log_2$ with base 2 to the inequality we are after, transforming it into the following equivalent statement:

$$-N = \log_2(1/2^N) < \log_2(\epsilon).$$

Since $\log_2(\epsilon)$ is some finite real number—it may be negative!—there is perhaps some large positive integer N such that $-N < \log_2(\epsilon)$. Choose such an N. Then for any $n > N$,

$$-n = \log_2(1/2^n) < -N < \log_2(\epsilon)$$

and hence, exponentiating everything back, $1/2^n < \epsilon$. This shows that the sequence in question converges to 0.

Now for some examples of sequences that diverge. For a first simple one, consider the sequence $c_n = n! = n \times (n-1) \times \cdots \times 2 \times 1$. This diverges to infinity. That is, for any positive integer M, we can find some N such that $n > N$, $c_n > M$—a precise way of saying that this sequence keeps growing larger and larger. Indeed, given a positive integer M, a way to guarantee c_n is larger than M is to take $N = M+1$. Then for any $n > N$,

$$c_n = n! > N! = (M+1)! > M$$

and this shows that this sequence diverges to infinity.

Another example of a sequence that is divergent might be one that constantly oscillates and never stabilizes to any value. One example might be the sequence $d_n = \cos(n)$. To justify precisely that this sequence diverges is somewhat tricky, but the idea is that as n varies, these numbers keep spreading out in the entire interval $[-1, 1]$, and it never stabilizes on anything specific. Let me not try to make this precise here. ■

2. Limits of sequences can be handled using things you know about limits of functions. For instance, given a function $f(x)$, we can define a sequence $\{a_n\}_{n=1}^{\infty}$ by sampling f at, say, the integers; that is, set

$$a_n := f(n).$$

Graphically, this means that the a_n are dots on the graph of f above the points $x = n$. Given this, one sees that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

Try to use this to determine whether the following sequences are convergent, and if they are, what their limits are.

(i) $a_n = \dfrac{5n^3+2n^2+1}{n^3+3}$, (iii) $a_n = \left(1+\dfrac{1}{n}\right)^n$,

(ii) $a_n = \dfrac{n^3}{n+1}$, (iv) $a_n = \dfrac{\log(n)^2}{n}$.

Solutions. For (i), the sequence a_n is generated by the function

$$f(x) := \frac{5x^3+2x^2+1}{x^3+3}$$

so by the fact above,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5x^3+2x^2+1}{x^3+3}.$$

The limit on the far right is an indeterminate form, and thus we may apply l'Hôpital's rule to compute it; the result will be that the limit is 5.

For (ii), the function giving rise to the a_n is $f(x) := x^3/(x+1)$. Taking this limit, maybe by using l'Hôpital again, we see that

$$\lim_{n\to\infty} \frac{n^3}{n+1} = \lim_{x\to\infty} \frac{x^3}{x+1} = \infty$$

and so that this sequence diverges.

For (iii), the relevant function is $f(x) := (1+1/x)^x$. Maybe a less funny way to write this function is

$$f(x) = \exp(x\log(1+1/x)).$$

Using this expression, we can find the limit of the sequence:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \exp(x\log(1+1/x)).$$

To compute the limit on the right, use the fact that the exponential is a continuous function and thus we can move the limit into the argument; we are therefore reduced to figuring out

$$\lim_{x\to\infty} x\log(1+1/x).$$

Well, when $x\to\infty$, $\log(1+1/x)\to\log(1) = 0$, and so we have another indeterminate form on our hands. One convenient form to put this into so that we may apply l'Hôpital's rule is

$$x\log(1+1/x) = \frac{\log(1+1/x)}{1/x}.$$

Now compute the limit:

$$\lim_{x\to\infty} \frac{\log(1+1/x)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

where the first equality comes from simplifying the expression resulting from taking the derivative of both the top and bottom individually. As a result, we see that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \exp(x\log(1+1/x)) = \exp\left(\lim_{x\to\infty} x\log(1+1/x)\right) = \exp(1) = e.$$

For (iv), the function in question is $f(x) := \log(x)^2/x$. Now compute:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\log(x)^2}{x} = \lim_{x\to\infty} \frac{2\log(x)}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$$

where l'Hôpital's rule has been applied twice to compute this limit. ■

3. Determine whether or not the following sequences converge or diverge. If they converge, determine their limit.

(i) $a_n = 1-(0.12)^n$, (iii) $a_n = (-1)^n$,
(ii) $a_n = \log(n+1)-\log(n)$, (iv) $\{0,1,0,0,1,0,0,0,1,\ldots\}$.

Solutions. For (i), we have a convergent sequence since

$$\lim_{n\to\infty}\left(1-(0.12)^n\right) = 1-\lim_{n\to\infty}(0.12)^n = 1-0 = 1$$

the second equality coming from the fact that if we take powers of a number that has absolute value strictly less than 1, we will obtain 0.

For (ii), we have a convergent sequence

$$\lim_{n\to\infty}\log(n+1)-\log(n) = \lim_{n\to\infty}\log\left(\frac{n+1}{n}\right) = \log\left(\lim_{n\to\infty}\frac{n+1}{n}\right) = \log(1) = 0$$

where I have used some properties of the logarithm function and also the fact that the logarithm is continuous.

For (iii), the sequence $a_n = (-1)^n$ diverges. To see this, by definition, I need to show that there does not exist a number L such that the definition of the limit is satisfied. Unwrapping that, I must show that if L is any number, then there is some $\epsilon > 0$ such that for any positive integer N, there is some $n > N$ such that $|L - a_n| > \epsilon$. If $L \neq \pm 1$, then take ϵ to be any positive number strictly between 0 and $\min\{|L-1|,|L+1|\}$. Then $|L-a_n| > \epsilon$ for all n. If $L = 1$, then take $\epsilon = 1/2$, for instance. Now for any positive integer N, take any odd integer $n > N$. Then we have

$$|L-a_n| = |1-(-1)^n| = |1-(-1)| = 2 > \frac{1}{2} = \epsilon.$$

Therefore L cannot be 1. Symmetrically, doing this argument but replacing n in the above by some even integer larger than N shows that L cannot be -1, either. This shows that a_n cannot converge to any number, so it is divergent.

For (iv), the sequence listed is divergent. We can see this from the definitions: if L is any number besides 0 or 1, then take ϵ to be any positive number strictly smaller than $\min\{|L|,|L-1|\}$. Then for any n, $|L-a_n| > \epsilon$. If $L = 0$, take $\epsilon = 1/2$. Then given a positive integer N, there is some $n > N$ such that $a_n = 1$, and therefore $|L-a_n| = |0-1| = 1 > 1/2 = \epsilon$. Finally, if $L = 1$, again take $\epsilon = 1/2$. Given a positive integer N, there is again some $n > N$ such that $a_n = 0$, and so $|L-a_n| = |1-0| = 1 > 1/2 = \epsilon$. All together, this shows that the sequence diverges. ■

4. The Squeeze Theorem is incredibly helpful for determining the convergence of certain sequences. Let's try to work in an example that is not as obvious as the ones given in class.

(i) Use the Pythagorean Theorem $\sin(x)^2 + \cos(x)^2 = 1$ together with the fact that $y^2 \geq 0$ for any real number y to justify the inequalities
$$-1 \leq \sin(x) \leq 1 \quad \text{and} \quad -1 \leq \cos(x) \leq 1.$$

(ii) Use (i) to show that for any numbers α and β, that
$$-\alpha - \beta \leq \alpha \sin(x) + \beta \cos(x) \leq \alpha + \beta$$
for any x.

(iii) Now use the Squeeze Theorem to show that the sequence
$$a_n = \frac{5\sin(n) + 7\cos(n)}{n^{3/5}}$$
is convergent. Find its limit along the way.

Solution. To show the inequalities at hand, observe that
$$1 = \sin(x)^2 + \cos(x)^2 \geq \sin(x)^2 + 0 = \sin(x)^2$$
where the first statement on the left is the Pythagorean Theorem; the inequality in the middle comes from the fact that $\cos(x)^2 \geq 0$—the square of a real number is non-negative. Taking square roots of the extreme sides of the inequality, we see that
$$|\sin(x)| = \sqrt{\sin(x)^2} \leq \sqrt{1} = 1.$$
But to say that the absolute value of some number is bounded by 1 amounts to meaning that the number inside the absolute value is bounded below by -1 and bounded above by 1, that is
$$-1 \leq \sin(x) \leq 1.$$
The inequality for $\cos(x)$ is shown analogously.

To get the inequalities in (ii), multiply the inequality involving sin by α and multiply the inequality involving cos by β, so
$$-\alpha \leq \alpha \sin(x) \leq \alpha \quad \text{and} \quad -\beta \leq \beta \cos(x) \leq \beta.$$
Adding these inequalities together gives the desired inequality.

Finally, observe that we have the inequalities
$$\frac{-5-7}{n^{3/5}} \leq \frac{5\sin(n) + 7\cos(n)}{n^{3/5}} \leq \frac{5+7}{n^{3/5}}$$
upon applying the inequality from (ii) using $\alpha = 5$ and $\beta = 7$. Now the sequences on the left and right extremes tend to 0 as $n \to \infty$, so by the Squeeze Theorem, $a_n \to 0$ as well. ■

5. Decide whether or not the sequence

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$$

is convergent. It may help to graph the sequence. Try to justify carefully, using the definition of limits given in class, your conclusion.

Solution. This sequence diverges. Write the terms of the sequence as follows:

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!} = \frac{1}{1} \cdot \frac{3}{2} \cdot \frac{5}{3} \cdot \cdots \cdot \frac{2n-1}{n}.$$

Note that for $n \geq 2$, the terms

$$\frac{2n-1}{n} \geq \frac{3}{2}$$

so we see from the expansion above that

$$a_n \geq \left(\frac{3}{2}\right)^{n-1}.$$

Now we can show that this sequence diverges to infinity: given a positive integer M, we need to find another positive integer N so that whenever $n > N$, $a_n > M$. But for this, take any N such that $(3/2)^{N-1} > M$—this is possible because powers of a fixed number larger than 1 are unbounded. Then for any $n > N$,

$$a_n \geq (3/2)^{n-1} > (3/2)^{N-1} > M$$

and this shows that sequence converges to infinity. ■

6. The Monotone Convergence Theorem is a powerful way to determine whether or not a sequence has a limit.

(i) Suppose you had a increasing sequence $\{a_n\}$ whose terms are numbers between 0 and 10. Carefully explain why this has a limit L.

(ii) What can you say about the limit L? That is, do you know some inequalities on L?

(iii) Let T_n be the average temperature of the universe at year n—whatever that means! Apparently, there is an absolute coldest temperature, and if we assume that the universe is finite, then there is also some absolute hottest temperature somewhere. Use this to conclude that as the average temperature of the universe must have a limit as time goes off to infinity.

Regarding (iii), you might enjoy this short story by Isaac Asimov.

Solutions. We have an increasing sequence, so it is monotonic. Moreover, it has values only between 0 and 10, so the sequence is bounded below by 0 and above by 10. So this sequence is bounded and monotonic, and thus the Monotone Convergence applies to show that the sequence converges.

The number L, being a limit of an increasing sequence with values between 0 and 10, must be strictly greater than 0—since terms are increasing—and at most L, since all the numbers in question are smaller than L. That is: $0 < L \leq 10$.

Okay, this doesn't work as well as I thought it would, so let me not try to carefully make this argument work. I hope the short story was nonetheless fun! ∎

7. Once you know that a sequence has a limit, you can use formal methods to find the actual limit. Here are some examples.

(i) Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$$

by writing $a_n = \sqrt{2a_{n-1}}$.

(ii) Find the limit of the sequence with $a_1 = 1$ and

$$a_n = 3 - \frac{1}{a_{n-1}} \quad \text{for } n > 1.$$

In fact, you can try to prove this one is convergent by trying to show that it is increasing and that $a_n < 3$ for all n.

(iii) Let $\{f_n\}_n$ be the Fibonacci sequence, so that $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Let $a_n = f_{n+1}/f_n$. Check that

$$a_n = 1 + \frac{1}{a_{n-1}} \quad \text{for } n \geq 1$$

and find its limit. This is the asymptotic growth rate of the Fibonacci sequence and is known as the golden ratio.

Proof. For (i), let the limit be L so

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{2a_{n-1}} = \sqrt{2 \cdot \lim_{n\to\infty} a_{n-1}} = \sqrt{2L}.$$

Squaring both sides, we get that L satisfies

$$L^2 - 2L = L(L-2) = 0.$$

Since the sequence is increasing, as you can see by writing the first few values, and starts with $\sqrt{2} > 0$, it follows that $L = 2$.

For (ii), we have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} 3 - \frac{1}{a_{n-1}} = 3 - \frac{1}{L}.$$

Clearing denominators and rearranging gives the quadratic equation

$$L^2 - 3L + 1 = 0 \quad \text{so} \quad L = \frac{3 \pm \sqrt{5}}{2}.$$

To decide which one, note that the sequence is increasing and is bounded below by 1, so since $(3-\sqrt{5})/2 < 1$, we see that $L = (3+\sqrt{5})/2$.

Finally, for (iii), to check the recurrence relation displayed, expand the right hand side of the claimed relation using the definition of the terms of the sequence:

$$1 + \frac{1}{a_{n-1}} = 1 + \frac{f_{n-1}f_n}{=} \frac{f_n + f_{n-1}}{f_n} = \frac{f_{n+1}}{f_n} = a_n.$$

Now to compute the limit:

$$\phi = \lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \frac{1}{a_{n-1}} = 1 + \frac{1}{\phi}$$

so we obtain the quadratic equation

$$\phi^2 - \phi - 1 = 0 \quad \text{so} \quad \phi = \frac{1 \pm \sqrt{5}}{2}.$$

Since $1 - \sqrt{5} < 0$ and our sequence consists of positive numbers, we conclude that $\phi = (1 + \sqrt{5})/2$. ■