CALCULUS II ASSIGNMENT 1 SOLUTIONS

1. Compute the following integrals using substitution:

(i) $\displaystyle\int_0^1 (2+4x)^3\,dx$, (iii) $\displaystyle\int_1^e \frac{\log(z)^4}{z}\,dz$,

(ii) $\displaystyle\int e^t\cos(e^t)\,dt$, (iv) $\displaystyle\int \frac{1+y}{1+y^2}\,dy$.

Solutions. For (i), we recognize that this is a power of a linear form, and life would be a lot simpler if we write $u = 2+4x$. In doing so, the differential of the two variables is related by $du = d(2+4x) = 4dx$, or $dx = \frac{1}{4}du$. Moreover, the bounds of integration change as follows:

- when $x = 0$ in the lower bound, $u = 2+4\cdot 0 = 2$; and
- when $x = 1$ in the upper bound, $u = 2+4\cdot 1 = 6$.

Therefore

$$\int_0^1 (2+4x)^3\,dx = \int_2^6 u^3\,\frac{du}{4} = \frac{1}{16}u^4\Big|_{u=2}^{u=6} = 81-1 = 80.$$

For (ii), the e^t is making it look more complicated than it needs to be, so let $u = e^t$ so that $du = e^t\,dt$. Then

$$\int e^t\cos(e^t)\,dt = \int \cos(u)\,du = \sin(u) = \sin(e^t).$$

For (iii), notice that $\frac{1}{z}$ is the derivative of $\log(z)$. This suggests the substitution $u = \log(z)$, so that $du = dz/z$. Doing this substitution, the integral simplifies to

$$\int_1^e \frac{\log(z)^4}{z}\,dz = \int_0^1 u^4\,du = \frac{1}{5}.$$

For (iv), we notice that if we use the substitution $u = 1+y^2$, then $du = 2y\,dy$ and we are able to get rid of a y in the numerator. So we should split the integral into two different integrals and perform each individually:

$$\int \frac{1+y}{1+y^2}\,dy = \int \frac{dy}{1+y^2} + \frac{1}{2}\int \frac{du}{u} = \arctan(y) + \frac{1}{2}\log(1+y^2)$$

using the direct knowledge of the integral of $1/(1+y^2)$. ■

2. Consider the function $f(x) = x^4$.

(i) Check that $f(x)$ is an even function.

(ii) Sketch a graph of $f(x)$ on the interval $[-2,2]$.

(iii) Compute the integral $\int_{-2}^{0} x^4 \, dx$ by substituting $u = -x$.

(iv) Use (iii) to compute the integral $\int_{-2}^{2} x^4 \, dx$.

This is definitely an easy enough example that you can just do directly, but maybe this indicates why observing that your function is even could be labour saving when doing more complicated integrals.

Solution. To see that $f(x) = x^4$ is an even function, just check the definition:

$$f(-x) = (-x)^4 = (-1)^4 x^4 = x^4 = f(x)$$

since (-1) to an even power is 1.

For (ii), ask Google or WolframAlpha to do this for you. The interesting thing to notice is that this graph is symmetric about the y-axis; this is basically what the evenness property means!

For (iii), if we substitute $u = -x$, then $du = -dx$, thanks to the chain rule. You also have to change the bounds of integration by flipping the signs. Therefore

$$\int_{-2}^{0} x^4 \, dx = -\int_{2}^{0} (-u)^4 \, du = \int_{0}^{2} u^4 \, du = \frac{32}{5}.$$

Finally, for (iv), we split the integral into two pieces by cutting it at 0:

$$\int_{-2}^{2} x^4 \, dx = \int_{-2}^{0} x^4 \, dx + \int_{0}^{2} x^4 \, dx = \int_{0}^{2} x^4 \, dx + \int_{0}^{2} x^4 \, dx = 2 \cdot \frac{32}{5} = \frac{64}{5}$$

where the second equality is thanks to (iii).

More generally, if $f(x)$ is any even function, then for any $a > 0$,

$$\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = 2\int_{0}^{a} f(x) \, dx$$

by performing the substitution $u = -x$ in the first integral and renaming. ∎

3. Consider the function $f(x) = \dfrac{\sin(x)}{1+x^6+x^{42}}$.

(i) Check that $f(x)$ is an odd function.

(ii) Mimic **2.**(iii) and **2.**(iv) to compute

$$\int_{-3}^{3} \frac{\sin(x)}{1+x^6+x^{42}}\,dx = 0.$$

Yes, seeing an odd function is odd is typically much more helpful than seeing an even function.

Solution. To see that $f(x)$ is odd, use the definition:

$$f(-x) = \frac{\sin(-x)}{1+(-x)^6+(-x)^{42}} = \frac{-\sin(x)}{1+x^6+x^{42}} = -f(x)$$

where I have used the property that $\sin(-x) = -\sin(x)$.

To evaluate this integral, notice that

$$\int_{-3}^{0} f(x)\,dx = -\int_{3}^{0} f(-x)\,dx = \int_{0}^{3} -f(x)\,dx = -\int_{0}^{3} f(x)\,dx$$

where the first equality is upon doing a substitution $x \mapsto -x$, the second from the oddness of f together with a flip of the bounds of integration from the first negative sign, and the last is just pulling out the inner negative sign. Therefore, splitting our total integral down the middle,

$$\int_{-3}^{3} f(x)\,dx = \int_{-3}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx = -\int_{0}^{3} f(x)\,dx + \int_{0}^{3} f(x)\,dx = 0$$

as required. No calculation required! ■

4. Evaluate the following integrals using integration by parts:

(i) $\displaystyle\int x\sin(4x)\,dx$, (iii) $\displaystyle\int_0^1 \frac{y}{e^{3y}}\,dy$,

(ii) $\displaystyle\int \log(s)^2\,ds$, (iv) $\displaystyle\int e^{2\theta}\sin(3\theta)\,d\theta$.

Solutions. For (i), I know how to integrate $\sin(4x)$ and differentiating that leading x will simplify matters, so

$$\int x\sin(4x)\,dx = -\frac{1}{4}x\cos(4x) + \frac{1}{4}\int \cos(4x)\,dx = -\frac{1}{4}x\cos(4x) + \frac{1}{16}\sin(4x).$$

For (ii), I don't really know how to integrate a logarithm, so the game is to see that constant function 1 hiding in the notation:

$$\int \log(s)^2\,ds = s\log(s)^2 - \int s\cdot\frac{2\log(s)}{s}\,ds = s\log(s)^2 - 2\int \log(s)\,ds = s\log(s)^2 - 2s\log(s) + s$$

where I have done the integration by parts once more.

For (iii), what one has to do is perhaps more clearly seen upon writing the integrand as ye^{-3y}; then I know how to integrate the exponential and life is simpler without the y in front, so

$$\int_0^1 \frac{y}{e^{3y}}\,dy = -\frac{1}{3}\frac{y}{e^{3y}}\Big|_{y=0}^{y=1} + \frac{1}{3}\int_0^1 \frac{dy}{e^{3y}} = -\frac{1}{3e^3} - \frac{1}{9e^3} + \frac{1}{9} = \frac{e^3-4}{9e^3}.$$

For (iv), well, let's just try to integrate the exponential, say, and see where we go:

$$\begin{aligned}\int e^{2\theta}\sin(3\theta)\,d\theta &= \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{2}\int e^{2\theta}\cos(3\theta)\,d\theta \\ &= \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{4}e^{2\theta}\cos(3\theta) - \frac{9}{4}\int e^{2\theta}\sin(3\theta)\,d\theta.\end{aligned}$$

The integral that I want to compute has shown up again! So treating the integral I want to compute as an expression in itself, rearrange to obtain the equation

$$\Big(1+\frac{9}{4}\Big)\int e^{2\theta}\sin(3\theta)\,d\theta = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{4}e^{2\theta}\cos(3\theta).$$

Dividing both sides by $1+\frac{9}{4} = \frac{13}{4}$ yields

$$\int e^{2\theta}\sin(3\theta)\,d\theta = \frac{2}{13}e^{2\theta}\sin(3\theta) - \frac{3}{13}e^{2\theta}\cos(3\theta).$$

Wonderful! ∎

5. Evaluate the following integrals by first making a substitution and then integrating by parts:

(i) $\displaystyle\int x\log(5+7x)\,dx$, (ii) $\displaystyle\int \sin(\log(x))\,dx$.

Solutions. For (i), consider the substitution $u = 5+7x$. Then $du = 7dx$ and, also, $x = \frac{u-5}{7}$. Therefore

$$\int x\log(5+7x)\,dx = \frac{1}{7}\int u\log(u)\,du - \frac{5}{7}\log(u)\,du.$$

Now each of the remaining integrals can be determined via integration by parts:

$$\int u\log(u)\,du = \frac{1}{2}u^2\log(u) - \frac{1}{2}\int \frac{u^2}{u}\,du = \frac{1}{2}u^2\log(u) - \frac{1}{4}u^2$$

and

$$\int \log(u)\,du = u\log(u) - \int \frac{u}{u}\,du = u\log(u) - u.$$

Putting everything together yields

$$\int x\log(5+7x)\,dx = \frac{1}{14}(5+7x)^2\Big(\log(5+7x) - \frac{1}{2}\Big) - \frac{5}{7}(5+7x)\big(\log(5+7x) - 1\big).$$

For (ii), consider the substitution $u = \log(x)$. The differential of this substitution then is $du = dx/x$, and this involves x on the right hand side. To make this useful, notice that the substitution in question can also be written $x = e^u$, so that we may rearrange the equation for the differential to obtain $dx = e^u\,du$ Therefore

$$\int \sin(\log(x))\,dx = \int e^u\sin(u)\,du = e^u\sin(u) - \int e^u\cos(u)\,du = e^u\sin(u) - e^u\cos(u) - \int e^u\sin(u)\,du.$$

Comparing the second expression from the left with the expression on the far right, we may rearrange to obtain

$$\int e^u\sin(u)\,du = \frac{e^u\sin(u) - e^u\cos(u)}{2}.$$

Substituting $u = \log(x)$ back in and using the fact $e^{\log(x)} = x$,

$$\int \sin(\log(x))\,dx = \frac{x\sin(\log(x)) - x\cos(\log(x))}{2}.$$

And that is a wrap! ■