Introduction to Stacks

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We shall introduce stacks from the perspective of moduli spaces following a mixture of [2], [3], and [4]. (For more background, see [1].) The claim is that stacks are a more natural class of objects to consider in moduli problems. Here are two examples.

1. Over $\mathbb{C}$, two elliptic curves E and E' are isomorphic iff $j(E) = j(E')$. But putting $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec}\mathbb{C}[j]$ as the moduli space is "too coarse" in that the moduli space has forgotten about many non-trivial automorphisms, e.g. the $[-1]$ automorphism possessed by every elliptic curve, or the extra automorphisms of $j = 0$ and $j = 1728$.

2. If X is a variety and G a group acting on X, the quotient X/G may not exist in the category of varieties or even schemes. But it does as a stack. Quotients in general are more natural for stacks.

1 Construction

Definition 1.1. Let S be a scheme and Sch/S be the category of S-schemes. A **category over** S is a category $\mathcal{F}$ equipped with a **projection functor** $p\colon \mathcal{F} \to \mathsf{Sch}/S$; we refer to $\mathcal{F}$ as an S**-category**. We view Sch/S as our **base category**.

Definition 1.2. A category $\mathcal{F}$ over S is **fibered in groupoids** if:

1. (arrow lifting) every arrow $\phi\colon U \to V$ in Sch/S **lifts** to an arrow $f\colon x \to y$ in $\mathcal{F}$, i.e. $p(f) = \phi$, for every choice of $y \in p^{-1}(V)$;

2. (diagram lifting) for all diagrams $x \xrightarrow{f} y \xleftarrow{g} z$ in $\mathcal{F}$ with image $U \xrightarrow{\phi} W \xleftarrow{\psi} V$ under p, for every $\chi\colon U \to V$ completing the image diagram there exists a unique **lift** $h\colon x \to y$ of χ completing the original diagram.

Remark 1.3. Note that condition 1 does not seem to specify any conditions on x. However, condition 2 implies the $f\colon x \to y$ in condition 1 is unique up to unique isomorphism, since if there were another such arrow $f'\colon x' \to y$ then we can form the diagram $x \xrightarrow{f} y \xleftarrow{f'} x'$. The image diagram $U \xrightarrow{p(f)} V \xleftarrow{p(f')} U$ is completed by the identity id_U, so by condition 2 there exists a unique lift $x \to x'$. By swapping x and x' we get its inverse.

Remark 1.4. Condition 2 implies that an arrow f in $\mathcal{F}$ is an isomorphism iff $p(f)$ is an isomorphism. The non-trivial direction: let $\phi := p(f)\colon U \to V$ be an isomorphism and $\psi\colon V \to U$ its left inverse, so that the diagram $V \xrightarrow{\text{id}} V \xleftarrow{\phi} U$ is completed by ψ, which lifts to a left inverse $g\colon y \to x$ of f in $\mathcal{F}$. Repeating the argument with the right inverse shows that f is an isomorphism.

Definition 1.5. Let U be a scheme over S and $\mathcal{F}$ be an S-category. Then $\mathcal{F}(U)$, the **fiber of $\mathcal{F}$ over** U, is the category with objects $p^{-1}(U)$ and arrows $f\colon x \to y$ with $x, y \in p^{-1}(U)$ and $p(f) = \mathrm{id}_U$. From the preceding remark, every arrow in $\mathcal{F}(U)$ is an isomorphism since id_U is, i.e. $\mathcal{F}(U)$ is a **groupoid**.

Definition 1.6. Recall that a **Grothendieck topology** on a category $\mathcal{C}$ with fiber products is an assignment to each object U of $\mathcal{C}$ a collection of sets of arrows $\{U_i \to U\}$, called **coverings** of U, such that

1. (isomorphisms) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering,
2. (stability under base change) if $\{U_i \to U\}_i$ is a covering and $V \to U$ is any arrow, then $\{U_i \times_U V \to V\}_i$ is a covering, and
3. (local character) if $\{U_i \to U\}_i$ is a covering where for each i we have a covering $\{V_{ij} \to U_i\}_j$, then the composite $\{V_{ij} \to U_i \to U\}$ is a covering.

A category with a Grothendieck topology is a **site**.

Example 1.7. Let X be a topological space and X_{cl} be the category of open subsets of X with inclusion maps. The **site of the topological space** X defines a covering of an open subset $U \subset X$ to be an open covering of U, i.e. a set $\{U_i \to U\}_i$ with U_i open and $U \subset \bigcup_i U_i$. Note that in this case, $U_1 \times_U U_2 = U_1 \cap U_2$.

Example 1.8. We say a set of maps $\{X_i \to X\}$ is **jointly surjective** if the set-theoretic union of their images is X. The **(global) étale topology** on Sch/S defines a covering to be a jointly surjective set of étale morphisms. (Being **étale** is the algebraic analogue of being a local isomorphism, e.g. a morphism of smooth varieties is étale at a point iff the differential there is an isomorphism.)

Definition 1.9. Assume that the base category Sch/S has a Grothendieck topology: the étale topology, unless otherwise specified. Write $U_{ij} := U_i \times_U U_j$. A **stack** over a scheme S is a category $\mathcal{F}$ fibered in groupoids over S such that the assignment

$$\mathsf{Sch}/S \to \mathsf{Set}, \quad U \mapsto \mathcal{F}(U) = p^{-1}(U)$$

is a **sheaf of groupoids** (in the topology of Sch/S). In other words:

1. for all $U \in \mathsf{Sch}/S$ and all $x, y \in \mathcal{F}(U)$, the functor

$$\begin{aligned} \mathrm{Isom}_U(x,y)\colon \mathsf{Sch}/U &\to \mathsf{Set} \\ V &\mapsto \{\alpha\colon x|_V \to y|_V \text{ is an isomorphism in } \mathcal{F}(V)\} \end{aligned}$$

   is a sheaf (in the topology of Sch/S), i.e. it satisfies

- (**sheaf axiom**) for all $x, y \in \mathcal{F}(U)$, all open covers $\{U_i \to U\}_i$ of U, and all isomorphisms $\alpha_i\colon x|_{U_i} \to y|_{U_i}$ such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$, there exists a unique isomorphism $\alpha\colon x \to y$ such that $\alpha|_{U_i} = \alpha_i$;

2. the **descent datum** is **effective**, i.e. for all open covers $\{U_i \to U\}_i$, all $x_i \in \mathcal{F}(U_i)$, and all $\alpha_{ij}\colon x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ such that $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ over U_{ijk}, there exists an $x \in \mathcal{F}(U)$ and $\alpha_i\colon x|_{U_i} \to x_i$ in $\mathcal{F}(U_i)$ such that $\alpha_{ij} = \alpha_j|_{U_{ij}} \circ (\alpha_i|_{U_{ij}})^{-1}$.

If $\mathcal{F}$ satisfies only condition 1, it is a **prestack**. (There are quite a few undefined terms here; we shall define them later.)

Example 1.10 (Vector bundles)**.** Let Bund_r/S be the category of line bundles of rank r on an S-scheme U. The morphisms are pullbacks. The **forgetful functor** mapping a line bundle $L \to U$ to its underlying S-scheme U puts an S-category structure on Bund_r/S. We check that Bund_r/S is fibered in groupoids:

1. (arrow lifting) every $\phi\colon U \to V$ induces a pullback ϕ^*, giving the map $\phi^* L \to L$ for any line bundle L over V;

2. (diagram lifting) given line bundles L' and L'' over U' and U'' respectively that map into a line bundle $L \to U$, a commuting morphism $\chi\colon U' \to U''$ induces a pullback χ^* which is unique up to unique isomorphism, completing the diagram of line bundles.

Now we verify that the category Bund_r/S is a stack over S:

1. condition 1 says that local (wrt an open cover) isomorphisms between two bundles agreeing on overlaps can be glued together to get a (global) bundle isomorphism;

2. condition 2 says that local (wrt an open cover) line bundles can be glued together by specifying transition maps α_{ij} that satisfy the cocycle condition.

Example 1.11 (Schemes and functors)**.** A scheme is a stack via its functor of points. Let X be an S-scheme. The morphism

$$p\colon \mathsf{Sch}/X \to \mathsf{Sch}/S, \quad (U \to X) \mapsto (U \to X \to S)$$

gives Sch/X the structure of an S-category. The structure of the category $\mathcal{F}(U)$ for $U \in \mathsf{Sch}/S$ is particularly simple:

1. the objects are $\mathcal{F}(U) = \mathrm{Hom}_S(U, X)$ since an object in $\mathcal{F}(U)$ is an X-scheme Z such that $Z \to X \to S = U \to S$, i.e. $Z = U$;

2. the only morphisms in $\mathcal{F}(U)$ are the identity morphisms, since an X-morphism $U \to U$ projects under p to an S-morphism $U \to U$, which is the identity iff the original X-morphism is also the identity, by 1.4.

Hence we can view $\mathcal{F}(U) = \mathrm{Hom}_S(U, X)$ as a set. Now we check the stack conditions:

1. $\mathrm{Isom}_U(x, y)(V)$ has either one element or none, depending on whether the objects $x, y \colon U \to X$ satisfy $x|_V = y|_V$, so locally it is either the empty sheaf or the constant sheaf and trivially satisfies the sheaf condition;

2. fact: the **functor of points** $\mathrm{Hom}_S(-, X)$ is a sheaf in the étale topology on S.

Henceforth we use X to represent both the S-scheme and the stack $p \colon \mathsf{Sch}/X \to \mathsf{Sch}/S$.

2 Representability

Definition 2.1. A **morphism of stacks** is a functor $F \colon \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{F}} = p_{\mathcal{G}} \circ F$. The morphisms from a stack $\mathcal{F}$ to a stack $\mathcal{G}$ form a category: the arrows in $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ are natural transformations of functors. Hence we say the category of stacks is a **2-category**.

Definition 2.2. A stack $\mathcal{F}$ is **representable** if there exists an S-scheme X and a stack isomorphism $\mathcal{F} \to X$. (Hence when we say isomorphism here, we mean in the sense of an equivalence of categories; the composition of the two functors need not be the identity functor.)

Proposition 2.3. *Let $F \colon \mathcal{F} \to \mathcal{H}$ and $G \colon \mathcal{G} \to \mathcal{H}$ be morphisms of stacks over S. The fiber product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ (defined categorically, as a limit) consists of:*

1. *objects (x, y, α) where $x \in \mathcal{F}$, $y \in \mathcal{G}$, and $\alpha \colon F(x) \to G(y)$ is a morphism in a fiber of $\mathcal{H}$, i.e. $p_{\mathcal{H}}(F(x)) = p_{\mathcal{F}}(x)$ is the same element of Sch/S as $p_{\mathcal{H}}(G(y)) = p_{\mathcal{G}}(y)$, and $p_{\mathcal{H}}(\alpha) = \mathrm{id}$;*

2. *morphisms $(x, y, \alpha) \to (x', y', \alpha')$ given by pairs $(\phi \colon x \to x', \psi \colon y \to y')$ in fibers of $\mathcal{F}$ and $\mathcal{G}$ such that*

$$G(\psi) \circ \alpha = \alpha' \circ F(\phi) \colon F(x) \to G(y').$$

The fiber of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ over U is precisely pairs $(x, y) \in \mathcal{F}(U) \times \mathcal{G}(U)$ such that $F(x) \cong G(y)$ in $\mathcal{H}(U)$.

Definition 2.4. Let $F \colon \mathcal{F} \to \mathcal{G}$ be a morphism of stacks. Then F is **representable** if for all S-schemes X and all morphisms of stacks $X \to \mathcal{G}$, the fiber product $\mathcal{F} \times_{\mathcal{G}} X$ is a scheme.

Example 2.5. A morphism of schemes is representable, because given schemes X, Y, Z, the fiber product $X \times_Z Y$ is still a scheme. More generally, a morphism $\mathcal{F} \to X$ with X a scheme is representable iff $\mathcal{F}$ is a representable stack.

3 Deligne–Mumford Stacks

Definition 3.1. Let $\mathscr{P}$ be a property of morphisms of schemes which is stable under base change. A representable morphism of stacks $F\colon \mathcal{F} \to \mathcal{G}$ **has property** $\mathscr{P}$ if for each scheme X and morphism of stacks $X \to \mathcal{G}$, the morphism of schemes $\mathcal{F} \times_{\mathcal{G}} X \to X$ has property $\mathscr{P}$.

Definition 3.2. Let $\mathcal{F}$ be a stack and $\Delta\colon \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ be the **diagonal** given by the two identity morphisms $\mathrm{id}_{\mathcal{F}}$. Then $\mathcal{F}$ is **Deligne–Mumford (DM)** (resp. **Artin** or **algebraic**) if:

1. the diagonal Δ is representable, quasi-compact, and separated;
2. there exists an S-scheme U, called an **atlas**, and an étale (resp. smooth) and surjective morphism $U \to \mathcal{F}$. (Think orbifold cover.)

By 3.5, the atlas morphism $U \to \mathcal{F}$ is automatically representable, so it makes sense to talk about it being étale or smooth.

Proposition 3.3 ([3, Prop. 7.15])**.** *The diagonal Δ of a Deligne–Mumford stack is unramified.*

Corollary 3.4 ([3, pg. 666])**.** *If $\mathcal{F}$ is a Deligne–Mumford stack and $U \in \mathsf{Sch}/S$ is quasi-compact, then any $x \in \mathcal{F}(U)$ has only finitely many automorphisms.*

The idea is that a DM stack is the algebraic analogue of an orbifold. The rest of this section is for interpreting and exploring the conditions of the definition of being a DM stack.

Proposition 3.5. *The diagonal Δ is representable iff every morphism from a scheme to $\mathcal{F}$ is representable.*

Proof. Suppose Δ is representable. Let $F\colon X \to \mathcal{F}$ be a morphism; we must show that for every $G\colon Y \to \mathcal{F}$, the stack $X \times_{\mathcal{F}} Y$ is a scheme. But

$$\begin{tikzcd} X \times_{\mathcal{F}} Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d, "{(F,G)}"'] \\ \mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

is a cartesian diagram, i.e. $X \times_{\mathcal{F}} Y = (X \times_S Y) \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F}$, by checking the universal property. (This diagram is sometimes called the **magic diagram** and holds in any category.) Since Δ is representable, $X \times_{\mathcal{F}} Y$ is a scheme, as desired.

Conversely, suppose every morphism from a scheme to $\mathcal{F}$ is representable. Let $H\colon X \to \mathcal{F} \times_S \mathcal{F}$ be a morphism; we must show that $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X$ is a scheme. By the universal property of fiber products, $H = (F, G) \circ \Delta_X$, so

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X \arrow[r] \arrow[d] & X \times_{\mathcal{F}} X \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"'] \\ X \arrow[r, "\Delta_X"] & X \times_S X \arrow[r, "{(F,G)}"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

consists of two cartesian squares using the magic diagram. Clearly $X \times_{\mathcal{F}} X$ is a scheme, since the morphism $X \to \mathcal{F}$ is representable by hypothesis. But Δ_X in the left square is just a morphism of schemes, which is representable, so since $X \times_{\mathcal{F}} X$ is a scheme, so is $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X$, as desired. $\square$

Example 3.6. Let X be a scheme. Then X is a DM stack, since clearly every morphism from a scheme to X is representable, and we can let X be its own atlas, with the identity map $X \to X$ both étale and surjective.

Example 3.7. The stack Bund_r/S is not a DM stack for $r \geq 1$, because in general the group of automorphisms of a vector bundle is not finite, violating 3.4.

References

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[3] Vistoli, A. *Intersection theory on algebraic stacks and on their moduli spaces.* Invent. Math. 97(3), 613–670 (1989).

[4] Voight, J. *Introduction to algebraic stacks.* `https://math.dartmouth.edu/~jvoight/notes/moduli-red-harvard.pdf`.