CALCULUS II ASSIGNMENT 3 SOLUTIONS

1. Let's do a few more trigonometric substitutions, just to get a feel for what sort of integration problems they might be useful for:

(i) Use the substitution $y = 3\sin(\phi)$ to compute $\int \sqrt{9-y^2}\,dy$.

(ii) Use the substitution $x = 2\tan(\theta)$ to compute $\int \frac{1}{x^2+4}\,dx$.

(iii) Use the substitution $z = 5\sec(\psi)$ to compute $\int \frac{1}{\sqrt{z^2-25}}\,dz$.

Solutions. The substitution $y = 3\sin(\phi)$ gives a differential $dy = 3\cos(\phi)\,d\phi$, so that

$$\begin{aligned}\int \sqrt{9-y^2}\,dy &= \int \sqrt{9(1-\sin(\phi)^2)}\cdot 3\cos(\phi)\,d\phi\\ &= 9\int \cos(\phi)^2\,d\phi\\ &= \frac{9}{2}\int 1-\cos(2\phi)\,d\phi\\ &= \frac{9}{2}\phi - \frac{9}{4}\sin(2\phi)\\ &= \frac{9}{2}\arcsin(y/3) - \frac{9}{4}\sin(2\arcsin(y/3)).\end{aligned}$$

The substitution $x = 2\tan(\theta)$ gives a differential $dx = 2\sec(\theta)^2\,d\theta$, so that

$$\int \frac{dx}{x^2+4} = \int \frac{2\sec(\theta)^2}{4(\tan(\theta)^2+1)}\,d\theta = \int \frac{d\theta}{2} = \frac{\theta}{2} = \frac{\arctan(x/2)}{2}.$$

Finally, the substitution $z = 5\sec(\psi)$ gives a differential $dz = 5\tan(\psi)\sec(\psi)\,d\psi$ so that

$$\begin{aligned}\int \frac{dz}{\sqrt{z^2-25}} &= \int \frac{5\tan(\psi)\sec(\psi)}{\sqrt{25(\sec(\psi)^2-1)}}\,d\psi\\ &= \int \sec(\psi)\,d\psi\\ &= \log(\sec(\psi)+\tan(\psi))\\ &= \log\big(\sec(\arccos(5/z)) + \tan(\arccos(5/z))\big).\end{aligned}$$

In the end, each trigonometric substitution is well-suited to a form where there is some quadratic polynomial; you can figure out which one is probably useful by looking at various forms of the Pythagorean identity. ■

2. Here is some practice for integrals of rational functions:

(i) $\displaystyle\int \frac{x^4}{x-1}\,dx$,

(ii) $\displaystyle\int \frac{y}{(y-3)(2y+1)}\,dy$,

(iii) $\displaystyle\int \frac{t^2+t+2}{t^2-1}\,dt$,

(iv) $\displaystyle\int \frac{1}{z^2-2z}\,dz$,

(v) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$,

(vi) $\displaystyle\int \frac{u+2}{u^4+3u^3+3u^2+u}\,du$,

(vii) $\displaystyle\int \frac{1}{1+e^y}\,dy$,

(viii) $\displaystyle\int \frac{4v+2}{v(v^2+1)^2}\,dv$.

You may have needed to factor a quadratic polynomial, perhaps using the quadratic formula...

Solutions. For (i), we need to divide the numerator by the denominator to proceed; one could either proceed via long division, or else by knowing facts like $x^4-1=(x-1)(x^3+x^2+x+1)$:

$$\int \frac{x^4}{x-1}\,dx = \int \frac{(x^4-1)+1}{x-1}\,dx = \int (x^3+x^2+x+1)+\frac{1}{x-1}\,dx = \frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\log(x-1).$$

For (ii), we perform partial fraction decomposition to the integrand; that is, we need to find numbers A and B so that

$$\frac{y}{(y-3)(2y+1)} = \frac{A}{y-3}+\frac{B}{2y+1}.$$

Clearing denominators, this gives the equation

$$y = A(2y+1)+B(y-3) = (2A+B)y+(A-3B)$$

so that comparing coefficients gives the system of equations

$$2A+B=1 \quad \text{and} \quad A-3B=0.$$

Solving this system of equations gives $B=1/7$ and $A=3/7$. Therefore

$$\int \frac{y}{(y-3)(2y+1)}\,dy = \frac{3}{7}\int\frac{dy}{y-3}+\frac{1}{7}\int\frac{dy}{2y+1} = \frac{3}{7}\log(y-3)+\frac{1}{7}\log(2y+1).$$

For (iii), we need to first reduce the degree of the numerator before performing partial fraction decompositions:

$$\frac{t^2+t+2}{t^2-1} = 1+\frac{t+3}{t^2-1}.$$

Now to decompose the second term: notice that $t^2-1=(t-1)(t+1)$ so that we are looking for numbers A and B so that

$$\frac{t+3}{t^2-1} = \frac{A}{t-1}+\frac{B}{t+1}.$$

Clearing denominators and comparing coefficients yields the system of equations

$$A+B=1 \quad \text{and} \quad A-B=3.$$

Solving this system of equations gives $A=2$ and $B=-1$. Putting everything together:

$$\int \frac{t^2+t+2}{t^2-1} = \int 1+\frac{2}{t-1}-\frac{1}{t+1}\,dt = t+2\log(t-1)-\log(t+1).$$

For (iv), the denominator factors as $z^2-2z=z(z-2)$ so that we are looking for a partial fraction decomposition

$$\frac{1}{z^2-2z} = \frac{A}{z}+\frac{B}{z-2}.$$

Clearing denominators, comparing coefficients, and solving the resulting system of equations gives $A = -1/2$ and $B = 1/2$. Therefore

$$\int \frac{dz}{z^2 - 2z} = -\frac{1}{2}\int \frac{dz}{z} + \frac{1}{2}\int \frac{dz}{z+2} = -\frac{1}{2}\log(z) + \frac{1}{2}\log(z+2).$$

For (v), this looks like a rational function in the funny looking variable e^x. To make this a more honest rational function, consider the substitution $u = e^x$ so that $du = e^x\,dx$. With this substitution, the integrand becomes

$$\frac{u}{u^2 + 3u + 2} = \frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$$

where I have gone ahead and named the numbers arising in the partial fraction decomposition of the rational function. Clearing denominators and comparing coefficients gives the system of equations $A + B = 1$ and $2A + B = 0$, so that $A = -1$ and $B = 2$. Therefore

$$\int e^{2x}e^{2x} + 3e^x + 3\,dx = \int \frac{u}{u^2 + 3u + 3}\,du = -\int \frac{du}{u+1} + 2\int \frac{du}{u+2} = -\log(e^x + 1) + 2\log(e^x + 2).$$

For (vi), we have a long partial fraction decomposition problem. First factor the denominator:

$$u^4 + 3u^3 + 3u^2 + u = u(u^3 + 3u^2 + 3u + 1) = u(u+1)^3$$

so that the partial fraction decomposition we are after is of the form

$$\frac{u+2}{u(u+1)^3} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{(u+1)^2} + \frac{D}{(u+1)^3}.$$

Clearing denominators and expanding gives:

$$\begin{aligned} u + 2 &= A(u+1)^3 + Bu(u+1)^2 + Cu(u+1) + Du \\ &= (A+B)u^3 + (3A + 2B + C)u^2 + (3A + B + C + D)u + A. \end{aligned}$$

Comparing coefficients gives the system of equations

$$\begin{aligned} A + B &= 0, \\ 3A + 2B + C &= 0, \\ 3A + B + C + D &= 1, \\ A &= 2. \end{aligned}$$

Since $A = 2$, the first equation gives $B = -2$, from which we deduce that $C = -2$ as well. Putting this all into the third equation we get $D = -1$. Therefore

$$\begin{aligned} \int \frac{u+2}{u^4 + 3u^3 + 3u^2 + u}\,du &= \int \frac{2}{u} - \frac{2}{u+1} - \frac{2}{(u+1)^2} - \frac{1}{(u+1)^3}\,du \\ &= 2\log(u) - 2\log(u+1) + \frac{2}{u+1} + \frac{1}{2(u+1)^2}. \end{aligned}$$

For (vii), consider the substitution $u = e^y$, so that $du = e^y\,dy = u\,dy$. We then have

$$\int \frac{dy}{1 + e^y} = \int \frac{du}{u(1+u)} = \int \frac{1}{u} - \frac{1}{1+u}\,du = \log(u) - \log(1+u) = y - \log(1 + e^y).$$

For (viii), first notice that $v^2 + 1$ is an irreducible quadratic polynomial, so that we should perform partial fraction just with that. So, we are looking for numbers A, B, C, D, E so that

$$\frac{4v + 2}{v(v^2+1)^2} = \frac{A}{v} + \frac{Bv + C}{v^2 + 1} + \frac{Dv + E}{(v^2+1)^2}.$$

Clearing denominators, this yields the equation

$$\begin{aligned} 4v+2 &= A(v^2+1)^2 + (Bv+C)v(v^2+1) + (Dv+E)v \\ &= (A+B)v^4 + Cv^3 + (2A+B+D)v^2 + (C+E)v + A. \end{aligned}$$

Comparing coefficients, we get the system of equations

$$\begin{aligned} A+B &= 0, \\ C &= 0, \\ 2A+B+D &= 0, \\ C+E &= 4, \\ A &= 2. \end{aligned}$$

From the fifth then the first equation, we see that $A = 2$ and $B = -2$; from the second then the fourth equation, we get $C = 0$ and $E = 4$; finally, the third equation gives $D = -2$. Therefore

$$\begin{aligned} \int \frac{4v+2}{v(v^2+1)^2}\,dv &= \int \frac{2}{v}\,dv - \int \frac{2v}{v^2+1}\,dv - \int \frac{2v-4}{(v^2+1)^2}\,dv \\ &= 2\log(v) - \int \frac{2v}{v^2+1}\,dv - \int \frac{2v}{(v^2+1)^2}\,dv + 4\int \frac{dv}{(v^2+1)^2}. \end{aligned}$$

To integrate the middle two terms above, consider the substitution $x = v^2+1$, so that $dx = 2v\,dv$:

$$\begin{aligned} \int \frac{2v}{v^2+1}\,dv &= \int \frac{dx}{x} = \log(x) = \log(v^2+1), \\ \int \frac{2v}{(v^2+1)^2}\,dv &= \int \frac{dx}{x^2} = -\frac{1}{x} = -\frac{1}{v^2+1}. \end{aligned}$$

For the last term, perform the trigonometric substitution $v = \tan(\theta)$ so that $dv = \sec(\theta)^2\,d\theta$:

$$\begin{aligned} \int \frac{dv}{(v^2+1)^2} &= \int \frac{\sec(\theta)^2}{(\tan(\theta)^2+1)^2}\,d\theta \\ &= \int \frac{\sec(\theta)^2}{\sec(\theta)^4}\,d\theta \\ &= \int \cos(\theta)^2\,d\theta \\ &= \frac{1}{2}\int 1+\cos(2\theta)\,d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = \frac{1}{2}\arctan(v) + \frac{1}{4}\sin(2\arctan(v)). \end{aligned}$$

Putting everything together, we finally obtain

$$\int \frac{4v+2}{v(v^2+1)^2}\,dv = 2\log(v) - \log(v^2+1) + \frac{1}{v^2+1} + \frac{1}{2}\arctan(v) + \frac{1}{4}\sin(2\arctan(v)).$$

Phew! That was a lot of integrals! ■

3. Let's do something fun with polar coordinates!

(i) Sketch the curve defined in polar coordinates by

$$r = 1 - \cos(\theta).$$

Feel free to ask your computer for help.

(ii) Compute the integral

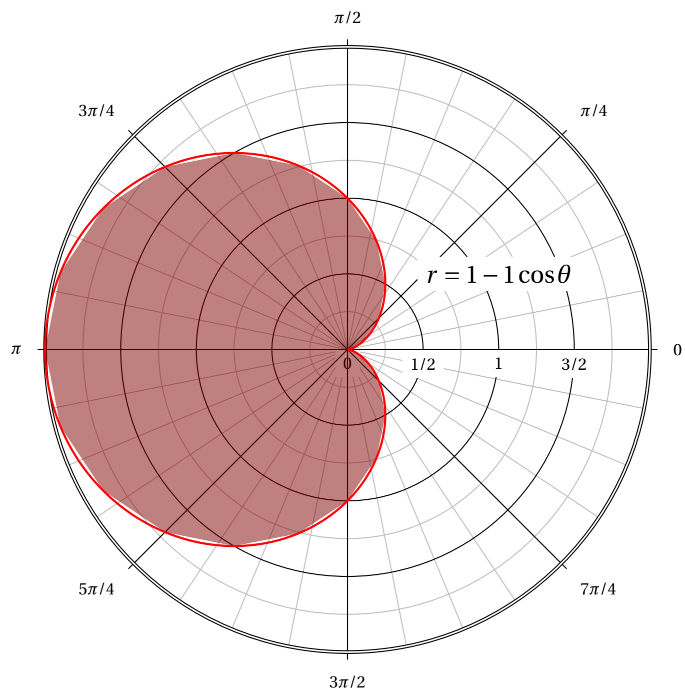
$$A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta$$

where r is the function of θ defined in (i).

(iii) Explain informally in your own words why the quantity $2A$ is the area of the figure drawn in (i). For somewhat a formal explanation, see here. It may be helpful to know the area of the sector of a circle is $\frac{1}{2}r^2\theta$.

This figure is called a cardioid. I think it's rather pretty. Happy Valentines Day!

Solution. Here's a sketch of the curve:



To compute the integral in (ii), substitute in the function $r = r(\theta)$ defined in (i):

$$A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} (1-\cos(\theta))^2\, d\theta = \frac{1}{2}\pi + \frac{1}{4}\pi = \frac{3}{4}\pi.$$

The reason that $2A$ is the area of the red shaded figure in (i) is that the integral computes the sum of the area segments that little sectors swept out by a radius arm. Summing all these infinitesimal contributions then covers the figure and gives the actual area. ■