CALCULUS II ASSIGNMENT 9 SOLUTIONS

1. Suppose that f is a function such that $f^{(n)}(0) = (n+1)!)$ for all nonnegative integers n. Find the Taylor series of f centred at 0 and find its radius of convergence.

Solution. Recall that the Taylor series of f centred at 0 is given by the formula

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Plugging in the fact $f^{(n)}(0) = (n+1)!$, we see that the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

To find the radius of convergence, begin thinking about the computation you would do with the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = \lim_{n\to\infty} \frac{n+2}{n+1}|x| = |x|.$$

Now the Ratio Test tells us that the series converges whenever $|x| < 1$, i.e. whenever $-1 < x < 1$. This means that the radius of convergence of this series is 1.

Incidentally, we can find a nicer formula for this by noticing that

$$T(x) = \sum_{n=0}^{\infty} (n+1)x^n = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}.$$

So the series $T(x)$ represents the rational function $\frac{1}{(1-x)^2}$ on the interval $(-1, 1)$. ■

2. Find the Taylor series of $f(x) := 1/x$ centred at $a = -3$. What is the radius of convergence of this representation?

Solution. There are multiple ways of doing this. One could directly use the definition of the Taylor series and compute the derivatives:

$$f^{(n)}(x) = \frac{d^n}{dx^n}\frac{1}{x} = \frac{d^n}{dx^n}x^{-1} = (-1)^n n! x^{-n-1}$$

so that

$$f^{(n)}(-3) = (-1)^n n!(-3)^{-n-1} = (-1)^{2n+1} n! \frac{1}{3^{n+1}} = -\frac{n!}{3^{n+1}}.$$

Therefore the Taylor series of $f(x)$ centred at $a = -3$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x-a)^n = \sum_{n=0}^{\infty} -\frac{n!}{n!3^{n+1}}(x+3)^n = -\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+3)^n.$$

We can now find the radius of convergence by the usual computation involving the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{-(x+3)^{n+1}/3^{n+2}}{-(x+3)^n/3^{n+1}}\right| = \lim_{n\to\infty}\frac{1}{3}|x+3| = \frac{1}{3}|x+3|.$$

Now the series converges whenever

$$\frac{1}{3}|x+3| < 1 \quad \text{or equivalently} \quad |x+3| < 3.$$

This means that the series converges when $-3 < x+3 < 3$, so the radius of convergence is 3.

Alternatively, our function sort of looks like something that we could apply the geometric series formula to. So we could try to manipulate it to find a relation with the geometric series $\frac{1}{1-y}$:

$$\frac{1}{x} = \frac{1}{(x+3)-3} = -\frac{1}{3}\frac{1}{1-\frac{1}{3}(x+3)} = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}(x+3)^n = -\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x+3)^n.$$

From this, we can also easily get the radius of convergence: we used the geometric series representation of $\frac{1}{1-y}$ where we substituted $y = \frac{1}{3}(x+3)$. Now the geometric series for $\frac{1}{1-y}$ converges when $|y| < 1$, so putting in our substitution, our series will converge when $\frac{1}{3}|x+3| = |y| < 1$. Multiplying both sides by 3, our series converges when $|x+3| < 3$, meaning that the radius of convergence is 3. ∎

3. Let $f(x) := (1+x)^\alpha$, where α is any fixed real number—in particular, it does not have to be an integer!

(i) Compute $f^{(n)}(x)$ and $f^{(n)}(0)$ for $n = 0, 1, 2, 3, 4$.
(ii) Guess a pattern for $f^{(n)}(0)$ and try to justify it.
(iii) Use your guess in (ii) to write down a Taylor series for $f(x)$ centred at 0.
(iv) Find the radius of convergence of your power series expansion.
(v) Use your power series to find a power series expansion of $f(x) := (1-x)^{3/4}$.

It might be helpful to use the following notation: for any α and any positive integer n, set

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Solution. For (i), we have

$$\begin{aligned}
f^{(0)}(x) &= (1+x)^\alpha, & f^{(0)}(0) &= 1,\\
f^{(1)}(x) &= \alpha(1+x)^{\alpha-1}, & f^{(1)}(0) &= \alpha,\\
f^{(2)}(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, & f^{(2)}(0) &= \alpha(\alpha-1),\\
f^{(3)}(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, & f^{(3)}(0) &= \alpha(\alpha-1)(\alpha-2),\\
f^{(4)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-3}, & f^{(4)}(0) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3).
\end{aligned}$$

Looking at the coefficients, it may not be too far a stretch to see that there are products of n copies of α, with them stepping down by one each time. That is,

$$f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1).$$

Using this, a Taylor series for $f(x)$ is given by

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

To get the radius of convergence of this series, we would like to perform the Ratio Test computation. Though to make careful sense of this computation, we should assume that α is not a positive integer—if α is a positive integer, then the series above terminates and is a polynomial, so the radius of convergence is ∞. Anyway, proceeding with the assumption that α is not a positive integer, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)/(n+1)!}{\alpha(\alpha-1)\cdots(\alpha-n+1)/n!}\right||x| = \lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right||x| = |x|$$

since α is a fixed number. Thus the series converges when $|x| < 1$, and so the radius of convergence is 1.

Finally, for (v), we now plug in $\alpha = 3/4$ and $-x$ into the series derived above to get

$$(1-x)^{3/4} = \sum_{n=0}^{\infty} (-1)^n \binom{3/4}{n} x^n.$$

With a lot of effort, the funny binomial coefficient can be simplified and expressed as a product of other regular binomial coefficients, but this is a fine expression. ■

4. Let f_n denote the n^{th} Fibonacci number. Recall that f_n is defined recursively by setting $f_0 = f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Let

$$F(x) := \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots.$$

(i) Find the radius of convergence of $F(x)$. It might be helpful to look at **7**.(iii), HW5.

(ii) Use the recurrence relation for the Fibonacci numbers to show that

$$F(x) = 1 + xF(x) + x^2 F(x).$$

(iii) Rearrange the relation in (ii) to show that

$$F(x) = \frac{1}{1 - x - x^2},$$

at least within the interval of convergence of $F(x)$.

(iv) Let

$$1 - x - x^2 = (\phi_+ - x)(\phi_- - x) \quad \text{where } \phi_\pm := \frac{1 \pm \sqrt{5}}{2}.$$

Find the partial fraction expansion of $\frac{1}{1-x-x^2}$ in terms of $\frac{1}{\phi_+ - x}$ and $\frac{1}{\phi_- - x}$

(v) Use the geometric series formula to find a power series expansion of $\frac{1}{\phi_\pm - x}$.

(vi) Put together (iii)–(v) to find an explicit formula for the n^{th} Fibonacci number f_n.

Solution. To find the radius of convergence of the series, perform the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{f_{n+1} x^{n+1}}{f_n x^n} \right| = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} |x| = \frac{1+\sqrt{5}}{2} |x|$$

Therefore, the series will converge when

$$\frac{1+\sqrt{5}}{2}|x| < 1 \quad \text{which means} \quad |x| < \frac{2}{1+\sqrt{5}}.$$

So the radius of convergence of this series is $\frac{1+\sqrt{5}}{2}$.

For (ii), we manipulate the right hand side of the claimed relationship to obtain $F(x)$:

$$\begin{aligned} 1 + xF(x) + x^2 F(x) &= 1 + x \sum_{n=0}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n \\ &= 1 + \sum_{n=1}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\ &= 1 + f_0 x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\ &= 1 + x + \sum_{n=2}^{\infty} f_n x^n \\ &= \sum_{n=0}^{\infty} f_n x^n \\ &= F(x) \end{aligned}$$

as desired.

Now using the relationship from (ii), rearrange so that all the appearances of $F(x)$ appear on one side:

$$1 = F(x) - xF(x) - x^2F(x) = (1 - x - x^2)F(x)$$

so upon dividing through by $F(x)$, we get the relationship in (iii).

Now let's attempt to find a partial fraction expansion

$$F(x) = \frac{1}{1-x-x^2} = \frac{A}{\phi_+ - x} + \frac{B}{\phi_- - x}$$

for some numbers A and B. Clearing denominators, this gives the equation

$$1 = A(\phi_- - x) + B(\phi_+ - x) = -(A+B)x + (A\phi_- + B\phi_+)$$

so upon equating coefficients of x and 1 on both sides, we obtain the system of linear equations

$$\begin{cases} A + B = 0, \\ A\phi_- + B\phi_+ = 1, \end{cases} \qquad \text{so} \qquad \begin{cases} B = -A, \\ A(\phi_- - \phi_+) = 1, \end{cases}$$

and therefore

$$-B = A = \frac{1}{\phi_- - \phi_+} = \frac{2}{(1-\sqrt{5}) - (1+\sqrt{5})} = -\frac{1}{\sqrt{5}}.$$

So we get a partial fraction expansion

$$F(x) = \frac{1}{1-x-x^2} = \frac{1}{\sqrt{5}}\Big(\frac{1}{\phi_- - x} - \frac{1}{\phi_+ - x}\Big).$$

Now to find a power series of the function on the right, let's do it for each piece:

$$\frac{1}{\phi_\pm - x} = \frac{1}{\phi_\pm}\frac{1}{1 - \frac{1}{\phi_\pm}x} = \frac{1}{\phi_\pm}\sum_{n=0}^{\infty}\frac{1}{\phi_\pm^n}x^n = \sum_{n=0}^{\infty}\frac{1}{\phi_\pm^{n+1}}x^n.$$

Putting everything together, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n x^n = F(x) &= \frac{1}{1-x-x^2} \\
&= \frac{1}{\sqrt{5}}\Big(\frac{1}{\phi_- - x} - \frac{1}{\phi_+ - x}\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\sum_{n=0}^{\infty}\frac{1}{\phi_-^{n+1}}x^n - \sum_{n=0}^{\infty}\frac{1}{\phi_+^{n+1}}x^n\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\Big(\frac{1}{\phi_-^{n+1}} - \frac{1}{\phi_+^{n+1}}\Big)x^n.
\end{aligned}$$

Comparing coefficients of x^n on both sides, we obtain

$$f_n = \frac{1}{\sqrt{5}}\Big(\frac{1}{\phi_-^{n+1}} - \frac{1}{\phi_+^{n+1}}\Big),$$

and hence an explicit formula for the Fibonacci numbers! ■