CALCULUS II ASSIGNMENT 6 SOLUTIONS

1. Justify why the following series converge and find their sums.[1],

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$, (iv) $\displaystyle\sum_{n=1}^{\infty} \big(\sin(1/n) - \sin(1/(n+1))\big)$.

Solutions. The series in (i) converges by comparison to the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$: indeed, for each positive integer n, $n^4 + n^2 \geq n^4$, so upon taking reciprocals, we obtain

$$\frac{1}{n^4+n^2} \leq \frac{1}{n^4}.$$

Since the series $\sum \frac{1}{n^4}$ converges, so does this one. Originally, I thought that this series would telescope some more, but alas, it does not, making the sum beyond our reach for the moment.

The series in (ii) is a geometric series: looking at the first few terms

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} + \frac{1}{4}\cdot\frac{-3}{4} + \frac{1}{4}\cdot\left(\frac{-3}{4}\right)^2 + \cdots$$

we see that this geometric series has initial term $a = \frac{1}{4}$ and the common ratio $r = \frac{-3}{4}$. Since $|r| < 1$, this geometric series is convergent and has sum

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1/4}{1-\frac{-3}{4}} = \frac{1}{7}.$$

The series in (iii) is again a geometric series: writing out the first few terms,

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{12}{-5} + \frac{12}{-5}\cdot\frac{1}{-5} + \frac{12}{-5}\cdot\left(\frac{1}{-5}\right)^2 + \cdots$$

so the initial term is $a = \frac{12}{-5}$ and the common ratio is $r = \frac{1}{-5}$. Since $|r| < 1$, this geometric series converges and has sum

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{12/-5}{1-\frac{1}{-5}} = \frac{12}{-6} = -2.$$

Finally, to see how the series in (iv) behaves, write out the first few terms:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right) &= \left(\sin(1) - \sin\frac{1}{2}\right) + \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \left(\sin\frac{1}{3} - \sin\frac{1}{4}\right) + \cdots \\ &= \sin(1) - \left(\sin\frac{1}{2} - \sin\frac{1}{2}\right) - \left(\sin\frac{1}{3} - \sin\frac{1}{3}\right) - \cdots\end{aligned}$$

[1] March 7: I misjudged **1.**(i) and the sum is not so easy to compute. Simply justify why it converges, please!

so we seem to see some telescoping happening in the series, suggesting that the sum will simply be $\sin(1)$. To perhaps say this slightly more carefully, let's go back to the definition of what the value of the series is: we need to compute the sequence (s_n) of partial sums:

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \left(\sin(1) - \sin\frac{1}{2}\right) + \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \cdots + \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right). \end{aligned}$$

Upon regrouping the terms, we see that the intermediate terms pair off and cancel out, leaving

$$\begin{aligned} s_n &= \sin(1) - \left(\sin\frac{1}{2} - \sin\frac{1}{2}\right) - \cdots - \left(\sin\frac{1}{n} - \sin\frac{1}{n}\right) - \sin\frac{1}{n+1} \\ &= \sin(1) - \sin\frac{1}{n+1}. \end{aligned}$$

Therefore the value of the series is

$$\begin{aligned} \sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right) &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} \sin(1) - \sin\frac{1}{n+1} = \sin(1) - \sin(0) = \sin(1) \end{aligned}$$

where I have used the fact that sin is a continuous function so that I can compute the limit by simply taking the limit of its argument. ■

2. Determine whether or not the following series converge or diverge. If they converge, determine their sum.

(i) $\displaystyle\sum_{n=1}^{\infty} \cos(n)$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \sin(100)^k$,

(iii) $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m^3-m}$,

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$,

(v) $\displaystyle\sum_{i=1}^{\infty} \frac{3^{i+1}}{(-2)^i}$,

(vi) $\displaystyle\sum_{\ell=1}^{\infty} \frac{1}{1+(2/3)^\ell}$.

Solutions. The series in (i) does not converge: indeed, the limit of the terms

$$\lim_{n\to\infty} \cos(n)$$

does not exist as $\cos(n)$ oscillates around the interval $[-1,1]$ and does not stabilize on any particular value. Therefore, by the Divergence Test, the series diverges.

The series in (ii) is a geometric series:

$$\sum_{k=1}^{\infty} \sin(100)^k = \sin(100) + \sin(100)^2 + \sin(100)^3 + \cdots$$

with $a = \sin(100)$ and $r = \sin(100)$. Note that $|r| = |\sin(100)| \approx 0.5 < 1$ so that this is a convergent geometric series with sum

$$\sum_{k=1}^{\infty} \sin(100)^k = \frac{\sin(100)}{1-\sin(100)}.$$

Note that we could have seen that $|\sin(100)| < 1$ without computing its actual value: from last week's assignment, we already know that $|\sin(x)| \le 1$ for any real number x; now the only time that $|\sin(x)| = 1$ is when x is an integral multiple of $\frac{\pi}{2}$; since 100 is not an integral multiple of $\frac{\pi}{2}$, $\sin(100) \neq \pm 1$ and therefore $|\sin(100)| < 1$.

To deal with the series in (iii), begin by rewriting the terms as

$$\frac{1}{m^3-m} = \frac{1}{(m-1)m(m+1)} = \frac{1}{2(m-1)} - \frac{1}{m} + \frac{1}{2(m+1)}.$$

Staring at the form of the general term, there appears to be some conspiracy amongst the terms that might cause for some miraculous cancellation; indeed, tinkering with different ways of writing the initial terms of the series, one might be led to writing

$$\begin{array}{lllllll}
\displaystyle\sum_{m=2}^{\infty} \frac{1}{m^3-m} = \dfrac{1}{2\cdot 1} & -\dfrac{1}{2} & +\dfrac{1}{2\cdot 3} & & & & \\
 & +\dfrac{1}{2\cdot 2} & -\dfrac{1}{3} & +\dfrac{1}{2\cdot 4} & & & \\
 & & +\dfrac{1}{2\cdot 3} & -\dfrac{1}{4} & +\dfrac{1}{2\cdot 5} & & \\
 & & & +\dfrac{1}{2\cdot 5} & -\dfrac{1}{5} & +\dfrac{1}{2\cdot 6} & \\
 & & & & +\dfrac{1}{2\cdot 5} & -\dfrac{1}{6} & +\dfrac{1}{2\cdot 7} + \cdots
\end{array}$$

from which we see that the terms enclosed in red boxes sandwich together and cancel out in the sum. Thus the only terms that remain upon this telescoping process are the left three terms, yielding

$$\sum_{m=2}^{\infty} \frac{1}{m^3 - m} = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4}.$$

The series (iv) looks divergent: when dealing with series with rational combinations of n as terms, the feature that determines convergence is the ratio of the leading term of the numerator with the leading term of the denominator. In this case, both are n^2, so the ratio is 1, suggesting divergence. Indeed, to make this precise, consider the limit of the terms of the series:

$$\lim_{n\to\infty} \frac{n^2}{n^2 - 2n + 5} = \lim_{n\to\infty} \frac{1}{1 - 2\frac{1}{n} + 5\frac{1}{n^2}} = 1$$

which is nonzero. Thus this series diverges by the divergence test.

The series (v) is a geometric series: writing out the first few terms

$$\sum_{i=1}^{\infty} \frac{3^{i+1}}{(-2)^i} = \frac{9}{-2} + \frac{9}{-2} \cdot \frac{3}{-2} + \frac{9}{-2} \cdot \left(\frac{3}{-2}\right)^2 + \cdots$$

we see that $a = -\frac{9}{2}$ and $r = -\frac{3}{2}$. In particular, $|r| > 1$ and hence this geometric series is divergent.

Finally, the series in (vi) is also divergent. Consider the limit of the terms:

$$\lim_{\ell\to\infty} \frac{1}{1 + (2/3)^\ell} = \frac{1}{1} = 1$$

since the $(2/3)^\ell$ are powers of a number that has absolute value strictly less than 1. In particular, the terms of the series do not go to 0 and so the series diverges by the Divergence Test. ■

3. The Comparison Test we discussed in class says that given two series $\sum a_i$ and $\sum b_i$ with positive terms, then

(i) if $\sum b_i$ is convergent and $a_n \leq b_n$ for all sufficiently large indices n, then $\sum a_i$ is convergent; and

(ii) if $\sum b_i$ is divergent and $a_n \geq b_n$ for all sufficiently large indices n, then $\sum a_i$ is divergent.

Formulate analogues of statements (i) and (ii) for series $\sum a_i$ and $\sum b_i$ with negatives terms. Try to justify your statements using (i) and (ii) above.

Solution. The analogues for the above statements when our series have negative terms are as follows:

Theorem. — Given series $\sum a_i$ and $\sum b_i$ with negative terms, then

(i) if $\sum b_i$ is convergent and $a_n \geq b_n$ for all sufficiently large indices n, then $\sum a_i$ is convergent; and

(ii) if $\sum b_i$ is divergent and $a_n \leq b_n$ for all sufficiently large indices n, then $\sum a_i$ is divergent.

To prove these statements given the Comparison Test above, take the negative of the series. Precisely, if $\sum b_i$ is convergent, consider the series $\sum(-b_i)$; this is a scalar multiple of a convergent series, and hence it is convergent. Since the b_i were negative, the numbers $-b_i$ are positive. Now the inequality $a_n \geq b_n$ is equivalent to the inequality $-a_n \leq -b_n$ obtained by multiplying through by -1—inequalities flip upon scaling by negative numbers. But we now the following situation: We have series $\sum(-a_i)$ and $\sum(-b_i)$ with positive terms such that $\sum(-b_i)$ is convergent and $-a_n \leq -b_n$ holds for all sufficiently large indices n. Thus, by case (i) of the usual Comparison Test, we deduce that $\sum(-a_i)$ is convergent. Now scaling this series by -1 again, we deduce that $\sum a_i$ is convergent, as required. Statement (ii) is proved in an analogous fashion. ∎

4. Use the Comparison Test to determine whether or not the following series converge or diverge:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$,

(ii) $\displaystyle\sum_{m=1}^{\infty} \frac{\log(m)}{m}$,

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$,

(iv) $\displaystyle\sum_{\ell=1}^{\infty} \frac{e^{1/\ell}}{\ell}$,

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(vi) $\displaystyle\sum_{m=1}^{\infty} \frac{9^m}{3+10^m}$.

In some of the comparisons above, it might be useful to know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges when $p > 1$ and diverges when $p \leq 1$; note that the $p = 1$ case is the Harmonic Series, which I mentioned in class. We will see why these statements are true, soon!

Solutions. The series in (i) looks divergent: as above, this is basically a ratio of two polynomials in n, so convergence or divergence of this series is basically determined by the ratio of the leading terms of the numerator and denominator. In this case, they are n and $n\sqrt{n} = n^{3/2}$, respectively. Hence their ratio is $1/n^{1/2}$, so this sum looks like $\sum \frac{1}{n^{1/2}}$: this is a p-series with $p = 1/2$ and thus is divergent. Okay, this suggests that we should compare from below with $\sum \frac{1}{n^{1/2}}$, that is, we should ask whether or not the inequality

$$\frac{1}{\sqrt{n}} \overset{?}{\leq} \frac{n+1}{n\sqrt{n}}$$

holds for maybe sufficiently large n. One could cross multiply the denominators and manipulate the resulting expression to verify this inequality. Another method would be to observe as follows:

$$\frac{1}{\sqrt{n}} = \frac{n}{n\sqrt{n}} \leq \frac{n+1}{n\sqrt{n}}$$

where in the first equality I multiplied by $1 = \frac{n}{n}$, and in the next inequality, I used the fact that $n \leq n+1$. This shows that $\frac{1}{\sqrt{n}} \leq \frac{n+1}{n\sqrt{n}}$ for all n, and since $\sum \frac{1}{\sqrt{n}}$ diverges, the Comparison Test implies that our series in (i) also diverges.

The series in (ii) also diverges: the terms basically looks like those in the harmonic series $\sum \frac{1}{n}$, which we know diverges, except that the numerators of the terms are scaled by $\log(n)$. However, when $n \geq 3$, $\log(n) > 1$, we have the inequality $\frac{1}{n} < \frac{\log(n)}{n}$, at least when $n \geq 3$. Therefore, since the harmonic series diverges, the Comparison Test implies that the series in (ii) also diverges.

The terms in (iii) seem to shrink really quickly; in particular, one thing that you might observe is that for $k \geq 2$, $k^k \geq k^2$, so upon taking reciprocals, $\frac{1}{k^k} \leq \frac{1}{k^2}$. Thus we can compare $\sum \frac{1}{k^k}$ with the series $\sum \frac{1}{k^2}$ and since the latter is convergent, the Comparison Test shows that the former is also convergent.

For the series in (iv), note that this vaguely looks like the harmonic series. Indeed, $e^{1/\ell} > 1$ for all positive integers ℓ. One way to see this is to note that

$$\lim_{\ell \to \infty} e^{1/\ell} = e^0 = 1$$

and that the sequence $(e^{1/\ell})$ is decreasing. Therefore $\frac{1}{\ell} < \frac{e^{1/\ell}}{\ell}$ for all positive integers ℓ and we see that the series in (iv) diverges upon applying Comparison with the harmonic series.

The series in (v) might take some deciding to get a feel for whether or not it converges or diverges. Maybe upon playing with some of the terms, one might feel that $n! = n \times (n-1) \times \cdots \times 2 \times 1$ grows quite a bit slower than n^n since both numbers are products of n numbers, but the numbers in n^n are all growing with n, whereas only the largest number, in a sense, is growing in the expression for $n!$. More generally, one might feel that $n!$ eventually grows slower than n^{n-k} for any fixed positive number k. This all might lead one to intuit that the series in question is convergent and thus we should find something that bounds the terms $\frac{n!}{n^n}$ from above and in such a way that we obtain an obviously convergent series. Tinkering a bit more, and perhaps having $\sum \frac{1}{n^2}$ as an obviously convergent series in mind, this could lead you to try to make the term-wise comparison

$$\frac{n!}{n^n} \overset{?}{\leq} \frac{n^{n-2}}{n^n}.$$

Clearing denominators, this is equivalent to the inequality

$$n! \overset{?}{\leq} n^{n-2}.$$

Alright, might this inequality be true? The first thing that one might do is to try the first few cases:

$$1 \overset{?}{\leq} 1,$$

$$2 \overset{?}{\leq} 1,$$

$$6 \overset{?}{\leq} 3,$$

$$24 \overset{?}{\leq} 16,$$

$$120 \overset{?}{\leq} 125,$$

$$\vdots$$

and so with some courage, we see that although the first few inequalities do not quite hold, the inequality that we want seems to hold for $n \geq 5$—at least certainly we see that $n! \leq n^{n-2}$ when $n = 5$. Given this, we can further prove that the inequality continues to hold for all $n \geq 5$: supposing that we have verified that $(n-1)! \leq (n-1)^{(n-1)-2}$, we have that

$$n! = (n-1)! \cdot n \leq (n-1)^{n-3} \cdot n \leq n^{n-2}.$$

Thus once we know the inequality starts holding, it will continue to hold forevermore. In summary, this shows that $n! \leq n^{n-2}$ for all $n \geq 5$, and thus

$$\frac{n!}{n^n} \leq \frac{n^{n-2}}{n^n} = \frac{1}{n^2}$$

for all $n \geq 5$. Thus by Comparison with $\sum \frac{1}{n^2}$, we see that the series $\sum \frac{n!}{n^n}$ is convergent.

Finally, for (vi), upon ignoring the pesky superfluous 3 in the denominator, the terms of our series looks like $\frac{9^m}{10^m}$, and thus our series sort of looks like a geometric series. This suggests that the series is convergent and that we should compare with the geometric series $\sum \frac{9^m}{10^m}$. This amounts to asking whether or not the inequality

$$\frac{9^m}{3+10^m} \stackrel{?}{\leq} \frac{9^m}{10^m}$$

holds true for all m. Dividing through by 9^m and cross multiplying, the above inequality is equivalent to the inequality

$$10^m \stackrel{?}{\leq} 3+10^m$$

and this one is evidently true: when I add 3 to 10^m, it does become a larger number. Thus the terms of our series are bounded by those of the convergent geometric series $\sum \frac{9^m}{10^m}$, and so our series also converges by the Comparison Test. ■