N. Piskunov_Differential_and_Integral_Calculus kitabından Türev Konusunda Alıştırmalar

Exercises on Chapter III

Find the derivatives of functions using the definition of a derivative. 1. $y=x^3$. *Ans.* $3x^2$. 2. $y=\frac{1}{x}$. *Ans.* $-\frac{1}{x^2}$. 3. $y=\sqrt{x}$. *Ans.* $\frac{1}{2\sqrt{x}}$. 4. $y=\frac{1}{\sqrt{x}}$. *Ans.* $-\frac{1}{2x\sqrt{x}}$. 5. $y=\sin^2 x$. *Ans.* $2\sin x\cos x$. 6. $y=2x^2-x$. *Ans.* $4x-1$.

Determine the tangents of the angles of inclination of tangents to the curves: 7. $y=x^3$. a) When $x=1$. *Ans.* 3. b) When $x=-1$. *Ans.* 3. Make a drawing. 8. $y=\frac{1}{x}$. a) When $x=\frac{1}{2}$. *Ans.* -4. b) When $x=1$. *Ans.* -1. Make a drawing. 9. $y=\sqrt{x}$ when $x=2$. *Ans.* $\frac{1}{2\sqrt{2}}$.

Find the derivatives of the functions: 10. $y=x^4+3x^2-6$. *Ans.* $y'=4x^3+6x$. 11. $y=6x^3-x^2$. *Ans.* $y'=18x^2-2x$. 12. $y=\frac{x^5}{a+b}-\frac{x^2}{a-b}-x$. *Ans.* $y'=\frac{5x^4}{a+b}-\frac{2x}{a-b}-1$. 13. $y=\frac{x^3-x^2+1}{5}$. *Ans.* $y'=\frac{3x^2-2x}{5}$. 14. $y=2ax^3-\frac{x^2}{b}+c$. *Ans.* $y'=6ax^2-\frac{2x}{b}$. 15. $y=6x^{\frac{7}{2}}+4x^{\frac{5}{2}}+2x$. *Ans.* $y'=21x^{\frac{5}{2}}+10x^{\frac{3}{2}}+2$. 16. $y=\sqrt{3x}+\sqrt[3]{x}+\frac{1}{x}$. *Ans.* $y'=\frac{\sqrt{3}}{2\sqrt{x}}+\frac{1}{3\sqrt[3]{x^2}}-\frac{1}{x^2}$.

17. $y=\frac{(x+1)^3}{x^{\frac{3}{2}}}$. *Ans.* $y'=\frac{3(x+1)^2(x-1)}{2x^{\frac{5}{2}}}$. 18. $y=\frac{x}{m}+\frac{m}{x}+\frac{x^2}{n^2}+\frac{n^2}{x^2}$.

Ans. $y'=\frac{1}{m}-\frac{m}{x^2}+\frac{2x}{n^2}-\frac{2n^2}{x^3}$. **19.** $y=\sqrt[3]{x^2}-2\sqrt{x}+5$. *Ans.* $y'=\frac{2}{3}\frac{1}{\sqrt[3]{x}}-$ $-\frac{1}{\sqrt{x}}$. **20.** $y=\frac{ax^2}{\sqrt[3]{x}}+\frac{b}{x\sqrt{x}}-\frac{\sqrt[3]{x}}{\sqrt{x}}$. *Ans.* $y'=\frac{5}{3}ax^{\frac{2}{3}}-\frac{3}{2}bx^{-\frac{5}{2}}+$ $+\frac{1}{6}x^{-\frac{7}{6}}$.

21. $y=(1+4x^3)(1+2x^2)$. *Ans.* $y'=4x(1+3x+10x^3)$. **22.** $y=x(2x-1)(3x+2)$. *Ans.* $y'=2(9x^2+x-1)$. **23.** $y=(2x-1)(x^2-6x+3)$. *Ans.* $y'=6x^2-26x+12$. **24.** $y=\frac{2x^4}{b^2-x^2}$. *Ans.* $y'=\frac{4x^3(2b^2-x^2)}{(b^2-x^2)^2}$. **25.** $y=\frac{a-x}{a+x}$. *Ans.* $y'=-\frac{2a}{(a+x)^2}$.

26. $f(t)=\frac{t^3}{1+t^2}$. *Ans.* $f'(t)=\frac{t^2(3+t^2)}{(1+t^2)^2}$. **27.** $f(s)=\frac{(s+4)^2}{s+3}$. *Ans.* $f'(s)=$ $=\frac{(s+2)(s+4)}{(s+3)^2}$. **28.** $y=\frac{x^3+1}{x^2-x-2}$. *Ans.* $y'=\frac{x^4-2x^3-6x^2-2x+1}{(x^2-x-2)^2}$.

29. $y=\frac{x^p}{x^m-a^m}$. *Ans.* $y'=\frac{x^{p-1}[(p-m)x^m-pa^m]}{(x^m-a^m)^2}$. **30.** $y=(2x^2-3)^2$. *Ans.* $y'=8x(2x^2-3)$. **31.** $y=(x^2+a^2)^5$. *Ans.* $y=10x(x^2+a^2)^4$. **32.** $y=$ $=\sqrt{x^2+a^2}$. *Ans.* $y'=\frac{x}{\sqrt{x^2+a^2}}$. **33.** $y=(a+x)\sqrt{a-x}$. *Ans.* $y'=\frac{a-3x}{2\sqrt{a-x}}$.

34. $y=\sqrt{\frac{1+x}{1-x}}$. *Ans.* $y'=\frac{1}{(1-x)\sqrt{1-x^2}}$. **35.** $y=\frac{2x^2-1}{x\sqrt{1+x^2}}$. *Ans.* $y'=$ $=\frac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}$. **36.** $y=\sqrt[3]{x^2+x+1}$. *Ans.* $y'=\frac{2x+1}{3\sqrt[3]{(x^2+x+1)^2}}$. **37.** $y=$ $=(1+\sqrt[3]{x})^3$. *Ans.* $y'=\left(1+\frac{1}{\sqrt[3]{x}}\right)^2$. **38.** $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$. *Ans.* $y'=$ $=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$. **39.** $y=\sin^2 x$. *Ans.* $y'=\sin 2x$. **40.** $y=2\sin x+\cos 3x$. *Ans.* $y'=2\cos x-3\sin 3x$. **41.** $y=$ $=\tan(ax+b)$. *Ans.* $y'=\frac{a}{\cos^2(ax+b)}$. **42.** $y=\frac{\sin x}{1+\cos x}$. *Ans.* $y'=\frac{1}{1+\cos x}$. **43.** $y=\sin 2x\cdot\cos 3x$. *Ans.* $y'=2\cos 2x\cos 3x-3\sin 2x\sin 3x$. **44.** $y=\cot^2 5x$. *Ans.* $y'=-10\cot 5x\csc^2 5x$. **45.** $y=t\sin t+\cos t$. *Ans.* $y'=t\cos t$. **46.** $y=$ $=\sin^3 t\cos t$. *Ans.* $y'=\sin^2 t(3\cos^2 t-\sin^2 t)$. **47.** $y=a\sqrt{\cos 2x}$. *Ans.* $y'=$ $=-\frac{a\sin 2x}{\sqrt{\cos 2x}}$. **48.** $r=a\sin^3\frac{\varphi}{3}$ *Ans.* $r'_\varphi=a\sin^2\frac{\varphi}{3}\cos\frac{\varphi}{3}$. **49.** $y=\frac{\tan\frac{x}{2}+\cot\frac{x}{2}}{x}$.

Ans. $y'=-\dfrac{2x\cos x+\sin^2 x\left(\tan\frac{x}{2}+\cot\frac{x}{2}\right)}{x^2\sin^2 x}$. 50. $y=a\left(1-\cos^2\frac{x}{2}\right)^2$. *Ans.* $y'=2a\sin^3\frac{x}{2}\cos\frac{x}{2}$. 51. $y=\frac{1}{2}\tan^2 x$. *Ans.* $y'=\tan x\sec^2 x$. 52. $y=\ln\cos x$. *Ans.* $y'=-\tan x$. 53. $y=\ln\tan x$. *Ans.* $y'=\dfrac{2}{\sin 2x}$. 54. $y=\ln\sin^2 x$. *Ans.* $y'=2\cot x$. 55. $y=\dfrac{\tan x-1}{\sec x}$. *Ans.* $y'=\sin x+\cos x$. 56. $y=\ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$. *Ans.* $y'=\dfrac{1}{\cos x}$. 57. $y=\ln\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$. *Ans.* $y'=\dfrac{1}{\cos x}$. 58. $y=\sin(x+a)\times\cos(x+a)$. *Ans.* $y'=\cos 2(x+a)$. 59. $f(x)=\sin(\ln x)$. *Ans.* $f'(x)=\dfrac{\cos(\ln x)}{x}$. 60. $f(x)=\tan(\ln x)$. *Ans.* $f'(x)=\dfrac{\sec^2(\ln x)}{x}$. 61 $f(x)=\sin(\cos x)$. *Ans.* $f'(x)=-\sin x\cos(\cos x)$. 62. $r=\frac{1}{3}\tan^3\varphi-\tan\varphi+\varphi$. *Ans* $\dfrac{dr}{d\varphi}=\tan^4\varphi$. 63. $f(x)=(x\cot x)^2$. *Ans.* $f'(x)=2x\cot x(\cot x-x\csc^2 x)$. 64. $y=\ln(ax+b)$. *Ans.* $y'=\dfrac{a}{ax+b}$. 65. $y=\log_a(x^2+1)$. *Ans.* $y'=\dfrac{2x}{(x^2+1)\ln a}$. 66. $y=\ln\dfrac{1+x}{1-x}$. *Ans.* $y'=\dfrac{2}{1-x^2}$. 67. $y=\log_3(x^2-\sin x)$. *Ans.* $y'=\dfrac{2x-\cos x}{(x^2-\sin x)\ln 3}$. 68. $y=\ln\dfrac{1+x^2}{1-x^2}$. *Ans.* $y'=\dfrac{4x}{1-x^4}$; 69. $y=\ln(x^2+x)$. *Ans.* $y'=\dfrac{2x+1}{x^2+x}$. 70. $y=\ln(x^3-2x+5)$. *Ans.* $y'=\dfrac{3x^2-2}{x^3-2x+5}$. 71. $y=x\ln x$. *Ans.* $y'=\ln x+1$ 72. $y=\ln^3 x$. *Ans.* $y'=\dfrac{3\ln^2 x}{x}$. 73. $y=\ln(x+\sqrt{1+x^2})$. *Ans.* $y'=\dfrac{1}{\sqrt{1+x^2}}$ 74. $y=\ln(\ln x)$. *Ans.* $y'=\dfrac{1}{x\ln x}$. 75. $f(x)=\ln\sqrt{\dfrac{1+x}{1-x}}$. *Ans.* $f'(x)=\dfrac{1}{1-x^2}$. 76. $f(x)=\ln\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2-1}+x}$. *Ans.* $f'(x)=-\dfrac{2}{\sqrt{1+x^2}}$. 77. $y=\sqrt{a^2+x^2}-a\ln\dfrac{a+\sqrt{a^2+x^2}}{x}$. *Ans.* $y'=\dfrac{\sqrt{a^2+x^2}}{x}$. 78. $y=\ln(x+\sqrt{x^2+a^2})-\dfrac{\sqrt{x^2+a^2}}{x}$. *Ans.* $y'=\dfrac{\sqrt{x^2+a^2}}{x^2}$. 79. $y=-\dfrac{\cos x}{2\sin^2 x}+\dfrac{1}{2}\ln\tan\dfrac{x}{2}$. *Ans.* $y'=\dfrac{1}{\sin^3 x}$. 80. $y=\dfrac{\sin x}{2\cos^2 x}$. *Ans.* $y'=\dfrac{1+\sin^2 x}{2\cos^3 x}$. 81. $y=\frac{1}{2}\tan^2 x+\ln\cos x$. *Ans.* $y'=\tan^3 x$. 82. $y=e^{ax}$. *Ans.* $y'=ae^{ax}$. 83. $y=e^{4x+5}$. *Ans.* $y'=4e^{4x+5}$. 84. $y=a^{x^2}$. *Ans.* $2x\,a^{x^2}\ln a$. 85. $y=7^{x^2+2x}$. *Ans.* $y'=2(x+1)\,7^{x^2+2x}\ln 7$. 86. $y=c^{a^2-x^2}$. *Ans.* $y'=-2xc^{a^2-x^2}\ln c$. 87. $y=ae^{\sqrt{x}}$. *Ans.* $y'=\dfrac{a}{2\sqrt{x}}e^{\sqrt{x}}$.

88. $r=a^{\theta}$. *Ans.* $r'=a^{\theta}\ln a$. 89. $r=a^{\ln\theta}$. *Ans.* $\frac{dr}{d\theta}=\frac{a^{\ln\theta}\ln a}{\theta}=\theta^{\ln a-1}\ln a$.

90. $y=e^x(1-x^2)$. *Ans.* $y'=e^x(1-2x-x^2)$. 91. $y=\frac{e^x-1}{e^x+1}$. *Ans.* $y'=\frac{2e^x}{(e^x+1)^2}$.

92. $y=\ln\frac{e^x}{1+e^x}$. *Ans.* $y'=\frac{1}{1+e^x}$. 93. $y=\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$. *Ans.* $y'=$ $=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$. 94. $y=e^{\sin x}$. *Ans.* $y'=e^{\sin x}\cos x$. 95. $y=a^{\tan nx}$. *Ans.* $y'=$ $=na^{\tan nx}\sec^2 nx\ln a$. 96. $y=e^{\cos x}\sin x$. *Ans.* $y'=e^{\cos x}(\cos x-\sin^2 x)$.
97. $y=e^x\ln\sin x$. *Ans.* $y'=e^x(\cot x+\ln\sin x)$. 98. $y=x^n e^{\sin x}$. *Ans.* $y'=$ $=x^{n-1}e^{\sin x}(n+x\cos x)$. 99. $y=x^x$. *Ans.* $y'=x^x(\ln x+1)$. 100. $y=x^{\frac{1}{x}}$. *Ans.* $y'=x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right)$. 101. $y=x^{\ln x}$. *Ans.* $y'=x^{\ln x-1}\ln x^2$. 102. $y=e^{x^x}$. *Ans.* $y'=e^{x^x}(1+\ln x)\,x^x$. 103. $y=\left(\frac{x}{n}\right)^{nx}$. *Ans.* $y'=n\left(\frac{x}{n}\right)^{nx}\left(1+\ln\frac{x}{n}\right)$.

104. $y=x^{\sin x}$. *Ans.* $y'=x^{\sin x}\left(\frac{\sin x}{x}+\ln x\cos x\right)$. 105. $y=(\sin x)^x$. *Ans.* $y'=$ $=(\sin x)^x(\ln\sin x+x\cot x)$. 106. $y=(\sin x)^{\tan x}$. *Ans.* $y'=(\sin x)^{\tan x}\times$ $\times(1+\sec^2 x\ln\sin x)$. 107. $y=\tan\frac{1-e^x}{1+e^x}$. *Ans.* $y'=-\frac{e^{2x}}{(1+e^x)^2}\frac{1}{\cos^2\frac{1-e^x}{1+e^x}}$.

108. $y=\sin\sqrt{1-2^x}$. *Ans.* $y'=-\frac{\cos\sqrt{1-2x}}{2\sqrt{1-2^x}}2^x\ln 2$. 109. $y=10^{x\tan x}$. *Ans.* $y'=$ $=10^{x\tan x}\ln 10\left(\tan x+\frac{x}{\cos^2 x}\right)$.

Find the derivatives of the functions after first taking logarithms of these functions:

110. $y=\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}$. *Ans.* $y'=\frac{1}{3}\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}\left(\frac{1}{x}+\frac{2x}{x^2+1}+\frac{2}{x-1}\right)$.

111. $y=\frac{(x+1)^3\sqrt[4]{(x-2)^3}}{\sqrt[5]{(x-3)^2}}$. *Ans.* $y'=\frac{(x+1)^3\sqrt[4]{(x-2)^3}}{\sqrt[5]{(x-3)^2}}\left(\frac{3}{x+1}+\frac{3}{4(x-2)}-\right.$ $\left.-\frac{2}{5(x-3)}\right)$. 112. $y=\frac{(x+1)^2}{(x+2)^3(x+3)^4}$. *Ans.* $y'=-\frac{(x+1)(5x^2+14x+5)}{(x+2)^4(x+3)^5}$.

113. $y=\frac{\sqrt[5]{(x-1)^2}}{\sqrt[4]{(x-2)^3}\sqrt[3]{(x-3)^7}}$. *Ans.* $y'=\frac{-161x^2+480x-271}{60\sqrt[5]{(x-1)^3}\sqrt[4]{(x-2)^7}\sqrt[3]{(x-3)^{10}}}$.

114. $y=\frac{x(1+x^2)}{\sqrt{1-x^2}}$. *Ans.* $y'=\frac{1+3x^2-2x^4}{(1-x^2)^{\frac{3}{2}}}$. 115. $y=x^5(a+3x)^3(a-2x)^2$. *Ans.* $y'=5x^4(a+3x)^2(a-2x)(a^2+2ax-12x^2)$. 116. $y=\arcsin\frac{x}{a}$. *Ans.* $y'=\frac{1}{\sqrt{a^2-x^2}}$. 117. $y=(\arcsin x)^2$. *Ans.* $y'=\frac{2\arcsin x}{\sqrt{1-x^2}}$. 118. $y=\operatorname{arc\,cot}(x^2+1)$. *Ans.* $y'=\frac{2x}{1+(x^2+1)^2}$. 119. $y=\operatorname{arc\,cot}\frac{2x}{1-x^2}$. *Ans.* $y'=\frac{2}{1+x^2}$. 120. $y=\arccos(x^2)$. *Ans.* $y'=\frac{-2x}{\sqrt{1-x^4}}$. 121. $y=\frac{\arccos x}{x}$. *Ans.* $y'=\frac{-(x+\sqrt{1-x^2}\arccos x)}{x^2\sqrt{1-x^2}}$. 122. $y=\arcsin\frac{x+1}{\sqrt{2}}$. *Ans.* $y'=\frac{1}{\sqrt{1-2x-x^2}}$. 123. $y=x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$. *Ans.* $y'=2\sqrt{a^2-x^2}$. 124. $y=\sqrt{a^2-x^2}+a\arcsin\frac{x}{a}$. *Ans.* $y'=\sqrt{\frac{a-x}{a+x}}$. 125. $u=\operatorname{arc\,cot}\frac{v+a}{1-av}$. *Ans.* $\frac{du}{dv}=\frac{1}{1+v^2}$. 126. $y=\frac{1}{\sqrt{3}}\operatorname{arc\,cot}\frac{x\sqrt{3}}{1-x^2}$. *Ans.* $y'=\frac{x^2+1}{x^4+x^2+1}$. 127. $y=x\arcsin x$. *Ans.* $y'=\arcsin x+\frac{x}{\sqrt{1-x^2}}$. 128. $f(x)=\arccos(\ln x)$. *Ans.* $f'(x)=-\frac{1}{x\sqrt{1-\ln^2 x}}$. 129. $f(x)=\arcsin\sqrt{\sin x}$. *Ans.* $f'(x)=\frac{\cos x}{2\sqrt{\sin x-\sin^2 x}}$. 130. $y=\operatorname{arc\,cot}\sqrt{\frac{1-\cos x}{1+\cos x}}$ $(0\leqslant x<\pi)$. *Ans.* $y'=\frac{1}{2}$. 131. $y=e^{\operatorname{arc\,cot} x}$. *Ans.* $y'=\frac{e^{\operatorname{arc\,cot} x}}{1+x^2}$. 132. $y=\operatorname{arc\,cot}\frac{e^x-e^{-x}}{2}$. *Ans.* $y'=\frac{2}{e^x+e^{-x}}$. 133. $y=x^{\arcsin x}$. *Ans.* $y'=x^{\arcsin x}\left(\frac{\arcsin x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)$. 134. $y=\arcsin(\sin x)$. *Ans.* $y'=\frac{\cos x}{|\cos x|}=\begin{cases}+1 \text{ in 1st and 4th quadrants.}\\ -1 \text{ in 2d and 3d quadrants.}\end{cases}$ 135. $y=\operatorname{arc\,cot}\frac{4\sin x}{3+5\cos x}$. *Ans.* $y'=\frac{4}{5+3\cos x}$. 136. $y=\operatorname{arc\,cot}\frac{a}{x}+\ln\sqrt{\frac{x-a}{x+a}}$. *Ans.* $y'=\frac{2a^3}{x^4-a^4}$. 137. $y=\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\frac{1}{2}\arctan x$. *Ans.* $y'=\frac{x^2}{1-x^4}$. 138. $y=\frac{3x^2-1}{3x^3}+\ln\sqrt{1+x^2}+\arctan x$. *Ans.* $y'=\frac{x^5+1}{x^6+x^4}$. 139. $y=\frac{1}{3}\ln\frac{x+1}{\sqrt{x^2-x+1}}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

Ans. $y'=\frac{1}{x^3-1}$. **140.** $y=\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+2\arctan\frac{x\sqrt{2}}{1-x^2}$. *Ans.* $y'=\frac{4\sqrt{2}}{1+x^4}$.

141. $y=\arccos\frac{x^{2n}-1}{x^{2n}+1}$. *Ans.* $-\frac{2n\,|x|^n}{x(x^{2n}+1)}$.

Differentiation of Implicit Functions

Find $\frac{dy}{dx}$ if: **142.** $y^2=4px$. *Ans.* $\frac{dy}{dx}=\frac{2p}{y}$. **143.** $x^2+y^2=a^2$. *Ans.* $\frac{dy}{dx}=-\frac{x}{y}$. **144.** $b^2x^2+a^2y^2=a^2b^2$. *Ans.* $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$. **145.** $y^3-3y+2ax=0$. *Ans.* $\frac{dy}{dx}=\frac{2a}{3(1-y^2)}$. **146.** $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$. *Ans.* $\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$. **147.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$. **148.** $y^2-2xy+b^2=0$. *Ans.* $\frac{dy}{dx}=\frac{y}{y-x}$.

149. $x^3+y^3-3axy=0$. *Ans.* $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$. **150.** $y=\cos(x+y)$. *Ans.* $\frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$. **151.** $\cos(xy)=x$. *Ans.* $\frac{dy}{dx}=-\frac{1+y\sin(xy)}{x\sin(xy)}$.

Find $\frac{dy}{dx}$ of functions represented parametrically:

152. $x=a\cos t$; $y=b\sin t$. *Ans.* $\frac{dy}{dx}=-\frac{b}{a}\cot t$. **153.** $x=a(t-\sin t)$; $y=a(1-\cos t)$. *Ans.* $\frac{dy}{dx}=\cot\frac{t}{2}$. **154.** $x=a\cos^3 t$; $y=b\sin^3 t$. *Ans.* $\frac{dy}{dx}=-\frac{b}{a}\tan t$. **155.** $x=\frac{3at}{1+t^2}$; $y=\frac{3at^2}{1+t^2}$. *Ans.* $\frac{dy}{dx}=\frac{2t}{1-t^2}$. **156.** $u=2\ln\cot s$; $v=\tan s+\cot s$. Show that $\frac{du}{dv}=\tan 2s$.

Find the tangents of angles of the slopes of tangent lines to curves:

157. $x=\cos t$, $y=\sin t$ at the point $x=-\frac{1}{2}$, $y=\frac{\sqrt{3}}{2}$. Make a drawing *Ans.* $\frac{1}{\sqrt{3}}$. **158.** $x=2\cos t$, $y=\sin t$ at the point $x=1$, $y=-\frac{\sqrt{3}}{2}$. Make a drawing. *Ans.* $\frac{1}{2\sqrt{3}}$ **159.** $x=a(t-\sin t)$, $y=a(1-\cos t)$ when $t=\frac{\pi}{2}$. Make a drawing. *Ans.* 1. **160.** $x=a\cos^3 t$, $y=a\sin^3 t$ when $t=\frac{\pi}{4}$. Make a drawing. *Ans.* -1. **161.** A body thrown at an angle α to the horizon (in airless space) described a curve, under the force of gravity, whose equations are: $x=$

$=v_0\cos\alpha t$, $y=v_0\sin\alpha t-\frac{gt^2}{2}$ ($g=9.8$ m/sec^2). Knowing that $\alpha=60°$, $v_0=$ $=50$ m/sec, determine the direction of motion when: 1) $t=2$ sec; 2) $t=7$ sec. Make a drawing. *Ans.* 1) $\tan\varphi_1=0.948$, $\varphi_1=43°30'$; 2) $\tan\varphi_2=-1.012$, $\varphi_2=+134°7'$.

Find the differentials of the following functions:

162. $y=(a^2-x^2)^5$. *Ans.* $dy=-10x(a^2-x^2)^4\,dx$. **163.** $y=\sqrt{1+x^2}$. *Ans.* $dy=$ $=\frac{x\,dx}{\sqrt{1+x^2}}$. **164.** $y=\frac{1}{3}\tan^3 x+\tan x$. *Ans.* $dy=\sec^4 x\,dx$. **165.** $y=\frac{x\ln x}{1-x}+$ $+\ln(1-x)$. *Ans.* $dy=\frac{\ln x\,dx}{(1-x)^2}$.

Calculate the increments and differentials of the functions:

166. $y=2x^2-x$ when $x=1$, $\Delta x=0.01$. *Ans.* $\Delta y=0.0302$, $dy=0.03$. **167.** Given $y=x^3+2x$. Find Δy and dy when $x=-1$, $\Delta x=0.02$. *Ans.* $\Delta y=0.098808$, $dy=0.1$. **168.** Given $y=\sin x$. Find dy when $x=\frac{\pi}{3}$, $\Delta x=\frac{\pi}{18}$. *Ans.* $dy=$ $=\frac{\pi}{36}=0.00873$. **169.** Knowing that $\sin 60°=\frac{\sqrt{3}}{2}=0.866025$; $\cos 60°=\frac{1}{2}$, find the approximate values of $\sin 60°3'$ and $\sin 60°18'$. Compare the results with tabular data. *Ans.* $\sin 60°3'\approx 0.866461$; $\sin 60°18'\approx 0.868643$. **170.** Find the approximate value of $\tan 45°4'30''$. *Ans.* 1.00262. **171.** Knowing that $\log_{10}200=2\ 30103$ find the approximate value of $\log_{10}200.2$. *Ans.* 2.30146. Derivatives of different orders. **172.** $y=3x^3-2x^2+5x-1$. Find y''. *Ans.* $18x-4$. **173.** $y=\sqrt[5]{x^3}$. Find y'''. *Ans.* $\frac{42}{125}x^{-\frac{12}{5}}$. **174.** $y=x^6$. Find $y^{(6)}$. *Ans.* 6!. **175.** $y=\frac{C}{x^n}$. Find y''. *Ans.* $\frac{n(n+1)C}{x^{n+2}}$. **176.** $y=\sqrt{a^2-x^2}$. Find y''. *Ans.* $-\frac{a^2}{(a^2-x^2)\sqrt{a^2-x^2}}$. **177.** $y=2\sqrt{x}$. Find $y^{(4)}$. *Ans.* $-\frac{15}{8\sqrt{x^7}}$. **178.** $y=$ $=ax^2+bx+c$. Find y'''. *Ans.* 0. **179.** $f(x)=\ln(x+1)$. Find $f^{\mathrm{IV}}(x)$. *Ans.* $-\frac{6}{(x+1)^2}$. **180.** $y=\tan x$. Find y'''. *Ans.* $6\sec^4 x-4\sec^2 x$. **181.** $y=\ln\sin x$. Find. y'''. *Ans.* $2\cot x\csc^2 x$. **182.** $f(x)=\sqrt{\sec 2x}$. Find. $f''(x)$. *Ans.* $f''(x)=$ $=3[f(x)]^5-f(x)$. **183.** $y=\frac{x^3}{1-x}$. Find $f^{(4)}(x)$. *Ans.* $\frac{4!}{(1-x)^5}$. **184.** $p=$ $=(q^2+a^2)\arctan\frac{q}{a}$. Find $\frac{d^3p}{dq^3}$. *Ans.* $\frac{4a^3}{(a^2+g^2)^2}$. **185.** $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. Find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{y}{a^2}$. **186.** $y=\cos ax$. Find $y^{(n)}$. *Ans.* $a^n\cos\left(ax+n\frac{\pi}{2}\right)$. **187.** $y=a^x$. Find $y^{(n)}$. *Ans.* $(\ln a^n)\,a^x$. **188.** $y=\ln(1+x)$. Find $y^{(n)}$. *Ans.* $(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$. **189.** $y=\frac{1-x}{1+x}$. Find $y^{(n)}$. *Ans.* $2(-1)^n\frac{n!}{(1+x)^{n+1}}$. **190.** $y=e^x x$. Find $y^{(n)}$. *Ans.* $e^x(x+n)$. **191.** $y=x^{n-1}\ln x$. Find $y^{(n)}$, *Ans.* $\frac{(n-1)!}{x}$. **192.** $y=\sin^2 x$. Find $y^{(n)}$. *Ans.* $-2^{n-1}\cos\left(2x+\frac{\pi}{2}n\right)$. **193.** $y=x\sin x$. Find $y^{(n)}$. *Ans.* $x\sin\left(x+\frac{\pi}{2}n\right)-n\cos\left(x+\frac{\pi}{2}n\right)$. **194.** If $y=e^x\sin x$, prove that $y''-2y'+2y=0$. **195.** $y^2=4ax$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{4a^2}{y^3}$. **196.** $b^2x^2+a^2y^2=a^2b^2$. Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *Ans.* $-\frac{b^4}{a^2y^3}$; $-\frac{3b^6x}{a^4y^5}$. **197.** $x^2+y^2=r^2$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{r^2}{y^3}$. **198.** $y^2-2xy=0$. Find $\frac{d^3y}{dx^3}$. *Ans.* 0. **199.** $\varrho=\tan(\varphi+\varrho)$. Find $\frac{d^3\varrho}{d\varphi^3}$. *Ans.* $-\frac{2(5+8\varrho^2+3\varrho^4)}{\varrho^8}$. **200.** $\sec\varphi\cos\varrho=C$. Find $\frac{d^2\varrho}{d\varphi^2}$. *Ans.* $\frac{\tan^2\varrho-\tan^2\varphi}{\tan^3\varrho}$. **201.** $e^x+x=e^y+y$. Find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{(1-e^{x+y})(e^x-e^y)}{(e^y+1)^3}$. **202.** $y^3+x^3-3axy=0$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{2a^3xy}{(y^2-ax)^3}$. **203.** $x=a(t-\sin t)$, $y=a(1-\cos t)$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{1}{4a\sin^4\left(\frac{t}{2}\right)}$. **204.** $x=a\cos 2t$, $y=b\sin^2 t$. Show that $\frac{d^2y}{dx^2}=0$. **205.** $x=a\cos t$, $y=a\sin t$. Find $\frac{d^3y}{dx^3}$. *Ans.* $-\frac{3\cos t}{a^2\sin^5 t}$. **206.** Show that $\frac{d^{2n}}{dx^{2n}}(\sinh x)=\sinh x$; $\frac{d^{2n+1}}{dx^{2n+1}}(\sinh x)=\cosh x$.

Equations of a Tangent and Normal. Lengths of a Subtangent and a Subnormal

207. Write the equations of the tangent and normal to the curve $y=x^3-3x^2-x+5$ at the point $M(3, 2)$. *Ans.* The tangent is $8x-y-22=0$; the normal, $x+8y-19=0$. **208.** Find the equations of the tangent and normal of the length of the subtangent and subnormal of the circle $x^2+y^2=r^2$ at the point $M(x_1, y_1)$. *Ans.* The tangent is $xx_1+yy_1=r^2$; the normal is $x_1y-y_1x=0$; $s_T=\left|-\frac{y_1^2}{x_1}\right|$; $s_N=|-x_1|$.

209. Show that the subtangent of the parabola $y^2=4px$ at any point is divided into two by the vertex, and the subnormal is constant and equal to $2p$. Make a drawing.

210. Find the equation of a tangent at the point $M(x_1, y_1)$:

a) To the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Ans.* $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

b) To the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *Ans.* $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

211. Find the equations of the tangent and normal to the Witch of Agnesi $y=\frac{8a^3}{4a^2+x^2}$ at the point where $x=2a$. *Ans.* The tangent is $x+2y=4a$; the normal is $y=2x-3a$.

212. Show that the normal to the curve $3y=6x-5x^3$ drawn to the point $M\left(1, \frac{1}{3}\right)$ passes through the coordinate origin.

213. Show that the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point $M(a, b)$ is $\frac{x}{a}+\frac{y}{b}=2$.

214. Find the equation of that tangent to the parabola, $y^2=20x$, which forms an angle of 45° with the x-axis. *Ans.* $y=x+5$ [at the point (5, 10)].

215. Find the equations of those tangents to the circle $x^2+y^2=52$, which are parallel to the straight line $2x+3y=6$. *Ans.* $2x+3y\pm 26=0$.

216. Find the equations of those tangents to the hyperbola $4x^2-9y^2=36$, which are perpendicular to the straight line $2y+5x=10$. *Ans.* There are no such tangents.

217. Show that the segment (lying between the coordinate axes) of the tangent to the hyperbola $xy=m$ is divided into two by the point of tangency.

218. Prove that the segment (between the coordinate axes) of a tangent to the asteroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is of constant length.

219. At what angle α do the curves $y=a^x$ and $y=b^x$ intersect? *Ans.* $\tan\alpha=\frac{\ln a-\ln b}{1+\ln a\cdot\ln b}$.

220. Find the lengths of the subtangent, subnormal, tangent and normal of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ at the point at which $\theta=\frac{\pi}{2}$. *Ans.* $s_T=a$; $s_N=a$; $T=a\sqrt{2}$; $N=a\sqrt{2}$.

221. Find the quantities s_T, s_N, T and N for the hypocycloid $x=4a\cos^3 t$, $y=4a\sin^3 t$. *Ans.* $s_T=-4a\sin^2 t\cos t$; $s_N=-4a\frac{\sin^4 t}{\cos t}$; $T=4a\sin^2 t$; $N=4a\sin^2 t\tan t$.

Miscellaneous Problems

Find the derivatives of the following functions: 222. $y=\frac{\sin x}{2\cos^2 x}-\frac{1}{2}\times\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$. *Ans.* $y'=\frac{1}{\cos^3 x}$. 223. $y=\arcsin\frac{1}{x}$. *Ans.* $y'=\frac{1}{|x|\sqrt{x^2-1}}$.

224. $y=\arcsin(\sin x)$. *Ans.* $y'=\frac{\cos x}{|\cos x|}$. 225. $y=\frac{2}{\sqrt{a^2-b^2}}\times\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$ $(a>0,\ b>0)$. *Ans.* $y'=\frac{1}{a+b\cos x}$. 226. $y=|x|$. *Ans.* $y'=\frac{x}{|x|}$. 227. $y=\arcsin\sqrt{1-x^2}$. *Ans.* $y'=-\frac{x}{|x|}\frac{1}{\sqrt{1-x^2}}$.

228. From the formulas for the volume and surface of a sphere,

$$v=\frac{4}{3}\pi r^3 \text{ and } s=4\pi r^2$$

it follows that $\frac{dv}{dr}=s$. Explain the geometric significance of this result. Find a similar relationship between the area of a circle and the length of the circumference.

229. In a triangle ABC, the side a is expressed in terms of the other two sides b, c and the angle A between them by the formula

$$a=\sqrt{b^2+c^2-2bc\cos A}.$$

For b and c constant, side a is a function of the angle A. Show that $\frac{da}{dA}=h_a$, where h_a is the altitude of the triangle corresponding to the base a. Interpret this result geometrically.

230. Using the differential concept, determine the origin of the approximate formulas

$$\sqrt{a^2+b^2}\approx a+\frac{b}{2a}, \quad \sqrt[3]{a^3+b}\approx a+\frac{b}{3a^2}$$

where $|b|$ is a number small compared with a.

231. The period of oscillation of a pendulum is computed by the formula

$$T=\pi\sqrt{\frac{l}{g}}.$$

In calculating the period T, how will the error be affected by an error of 1% in the measurement of: 1) the length of the pendulum l; 2) the acceleration of gravity g? *Ans.* 1) $\approx 1/2\%$; 2) $\approx 1/2\%$.

232. The tractrix has the property that for any point of it, the segment of the tangent T remains constant in length. Prove this on the basis of: 1) the equation of the tractrix in the form

$$x=\sqrt{a^2-y^2}+\frac{a}{2}\ln\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}} \quad (a>0);$$

2) the parametric equations of the curve

$$x=a\left(\ln\tan\frac{t}{2}+\cos t\right),$$

$$y=a\sin t.$$

233. Prove that the function $y=C_1e^x+C_2e^{-2x}$ satisfies the equation $y''+3y'+2y=0$ (here C_1 and C_2 are constants).

234. Putting $y=e^x\sin x$, $z=e^x\cos x$ prove the equalities $y''=2z$, $z''=-2y$.

235. Prove that the function $y=\sin(m\arcsin x)$ satisfies the equation $(1-x^2)y''-xy'+m^2y=0$.

236. Prove that if $(a+bx)e^{\frac{y}{x}}=x$, then $x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$.