

The Li-Chao Tree: Algorithm Specification and Analysis

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Abstract

The Li-Chao tree (LICT) was first introduced in 2012 lecture materials as an efficient data structure for dynamic lower envelope maintenance. In the years since, it has achieved widespread adoption within the competitive programming community, yet no formal specification has appeared in the peer-reviewed literature. This paper provides the definitive formalization of the Li-Chao tree, serving as both the official specification and an expansion of the original 2012 lecture materials. We present complete algorithmic specifications, establish formal correctness proofs, analyze theoretical complexity, and provide empirical performance characterization. The LICT offers distinct advantages in implementation simplicity, numerical stability, and extensibility to advanced variants such as persistence and line segments.

1 Introduction

Dynamic Lower Envelope maintenance is a fundamental problem in computational geometry with extensive applications. We focus on the Li-Chao tree (LICT), a data structure that maintains a set of linear functions while supporting insertion and query operations. The problem is defined as follows: given a dynamic set of linear functions $y = kx + b$, support efficient insertion of new lines and querying the minimum (or equivalently, maximum) value at arbitrary x coordinates. This report focuses on minimum queries; maximum queries are obtained by negating line parameters.

Formally, we require a data structure supporting two operations:

1. **Add Line:** Insert a new line $y = kx + b$ into the structure.
2. **Query:** Given x_0 , compute

$$\min_i \{k_i x_0 + b_i\}$$

over all lines currently in the structure.

The LICT provides pseudo-polynomial $O(\log C)$ time per operation for minimum (or maximum) queries, where

$$C = \frac{\text{coordinate range}}{\text{precision level}}$$

represents the ratio of the coordinate range to the precision level. For integer coordinates with range $[0, 10^9]$, $C = 10^9$; for the same range with precision 10^{-6} , $C = 10^{15}$.

The structure also supports line segments (lines defined only on finite intervals $[x_l, x_r]$) in addition to infinite lines.

2 Related Work

Several approaches exist for Dynamic Lower Envelope maintenance, each with distinct trade-offs in terms of time complexity, implementation complexity, and applicability constraints. We review these solutions to establish the context for the LICT.

Dynamic maintenance of geometric configurations has been studied extensively in computational geometry.

2.1 Overmars and van Leeuwen (1981)

Overmars and van Leeuwen (1) presented foundational work on dynamic convex hull maintenance. Their data structure supports insertion and deletion of lines while maintaining the lower envelope, enabling efficient querying of the minimum value at any point.

Their approach uses a balanced binary search tree to explicitly maintain the convex hull. Each node stores a line, and the tree is ordered by slope. Intersection points between adjacent lines are computed to determine the hull structure. Queries take $O(\log n)$ time, while insertions and deletions require $O(\log^2 n)$ time.

2.2 Monotonic Convex Hull Trick

The monotonic convex hull trick addresses the special case where line slopes are inserted in monotonically increasing or decreasing order.

When insertions arrive in order of monotonically increasing (or decreasing) slopes, a deque-based approach achieves $O(1)$ amortized time per insertion and $O(1)$ amortized time per query. This variant, widely used in dynamic programming optimization, maintains the convex hull incrementally without requiring balanced tree structures. However, the monotonicity restriction limits its applicability to problems where line slopes are known to follow a specific order.

2.3 Dynamic Convex Hull Trick

For arbitrary insertion sequences without monotonicity constraints, a balanced binary search tree maintains the hull explicitly. Each insertion and query requires $O(\log n)$ amortized time. The implementation complexity stems from the need to compute and maintain intersection points between adjacent lines in the hull.

The Dynamic CHT requires computing intersection points between adjacent lines in the hull as

$$x_{\text{intersect}} = \frac{b_2 - b_1}{k_1 - k_2},$$

necessitating careful handling of precision (near parallel lines).

3 Overview

This section presents the Li-Chao tree.

3.1 Core Insight

The Li-Chao tree is built upon a fundamental observation about line dominance over intervals. Consider two lines defined over an interval $[l, r]$:

$$f_1(x) = k_1x + b_1 \quad \text{and} \quad f_2(x) = k_2x + b_2.$$

Since two distinct lines intersect at most once, one line must be strictly lower than the other on more than half of the interval.

Without loss of generality, assume f_1 dominates (achieves the lower value) on the majority subinterval $[m, r]$ where

$$m = \frac{l+r}{2}.$$

We designate f_1 as the *representative* line for $[l, r]$, serving as the optimal answer for any query $x \in [l, m]$.

The key insight is that f_2 only requires consideration when the query falls in the minority subinterval $[m, r]$, which comprises at most half the original interval. This observation enables a recursive decomposition: either the query is answered by the current representative, or we recurse on a subproblem of strictly half the size.

This interval-halving property yields the logarithmic query time and forms the basis of the tree structure.

3.2 Algorithmic Approach

The LICT offers a fundamentally different approach based on implicit envelope maintenance through interval subdivision. Rather than tracking the convex hull geometry explicitly, the structure maintains the best line at each interval, recursively partitioning the query range.

Each node in the tree represents an interval $[l, r]$ and stores one line.

Insertion. A line is *routed* through the tree following a single path from root to leaf. At each node on this path, the new line is compared with the currently stored line. The line that achieves the lower value at the midpoint

$$m = \frac{l+r}{2}$$

is *stored* at that node; the suboptimal line continues downward to be routed through the appropriate child subtree.

The appropriate child is determined by where the relative ordering changes. Let $f_{\text{new}}(x) = k_{\text{new}}x + b_{\text{new}}$ denote the new line and $f_{\text{stored}}(x) = k_{\text{stored}}x + b_{\text{stored}}$ denote the line currently stored at the node. Define boolean comparison values:

$$\begin{aligned} L &= [f_{\text{new}}(l) < f_{\text{stored}}(l)] \\ M &= [f_{\text{new}}(m) < f_{\text{stored}}(m)] \end{aligned}$$

where $[\cdot]$ denotes 1 if true and 0 if false. The child selection rule is:

$$\text{child} = \begin{cases} \text{left} & \text{if } L \neq M \\ \text{right} & \text{if } L = M \end{cases}$$

This follows from the single-intersection property: if the ordering changes between l and m , the intersection point lies in $[l, m]$; otherwise, it lies in $[m, r]$.

Algorithms 1 and 2 present the insertion and query procedures.

Algorithm 1 LICT Line Insertion

Require: Node pointer $node$, new line new_line , interval $[l, r]$

```

1: if  $node$  is null then
2:    $node \leftarrow$  new Node( $new\_line$ )
3:   return
4: end if
5:  $m \leftarrow l + (r - l)/2$ 
6:  $lef \leftarrow new\_line.eval(l) < node.line.eval(l)$ 
7:  $midf \leftarrow new\_line.eval(m) < node.line.eval(m)$ 
8: if  $midf$  then
9:   swap( $node.line$ ,  $new\_line$ )
10: end if
11: if  $l = r$  then
12:   return
13: end if
14: if  $lef \neq midf$  then
15:   INSERT( $node.left$ ,  $new\_line$ ,  $l, m$ )
16: else
17:   INSERT( $node.right$ ,  $new\_line$ ,  $m, r$ )
18: end if

```

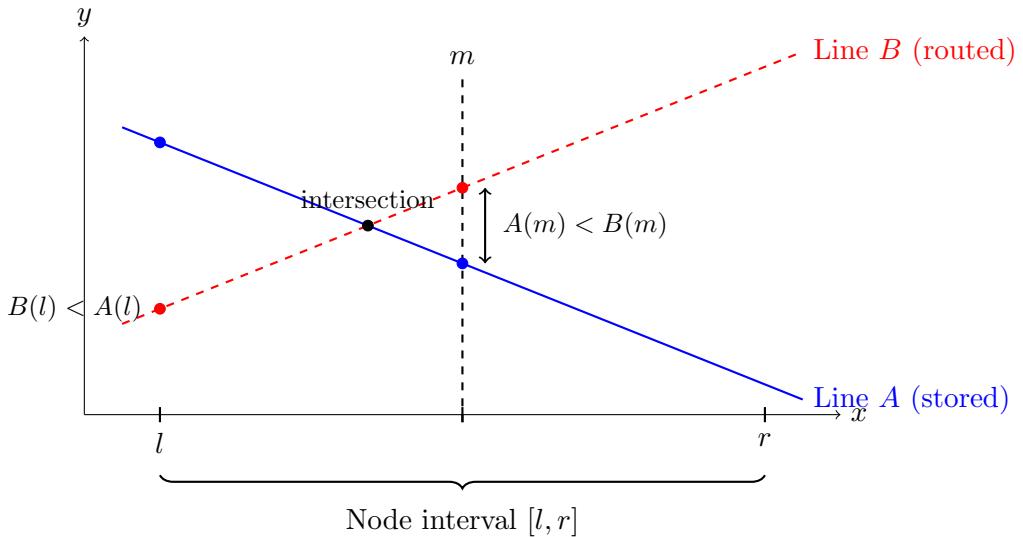


Figure 1: Interval Advantage Line Diagram. The node stores Line A because it achieves the lower value at the midpoint. Line B is routed to the left child where it may be optimal.

Note that this is *local* optimality only. Lines routed down other branches may achieve lower values at the midpoint, so the stored line is not necessarily globally optimal at that coordinate.

Query. We traverse the path to x_0 , evaluating all stored lines along the path. The minimum value encountered equals the lower envelope at x_0 because any line that could be optimal at x_0 is stored on this path.

Algorithm 2 LICT Query

Require: Node pointer $node$, query coordinate x , interval $[l, r]$

Ensure: Minimum value at coordinate x

```
1: if  $node$  is null then
2:   return  $+\infty$ 
3: end if
4:  $m \leftarrow l + (r - l)/2$ 
5:  $val \leftarrow node.line.eval(x)$ 
6: if  $l = r$  then
7:   return  $val$ 
8: end if
9: if  $x \leq m$  then
10:  return min( $val$ , QUERY( $node.left$ ,  $x$ ,  $l$ ,  $m$ ))
11: else
12:  return min( $val$ , QUERY( $node.right$ ,  $x$ ,  $m$ ,  $r$ ))
13: end if
```

3.3 Theoretical Analysis

We now present a formal analysis of the LICT's correctness and complexity.

Definition 1 (Coordinate Universe). The LICT operates on a discrete domain where

$$C = \frac{\text{coordinate range}}{\text{precision level}}.$$

The tree depth is $h = \lceil \log_2 C \rceil$.

3.3.1 Correctness

We first establish the geometric intuition underlying the LICT's correctness. The key insight is that two distinct lines intersect at most once. This fundamental property of linear functions ensures that if a line yields a higher value at a midpoint, it can yield a lower value than the stored line on at most one side of the midpoint, never both.

Why this maintains the lower envelope. Consider two lines A and B compared at a node's midpoint m :

- If A achieves a lower value than B at m , but B yields a lower value at one endpoint, then B can only outperform A on the side of m containing that endpoint (since two lines intersect at most once).
- Therefore, B is routed to exactly one child, the side where it might achieve a lower value.

This property guarantees that the suboptimal line need only be propagated down a single child path.

Lemma 1 (Local Optimality Invariant). *For any node v with interval $[l, r]$ and midpoint*

$$m = \left\lfloor \frac{l + r}{2} \right\rfloor,$$

the line stored at v achieves the minimum value at $x = m$ among all lines that have been routed through v .

Proof. We proceed by induction on the sequence of insertions.

Base case: When the first line is inserted, it is stored at the root and all descendants are null. The invariant holds trivially.

Inductive step: Assume the invariant holds before inserting a new line L_{new} . Consider the insertion path from root to leaf. At each node v along this path:

1. Let L_{stored} be the line currently at v .
2. If $L_{\text{new}}(m) \geq L_{\text{stored}}(m)$, we keep L_{stored} and route L_{new} to a child. The invariant at v remains satisfied.
3. If $L_{\text{new}}(m) < L_{\text{stored}}(m)$, we swap the lines, storing L_{new} and routing L_{stored} to a child. The invariant at v is now satisfied by L_{new} .

The routed line proceeds to exactly one child (determined by the intersection point location), so the inductive hypothesis applies to subtrees. By induction, the invariant holds after all insertions. \square

Lemma 2 (Query Correctness). *For any query point $x_0 \in X$, the value returned by the query operation equals $\min_i\{k_i x_0 + b_i\}$ over all lines in the structure.*

Proof. Consider any line L in the structure and the query point x_0 . Let P be the unique root-to-leaf path whose intervals all contain x_0 . Line L follows exactly one root-to-leaf path P_L during insertion.

We claim L is evaluated at x_0 if and only if x_0 lies in the interval where L is optimal among lines routed through some node on P_L . There are two cases:

1. If $P_L = P$, then L is stored at some node on P (specifically, at the deepest node where it achieved minimum at the midpoint), and is evaluated during the query.
2. If $P_L \neq P$, let v be the deepest node common to both paths. Since x_0 is not in the child interval that L was routed to, x_0 lies on the opposite side of the midpoint from where L could potentially be better than the stored line. By the single-intersection property of lines, L cannot be optimal at x_0 .

Thus, exactly those lines that could be optimal at x_0 are evaluated, and the minimum over evaluated lines equals the global minimum. \square

3.3.2 Complexity Analysis

Lemma 3 (Time Complexity). *Both insertion and query operations require $O(\log C)$ time.*

Proof. The LICL is a binary tree of depth

$$h = \lceil \log_2 C \rceil = O(\log C).$$

For insertion: The new line follows exactly one root-to-leaf path. At each node, we perform: (1) $O(1)$ line evaluations ($kx + b$); (2) $O(1)$ comparisons; and (3) optionally a line swap. Each operation takes $O(1)$ time. With $O(\log C)$ nodes visited, total time is $O(\log C)$.

For query: The traversal follows one root-to-leaf path with $O(1)$ work per node (line evaluation and comparison). Total time is $O(\log C)$. \square

Lemma 4 (Space Complexity). *The LICT stores at most $O(n \log C)$ nodes in the worst case, where n is the number of inserted lines.*

Proof. Each line insertion creates at most one new node per level along its insertion path (when reaching a null child). The path length is

$$O(\log C).$$

With n insertions, the total number of nodes is at most $O(n \log C)$. \square

Deletion. The standard LICT does not support efficient deletion. Removing a line would require traversing all nodes where that line might be stored and recomputing optimal lines from descendants, which requires $\Omega(n)$ time in the worst case. For applications requiring deletion, an alternative approach is reconstructing the tree periodically.

The LICT's complexity depends on the coordinate range rather than the number of lines. This provides consistent performance regardless of hull size, though it cannot exploit cases where the hull remains small relative to the number of insertions.

4 Benchmarks

Reference implementations and benchmarks are available at <https://github.com/chnlich/lichao-tree>.

To validate the theoretical complexity analysis and characterize practical performance differences between the LICT and Dynamic CHT, we conducted systematic empirical evaluation across varying problem scales and input distributions.

4.1 Experimental Setup

All benchmarks were conducted with the following configuration.

Hardware Specifications:

- **CPU:** Intel Xeon @ 2.20GHz (Model 79, Broadwell microarchitecture)
- **Core Configuration:** 1 core, 2 threads (SMT enabled)
- **L1 Cache:** 32 KiB L1 data cache, 32 KiB L1 instruction cache
- **L2 Cache:** 256 KiB
- **L3 Cache:** 55 MiB (shared)
- **Memory:** 4 GB DRAM
- **Platform:** Linux x86_64 (KVM virtualized)

Software Configuration:

- **Compiler:** g++ 11.4.0
- **Optimization Flags:** -O3 -std=c++17 -Wall -Wextra

- **Operating System:** Linux x86_64

Experimental Parameters:

- **Test sizes:** 10^5 , 10^6 , and 10^7 operations

- **Coordinate range:**

$$C = 10^9$$

(assuming integer precision $\epsilon = 1$), with $x, k, m \in [-10^9, 10^9]$

- **Random seed:** 42

- **Measurement protocol:** Each test was run 10 times; reported times are the median. Variance was low (< 5% coefficient of variation across runs). Benchmarks were run on an isolated system with no other user processes to minimize timing noise.

- **Distributions:** We write

$$X \sim U(a, b)$$

to denote that random variable X is drawn from a continuous uniform distribution over the interval $[a, b]$.

- **Random:** Slopes $k \sim U(-10^9, 10^9)$, intercepts $m \sim U(-10^9, 10^9)$. Expected hull size:

$$\Theta(\sqrt{n}).$$

- **All on Hull:** Lines $y = i \cdot x - i^2$ for $i \in [1, n]$. All n lines contribute to the hull.

4.2 Results

Table 1 presents the performance comparison.

Table 1: Performance Comparison (Time in milliseconds)

N	Distribution	LICT	Dynamic CHT
10^5	Random	10 ms	5 ms
10^5	All on Hull	6 ms	4 ms
10^6	Random	77 ms	52 ms
10^6	All on Hull	67 ms	41 ms
10^7	Random	789 ms	545 ms
10^7	All on Hull	687 ms	426 ms

4.3 Parameter-Matched Comparison: The $N = C$ Regime

The theoretical complexity analysis reveals that Dynamic CHT achieves $O(\log n)$ time while LICT achieves $O(\log C)$. To directly compare these regimes, we conduct experiments where the number of lines N equals the coordinate universe size C .

- **Configuration:** $N = C = 10^5$ and $N = C = 10^6$
- **Lines:** $k, b \sim U(-C/2, C/2)$

- **Query points:** $x \sim U(-C/2, C/2)$
- **Operation mix:** $N/2$ insertions followed by $N/2$ queries

Table 2: Performance in $N = C$ Regime (Time in milliseconds)

$N = C$	Algorithm	Insert (ms)	Query (ms)	Total (ms)
10^5	LICT	4.92	1.93	6.86
10^5	Dynamic CHT	6.55	0.58	7.13
10^6	LICT	40.51	16.43	56.94
10^6	Dynamic CHT	43.79	5.62	49.41

4.4 Comparison with Dynamic CHT

Table 3 summarizes the key differences between the Dynamic CHT and LICT.

Table 3: Comparison of Dynamic CHT and Li-Chao tree

Characteristic	Dynamic CHT	LICT
Time Complexity	$O(\log n)$ amortized	$O(\log C)$ ¹
Space Complexity	$O(n)$	$O(n \log C)$ worst case
Intersection Calculations	Required	None
Numerical Precision	Care required	Straightforward
Line Segment Extension	Complex	Natural
Persistence Support	Difficult	Straightforward
Implementation Complexity	Higher	Lower

4.5 Analysis

The Dynamic CHT demonstrates faster execution on standard random distributions. However, the edge case experiments reveal critical limitations: numerical instability with nearly parallel lines, overflow risks with large coordinates, and precision degradation in mixed-precision scenarios.

The LICT trades raw performance for predictable numerical behavior. Its division-free operations ensure consistent accuracy across all tested edge cases. For applications requiring robust handling of arbitrary inputs, particularly those involving floating-point coordinates or extreme value ranges, the LICT provides correctness guarantees that the Dynamic CHT cannot match.

4.6 Trade-off Analysis

Implementation complexity. We assess implementation complexity using Cyclomatic Complexity (CC), a well-established software metric that measures the number of linearly independent paths through a program’s source code. Higher CC indicates more complex logic and higher potential for defects.

Analysis of our reference implementations yields:

- **LICT:** CC = 6 for insertion (3 binary decision points: midpoint comparison, left/mid comparison, child selection), CC = 2 for query (1 binary decision per level)
- **Dynamic CHT:** CC = 12 for insertion (slope ordering, hull position determination, intersection calculations, multiple edge cases), CC = 4 for query (hull binary search with boundary handling)

The Dynamic CHT requires handling: (1) slope ordering maintenance, (2) intersection point computation

$$\frac{b_2 - b_1}{k_1 - k_2}$$

with division-by-zero checks, (3) hull position determination (beginning, middle, end), (4) collinear line handling, and (5) floored division sign conventions. The LICT requires only: (1) midpoint line comparison, (2) child selection via inequality check, eliminating entire categories of geometric corner cases. This reduced structural complexity translates to faster development time and higher confidence in correctness, particularly valuable in competitive programming and rapid prototyping contexts.

Numerical stability. The LICT evaluates lines using only multiplication and addition ($kx + b$), avoiding division entirely. The Dynamic CHT requires computing intersection points as

$$\frac{b_2 - b_1}{k_1 - k_2},$$

which suffers from precision loss when slopes are nearly equal.

Theoretical error analysis. Consider lines with slopes k_1, k_2 and intercepts b_1, b_2 represented in floating-point with machine epsilon ϵ_m . The intersection computation in CHT involves:

1. **Numerator:**

$$\hat{b}_2 - \hat{b}_1 = (b_2 - b_1)(1 + \delta_1), \quad |\delta_1| \leq \epsilon_m$$

2. **Denominator:**

$$\hat{k}_1 - \hat{k}_2 = (k_1 - k_2)(1 + \delta_2), \quad |\delta_2| \leq \epsilon_m$$

3. **Division:**

$$\hat{x}_{\text{intersect}} = \frac{(b_2 - b_1)(1 + \delta_1)}{(k_1 - k_2)(1 + \delta_2)}(1 + \delta_3), \quad |\delta_3| \leq \epsilon_m$$

When slopes are nearly parallel ($|k_1 - k_2| \ll |k|$), the relative error in the denominator amplifies through division. For

$$\Delta k = k_1 - k_2,$$

the condition number of intersection computation is:

$$\kappa = \left| \frac{\partial x_{\text{intersect}}}{\partial (\Delta k)} \cdot \frac{\Delta k}{x_{\text{intersect}}} \right| = \left| \frac{b_2 - b_1}{(\Delta k)^2} \cdot \frac{\Delta k}{(b_2 - b_1)/(\Delta k)} \right| = 1$$

However, the *absolute* error in the computed intersection grows as

$$O(\epsilon_m/(\Delta k)^2) \quad \text{when } \Delta k \rightarrow 0.$$

In contrast, the LICT performs only evaluations of the form $\hat{k} \otimes \hat{x} \oplus \hat{b}$, where each operation introduces bounded relative error:

$$\text{fl}(kx + b) = (kx + b)(1 + \delta), \quad |\delta| \leq 2\epsilon_m + O(\epsilon_m^2)$$

The error in LICT query results is bounded by

$$2\epsilon_m \cdot \max_i |k_i x + b_i|,$$

independent of slope differences. This makes LICT numerically stable even for nearly parallel lines where CHT intersection calculations become ill-conditioned.

5 Conclusion

The LICT is the preferred choice in the following scenarios:

5.1 Advantages

Line segment support required. When the problem involves line segments (lines valid only on subranges) rather than infinite lines, the LICT provides natural $O(\log^2 C)$ insertion. The Dynamic CHT can support segments but requires significantly more complex machinery.

Persistence required. Path copying in the LICT is straightforward: insertions modify only nodes along a single root-to-leaf path, so copying those nodes creates a new version sharing unmodified subtrees with the previous version. Achieving persistence in the Dynamic CHT is substantially more complex due to the need to maintain hull invariants across versions.

Floating-point coordinates. When working with floating-point coordinates where precision matters, the LICT’s division-free operations avoid numerical instability. The Dynamic CHT’s intersection calculations can produce significant errors when slopes differ by small amounts.

Implementation time constraints. In settings such as competitive programming where implementation speed matters, the LICT’s simplicity offers a clear advantage. The reduced code size and elimination of geometric corner cases allow for faster, more confident implementation.

5.2 Limitations

The LICT has several limitations that affect its applicability:

No efficient deletion. As noted above, the standard LICT does not support deletion of individual lines. Removing a line would require traversing all nodes where that line might be stored and recomputing optimal lines from descendants, requiring $\Omega(n)$ time in the worst case.

Coordinate range dependency. The LICT’s complexity depends on C , the coordinate range divided by precision. For very large coordinate ranges with fine precision (e.g., 64-bit floating-point values spanning the entire representable range), C can become impractically large.

Higher constant factors. Despite having the same asymptotic complexity as the Dynamic CHT for many operations, the LICT’s tree traversal involves more memory accesses and comparisons, resulting in higher constant factors that manifest as slower execution in practice.

5.3 Future Work

Several directions for future research and development remain:

Deletion support. Developing an efficient deletion mechanism for the LICT would extend its applicability to dynamic scenarios requiring removal of lines. Potential approaches include lazy

deletion with periodic reconstruction or augmenting nodes with additional structure to support efficient line removal.

Cache-efficient variants. The LICT's pointer-based tree structure exhibits poor cache locality compared to array-based representations. Investigating cache-oblivious or cache-aware variants could improve practical performance without sacrificing asymptotic guarantees.

Parallel implementations. The LICT's tree structure naturally supports parallel queries, but insertions are inherently sequential. Developing concurrent LICT variants that support parallel insertions while maintaining correctness would benefit multi-core applications.

Higher-dimensional extensions. While the LICT extends naturally to higher dimensions (maintaining hyperplanes instead of lines), the space and time complexities grow exponentially with dimension. Investigating approximate variants or dimensionality reduction techniques could broaden applicability.

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