CSDS 440: Assignment 7

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Problem 28

Rewrite the $Ax \geq b$ to $-(Ax - b) \leq 0$ of $\min_x c^T x$. The largrangian for primal will be $\ell(x, u) = c^T x - u^T(Ax - b)$. Then

$$
\begin{aligned}
\nabla \ell(x, u) &= \nabla(c^T x - u^T(Ax - b)) = 0 \\
&= c^T - u^T A = 0 \\
\Longrightarrow & c = A^T u
\end{aligned}
$$

Now for the dual $\max_{u:u \geq 0} D(u) - b^T A$. $\nabla \ell(x, u) \geq 0$ suggests the dual is stilling moving towards the maximum direction of $b^T A$. So we have obtained the asked $\max b^T A \quad s.t. \ A^T u \leq c, u \geq 0$.

Problem 29

We try to show that the new $K = aK_1 + bK_2$ is a valid kernel as it compliants to the Mercer's conditions.

$$\begin{aligned} K(x,y) &= aK_1(x,y) + bK_2(x,y) \\ &= aK_1(y,x) + bK_2(y,x) \quad \text{as } K_1 \text{ and } K_2 \text{ are valid kernels.} \\ &= K(y,x) \end{aligned}$$

So K is symmetry. Now suppose $\forall v \neq 0$, we have:

$$\begin{aligned} v^T \cdot Kv &= v^T(aK_1 + bK_2)v \\ &= a\underbrace{(v^T K_1 v)}_{\geq 0} + b\underbrace{v(v^T K_2 v)}_{\geq 0} \\ &\geq 0 \end{aligned}$$

So K is also PSD. We may say K is a valid kernel as both of the Mercer's conditions are met.

Problem 30

For K_1 to be a valid kernel, we must have $K_1(x,y) = \varphi_1(x)\varphi_1(y)$ where ϕ is a non-linear mapping of its input. Due to the same principle, we also have $K_2(x,y) = \varphi_2(x)\varphi_2(y)$. Which gives us:

$$
\begin{aligned}
K(x,y) &= aK_1(x,y)K_2(x,y) \\
&= a(\varphi_1(x)\varphi_1(y) \cdot \varphi_2(x)\varphi_2(y) \\
&= a[(\sum_{i=1}^{n} \varphi_{1i}(x)\varphi_{1i}(y)) \cdot (\sum_{j=1}^{n} \varphi_{2j}(x)\varphi_{2j}(y))] \\
&= a\sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_{1i}(x)\varphi_{1i}(y)\varphi_{2j}(x)\varphi_{2j}(y) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (\sqrt{a}\ \varphi_{1i}(x)\varphi_{2j}(x)) \cdot (\sqrt{a}\ \varphi_{1i}(y)\varphi_{2j}(y)) \\
&= \varphi(x)\varphi(y) \begin{cases} \varphi(x) = \sqrt{a}\ \varphi_{1i}(x)\varphi_{2j}(x) \\ \varphi(y) = \sqrt{a}\ \varphi_{1i}(y)\varphi_{2j}(y) \end{cases}
\end{aligned}
$$

As the above φ is created upon multiplying a constant $\sqrt{a}$ with non-linear mapping φ_1 and φ_2, it is also a non-linear mapping; and K is therefore a valid kernel.

Problem 31

$$
\begin{aligned}
K(x,y) &= (x \cdot y + c)^3 \\
&= (xy)^3 + 3(xy)^2c + 3(xy)c^2 + c^3 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_iy_i)(x_jy_j)(x_ky_k) + 3c\sum_{i=1}^{n}\sum_{j=1}^{n}(x_iy_i)(x_jy_j) + 3c^2\sum_{i=1}^{n}\sum_{j=1}^{n}(x_iy_i)(x_jy_j) + c^3 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_ix_jx_k)(y_iy_jy_k) + \sum_{i=1}^{n}\sum_{j=1}^{n}(x_ix_j \cdot \sqrt{3c})(y_iy_j \cdot \sqrt{3c}) \\
&\quad + \sum_{i=1}^{n}(x_i \cdot c\sqrt{3})(y_i \cdot c\sqrt{3}) + \sqrt{c^3}\sqrt{c^3} \\
&= \varphi(x)\varphi(y)
\end{aligned}
$$

Where

$$\varphi(a) = [a_1a_1a_1, \ldots, a_ia_ja_k, \ldots, a_na_na_n,$$
$$a_1a_1\sqrt{3c}, \ldots, \sqrt{2}a_ia_j\sqrt{3c}, \ldots, a_na_n\sqrt{3c},$$
$$a_1c\sqrt{3}, \ldots, a_ic\sqrt{3}, \ldots, a_nc\sqrt{3},$$
$$\sqrt{c^3}]$$

which is a non-linear mapping of input a. So we have showed $K(x, y) = \varphi(x)\varphi(y)$, we will continue to show that K is also symmetric and PSD in *Question 32*.

Problem 32

First it is clear that $K(x, y) = K(y, x)$ as $(xy + c)^3 = (yx + c)^3$, so K is symmetric.

Now to show that K is also PSD, we need that for $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, K is valid for all $\{x^{(1)}, \ldots, x^{(m)}\}$. With such information on x we may form the equality this way:

$$
\begin{aligned}
v^T K v &= \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j K(x^{(i)} x^{(j)}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j (x^{(i)} x^{(j)} + c)^3 \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j (y^{(i)} y^{(j)})^3 \quad \text{absorb } +c \text{ to } x \text{ and make } y \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j (\sum_{a=1}^{n} y_a^{(i)} y_a^{(j)}) \cdot (\sum_{b=1}^{n} y_b^{(i)} y_b^{(j)}) \cdot (\sum_{c=1}^{n} y_c^{(i)} y_c^{(j)}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} (y_a^{(i)} y_b^{(i)} y_c^{(i)})(y_a^{(j)} y_b^{(j)} y_c^{(j)}) \\
&= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{i=1}^{m} v_i (y_a^{(i)} y_b^{(i)} y_c^{(i)}) \quad \sum_{j=1}^{m} v_j (y_a^{(j)} y_b^{(j)} y_c^{(j)}) \\
&= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \big(\sum_{i=1}^{m} v_i (y_a^{(i)} y_b^{(i)} y_c^{(i)}) \big)^2
\end{aligned}
$$

Since whole equation is inside a square, K is PSD. And combined with the above finding of K being symmetric, K is a valid kernel.