CSDS 440: Assignment 4

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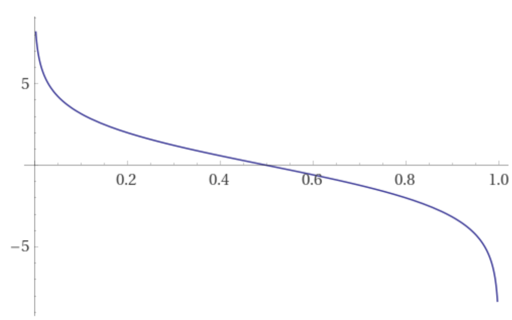
Problem 15

Say we have X being a Bernoulli r. v. Let $P(X = 1) = p$ and $P(X = 0) = 1 - p$ we know that its entropy would be:

$$H(X) = H_b(p) = -p\log_2(p) - (1-p)\log_2(1-p)$$
$$\Rightarrow H_b'(p) = \log_2(1-p) - \log_2(p)$$

Now plot the function.



With the first derivative being a decreasing function, we know the function is concave.

Problem 16

Pick two points, x_1 and x_2 in R^n where $Ax_1, Ax_2 \geq b$. W.T.S. for any point x between the line of x_1 and x_2, we have $Ax \geq b$.

$$\begin{aligned}
Ax &= A(\lambda x_1 + (1-\lambda)x_2) \\
&= \lambda A x_1 + (1-\lambda)Ax_2 \\
&\geq \lambda b + (1-\lambda)b = \lambda b + b\lambda b = b \\
\implies Ax &\geq b
\end{aligned}$$

As x in above case can be any point from of $\{x \mid Ax \geq b\}$, we have proven the set is convex.

Problem 17

Proof. To prove by contradiction. Assume we have a local minimum x in a convex function f but there is another global minimum x', where $f(x') < f(x)$.

Since f is convex, by Jensen's inequality we must have:

$$\begin{aligned}
f(\lambda x + (1-\lambda)x') &\leq \lambda f(x) + (1-\lambda)f(x') \\
&< \lambda f(x) + (1-\lambda)f(x) \\
&< f(x)
\end{aligned}$$

Let $\lambda = 1$, we will have the below contradiction:

$$f(x) < f(x)$$

Thus, by contradiction, the local minimum of a convex function is always the global minimum.

□

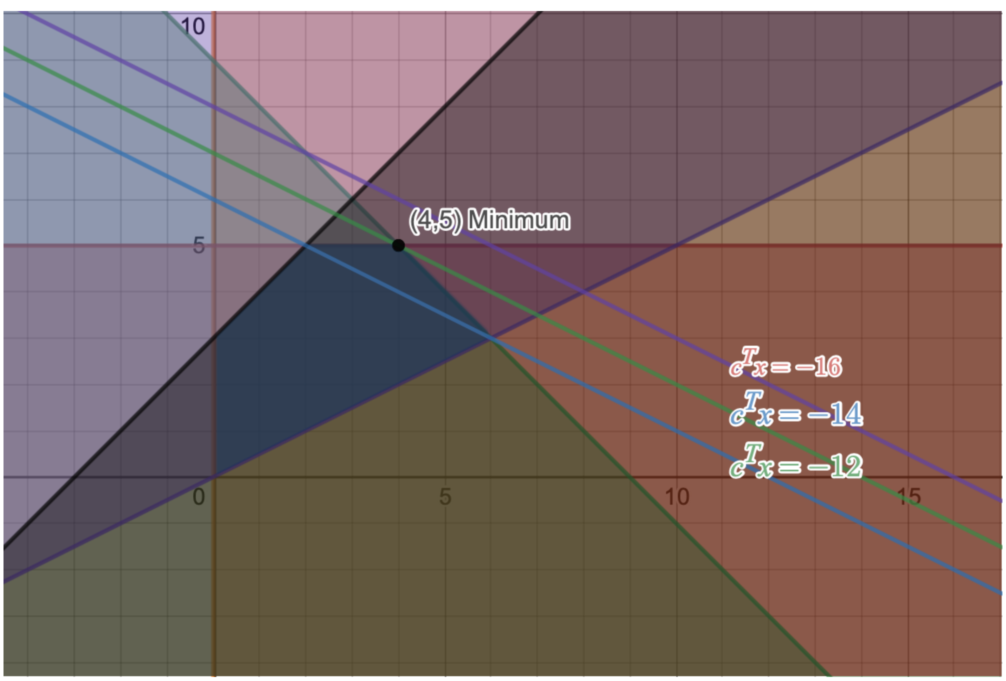
Problem 18

For the ease of description, we denotes elements in A as $\begin{pmatrix} x & y \\ x & y \\ \dots & \end{pmatrix}$. We have the following for $Ax \geq b$:

$$\begin{aligned} 0x - y &\geq -5 \\ -x - y &\geq -9 \\ -x + 2y &\geq 0 \\ x - y &\geq -1 \end{aligned}$$

Similarly, we also have $c^T x = -x - 2y$.

(a) (b)



The area shaded by dark blue is the feasible region, the three $c^T x$ contours are labelled accordingly.

Yes. The minimun did go through a vertex of the feasible region at $(4, 5)$.

It is because to optimize a line, either the oprimized line will "overlap" with an edge of the feasible region, or it will intersect an edge / vertex of the feasible region. In the former case, any point on the "overlapped" edge is able to provide the optimized solution; and since an edge includes two vertices, the minimum can be on a vertex. In the latter case, the line will always find a direction (along the edge it intersects) to further optimize untill it reaches a vertex, so the minimum will also be on a vertex.

In this particular problem setting, among three contours the "higher" one (one with larger y-axis intersection value) on graph will have a smaller value. Since $c^T x = -16$ is not in the feasible region, we have $c^T x = -14$ being the minimum.