CSDS 455: Homework 20

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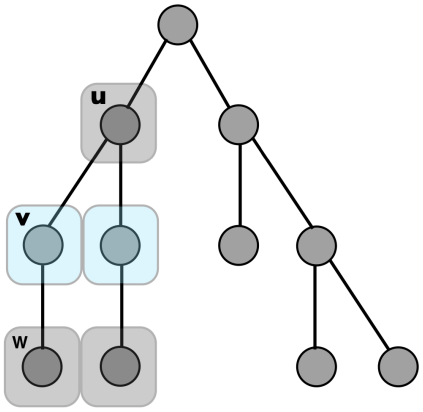
I have consulted Yige Sun and Yuhui Zhang for the following problems.

Problem 1

I have consulted and cited visual aids from https://cseweb.ucsd.edu/classes/sp11/cse202-a/lecture7-final.pdf.

Let `I(u)` to be the maximum independent set of the subtree rooted to node `u`. We will have two cases:



- `I(u)` includes `u`.
- `I(u)` does not `u`, but includes children of `u`.

Since the goal is to maximize the size of the set, we have `I(u)` to be:

$$I(u) = \max\{\sum_{\text{child } v \text{ of } u} I(v),\ \ 1 + \sum_{\text{grandchild } w \text{ of } u} I(w)\}$$

Then we may simply do a postorder traversal from leaves to the root of the tree, the runtime will therefore be $O(|T|)$.

Problem 2

We will try to make T a tree composition out of k-tree T_k. We first identify all the verticies with degree k, add them as leaves of T; we call them the "first group". Then we update G by removing

all the vertices and edges of vertices in T, and find the next (second) group of verticies with degree k – if two verticies out of the two (different) groups has an edge connected to them, they are child-parent (according to the order of deletion) in T. We may do this procedure recursively until we are left with a k or $k+1$ clique, which will be the root of T.

We know that T will be a tree, as no vertices with degree k is connected to more than one vertex with degree k with all verticies of the first group removed (namely, no verticies in the "first group" is connected to two verticies in the "second group"). This procedure may potentialy take $O(|T_k| \cdot k)$ as we have to evaluate almost every verticies in T_k meanwhile each of the verticies has k potential "parent" to evaluate with.

We then run the algorithm introduced in *Problem 1* on T, which will take $O(|T|)$ time which will be absorbed to $O(|T_k| \cdot k)$.

Problem 3

I have consulted https://en.wikipedia.org/wiki/Tree_decomposition.

Let T be the tree composition of G with treewith k. We denote X_i to be a node (bag) of T and X_j to be one of the child nodes of X_i (if we traverse T from an arbitary root). Let $S \subset X_i$ and $S' \subset (X_i \cap X_j)$, we further denote:

- $A(S, i)$ to be the largest independent set I among all X_j such that $I \cap X_i = S$.
- $B(S', i, j)$ to be the largest independent set I among all X_j such that $I \cap X_i \cap X_j = S'$.

We have the DP structure of (starts from a leave of tree T):

$$A(S, i) = S + \sum_j (B(S \cap X_j, i, j) - S \cap X_j)$$

$$B(S, j, i) = \max(|A(S', i)|) \mid \text{ for } S' \subset X_j, S = S' \cap X_i$$

The algorithm works because if we want $S \subset X_i$ to be part of the maximum independent set M of G, then we need to find all of the verticies that are independent to S to add to M – from a DP standpoint, we will need to find how many vertices in the child nodes of X_i can be add to M along with S. $B(S \cap X_j, i, j)$ is an independent set I' where $I' \cap X_i \cap X_j = S \cap X_j$. This means I' can be add to M along with S as nothing in I' is connected to $S \cap X_j$ since $S \cap X_j$ is in I', and nothing in I' is connected to $S - S \cap X_j$ since $S - S \cap X_j$ are in X_i but not in X_j, and I' was exclusively selected from X_j. The only problem is I' will be double-counting $S \cap X_j$ which already added to $A(S, i)$ with the leading S, so we remove them.

By doing this recursively on all X_j, we may find the M with respect to S.

The runtime of calculating this M is $O(|j|k)$ as we need $O(k)$ to figure out if the picked S is an independent set, and there are $|j|$ children of X_i to run the above algorithm with. Which is $\approx O(|G|)$ as we know that $|j|$ is at most $|T| - 1$ and $O((|T| - 1)k) < O(|G|)$.

Now we still need to find the best S for X_i. Since $|X_i| \leq k + 1$, there are $2^{k+1} \approx 2^k$ subsets to consider. Combine the runtime of the procedure above together, we have $O(2^k|G|)$.