CSDS 455: Homework 1

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1 Problem 1

- **Proposition 1**: There exists a self-complementary graph G with n vertices .
- **Proposition 2**: For $n \equiv 0$ or $n \equiv 1 \mod 4$.

Proof. **Proposition 1** $\longrightarrow$ **Proposition 2**, to prove by direct proof:

Let graph G be a self-complementary graph with n vertices, and let G' be a completed graph based on the vertices of G. It is known that G must have half of the edges of G' to be self-complementary. Thus, we may tell G has:

$$\frac{\binom{n}{2}}{2} = \frac{n(n-1)}{4}$$

number of edges.

Since the number of edges must be $\in \mathbb{Z}$, we must have n or $(n-1)$ being divisible by 4. Therefore, for a self-complementary graph with n vertices, we must have $n \equiv 0$ or $n \equiv 1 \mod 4$. $\square$

Proof. **Proposition 2** $\longrightarrow$ **Proposition 1**, to prove by induction:

It is known that we have P_4 (a line that connects four vertices) and C_5 (a pentagon that connects five vertices) being self-complementary graphs, we call these graphs α. Now we have another β graph of P_4, and we connects all the vertices of graph α to the two vertices with a degree of 2 from graph β to form a new graph α'.

This new graph α' will still be self-complementary since the α graph is self-complementary, the β graph is self-complementary; and the connections between α and β will also be self-complementary due to the fact that the two degree-1 vertices in β that did NOT joined α will be the degree-2 vertices in $\bar{\beta}$ that joining $\bar{\alpha}$ with $\bar{\beta}$ – which is isomorphic to the structure of α' in terms of the connections between α and β.

Following this process, we can always contruct a self-complementary graph with n vertices with $n \equiv 0 \mod 4$ (building from P_4) or $n \equiv 1 \mod 4$ (building from C_5). Thus, for graph G with n vertices where $n \equiv 0$ or $n \equiv 1 \mod 4$, there exists a self-complementary graph.

2 Problem 2

Proof. To prove by contradiction:
Assume there exists two disjoined paths, P and Q, both being the longest paths in a connected graph G (with length L), where we have vertices $p \in P$ and $q \in Q$. Since P and Q have no shared vertex, also since G is a connected graph, there must be a path R which connects vertex p to q, where the length of R must be ≥ 1.

Make p and q being the mid-point of P and Q respectively, we may have a new path S travels from the first half of P (till vertex p), then go through R (to vertex q), then travels thought the second half of Q. This path S will have a length of `1/2 len(P)` $+$ `len(R)` $+$ `1/2 len(Q)` which is at least $\frac{1}{2}L + 1 + \frac{1}{2}L$, which is $\geq L$. Thus, by contradiction, if P and Q are the longest paths in a connected graph, they must have a common vertex.

$\square$

3 Problem 3

Proof. Set *(i)* as base.

(i) $\longrightarrow$ *(ii)*, to prove by contradiction:

Assume there are two paths in T between two arbitrarily selected nodes u and v, then we can form a cycle on nodes (u, v), which is against the definition of a tree. Thus, by contradiction, the statement is proven

(ii) $\longrightarrow$ *(i)*, to prove by contrapositive:

Assume T is not a tree, which means it is not a connected acyclic graph, it must has a cycle of same fashion or some nodes of T is not conncted to others. Both of the situations void proposition *(ii)*, as if there is a cycle then the path between a certain two-node are no long unique; and if some nodes are not conncted, then there is no path (not to mention "unique path") at all. Thus, by contrapositive, the statement is proven.

(ii) $\longrightarrow$ *(iii)*, to prove by contradiction:

Let e to be the edge of (u, v) in T. If $T - e$ is disconncted, this means there is another path to connect u with v. Which is a contradiction to proposition *(ii)*. Thus, the statement is proven.

(iii) $\longrightarrow$ *(ii)*, to prove by contrapositive:

Assume there are two path connecting nodes u, v in T (a negation of proposition *(ii)*), while one of them is a direct edge (u, v), lets set it to be e. Then when we delete e, we may have another path still connecting u, v, which makes the new T' still being conncted (a negation of proposition *(iii)*). Thus, the statement is proven by contrapositive.

(ii) $\longrightarrow$ *(iv)*, to prove by direct proof:

For every two nodes in T to be uniquely conncted, T is connected, has no cycle, and any two non-adjacent nodes u, v in T are not directly connected – as otherwise node u and v

can have at aleast two paths between them (the direct connection and indirect connection $< u, ..., v >$ since T is conncted) and this forms a cycle. Let xy to be edge (u, v), a cycle is formed with this edge (u, v) and original path between u, v in T. Thus, the statement is proven.

(iv) $\longrightarrow$ *(ii)*, to prove by direct proof:

For a T which does not contain any cycle, and we can find a cycle for any two non-adjacent nodes x, y in $T + xy$. We can tell that without the direct edge of (x, y), node x and y are still conncted in T. This is because a cycle of x, y needs a minimum of two paths between node x and y, so there must be another path from x to y exists in T. Since x and y can be any two non-adjacent nodes, and since any two adjacent nodes are intrinsically connected, we may say that all nodes in T are connected. With T being connected and having no cycle, we may say that every two nodes in T are linked by a unique path. Thus, the statement is proven with direct proof.

□

4 Problem 4

Proof. To prove by induction:

For T_1 being a set of trees with 1 node (in this case, there is only one tree). Any tree from T_1 must be a subgraph of graph G_1, where G_1 represents all the graphs with $\delta(G_1) \geq 0$ – as both the tree and the graph is essentially a single node with no edges.

Assume a random tree T_k from set T_K, a set of trees with k nodes, such tree must be a subgraph of a graph G_k from set G_K, where $\delta(G_k) \geq k - 1$. Such randomly selected T_k will also be a subgraph of a certain graph from set G_{K+1}; since G_{K+1}, with $\delta(G_(k + 1)) \geq k$, is included in set G_K due to $\delta(G_(k + 1)) \geq \delta(G_k) \geq k - 1$.

If we locate an arbitary node p from tree T_k, then add a child node q with a corresponding edge to p (namely, edge (p, q)), we have a new tree T'. This T' will be a potential T_{k+1} tree from set T_{K+1} – since with the newly added node q, now it has $k + 1$ nodes.

Now take graph G_k and add a new node q' that is connected to all nodes of G_k, we call this modified version of G_k as G'_k. Such graph G'_k is considered to be a graph from set G_{K+1}. Since node q' is conncted to every node of G_k, thus it increases $\delta(G'_k)$ by 1 to be at least $\delta(G_k) + 1$, namely $\delta(G'_k) \geq k$ – which is the exact requirement for $\delta(G_{K+1})$.

We now may show T' is a subgraph of G'_k. We may locate the subgraph of T_k in G'_k, and find the projection of node p from T_k in G'_k (let's call it "p'"). This p' node is connected to the q' node we just added to G'_k, and the subgraph of T_k in G'_k plus this p' node will form a subgraph that is isomorphic to T' – since T' is essentially T_k with an extra node q connected to p.

Since the node p in T' is an arbitrarily selected node in randomly selected tree from set T_K, this T' can essentially be any tree in set T_{K+1}. Thus, any tree from set T_{K+1} may have a corresponding graph from set G_{K+1}, where the tree will be a subgraph of the graph, namely $T_{k+1} \subseteq G_{k+1}$; we have proven the statement by induction.

$\square$