CSDS 455: Homework 4

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Due and submitted on 09/07/2020
CSDS 455, Dr. Connamacher

1 Problem 1

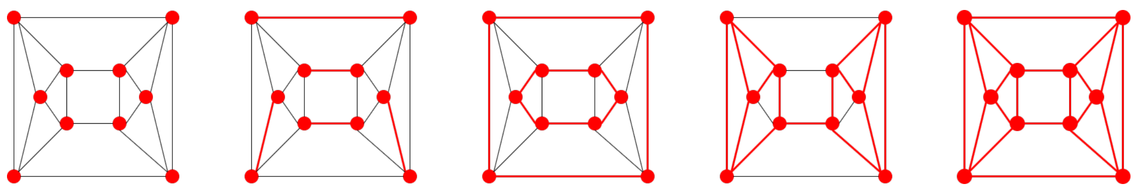

Figure 1: Visulization of 0- to 4-factor (respectively presented)

2 Problem 2

For this question I have consulted:

- https://math.stackexchange.com/questions/1805181
- https://math.stackexchange.com/questions/2422069
- https://math.ryerson.ca/~danziger/professor/MTH607/W08/Labs/lab12-soln.html
- https://math.stackexchange.com/questions/3681587

W.T.S. If G is a k-regular, bipartite graph that can be decomposed into r factors, then r divides k.

Proof. Since we know that G can be decomposed into r factors, namely, G is r-factorable. We may therefore assume that G has n disjoint r-factors for $n \in \mathbb{Z}^+$, and the union of these r-factors may yield a graph where all of its vertices have a degree of nr. This nr-regular graph will be the same graph as G (since it is G to be decomposed into n of r-factor subgraphs), this means G is a nr-regular graph. Since G is known to be $k - regular$, there must be $k = nr$ and the relationship of $r|k$ is demonstrated. □

W.T.S. If r divides k, then G is a k-regular, bipartite graph that can be decomposed into r factors.

Proof.

Lemma: For G being a k-regular bipartite graph, G must have a perfect matching M_1.

Proof. It is because the number of edges connected to each vertex in G is k, and for G being a bipartite graph with partition U and V, the number of edges associated with set U and V must be $k|U|$ and $k|V|$ respectively. Since every edge is connected from a vertex in U to a vertex in V, the total number of edges of from U to V is the same as the number of edges from V to U – this suggest $k|U| = k|V|$, thus $|U| = |V|$.

Now we want to show that we may find a matching M_1 in G by Hall's theorem, and such matching is also a perfect matching. Assume we have $S \subseteq U$ and let $N(S)$ denotes the neighbors of S in V. Since every edges starts from S and ends in $N(S)$, denotes $E(S)$ and $E(N(S))$ to be the edge set of edges connected to S and $N(S)$ respectively. Knowing the total number of edges from S to V (namely, to $N(S)$) is $k|S|$ and the total number of edges from $N(S)$ to U is $k|N(S)|$, there must be $k|N(S)| \geq k|S|$ since edges from $N(S)$ to U includes edges from $S \subseteq U$ to $N(S)$.

This implies $|S| \leq |N(S)|$, then the Hall's condition is achieved and we have a matching of M_1 with $|E(M_1)| = |U|$. Since we have previously proven that $|U| = |V|$, and the union of U and V yields all vertices of G; M_1 has matched every vertex of G and it is therefore a perfect matching of G.

□

Now with the lemma proven, by the nature of perfect matching M_1 must be a 1-factor of G. We denote $G_1 = G - M_1$. This G_1 is a $k-1$-regular graph since every vertex of G has decrease a degree of 1; and this G_1 is still bipartite as deletion of edges will not affect the bipartite property. Refer to the above lemma, this G_1, being a $k-1$-regular bipartite graph, must have a perfect matching as well (denotes as M_2). Following the induction, we may have $G_2 = G_1 - M_2$, $G_3 = G_2 - M_3$... till $G_r = G_{r-1} - M_r = G - M_1 - M_2 - ... - M_r$ being a $(k-r)$-regular graph.

Since M_1, M_2, ... , M_r are all 1-factors of G, a union of these Ms may yield a r-factor of G and we have showed G has a r-factor. Since we know that $r|k$ for $nr = k$ for $n \in \mathbb{Z}^+$, now we keep removing r-factors from this G_r (by removing r number of 1-factors at each time), there must be a $E(G_{nr}) = \emptyset$ with n number of r-factors being removed from G. This is same as saying an union of n number of r-factors may yield G, and we have therefore showed that for $r|k$, G is a k-regular, bipartite graph that can be decomposed into r factors.

□

3 Problem 3

For this question I consulted https://math.stackexchange.com/questions/520203. I also borrowed the below visual aid from *Elchanan Solomon* who contributed to the above webpage.

Proof.

To construct a k-regular graph with no prefect matching. For k being even, we can simply construct a complete graph of $k+1$ vertices – since every vertex is connected to k other vertices, this is a k-regular graph – as the graph has $k+1$ vertices, it has no prefect matching.

For k being odd and $k > 1$, we will need help from the below lemma.

Lemma - Tutte's Theorem (simplified, one direcons): If a simple graph G has a 1-factor, then there must be $o(G-S) \leq |S|$ for any $S \subseteq V(G)$; where $o(G-S)$ denotes the number of odd components in graph $G-S$.

Proof. If G has a 1-factor M, then for every odd component G_o of $G-S$ (for $S \subseteq V(G)$), there must be $o(G-S) \leq |S|$. It is because any G_o cannot have any prefect matching (as perfect matching requires even number of vertices), then there must be a vertex w in each G_o that is connected to a vertex in S (denotes this vertex as S_w). And this edge $< w, S_w >$ must be in M as otherwise this w will be unmatched.

As matching is an one-to-one relationship, $o(G-S)$ number of ws must be matched to $o(G-S)$ number of S_ws. This implies $o(G-S) \leq |S|$. □

Now to construct the graph G. We start with an initial node u and branch out k edges out of it, thus we have $|N(u)| = k$. Now for each $v \in N(u)$, we branch out $k-1$ edges out of it. Then for each $w \in N(v)$, we make a single vertex w' $((q) = 0)$ along side the w. After all w's are made under a v, we fully connect ws with w's and then internaly connect each w' to another w'. We do it repreatly for next v until this is done to all $v \in N(u)$.

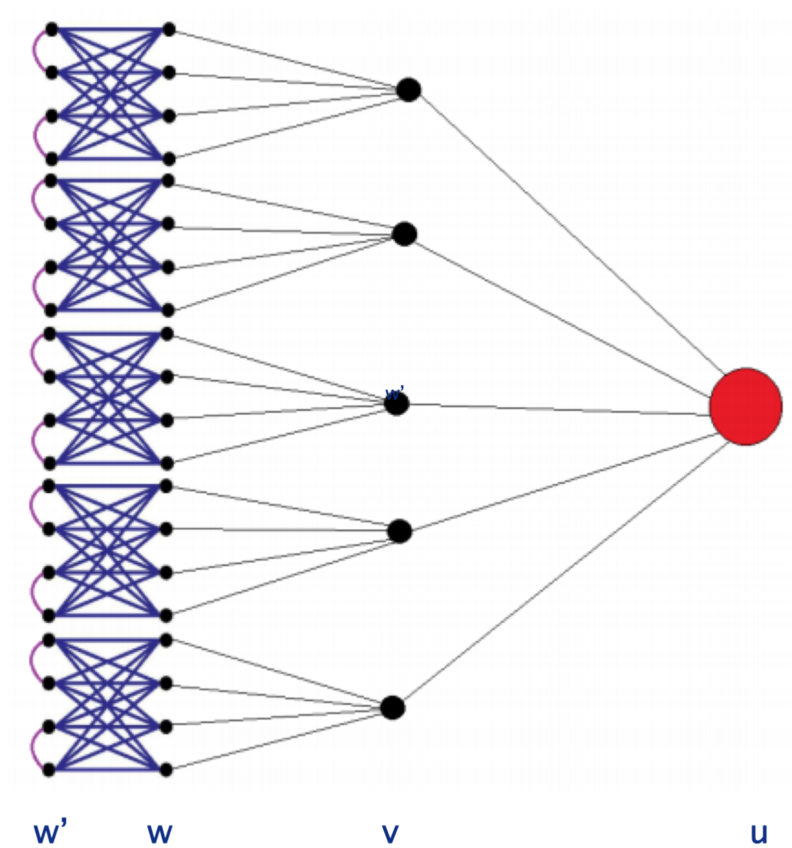

Figure 2: Demo of G for $k = 5$ (modified work based on Elchanan Solomon's diagram)

This G will be a k-regular graph because: u branch out for k edges, $\delta(u) = k$; every v branch out $k-1$ edges and connected to u, therefore $\forall\delta(v) = k$; every w is connected to $k-1$ number of

w' and also to a v, so $\forall\delta(w) = k$; finally, every w' is connected to $k-1$ number of w and also to another w', therefore $\forall\delta(w') = k$.

Now if we let this u to be S, and for $G - S$ we have all k compoments left. All of these components (lead by $v \in N(u)$) are odd components, as:

$$k - 1 + k - 1 + 1 = 2k - 1 \tag{1}$$

For k being odd, $2k - 1$ must be odd. This voids the above lemma since the contrapositive of lemma "*G has a 1-factor $\longrightarrow o(G - S) \leq |S|$ for any $S \subseteq V(G)$*" is "*$o(G - S) > |S|$ for any $S \subseteq V(G) \longrightarrow G$ has a no 1-factor.*"

Now we have $|S| = 1$ but $o(G-S) = k$ (known that $k > 1$), thus G has no 1-factor and therefore has no perfect matching.

We have demonstrated there is a way to construct a k-regular graph G for both k being even and odd (for $k > 1$), thus there is a way to construct such graph G for all $k > 1$ for $k \in \mathbb{Z}^+$.

$\square$