CSDS 455: Homework 6

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Problem 1

Proof. We have $rad(G) \leq diam(G)$ proven by definition. Now locate vertices $u, v \in V(G)$ with $diam(G) = d(u, v)$, then locate a vertex c to the center of G. We know that $diam(G) = d(u, v) \leq d(u, c) + d(c, v) \leq 2\epsilon(c)$. Due to the nature of c being a center vertex, we also know that any vertex has a $\epsilon(c) = rad(G)$ by definition, thus we have $diam(G) \leq 2rad(G)$ and the statement is proven by direct proof. □

Problem 2

Proof. The question is essentially same as proving that the Prim's algorithm produced tree P is a minimum weight spanning tree of G. This is because if T is a minimum weight spanning tree of G with a weight of W, by doing the equal-weight-edge-replacements on T to make T', T'', T'''... we will eventually have a replaced version of T that is same as P, as both T and P (if being a minimum weight spanning tree of G) have a weight of W.

> **Lemma (Cut Property): Assume $S \in V(G)$ with G being a graph with distinct edge weights. If the minimum cost edge between $V - S$ and S is e, this edge e must be contained in the minimum weight spanning tree of G.**
>
> *Proof.* To prove by contradiction: Denote the minimum weight spanning tree of G to be M and assume edge e is not in it. This means there must be some other edge e' that connects V to $V - S$. Since we know that $w(e) < w(e')$ and M is connected, $M - e' + e'$ will another spanning tree of G and has a lower weight. This contradicts the assumption of M being a minimum weight spanning tree of G, and the statement is therefore proven. □

With the lemma proven. Now assume P is not a minimum weight spanning tree of G and traverse through the sequences of vertices of P and T. Let $P_{k+1} \subseteq P$ denotes the first tree that has an edge that is not contained in T (as in P, it went from its k-th vertex x to $(k+1)$-th vertex y; but in T, it went from its k-th vertex x to $(k+1)$-th vertex z).

Now we denotes $P_k = P_{k+1} - y$, and edge $< x, y >$ to be the minimum edge from P_k to $V - P_k$ (as it is defined by the mechanism of Prim's algorithm). By lemma, this edge $< x, y >$ must be in the minimum weight spanning tree T. However, $< x, y >$ is not in T as T should chose edge

$< x, z >$. This contradicts the setting and our assumption of P is not a minimum weight spanning tree of G is therefore voided.

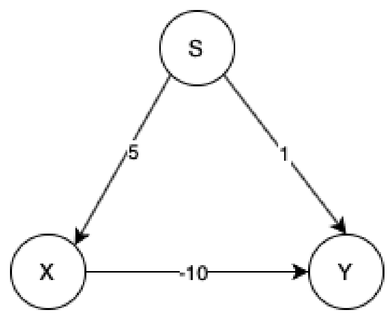
Since at the beginning we have estalished that proving P to be a minimum weight spanning tree of G is equivalent to the proposed statement. The statment is therefore proven.

Note the lemma is proved based on the assumption of a graph having distinct edge weights. However, this may not be true for the proposed G. Which would suggest a minimal cost edge e may not been chosen to connect V to $V - S$ as an alternative edge e'' with $w(e'') = w(e)$ is chosen. However, this won't effect the conclusion as as long as P has the same weight of a minimum weight spanning tree, we may convert T to P by doing the equal-weight-edge-replacement manuver.

□

Problem 3

Proof. To prove by counterexample: Assume we have the graph directed graph with negative edge weight of the following.



Staring from S as the source, the first iteration of DIJKSTRA will give us $S0;\ X : 5, S;\ Y : 1, S$ with S removed from unvisited nodes. As Y has the shortest cost from all unvisited nodes, we will start our second iteration from Y. Which will give us the same $S0;\ X : 5, S;\ Y : 1, S$, as Y has no neighbor, with X, Y removed from unvisited nodes. Now we start our third iteration with our only unvisited node X, since there are no other unvisited node, the iteration will stop with the same $S0;\ X : 5, S;\ Y : 1, S$.

This DIJKSTRA's algorithm produce a spanning tree of node sequence of $\{< S, Y >, < S, X >\}$ with a total tree weight of $5 + 1 = 6$. However, the actual shortest spanning tree should be $\{< S, X >, < X, Y >\}$ with a tree weight of $5 - 10 = -5$. As $-5 < 6$, this counterexample has proven the statement.

□

Problem 4

For this quetion I have consulted https://www.cs.cmu.edu/~ckingsf/class/02713-s14/lectures/lec11-bellman.pdf.

Proof. It is my understaning that I can produce a spanning tree of G with the BELLMAN-FORD algorithm. The detail procedure of such algorithm is omitted, but the general principle is from source node s, we set $d(s) = 0$ and $d(v) = \infty$ for all $v \in V(G) - s$. Then we find an edge (u, v)

where $d(u) + d(u, v) < d(v)$ and set $d(v) = d(u) + d(u, v)$, until we have $d(u) + d(u, v) \geq d(v)$ for every possible u, v pair in $V(G)$. We denotes the tree generated by this algorithm as T.

We want to show that for any arbitrarily selected $v \in V(G)$, there should be $w(P_{sv}) \geq d_T(v)$, where P_{sv} can be any path from s to v. We will show this be using induction on number of edges in P_{sv}.

For $n = 1$, the assumption is trivially true as there is only one egde from s to v in G, so of course there is $w(P_s v_1) = d_T(v)$. Assume this is also true for k edges: which means if the last of k-edged P_{sv} is v_k, we have $w(P_{sv_k}) \geq d_T(v_k)$.

Now let $P_{sv_{k+1}}$ to be a path with $k+1$ edges, where $P_{sv_{k+1}} = P_{sv_k} + (v_k, v_{k+1})$. There must be:

$$w(P_{sv_{k+1}}) = w(P_{sv_k}) + d(v_k, v_{k+1}) \geq d_T(v_k) + d(v_k, v_{k+1}) \geq d_T(v_{k+1})$$

As otherwise we void the $d(u) + d(u, v) \geq d(v)$ requirement and should keep iterating the Bellman-Ford algorithm with more Ford step(s). As v_k and v_{k+1} can be any two adjacent nodes in $V(G) - s$, and with the statement of $w(P_{sv_k}) \geq d_T(v_k)$ to be true for both 1, k, and $k+1$ cases, we have proven the statement to be true.

$\square$