CSDS 455: Homework 13

Shaochen (Henry) ZHONG, sxz517

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Fall 2020, Dr. Connamacher

Problem 1

I consulted https://www.youtube.com/watch?v=otky1bBhwgM *for this problem.*

Proof. We will have the following diagram for a base graph:

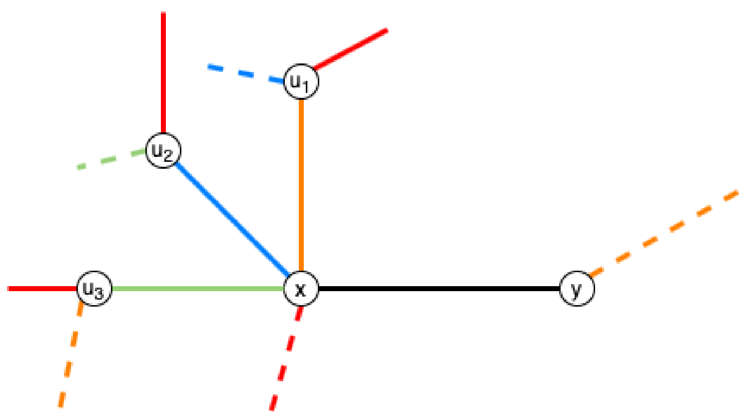


Figure 1: Base graph for *Problem 1*

For better clarity, we have $u_1, u_2, ..., u_i \in N(x)$ in $G - xy$. Known that $\chi'(G - xy) = \Delta + 1$, every vertex in $G - xy$ must be missing[1] at least one color from the $\Delta + 1$ colors, we use dashed lines to represent such colors. We also use solid lines to represent the color of edges under a $\Delta + 1$ edge-coloring of $G - xy$. Note we use some arbitary actual color names instead of $c_1, c_2, ...$ to help understanding, but they are essentially the same.

We want to show that other than having an 2-color alternating path between x and y, we will always have $\chi'(G) = \Delta + 1$, thus a contradiction.

Proposition 1. x and y can't miss the same color.

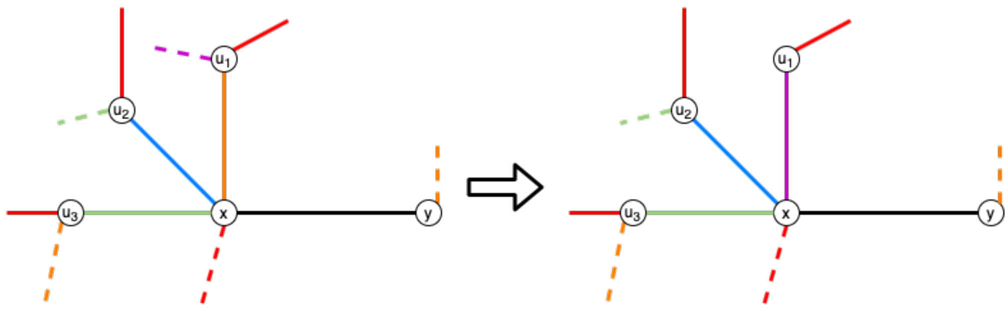
This observation is almost trivial as otherwise we may just color edge xy with this color, then we have a $\Delta + 1$ coloring of G – a contradiction.

[1] "Missing" in this context means not having an edge of a certain color. "Vertex v is missing color c" means there is no edge of color c connected to v.

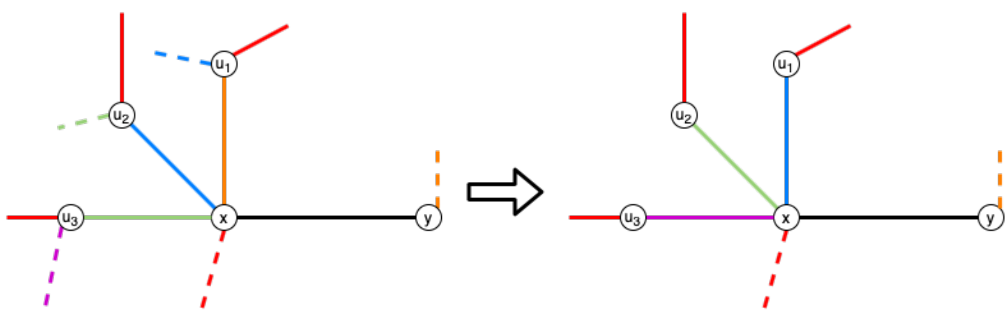
Proposition 2. x **must include every missing color of** $N(x)$**.**

Say we have a set S which include all the missing colors of $N(x)$ in $G - xy$, such S must be a subset of the colors connected to x. This is because if we have a vertex $u_i \in N(x)$ which has a missing color of purple which is not included in the connected colors of x, we have the following two cases.

First, u_i is u_1, where edge u_1x has the color which y misses, then we may recolor the graph as following. This will make x and y both missing orange; we can therefore color xy with orange and have a $\Delta + 1$ coloring of G, a contradiction.



Second, we may have u_ix not missing the color which y misses. We can recolor u_ix to its missing color (purple), then recolor the original blue u_jx (u_j is another neighbor vertex of x which is missing green) to be green, and recolor another neighbor vertex of x which is missing the color of u_jx... we repeat it until a contradiction is found. Take $u_3 = u_i$, $u_2 = u_j$ as an example, we will have:

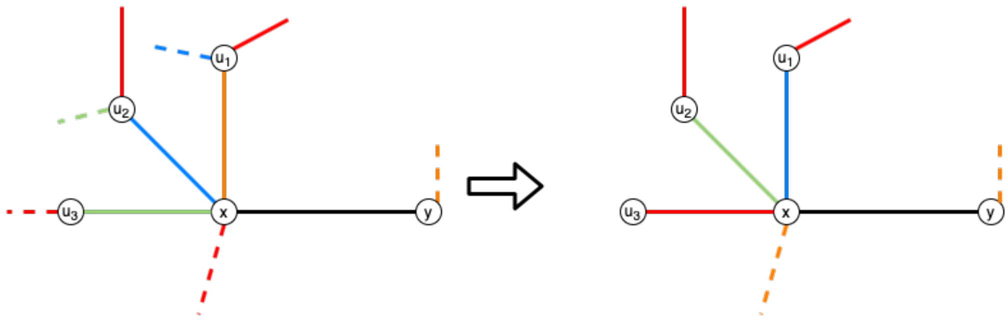


We know this recoloring method works since $N(x)$ is finite, so we will always able to recolor u_1x (orange, which y misses) to the missing color of u_1 (blue), then recolor the blue edge u_jx to its missing color, then .... we will eventually reaches u_i and recolor u_ix to its missing color purple. Then we will have x and y both missing orange and thus a contradiction.

Now we have showed that every missing color of $N(x)$ must show on an edge of ux for $u \in N(x)$. Be familiar with this recoloring method as we will use it later.

Proposition 3. If x **misses red. Every** $u \in N(x)$ **must have an red edge connected to** u**.**

Because otherwise, by using the recolring method introduced in **Proposition 2**, we may have x and y both missing orange and thus a contradiction. The following is an example assuming u_3 missing red, but it can be any $u \in N(x)$ in $G - xy$.

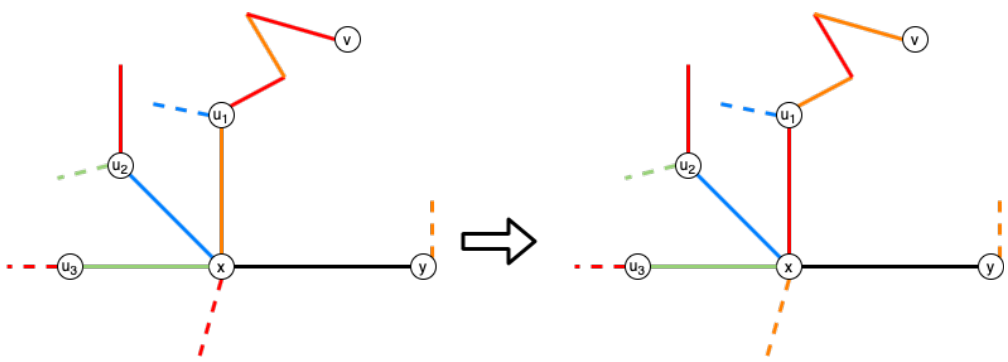


Combining the finding of **Proposition 1, 2, 3**, we will have a graph like [Figure 1].

Proposition 4. If x misses red and y misses orange, the maximal orange-red alternating path from x can't end on a vertex $\notin N(x)$ and not y.

If we consider a subgraph G' with only orange and red edges by the coloring of $G - xy$, we know that every vertex in G' will have a degree of 0, 1, or 2. Now we take a orange-red alternating walk P from x, until it reaches a vertex $v \in V(G)$ where we can't extend the walk any farther (which implies $d(v) = 1$). We let $v \notin N(x)$ and $v \neq y$ in this case.

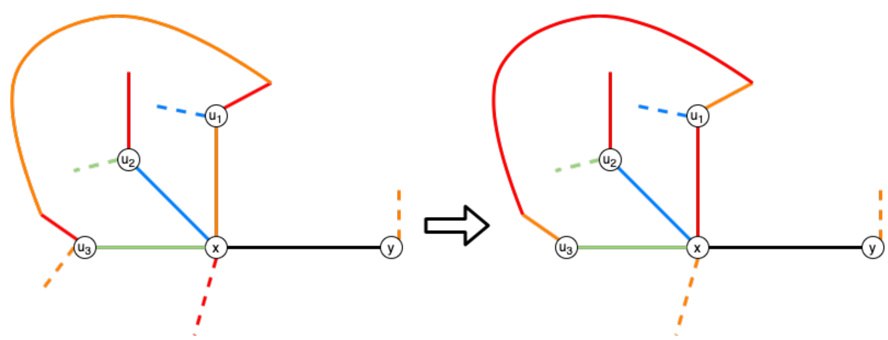
We can't have this kind of v as by simply interchange the orange-red coloring of P, we may then have x and y both missing orange, thus a contradiction. So we know a orange-red alternating walk P from x can't end on a vertex outside of $N(x)$ and y in $G - xy$.



Proposition 5. If x misses red and y misses orange, the maximal orange-red alternating path from x can't end on a vertex $\in N(x)$.

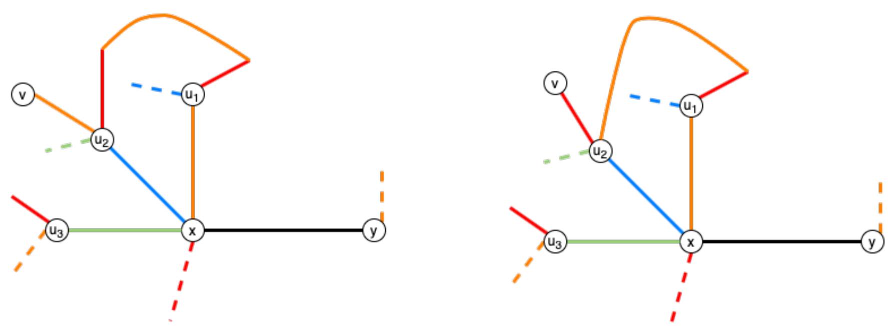
It is trivial to tell a orange-red alternating walk P from x can't end on u_1 (where xu_1 is the first edge of P). P may either end on an u_i where u_i misses orange (like u_3 in the below graph); or P will visit another $u_j \in N(x)$ which has orange edge connected to it, but in this case this P will not end on $N(x)$.

We first show the case of P ending on a vertex $u_i \in N(x)$ where u_i misses the same color as y (orange). In this case we interchange the color in P as following and we have have x and y both missing orange – a contradiction.



In the case of P ending on a vertex $u_j \in N(x)$ where u_j has orange edge connected to it. We can tell P will not end on $N(x)$ as:

1. If $u_{j-1}u_j \in E(P)$ is red, in this case we may extend this P farther by including the orange edge connected to u_j.
2. If $u_{j-1}u_j \in E(P)$ is orange, then we may extend this P farther to include the red edge connected to u_j.



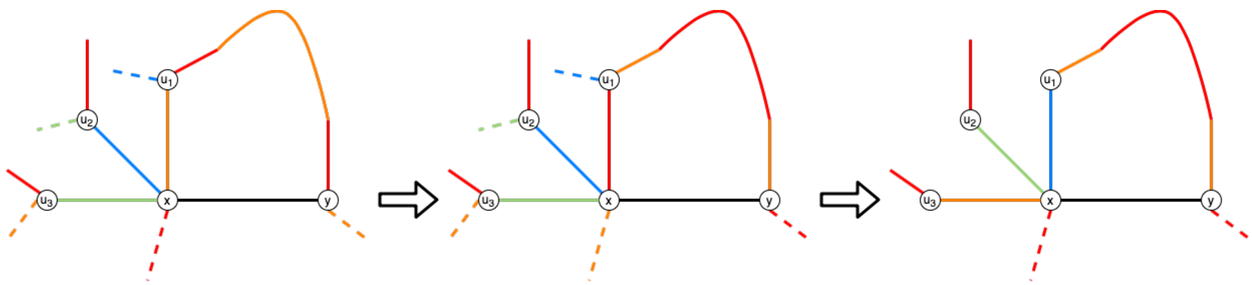
Either of the case will make P end on a vertex v outside of $N(x)$ and y. According to **Proposition 2**, this will give us a contradiction.

Conclusion: P must end on y.

By **Proposition 4, 5**, we know that a orange-red alternating walk P from x can't end on $N(x)$, can't end on $G - y$, and obviously it can't end on x itself as x misses red. So the only left option it to have P ends on y, then the statement in question is therefore proven.

$\square$

In the following diagram we will show with P ends on y, by having a series of recoloring operations, we may still make x and y both missing the same color (red) and form a contradiction. But this is simply because the assumption of the problem that $\chi'(G) > \Delta(G) + 1$ is false (Vizing's theorm says $\chi'(G) \leq \Delta(G) + 1$), so a skewed conclusion is reached.



In fact, by showing that regardless where P ends, we may always color the extra edge xy with the existing $\Delta + 1$ colors used in coloring $G - xy$ proven the VIZING's theorm.

Problem 2

I have consulted this problem with Danial Shao.

To prove by induction. For G with $\Delta = 2$, each vertex have two edges connected to it. We can color all vertices with 2-color untill an odd cycle is reached. In this case we introduced the thrid color and we are done. Thus $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Assume it holds true for G with $\Delta = k$. Now we have G' with $\Delta = k + 2$. Known that G is already properly colored, we inspect $E(G') - E(G)$ as thoese are the edges we need to color. It is observable that this is just a $\Delta = 2$ multigraph and we have showned this can be colored with 3 new colors.

Now we have $\chi'(G') \leq \chi'(G') + 3$, where we know that $\chi'(G') + 3 = \frac{3k}{2} + 3 = \frac{3(k+2)}{2}$. Thus, with $n = 2$, $n = k$, $n + 2$ all being true, we have shown for G with $\Delta = k$, $\chi'(G') \leq \frac{3k}{2}$.