CSDS 455: Homework 5

Shaochen (Henry) ZHONG, sxz517

Due and submitted on 09/09/2020
CSDS 455, Dr. Connamacher

1 Problem 1

For this quetion I have consulted http://ion.uwinnipeg.ca/~ychen2/advancedAD/notes-March15.pdf *and* https://math.la.asu.edu/~andrzej/teach/mat416/proofs3.pdf.

We assume G to be the maximal counterexample as it has no 1-factor, but having $G' = G + e$ will have a 1-factor. Since $q(G' - S) \leq q(G - S)$ (we know this from *Class 5 Pratice: Q1*), the Tutte's condition is still satisfied. We want to show a contradiction of if G' can have a 1-factor, so does G.

Let U be the set of vertices in G with degree $|V(G) - 1$ (every vertex if V connected to every other vertices in G). If $G-U$ are consist of disjoint complete graphs, then $q(G-U) \leq |U|$ (because a component is formed by removing a vertex $\in U$ from the connection). We can find a perfect matching out of it by finding 1-factor from each even component, and connect 1 unmatched vertex in each odd component to one vertext U, then connect the unmacthed vertices in U to each other (we know this from *Class 5 Pratice: Q2*).

If $G - U$ are not disjoint union of complete graphs, there must be vertices u, v which are in the same component of $G - U$, both conncted to a vertex w, but not adjacent to each other. This means we have $uw, uw \in E(G)$, but not uv. Now we locate a vertex z under the same component of $\in G - U$ and there is no edge wz, there must be one as otherwise z will be in U.

Since we asssume G' has a one factor, denotes a 1-factor M_1 for $G + uv$, anda 1-factor M_2 for $G + wz$. Let $F = M_1 \Delta M_2$, we must have $uv, wz \in F$ as they are only in M_1 or M_2, never both. This suggests F is consisted by every even cycle of G which traversed through all $V(G)$.

Let even cycle $C_1 \in F$ contains uv but not wz, and even cycle $C_2 \in F$ contains wz but not uv. Due to the even cycle nature, we can always find a 1-factor of C_1 without taking uv as a match (say the 1-factor of C_1 with uv being a matched edge is M_{uv}, the alternative 1-factor would be $C_1 - M_{uv}$); same goes to C_2 by not taking wz as a match. Since we can do this to every $C \in F$, this suggests if G' may have a 1-factor with an extra edge e, so does G – the contradiction is found in this case.

However, it is possible to have an even cycle $C \in F$ which contains both uv and wz at the same time. Known that there is $uw, vw \in E(G)$, let path $P_1 \in C$ to be between w and u, and $P_2 \in C$ to be between w and v. For $N_1 = E(P_2) \cap M_1$ and $N_2 = E(P_1) \cap M_2$, we have $(N_1 \cup_2 \cup \{wu\}) \ \cup \ (M_1 - E(C))$ being a 1-factor of G. Again, since we can do this to every compunent $C \in F$, we can always find a 1-factor of G in this case.

Since a contradiction can be found in all possible cases, the statement is proven by contradiction.

2 Problem 2

I worked with – or technically, I learned from – Yige Sun for this question.

$\Longrightarrow$: For G containing a k-factor, every $A(v)$must be connected to k other vertices in other A partitions. Thus, $d(v) - k$ vertices will have a perfect matching with vertices in $B(v)$. Combine edges of partitions by traverse through different vertices in G, we have a perfect matching for H, thus H has a 1-factor.

$\Longleftarrow$: For H having a 1-factor, every vertices in every $B(v)$ will matched to $d(v) - k$ vertices in its corresponding $A(v)$. By removing all the matched edges and vertcies, we have $|A(v)| = d(v) - (d(v) - k) = k$ for every $v \in G$. This implies that every vertex in G are at least conncted to k other vertices in G. Taking these connections, we will have k-factor in G.