CSDS 455: Homework 18

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I have consulted Yige Sun and https://www.researchgate.net/publication/229747655 *for the following problems.*

Problem 1

If the contracted edge e is a part of a chordal (3-edge) circle in G, then after the contraction every chordal circle including this e will be "collapsed" to a line, and such collapse will not creating any non-chordal structure, thus the graph is still chordal. If the contracted edge e is not part of a chordal circle, then a contraction of e won't affect the chordal property of the graph.

Since the graph is still chordal regardless which edge e is contracted, the statement is therefore proven.

Problem 2

Known that every planar graph can be drawn in a plane graph format, we convert our planar G to plane G – we may say that if a minor of plane G is still plane, it must be planar.

It is trivial that edge or vertex deletion will not make a plane graph no longer plane. For edge contraction, say we are contraction edge of xy into vertex x'. Due to the plane graph property, edge coming out of x or y are not crossing any other edges, and therefore so do edges comming out of x' after the contraction.

Since we have showed a plane graph G will still be plane graph after all three possible minor-manuvers, a minor of G must be a planar graph.

Problem 3

Base on the given instruction, of $\chi(G') \geq k$, then the proof is trivial as we know that such G' will have a K_k minor, and therefore by removing a vertex out of this K_k minor, ad have a K_{k-1} minor.

If $\chi(G') \ngeq k$, in other another word $\chi(G') = k - 1$, we now add a vertex v to G' where v is connected to all vertices of G'. Now there must be $\chi(G' + v) = k$ and therefore has a K_k minor, we then proceed to solve it by cases:

- If v is part of the K_k minor of $G' + v$. Then this suggests there must be a $K - 1$ vertices of the K_k minor in G', where any two of them a connected. So we can identify these $K - 1$ vertices inside G' and deletion/contraction according to achieve a K_{k-1} minor out of G'.

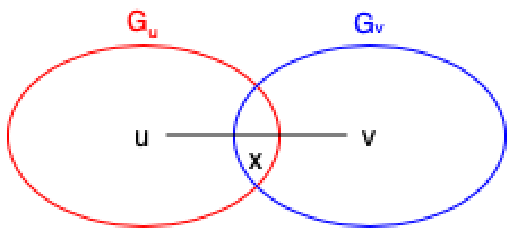
- If v is not in the K_k minor of $G' + v$ due to deletion or simply not included in the K_k minor, that means G' already has a K_k in itself and we can make a K_{k-1} minor out of K_k minor by removing one vertex (this is in fact not a case of $\chi(G' + v) = k$).
- If v is not in the K_k minor of $G' + v$ due to edge contraction. Say v is contracted with vertex u to make v' and this v' is in the K_k minor of $G' + v$. Then we can simply remove this v' to make a K_{k-1} minor.

Since we may obtain a K_{k-1} minor out of G' in every case, the statement is therefore proven.

Problem 4

The base case will be $|V(G)| = 1$, which does not contain K_4 minor and is certainly 3-colorable. We assume that this is true for $|V(G)| \leq k - 1$.

For any graph with $|V(G)| = k$ (we only consider connected non-tree graph, as every tree is 2-colorable), we pick an edge e between verticies u, v. Denotes the minimum vertex cut between u, v to be x, we know this x must be a independent set (i.e. no verteices within x are internally connected), as otherwise (assuming x_1, x_2 are connected) we have path of $uv, ux_1, u_x2, v_1, v_x2, x_1x_2$ which can produce a K_4 minor.



After the vertex cut between u, v, we now denote the partition including u plus x to be G_u, and likewise the parition including v plus v to be G_v. Namely, we have $G_u \cap G_v = x$ and $G_u \cup G_v = G - e$. Now we do edge contraction to G_u so that every verteices of G_u is contracted to one vertex u', we denote the graph G with contracted G_u as G'_u; and likewise we denote the graph G with contracted G_v to one vertex v' as G'_v.

As $|V(G'_u)|, |V(G'_v)| < k$ and neither of them has a K_4 minor, by the induction assumption they are 3-colorable. This means, the x portion of G'_u can be colored as c_1, and the leftover $G'_u - x$ can be colored as c_1, c_2, c_3; similarily, x portion of G'_v can be colored as c_1, and the leftover $G'_v - x$ can be colored as c_1, c_2, c_3. Note the

Then for G, we may color the x portion of it as c_1, then we color vertices in $G_u - x$ exactly as their corresponding vertices in $G'_v - x$; similarly, we color vertices in $G_v - x$ exactly as their corresponding vertices in $G'_u - x$. This will give us a 3-color of $G - e$ as $G_u - x$ are not connected to $G_v - x$, so as long as each of them are 3-colorable and x using one of the three colors, we have a 3-colored $G - e$.

Note it is possible that we may have u, v colored as the same color. We know this color must be c_2 or c_3, as in G'_u (W.L.O.G.) v is connected to u' (which includes all verticies in x), so v must be using a different color to x. So if u, v happen to be using the same color, say c_2, we may simply shift swap c_2-colored vertices to c_3 in $G_v - x$. With u, v using different colors, we may add the e

back and have a 3-colored G.

Since we haved showed that for G with $|V(G)| = 1, k-1, k$ having no K_4 minor, such G is 3-colorable, we may say that every G that does not contain a K_4 minor is three colorable.