CSDS 455: Homework 9

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Problem 1

Proof. Suppose we have k edge-disjoint $a \to b$ paths in G, this suggest there are at least k entirely distinct ways of getting from vertex a to b. So intuitively, we need to cut out all of these paths to ensure a to be disconnected from b. And it is known that to cut off a path, at least 1 edge needs to be removed, thus for a graph with k edge-disjoint $a \to b$ paths, we need to remove k edges to make a to be disconnected from b.

Note when selecting an edge to remove from an edge-disjoint path, such edge must be an common edge of all paths from $a \to b$ which share edge(s) with this particular edge disjoint path. □

Problem 2

I consulted https://math.stackexchange.com/questions/3113602/ *for this problem.*

Proof.

> *Proof.* **Menger's Theorem: For a connected, finite undirected graph G. The minimum vertex cut for $u, v \in V(G)$ is equal to the maximum number of vertex-disjoint paths from u to v.**
>
> Say we have k vertex-disjoint paths from u to v. We will need to remove a vertex to break a path (similar to *Problem 1*, we will need to remove the common vertex of all paths from u to v where these path has a shared vertex with the path we removing vertex from), so we must remove k vertices to disconnect u to v.
>
> **By promoting this proof to all vertex pairs, this means any k-connected graph will have k (internally) vertex-disjoint paths from any vertex pair in G.**
>
> □

With the lemma proven, we may arbitrarily select k desired vertices and find a cycle C of G which has j common vertices with the k desired vertex set (we call this set D_k), say $v_1, v_2, ..., v_j$. If $j = k$, then the statment is automatically proven.

If $j < k$ but $|C| \geq k$, by the lemma and knowing that G is a k-connected graph, we know that there must be k vertex-disjoint paths from C to v_k, where $v_k \in D_k$ and $\notin C$. We also know that these k vertex-disjoint paths from C to v_k can end on k different vertices on C.

Thus, we may find two adjacent vertices on C (denotes them v_i and v_{i+1}) and replace the edge between them with a path[1] of $v_i \to v_k \to v_{i+1}$. Since there are k paths between C to any vertex in D_k, we may do replace-edge-with-a-path manuver entil all k vertices in D_k is included in the cycle.

If $j < k$ but $|C|$ =k−1, there must be $k-1$ vertex-disjoint paths from C to v_k, each ending on a different vertex on C. We know that $|C|$ must be 2 as otherwise C will not be a cycle, so we may always find two adjacent vertices v_i and v_{i+1} on C. Replace the edge between them with the path $v_i \to v_k \to v_{i+1}$, now we have included the only leftover desired vertex v_k into the cycle.

With $\geq k$ and $k-1$ both being true, and known that $k = 2$ is trivially true[2], we have proven the statement by induction.

□

Problem 3

I intensively discussed with Jiaqi Yu and Yuhui Zhang on this problem.

$\Longrightarrow$ To prove by contrapositive, W.T.S. that if $G\{x, y\}$ is not 2-connected, then G/xy is not 3-connected.

Proof. For $G - \{x, y\}$ to be not 2-connected, there must be $\kappa(G - \{x, y\}) < 2$. Since G is 3-connected, we can't have $\kappa(G - \{x, y\}) = 0$, so the only left option would be $\kappa(G - \{x, y\}) = 1$; we want to show that $\kappa(G/xy)$ is at most 2 in this case.

This is because regardless what the v_{xy} will be in G/xy, we can simply remove it, which will give us the 1-connectivity $G - \{x, y\}$ graph back, then we shall remove that 1 cut-off vertex in $G - \{x, y\}$ which will disconnect G/xy. Since we have at most $\kappa(G/xy) = 2$, G/xy can't be 3-connected.

Since the contrapositive is fulfilled, we have proven the forwards direction.

□

$\Longleftarrow$ To prove by contrapositive, W.T.S. that if G/xy is not 3-connected, then $G\{x, y\}$ is not 2-connected.

Proof. Same as above, a not 3-connected G/xy implies $\kappa(G/xy)$ is at most 2. Also known that G is 3-connected, which implies at least $\kappa(G) = 3$. Combine them together, we know that vertices x, y must be both removed to disconnect G.

We know this because if none of x, y need to be removed to have a disconnected G where $\kappa(G) = 3$, then x, y and also v_{xy} are not influential to the connectivity of their graphs, so we won't have a $\kappa(G/xy) \leq 2$ as the same 3 vertices also need to be removed in G/xy to make G/xy disconnect, a contradiction.

If one of x, y but not both needs to be removed to to disconnect G where $\kappa(G) = 3$. W.L.O.G. say y is the one needs to be removed, then there must be two other vertices (say A, B) need to be removed to disconnect G. Which implies for vertices u, v (where u and v are from different

[1] This path must exist as there will be at least k distinct vertices connecting v_i or v_{i+1} to v_k, so we can always connect v_i to one of the vertex, reach v_k via a path, and get back to v_{i+1} from another vertex via another path. Notices we use two intermidiate vertices here, so the graph must be at least 2-connected

[2] Due to any two vertices in a 2-connected graph will have 2 vertex-disjoint paths between them. So by identifying the two desired the vertices and connecting the two vertex-disjoint paths between them, we will automatically have a cycle containing them.

components of the the disconnected G), path $u \to x \to v$ must internally shares vertex(ices) with path $u \to A \to v$, $u \to B \to v$, or path $u \to y \to v$. Then in G/xy we still need to break these 3 internally-vertex-disjoint paths to disconnect G/xy, which yields $\kappa(G/xy) = 3$, also a contraction to the setup.

So only if vertices x, y must be both removed to disconnect G, we shall then have a G/xy with $\kappa(G/xy) \leq 2$ as now we can just removed the contracted v_{xy} and have a $\kappa(G/xy) = 2$. However, in $G - \{x, y\}$, vertices x and y are already removed; this means $\kappa(G - \{x, y\}) = 1$ and $G - \{x, y\}$ is not 2-connected. This fulfills the contrapositive and proven the backwards direction.

$\square$