CSDS 455: Homework 14

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I have discussed with Yige Sun for this assignment.

Problem 1

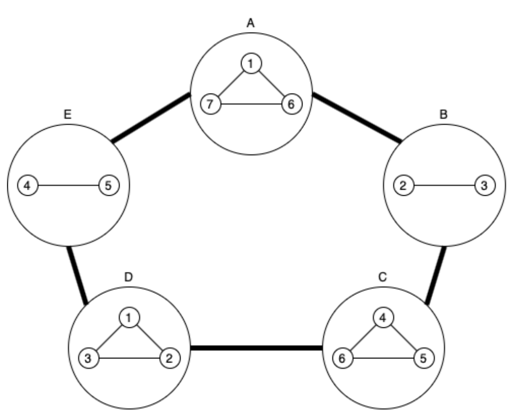
Proof.

Proof. **Lemma: For a graph with $\alpha(G)$ number of independent vertices, we have $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.**

Say we color such graph G with $\chi(G)$ colors and we group vertices according to their color. It is intuitive to tell that $\chi(G)$ will size of the color group with largest cardinality. Since vertices from different color group are not adjacent by definition, we know there will be $\alpha(G)$ color groups under the $\chi(G)$-colored G. Thus, it is save to say $|V(G)| \leq \alpha(G) \cdot \chi(G)$ as each color group can have at most $\chi(G)$ vertices and we have at most $\alpha(G)$ color groups. By doing a simple transformation we have $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

□

For the graph in question we have $|V(G)| = 3 \cdot 3 + 2 \cdot 2 = 13$, and we know that $\alpha(G) = 2$ as only two non-adjacent component will be independent from eachother, which implies $\chi(G) \geq \frac{13}{2} = 6.5 \implies \chi(G) \geq 7$. The graph is 7-chromatic as demonstrated below (each number represents a color):



Now we want to show that this graph does not containa subdivision of 7-clique. Note that each "component" of G has either 3 or 2 vertices and we know that a 7-clique has 7 vertices, this suggests that there should be at least two vertices in this 7-clique that are not adjacent to each other, but have 6-vertex-disjoint path between them.

So now we are picking "2-component" pairs out of the graph, we have A with C/D (namely a triangle with another triangle), C/D with B/E (a triangle with an edge), or E with B.

In the first case, we have 4 vertex-disjoint path from a vertex in A to a vertex in C/D (2 to B and 2 to E), $4 < 6$ so this case won't hold up. Similiarily we have 5 vertex-disjoint path in the second case (3 from D/C to E/B, 2 internally between D and C), $5 < 6$ so the second case won't hold up either.

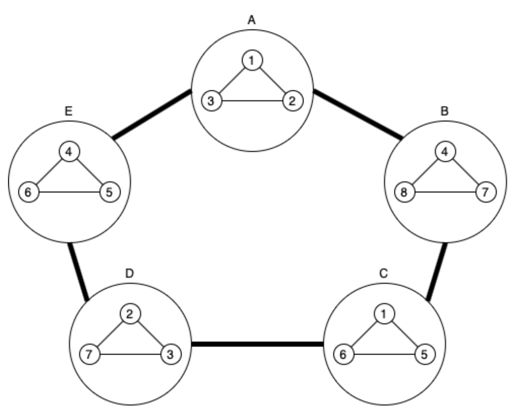
The third case has 6 vertex-disjoint path (as the edge is adjacent to two triangles). We need 7 of these vertices and we can find them by counting all vertices in "components" A, B, E. Note as we were asked for subdivision of 7-clique, so we will need at least one extra vertex between between the above 7 vertices, however all vertices in D, C (which are the only vertices left to use) do not meed this requirement as regardless which vertex you pick, if it is in D is must go through C and vice versa. Therefore this case eventually also won't hold up.

And as we have showed the graph cannot have a subdivision of a 7-clique, we have completed the prove of the statement.

$\square$

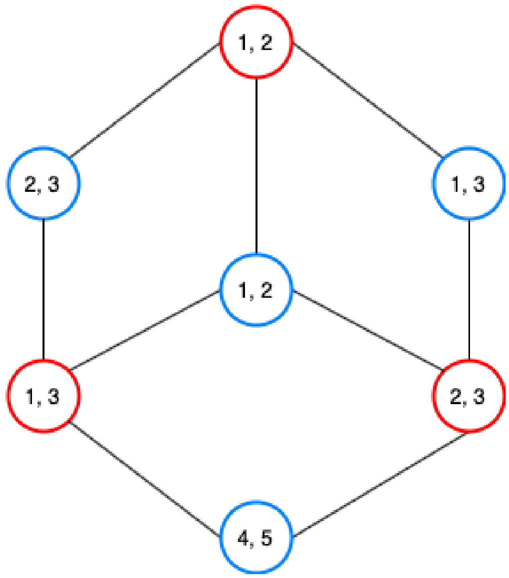
Problem 2

We first inspect the *lemma* introduced in problem 1. We have $|V(G)| = 3 \cdot 5 = 15$, and we also have $\alpha(G) = 2$ as only two non-adjacent "components" will be independent from eachother, which implies $\chi(G) \geq \frac{15}{2} = 7.5 \Longrightarrow \chi(G) \geq 8$. So the graph has the possibility for being 8-chromatic, we have provided one way of doing so below:



Now to show this graph does not contain a subdivision of an 8-clique. Knowing that an 8-clique need 8 vertices with 7 vertex-disjoint paths between every two of them, however for any vertex v from the graph, $d(v) = 6$. Therefore we have no subdivision of an 8-clique in this graph.

Problem 3



The shown graph is certainly 2-colorable (as colored in blue and red), but it is not 2-list colorable.

If we choose A to be 1, we will have $B = 3$, $C = 2$, and we are left with G with a $\{1, 2\}$ list but connected to both 1 and 2. Thus, the graph is not 2-list colorable in this case.

Similiarily, if we choose A to be 2, we will have $F = 3$, $C = 1$, and we are again left with G with a $\{1, 2\}$ list but connected to both 1 and 2. Thus, the graph is not 2-list colorable in every possible cases.

Problem 4

If a graph is k-list colorable, it means as long as we assign a k-sized color list to every vertex of the graph, we should be able to generate a proper coloring base on the lists. So, let us arbitrarily set the list for every vertex of a k-list colorable graph to be $\{c_1, c_2, ..., c_k\}$, by definition we should be able to get a proper coloring base on it. Since all lists only uses k colors, the graph is therefore k-colorable.