CSDS 455: Applied Graph Theory, Final Exam
Due Thursday, December 10 at 3pm EST

You have 3 hours to take this test. The test is closed book and closed notes. The two questions with the lowest grades will be dropped from your total.

At the end of the test, please write "I have neither given nor received aid on this examination, and I did not exceed the allowed time" and sign your name.

Problem 1:
Prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

Problem 2:
Prove that every regular bipartite graph has a perfect matching.

Problem 3:
Let G be a k-connected graph that is not a complete graph. Prove that for every edge e of G, if the result of contracting that edge is a graph that is no longer k-connected, then $G - e$ is a k-connected graph.

Problem 4:
Let T be a tree (undirected). Suppose we have k vertex disjoint paths in T such that path i goes from leaf u_i to leaf v_i. Consider two leaves a and b of T different from $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$, and consider the path from a to b in T. Suppose that the intersection of this a–b path with path i, for each i, is either $\emptyset$ or contains at least two different vertices. Prove that there exists $k + 1$ vertex disjoint paths in T between the leaves $\{a, b, u_1, \ldots, u_k, v_1, \ldots, v_k\}$.

Problem 5:
Let G be a simple plane graph in which all vertices have a degree that is a prime number $(2, 3, 5, 7, \ldots)$. Furthermore, assume that G requires 4 colors to properly color its vertices but any subgraph of G requires only three colors. Prove that G must have either a vertex of degree 3 on a face of size 5 or less or a vertex of degree 5 incident to 4 faces of size 3.

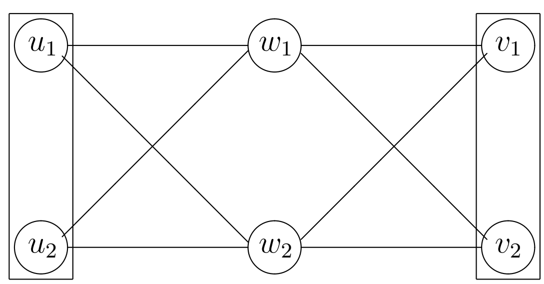
Problem 6:
Let graph G require $\Delta(G) + 1$ colors to properly color its edges. Assume G is critical for this coloring. Prove that G is bridgeless.

Problem 7:
Let G be a directed graph, let xy be an edge and assume $G - xy$ is bridgeless. Prove that if G has a nowhere-zero 2-flow then $G - xy$ has a nowhere-zero 3-flow.

Problem 8:
Prove that having treewidth at most k is a hereditary property.

Problem 9:
We know that if each graph in a set of graphs has treewidth at most k, for some constant k, then we can solve problems such as Hamiltonian cycle, graph isomorphism, vertex cover, and 3-color on these graphs using dynamic programming where the running time of the algorithm is a polynomial on the size the graph. Explain why dynamic programming is able to solve these problems and can work in polynomial time.

Problem 10:
Let α be the following shape.



Describe entries of the matrix M_α representing this shape on a Erdös-Rényi random graph $G(n, 1/2)$.