CSDS 455: Homework 16

Shaochen (Henry) ZHONG, `sxz517`

Due and submitted on 10/19/2020
Fall 2020, Dr. Connamacher

Problem 1

I have consulted https://www.cs.cmu.edu/~ckingsf/bioinfo-lectures/matching.pdf *for this problem.*

Proof. Given a bipartite graph G with partitions U, V. We will make a G' with two new verticies s, t where S is directionally connected to all the verticies in U, likewise, t is directionally connected to all the verticies in V. We direct all the edges from U to V, then make all edges in this G' with capacity of 1. We will find the maximum flow f of this G', where edges used in f that is also in G is the larges possible matching in G.

We denote the edges used in f that is also in G as M. We know that M is a matching as every edge from $U \to t$ in f has a capacity of 1, since the input and output flow of a vertex is balanced, this menas if we have multiple verticies from U that is connected to a same vertex in $v \in V$ in M, then we will know that edge vt will have a > 1 flow in f, which is a contradiction to our flow capacity setup. So M must be a matching of G.

We also know that M is a maximum matching of G. Known M's corresponding f is the maximum flow of G' and considered every edge in G' has a flow value of 1. If we ever have a M' that is a bigger matching than M, then it must have a corresponding f' that has a bigger flow value than f – which is, again, a contradiction to the setting of f. So, M must be the maximum matching of G.

$\square$

Problem 2

I have consulted https://www.cs.princeton.edu/courses/archive/spr04/cos226/lectures/maxflow.4up.pdf *for this problem.*

Proof. Define two non-adjacent vertices $\in V(G)$ as s, t, then find out the set of vertex-disjoint paths from s to t as P. We define our flow f as a directed vertion of P with each edge having a flow capacity of 1. Also when ever we add a vertex $v \in V(P)$ to our flow, we erase all the edges entering v that are not in $E(P)$; we denote the graph after the flow is fully constructed (and erasion is fully done) as G_f.

We know f is a legal flow construction for G_f as (for u not being s, t) if we have an $u \in V(G_f)$ but $\not\in V(P)$, then there is no input nor output flow on u; likewise if we have an $u \in V(G_f)$ and also in $\in V(p)$ for $p \in P$, since p is a vertex-disjoint path so this u must be degree 2 with 1 flow input and 1 flow output. The input and output of flow is balanced in both cases.

We also know that f is a maximum flow for G_f, as otherwise if there is an augumenting path q from s to t, such q must not share any edge with f (as edges in f are at their full capacities), it will also not share any verticies in f (as other edges connected to verticies in P are removed). So q must be a path from s to t on vertices outside of $V(P)$, this is a contradiction to the setup of f as f should be the directed version of all vertex-disjoint pathes from s to t.

Proof. **Lemma (max-flow-min-cut theorem): The value of the max flow is equal to the capacity of the min cut.**

Denotes the maximal s,t flow to be $f(s,t)$. For a random $s-t$ cut, we denote the flow value of the cut to be $c(s,t)$ and the capacity of the cutted edges to be $C(s,t)$. Then we must have:

$$c(s,t) \leq f(s,t) \leq C(s,t) \tag{1}$$

This suggest, by doing a "min cut" – making a $s-t$ cut by removing minimum necessary edges where each edge's flow value is equal to its capacity – after a max flow is achieved, we have $c(s,t) = C(s,t)$ and will therefore also equals to $f(s,t)$. Thus the statement is proven. □

Known that the f constructed based on P is a max flow where all of its edges have a 1/1 value/capacity status. So by removing an edge from each vertex-disjoint path's projection in f, we have obtained a "min cut" of the graph with the value of the cut being equal to the number of disjoint pathes in p – which is also how MENGER's theorem determines the minimum cut required to disconnect the graph.

□

Problem 3

I have discussed with Yige Sun for this problem.

We define:

- $a = cap(S \cap T, S \cap \bar{T})$
- $b = cap(S \cap T, \bar{S} \cap \bar{T})$
- $c = cap(S \cap T, \bar{S} \cap T)$
- $d = cap(S \cap \bar{T}, \bar{S} \cap \bar{T})$
- $e = cap(\bar{S} \cap T, \bar{S} \cap \bar{T})$
- $f = cap(S \cap \bar{T}, \bar{S} \cap T)$
- $g = cap(\bar{S} \cap T, S \cap \bar{T})$

First we analyse $cap(S \cup T, \overline{S \cup T})$, it is basically asking about edges going from $S \cup T$ to $\bar{S} \cap \bar{T}$. So we have:

$$\begin{aligned} cap(S \cup T, \overline{S \cup T}) = \; & cap(S \cap T, \bar{S} \cap \bar{T}) \\ & + cap(S \cap \bar{T}, \bar{S} \cap \bar{T}) \\ & + cap(\bar{S} \cap T, \bar{S} \cap \bar{T}) \\ = \; & b + d + e \end{aligned}$$

Similarily, we have $cap(S \cap T, \overline{S \cap T})$ to be edges from $S \cap T$ to $1 - (S \cap T)$

$$\begin{aligned} cap(S \cap T, \overline{S \cap T}) = \; & cap(S \cap T, \bar{S} \cap \bar{T}) \\ & + cap(S \cap T, \bar{S} \cap T) \\ & + cap(S \cap T, S \cap \bar{T}) \\ = \; & b + c + a \end{aligned}$$

So we now that $cap(S \cup T, \overline{S \cup T}) + cap(S \cap T, \overline{S \cap T}) = a + 2b + c + d + e$. Then for $cap(S, \bar{S}) + cap(T, \bar{T})$, we have:

$$\begin{aligned} cap(S, \bar{S}) + cap(T, \bar{T}) &= (b + c + d + f) + (a + b + e + g) \\ &= a + 2b + c + d + e + g \\ &\geq a + 2b + c + d + e \\ \implies \; & cap(S \cup T, \overline{S \cup T}) + cap(S \cap T, \overline{S \cap T}) \leq cap(S, \bar{S}) + cap(T, \bar{T}) \end{aligned}$$

The statement is therefore proven.