CSDS 455: Take Home Midterm

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Problem 1

Fundation: every complete even graph has a perfect macthing by itself.[1]

We first observe that the number of odd components of $G - X$ must be less than (not equal to) $|X|$. Knowing that vertices in X will connect to every other verticies in G, we may group every odd component of $G - X$ with one vertex in X and find a perfect matching out of them (as one odd component should have a matching in itself except one vertex, then by matching this leftover vertex to a vertex in X, we have a perfect matching). If the number of odd components of $G - X$ equals to $|X|$, then with the just mentioned grouping operation we will have $|X|$ even components and couple even components in the original $G - X$ set. Then G can't have odd number of vertices, a contradition of the setting.

In the case of the number of odd components of $G - X$ is less than $|X|$. We denote odd components in $G - X$ as $O = \{O_1, O_2, ...\}$ and likewise $E = \{E_1, E_2, ...\}$ for even components. To remove an edge v, v must be among X, O, or E.

For $v \in X$, with the just mentioned grouping operation between X and Os, we will have couple even complete components (all have a perfect matching by themself) and the leftover of verticies of X. This means the leftover of X (after grouping) must be odd in number of verticies, then by removing one v now the leftover of X will be complete and even and therefore has a perfect matching. As every component in $G - v$ now has a perfect matching, $G - v$ has a 1-factor.

For $v \in O$ then we have one odd component O_i become even. By doing the same grouping operation between X and rest of the Os. We have Es, some even components created by X and Os, $O_i - v$, and leftover of X (must be even in verticies as all others are even and $G - v$ is even) – since all of them are complete and even, an 1-factor can be found.

For $v \in E$, then we have one even component E_i become odd. Since we know $|X| > |O|$, so even with one more odd component we can do the grouping operation between X and all the Os. Now we have Es, some even components created by X with Os or E_i, and leftover of X (if any). Since the formor two have even cardinality, the leftover of X must be even. Thus, as all of them are complete and even there will be an 1-factor for $G - v$.

We have showed the statement to be true by justifying all cases.

[1] Because every vertex is connected to all other vertices, we may therefore find an arbitrary vertex order of graph G like $\{v_1, v_2, v_3, ..., v_k\}$; by choosing the edges of $v_1v_2, v_3v_4, ...$ we will have a perfect matching of G.

Problem 2

Refer to Dr. Connamacher's MDST algorithm, we have learned that for vertices x, y in a spanning tree T where $d_T(x, y) = diam(T)$, there will always be a midpoint of $x \to y$ path χ on the edge between s_1 and s_2. This implies if we try different χs on different edges and check on their $d_T(x, y) = diam(T)$ respectively, we should be able to locate a tree that has the minimum $diam(T)$.

To generate a SPT rooting from χ on edge s_1s_2, we may have a d_T with the minimum $\max d_T(s_1, u) + \max d_T(s_2, v) = diam(T)$ for $u, v \in V(G)$[2] as this is the nature of SPT. We learned that there are at most $|V| + 1$ different SPTs can be produced (in a sense of $d_T(s_1, v) \neq d'_T(s_1, v)$ W.L.O.G.) by position χ differently on s_1s_2. Known there are $|E|$ edges in G, we will create $(|V| + 1)|E| = O(VE)$ SPTs.

Use Dijikstra to create SPTs, which has a $O(E \log(V))$ runtime with adjacency list implemented. Also to inspect the $diam(T)$ of each produced SPT, an $O(V + E)$ BFS is required. So to position, create, and inspect all SPTs, we will have a time complexity of $O(VE^2 \log(V) \cdot (V + E))$ in total, a polynomial of number of vertices and edges of G.

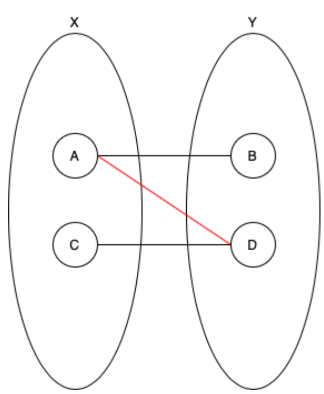
Problem 3

By Menger's theorem we know that for a connected undirecrt graph G, the minimum vertex cut for $u, v \in V(G)$ is equal to the maximum number of vertex-disjoint paths from u to v. By promoting this theorem to all vertices pairs in G, it implies a k-connected graph will have k vertex-disjoint pathes between any vertices pair in G.

So for a $k + 2$ vertex connectivity graph (which is also $k + 2$ connected), there will be $k + 2$ vertex-disjoint pathes between any vertices pair in G. And as the removal of k verticies can at most break k vertex-disjoint pathes, 2 more pathes are left and we may therefore have a cycle between any vertices pair in G. So $k + 2$ is the minimum vertex connectivity needed for a k-resilient graph.

Problem 4

By Hall's theorem we know there will be a matching of size $|X|$ by the given condition, but I don't think it will be the case that every edge of G is part of some matching of size $|X|$. Considering the following counter example:



[2] They also must be leaves of d_T as otherwise we can make the path longer by proceeding to an leave.

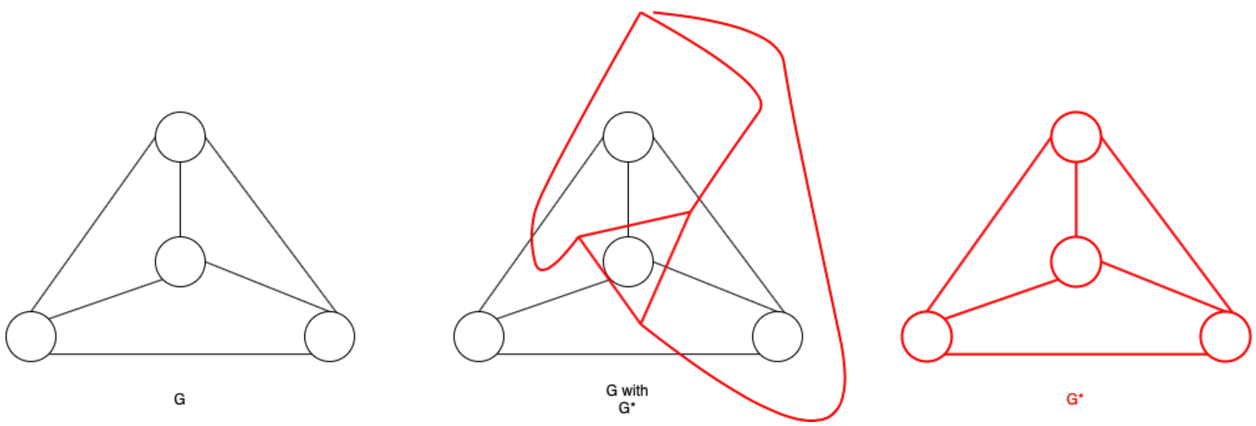
By selecting AD as a matching edge, we have vertex $A \subseteq X$, we have $|N(A)| = 2$ and $|A| = 1$ therefore $|N(A)| \geq |A|$, so this is a legit graph per requirements. However, edge AD is not in a matching of $|X| = 2$ as it will be the only edge in this matching.

My time is running short but I think this question is provable with the adjustment of $|N(A)| > |A|$ (but not equal). This is because for $A \subseteq X$ we might have an at most $|X|$ matching; and since $|N(A)| > |A|$, when $A = X$ this means we have more edges (than vertices in A) available to choose to form a $|X|$ matching. Since the graph is bipartite and $A = X$, all edges in G coorsponding to an $(A, N(A))$ pair. And we can therefore pick any desired edge first and pick the rest $|X| - 1$ edges between the two partition to fullfill a $|X|$ matching.

Problem 5

By the *Eular* therom we have $n - e + f = 2$ for n, e, f representing the number of verticies, edges, and faces of a plane graph G. For dual G^* we have $n^* = f$, for isomorphic to G we have $f^* = f$; thus $n^* - e^* + f^* = 2f - e^* = 2$, which implies $e = 2f - 2 = 2n^* - 2$ and we have $e = 2n - 2$ again due to isomorphic.

An example will be the following:



Note G has 4 verticies and $2 \cdot 4 - 2 = 6$ edges.

I have neither given nor received aid on this examination, and I did not exceed the allowed time. – HZ.