CSDS 455: Homework 13

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Problem 1

I consulted https://www.youtube.com/watch?v=otky1bBhwgM *for this problem.*

Proof. We will have the following diagram for a base graph:

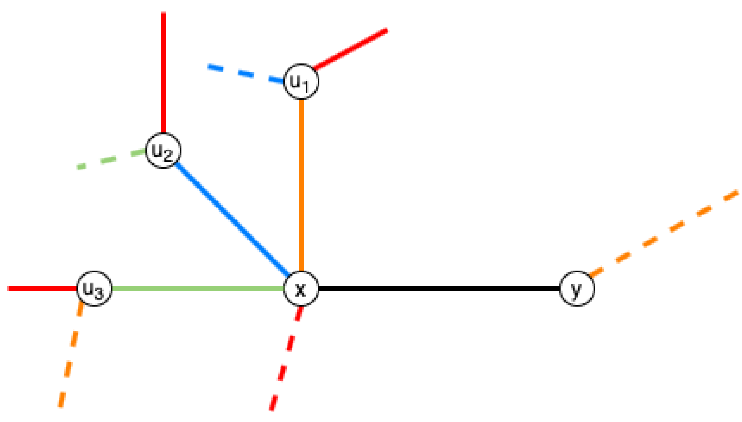


Figure 1: Base graph for *Problem 1*

For better clarity, we have $u_1, u_2, ..., u_i \in N(x)$ being the adjacent vertices of x excluding y. Known that $\chi'(G - xy) = \Delta + 1$, every vertex in $G - xy$ must be missing[1] at least one color from the $\Delta + 1$ colors, we use dashed lines to represent such colors. We also use solid lines to represent the color of edges under a $\Delta + 1$ edge-coloring of $G - xy$. We will also use some arbitary actual color names to help understanding.

We want to show that other than having an 2-color alternating path between x and y, we will always have $\chi'(G) = \Delta + 1$, thus a contradiction.

Proposition 1. x and y can't miss the same color.

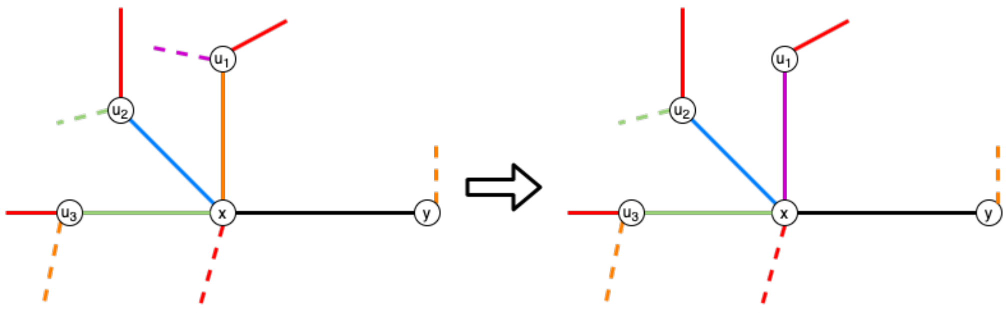
This observation is almost trivial as otherwise we may just color edge xy with this color, then we have a $\Delta + 1$ coloring of G – a contradiction.

[1] "Missing" in this context means not having a edge of a certain color. "Vertex v is missing color c" means there is no edge of color c connected to v

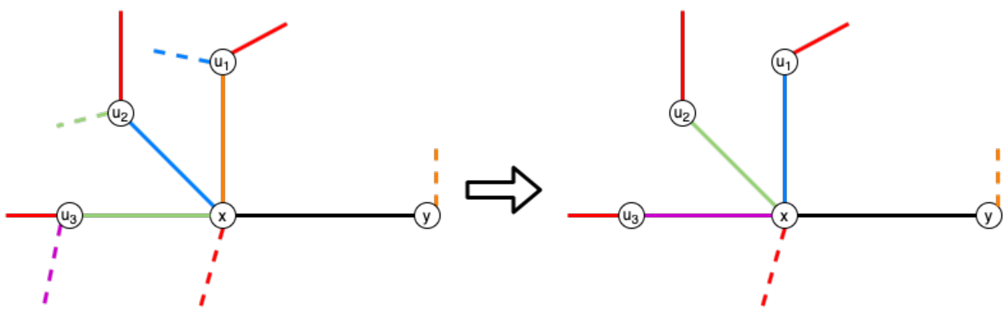
Proposition 2. ***x* must include every missing color of *$N(x)$*.**

Say we have a set S which include all the missing colors of $N(x)$, such S must be a subset of the colors connected x. This is because if we have a vertex $u_i \in N(x)$ which has a missing color of purple which is not included in the connected colors of x, we have two cases.

First, u_i is u_1, where edge u_1x has the color which y misses, then we may recolor the graph as following. Which will make x and y both missing orange; we can therefore color xy with orange and have a $\Delta + 1$ coloring of G – a contradiction.



Second, we may have u_ix not having the color which y misses. We can then recolor u_ix to be purple, and recolor the original blue u_jx (u_j is another neighbor vertex of x which is missing green) to be green, and recolor another neighbor vertex of x which is missing the color of u_jx... Take $u_3 = u_i$ as an example, we will have:

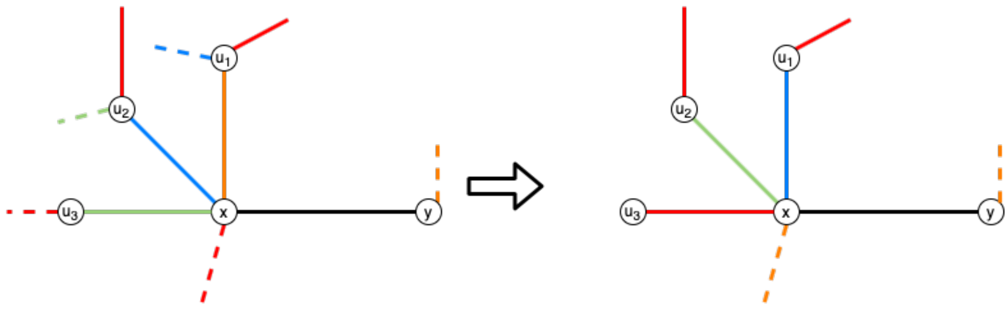


We know this recoloring method works since $N(x)$ is finite, so we will always able to recolor u_1x (orange, which y misses) to the missing color of u_1 (blue), then recolor the blue edge u_jx to its missing color, then .... we will eventually reaches u_i and recolor u_ix to its missing color purple. Then we will have x and y both missing orange and thus a contradiction.

Now we have showed that every missing color of $N(X)$ must show on an edge of ux for $u \in N(x)$. Be familiar with this recoloring method as we will use it later.

Proposition 3. **If *x* misses red. Every *$u \in N(x)$* must have an red edge connected to *u*.**

Because otherwise, by using the recolring method introduced in **Proposition 2**, we may have x and y both missing orange and thus a contradiction. The following is an example assuming u_3 missing red, but it can be any $u \in N(x)$.

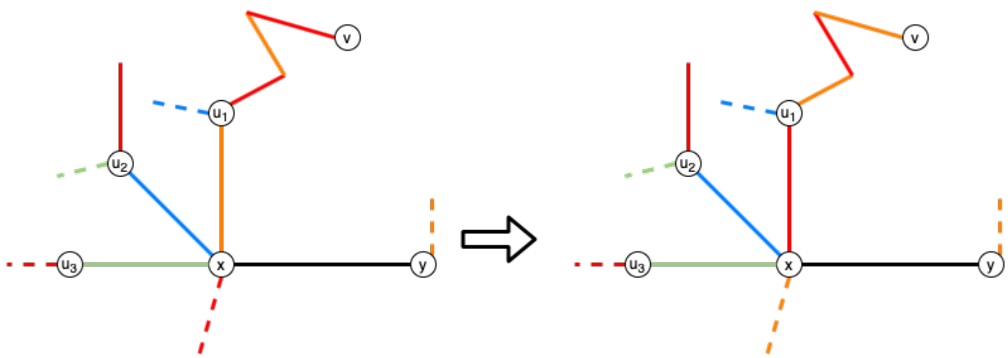


Combining the finding of **Proposition 1, 2, 3**, we will have a graph like [Figure 1].

Proposition 4. If x misses red and y misses orange, the maximal orange-red alternating path from x can't end on a vertex $\notin N(x)$ and not y.

If we consider a subgraph G' with only orange and red edges by the coloring of $G - xy$, we know that every vertex in G' will have a degree of 0, 1, or 2. Now we take a orange-red alternating walk P from x, until it reaches a vertex $v \in V(G)$ where we can't extend the walk any farther (which implies $d(v) = 1$). We let $v \notin N(x)$ and $v \neq y$ in this case.

We can't have this kind of v as we may simply interchange the orange-red coloring of P. We may then have x and y both missing orange and thus a contradiction. So we know a orange-red alternating walk P from x can't end on a vertex outside of $N(x)$ and y.



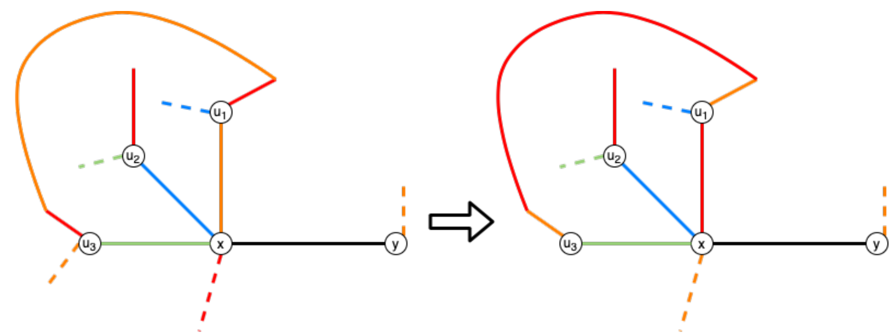
Proposition 4. If x misses red and y misses orange, the maximal orange-red alternating path from x can't end on a vertex $\in N(x)$.

It is trivial to tell a orange-red alternating walk P from x can't end on u_1 (where xu_1 is the first edge of P). P may either end on a u_i where u_i misses orange (like u_3 in the below graph); or P will visit another $u_j \in N(x)$ which has orange edge connected to it, but in this case this P will not end on $N(x)$.

We first show the case of P ending on a vertex $u_i \in N(x)$ where u_i misses the same color as y (orange). In this case we interchange the color in P as following and we have have x and y both missing orange – a contradiction.

In the case of P ending on a vertex $u_j \in N(x)$ where u_j has orange edge connected to it. We can tell P will not end on $N(x)$ as:

1. $u_{j-1}u_j \in E(P)$ is orange, then we may extend this P farther to include the red edge connected to u_j.
2. $u_{j-1}u_j \in E(P)$ is red, in this case we may extend this P farther by including the orange edge connected to u_j.

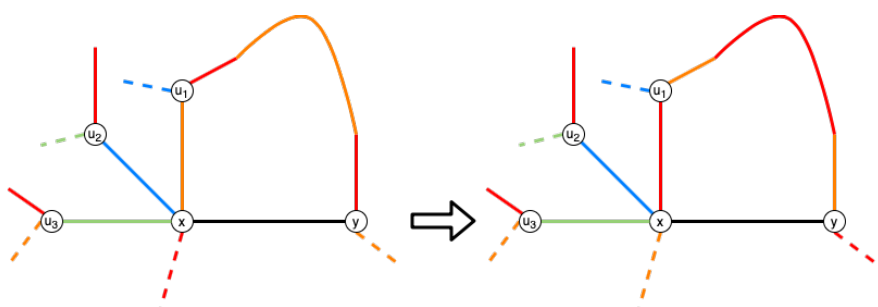


Either of the case will make P end on a vertex v outside of $N(x)$ and y. According to **Proposition 2**, this will give us a contradiction.

Conclusion: P must end on y.

By **Proposition 3, 4**, we know that a orange-red alternating walk P from x can't end on $N(x)$, can't end on $G - y$, and obviously it can't end on x itself as x misses red. So the only left option it to end on y.

In the following diagram we will show by having P ends on y, we can't make a contradiction by doing the recoloring operation.



It is clear that although an interchange recoloring of P is possible, we only swapped the missing color between x and y, so we can't color xy with an existing color and no contradiction is made.

The statement in question is therefore proven.

□

Problem 2