CSDS 455: Homework 3

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CSDS 455, Dr. Connamacher

1 Problem 1

Proof. To prove by induction:

Due to the given $|S| \leq |N(S)|$, there must be $|A| \leq |B|$. Since we know there is a free vertex $a \in A$ and $\lfloor |B| \rfloor = |A|$, we know that there must be at least a free vertex in B. It is because if all vertices in B are matched, all vertices must be matched in A as well – which is a contradiction to the exsistence of a. We denote the set of free vertex(ies) in B as F_b.

We also know that some free vertex $b \in B$ is connected to at least a vertex in A. This is because if we take all matched vertices in A (denote as M_A) plus a free vertex $a \in A$ as S, we should have at least $|M_A| + 1$ vertices in B connected to this S. Since there are only $|M_A|$ matched vertices in B, there must be a free vertex b connected to at least a vertex in A, we denote this vertex in A as A_k.

If A_k is a free vertex, edge $< A_k, b >$ will be a single-edge argumenting path connecting two free vertex. We then set A_k to be a, the statement is trivially proven as an argumenting path from a is found.

However, if A_k is matched vertex (to B_k), and there is no single-edge argumenting path in G (otherwise the statement is instantly proven), then we will have the following diagram:

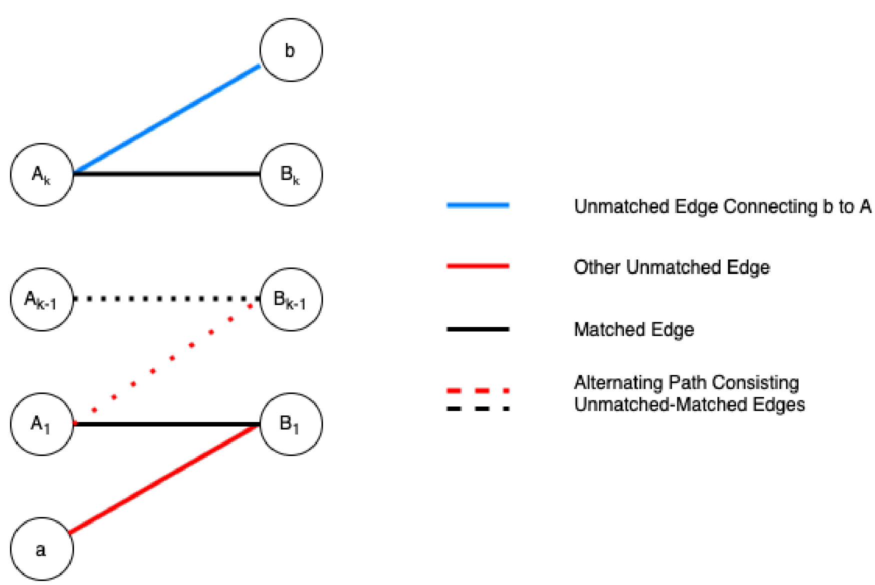

Figure 1: Structure of G without any single-edge argumenting path

With B_k being the matched vertex connected to A_k, we inspect the longest alternating path P_{k-1} starting from a. Assume we reached a matched vertex $A_{k-1} \in A$ as a stopping point of P_{k-1}. We will try to show that A_{k-1} must connect to B_k. Note we delete matched edges that are isolated to the rest of the graph, like matched edge $< A_o, B_o >$ where both $A_o \in A$ and $B_o \in B$ have a degree of 1; as they will never be part of an argumenting path, nor have any effect on any potential argumenting pathes.

Inspect a path P_1 from a to A_1, we know that A_1 must be conncted to a vertex in B. This is because $|P_1 \cap A| = 2$, but $P_1 \cap A$ is only conncted with 1 vertex $\in B$ (B_1). This means at least a vertex in $P_1 \cap A$ must have connected with a vertext $\in B$ other than B_1, we denote this vertex as A_T.

- If A_T is free vertex and is conncted to a free vertex $B_T \in B$, we have a single-edge argumenting path $< a, B_T >$ and the statement is solved. This case is avoided in setup since it is trivial.
- If A_T is a matched vertex and conncted to a free vertex $B_T \in B$, we have a argumenting path of $< a, ..., A_T, B_T >$ (in this case, $< a, B_1, A_1, B_T >$). The statement is proven.
- If A_T is a matched vertex and conncted to a matched vertex $B_T \in B$, we let this B_T be B_2, connect this $< B_T, B_2 >$ with an unmatched edge and continuing the induction.

We showed that for an alternating path P_1 starting from a and stops on A_1, we must either be able to find an argumenting path upon this P_1, **OR** P_1 can be extended by connecting to a matched node in B but not in P_1. Assume it is also true for P_{k-1}, and say there are only k matched edges in M (isolated matched edges excluded).

We know that this alternating path P_{k-1} has traversed k vertices in A, but due to a is connected to a matched vertex in B (as otherwise we have a single-edge argumenting path direcly), there are only $k-1$ vertices in $P_{k-1} \cap B$. By making this $P_{k-1} \cap A$ as S, there must be at least k vertices conncted to this S – which means there must be a vertex in $P_{k-1} \cap A$ that is conncted to some other vertex in B that are not in $P_{k-1} \cap B$, we again denote this vertex as A_T.

The first two cases are essentially same as above, that we can find an argumenting path upon this P_{k-1}. The interesting part is in case of A_T being a matched vertex conncted to a matched vertex in B, but not in $P_{k-1} \cap B$. Since there are only k non-isolated matched edge in G, this A_T can only connect to B_k with an unmatched edge. Since we have a `matched - unmatched` path on $< B_k, A_k, b$, by connecting this P_{k-1} to B_k with an unmatched graph, we have an argumenting path of $< a, B_1, A_1, ..., A_T, B_k, A_k, b$. Since $< a, B_1, A_1, ..., A_T, B_k, A_k >$ is P_k, and we must able to build an argumenting path upon P_k, the induction logic is maintained. The statement is therefore proven.

$\square$

2 Problem 2

I worked with Yuhui Zhang on this problem.

We denote $M(G)$ to be the maximum matching of G, and $|M(G)|$ to be cardinality of edges of $M(G)$. Let $def(S)$ to be the deficiency of any S, which implies $Def(A) = \max_{S \subseteq A} def(S)$.

Showing $|M(G)| \leq |A| - Def(A)$

Proof. Let $S' \subseteq A$ to be a set with maximum deficiency. Which implies for any matching M, there will be at least $def(S')$ unmatched vertices $\in S'$ (and therefore also $\in A$). This suggests: $|M(G)| \leq |A| - def(S') \Longrightarrow |M(G)| \leq |A| - Def(A)$.

$\square$

Showing $|M(G)| \geq |A| - Def(A)$

Proof. Let $M(G) = A - k$ where k being the unmatched vertices in A. We may have a $S' \subseteq A$ where $def(S') = k$. Since this is just a single $S' \subseteq A$, there might be another $S'' \subseteq A$ or $S''' \subseteq A$ with greater deficiency. Thus we have:

$$\begin{aligned}
\max_{S \subseteq A} def(S) &\geq k \\
|A| - k &\geq |A| - \max_{S \subseteq A} def(S) \\
|A| - k &\geq |A| - Def(A) \\
|M(G)| &\geq |A| - Def(A)
\end{aligned}$$

$\square$

Since we have $|M(G)| \leq |A| - Def(A)$ and $|M(G)| \geq |A| - Def(A)$, there must be $|M(G)| = |A| - Def(A)$. The statement is therefore proven.

3 Problem 3

I consulted http://www.sfu.ca/~mdevos/345/homework6_sol.pdf *for this problem.*

Proof. Proving forward of *iff*:

By definition, a graph with a perfect matching must have even number of vertices. This means $|T|$ must be even. Removing a vertex $v \in V(T)$ from this T will left us a connected tree T' (thus one component) with odd number of vertices. This mean there will be $q(T-v) = 1$ for all $v \in V(T)$. $\square$

Proof. Proving backward of *iff*:

We know that $|V(T)|$ must be even, as otherwise we will have $q(T-v) = 0$, which is a contradiction of the condition. We represent this finding as $|V(T)| = 2k$, $k \in \mathbb{Z}^+$. For $k = 1$, we have $|V(T)| = 2$, which clearly have a perfect matching.

From the base case, we assume that there will be at least one perfect matching for every $|V(T)| < 2k$ ($k \in \mathbb{Z}^+$). We want to show that this is also the true for $|V(T)| = 2k$.

We locate a leaf ℓ of v for all $v \in V(T)$, we know that $|T - v - \ell|$ will have even number of components because $|T|$ is $2k$. This means $|T - v - \ell| = 2k - 2 = 2(k-1) < 2k$, thus $T - v - \ell$ has a perfect matching by it own. Now we inspect vertex v and ℓ, by connecting an edge $< v, \ell >$, this will be another perfect matching. Now every vertex in $T - v - \ell$ are matched in a perfect matching, plue vertex v and ℓ are also matached in a perfect matching. We combine these two perfect matching, there will be a perfect matching for every $|V(T)| = 2k$.

Since a perfect is found for $|V(T)| < k$ and $|V(T)| = 2k$, with $k \in \mathbb{Z}^+$ and $q(T-v) = 1$. We conclude that there will be a perfect matching for every T with $q(T-v) = 1$ by induction.

$\square$