CSDS 455: Homework 7

Shaochen (Henry) ZHONG, sxz517

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Problem 1

I consulted https://math.stackexchange.com/questions/666997/ *for this problem.*

With the Cayley's Formula (learned on Monday's class), we know that a K_n graph will have n^{n-2} spanning trees. We may create a bipartite graph with partitions T, E where each node $t \in T$ represent a spanning tree of K_n, and each $e \in E$ represent an edge in K_n. Then we connect these two partitions if $e \in E(t)$.

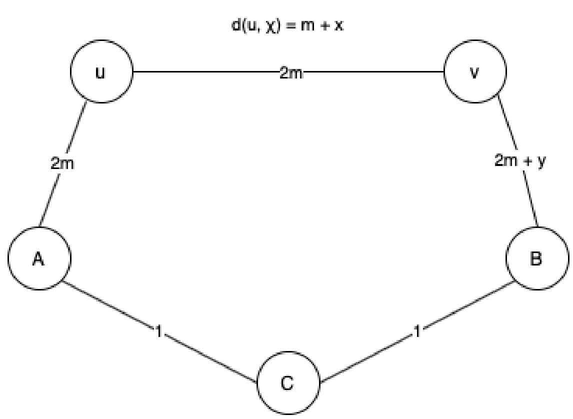
It is known that each spanning tree of K_n has $n-1$ edges, and we know that there are n^{n-2} nodes $\in T$. Therefore the bipartite graph will have $(n-1) \cdot n^{n-2}$ edges.

Due the nature of K_n, every edge of it is equivalent to another; this implies the number of spanning trees containing an edge will be the same as the number of spanning trees containing any other edge. Therefore each edge is contained by $\frac{(n-1) \cdot n^{n-2}}{\binom{n}{2}} = 2n^{n-3}$ (as there are $\binom{n}{2}$ edges in K_n). We substract this number from the total number of spanning tree of K_n, which is same as removing an edge e from K_n, and there will be $n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$ spanning tree left in $K_n - e$.

Problem 2

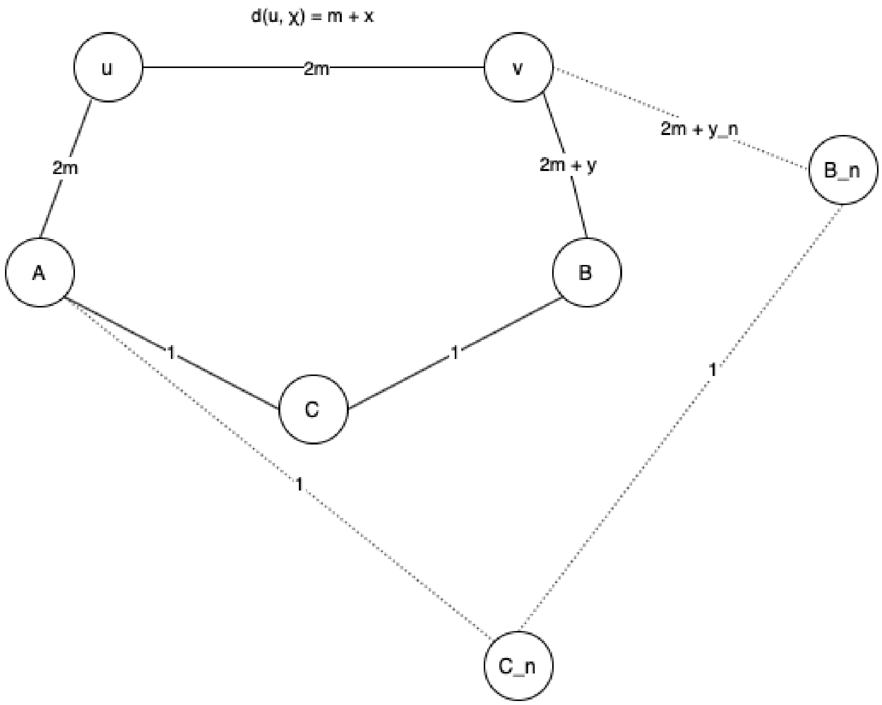
There will be infinite number of classes. First it is easy to tell if edge uv is not in the spanning tree, we may have two classes of spanning trees with the first class starts from χu, and the second from χv.

However the interesting part is if uv is contained in the spanning tree. For the ease of description let $d(uv) = 2m$, $d(u, \chi) = m + x$ $(x \geq 0)$, and $y > 0$, we may have the below base cases:



For T_1, assume χ is in the middle of uv (which implies $x = 0$), in this case we shall have $d(\chi \to u \to A \to C) = m+2m+1 = 3m+1$ and $d(\chi \to v \to B \to C) = m+2m+y+1 = 3m+y+1$. So clearly we should connect C to A, and we have $d_{T_1}(u, C) = 2m + 1$.

However, if we let $d(u, \chi) > d(v, \chi)$ on uv, to a point that $x > y$ in T_2. In this case we shall have $d(\chi \to u \to A \to C) = 3m+x+1$ and $d(\chi \to v \to B \to C) = m-x+2m+y+1 = 3m-x+y+1 < 3m + 1$. So we should connect C to B, and $d_{T_2}(u, C) = 2m + 2m + y + 1 = 4m + 1 \neq d_{T_1}(u, C)$



Since it is known that once this $x > y$ condition is established, the shortest path tree must detach C from A and connect it to B, this will make a new shortest path tree T_n with $d_{T_n}(u, C) \neq d_{T_1}(u, c)$.

Note that we may have infinite edges like vB and vertices like C; e.g. vB_n and C_n, with a length of $d(v, B_n) = 2m + y_n$, $d(B_n, C_n) = 1$, and $d(A, C_n) = 1$. And as we increase the value of x on $d(u, \chi)$ (a.k.a moving χ passes midpoint of uv and towards v to make $\chi', \chi'', \chi''', ..., \chi_n$) to be x_n, we may find infinite pairs of (x_n, y_n) where $x_n > y_n$, and thus we have infinite classes of non-equivalent spanning trees.