CSDS 455: Homework 12

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Problem 1

Assume we have graph G' which is identical with G, but already colored by $\chi(G)$ colors. We may have a vertex ording of $v_1, v_2, ..., v_n$ (assume $|V(G)| = n$), where v_1 to v_i have color c_1, v_{i+1} to v_j have color c_2, ..., and v_k to v_n have color $c_{\chi(G)}$ (for $i < j < k < n$).

This votex odering will guarteen a $\chi(G)$-coloring of G. As if the a vertex v has a color of c in G', then the next vertex v' from the above ordering will either have the same color as v in G', in this case the greedy algorithm will color v' the same color as v; or it will have a different color, in this case the greedy algorithm will try all "used" colors[1] – which obviously not going to work as G' represents the minimum colored graph of G – then the algorithm will give v' a new color. Following this order the algorithm will color G just as G', which contains $\chi(G)$ colors.

Problem 2

Let G being a bipartite graph, where each of its part partision has n vertices (so $2n$ vertices in total). We denote the vertices in partition U to be $u_1, u_2, ..., u_n$, and likewise for vertices in partition V.

For a vertex $u_i \in V(U)$, we connect it to all vertices in U except to v_i; we do the same to all vertices in U. The order that produce a $2n$-coloring of G would be $u_1, v_1, u_2, v_2, ..., u_n, v_n$.

It is because if a u vertex and its next v vertex have the same subscript, they will not be connect and the greedy algorithm may assign the same color to them. However, if a v vertex and its next u vertex have the different subscripts (specifically, v_i and u_j where $j = i + 1$), the u vertex will be connected too all currently colored v vertices, this means the greedy algorithm will have to assign a new color for it. Since we have n pairs of neighbor (v_i, u_j) where $j = i + 1$, n new colors are were assigned and the a n-coloring of G is obtained.

Problem 3

We first want to show for any k-chromatic graph has a k-critical subgraph: For any given k-chromatic graph G, if it is a critical k-critical graph by itself, then we are done. If G itself is not a k-critical graph, then there must be a $v \in VG$ where $\chi(G - v) \geq \chi(G)$. Since removing a vertex will not increase the minimum coloring of a graph, for this vertex we must have $\chi(G - v) = \chi(G)$. If the new G (with v removed) is still not k-critical, we will remove another v where $\chi(G - v) = \chi(G)$,

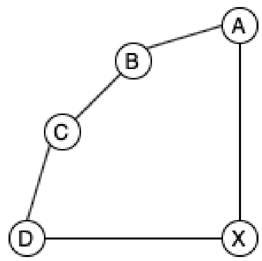
[1] Colors used on vertices before v' in the listed ordering

untill we have a k-critical subgraph out of G.

We will then show every vertex in k-critical graph G' has a minimum degree of $k-1$. Assume there is $v \in V(G')$ where $d(v) = k-2$. Due to the property of being a k-critical graph, we must have $\chi(G'-v) <= k-1$. Since x has only $k-2$ neighbors, we may always color x in G' with the $k-1$-th color used in $G'-v$. This suggests G' is $k-1$ colorable in stead of k, which is contradiction to the setup of G'. Thus, if G' is k-critical, $\delta(G') \geq k-1$.

Problem 4

We will introduce a way to construct graph G that fits the question requirements. We first draw a vertex X, then we draw $\Delta(G)$ number of vertices (we denotes them alphabetically). In the base case of $\Delta(G) = 4$, we may have the following graph for fundation.



We will notice between the first and last alphabetically named vertices, there will be $\Delta(G) - 2$ vertices in between (non-inclusive), we call the path between them as P. We then remove the first and last vertices of P to make P', and connect the $V(P')$ to X. In this case, P will be B, C; and since P has only 2 vertices, we have $|V(P')| = 0$, thus nothing from P' conncted to X.

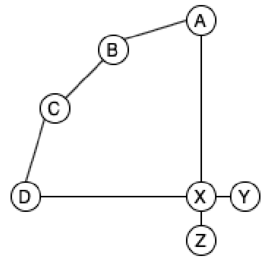
We then try to draw 3 $\Delta(G) - 2$ cliques among the alphabetically named vertices. In this case it will be:

$$
\begin{gathered}
A\ B \\
B\ C \\
C\ D
\end{gathered}
$$

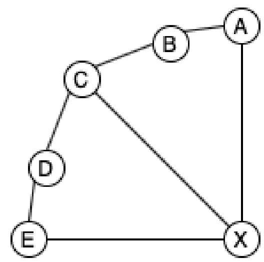
Where AB, BC, CD each formed a 2-clique. We may tell there must be $\chi(G) = 3$. As the second $\Delta(G) - 2$-clique ($B\ C$) must use 2 colors, then A and D must have a different color, this make X must have the 3-rd color; thus accomplishing $\chi(G) = 3$ and satisfies the $\chi(G) \geq \Delta(G) - 1$ requirement.

We also know that this graph does not contain any $\Delta(G) - 1$ clique as each two rows of clique will have at least 1 different vertex.

We also know that this graph will not be have a maximum degree ≥ 4, as there are only 5 vertices in it. Actually, to make $\Delta(G) = 4$, we must place some dummy vertices connected to X (which won't affect the $\chi(G)$). The below graph is a solution to $\Delta(G) = 4$.



Since $\Delta(G) = 4$ has no P', here is $\Delta(G) = 5$ to help demonstrating the idea (in this case P' is C):



Which will have $\Delta(G) - 2 = 3$-cliques on:

$$A\ B\ C$$
$$B\ C\ D$$
$$C\ D\ E$$

Where $B\ C\ D$ willl use up 3 color, and A, E (for the sake of having minimum coloring), must use the color of D, B. However, since A, E, C are all connected to X, X must have the 4-th color, making $\chi(G) = 4$. The graph has 6 vertices so we can at most $\Delta(G) = 5$; we may always have $\Delta(G) = 5$ by adding dummy vertices, although it is not necessary at this case since $d(C) = 5$. And the graph conyains no $\Delta(G) - 1 = 4$ cliques as each two rows of clique will have at least 1 different vertex.

Now we like to show this method also works for $\Delta(G) = k$. We may first have X and k alphabetically named vertices, all connected as a cycle. We then identify the P' and connect all $V(P')$ to X. We then form 3 $\Delta(G) - 2$-cliques on the alphabetically named vertices, which will be like (Note k means the $k - th$ alphabetically named vertex):

$$A\ B\ C\ D\ E\ ...\ k - 2$$
$$B\ C\ D\ E\ F\ ...\ k - 1$$
$$C\ D\ E\ F\ G\ ...\ k$$

We know that the second row of clique will use up $k - 2$ colors and A, k won't share a color. We also know that $k - 4$ vertices ($\in V(P')$) will be connected to X along with A and k, where all of them have different colors. This suggest X must have the $k - 1$-th color.

We have showed $\chi(G) = k - 1$. We now this G contains no $k - 1$ cliques as every 2 rows of $k - 2$-cliques have at least 1 vertex differ. We also know that this graph will not have a maximum degree $\geq k$, as it has only $k + 1$ vertices.

We have showed this graph constructing method works for any arbitary G with $\Delta(G) \geq 4$.