

EECS 491 Assignment 1

Due Mon Feb 22 before midnight. 140 points total.

Submitting assignments to Canvas

- For jupyter notebooks, submit the .ipynb file and a pdf export of the notebook.
- Make sure you check that the pdf export represents the latest state of your notebook and that the equations and figures are properly rendered.
- If your are not using notebooks, writeup your assignment using latex and submit a pdf with your code. The writeup should include relevant code with description if it can fit on a page.
- Use the following format for filenames:
 - EECS491-A1-yourcaseid.ipynb
 - EECS491-A1-yourcaseid.pdf
- If you have more than these two files, put any additional files in a directory named EECS491-A1-yourcaseid . Do not include binaries or large data files. Then zip this directory and submit it with the name EECS491-A1-yourcaseid.zip . Do not use other compression formats. The .ipynb file can be included in the zipped directory, but make sure you submit the .pdf file along with the .zip file. This is so it appears at the top level on canvas, which allows for easier grading.

Some of questions below aren't specified in great detail and you may need to spend sometime making sense of the questions themselves, which you can do from the reads and other sources. You also might need to fill in some blanks or make some assumptions. The spirit behind this approach is explained in [The Problem with Problems](#) by Eric Mazur, which I encourage everyone to read.

Put your name and Case ID here

Q1. Basic probability (10 pts)

In the proofs below you should use general probability distributions (as opposed to specific examples) and the basic laws of probability. Be concise and clear. The proof should be in terms of mathematical facts of probability theory.

1.1. Prove (5 pts)

$$p(x,y|z) = p(x|z)p(y|x,z)$$

1.2. Prove (5 pts)

$$p(x|y,z) = \frac{p(y|x,z)p(x|z)}{p(y|z)}$$

Q2. Independence (10 pts)

Again these proofs should use general probability distributions and the basic laws of probability. Note that the proof should be in terms of mathematical facts. It should not be an argument that depends on real-world knowledge. The example should use common real-world knowledge and interpretation should convey the ideas of the proof.

2.1 Prove that independence is not transitive, i.e. $a \perp b \wedge b \perp c \not\Rightarrow a \perp c$. Define a joint probability distribution $p(a,b,c)$ for which the previous expression holds and provide an example with an interpretation. (5 pts)

2.2 Prove that conditional independence does not imply marginal independence, i.e. $a \perp b|c \not\Rightarrow a \perp b$. Again provide an example that illustrates the statement. (5 pts)

Q3. Inspector Clouseau re-revisited (20 pts)

3.1 Write a program to evaluate $p(B|K)$ in Example 1.3 in Barber. Write your code and choose your data representations so that it is easy to use it to solve the remaining questions. Show that it correctly computes the value in the example. (5 pts)

3.2 Define a different distribution for $p(K|M,B)$. Your new distribution should result in the outcome that $p(B|K)$ is either < 0.1 or > 0.9 , i.e. reasonably strong evidence. Use the original values of $p(B)$ and $p(M)$ from the example. Provide (invent) a reasonble justification for the value of each entry in $p(K|M,B)$. (5 pts)

3.3 Derive the equation for $p(M|K)$. (5 pts)

3.4 Calculate it's value for both the original $p(K|M,B)$ and the one you defined yourself. Is it possible to provide a summary of the main factors that contributed to the value? Why/Why not? Explain. (5 pts)

Q4. Biased views (20 pts)

4.1 Write a program that calculates the posterior distribution of the θ (probability of heads) from the Binomial distribution given y heads out of n trials. Feel to use a package where the necessary distributions are defined as primitives. (5 pts)

4.2 Imagine three different views on the coin bias:

- "I believe strongly that the coin is biased to either mostly heads or mostly tails."
- "I believe strongly that the coin is unbiased".
- "I don't know anything about the bias of the coin."

Define and plot prior distributions that expresses each of these beliefs. Provide a brief explanation. (5 pts)

4.3 Perform Bernoulli trials where one of these views is correct. Show how the posterior distribution of θ changes for each view for $n=0, 1, 2, 5, 10$, and 100 . Each view should have its own plot, but with the curves of the posterior after different numbers of trials overlayed. (5 pts)

4.4 Is it possible that each view will always arrive at an accurate estimate of θ ? How might you determine which view is most consistent with the data after n trials? (5 pts)

Q5. Inference using the Poisson distribution (20 pts)

Suppose you observe for 3 seconds and detect a series of events that occur at the following times (in seconds): 0.53, 0.65, 0.91, 1.19, 1.30, 1.33, 1.90, 2.01, 2.48.

5.1 Model the rate at which the events are produced using a Poisson distribution where λ is the number of events n observed per unit time (1 second). Show the likelihood equation and plot it for three different values of λ : less, about equal, and greater than what you estimate (intuitively) from the data. (5 pts)

5.2 Derive the posterior distribution of λ assuming a Gamma prior (usually defined with parameters α and β). The posterior should have the form $p(\lambda|n,T,\alpha,\beta)$ where T is the total duration of the observation period and n is the number of events observed within that period. (5 pts)

5.3 Show that the Gamma distribution is a *conjugate prior* for the Poisson distribution, i.e. it is also a Gamma distribution, but defined by parameters α' and β' that are functions of the prior and likelihood parameters. (5 pts)

5.4 Plot the posterior distribution for the data above at times $T = 0, 0.5$, and 1.5 . Overlay the curves on a single plot. Comment how it is possible for your beliefs to change even though no new events have been observed. (5 pts)

Q6. Probability Distribution Example (20 pts)

In this problem you will illustrate a probability distribution in a settings of your choosing. It can be discrete or continuous. This is meant to be a simpler version of the letter sequeunce example shown in class (so don't use that). Your example should use two random variables that each have at least *three* distinct values (if it is discrete), i.e. don't use binary variables. The variables should not be independent, in other words, the setting you are modeling should have structure, and ideally structure that is interesting and interpretable in some way. Your example should include the following:

- a decription of the setting
- an illustration of the joint probability and how it captures the structure
- an illustration of a conditional probability
- an illustration of marginal probability

Note that "illustration" here means to explain with tables or figures that convey the ideas of the mathematical operations. The motivation behind this exercise is to help you develop a better understanding of how joint probability distributions model probabilistic structure in a simplified setting, so try to choose something you are very familiar with. If find this is getting too long, you can continue it as part of the exploration, but there you will also need to add and inference problem.

Exploration (40 pts)

In these problems, you are meant to do creative exploration. Define and explore:

E.1 A discrete inference problem (20 pts)

E.2 A continuous inference problem (20 pts)

This is meant to be open-ended; you should not feel the need to write a book chapter; but neither should you just change the numbers in one of the problems above. After doing the readings and problems above, you should pick a concept you want to understand better or an simple modeling idea you want to try out. You can also start to explore ideas for your project. The general idea is for you to teach yourself (and potentially a classate) about a concept from the assignments and readings or solidify your understanding of required technical background. For additional guidance, see the grading rubric below.

You can use the readings and other sources for inspiration, but here are a few ideas:

- An inference problem using categorical data
- A disease for which there are two different tests
- A two-dimensional continuous inference problem
- The idea of a conjugate prior

Exploration Grading Rubric

Exploration problems will be graded according the elements in the table below. The scores in the column headers indicate the number of points possible for each rubric element (given in the rows). A score of zero for an element is possible if it is missing entirely.

	Substandard (+1)	Basic (+2)	Good (+3)	Excellent (+5)
Pedagogical Value	No clear statement of idea or concept being explored or explained; lack of motivating questions.	Simple problem with adequate motivation; still could be a useful addition to an assignment.	Good choice of problem with effective illustrations of concept(s). Demonstrates a deeper level of understanding.	Problem also illustrates or clarifies common conceptual difficulties or misconceptions.
Novelty of Ideas	Copies existing problem or makes only a trivial modification; lack of citation(s) for source of inspiration.	Concepts are similar to those covered in the assignment but with some modifications of an existing exercisce.	Ideas have clear pedagogical motivation; creates different type of problem or exercise to explore related or foundational concepts more deeply.	Applies a technique or explores concept not covered in the assignment or not discussed at length in lecture.
Clarity of Explanation	Little or confusing explanation; figures lack labels or useful captions; no explanation of motivations.	Explanations are present, but unclear, unfocused, wordy or contain too much technical detail.	Clear and concise explanations of key ideas and motivations.	Also clear and concise, but includes illustrative figures; could be read and understood by students from a variety of backgrounds.
Depth of Exploration	Content is obvious or closely imitates assignment problems.	Uses existing problem for different data.	Applies a variation of a technique to solve a problem with an interesting motivation; explores a concept in a series of related problems.	Applies several concepts or techniques; has clear focus of inquiry that is approached from multiple directions.