CSDS 491: Assignment 1

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Spring 2021, Dr. Lewicki

Q1

1.1.

$$\begin{aligned} p(x, y \mid z) &= \frac{p(x, y, z)}{p(x)} \\ &= \frac{p(x, y, z) \cdot p(x, z)}{p(x) \cdot p(x, z)} \\ &= \frac{p(x, z)}{p(z)} \cdot \frac{p(x, y, z)}{p(x, z)} \\ &= p(x \mid z) p(y \mid x, z) \end{aligned}$$

1.2.

$$\begin{aligned} p(x \mid y, z) &= \frac{p(x, y, z)}{p(y, z)} \\ &= \frac{\frac{p(x,y,z)}{p(z)}}{\frac{p(y,z)}{p(z)}} = \frac{\frac{p(x,y,z)}{p(x,z)} \frac{p(x,z)}{p(z)}}{p(y \mid z)} \\ &= \frac{p(y \mid x, z) p(x \mid z)}{p(y \mid z)} \end{aligned}$$

Q1

2.1.

Assume $a = \{1, 2, 3\}$, $b = \{1, 4\}$, $c = \{1, 2, 6\}$ from a set of $\{1, 2, 3, 4, 5, 6\}$, we have:

$$
\begin{aligned}
p(a,b) &= \frac{\{1\}}{\{1,2,3,4,5,6\}} = \frac{1}{6} = p(a)p(b) = \frac{1}{2}\cdot\frac{1}{3} \Longrightarrow a \perp b \\
p(b,c) &= \frac{\{1\}}{\{1,2,3,4,5,6\}} = \frac{1}{6} = p(b)p(c) = \frac{1}{2}\cdot\frac{1}{3} \Longrightarrow b \perp c \\
p(a,c) &= \frac{\{1,2\}}{\{1,2,3,4,5,6\}} = \frac{1}{3} \\
p(a)p(c) &= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \\
\Longrightarrow p(a,c) &\neq p(a)p(c)
\end{aligned}
$$

This suggests $a \perp b \wedge b \perp c \not\Rightarrow a \perp c$. For an example, we may have

- **a**: Rainning tomorrow.
- **b**: Had pasta as dinner today.
- **c**: Not rainning tomorrow.

By common sense we know that $a \perp b \wedge b \perp c$ holds true, but we can't have $a \perp c$ as it must be one way or another in terms rainning tomorrow or not.

2.2.

Assume $a \perp b \mid c$, we have:

$$
\begin{aligned}
p(a,(b \mid c)) &= p(a)p(b \mid c) \\
&= p(a)\frac{p(b,c)}{p(c)}
\end{aligned}
$$

To have $a \perp b$, we must show $p(a,b) = p(a)p(b)$. The only way to convert the above equation to such format is to assume $b \perp c$ so that we can have:

$$
\begin{aligned}
p(a,(b \mid c)) &= p(a)\frac{p(b,c)}{p(c)} \\
p(a,b) &= p(a)\frac{p(b)p(c)}{p(c)} \\
p(a,b) &= p(a)p(b)
\end{aligned}
$$

But $b \perp c$ is an assumption which cannot be guaranteed.

Q3

Please refer to `code/q3.py` for code.

3.1.

```
0.7281553398058251
```

3.2.

`0.9765625`

I am aming to have $p(B \mid K) > 0.9$, thus I will need to lower the possibility of scenarios where the butler is not the killer. I found $p(K \mid B = F, M = F)$ and $p(K \mid B = F, M = T)$ to be more "explainable" as for the former we may simply say the inspector had more on-scence info indicating the killer is very likely to be still in the house; for the latter we may also just let the scence implies that the victim is strong and tall but the maid is petite with no sign of combat – thus lowering the possibility of $M = T$ along.

3.3.

$$
\begin{aligned}
p(M \mid K) &= \sum_b p(b, M \mid K) = \sum_b \frac{p(b, M, K)}{p(K)} \\
&= \frac{\sum_b p(K \mid b, M) p(b, M)}{\sum_{b,m} p(K \mid b, m) p(b, m)} = \frac{p(M) \sum_b p(K \mid b, M) p(b)}{\sum_m p(m) \sum_b p(K \mid b, m) p(b)}
\end{aligned}
$$

3.4.

With the setup proposed by the textbook, we have $p(M \mid K)$ being `0.06796116504854369`; with the setup defined by me, we have $p(M \mid K)$ being `0.05208333333333333`.

The main contributing factor is $p(K \mid B = F, M = T)$, as we have lowered it in the second setup, we directly decrease the chances of the maid being the murder and thus potentially lowered $p(M \mid K)$. We may also confirm this guessing by calculating only with the lowered $p(K \mid B = F, M = F)$ and only with the lowered $p(K \mid B = F, M = T)$. We got `0.08771929824561403` with only the lowered $p(K \mid B = F, M = F)$ and `0.04` with only the lowered $p(K \mid B = F, M = T)$ – suggesting that knowing the maid is very unlikely to be the murder herslef ($\downarrow p(K \mid B = F, M = T)$) has more of an impact on $p(M \mid K)$ than knowing the murder is more likely to be between bulter and maid ($\downarrow p(K \mid B = F, M = F)$).

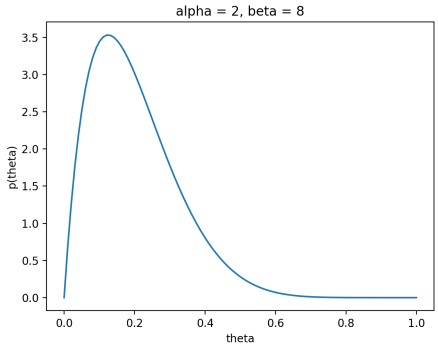
Q4

Please refer to `code/q4.py` for complete code.

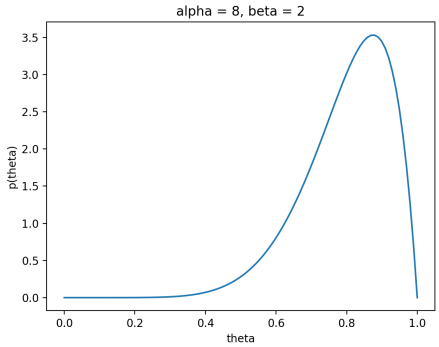
4.1.

```python
def p_theta__beta(theta, y, n, alpha, beta):
    return stats.beta.pdf(theta, alpha + y, beta + n - y)
```

4.2.

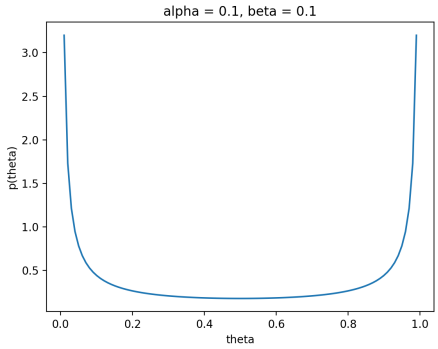


Biased coin, more likely tail.

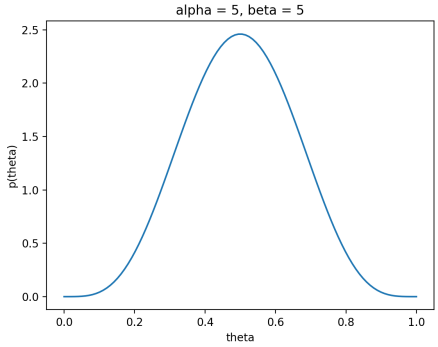


Biased coin, more likely head.

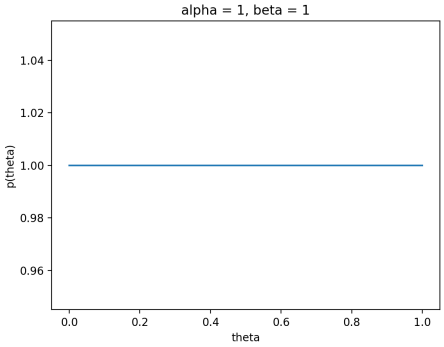
Alternatively we may have it on the same plot with $\alpha < 1, \beta < 1$:



Biased coin (either way).

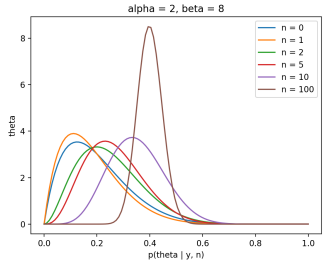


Unbiased coin

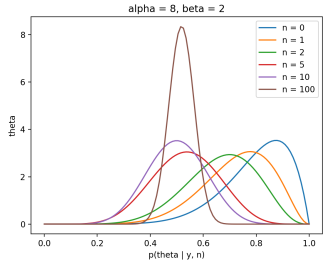


No info coin

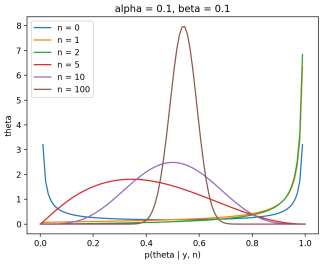
4.3.



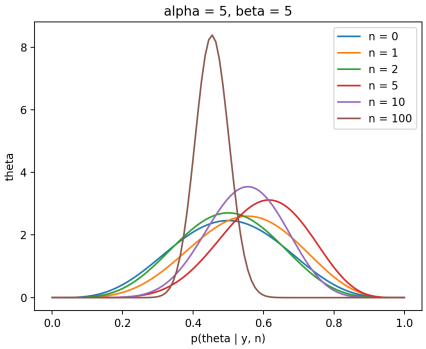
Biased coin, more likely tail.



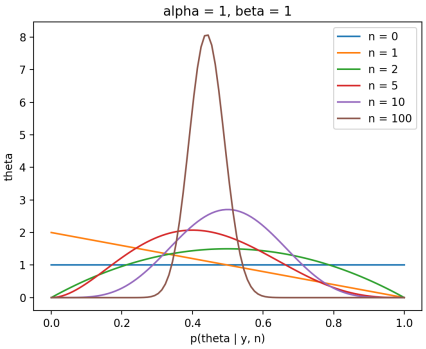
Biased coin, more likely head.



Biased coin (either way).



Unbiased coin



No info coin

4.4.

Yes, as n getting large we may more info on the likelihood and therefore result in an more accurate estimation. We may confirm this by checking at $n = 100$, regardless which prior we used, the prediction came out to be unbiased and accurate (around $\theta = 0.5$).

Q5

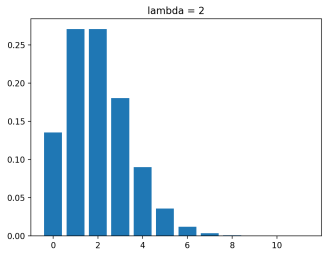
Please refer to `code/q5.py` for code.

5.1.

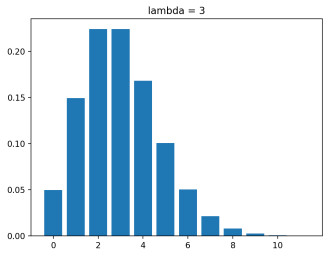
Let x to be the actual occurence of events per unit time, we have:

$$p(x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

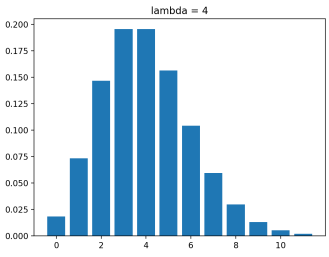
Since we have 9 events occured in a duration of 3 seconds, we estinate $\lambda = 3$. I will also plut $\lambda = 2$ and $\lambda = 4$ as examples of "less" and "greater than" of my estimation.



$\lambda = 2$



$\lambda = 3$



$\lambda = 4$

5.2.

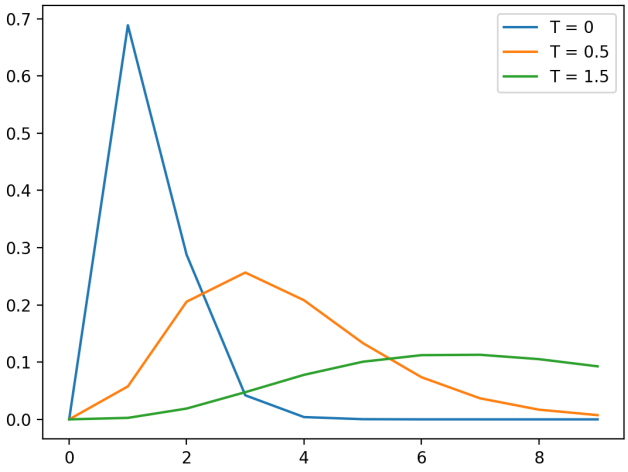
$$
\begin{aligned}
p(\lambda \mid n, T, \alpha, \beta) &= \frac{p(n, T \mid \lambda, \alpha, \beta) p(\lambda \mid \alpha, \beta) p(\alpha, \beta)}{p(n, T \mid \alpha, \beta) p(\alpha, \beta)} \\
&= \frac{p(n, T \mid \lambda, \alpha, \beta) \Gamma(\lambda; \alpha, \beta)}{p(n, T \mid \alpha, \beta)} \\
&= \frac{p(n, T \mid \lambda, \alpha, \beta) \Gamma(\lambda; \alpha, \beta)}{\int_0^\infty p(n, T \mid \lambda, \alpha, \beta) p(\lambda \mid \alpha, \beta) \cdot d\lambda} \\
&= \frac{\frac{e^{-\lambda T}(\lambda T)^n}{n!} \cdot \Gamma(\lambda; \alpha, \beta)}{\int_0^\infty (\frac{e^{-\lambda T}(\lambda T)^n}{n!} \cdot \Gamma(\lambda; \alpha, \beta)) d\lambda} \\
&= \frac{e^{-\lambda T}(\lambda T)^n \cdot \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_0^\infty e^{-\lambda(\beta+T)} (T\lambda)^n \lambda^{\alpha-1} \cdot d\lambda}
\end{aligned}
$$

5.3.

$$
\begin{aligned}
p(\lambda \mid n, T, \alpha, \beta) &= \frac{e^{-\lambda T}(\lambda T)^n \cdot \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_0^\infty e^{-\lambda(\beta+T)} (T\lambda)^n \lambda^{\alpha-1} \cdot d\lambda} \\
&= \frac{\frac{(\beta+T)^{\alpha+n} T^n \lambda^n e^{-\lambda(\beta+T)} \lambda^{\alpha-1}}{\Gamma(a+n)}}{T^n \cdot \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\beta^\alpha}{(T+\beta)^{a+n}}} \\
&= \frac{(\beta+T)^{\alpha+n} \lambda^{\alpha+n-1} e^{-\lambda(\beta+T)}}{\Gamma(a+n)} \\
&= \Gamma(\lambda; \alpha+n, \beta+T)
\end{aligned}
$$

Thus, the Gamma distribution is the conjugate prior of Possion distribution with $\alpha' = \alpha + n$ and $\beta' = \beta + T$.

5.4.



Although no new event has occured, the fact that no event has occured bring in useful information to the estimation.

Q6

The following example is altered from the book.

Assuem we have two 3-side dice to roll where we the first dice being $X \in \{1, 2, 3\}$ and Y being the sum of two dices. We have the joint probability being:

$p(x, y)$	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	1/9	0	0
$Y = 3$	1/9	1/9	0
$Y = 4$	1/9	1/9	1/9
$Y = 5$	0	1/9	1/9
$Y = 6$	0	0	1/9

I capture the structure in such a way that if $y - x > 3$, the probability will be 0 as the max roll from second dice is only 3. If $y - x \leq 3$, this means we can only have one possible second dice outcome to match with the first dice, so the join probability will bt $(\frac{1}{3})^2 = \frac{1}{9}$.

For conditional probablity, we have $p(y|x) = \frac{p(x,y)}{x}$. But since given a certain x, we will have only one option, $(y - x)$, for the second dice; thus, we have $p(y|x) = \frac{1}{3}$ if such (x, y) combination is at all possible. On the othe hand we have $p(x|y) = \frac{p(x,y)}{y}$. Since we only have one first-second dice combination given a possible (x, y), $p(x|y) = p(x) \cdot p(\text{second dice outcome}) = \frac{1}{9}$.

For marginal probablity for $P(x)$, is it universially $\frac{1}{3}$ due to the nature of being a fair dice. But for $P(Y)$ we have:

$$P(y) = \sum_j P(x_j, y)$$

	$Y=2$	$Y=3$	$Y=4$	$Y=5$	$Y=6$
$P(Y)$	1/9	2(1/9)	3(1/9)	2(1/9)	1/9

The intuition behind this equation is to get a certain Y, you must add the all possible dice combination to form such Y by chances. Which means we may simply add up each row of the joint probablity table to form a table for all $P(Y)$.

Q7: Exploration

Discrete Inference Problem

I overhead the idea of "draw a dice from a jar (of different dices) and roll it without checking the dice" in classroom and thought it would be interesting. Although the proposed question in classroom was seemingly unsolvable, many interesting questions can be made base on this set up.

Assuming we have a jar of three dices, $\{D4, D6, D8\}$ representing a 4-, 6-, 8-side fair dice respectively. We may draw a dice from the jar, roll it, record the outcome and put it back. Say we got an output sequence of

$$\{1, 6, 3\}$$

What is the most likely sequence of the dice rolled?

This question will be easy if the output sequence is $\{8, 8, 8\}$ so that we know the dice rolling sequence must be $\{D8, D8, D8\}$ as no other dice can generate an outcome of 8. This implies for $P(D \mid 8)$, we have $P_1(D4 \mid 8) = P(D6 \mid 8) = 0$ and $P(D8 \mid 8) = 1$

Although we can definitively tell what will $P_1(D_{\{4,6,8\}} \mid 1)$ be, we may infer base on the structure of the dice:

$$P_1(D4 \mid 1) = \frac{1}{4}$$
$$P_1(D6 \mid 1) = \frac{1}{6}$$
$$P_1(D8 \mid 1) = \frac{1}{8}$$

So we may safely conclude that for the first outcome of the sequence $\{1\}$, it is most likely generated by $D4$. Similarily, for $\{1, 6\}$, we may find the dice-rolling sequence that has the maximal possibility of generating this output sequence. We already know the first dice is most likely to be $D4$, then we have

$$P_2((D4 \mid 1)) = P(D4) \cdot P(1 \mid D4) \cdot P(D4 \to D4) \cdot \cdot P(6 \mid D4)$$
$$P_2((D6 \mid 1)) = P(D4) \cdot P(1 \mid D4) \cdot P(D4 \to D6) \cdot \cdot P(6 \mid D6)$$
$$P_2((D8 \mid 1)) = P(D4) \cdot P(1 \mid D4) \cdot P(D4 \to D8) \cdot \cdot P(6 \mid D8)$$

In this case, we have $P_2((D6 \mid 1))$ yieling the largest probability, so the dice-rolling sequence is mostly likely to be $\{D4, D6\}$... and using the same idea we will have an answer of $\{D4, D6, D4\}$.

This question is simple because we have a short output sequence to decode, and we may even simply brute force this problem by trying out every combinations. But the idea of calculating the most likely dice one by one can be extend to sequence of any length, and even with many more add on restructions like:

- Not to use the same dice twice in a row: we may simply set $P(D_x \to D_x) = 0$.
- Got mutiple dice of same structure in the jar: we may simply $P(D_x \to D_y) = \frac{\text{left over } Dy \text{ dices}}{\text{all dices - number of } Dx \text{ dices}}$.
- There is uneven transition possibility between $D_x \to D_y$: we set $P(D_x \to D_x)$ accordingly.

This question will eventually become a dynamic programming problem to solve.

Continuous Inference Problem

I will try to discuss how gramma distribution is a conmjugate prior for exponential distribution.

For $\theta \sim Gamma(\alpha, \beta)$ we have a prior of:

$$f_\Theta(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}$$

It's likelihood is a exponential distribution:

$$f(X|\theta) = \theta^n e^{-\theta \sum x_i}$$

Now to find the posterior (likelihood times prior):

$$\begin{aligned} f_{\Theta|X}(\theta|x) &\propto f_{x|\Theta}(x|\theta) \times f_\Theta(\theta) \\ f_{\Theta|X}(\theta|x) &\propto \theta^n e^{-\theta \sum x_i} \times \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)} \\ &= \theta^{n+\alpha-1} e^{-\theta(\sum x_i + \beta)} \end{aligned}$$

We have a gamma distribution of $\Gamma(\alpha' = \alpha + n, \beta' = \beta + \sum x_i)$ being the conjugate prior.