EECS 340: Assignment 2

Shaochen (Henry) ZHONG, `sxz517`
Zhitao (Robert) CHEN, `zxc325`

Due and submitted on 02/03/2020
EECS 340, Dr. Koyutrk

Problem 1

(a) $max\{f(n), g(n)\} = \Theta(f(n) + g(n))$

Since it is known that $f(n) \geq 0$, $g(n) \geq 0$, and $c > 0$; we must have:

$$f(n) \leq f(n) + g(n)$$
$$g(n) \leq f(n) + g(n)$$
$$\Rightarrow \max(f(n), g(n)) \in O(f(n) + g(n)) \quad \text{for} \quad \begin{cases} c = 1 \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{1}$$

Since it is also known that $f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$, we may therefore infer:

$$\max(f(n), g(n)) \in \Omega(f(n) + g(n)) \quad \text{for} \quad \begin{cases} c = \frac{1}{2} \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{2}$$

Since both the O- and Ω-notation are established, we may therefore conclude:

$$\max(f(n), g(n)) \in \Theta(f(n) + g(n)) \tag{3}$$

(b1) $f(n) + d = O(f(n))$.

False

For the seek of disambiguation, we rewrite the questioned equation as $f(n) + d = O(f(n))$ by d replacing c for $d > 0$.

To prove the statement to be valid, we need to show:

$$0 \leq f(n) + d \leq cf(n) \tag{4}$$

For $f(n) = 0$, we cannot find any c which satisfies the above equation since:

$$O(f(n)) = cf(n) = 0$$
$$\Rightarrow 0 \leq 0 + d \leq c(0) \quad \text{for } d > 0 \tag{5}$$

As the equation $0 + d \leq 0$ leads to a contradiction, the statement is invalid.

(b2) If $f(n) \geq 1$, then $f(n) + d = O(f(n))$.

For the seek of disambiguation, we rewrite the questioned equation as $f(n) + d = O(f(n))$ by d replacing c for $d > 0$.

According to the defination of O-notation, we have:

$$f(n) = O(f(n))$$
$$\exists c, n_0 > 0 \ \text{ s.t. } \ 0 \leq f(n) \leq cf(n) \quad \text{for } n \geq n_0 \tag{6}$$
$$\exists n' > 0 \ \text{ s.t. } \ f(n) \geq f(n') \quad \text{for } n, n' \geq n_0 \tag{7}$$
$$\Rightarrow 0 \leq f(n) + d \leq cf(n) + d \quad \text{for } n \geq n_0 \tag{8}$$

Thus, we may rewrite the above equation as:

$$0 \leq f(n) + d \leq (c + \frac{d}{f(n)})f(n) \quad \text{for } f(n) \geq 1 \tag{9}$$

$$\Rightarrow 0 \leq f(n) + d \leq c'f(n) \quad \text{for } \begin{cases} c' = c + \frac{d}{f(n')} \\ n, n' \geq n_0 \end{cases} \tag{10}$$

Therefore, we may conclude if $f(n) \geq 1$, then $f(n) + d = O(f(n))$ via direct proof.

(c1) If $f(n) = O(g(n)), \log(f(n)) \geq 0$ and $\log(g(n)) \geq 0$, then $log(f(n)) = O(\log(g(n))$.

False

For $f(n) = 2$ and $g(n) = 1$, then $\log f(n) = 1$ and $\log g(n) = 0$. Since we cannot find any constant c where:

$$0 \leq 1 \leq c(0) \tag{11}$$

Thus the statement is invalid.

(c2) If $f(n) = O(g(n))$, $\log(f(n)) \geq 0$ and $\log(g(n)) \geq 1$, then $log(f(n)) = O(\log(g(n))$.

According to the defination of O-notation, we must have:

$$\exists c, n_0 > 0 \quad \text{s.t.} \quad f(n) \leq cg(n) \quad \text{for } n \geq n_0 \tag{12}$$

Since it is given that $\log(f(n)) \geq 0$, $\log(g(n)) \geq 1$, thus we must have:

$$\log(f(n)) \leq \log(c(g(n))) \quad \text{for } n \geq n_0$$
$$\Rightarrow \log(f(n)) \leq \log c + \log(g(n)) \quad \text{for } n \geq n_0 \tag{13}$$

As c, n_0 are constants, there must be a constant c' s.t.

$$c' \geq \frac{\log c}{\log(g(n_0))} + 1 \tag{14}$$
$$\Rightarrow (c'-1)\log(g(n)) \geq (c'-1)\log(g(n_0)) \geq \log c \quad \text{for } n \geq n_0 \tag{15}$$
$$\exists c, n_0 > 0 \quad \text{s.t.}$$
$$\log(f(n)) \leq \log c + \log(g(n)) \leq (c'-1)\log(g(n)) + \log(g(n)) \quad \text{for } n \geq n_0 \tag{16}$$
$$\Rightarrow \log(f(n)) \leq c' \log(g(n)) \tag{17}$$

Thus we may conclude $log(f(n)) = O(\log(g(n))$, the statement is therefore proven to be valid.

(d1) $f(2n) = \Theta(f(n))$

False

For $f(n) = 2^n$, we have $f(2n) = 4^n$. Where $f(2n) \neq \Theta(f(n))$ due to the LHS has a higher asymptotic order, and therefore we can't find any constant c_1, c_2 to form a relation of $c_1 \cdot 2^n \leq 4^n \leq c_2 \cdot 2^n$. Thus, the statement is invalid.

(d2) If $f(n) = O(n^k)$, then $f(2n) = O(n^k)$

For the seek of disambiguation, we rewrite the questioned equation as: if $f(n) = O(n^k)$, then $f(2n) = O(n^k)$ by k replacing c for $k > 0$.

Since $f(n) = O(n^k)$, we must have $0 \leq f(n) \leq cn^k$ for $n \geq n_0$ and a c for $c \in \mathbb{R}^+$. Now substitute n as $2n$, we may have:

$$0 \leq f(2n) \leq c(2n)^k$$
$$\Rightarrow 0 \leq f(2n) \leq c \cdot (2)^k \cdot n^k \tag{18}$$
$$\Rightarrow 0 \leq f(2n) \leq c' \cdot n^k \quad \text{where } c' = c \cdot (2)^k \tag{19}$$

Thus we may conclude if $f(n) = O(n^k)$, then $f(2n) = O(n^k)$.

(d3) If $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$

For the seek of disambiguation, we rewrite the questioned equation as: if $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$.

Since $f(n) = \Theta(n^k)$, we must have $0 \leq c_1 \cdot n^k \leq f(n) \leq c_2 \cdot n^k$ for $n \geq n_0$ and c_1, c_2 for $c_1, c_2 \in \mathbb{R}^+$. Now substitute n as $2n$, we may have:

$$0 \leq c_1(2n)^k \leq f(2n) \leq c_2(2n)^k$$
$$\Rightarrow 0 \leq c_1 \cdot 2^k \cdot n^k \leq f(2n) \leq c_2 \cdot 2^k \cdot n^k \tag{20}$$

From $0 \leq c_1 \cdot n^k \leq f(n) \leq c_2 \cdot n^k$, we may also infer:

$$\frac{f(n)}{c_1} \geq k \geq \frac{f(n)}{c_2} \tag{21}$$

$$\Rightarrow c_1 \cdot 2^k \cdot \frac{f(n)}{c_2} \leq f(2n) \leq c_2 \cdot 2^k \cdot \frac{f(n)}{c_1} \tag{22}$$

$$\Rightarrow 0 \leq c_1' \cdot f(n) \leq f(2n) \leq c_2' \cdot f(2n) \quad \text{for} \begin{cases} c_1' = \frac{c_1 \cdot 2^k}{c_2} \\ c_2' = \frac{c_2 \cdot 2^k}{c_1} \end{cases} \tag{23}$$

Thus we may conclude if $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$.

Problem 2

Answer

$$\frac{1}{n^a} \ll \frac{1}{n^\epsilon} \ll \log_{\frac{1}{\epsilon}}(n) \ \ (\text{when } 0 < \epsilon < \frac{1}{2}) \ll \log(n^\epsilon) \equiv \log(bn) \equiv \log(n^a) \tag{24}$$

$$\equiv \log(n^b) \leq \log_{\frac{1}{\epsilon}}(n) \ \ (\text{when } \epsilon \geq \frac{1}{2}) \ll (\log n)^a \ll n^\epsilon \tag{25}$$

$$\ll a^{\log_a(n)} \equiv \epsilon n \equiv \frac{n}{a} \ll n^a \equiv (n+b)^a \tag{26}$$

$$\ll (n+a)^b \ll n^{a+b} \ll \epsilon^n \ll a^{\epsilon n} \equiv a^n \ll b^n \tag{27}$$

Justification of $\log(n^\epsilon) \equiv \log(bn) \equiv \log(n^a) \equiv \log(n^b)$
The above equations can be rewrite as:

$$\log(n^\epsilon) = \epsilon \log(n) \tag{28}$$
$$\log(n^a) = a \log(n) \tag{29}$$
$$\log(bn) = \log(b) + \log(n) \tag{30}$$
$$\log(n^b) = b \log(n) \tag{31}$$

Where all of them can be generalized as $\Theta(\log(n))$, as it is known that $n^\epsilon \leq bn \leq n^a$ due to the decending of *power(s)*.

Justification of $\log_{\frac{1}{\epsilon}}(n)$ **(when** $0 < \epsilon < \frac{1}{2}$**)** $\equiv \log(n^b) \leq \log_{\frac{1}{\epsilon}}(n)$ **(when** $\epsilon \geq \frac{1}{2}$**)**

It is known that when $\epsilon < \frac{1}{2} \Rightarrow \frac{1}{\epsilon} \geq 2$; when $\epsilon \geq \frac{1}{2} \Rightarrow \frac{1}{\epsilon} \leq 2$. Since it is observable that $\log_x(n) \gg \log_{x'}(n)$ for $x < x'$, such (in)equaility is valid.

Justification of $a^{\log_a(n)} \equiv \epsilon n \equiv \frac{n}{a}$

The above equations can be rewrite as:

$$a^{\log_a(n)} = n^1 = \Theta(n) \tag{32}$$
$$\epsilon n = \Theta(n) \tag{33}$$
$$fracna = \frac{1}{a}n = \Theta(n) \tag{34}$$

Thus, the above equality is justified.

Justification of $a^{\epsilon n} \equiv a^n$

We may rewrite $a^{\epsilon n}$ as the following, since there must be a c for $c \in \mathbb{R}^+$ which satisfy the equality.

$$a^\epsilon \cdot a^n = c \cdot a^n = \Theta(a^n) \tag{35}$$

Since it is known that $a^n = \Theta(a^n)$, these two expressions are considered equivalent.