EECS 340: Assignment 2

Shaochen (Henry) ZHONG, `sxz517`
Zhitao (Robert) CHEN, `zxc325`

Due and submitted on 02/03/2020
EECS 340, Dr. Koyutürk

Problem 1

(a) $max\{f(n), g(n)\} = \Theta(f(n) + g(n))$

True

Since it is known that $f(n) \geq 0$, $g(n) \geq 0$, and $c > 0$; we must have:

$$\begin{aligned} f(n) &\leq f(n) + g(n) \\ g(n) &\leq f(n) + g(n) \end{aligned}$$

$$\Rightarrow \max(f(n), g(n)) \in O(f(n) + g(n)) \quad \text{for } \begin{cases} c = 1 \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{1}$$

Since it is also known that $f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$, we may therefore infer:

$$\max(f(n), g(n)) \in \Omega(f(n) + g(n)) \quad \text{for } \begin{cases} c = \frac{1}{2} \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{2}$$

Since both the O- and Ω-notations are established, we may therefore conclude:

$$\max(f(n), g(n)) \in \Theta(f(n) + g(n)) \tag{3}$$

(b1) $f(n) + d = O(f(n))$

False

For the sake of disambiguation, we rewrite the questioned equation as $f(n)+d = O(f(n))$ by d replacing c for $d > 0$.

To prove the statement to be valid, we need to show:

$$0 \leq f(n) + d \leq cf(n) \tag{4}$$

For $f(n) = 0$, we cannot find any c which satisfies the above equation since:

$$\begin{aligned} O(f(n)) &= cf(n) = 0 \\ \Rightarrow 0 \leq 0 + d &\leq c(0) \quad \text{for } d > 0 \end{aligned} \tag{5}$$

As the equation $0 + d \leq 0$ leads to a contradiction, the statement is invalid.

(b2) If $f(n) \geq 1$, then $f(n) + d = O(f(n))$

True

For the sake of disambiguation, we rewrite the questioned equation as $f(n)+d = O(f(n))$ by d replacing c for $d > 0$.

According to the defination of O-notation, we have:

$$f(n) = O(f(n))$$

$$\exists c, n_0 > 0 \ \text{ s.t. } 0 \leq f(n) \leq cf(n) \quad \text{ for } n \geq n_0 \tag{6}$$

$$\exists n' > 0 \ \text{ s.t. } f(n) \geq f(n') \quad \text{ for } n, n' \geq n_0 \tag{7}$$

$$\Rightarrow 0 \leq f(n) + d \leq cf(n) + d \quad \text{for } n \geq n_0 \tag{8}$$

Thus, we may rewrite the above equation as:

$$0 \leq f(n) + d \leq (c + \frac{d}{f(n)})f(n) \quad \text{for } f(n) \geq 1 \tag{9}$$

$$\Rightarrow 0 \leq f(n) + d \ \leq \ c'f(n) \quad \text{for } \begin{cases} c' = c + \frac{d}{f(n')} \\ n, n' \geq n_0 \end{cases} \tag{10}$$

Therefore, we may conclude if $f(n) \geq 1$, then $f(n) + d = O(f(n))$ via direct proof.

(c1) If $f(n) = O(g(n)), \log(f(n)) \geq 0$ and $\log(g(n)) \geq 0$, then $log(f(n)) = O(\log(g(n))$

False

For $f(n) = 2$ and $g(n) = 1$, then $\log f(n) = 1$ and $\log g(n) = 0$. Since we cannot find any constant c where:

$$0 \leq 1 \leq c(0) \tag{11}$$

Thus the statement is invalid.

(c2) If $f(n) = O(g(n)), \log(f(n)) \geq 0$ and $\log(g(n)) \geq 1$, then $log(f(n)) = O(\log(g(n))$

True

According to the definition of O-notation, we must have:

$$\exists c, n_0 > 0 \ \text{ s.t. } \ f(n) \leq cg(n) \quad \text{for } n \geq n_0 \tag{12}$$

Since it is given that $\log(f(n)) \geq 0$, $\log(g(n)) \geq 1$, thus we must have:

$$\begin{aligned} &\log(f(n)) \leq \log(c(g(n))) \quad \text{for } n \geq n_0 \\ \Rightarrow &\log(f(n)) \leq \log c + \log(g(n)) \quad \text{for } n \geq n_0 \end{aligned} \tag{13}$$

As c, n_0 are constants, there must be a constant c' s.t.

$$c' \geq \frac{\log c}{\log(g(n_0))} + 1 \tag{14}$$

$$\Rightarrow (c' - 1)\log(g(n)) \ \geq \ (c' - 1)\log(g(n_0)) \ \geq \ \log c \quad \text{for } n \geq n_0 \tag{15}$$

$$\exists c, n_0 > 0 \ \text{ s.t.}$$

$$\log(f(n)) \ \leq \ \log c + \log(g(n)) \ \leq \ (c' - 1)\log(g(n)) + \log(g(n)) \quad \text{for } n \geq n_0 \tag{16}$$

$$\Rightarrow \log(f(n)) \leq c' \log(g(n)) \tag{17}$$

Thus we may conclude $log(f(n)) = O(\log(g(n))$, the statement is therefore proven to be valid.

(d1) $f(2n) = \Theta(f(n))$

False

For $f(n) = 2^n$, we have $f(2n) = 4^n$. Where $f(2n) \neq \Theta(f(n))$ due to the LHS has a higher asymptotic order, and therefore we can't find any constant c_1, c_2 to form a relation of $c_1 \cdot 2^n \leq 4^n \leq c_2 \cdot 2^n$. Thus, the statement is invalid.

(d2) If $f(n) = O(n^k)$, then $f(2n) = O(n^k)$

True

For the sake of disambiguation, we rewrite the questioned equation as: if $f(n) = O(n^k)$, then $f(2n) = O(n^k)$ by k replacing c for $k > 0$.

Since $f(n) = O(n^k)$, we must have $0 \leq f(n) \leq cn^k$ for $n \geq n_0$ and a c for $c \in \mathbb{R}^+$. Now substitute n as $2n$, we may have:

$$
\begin{aligned}
& 0 \leq f(2n) \leq c(2n)^k \\
& \Rightarrow 0 \leq f(2n) \leq c \cdot (2)^k \cdot n^k && (18) \\
& \Rightarrow 0 \leq f(2n) \leq c' \cdot n^k \quad \text{where } c' = c \cdot (2)^k && (19)
\end{aligned}
$$

Thus we may conclude if $f(n) = O(n^k)$, then $f(2n) = O(n^k)$.

(d3) If $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$

True For the sake of disambiguation, we rewrite the questioned equation as: if $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$.

Since $f(n) = \Theta(n^k)$, we must have $0 \leq c_1 \cdot n^k \leq f(n) \leq c_2 \cdot n^k$ for $n \geq n_0$ and c_1, c_2 for $c_1, c_2 \in \mathbb{R}^+$. Now substitute n as $2n$, we may have:

$$
\begin{aligned}
& 0 \leq c_1(2n)^k \leq f(2n) \leq c_2(2n)^k \\
& \Rightarrow 0 \leq c_1 \cdot 2^k \cdot n^k \leq f(2n) \leq c_2 \cdot 2^k \cdot n^k && (20)
\end{aligned}
$$

From $0 \leq c_1 \cdot n^k \leq f(n) \leq c_2 \cdot n^k$, we may also infer:

$$
\begin{aligned}
& \frac{f(n)}{c_1} \geq k \geq \frac{f(n)}{c_2} && (21) \\
& \Rightarrow c_1 \cdot 2^k \cdot \frac{f(n)}{c_2} \leq f(2n) \leq c_2 \cdot 2^k \cdot \frac{f(n)}{c_1} && (22) \\
& \Rightarrow 0 \leq c_1^{'} \cdot f(n) \leq f(2n) \leq c_2^{'} \cdot f(2n) \quad \text{for} \begin{cases} c_1^{'} = \frac{c_1 \cdot 2^k}{c_2} \\ c_2^{'} = \frac{c_2 \cdot 2^k}{c_1} \end{cases} && (23)
\end{aligned}
$$

Thus we may conclude if $f(n) = \Theta(n^k)$, then $f(2n) = \Theta(f(n))$.

Problem 2

Conclusion

$$
\begin{aligned}
& n^{-a} \ll n^{-\epsilon} \ll \epsilon^n && (24) \\
& \ll \log(n^\epsilon) \equiv \log(bn) \equiv \log(n^a) \equiv \log(n^b) \equiv \log_{\frac{1}{\epsilon}}(n) && (25) \\
& \ll (\log n)^a \ll n^\epsilon \ll a^{\log_a(n)} && (26) \\
& \equiv \epsilon n \equiv \frac{n}{a} && (27) \\
& \ll n^a \equiv (n+b)^a \ll (n+a)^b \ll n^{a+b} && (28) \\
& \ll a^{\epsilon n} \ll a^n \ll b^n && (29)
\end{aligned}
$$

Justifications

Justification of $n^{-a} \ll n^{-\epsilon} \ll \epsilon^n$

$$\lim_{n\to\infty} \frac{\epsilon^n}{n^{-\epsilon}} = \lim_{n\to\infty} \frac{n\log\epsilon}{-\epsilon\log n} = \lim_{n\to\infty} \frac{\log\epsilon}{-\epsilon} \cdot \frac{n}{\log n} \tag{30}$$

$$\text{Since } \epsilon \in (0,1) \Rightarrow \log\epsilon < 0 \Rightarrow \frac{\log\epsilon}{\epsilon} > 0$$

$$\lim_{n\to\infty} \frac{\epsilon^n}{n^{-\epsilon}} = \lim_{n\to\infty} c \cdot \frac{n}{\log n} = \infty \tag{31}$$

$$\Rightarrow \epsilon^n \gg n^{-\epsilon} \tag{32}$$

$$\text{Since } 0 < \epsilon < 1 < a \tag{33}$$

$$\lim_{n\to\infty} \frac{n^{-a}}{n^{-\epsilon}} = \lim_{n\to\infty} n^{\epsilon-a} = 0 \tag{34}$$

$$\Longrightarrow n^{-a} \ll n^{-\epsilon} \tag{35}$$

Combine the above two set of equations together, we have $n^{-a} \ll n^{-\epsilon} \ll \epsilon^n$.

Justification of $\epsilon^n \ll \log(n^\epsilon)$

Since $\lim_{n\to\infty} \epsilon^n = 0$, where $0 < 1 < \log(n^\epsilon)$, we may say the above inequality if true.

Justification of $\log(n^\epsilon) \equiv \log(bn) \equiv \log(n^a) \equiv \log(n^b) \equiv \log_{\frac{1}{\epsilon}}(n)$

The above equations can be losely rewritten as:

$$\log(n^\epsilon) = \epsilon\log(n) \tag{36}$$

$$\log(n^a) = a\log(n) \tag{37}$$

$$\log(bn) = \log(b) + \log(n) \tag{38}$$

$$\log(n^b) = b\log(n) \tag{39}$$

$$\tag{40}$$

Where all of them can be generalized as $\Theta(\log(n))$, as it is known that $n^\epsilon \leq bn \leq n^a$ due to the decending of *power(s)*.

Note the relationship of the above expressions with $\log_{\frac{1}{\epsilon}}(n)$ can be a bit unintuitive, we may show they are considered equivalent with:

$$\lim_{n\to\infty} \frac{\log(n^{\epsilon})}{\log_{\frac{1}{\epsilon}}(n)} = \lim_{n\to\infty} \frac{\epsilon \log_2 n}{\frac{\log_2 n}{\log_2(\frac{1}{\epsilon})}} = \lim_{n\to\infty} \epsilon \log_2(\frac{1}{\epsilon}) = c$$
$$\Rightarrow \log(n^{\epsilon}) \equiv \log_{\frac{1}{\epsilon}}(n) \tag{41}$$

Justification of $\log(n) \ll (\log(n))^a$

$$\lim_{n\to\infty} \frac{(\log(n))^a}{\log(n)} = \lim_{n\to\infty} (\log(n))^{a-1} = \infty$$
$$\Rightarrow \log(n) \ll (\log(n))^a \tag{42}$$

Considered $\log(n)$ is a generalized form of the previous group of functions, $(\log(n))^a$ is proven to be greater than the previous group.

Justification of $(\log n)^a \ll n^{\epsilon} \ll a^{\log_a(n)}$

$$\lim_{n\to\infty} \frac{\log((\log(n)^a))}{\log(n^{\epsilon})} = \lim_{n\to\infty} \frac{a \cdot \log((\log(n)))}{\epsilon \log(n)} = 0$$
$$\Rightarrow (\log n)^a \ll n^{\epsilon} \tag{43}$$

And it is apparent to see $n^{\epsilon} \ll a^{\log_a(n)}$ since $a^{\log_a(n)} = n$; as $\epsilon \in (0, 1)$, there will always be $n^{\epsilon} \ll n$.

Justification of $a^{\log_a(n)} \equiv \epsilon n \equiv \frac{n}{a}$

The above equations can be losely rewritten as:

$$a^{\log_a(n)} = n^1 = \Theta(n) \tag{44}$$
$$\epsilon n = \Theta(n) \tag{45}$$
$$\frac{n}{a} = \frac{1}{a} n = \Theta(n) \tag{46}$$

Thus, the above equality is justified.

Justification of $n^a \equiv (n + b)^a \ll (n + a)^b \ll n^{a+b}$

These are all polynomial functions with the greatest power > 1; thus they are considered greater than the previous group (functions with the greatest power $= 1$), and ordered according to ascending of *power(s)*.

Justification of $n^{a+b} \ll a^{\epsilon n}$

n^{a+b} is considered much greater than $a^{\epsilon n}$ due to its nature of being an exponential function.

Justification of $a^{\epsilon n} \ll a^n \ll b^n$

$$\begin{aligned} &\lim_{n\to\infty} \frac{a^{\epsilon n}}{a^n} = 0 \\ &\Rightarrow a^{\epsilon n} \ll a^n \end{aligned} \tag{47}$$

Also it is apparent to see that $a^n \ll b^n$ due to $1 < a < b$. Thus, the inequality is justified.