EECS 340: Assignment 2

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Problem 1

(a) $max\{f(n), g(n)\} = \Theta(f(n) + g(n))$

Since it is known that $f(n) \geq 0$, $g(n) \geq 0$, and $c > 0$; we must have:

$$f(n) \leq f(n) + g(n)$$
$$g(n) \leq f(n) + g(n)$$
$$\Rightarrow \max(f(n), g(n)) \in O(f(n) + g(n)) \quad \text{for } \begin{cases} c = 1 \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{1}$$

Since it is also known that $f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$, we may therefore infer:

$$\max(f(n), g(n)) \in \Omega(f(n) + g(n)) \quad \text{for } \begin{cases} c = \frac{1}{2} \\ \forall n_0 \in \mathbb{R} \end{cases} \tag{2}$$

Since both the O- and Ω-notation are established, we may therefore conclude:

$$\max(f(n), g(n)) \in \Theta(f(n) + g(n)) \tag{3}$$

(b1) $f(n) + d = O(f(n))$.

For the seek of disambiguation, we rewrite the questioned equation as $f(n) + d = O(f(n))$ by d replacing c for $d > 0$.

According to the defination of O-notation, we have:

$$
\begin{aligned}
f(n) &= O(f(n)) \\
\exists c, n_0 > 0 \ \text{ s.t. } 0 \le f(n) &\le cf(n) \quad \text{ for } n \ge n_0 \qquad (4)\\
\exists n' > 0 \ \text{ s.t. } f(n) &\ge f(n') \quad \text{ for } n, n' \ge n_0 \qquad (5)\\
\Rightarrow 0 \le f(n) + d &\le cf(n) + d \quad \text{for } n \ge n_0 \qquad (6)
\end{aligned}
$$

Due to *Equation 5*, we may rewrite *Euqation 6* as:

$$
0 \le f(n) + d \le (c + \frac{d}{f(n)})f(n) \tag{7}
$$

$$
\Rightarrow 0 \le f(n) + d \ \le \ c'f(n) \quad \text{for } \begin{cases} c' = c + \frac{d}{f(n')} \\ n, n' \ge n_0 \end{cases} \tag{8}
$$

Based *Euqation 8*, we may conclude $f(n) + d = O(f(n))$ via direct proof.

(b2) If $f(n) \ge 1$, then $f(n) + c = O(f(n))$.

Please refer to proof at **b1** as it provides a broader proof base on $f(n)$ regardless $f(n) \ge 1$ or not.

(c1) If $f(n) = O(g(n)), \log(f(n)) \ge 0$ and $\log(g(n)) \ge 0$, then $log(f(n)) = O(\log(g(n))$.

According to the defination of O-notation, we must have:

$$
\exists c, n_0 > 0 \ \text{ s.t. } \ f(n) \le cg(n) \quad \text{for } n \ge n_0 \tag{9}
$$

Since it is given that $\log(f(n)), \log(g(n)) \ge 0$, thus we must have:

$$
\begin{aligned}
\log(f(n)) &\le \log(c(g(n))) \quad \text{for } n \ge n_0 \\
\Rightarrow \log(f(n)) &\le \log c + \log(g(n)) \quad \text{for } n \ge n_0 \qquad (10)
\end{aligned}
$$

As c, n_0 are constants, there must be a constant c' s.t.

Case 1 Assume $\log(g(n_0)) \neq 0$:

$$c' \geq \frac{\log c}{\log(g(n_0))} + 1 \tag{11}$$

$$\Rightarrow (c'-1)\log(g(n)) \;\geq\; (c'-1)\log(g(n_0)) \;\geq\; \log c \quad \text{for } n \geq n_0 \tag{12}$$

$$\exists c, n_0 > 0 \;\; \text{s.t.}$$

$$\log(f(n)) \;\leq\; \log c + \log(g(n)) \;\leq\; (c'-1)\log(g(n)) + \log(g(n)) \quad \text{for } n \geq n_0 \tag{13}$$

$$\Rightarrow \log(f(n)) \leq c' \log(g(n)) \tag{14}$$

Thus we may conclude $log(f(n)) = O(\log(g(n))$ for this case.

Case 2 Assume $\log(g(n_0)) = 0$:

Since $\log(g(n_0)) = 0$, we shall infer that $g(n_0) = 1$. We may arbitrarily pick some constants c, c' where:

$$\log c \leq 0$$
$$\log c \leq (c'-1) \cdot 0$$
$$\log c \leq (c'-1) \cdot \log(g(n_0)) \tag{15}$$

Since $n \geq n_0$ by defination, and known that $g(n) \geq 1$ due to $\log(g(n)) \geq 0$; therefore there must be $g(n) \geq g(n_0)$. Putting this into the context of *Equation 10*, we may have:

$$\log(f(n)) \leq \log c + \log(g(n)) \quad \text{for } n \geq n_0$$
$$\Rightarrow \log(f(n)) \leq (c'-1) \cdot \log(g(n_0)) + \log(g(n))$$
$$\Rightarrow \log(f(n)) \leq (c'-1) \cdot \log(g(n)) + \log(g(n))$$
$$\Rightarrow \log(f(n)) \leq c' \log(g(n)) \tag{16}$$

Thus we may conclude $log(f(n)) = O(\log(g(n))$ for this case.

Since both cases reach to the conclusion of $log(f(n)) = O(\log(g(n))$, we have proven the statement to be valid.

(c2) If $f(n) = O(g(n)), \log(f(n)) \geq 0$ and $\log(g(n)) \geq 1$, then $log(f(n)) = O(\log(g(n))$.

According to the defination of O-notation, we must have:

$$\exists c, n_0 > 0 \;\; \text{s.t.} \quad f(n) \leq cg(n) \quad \text{for } n \geq n_0 \tag{17}$$

Since it is given that $\log(f(n)) \geq 0$, $\log(g(n)) \geq 1$, thus we must have:

$$
\begin{aligned}
&\log(f(n)) \leq \log(c(g(n))) \quad \text{for } n \geq n_0 \\
\Rightarrow &\log(f(n)) \leq \log c + \log(g(n)) \quad \text{for } n \geq n_0
\end{aligned} \tag{18}
$$

As c, n_0 are constants, there must be a constant c' s.t.

$$c' \geq \frac{\log c}{\log(g(n_0))} + 1 \tag{19}$$

$$\Rightarrow (c'-1)\log(g(n)) \;\geq\; (c'-1)\log(g(n_0)) \;\geq\; \log c \quad \text{for } n \geq n_0 \tag{20}$$

$$\exists c, n_0 > 0 \;\text{ s.t.}$$

$$\log(f(n)) \;\leq\; \log c + \log(g(n)) \;\leq\; (c'-1)\log(g(n)) + \log(g(n)) \quad \text{for } n \geq n_0 \tag{21}$$

$$\Rightarrow \log(f(n)) \leq c' \log(g(n)) \tag{22}$$

Thus we may conclude $log(f(n)) = O(\log(g(n))$, the statement is therefore proven to be valid.

Problem 2