MATH 307: Individual Homework 4

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Problem 1

Textbook page 40, problem 1.

For W to be a vector space, we must have $(u+v) \in W$ for $u, v \in W$. However for $u = v = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$, we have $(u+v) = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$; which is $\notin W$ as $|x_j| \nless 3$ for $i \leq j \leq 4$. Thus, we may conclude that W is not a vector space.

Problem 2

Textbook page 40, problem 5.

For $W \in \mathbb{R}^3 : x + 20y - 12z - 1 = 0$ for x, y, z as elements of W to be a vector space, we must have a zero vector 0 where $0 + u = u$ for $u \in W$. In this case the zero vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ but it is $\notin W$ as $0 + 20(0) - 12(0) - 1 \neq 0$. Thus, we may conclude that W is not a vector space.

Problem 3

See HW instruction.

For matrices $M, N, U \in F^{n \times n}$, we know that $tr(M) = tr(N) = tr(U) = 0$. Since the trace of a matrix is only about its diagonal elements, lets assume we have the $tr(M) = M_{11} + M_{22} + ... + M_{nn} = 0$, $tr(N) = N_{11} + N_{22} + ... + N_{nn} = 0$, and same for $tr(U)$; where the subscript is the index of element. Also assume we have scalar $\lambda, \mu \in F$, We have:

1. $M + N \in F^{n \times n}$ as $tr(M+N) = (M_{11} + N_{11}) + (M_{22} + N_{22}) + ... + (M_{nn} + N_{nn}) = 0$.
2. $M + N = N + M = \begin{bmatrix} M_{11} + N_{11} & M_{12} + N_{12} & \cdots & M_{1n} + N_{1n} \\ M_{21} + N_{21} & M_{22} + N_{22} & \cdots & M_{2n} + N_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} + N_{n1} & M_{n2} + N_{n2} & \cdots & M_{nn} + N_{nn} \end{bmatrix}$

3. $U+(M+N) = (U+M)+N = \begin{bmatrix} M_{11}+N_{11}+U_{11} & M_{12}+N_{12}+U_{12} & \cdots & M_{1n}+N_{1n}+U_{1n} \\ M_{21}+N_{21}+U_{21} & M_{22}+N_{22}+U_{22} & \cdots & M_{2n}+N_{2n}+U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1}+N_{n1}+U_{n1} & M_{n2}+N_{n2}+U_{n2} & \cdots & M_{nn}+N_{nn}+U_{nn} \end{bmatrix}$

4. $0 \in F^{n\times n}$ where $0 + M = M$. We know this is true as as the element on ij index on LHS is M_{ij} which is equals to the ij-indexed element on RHS. We also know that a $n \times n$ matrix filled with 0s is $\in F^{n\times n}$.

5. $\forall M \in F^{n\times n}$, we have $M + -M = 0$ where the element on ij index in $-M$ is simply $-1 \cdot M_{ij}$, so we must have $M_{ij} + (-M_{ij}) = 0$ and therefore $M + -M = 0$.

6. $\lambda M \in F^{n\times n}$ as the element on ij index in λM is simply $\lambda \cdot M_{ij}$ which is still in F, we then have $\lambda M \in F^{n\times n}$.

7. $\lambda(M + N) = \lambda M + \lambda N$ as the element on ij index on LHS is $\lambda(M_{ij} + N_{ij}) = \lambda M_{ij} + \lambda N_{ij}$, which is equals to the ij-indexed element on RHS.

8. $(\lambda + \mu)M = \lambda M + \mu M$ as the element on ij index on LHS is $(\lambda + \mu)M_{ij} = \lambda M_{ij} + \mu M_{ij}$, which is equals to the ij-indexed element on RHS.

9. $\lambda(\mu M) = (\lambda\mu)M$ as the element on ij index on LHS is $\lambda \cdot \mu \cdot M_{ij}$ which is equals to the ij-indexed element on RHS.

10. $1 \cdot M = M$ as the element on ij index on LHS is $1 \cdot M_{ij}$ which is equals to the ij-indexed element on RHS.

As all ten axioms are proven to be valid, we may conclude that $F^{n\times n}$ is vector field over F.