MATH 307: Individual Homework 8

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Due and submitted on 03/03/2021
Spring 2021, Dr. Guo

Problem 1

See HW instruction.

For the ease of expression let the following $\|x\|$ be $\|x\|_1$ in this problem.

We know $\sum_i^n |x_i|$ is a norm as it shows *non-negativity* as we always have $\sum_i^n |x_i| \geq 0$ as $|x_i| = \sqrt{Re(x_i)^2 + Im(x_i)^2} \geq 0$.

It also shows *scaling* as for $\|\lambda x\|$ for all $\lambda \in \mathbb{R}$ we have:

$$
\begin{aligned}
\|\lambda x\| &= \sum_i^n |\lambda x_i| \\
&= \sum_i^n \left| \lambda(\sqrt{Re(x_i)^2 + IM(x_i)^2}) \right| \\
&= \lambda \sum_i^n |x_i| = |\lambda| \, \|x\|
\end{aligned}
$$

For *Triangle inequality*, assume we have a $y = [y_1, y_2, y_3, \ldots, y_n]$ for $y_i \in \mathbb{C}$. We must have:

$$
\begin{aligned}
\|x + y\| = \left\| \sum_i^n |x_i + y_i| \right\| &= \sum_i^n |x_i + y_i| \\
\sum_i^n |x_i + y_i| \leq \sum_i^n |x_i| + |y_i| &= \|x\| + \|y\| \\
\|x + y\| \leq \|x\| + \|y\|
\end{aligned}
$$

As all conditions for norm are proven, we may confirm $\sum_i^n |x_i|$ is a norm.

Problem 2

See HW instruction.

For the ease of expression we let $|x_k|$ to have the maximum value among all $|x_i|$. We first to show $\|x\|_\infty \leq \|x\|_1$:

$$
\begin{aligned}
\|x\|_\infty &= |x_k| \\
\|x\|_1 &= \sum_i^n |x_i| = |x_1| + |x_2| + \cdots + |x_k| + \cdots + |x_n| \\
\text{Since all } &|x_i| \geq 0 \\
\implies \|x\|_\infty &\leq \|x\|_1
\end{aligned}
$$

Now to show that $\|x\|_1 \leq n \|x\|_\infty$:

$$
\begin{aligned}
\|x\|_1 &= \sum_i^n |x_i| = \underbrace{|x_1| + |x_2| + \cdots + |x_k| + \cdots + |x_n|}_{n \text{ elements}} \\
n \|x\|_\infty &= n \cdot |x_k| \\
\text{Since } &|x_k| \geq |x_i| \text{ for all } |x_i| \\
\implies \|x\|_1 &\leq n \|x\|_\infty
\end{aligned}
$$

By combining the above two findings together we have $\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$. Thus, the 1 and ∞ norm are equivalent in $\mathbb{C}^n$.