MATH 307: Group Homework 2

Group 8
Shaochen (Henry) ZHONG, Zhitao (Robert) CHEN, John MAYS, Huaijin XIN
{sxz517, zxc325, jkm100, hxx200}@case.edu

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Problem 1

Prove that $\mathbb{C}$, the set of all complex valued numbers with the regular addition and multiplication of complex numbers is a field.

For the simplicity of discussion, say we have three complex numbers: $z_1, z_2, z_3 \in \mathbb{C}$ that can be represented respectively as:

$$z_1 = a + ib$$
$$z_2 = c + id$$
$$z_3 = e + if$$

where $i^2 = -1$ and $a, b, c, d, e, f \in \mathbb{R}$.

$(\mathbb{C}, +, \times)$ is a field, because it satisfies all of the conditions for a field:

1. $(\mathbb{C}, +)$ **is an Abelian group:**

1. Closure: $\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 = (a + ib) + (c + id) = (a + b) + i(c + d) \in \mathbb{C}$

2. Associativity: $\forall z_1, z_2, z_3 \in \mathbb{C}, z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
   Proof:
   $z_1 + (z_2 + z_3) = a + ib + ((c + e) + i(d + f)) = (a + c + e) + i(b + d + f)$
   $(z_1 + z_2) + z_3 = ((a + c) + i(b + d)) + e + if = (a + c + e) + i(b + d + f)$

3. Identity: $\forall z \in \mathbb{C}, \exists id = 0 + i0$ s.t $z + id = id + z = z$
   Proof:
   $z + (0 + i0) = (a + 0) + i(b + 0) = a + ib = z$
   $(0 + i0) + z = (0 + a) + i(0 + b) = a + ib = z$

4. Inverse: $\forall z \in \mathbb{C}, \exists - z$ s.t $z + (-z) = 0 + i0$
   Proof:
   $z + (-z) = a + ib - (a + ib) = (a - a) + i(b - b) = 0 + 0i$

5. Commutativity: $\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 = z_2 + z_1$
   Proof:

$z_1 + z_2 = a + ib + c + id = (a + c) + i(b + d)$
$z_2 + z_1 = c + id + a + ib = (a + c) + i(d + b)$

2. $\mathbb{C}, \times$ **is a commutative monoid:**

1. Closure: $\forall z_1, z_2 \in \mathbb{C}, z_1 \times z_2 = (a + ib) \times (c + id) = (ac - bd) + i(bc + ad) \in \mathbb{C}$

2. Associativity: $\forall z_1, z_2, z_3 \in \mathbb{C}, z_1 \times (z_2 \times z_3) = (z_1 \times z_2) \times z_3$
   Proof:
   $z_1 \times (z_2 \times z_3) = (a + ib) \times ((c + id) \times (e + if) = (a + ib) \times ((ce - df) + i(de + cf)) = (ace - adf - bde - bcf) + i(bce - dfb + ade + acf)$
   $(z_1 \times z_2) \times z_3 = ((a + ib) \times (c + id)) \times (e + if) = ((ac - bd) + i(bc + da)) \times (e + if) = (eac - ebd - fbc - fda) + i(bce + dae + fac - fbd)$

3. Identity: $\forall z \in \mathbb{C}, \exists id = 1 + i0$ s.t $z \times id = id \times z = z$
   Proof:
   $z \times id = (a + ib) \times (1 + i0) = (a - 0) + i(b + 0) = a + ib = z$
   $id \times z = (1 + i0) \times (a + ib) = (a - 0) + i(b + 0) = a + ib = z$

4. Commutativity: $\forall z_1, z_2 \in \mathbb{C}, z_1 \times z_2 = z_2 \times z_1$
   Proof:
   $z_1 \times z_2 = (a + ib) \times (c + id) = (ac - bd) + i(cb + da)$
   $z_2 \times z_1 = (c + id) \times (a + ib) = (ac - bd) + i(cb + da)$

3. $\forall z \in \mathbb{C} \backslash \{0 + i0\}, \exists z^{-1}$ **s.t.** $z \times z^{-1} = 1$

Proof:
Assume $z = a + ib$:
$z^{-1} = \frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{(a+ib)\times(a-ib)} = \frac{a-ib}{a^2+b^2}$
Now, $z \times z^{-1} = a + ib \times \frac{a-ib}{a^2+b^2} = \frac{(a-ib)\times(a+ib)}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$

4. $\times$ **is distributive over** $+$ **s.t.** $\forall z_1, z_2, z_3 \in \mathbb{C}$, **it holds that** $z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$

Proof:
$z_1 \times (z_2 + z_3) = (a + ib) \times ((c + id) + (e + if)) = (a + ib) \times ((c + e) + i(d + f)) = (ac + ae - bd - bf) + i(bc + be + ad + af) = (ac - bd) + i(bc + ad) + (ae - bf) + i(be + fa) = (a + ib) \times (c + id) + (a + ib) \times (e + if) = z_1 \times z_2 + z_1 \times z_3$

Problem 2

Compute the multiplicative inverse of $z = 3 - 7i$ and express it in the form $a + ib$.

Known that for $a + bi \neq 0$, we have its multiplicative inverse being $\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.

$$\begin{aligned}(3-7i)^{-1} &= \frac{3}{3^2+7^2} - i\frac{-7}{3^2+7^2} \\ &= \frac{3}{58} + i\frac{7}{58}\end{aligned}$$

Problem 3

Write the complex number $z = -1 - i$ in polar form, then compute z^4.

We have the polar form of $z = -1 - i$ being:

$$\begin{aligned}z &= \sqrt{1^2+1^2}\left(\frac{-1}{1^2+1^2} - i\frac{-1}{1^2+1^2}\right) \\ &= \sqrt{2}(\cos(\pi + \tan^{-1}(\frac{-1}{-1})) + i\sin(\pi + \tan^{-1}(\frac{-1}{-1}))) \\ &= \sqrt{2}(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}))\end{aligned}$$

Now to calculate z^4 with De Moivre's formula:

$$\begin{aligned}z^4 &= (\sqrt{2})^4(\cos(4\cdot\frac{5\pi}{4}) + i\sin(4\cdot\frac{5\pi}{4})) \\ &= 4(\cos 5\pi + i\sin 5\pi) \\ &= 4(-1+0) = -4\end{aligned}$$

Problem 4

Using the polar form, show that the product of a complex number $z = a + ib$ and its complex conjugate must be a real number greater than or equal to zero.

For a complex number $z = a + ib$, let $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$, we may express its as polar form as $r(\cos\theta + i\sin\theta)$.

Then we have:

$$\begin{aligned}\overline{z} &= \overline{r}\cdot\overline{(\cos\theta + i\sin\theta)} \\ &= r\cdot\overline{(\cos\theta + i\sin\theta)} \\ &= r(\cos\theta - i\sin\theta)\end{aligned}$$

Then for $z\cdot\overline{z}$, we have:

$$
\begin{aligned}
z \cdot \overline{z} &= r(\cos\theta + i\sin\theta) \cdot r(\cos\theta - i\sin\theta) \\
&= (r\cos\theta + ri\sin\theta) \cdot (r\cos\theta - ri\sin\theta) \\
&= r^2(\cos\theta)^2 - r^2\cos\theta i\sin\theta + r^2 i\sin\theta\cos\theta - r^2(i\sin\theta)^2 \\
&= r^2(\cos\theta)^2 + r^2(\sin\theta)^2 \\
&= r^2
\end{aligned}
$$

As there is no imaginary part in the resultant equation, and as $r^2 = (a^2 + b^2)$ is always ≥ 0, we may conclude that $z \cdot \overline{z}$ must be a real number greater than or equal to zero