MATH 307: Group Homework 3

Group 8
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Problem 1

For $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $w = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. We may obtain $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -w$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = v + w$. So for any arbitrary $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ we have:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= yu + (y - x)w \\ &= y \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (y - x) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{aligned}$$

Thus, we have $span(u, v, w) \subset \mathbb{R}^2$.
For any arbitrary vector $d \in span(u, v, w)$, we have:

$$\begin{aligned} d &= \lambda_1 u + \lambda_2 v + \lambda_3 w \\ &= \begin{pmatrix} \lambda_1 + \lambda_2 - \lambda_3 \\ 2\lambda_1 + \lambda_2 \end{pmatrix} \end{aligned}$$

where $d \in \mathbb{R}^2$, thus we have $\mathbb{R}^2 \subset span(u, v, w)$ and therefore $\mathbb{R}^2 = span(u, v, w)$

Problem 2

$\forall p \in P^4 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ for $a \in \mathbb{R}$, we will have a $span(-x^4, x^3, -x^2, x, -1) \subset P^4$.

$$-a_0 \cdot (-1) + a_1 x - a_2(-x^2) + a_3 x^3 - a_4(-x^4)$$

which is equals to such p, which implies $span(-x^4, x^3, -x^2, x, -1) \subset P^4$.

$\forall q \in span(-x^4, x^3, -x^2, x, -1)$, we may express them as $q = b_4(-x^4) + b_3(x^3) + b_2(-x^2) + b_1(x) + (b_0)(-x^0)$ which is clearly in $\in P^4$, so we also have $P^4 \subset span(-x^4, x^3, -x^2, x, -1)$ and therefore $P^4 = span(-x^4, x^3, -x^2, x, -1)$.

Problem 3

(a)

No, it is linearly dependent as we may have:

$$0 = \lambda_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} + \lambda_2 \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$
$$\begin{cases} \lambda_1 + \lambda_2 - \lambda_2 i = 0 \\ \lambda_1 + \lambda_1 i - 2\lambda_2 = 0 \end{cases}$$
$$\implies \begin{cases} \lambda_1 = -3 + i \\ \lambda_2 = 2 + i \end{cases}$$

for $\lambda_1, \lambda_2 \in \mathbb{C}$, where the coefficients are not zeros.

(b)

No, it is linearly dependent as we may have:

$$0 = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\implies \lambda_1 = -\lambda_2 = \lambda_3$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, where the coefficients are not zeros.

(c)

No, it is linearly dependent as we may have:

$$0 = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \lambda_4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$
$$\implies \begin{cases} \lambda_1 & = -7_4 \\ \lambda_2 & = -4\lambda_4 \\ \lambda_3 & = \lambda_4 \end{cases}$$

Assume $\lambda_4 = 1$, we have $\lambda_3 = 4, \lambda_2 = -4, \lambda_1 = -7$ which is a non zero solution of the system.

Problem 4

For a vector space of $\mathbb{C}^{2\times3}$, we may have an arbitrary matrix like $\begin{bmatrix} a+bi & e+fi & j+ki \\ c+di & g+hi & l+mi \end{bmatrix}$ for $a, b, ..., h, j, ..., m \in \mathbb{R}$. So by having a the following system:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\cdots,\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}}_{\text{6 matrices}}$$

We first know they are linearly independent as to have $\lambda_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+\lambda_2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}+\cdots+\lambda_6\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}=0$ we must have $\lambda_1=\lambda_2=...=\lambda_6=0$.

Then for spanning, by simply assigning the scalar mutipliers of $(a+bi),(c+di),\cdots,(l+mi)$ to these matrices respectively, we shall produce any $\begin{bmatrix}a+bi&e+fi&j+ki\\c+di&g+hi&l+mi\end{bmatrix}\in\mathbb{C}^{2\times3}$. Thus, the proposed system is a basis of $\mathbb{C}^{2\times3}$.

Problem 5

Yes, first we know they are linearly independent as we may only have:

$$\begin{aligned}0&=\lambda_1\begin{bmatrix}1\\i\end{bmatrix}+\lambda_2\begin{bmatrix}0\\1\end{bmatrix}\\&=0\begin{bmatrix}1\\i\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

where the coefficients $\lambda_1=\lambda_2=0$.

Then to prove they are spanning over $\mathbb{C}^2$, we may have any arbitrary $\begin{bmatrix}a+bi\\c+di\end{bmatrix}$ for $a,b,c,d\in\mathbb{R}$. This means if we may individually produce $(1,0),(0,1),(i,0),(0,i)$, we shall simply put a,c,b,d as their scalar mutipliers respectively then add them together, we have any possible $\begin{bmatrix}a+bi\\c+di\end{bmatrix}$.

$$\begin{aligned}\begin{bmatrix}0\\1\end{bmatrix}&=\begin{bmatrix}0\\1\end{bmatrix}\\\begin{bmatrix}1\\0\end{bmatrix}&=\begin{bmatrix}1\\i\end{bmatrix}-i\begin{bmatrix}0\\1\end{bmatrix}\\\begin{bmatrix}i\\0\end{bmatrix}&=i\begin{bmatrix}1\\i\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}i\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\\\begin{bmatrix}0\\i\end{bmatrix}&=i\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

Thus $\begin{bmatrix}1\\i\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$ is also spanning $\mathbb{C}^2$ and therefore a basis of it.