# MATH 307: Individual Homework 2

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## Problem 1

*Prove that $Q(x)$, the set of all polynomials with rational coefficients with the regular polynomial multiplication and addition is a ring.*

For the simplicity of discussion, let's assume we have $f, g, k \in Q(x)$ with $f = a_0 + a_1x + a_2x^2 + ... + a_mx^m$, $g = b_0 + b_1x + b_2x^2 + ... + b_nx^n$, and $k = c_0 + c_1x + c_2x^2 + ... + c_jx^j$.
We have $(Q(x), +, 0)$ to be an Abelian group as:

- It is a closure as for $f + g = a_0 + b_0 + a_1x + b_1x + ... + a_mx^m + b_nx^n$ is also a polynomial with rational coefficient and therefore also $\in Q(x)$.
- It shows associativity as for $f, g, k \in Q(x)$, $f + (g + k) = (f + g) + k$.
- It has the (additive) identity of 0 for $f + 0 = f$.
- It has the inverse of $-f$ as $f + (-f) = 0$.
- It also shows commutativity with $f + g = g + f$.

On the other hand we have $(Q(x), \times, 1)$ to be a monoid as:

- It is a closure as we have $f \cdot g = a_0b_0 + a_1b_1x^2 + ... + a_mb_nx^{mn}$ to be a polynomial with rational coefficient and therefore also $\in Q(x)$.
- It shows associativity as for $f, g, k \in Q(x)$, $f \cdot (g \cdot k) = (f \cdot g) \cdot k$.
- It has the (multiplicative) identity of 1 for $f \cdot 1 = f$.

Now to check the distributive property, for $f \cdot (g + h)$ we have $a_0b_0 + a_1b_1x^2 + ... + a_mb_nx^{mn} + a_0c_0 + a_1c_1x^2 + ... + a_mc_jx^{mj} = (f \cdot g) + (f \cdot h)$. So the distributive property is proven and $(Q(x), +, \times)$ is therefore a ring.

## Problem 2

*Is $\mathbb{Z}$, the set of integers with the usual addition and multiplication, a field? Justify your answer.*

$(\mathbb{Z}, +, \times)$ is not a field. First it is clear that $(\mathbb{Z}, +, 0)$ is supposed to be the Abelian group and $(\mathbb{Z}, \times, 1)$ is supposed to be the commutative monoid – as we can't have a multiplicative inverse for every integers $\in \mathbb{Z}$. Which implies for $(\mathbb{Z}, +, \times)$ to be a field, it is required that every element in $\mathbb{Z}$ which is not the additive inverse (0) to have an inverse with respect to $\times$. But the only multiplicative inverse that are $\in \mathbb{Z}$ are $-1$ and 1. Say if we have 2, which is an integer that is not 0, but we can't have its multiplicative inverse $\frac{1}{2}$ to be $\in \mathbb{Z}$. Thus, $(\mathbb{Z}, +, \times)$ is not a field.