MATH 307: Individual Homework 5

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Problem 1

See HW instruction.

(a)

We know the purposed set is linearly independent as each of element of the set has a different power of x, and we can't raise or lower the power of x with scalar multiplications.

However the purposed set is not a spanning set of P^4 as we cannot represent x^4.

We may confirm this answer knowing that P^4 admits a finite basis of 5 vectors along the line of $\{x^0, x, x^2, x^3, x^4\}$; the purposed set only has 4 vectors and is therefore not a basis for P^4.

(b)

Again, we know the purposed set is linearly independent as each of element of the set has a different (highest) power of x. It is also a spanning set as we have:

$$\begin{aligned}
1 &= 1 \\
x &= (1 + x) - 1 \\
x^2 &= (1 + x + x^2) - (1 + x) \\
x^3 &= (x^2 + x^3) - x^2 \\
x^4 &= -(x^3 - x^4) + x^3
\end{aligned}$$

Then $\forall p \in P^4 = a + bx + cx^2 + dx^3 + ex^4$ we simply times $a, b, c, d \in \mathbb{R}$ to each of the above listed element respectively we may have a representation of p.

We may confirm this answer knowing that P^4 admits a finite basis of 5 vectors; and 5 vectors are provided in the purposed set.

(c)

Again, we know the purposed set is linearly independent as each of element of the set has a different (highest) power of x. It is also a spanning set as we have:

$$
\begin{aligned}
1 &= -(-1) \\
x &= x \\
x^2 &= -(-x^2) \\
x^3 &= x^3 \\
x^4 &= -(-x^4)
\end{aligned}
$$

Then $\forall p \in P^4 = a + bx + cx^2 + dx^3 + ex^4$ we simply times $a, b, c, d \in \mathbb{R}$ to each of the above listed element respectively we may have a representation of p.

We may confirm this answer knowing that P^4 admits a finite basis of 5 vectors; and 5 vectors are provided in the purposed set.

(d)

The purposed set is not linearly independent as we may have:

$$x^2 - 5 = (x^2 - x) + (x + 10) - 3(5)$$

and it is therefor also not a basis for P^4.

We may confirm this answer knowing that P^4 admits a finite basis of 5 vectors along the line of $\{x^0, x, x^2, x^3, x^4\}$; the purposed set only has 6 vectors and is therefore not a basis for P^4.

Problem 2

Textbook page 40, problem 6.

No. Assume we have the basis of R^4 being $\{v_1, v_2, v_3, v_4\}$ since knowing its dimension; and the supposely the 5 linearly independent vectors are $\{w_1, w_2, w_3, w_4, w_5\}$. By the EXCHANGE THEROME we know that we may swap a $v \in \{v_1, v_2, v_3, v_4\}$ with a $w \in \{w_1, w_2, w_3, w_4, w_5\}$ by doing the following the following four times (example showed for swapping v_1 for w_1):

$$
\begin{aligned}
w_1 &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\
\text{Assume } a_1 \neq 0 v_1 &= a_1^{-1}(w_1 - a_2 v_2 - a_3 v_3 - a_4 v_4) \\
v_1 \in span(w_1, v_2, v_3, v_4)
\end{aligned}
$$

Then we got $\{w_1, w_2, w_3, w_4\}$ to be a span of $\mathbb{R}^4$ and w_5, as it is also $\in \mathbb{R}^4$, must be linearly dependent to $\{w_1, w_2, w_3, w_4\}$.

Problem 3

Textbook page 40, problem 7.

Yes. as $\mathbb{R}^6$ has a dimension of 6, it must has a 6-vector basis where any 5 of them will be linearly independent. A simple example will be:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Problem 4

See HW instruction.

The dimension of $\mathbb{C}^{3\times 2}$ is 6 as it will need a 6-vector basis: $e_{11}, e_{12}, e_{21}, ..., e_{32}$ so for any $a + bi$ on any of the ij-th index, we may represent it with $a \cdot e_{ij} + bi \cdot e_{ij}$.

The proposed set is not a basis of $\mathbb{C}^{3\times 2}$ because the last element $e_{32} - e_{11}$ is not linearly independent to the othere while e_{32} and e_{11} is individually included in such set. Also basis of a vector space is an invariant equals to the vector space's dimension – in this case it shuold be 6 – but the proposed set got 7 vectors.