MATH 307: Individual Homework 10

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Problem 1

Textbook page 56, problem 6.

$$
\begin{aligned}
P_{v_1}(v_4) &= < v_4, \frac{v_1}{\|v_1\|} > \frac{v_1}{\|v_1\|} \\
&= < \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}{\sqrt{6}} > \cdot \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}{\sqrt{6}} = \frac{3}{\sqrt{6}} \cdot \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}{\sqrt{6}} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \\
&= [\frac{-1}{2}, \frac{1}{2}, 0, 1]
\end{aligned}
$$

$$
\begin{aligned}
P_{v_4}(v_1) &= < v_1, \frac{v_4}{\|v_4\|} > \frac{v_4}{\|v_4\|} \\
&= < \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} > \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{3}{\sqrt{3}} \cdot \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} \\
&= [0, 1, 1, 1]
\end{aligned}
$$

Problem 2

See HW instruction.

f	1	1	1	x	x	x	x^2	x^2	x^2
g	1	x	x^2	1	x	x^2	1	x	x^2
$< f, g >$	$\int_0^1 1dx$	$\int_0^1 xdx$	$\int_0^1 x^2dx$	$\int_0^1 xdx$	$\int_0^1 x^2dx$	$\int_0^1 x^3dx$	$\int_0^1 x^2dx$	$\int_0^1 x^3dx$	$\int_0^1 x^4dx$

By inspecting the $< f, g >$ of possible combinations, we have $\int_0^1 x^i dx$ for $i \in \{0, 1, 2, 3, 4\}$, which will yield results of x, $\frac{1}{2}x^2$, $\frac{1}{3}x^3$, $\frac{1}{4}x^4$, $\frac{1}{5}x^5$ respectively. It is clear that they are linearly independent as each resultant polynomial has a x of different power and we must have $\lambda_1 x + \lambda_2 \frac{1}{2}x^2 + \lambda_3 \frac{1}{3}x^3 + \lambda_4 \frac{1}{4}x^4 + \lambda_5 \frac{1}{5}x^5 = 0$ for all $\lambda = 0$.

Now to show they are not orthogonal, we may simply take an example, say $\int_0^1 1dx$, which equals to $1 \neq 0$ and therefore not orthogonal. In fact, all of these inner products are not orthogonal as they will have a result of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ respectively to the abovementioned orders.

Problem 3

See HW instruction.

$$
\begin{aligned}
P_{v_3}(v_1) &=< v_1, \frac{v_3}{\|v_3\|} > \frac{v_3}{\|v_3\|} \\
&=< \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} > \cdot \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= [\frac{1}{2}, \frac{1}{2}, 0]
\end{aligned}
$$

$$
\begin{aligned}
P_{v_3}(v_2) &=< v_2, \frac{v_3}{\|v_3\|} > \frac{v_3}{\|v_3\|} \\
&=< \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} > \cdot \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= [\frac{1}{2}, \frac{1}{2}, 0]
\end{aligned}
$$

Due to the linearity of inner product we must have:

$$
\begin{aligned}
P_{v_3}(2v_1 + v_2) &= 2 \cdot P_{v_3}(v_1) + P_{v_3}(v_2) \\
&= 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = [\frac{3}{2}, \frac{3}{2}, 0]
\end{aligned}
$$

Problem 4

See HW instruction.

To find the basis e_1, e_2 based on Gram-Schmidt, we denote $v_1 = 1$ and $v_2 = x$, where $e_1 = \frac{v_1}{\|v_1\|} = 1$.

Now to find e_2 base on v_2 for $e_2 = v_2 - p_{e_1}(v_2)$:

$$
\begin{aligned}
e_2 &= \frac{v_2 - < v_2, e_1 > e_1}{\|v_2 - < v_2, e_1 > e_1\|} \\
&= \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2}} \\
&= \sqrt{12}(x - \frac{1}{2})
\end{aligned}
$$

Since V has a dimension of 2 and we have 1 and $\sqrt{12}(x - \frac{1}{2})$ by the Gram-Schmidt, they are the orthonormal basis of V.